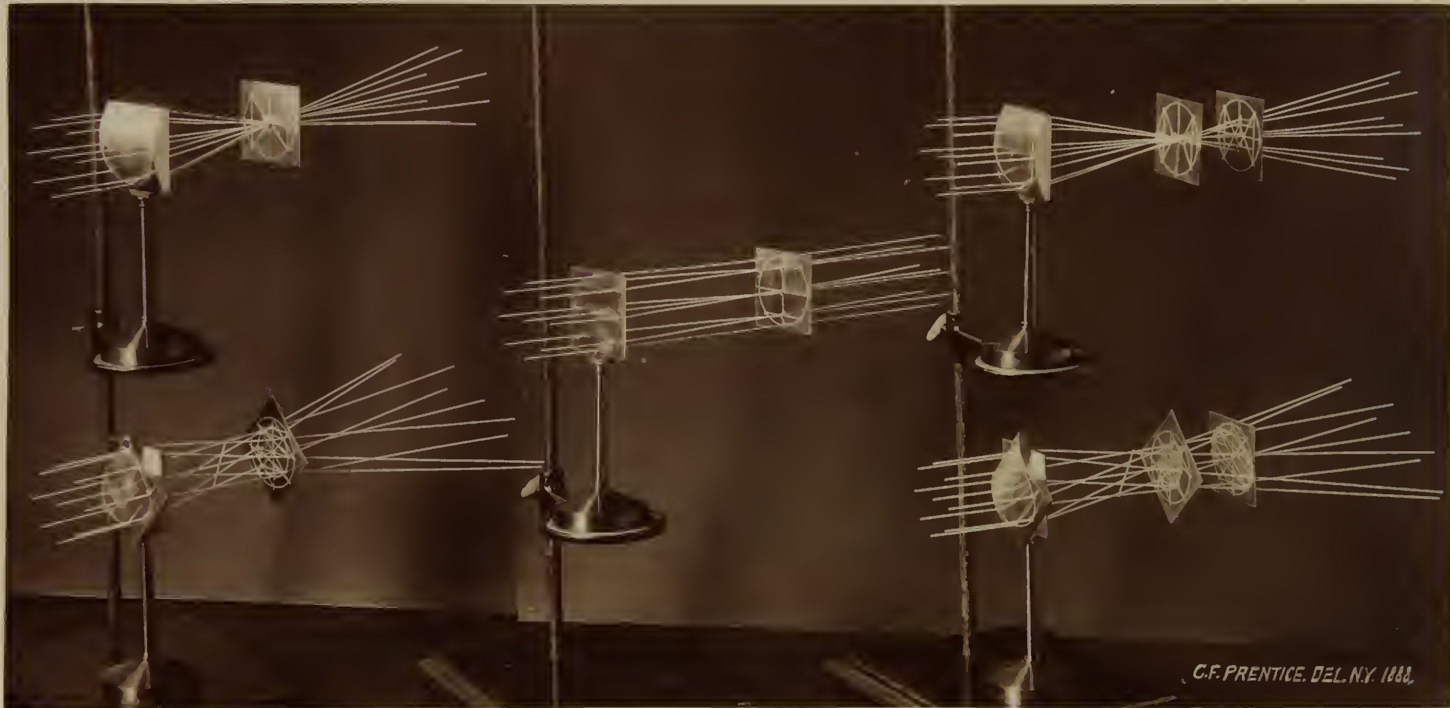
# DIOPTRIC FORMULÆ
# CYLINDRICAL LENSES

PRENTICE

DR. SWAN M. BURNETT'S MODELS, DEMONSTRATIVE OF CYLINDRICAL REFRACTION.




Equal Cylinders, 10 Dioptres, axial deviation 90°. Unequal Cylinders, 7 and 10 Dioptres, axial deviation 90°.

Plano-convex Cylinder, 7 Dioptres, axis vertical.

Equal Cylinders, 10 Dioptres, axial deviation 45°. Unequal Cylinders, 7 and 10 Dioptres, axial deviation 45°.

# DIOPTRIC FORMULÆ

FOR COMBINED

# CYLINDRICAL LENSES

APPLICABLE FOR

ALL ANGULAR DEVIATIONS OF THEIR AXES

With Six Original Diagrams and One Albertype Plate

BY

CHAS. F. PRENTICE

AUTHOR OF

"A TREATISE ON SIMPLE AND COMPOUND OPHTHALMIC LENSES"



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TO

DR. SWAN M. BURNETT,

PROFESSOR OF OPHTHALMOLOGY AND OTOLOGY IN THE UNIVERSITY OF GEORGETOWN;
OPHTHALMIC AND AURAL SURGEON TO THE GARFIELD HOSPITAL, AND
DIRECTOR OF THE OPHTHALMIC AND AURAL CLINIC AT
THE CENTRAL DISPENSARY AND EMERGENCY
HOSPITAL, WASHINGTON, D. C.,
IN GRATEFUL RECOGNITION OF HIS GENEROUS APPRECIATION
OF MY PREVIOUS EFFORTS, AND TO

DR. RICHMOND LENNOX,

ASSISTANT SURGEON TO THE BROOKLYN EYE AND EAR HOSPITAL,
AS A TOKEN OF MY REGARD FOR HIS VALUABLE TEACHINGS IN OPHTHALMOSCOPY,
THESE DIOPTRIC FORMULÆ ARE RESPECTFULLY
DEDICATED.

# PREFACE.

SHORTLY after publication of my "Treatise on Ophthalmic Lenses," Dr. Swan M. Burnett, of Washington, D. C., kindly suggested the execution of plastic models of combined cylindrical lenses, by placing a set of these, conceived and hastily prepared by himself, in my hands for further elaboration; with the request, if possible, also to produce two combinations in which the cylinders were to be united at angles other than right angles. As the result of my research, during the time devoted to the construction of the latter more especially, and with a view to establish confidence in the precision of these models, this mathematical demonstration is presented.

For convenience of reference, the subject has been divided under seemingly appropriate headings, liberty being taken to introduce the qualifying terms—*congeneric,* as implying cylinders of the same class, both being convex or concave, and *contra-generic,* coined by myself to designate cylinders of the opposite class, convex and concave.

In the theorem for combined congeneric cylinders, the full reduction of the formulæ is given, it being deemed sufficient, in the second theorem, merely to indicate the means by which the results have been obtained.

For the benefit of those indisposed to follow the subject in all its details, it has been thought befitting to append a series of values, calculated by the formulæ, which the reader may also easily verify by practical experiment.

While the diagrams have been prepared with great care, yet they are somewhat at variance with the laws of true perspective, it being my object, in the interest of greater clearness, to strictly preserve all important circles and right angles referred to in the text. Two of the plates have been printed upon detached cards to facilitate reference. A careful study of these diagrams is urgently advised, since it is to my truthful conception of them I so largely attribute my success in presenting these general formulæ, which, to my knowledge, are the first to be advanced as containing the known quantities of cylindrical foci and axial deviation *only*.

A more simple and convenient form may ultimately be given the formulæ, though as here presented it is believed they will prove sufficiently adequate when their limited application is considered. Their transformations, as adapted to the requirements of the metric system, which are given at the close, are also believed to suffice in expression of their terms in refraction.

The text having been somewhat hastily prepared, I feel obliged to ask the reader's kind indulgence for its deficiencies, in the hope that others, in the future, may give this subject, which contains so many points of interest hitherto unpublished, that consideration of which it is deserving.

This first publication is therefore confined to an exceedingly limited edition, particularly as it is likely to prove comprehensive and of advantage only to ophthalmic surgeons.

Suspecting my attempt to instruct, while in the capacity of an optician, may call forth unusual criticism, I trust the same will be mitigated when it is known that this effort is based upon the mere recollections of my earlier mathematical studies in Germany, which were prematurely terminated while in pursuit of a technical profession.

CHAS. F. PRENTICE.

NEW YORK, May, 1888.

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FOR COMBINED

# CONGENERIC CYLINDRICAL LENSES.

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## 1. RELATIVE POSITIONS OF THE PRIMARY AND SECONDARY PLANES OF REFRACTION.

In the following theorems, a prior knowledge of the established mathematical deductions applied to lenses, for parallel rays incident in the immediate vicinity of the optical axis, and in which the lenses' thicknesses are considered vanishing quantities in proportion to the focal distances, is taken for granted; as the formulæ here advanced are to be considered dependent upon those which have not been carried beyond *first approximations*. Practically, in almost all cases that occur, the thicknesses of the combined lenses are very small quantities compared to the other dimensions involved, so that we shall consider the cylinders to be so thin that their centres may be supposed to coincide, and in which case the focal distances are to be counted from a plane perpendicular to the optical axis, in the optical centre of the combined lenses.

In Plate I, two combined convex cylindrical lenses are shown, which, while somewhat at variance with the prescribed conditions of thickness, will, however, better serve to make our subject clear.

The dotted circle shown within the lenses, with its centre at the optical centre $o$, shall represent the plane above alluded to.

The *passive* or axial planes of the cylinders are shown by dotted parallelograms at $A$ and $a$, bisecting each other under the angle $Aoa = \gamma$ in the optical axis at $o$; and their *active* planes of refraction $C$ and $c$, which are of necessity at right angles to their correlative axial planes, similarly bisect each other at the same point. Hence, $< Coc = < Aoa = \gamma$.

The compound lens, thus presented, consists of two congeneric cylindrical elements, each of which, *independently considered,* will have its corresponding focal plane, which, for convenience, we may term an *elementary* focal plane of the combination. Thus, $E_1$ and $E_2$, at the focal distances $f_1$ and $f_2$, are the elementary focal planes for the cylinders $C$ and $c$, respectively. The cylinder $C$ will consequently have the property of deflecting a ray, incident at $D$, perpendicularly from $D_1$, in the plane $E_1$, to the point $Z_1$ of the axial plane $A_1 Z_1$, while the cylinder $c$ will have the property of deflecting a ray incident at the same point, perpendicularly from $D_2$, in the plane $E_2$, to the point $V_2$ of the axial plane $a_2 o_2$.

The greatest amplitude of deflection for $C$ will therefore be $D_1 Z_1$ in the plane $E_1$, and for $c$ will be $D_2 V_2$ in the plane $E_2$. It is further manifest that the refracted ray $D V_1 V_2$, contributed by $c$ only, in attaining to its greatest deflection $D_2 V_2$ in the plane $E_2$, would penetrate the plane $E_1$ at $V_1$, and in it present a proportionate deflection $D_1 V_1$.

$D_1 Z_1$ and $D_1 V_1$, being amplitudes of deflection *reduced to the same plane* $E_1$, will bear the same relation to each other as their corresponding refractions. Thus,

$$D_1 Z_1 : \frac{1}{f_1} = D_1 V_1 : \frac{1}{f_2};$$

or, $$D_1 Z_1 = \frac{1}{f_1}, \quad \text{when} \quad D_1 V_1 = \frac{1}{f_2},$$

which may easily be shown to be the case when the deflections are measured in a plane one inch from the lens.*

Conditional, therefore, that the deflections are measured, within the same plane, from a point $D_1$ of the same line of incidence $DD_1$, we may attain to the resultant of two deflections $D_1 Z_1$ and $D_1 V_1$, for any angular deviation existing between them at $D_1$, by the physical law governing similarly united forces. $D_1 M_1$, as the diagonal of the

* "Refraction and Accommodation of the Eye," by E. Landolt, M.D., Paris, translated by C. M. Culver, M.A., M.D., Philadelphia, 1886 (see page 58).

parallelogram $D_1 V_1 M_1 Z_1$, will consequently be the resultant deflection accruing from a combination of the cylinders $C$ and $c$.

As each cylinder contributes a plane of active and one of passive refraction, we shall evidently obtain two resultant principal planes for their combination, the one of greatest refraction, commonly called the *primary* plane, $DD_1o_1o$, intersecting the angle $Coc = \gamma$ between the active planes of refraction $C$ and $c$, and one of least refraction, termed the *secondary* plane, $dd_2o_2o$, intersecting the angle $Aoa = \gamma$ between the passive or axial planes $A$ and $a$.

The primary plane, in penetrating the plane $E_1$, will consequently divide the angle $C_1o_1c_1 = Coc = \gamma$ into $D_1o_1c_1 = \alpha$ and $D_1o_1C_1 = \beta$. In the plane $E_1$ we shall then find the angles $\alpha$ and $\beta$ to be directly dependent upon the associated deflections $D_1Z_1$ and $D_1V_1$ for the point $D_1$. In the plane $E_2$ a similar division of the angle $A_2o_2a_2$, by the secondary plane, will be rendered dependent upon $d_2v_2$ and $d_2z_2$ for the point $d_2$. As to this, the diagram is believed to be sufficiently clear, without further reference.

Since the resultants $D_1M_1$ and $d_2m_2$ will define the directions of the refracted rays $DM_1$ and $dm_2$, it is further evident that for $D$ and $d$ to be points of the primary and secondary planes, respectively, they will have to be so chosen that $D_1M_1$ and $d_2m_2$ shall be directed *to* the optical axis $oo_1o_2$; and as we shall later learn, this is but one of the restrictions which renders a diagram somewhat difficult of construction. The resultant deflections $D_1M_1$ and $d_2m_2$ are consequently shown as being in the primary plane, *coincident* with $D_1o_1$, and in the secondary plane *coincident* with $d_2o_2$, respectively.

For all intermediate points of the circle, we should find the resultant deflections to deviate *from* the optical axis. This has been taken advantage of in constructing Dr. Burnett's models, and in determining the directions of twelve refracted rays in each of the figures 2, Plates II and IV.

**The position of the primary plane** $DD_1o_1o$, shown as dividing the angle $C_1o_1c_1 = \gamma$ so that

$$\gamma = \alpha + \beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

will then be determined by fixing the relations existing between $\alpha$ and $\beta$.

In the plane $E_1$, from the triangle $D_1Z_1M_1$, we have

$$D_1Z_1 : Z_1M_1 = \sin < Z_1M_1D_1 : \sin < Z_1D_1M_1,$$

$$< Z_1M_1D_1 = < D_1o_1c_1 = \alpha,$$

by parallelism of $Z_1M_1$ and $c_1o_1$; and, for similar reasons,

$$< Z_1D_1M_1 = < D_1M_1V_1 = D_1o_1C_1 = \beta.$$

$$\therefore \quad D_1Z_1 : Z_1M_1 = \sin\alpha : \sin\beta,$$

$$Z_1M_1 = D_1V_1.$$

$$\therefore \quad D_1Z_1 : D_1V_1 = \sin\alpha : \sin\beta. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the oblique plane $DD_2V_2$ we find

$$D_1V_1 : D_2V_2 = DD_1 : DD_2 ;$$

or, as $DD_1$ and $DD_2$ are the focal distances $f_1$ and $f_2$ of the cylinders $C$ and $c$, respectively,

$$D_1V_1 : D_2V_2 = f_1 : f_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Multiplying the equations (2) and (3), we obtain,

$$\frac{D_1Z_1}{D_2V_2} = \frac{\sin\alpha}{\sin\beta}\,\frac{f_1}{f_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since $D_1o_1$ is the radius of the circle indicated, we may, for convenience, ascribe to it the value 1. We shall then have,

$$D_1Z_1 = \sin < D_1o_1Z_1,$$

$$< D_1o_1Z_1 = < C_1o_1Z_1 - < D_1o_1C_1.$$

$$\therefore \quad < D_1o_1Z_1 = 90^\circ - \beta.$$

$$\therefore \quad D_1Z_1 = \sin(90^\circ - \beta) = \cos\beta. \quad . \quad . \quad (5)$$

In the plane $E_2$ we similarly find,

$$D_2V_2 = \sin < D_2o_2V_2,$$

$$< D_2o_2V_2 = < V_2o_2c_2 - < D_2o_2c_2.$$

$$\therefore \quad < D_2o_2V_2 = 90^\circ - \alpha.$$

$$\therefore \quad D_2V_2 = \sin(90^\circ - \alpha) = \cos\alpha. \qquad (6)$$

Substituting the values for $D_1 Z_1$ and $D_2 V_2$ from (5) and (6) in the equation (4), we obtain,

$$\frac{\cos\beta}{\cos\alpha} = \frac{\sin\alpha}{\sin\beta}\,\frac{f_1}{f_2};$$

or, by multiplying both members of equation by 2 and transposing,

$$2\cos\beta\sin\beta = 2\cos\alpha\sin\alpha\,\frac{f_1}{f_2}.$$

$$\therefore\quad \sin 2\beta = \sin 2\alpha\,\frac{f_1}{f_2}. \quad \ldots\ldots \quad (7)$$

**The position of the secondary plane** $dd_2o_2o$, shown as dividing the angle $A_2o_2a_2 = \gamma$ into $d_2o_2a_2 = \alpha$ and $d_2o_2A_2 = \beta$, *provided* $d_2o_2$ *is perpendicular to* $D_2o_2$, will be determined by similarly fixing the relations between $\alpha$ and $\beta$.

In the plane $E_2$, from the triangle $d_2z_2m_2$, we have

$$d_2z_2 : z_2m_2 = \sin \angle z_2m_2d_2 : \sin \angle z_2d_2m_2,$$

$$\angle z_2m_2d_2 = \angle m_2d_2v_2,$$

by parallelism $z_2m_2$ and $d_2v_2$; or, as $\angle m_2d_2v_2 = \angle d_2v_2o_2 - \angle v_2o_2d_2 = 90^\circ - \alpha$,

$$\sin \angle z_2m_2d_2 = \sin(90^\circ - \alpha) = \cos\alpha.$$

Similarly, $\sin \angle z_2d_2m_2 = \sin(90^\circ - \beta) = \cos\beta.$

$$\therefore\quad d_2z_2 : z_2m_2 = \cos\alpha : \cos\beta,$$

$$z_2m_2 = d_2v_2.$$

$$\therefore\quad d_2z_2 : d_2v_2 = \cos\alpha : \cos\beta. \quad \ldots\ldots \quad (8)$$

In the oblique plane $dd_2z_2$, we find

$$d_1z_1 : d_2z_2 = dd_1 : dd_2;$$

or, as $dd_1 = f_1$ and $dd_2 = f_2$,

$$d_1z_1 : d_2z_2 = f_1 : f_2. \quad \ldots\ldots \quad (9)$$

Multiplying the equations (8) and (9), we obtain,

$$\frac{d_1 z_1}{d_2 v_2} = \frac{\cos \alpha}{\cos \beta} \frac{f_1}{f_2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and, since $d_2 o_2 = d_1 o_1 = \text{radius} = 1$,

$$d_1 z_1 = \sin < d_1 o_1 A_1 = \sin < d_2 o_2 A_2 = \sin \beta, \quad . \quad (11)$$

$$d_2 v_2 = \sin \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Substituting these values in (10),

$$\frac{\sin \beta}{\sin \alpha} = \frac{\cos \alpha}{\cos \beta} \frac{f_1}{f_2}.$$

$$\therefore \quad 2 \sin \beta \cos \beta = 2 \sin \alpha \cos \alpha \frac{f_1}{f_2};$$

or, as before,

$$\sin 2\beta = \sin 2\alpha \frac{f_1}{f_2}.$$

As the same relations, deduced from the deflections $d_1 z_1$ and $d_2 v_2$, *under provisions that* $d_2 o_2 \perp D_2 o_2$, are here shown to exist between $\alpha$ and $\beta$ as were obtained from $D_1 Z_1$ and $D_2 V_2$, we are to conclude that :

***1. The primary and secondary planes of refraction are at right angles to each other for any angular deviation of the axes of two combined congeneric cylindrical lenses.***

In a further consideration of the relation (7),

$$\sin 2\beta = \sin 2\alpha \frac{f_1}{f_2},$$

we observe the sines of double the angles, which are each always less than 90°, to differ merely by the co-efficient $\frac{f_1}{f_2}$.

If, therefore, $f_2 = f_1$, which is the case when the cylinders are of equal refraction, the $\sin 2\beta$ will be equal to the $\sin 2\alpha$, which can only be the case when $\alpha = \beta$, or, as $\alpha + \beta = \gamma$, when $\alpha = \beta = \frac{\gamma}{2}$; hence,

***2. For combined congeneric cylinders of equal refraction, the primary plane equally divides the angle between the active planes of the cylinders, and the secondary plane similarly divides the angle between the axial planes of the cylinders.***

In case, however, $f_2 > f_1$, which is the case when the refraction of the cylinder $C$ is greater than $c$, then $\sin 2\alpha > \sin 2\beta$, or, when $\alpha > \beta$, so that

***3. For combined congeneric cylinders of unequal refraction, the primary plane, in dividing the angle between the active planes of the cylinders, will be nearer to the active plane of the stronger cylinder, and the secondary plane consequently nearer to the axial plane of the same cylinder.***

This is also demonstrated in the diagram.

As, for a combination of two cylinders, $C$ and $c$, under given angular deviation of their axes, the only known quantities will be $f_1$, $f_2$, and $\gamma$, it will be necessary to express $\alpha$ and $\beta$ in terms of $f_1$, $f_2$, and $\gamma$.

This we accomplish through the equations

$$\sin 2\beta = \sin 2\alpha \frac{f_1}{f_2},$$

$$\alpha + \beta = \gamma;$$

and, as these also contribute elements of vital importance to future deductions, we shall seek to reduce in a manner adapted to ultimate reference by placing

$$\frac{f_1}{f_2} = k. \quad \ldots \ldots \quad (13)$$

The above equations may then be written

$$\sin 2\beta = k \sin 2\alpha, \quad \ldots \ldots \quad (14)$$

$$\beta = \gamma - \alpha. \quad \ldots \ldots \quad (15)$$

$$\therefore \quad \sin 2\beta = \sin 2\gamma \cos 2\alpha - \cos 2\gamma \sin 2\alpha = k \sin 2\alpha. \quad . \quad (16)$$

$$\therefore \quad \sin 2\gamma \cos 2\alpha = (k + \cos 2\gamma) \sin 2\alpha.$$

$$\therefore \quad \cos 2\alpha = \frac{k + \cos 2\gamma}{\sin 2\gamma} \sin 2\alpha. \quad . \quad . \quad . \quad (17)$$

$$\therefore \quad \cos^2 2\alpha = 1 - \sin^2 2\alpha = \frac{(k + \cos 2\gamma)^2}{\sin^2 2\gamma} \sin^2 2\alpha.$$

$$\therefore \quad \sin^2 2\alpha \left[ \frac{(k + \cos 2\gamma)^2}{\sin^2 2\gamma} + 1 \right] = 1.$$

$$\therefore \quad \sin^2 2\alpha = \frac{\sin^2 2\gamma}{(k + \cos 2\gamma)^2 + \sin^2 2\gamma} = \frac{\sin^2 2\gamma}{k^2 + 2k \cos 2\gamma + 1}.$$

$$\therefore \quad \sin 2\alpha = \frac{\sin 2\gamma}{\sqrt{k^2 + 2k \cos 2\gamma + 1}}.$$

Hence, from (17),

$$\cos 2\alpha = \frac{k + \cos 2\gamma}{\sqrt{k^2 + 2k \cos 2\gamma + 1}}.$$

For convenience, let $m = \sqrt{k^2 + 2k \cos 2\gamma + 1}$. . . . . (18)

$$\therefore \quad \sin 2\alpha = \frac{\sin 2\gamma}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\therefore \quad \cos 2\alpha = \frac{k + \cos 2\gamma}{m}. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

From (15), $\quad \cos 2\beta = \cos 2\gamma \cos 2\alpha + \sin 2\gamma \sin 2\alpha.$

Replacing $\cos 2\alpha$ and $\sin 2\alpha$ by their values from (20) and (19), gives,

$$\cos 2\beta = \frac{(k + \cos 2\gamma) \cos 2\gamma}{m} + \frac{\sin^2 2\gamma}{m} = \frac{k \cos 2\gamma + 1}{m}. \quad (21)$$

Resorting to the general formulæ $2 \cos^2 \alpha = 1 + \cos 2\alpha$ and $2 \sin^2 \alpha = 1 - \cos 2\alpha$, we may write :

$$\cos^2 \alpha = \tfrac{1}{2} + \tfrac{1}{2} \cos 2\alpha,$$

$$\sin^2 \alpha = \tfrac{1}{2} - \tfrac{1}{2} \cos 2\alpha.$$

Similarly, $\quad \cos^2 \beta = \tfrac{1}{2} + \tfrac{1}{2} \cos 2\beta,$

$$\sin^2 \beta = \tfrac{1}{2} - \tfrac{1}{2} \cos 2\beta.$$

Substituting in these for $\cos 2\alpha$ and $\cos 2\beta$ their values from (20) and (21) gives,

$$\cos^2 \alpha = \frac{1}{2} + \frac{1}{2} \frac{k + \cos 2\gamma}{m} = \frac{m + k + \cos 2\gamma}{2m}. \quad . \quad . \quad (22)$$

$$\sin^2 \alpha = \frac{1}{2} - \frac{1}{2} \frac{k + \cos 2\gamma}{m} = \frac{m - k - \cos 2\gamma}{2m}. \quad . \quad . \quad (23)$$

$$\cos^2 \beta = \frac{1}{2} + \frac{1}{2} \frac{1 + k \cos 2\gamma}{m} = \frac{m + 1 + k \cos 2\gamma}{2m}. \quad . \quad (24)$$

$$\sin^2 \beta = \frac{1}{2} - \frac{1}{2} \frac{1 + k \cos 2\gamma}{m} = \frac{m - 1 - k \cos 2\gamma}{2m}. \quad . \quad (25)$$

The angles $\alpha$ and $\beta$ may then be expressed in terms of $f_1$, $f_2$, and $\gamma$, by substituting, in the above formulæ, for $k$ and $m$, their values, as, for instance,

$$\cos^2 \alpha = \frac{1}{2} + \frac{1}{2} \frac{\frac{f_1}{f_2} + \cos 2\gamma}{\sqrt{\left(\frac{f_1}{f_2}\right)^2 + 2\frac{f_1}{f_2} \cos 2\gamma + 1}};$$

or, multiplying both terms of fraction by $f_2$,

$$\cos \alpha = \sqrt{\frac{1}{2} + \frac{1}{2} \frac{f_1 + f_2 \cos 2\gamma}{\sqrt{f_1{}^2 + 2f_1 f_2 \cos 2\gamma + f_2{}^2}}}. \quad . \quad . \quad (I)$$

It will be unnecessary to seek $\beta$ in the same manner, since, by (15), $\beta = \gamma - \alpha$.

When reducing the above formula, for any given value of $\gamma$, pursuant to reasons later given, it should be observed that $f_2 > f_1$, in which case $\alpha$, within the angle $\gamma$, is to be counted from the axis of the *weaker* cylinder.

## 2. POSITIONS OF THE PRIMARY AND SECONDARY FOCAL PLANES.

The plane $DD_1o_1o$ being the primary plane, it follows that all parallel rays incident in it between $D$ and $o$ will, after refraction, intercept the optical axis $oo_1$ at some point, which will be a point of the primary focal line. Thus, the final ray $DM_1M_2$, in attaining to its greatest deflection $D_1M_1$ in the elementary plane $E_1$, will establish the limiting position for the primary focal line by its intersection of the optical axis $oo_1$, at $O_1$.

For similar reasons, in the secondary plane, $O_2$ will be a point of the secondary focal line, this intersection of the final ray $dm_1m_2$ with the optical axis being more remote in consequence of the inferior deflection $d_2m_2$ in the plane $E_2$.

Like deflections, for opposite cardinal points of the circle within the lens, will define the directions of the corresponding final rays, which are shown as limiting the major and minor axes of the ellipses in the planes $E_1$ and $E_2$, and consequently also the magnitudes of the focal lines at $O_1$ and $O_2$. Thus, $O_2M_3$ represents one half of the secondary focal line at $O_2$. The primary focal line, in the *secondary* plane, perpendicular to $YO_1$ at $O_1$, has been omitted, to avoid possible misinterpretation of more important points of reference in this region. All rays parallel to the optical axis, incident at intermediate points of the circle within the lens, will, upon refraction, intersect the planes $E_1$ and $E_2$ at correlative points of the ellipses drawn.

The region of transition $T$, or circle of least confusion, will lie between the planes $E_1$ and $E_2$. (See Plate II, Fig. 2.) Its position may be determined through a simple formula advanced by Prof. W. Steadman Aldis, of the University College, Auckland, New Zealand, in his consideration of the "focal interval" resulting from rays obliquely incident upon a spherical lens.*

Our object being to determine the distances of the primary and secondary focal lines or planes from the principal plane within the combined cylinders, we may proceed as follows :

* Elementary Treatise on Geometrical Optics, W. S. Aldis, M.A., Cambridge, 1886 (see page 39).

In the primary plane $DD_1M_1$, we have

$$DY : DD_1 = YO_1 : D_1M_1.$$

Substituting, $DY = O_1o = F_1$ as the primary focus;

$$DD_1 = f_1;$$

$$YO_1 = D_1o_1 = \text{radius} = 1.$$

$$\therefore \quad F_1 = \frac{f_1}{D_1M_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In the parallelogram $D_1V_1M_1Z_1$, the angle between the forces, $D_1V_1$ and $D_1Z_1$, being equal to $< C_1o_1c_1 = \gamma$, we have, as the resultant deflection,

$$D_1M_1 = \sqrt{(D_1Z_1)^2 + (D_1V_1)^2 + 2\,(D_1V_1)\,(D_1Z_1)\cos\gamma}, \quad (27)$$

in conformity with the statical formula,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\gamma},$$

for forces $P$ and $Q$, acting at the same point, within the same plane, under the angle $\gamma$.

Substituting in (27) the value of $D_1Z_1 = \cos\beta$, from (5); and of $D_1V_1 = \frac{f_1}{f_2}D_2V_2$, from (3), $= \frac{f_1}{f_2}\cos\alpha$, from (6), we obtain,

$$D_1M_1 = \sqrt{\cos^2\beta + \left(\frac{f_1}{f_2}\right)^2\cos^2\alpha + 2\,\frac{f_1}{f_2}\cos\alpha\cos\beta\cos\gamma}.$$

Introducing this value for $D_1M_1$ in (26),

$$F_1 = \frac{f_1}{\sqrt{\cos^2\beta + \left(\frac{f_1}{f_2}\right)^2\cos^2\alpha + 2\,\frac{f_1}{f_2}\cos\alpha\cos\beta\cos\gamma}}. \quad (28)$$

Substituting here, as before, $\frac{f_1}{f_2} = k$,

$$F_1 = \frac{f_1}{\sqrt{\cos^2\beta + k^2\cos^2\alpha + 2k\cos\alpha\cos\beta\cos\gamma}}. \quad . \quad (29)$$

To reduce the third member under the radical, we deduce from (15),

$$\cos\beta = \cos(\gamma - \alpha) = \cos\gamma\cos\alpha + \sin\gamma\sin\alpha.$$

$$\therefore \quad \cos\alpha\cos\beta = \cos\gamma\cos^2\alpha + \sin\gamma\sin\alpha\cos\alpha$$

$$= \cos\gamma\cos^2\alpha + \tfrac{1}{2}\sin\gamma\sin 2\alpha\,;$$

and by substituting (22) and (19) for $\cos^2\alpha$ and $\sin 2\alpha$,

$$\cos\alpha\cos\beta = \frac{(m+k)\cos\gamma + \cos 2\gamma\cos\gamma}{2m} + \frac{\sin 2\gamma\sin\gamma}{2m}.$$

But $\qquad \cos 2\gamma\cos\gamma + \sin 2\gamma\sin\gamma = \cos(2\gamma - \gamma) = \cos\gamma.$

$$\therefore \quad \cos\alpha\cos\beta = \frac{(m+k+1)\cos\gamma}{2m}.$$

$$\therefore \quad \cos\alpha\cos\beta\cos\gamma = \frac{(m+k+1)\cos^2\gamma}{2m}.$$

$$\cos^2\gamma = \tfrac{1}{2}(1 + \cos 2\gamma).$$

$$\therefore \quad \cos\alpha\cos\beta\cos\gamma = \frac{(m+k+1)(1+\cos 2\gamma)}{4m}$$

$$= \frac{m + k + 1 + m\cos 2\gamma + k\cos 2\gamma + \cos 2\gamma}{4m}.$$

$$\therefore \; 2k\cos\alpha\cos\beta\cos\gamma = \frac{mk + k^2 + k + mk\cos 2\gamma + k^2\cos 2\gamma + k\cos 2\gamma}{2m}.$$

For the first two members under the radical, by substituting values from (24) and (22), we have

$$\cos^2\beta + k^2\cos^2\alpha = \frac{m + 1 + k\cos 2\gamma + mk^2 + k^3 + k^2\cos 2\gamma}{2m}.$$

Consequently, the entire value under the radical,

$$\cos^2\beta + k^2\cos^2\alpha + 2k\cos\alpha\cos\beta\cos\gamma$$

$$= \frac{mk^2 + mk\cos 2\gamma + mk + m + k^3 + 2k^2\cos 2\gamma + k + k^2 + 2k\cos 2\gamma + 1}{2m}$$

$$= \frac{(k^2 + k\cos 2\gamma)m + (k+1)m + k(k^2 + 2k\cos 2\gamma + 1) + k^2 + 2k\cos 2\gamma + 1}{2m}.$$

Since, by equation (18), $k^2 + 2k \cos 2\gamma + 1 = m^2$,

$$\cos^2 \beta + k^2 \cos^2 \alpha + 2k \cos \alpha \cos \beta \cos \gamma$$

$$= \frac{(k^2 + k \cos 2\gamma)\, m + (k + 1)\, m + km^2 + m^2}{2m}$$

$$= \frac{k\,(k + \cos 2\gamma) + (k + 1) + (k + 1)\, m}{2}$$

$$= \tfrac{1}{2}\,[k\,(k + \cos 2\gamma) + (k + 1)\,(1 + m)].$$

Substituting this under the radical in (29), we obtain,

$$F_1 = \frac{f_1}{\sqrt{\tfrac{1}{2}\,[k\,(k + \cos 2\gamma) + (k + 1)\,(1 + m)]}}.$$

Replacing $k$ and $m$ by their values from (13) and (18),

$$F_1 = \frac{f_1}{\sqrt{\frac{1}{2}\left[\frac{f_1}{f_2}\left(\frac{f_1}{f_2} + \cos 2\gamma\right) + \left(\frac{f_1}{f_2} + 1\right)\left(1 + \sqrt{\left(\frac{f_1}{f_2}\right)^2 + 2\frac{f_1}{f_2}\cos 2\gamma + 1}\right)\right]}}.$$

Multiplying both terms of fraction by $f_2$,

$$F_1 = \frac{f_1 f_2}{\sqrt{\tfrac{1}{2}\left[f_1(f_1 + f_2 \cos 2\gamma) + (f_1 + f_2)\left(f_2 + \sqrt{f_1{}^2 + 2 f_1 f_2 \cos 2\gamma + f_2{}^2}\right)\right]}}. \quad \ldots\ldots\ldots \quad (30)$$

Transforming, and substituting $1 - 2 \sin^2 \gamma$ for $\cos 2\gamma$, we may, for convenience in calculating, preferably write,

$$F_1 = \frac{f_1 f_2}{\sqrt{\frac{(f_1 + f_2)^2}{2} - f_1 f_2 \sin^2 \gamma + (f_1 + f_2)\sqrt{\frac{(f_1 + f_2)^2}{4} - f_1 f_2 \sin^2 \gamma}}}. \quad \ldots\ldots\ldots \quad (\text{II})$$

When the cylinders are of equal refraction, $f_1$ being equal to $f_2 = f$, the above formula, by adequate reduction, assumes the simple form,

$$F_1 = \frac{f}{1 + \cos \gamma}. \quad \ldots\ldots\ldots \quad (\text{IV})$$

In the secondary plane $dd_2XO_2$, we have

$$dX : dd_2 = XO_2 : d_2m_2.$$

Substituting, $dX = O_2o = F_2$ as the secondary focus;

$$dd_2 = f_2;$$

$$XO_2 = \text{radius} = 1.$$

$$\therefore \quad F_2 = \frac{f_2}{d_2m_2}. \quad \ldots \ldots \quad (31)$$

In the parallelogram $d_2v_2m_2z_2$, the angle between the forces, $d_2v_2$ and $d_2z_2$, being equal to $< v_2d_2z_2 = 180° - < A_2o_2a_2 = 180° - \gamma$,

$$d_2m_2 = \sqrt{(d_2z_2)^2 + (d_2v_2)^2 + 2\,(d_2v_2)\,(d_2z_2)\cos\,(180° - \gamma)}.$$

Substituting the value for $d_2z_2 = \frac{f_2}{f_1}\,d_1z_1$, from (9), $= \frac{f_2}{f_1}\sin\beta$, from (11); and for $d_2v_2 = \sin\alpha$, from (12), we obtain,

$$d_2m_2 = \sqrt{\left(\frac{f_2}{f_1}\right)^2 \sin^2\beta + \sin^2\alpha - 2\frac{f_2}{f_1}\sin\alpha\sin\beta\cos\gamma};$$

which introduced in (31) gives,

$$F_2 = \frac{f_2}{\sqrt{\left(\frac{f_2}{f_1}\right)^2 \sin^2\beta + \sin^2\alpha - 2\frac{f_2}{f_1}\sin\alpha\sin\beta\cos\gamma}}.$$

Multiplying numerator and denominator by $\frac{f_1}{f_2}$,

$$F_2 = \frac{f_1}{\sqrt{\sin^2\beta + \left(\frac{f_1}{f_2}\right)^2 \sin^2\alpha - 2\frac{f_1}{f_2}\sin\alpha\sin\beta\cos\gamma}};$$

and which may then be written,

$$F_2 = \frac{f_1}{\sqrt{\sin^2\beta + k^2\sin^2\alpha - 2k\sin\alpha\sin\beta\cos\gamma}}. \quad \cdot \quad (32)$$

To reduce the third member under the radical, we have, from (15),

$$\sin\beta = \sin\gamma\cos\alpha - \cos\gamma\sin\alpha.$$

$$\therefore\quad \sin\alpha\sin\beta = \sin\gamma\sin\alpha\cos\alpha - \cos\gamma\sin^2\alpha$$

$$= \tfrac{1}{2}\sin\gamma\sin 2\alpha - \cos\gamma\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\alpha\right)$$

$$= \tfrac{1}{2}\sin\gamma\sin 2\alpha + \tfrac{1}{2}\cos\gamma\cos 2\alpha - \tfrac{1}{2}\cos\gamma\,;$$

and by substituting (19) and (20) for $\sin 2\alpha$ and $\cos 2\alpha$,

$$\sin\alpha\sin\beta = \frac{\sin 2\gamma\sin\gamma}{2m} + \frac{\cos 2\gamma\cos\gamma + k\cos\gamma}{2m} - \tfrac{1}{2}\cos\gamma$$

$$= \frac{\cos(2\gamma-\gamma) + k\cos\gamma - m\cos\gamma}{2m}$$

$$= \frac{(1+k-m)\cos\gamma}{2m}.$$

$$\therefore\quad 2k\sin\alpha\sin\beta\cos\gamma = \frac{2(k+k^2-mk)\cos^2\gamma}{2m}.$$

But $$\cos^2\gamma = \tfrac{1}{2}(1+\cos 2\gamma)\,;$$

$$\therefore\ 2k\sin\alpha\sin\beta\cos\gamma = \frac{(k+k^2-mk)(1+\cos 2\gamma)}{2m}$$

$$= \frac{k+k^2-mk+k\cos 2\gamma+k^2\cos 2\gamma-mk\cos 2\gamma}{2m};$$

and for the first two members under the radical, through (25) and (23), we find,

$$\sin^2\beta + k^2\sin^2\alpha = \frac{m-1-k\cos 2\gamma+mk^2-k^3-k^2\cos 2\gamma}{2m}.$$

$$\therefore\quad \sin^2\beta + k^2\sin^2\alpha - 2k\sin\alpha\sin\beta\cos\gamma$$

$$= \frac{mk^2+mk\cos 2\gamma+mk+m-k^3-2k^2\cos 2\gamma-k-k^2-2k\cos 2\gamma-1}{2m}$$

$$= \frac{(k^2+k\cos 2\gamma)m+(k+1)m-k(k^2+2k\cos 2\gamma+1)-(k^2+2k\cos 2\gamma+1)}{2m}$$

$$= \frac{(k^2 + k \cos 2\gamma)\, m + (k + 1)\, m - km^2 - m^2}{2m}$$

$$= \frac{k\,(k + \cos 2\gamma) + (k + 1) - (k + 1)\, m}{2}$$

$$= \tfrac{1}{2}\,[k\,(k + \cos 2\gamma) + (k + 1)\,(1 - m)].$$

Substituting this under the radical in (32) and replacing $k$ and $m$ by their values, we obtain,

$$F_2 = \frac{f_1}{\sqrt{\frac{1}{2}\left[\frac{f_1}{f_2}\left(\frac{f_1}{f_2} + \cos 2\gamma\right) + \left(\frac{f_1}{f_2} + 1\right)\left(1 - \sqrt{\left(\frac{f_1}{f_2}\right)^2 + 2\frac{f_1}{f_2}\cos 2\gamma + 1}\right)\right]}}.$$

Multiplying both terms of fraction by $f_2$,

$$F_2 = \frac{f_1 f_2}{\sqrt{\tfrac{1}{2}\left[f_1\,(f_1 + f_2 \cos 2\gamma) + (f_1 + f_2)\left(f_2 - \sqrt{f_1^{\,2} + 2 f_1 f_2 \cos 2\gamma + f_2^{\,2}}\right)\right]}}. \qquad (33)$$

Substituting, $\cos 2\gamma = 1 - 2\sin^2\gamma$,

$$F_2 = \frac{f_1 f_2}{\sqrt{\frac{(f_1 + f_2)^2}{2} - f_1 f_2 \sin^2\gamma - (f_1 + f_2)\sqrt{\frac{(f_1 + f_2)^2}{4} - f_1 f_2 \sin^2\gamma}}}. \qquad \text{(III)}$$

This formula, reduced for cylinders of equal refraction, $f_1$ being equal to $f_2 = f$, becomes

$$F_2 = \frac{f}{1 - \cos\gamma}. \qquad \text{(V)}$$

It may be of interest to note that these formulæ differ from those given for $F_1$ merely by a minus sign in the denominator.

The preceding formulæ being alike applicable for combinations of convex or concave cylinders, the foci $f_1$ and $f_2$ are to be introduced as positive values, merely with the restriction that $f_2$ be greater than or equal to $f_1$, in either case.

## 3. RELATIONS BETWEEN THE PRIMARY AND SECONDARY FOCAL PLANES.

Since $F_1$ and $F_2$ have been shown to be dependent upon $f_1, f_2$, and $\gamma$, it is evident that, for fixed values of $f_1$ and $f_2$, the same will be rendered dependent upon successive values of the angle $\gamma$ only.

It is further obvious that the refraction of one cylinder will be affected most by the other when their axes coincide, or when $\gamma = 0°$, and least when their axes are at right angles to each other, or when $\gamma = 90°$.

We shall, consequently, fix upon the limits of $F_1$ and $F_2$ for these extremes of $\gamma$.

Introducing $\gamma = 0°$, and consequently $\cos 2\gamma = +1$, into the formulæ (30) and (33), we obtain, for $f_2 > f_1$,

$$F_1 = \frac{f_1 f_2}{\sqrt{\frac{1}{2}[f_1(f_1+f_2)+(f_1+f_2)(f_2+f_1+f_2)]}} = \frac{f_1 f_2}{f_1+f_2}.$$

$$F_2 = \frac{f_1 f_2}{\sqrt{\frac{1}{2}[f_1(f_1+f_2)+(f_1+f_2)(f_2-f_1-f_2)]}} = \frac{f_1 f_2}{0} = \infty.$$

$$\therefore \quad F_1 : F_2 = \frac{f_1 f_2}{f_1+f_2} : \infty. \quad . \quad . \quad . \quad . \quad . \quad (34)$$

For $F_1 = \dfrac{f_1 f_2}{f_1+f_2}$, we shall have as the refraction,

$$\frac{1}{F_1} = \frac{1}{f_1} + \frac{1}{f_2}; \quad \text{consequently,}$$

***4. When the axes of the congeneric cylinders coincide, the primary focal plane will correspond to that focal plane which is defined by the sum of the refractions of the cylinders, whereas the secondary focal plane will be at infinity.***

This is shown in Plate II, Fig. 1.

Introducing $\gamma = 90°$, and consequently $\cos 2\gamma = \cos 180° = -1$, into (30) and (33), we have, for $f_2 > f_1$,

$$F_1 = \frac{f_1 f_2}{\sqrt{\frac{1}{2}[-f_1(f_2 - f_1) + (f_1 + f_2)(f_2 + f_2 - f_1)]}} = \frac{f_1 f_2}{f_2} = f_1.$$

$$F_2 = \frac{f_1 f_2}{\sqrt{\frac{1}{2}[-f_1(f_2 - f_1) + (f_1 + f_2)(f_2 - f_2 + f_1)]}} = \frac{f_1 f_2}{f_1} = f_2.$$

$$\therefore \quad F_1 : F_2 = f_1 : f_2. \qquad (35)$$

As $f_1$ and $f_2$ correspond to the positions of the elementary planes $E_1$ and $E_2$, it follows that

***5. The primary and secondary focal planes coincide with their correlative elementary focal planes, when the axes of the congeneric cylinders of unequal refraction are at right angles to each other.***

This is demonstrated in Plate II, Fig. 2.

In the same relation (35), if $f_1 = f_2$, then $F_1 = F_2$, or

***6. The primary, secondary, and elementary focal planes all merge into one plane, when the axes of the congeneric cylinders of equal refraction are at right angles to each other.***

As in this case we have but one focal plane, the refraction corresponds to that of a spherical lens.

$F_1$ being adopted as signifying the primary focal distance, it will have to be less than $F_2$, yet if $f_1 > f_2$, we should find, as a consequence, by the relation (35), $F_1 > F_2$. To retain the significances of $F_1$ and $F_2$, it will therefore be convenient to substitute $f_2$ by the greater given value of cylindrical focus, and $f_1$ by the lesser, as stated under the formulæ, page 24.

By the previous considerations, between the limits of 0° and 90° for $\gamma$, we are then to conclude that $F_1$ will vary between $\dfrac{f_1 f_2}{f_1 + f_2}$ and $f_1$,

## PLATE II

THE REFRACTION BY COMBINED

CONGENERIC CYLINDRICAL LENSES

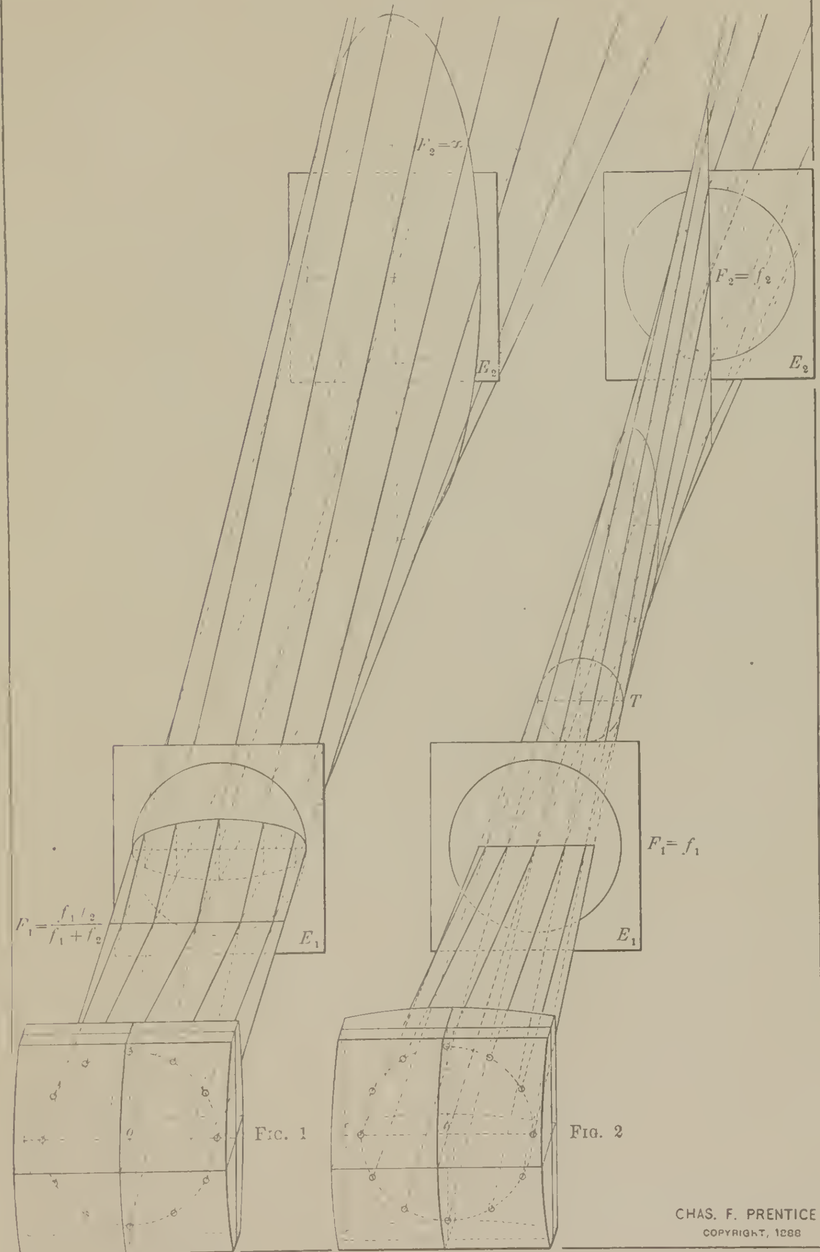


FIG. 1    FIG. 2

while $F_2$ varies between $\infty$ and $f_2$, as the nearest and most remote limits of focal distance.

As an illustration, let Fig. 1, Plate II, represent two combined convex cylinders of unequal refraction, with their axes coincident, and so united as to permit of the rotation of one of the cylinders upon the true planes of their faces, about the optical centre $o$.

In the position shown ($\gamma = 0°$), the limiting distance $F_1$ of the primary focal line will be $\frac{f_1 f_2}{f_1 + f_2}$, which corresponds to the combined refraction, $\frac{1}{f_1} + \frac{1}{f_2}$, of the cylinders in the active plane; and in the secondary plane, $F_2 = \infty$; consequently, $\frac{1}{F_2} = \frac{1}{\infty} = 0$, which corresponds to the refraction in the axial or passive plane of the cylinders.

The slightest change in the position of one of the cylindrical axes will give rise to a definite value of the angle $\gamma$ in the Formula III, thereby bringing $F_2$ within the limits of finite distance, while decreasing the value of $F_1$ in the Formula II.

For each successive increase in the angle $\gamma$, the primary focal plane, corresponding to $F_1$, will recede farther and farther from the lens towards $E_1$, while the secondary focal plane, corresponding to $F_2$, approaches nearer and nearer from $\infty$ to $E_2$, until $\gamma = 90°$, when $F_1$ will have reached $E_1$, and $F_2$ become merged into $E_2$, as shown in Plate II, Fig. 2.

Rotation of one of the cylinders is thus associated with corresponding changes in the distances $F_1$ and $F_2$, while the movements of their correlative focal planes will be in opposite directions to each other; and, as a consequence:

***7. The primary and secondary focal planes are conjugate planes, subject to variations of the angle between the axes of the congeneric cylinders.***

It being impossible to construct a truthful diagram, Plate I, without strictly adhering to the principles heretofore explained, it has been necessary to select elementary foci in marked disproportion to the curvatures or refractive indices of the cylinders, so as to bring $F_2$ within the limits of the space allotted.

# II. DIOPTRIC FORMULÆ

FOR COMBINED

# CONTRA-GENERIC CYLINDERS.

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## 1. RELATIVE POSITIONS OF THE PRINCIPAL POSITIVE AND NEGATIVE PLANES OF REFRACTION.

In a combination of convex and concave cylinders, we can no longer have the primary and secondary planes, which we have learned to consider as planes of greatest and least refraction, but, instead, we shall have a plane of greatest positive and one of greatest negative refraction, synonymously with the generally-adopted distinction between convex and concave lenses, designated by the signs + (plus) and — (minus), respectively. As the refractions by the convex and concave elements of the combination are opposing forces, the plane of greatest positive refraction will evidently lie between the active plane of the convex and the axial plane of the concave cylinder, while the plane of greatest negative refraction will be between the active plane of the concave and the axial plane of the convex cylinder.

In Plate III, therefore, the plane $DD_1o_1o$ of greatest positive refraction is shown between $c$ and $A$, and the plane $dd_1o_1o$ of greatest negative refraction between $C$ and $a$, these planes, by provision of their being at right angles to each other, dividing each of the angles $A_1o_1c_1$ and $C_1o_1a_1$ into $\alpha$ and $\beta$.

To establish the formulæ for combined contra-generic cylinders, we shall therefore have to ascribe another significance to the angles $\alpha$ and $\beta$.

The deviation of the axes $Aoa$ is equal to angle $A_1o_1a_1 = \gamma$, and, since $c_1o_1$ is perpendicular to $a_1o_1$, $\alpha + \beta + \gamma$ is equal to 90°; consequently,

$$\alpha + \beta = 90° - \gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The elementary focal planes $E_0$ and $E_1$, corresponding to the focal distances $f_0$ and $f_1$, respectively, are exhibited on opposite sides of the combined cylinders; since $E_0$, for the concave cylinder, will be virtual, and in the negative region before the lens, while $E_1$, for the convex cylinder, will be in the positive region behind the lens. Consequently, for the point D, the convex cylinder $c$ will contribute as its greatest amplitude of deflection $D_1Z_1$, perpendicular to $a_1o_1$ in the plane $E_1$, while the greatest amplitude of deflection for the concave cylinder $C$ will be $D_0V_0$, perpendicular to $A_0o_0$ in the virtual plane $E_0$. As the incident ray at $D$ will be refracted by the concave cylinder, as if emanating from a correlative point $V_0$ of the virtual axial line $V_0o_0$, it is evident that the direction of the ray refracted by it would be $V_0DV_1$. The proportionate deflection contributed by the concave cylinder, measured in the plane $E_1$, will consequently be $D_1V_1$.

Provided the point $D$ be properly chosen, it will be a point of the plane of greatest positive refraction, that is to say, when the resultant deflection $D_1M_1$, accruing from the associated deflections $D_1V_1$ and $D_1Z_1$ in the parallelogram of forces $D_1V_1M_1Z_1$, is directed *to* the optical axis.

To insure $D_1M_1$ being so directed, it is obvious that the associated deflections, $D_1Z_1$ and $D_1V_1$, must also be measured in the plane $E_1$, in the positive region behind the lens.

Similar reasoning will apply to the point $d$ as being in the plane $dd_1o_1o$ of greatest negative refraction. In this instance, $d_1m_1$ being a force directed from the optical axis, in the plane $E_1$, is to be taken negative, synonymously with the plane of greatest negative refraction.

The relations between $\alpha$ and $\beta$ are to be determined by an analogous method to the one given for congeneric cylinders, whereby we obtain

$$\sin 2\alpha = \sin 2\beta \frac{f_1}{f_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

as defining the positions of the planes of greatest positive and negative refraction, which are again at right angles to each other.

We here also find the sines of double the angles to differ by the co-efficient $\frac{f_1}{f_0}$. Hence, when $f_0 = f_1$, we shall have $\alpha = \beta = \frac{90^\circ - \gamma}{2}$, or,

***8. For combined contra-generic cylinders of equal refraction, the plane of greatest positive refraction equally divides the angle between the active plane of the convex and the axial plane of the concave cylinder; and the plane of greatest negative refraction similarly divides the angle between the active plane of the concave and the axial plane of the convex cylinder.***

In case $f_0 > f_1$, then $\beta > \alpha$; or,

***9. When the convex cylinder is stronger than the concave cylinder, the plane of greatest positive refraction will be nearer to the active plane of the convex, while the plane of greatest negative refraction will be proportionately farther from the active plane of the concave cylinder.***

In case $f_1 > f_0$, then $\alpha > \beta$; or,

***10. When the concave cylinder is stronger than the convex cylinder, the plane of greatest negative refraction will be nearer to the active plane of the concave, while the plane of greatest positive refraction will be proportionately farther from the active plane of the convex cylinder.***

This is manifest in the diagram.

The values of $\alpha$ and $\beta$ may be expressed in terms of $f_1$, $f_0$, and $\gamma$ in a similar manner to that shown in the previous theorem, by placing

$$\frac{f_1}{f_0} = k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

when, by (36) and (37), we shall have,

$$\cos 2\beta = \frac{k - \cos 2\gamma}{\sin 2\gamma} \sin 2\beta.$$

$$\sin 2\beta = \frac{\sin 2\gamma}{\sqrt{k^2 - 2k \cos 2\gamma + 1}}.$$

Substituting, in this case,

$$m = \sqrt{k^2 - 2k \cos 2\gamma + 1}. \quad . \quad . \quad . \quad . \quad (39)$$

$$\therefore \quad \sin 2\beta = \frac{\sin 2\gamma}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$\therefore \quad \cos 2\beta = \frac{k - \cos 2\gamma}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$\therefore \quad \cos 2\alpha = \frac{1 - k \cos 2\gamma}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Resorting to the general formulæ mentioned on page 16,

$$\cos^2 \alpha = \frac{m + 1 - k \cos 2\gamma}{2m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$\sin^2 \alpha = \frac{m - 1 + k \cos 2\gamma}{2m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$\cos^2 \beta = \frac{m + k - \cos 2\gamma}{2m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$\sin^2 \beta = \frac{m - k + \cos 2\gamma}{2m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Substituting for $k$ and $m$ their values, through (43) we obtain,

$$\cos \alpha = \sqrt{\frac{1}{2} + \frac{1}{2} \, \frac{f_0 - f_1 \cos 2\gamma}{\sqrt{f_0^2 - 2f_0 f_1 \cos 2\gamma + f_1^2}}}. \quad . \quad . \quad (\mathrm{VI})$$

and by equation (36), $\quad \beta = 90^\circ - (\gamma + \alpha)$;

the latter equations being all that is requisite to locate the positions of the principal planes of refraction; the angle $\alpha$ being counted from the axis of the *convex* cylinder.

## 2. POSITIONS OF THE POSITIVE AND NEGATIVE FOCAL PLANES.

The positions of the positive and negative focal planes will evidently here also be determined by the resultant rays, $DM_1$ and $dm_1$, and their correlative intersections with the optical axis at $O_1$ and $O_0$.

$O_1m_3$ will therefore represent one half the focal line in the positive region behind the lenses, and $O_0M_3$ one half the virtual focal line in the negative region before the same.

The ellipses shown in the planes $E_1$ and $E_0$ are of the same significance in this as in the preceding combination.

**In the plane of greatest positive refraction,** $DD_1YO_1$, we have

$$DY : DD_1 = YO_1 : D_1M_1.$$

Substituting, $DY = O_1o = F_1$ as the positive focus;

$$DD_1 = f_1;$$

$$YO_1 = Do = \text{radius} = 1.$$

$$\therefore \quad F_1 = \frac{f_1}{D_1M_1} \quad \ldots\ldots\ldots \quad (47)$$

In the parallelogram $D_1V_1M_1Z_1$, the angle between the forces, $D_1V_1$ and $D_1Z_1$, is equal to $180° - \gamma$, since $D_1Z_1 \perp Z_1o_1$, and $D_1V_1 \perp A_1o_1$.

$$\therefore \quad D_1M_1 = \sqrt{(D_1Z_1)^2 + (D_1V_1)^2 + 2(D_1Z_1)(D_1V_1)\cos(180° - \gamma)}.$$

In the oblique plane $D_0V_0DV_1D_1$, we find,

$$D_1V_1 : DD_1 = D_0V_0 : DD_0.$$

$$D_0V_0 = \sin < D_0o_0A_0 = \sin < D_1o_1A_1 = \sin\beta.$$

$$DD_0 = f_0.$$

$$\therefore \quad D_1 V_1 = \frac{f_1}{f_0} \sin \beta.$$

$$D_1 Z_1 = \sin (< Z_1 o_1 c_1 - < D_1 o_1 c_1) = \sin (90^\circ - \alpha) = \cos \alpha.$$

Substituting these values in the equation for $D_1 M_1$, formula (47) becomes,

$$F_1 = \frac{f_1}{\sqrt{\cos^2 \alpha + \left(\frac{f_1}{f_0}\right)^2 \sin^2 \beta - 2 \frac{f_1}{f_0} \sin \beta \cos \alpha \cos \gamma}};$$

and, by placing $\frac{f_1}{f_0} = k$, with the aid of the formulæ (39), (43), and (46), upon adequate reduction, we obtain,

$$F_1 = \frac{f_1}{\sqrt{\frac{1}{2} [k (k - \cos 2\gamma) + (1 - k)(1 + m)]}}.$$

Replacing $k$ and $m$ by their values, and multiplying both terms of fraction by $f_0$, gives,

$$F_1 = \frac{f_1 f_0}{\sqrt{\frac{1}{2}\left[f_1 (f_1 - f_0 \cos 2\gamma) + (f_0 - f_1)(f_0 + \sqrt{f_0^2 - 2 f_0 f_1 \cos 2\gamma + f_1^2})\right]}}. \quad \ldots \ldots \quad (48)$$

Substituting, $\cos 2\gamma = 1 - 2 \sin^2 \gamma$,

$$F_1 = \frac{f_1 f_0}{\sqrt{\frac{(f_0 - f_1)^2}{2} + f_0 f_1 \sin^2 \gamma + (f_0 - f_1)\sqrt{\frac{(f_0 - f_1)^2}{4} + f_0 f_1 \sin^2 \gamma}}}. \quad \ldots \ldots \quad (\text{VII})$$

This formula, when reduced for cylinders of equal positive and negative refraction, $f_0$ being equal to $f_1 = f$, assumes the simple form

$$F_1 = \frac{f}{\sin \gamma}. \quad \ldots \ldots \quad (\text{IX})$$

In the plane of greatest negative refraction, $d_1 m_1 d O_0 X$, we obtain,

$$dX : dd_1 = XO_0 : d_1 m_1.$$

Substituting, $dX = O_0 o = - F_0$ as the negative focus;

$$dd_1 = f_1;$$

$$XO_0 = do = \text{radius} = 1.$$

$$\therefore \quad - F_0 = - \frac{f_1}{d_1 m_1}; \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (49)$$

since $d_1 m_1$ is to be taken negative.

In the parallelogram $d_1 v_1 m_1 z_1$, the angle between the forces, $d_1 v_1$ and $d_1 z_1$, is again $180° - \gamma$; hence,

$$d_1 m_1 = \sqrt{(d_1 z_1)^2 + (d_1 v_1)^2 + 2\,(d_1 z_1)\,(d_1 v_1) \cos\,(180° - \gamma)}.$$

In the oblique plane $d_0 v_0 d v_1 d_1$, we find,

$$d_1 v_1 : dd_1 = d_0 v_0 : dd_0.$$

$$d_0 v_0 = \sin\,(< D_0 o_0 d_0 - < D_0 o_0 A_0) = \sin\,(90° - < D_1 o_1 A_1)$$

$$= \sin\,(90° - \beta) = \cos\beta.$$

$$dd_0 = f_0.$$

$$\therefore \quad d_1 v_1 = \frac{f_1}{f_0} \cos\beta.$$

$$d_1 z_1 = \sin < d_1 o_1 z_1 = \sin\alpha.$$

Substituting these values in the equations for $d_1 m_1$ and (49), we have,

$$- F_0 = - \frac{f_1}{\sqrt{\sin^2\alpha + \left(\frac{f_1}{f_0}\right)^2 \cos^2\beta - 2\frac{f_1}{f_0} \sin\alpha \cos\beta \cos\gamma}};$$

Resorting to the equations (38), (39), (44), and (45), the above may be given the form, $-F_0 =$

$$-\frac{f_1 f_0}{\sqrt{\frac{1}{2}[f_1(f_1-f_0\cos 2\gamma)+(f_0-f_1)(f_0-\sqrt{f_0^2-2f_0f_1\cos 2\gamma+f_1^2})]}}. \quad \ldots \ldots \quad (50)$$

$\therefore \quad -F_0 =$

$$-\frac{f_1 f_0}{\sqrt{\frac{(f_0-f_1)^2}{2}+f_0f_1\sin^2\gamma+(f_1-f_0)\sqrt{\frac{(f_0-f_1)^2}{4}+f_0f_1\sin^2\gamma}}}, \quad \ldots \ldots \quad (\text{VIII})$$

which differs from the formula given for $F_1$ merely by a transposition of the elements in the factor before the second radical, and, consequently, when reduced to cylinders of equal refraction, also becomes

$$-F_0 = -\frac{f}{\sin\gamma}. \quad \ldots \ldots \quad (\text{X})$$

The formulæ (IX) and (X) correspond to those applied to the Stokes Lens.

In reducing the preceding formulæ for given values of cylindrical foci, $f_0$ is to be substituted by the focus of the concave and $f_1$ by the focus of the convex cylinder, both being introduced as positive values.

## 3. RELATIONS BETWEEN THE POSITIVE AND NEGATIVE FOCAL PLANES.

As in this combination the cylinders likewise affect each other most when their axes coincide, and least when their axes are diametrically opposed, we may here also fix upon the limits of $F_1$ and $-F_0$ for $\gamma = 0°$ and $\gamma = 90°$, as in the previous theorem.

When $\gamma = 0°$, or $\cos 2\gamma = +1$, from the formulæ (48) and (50) we find, for $f_0 > f_1$,

$$F_1 = \frac{f_1 f_0}{\sqrt{\frac{1}{2}[-f_1(f_0-f_1)+(f_0-f_1)(f_0+f_0-f_1)]}} = \frac{f_1 f_0}{f_0-f_1}.$$

$$-F_0 = -\frac{f_1 f_0}{\sqrt{\tfrac{1}{2}[-f_1(f_0-f_1)+(f_0-f_1)(f_0-f_0+f_1)]}} = -\frac{f_1 f_0}{0} = -\infty.$$

$$\therefore \quad F_1 : -F_0 = \frac{f_1 f_0}{f_0 - f_1} : -\infty. \quad . \quad . \quad . \quad . \quad (51)$$

For $F_1 = \frac{f_1 f_0}{f_0 - f_1}$, we have as the refraction $\frac{1}{F_1} = \frac{1}{f_1} - \frac{1}{f_0}$; consequently,

***11. When the convex cylinder is of greater refraction than the concave, and their axes are coincident, the positive focal plane will coincide with that focal plane which is defined by the difference of the refractions of the cylinders,* whereas the negative focal plane will be at infinity.***

Placing $\gamma = 0°$, or $\cos 2\gamma = +1$, in the formulæ (48) and (50), we have, for $f_1 > f_0$,

$$F_1 = \frac{f_1 f_0}{\sqrt{\tfrac{1}{2}[f_1(f_1-f_0)-(f_1-f_0)(f_0+f_1-f_0)]}} = \frac{f_1 f_0}{0} = \infty.$$

$$-F_0 = -\frac{f_1 f_0}{\sqrt{\tfrac{1}{2}[f_1(f_1-f_0)-(f_1-f_0)(f_0-f_1+f_0)]}} = -\frac{f_1 f_0}{f_1 - f_0}.$$

$$\therefore \quad F_1 : -F_0 = \infty : -\frac{f_1 f_0}{f_1 - f_0}. \quad . \quad . \quad . \quad . \quad (52)$$

For $-F_0 = -\frac{f_1 f_0}{f_1 - f_0}$, we have as the refraction $-\frac{1}{F_0} = -\left(\frac{1}{f_0} - \frac{1}{f_1}\right)$; consequently,

***12. When the concave cylinder is of greater refraction than the convex, and their axes are coincident, the negative focal plane will coincide with that focal plane which is defined by the difference of the refractions of the cylinders,* whereas the positive focal plane will be at infinity.***

This is shown in Plate IV, Fig. 1.

* Or the sum of their refractions when taken as positive and negative elements.

# PLATE IV

THE REFRACTION BY COMBINED

CONTRA-GENERIC CYLINDRICAL LENSES

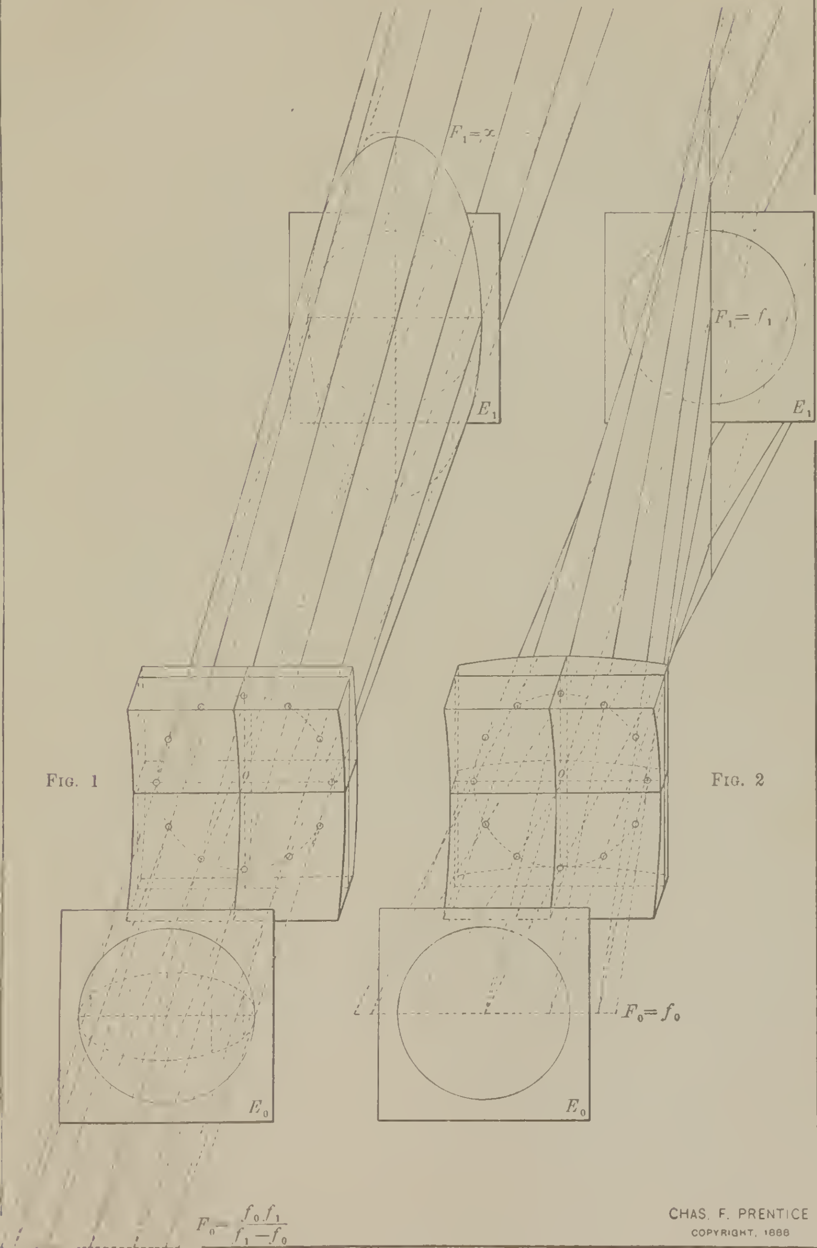


Fig. 1

Fig. 2

Introducing $\gamma = 90°$, or $\cos 2\gamma = \cos 180° = -1$ in the formulæ (48) and (50), we have, for $f_0 \gtreqless f_1$,

$$F_1 = \frac{f_1 f_0}{\sqrt{\frac{1}{2}[f_1(f_1+f_0)+(f_0-f_1)(f_0+f_0+f_1)]}} = \frac{f_1 f_0}{f_0} = f_1.$$

$$-F_0 = -\frac{f_1 f_0}{\sqrt{\frac{1}{2}[f_1(f_1+f_0)+(f_0-f_1)(f_0-f_0-f_1)]}} = -\frac{f_1 f_0}{f_1} = -f_0.$$

$$\therefore \quad F_1 : -F_0 = f_1 : -f_0. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53)$$

From which we deduce :

***13. The positive and negative focal planes coincide with their correlative elementary focal planes, when the axes of the contra-generic cylinders are at right angles to each other.***

This is demonstrated in Plate IV, Fig. 2.

Between the limits of 0° and 90°, for $f_0 > f_1$, we have consequently found $F_1$ to vary between the limits of $\frac{f_1 f_0}{f_0 - f_1}$ and $f_1$ behind the combined lenses, while $F_0$ varies between the limits of $\infty$ and $f_0$ on the incident side of the same.

The convex cylinder being stronger than the concave, when their axes coincide their combined refraction will evidently be equal to that of a periscopic convex cylinder, since $\frac{1}{F_1} = \frac{1}{f_1} - \frac{1}{f_0}$ in the active plane; and $\frac{1}{F_0} = \frac{1}{\infty} = 0$ in the passive plane.

Between the same limits, when $f_1 > f_0$, $F_0$ will vary between $\frac{f_1 f_0}{f_1 - f_0}$ and $f_0$ on the incident side of the combined cylinders, while $F_1$ varies between $\infty$ and $f_1$ behind the same. (See Plate IV.)

In this case, when the axes coincide, it is evident that the resultant refraction will be equal to that of a periscopic concave cylinder, since $-\frac{1}{F_0} = -\left(\frac{1}{f_0} - \frac{1}{f_1}\right)$ in the active plane; and $\frac{1}{F_1} = \frac{1}{\infty} = 0$ for the axial plane.

For an inequality in the refractive power of the cylinders, rotation of one of them, from 0° to 90°, will therefore be associated with corresponding changes in the positions of the resultant focal planes between the limits of infinity and the focus of the weaker cylinder on the one side, and between that focal plane which corresponds to the difference of their refractions and the focus of the stronger cylinder on the other. Since in this case the approach of one focal plane is accompanied by a corresponding recession of the other on the opposite side of the lenses, their movements are, as in the previous theorem, in opposite directions.

When the cylinders are of equal refractive power, $f_1$ being equal to $f_0$, it will follow, from the relation (53), that $F_1 = F_0$, so that, between the limits of 0° and 90°, $F_1$ will vary between infinity and $f_1$ on the positive side, while $F_0$ varies between infinity and $f_0$ on the negative or incident side of the combined cylinders.

Consequently, when the axes coincide, $+ F_1 = + \infty$ and $- F_0 = - \infty$. This is evident, since the refractions of equal convex and concave cylinders, under such circumstances, neutralize each other throughout.

By the previous considerations we therefore here also find :

***14. The positive and negative focal planes are conjugate planes, subject to variations of the angle between the axes of the contrageneric cylinders.***

The diagram, Plate III, has been constructed in accordance with the foregoing provisions.

# III. DIOPTRAL* FORMULÆ.

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As the task of reducing *dioptres* to their focal distances would render calculation by the preceding formulæ somewhat arduous, we may here introduce the formulæ, expressed in refraction, which will be found exceedingly convenient when applied to combinations of cylinders of the metric system more especially.

For the focal distance $F_1$ we have as the refraction $\frac{1}{F_1} = R_1$, and for $f_1$ and $f_2$, similarly, $\frac{1}{f_1} = r_1$ and $\frac{1}{f_2} = r_2$, which may be understood as signifying dioptres of refraction.

By these, and similar substitutions for other foci, we may then write :

### THE DIOPTRAL FORMULÆ FOR COMBINED CONGENERIC CYLINDERS.

$$\cos \alpha = \sqrt{\frac{1}{2} + \frac{1}{2}\,\frac{r_2 + r_1 \cos 2\gamma}{\sqrt{r_1^{\,2} + 2r_1 r_2 \cos 2\gamma + r_2^{\,2}}}} \quad . \quad . \quad . \quad (\mathrm{I}D)$$

$$R_1 = \sqrt{\tfrac{1}{2}(r_1 + r_2)^2 - r_1 r_2 \sin^2\gamma + (r_1 + r_2)\sqrt{\tfrac{1}{4}(r_1 + r_2)^2 - r_1 r_2 \sin^2\gamma}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{II}D)$$

$$R_2 = \sqrt{\tfrac{1}{2}(r_1 + r_2)^2 - r_1 r_2 \sin^2\nu - (r_1 + r_2)\sqrt{\tfrac{1}{4}(r_1 + r_2)^2 - r_1 r_2 \sin^2\gamma}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{III}D)$$

To retain the significances of $R_1$ and $R_2$, in calculating, $r_1$ should reoresent the greater cylindrical refraction.

$$R_1 = r\,(1 + \cos \gamma). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{IV}D)$$

$$R_2 = r\,(1 - \cos \gamma). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{V}D)$$

* The adaptation of this adjective would seem justifiable, since the unit "dioptre" has been chosen in distinction to "dioptric," which, though related, has another significance.

## THE DIOPTRAL FORMULÆ FOR COMBINED CONTRA-GENERIC CYLINDERS.

$$\cos \alpha = \sqrt{\frac{1}{2} + \frac{1}{2} \frac{r_1 - r_0 \cos 2\gamma}{\sqrt{r_1^2 - 2r_1 r_0 \cos 2\gamma + r_0^2}}}. \quad . \quad \text{(VI}D\text{)}$$

$$R_1 = \sqrt{\tfrac{1}{2}(r_1 - r_0)^2 + r_1 r_0 \sin^2 \gamma + (r_1 - r_0)\sqrt{\tfrac{1}{4}(r_1 - r_0)^2 + r_1 r_0 \sin^2 \gamma}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VII}D\text{)}$$

$$-R_0 = -\sqrt{\tfrac{1}{2}(r_1 - r_0)^2 + r_1 r_0 \sin^2\gamma + (r_0 - r_1)\sqrt{\tfrac{1}{4}(r_1 - r_0)^2 + r_1 r_0 \sin^2\gamma}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(VIII}D\text{)}$$

$$R_1 = r \sin \gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(IX}D\text{)}$$

$$-R_0 = -r \sin \gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(X}D\text{)}$$

If, in (II$D$) and (III$D$), the convex element $r_2$ be replaced by the concave element $-r_0$, we obtain (VII$D$) and (VIII$D$).

By the aid of these formulæ we may also arrive at the following significant facts.

The formulæ (II$D$) and (III$D$) may be written :

$$R_1^2 = \tfrac{1}{2}(r_1 + r_2)^2 - r_1 r_2 \sin^2 \gamma + (r_1 + r_2)\sqrt{\tfrac{1}{4}(r_1 + r_2)^2 - r_1 r_2 \sin^2 \gamma},$$

$$R_2^2 = \tfrac{1}{2}(r_1 + r_2)^2 - r_1 r_2 \sin^2 \gamma - (r_1 + r_2)\sqrt{\tfrac{1}{4}(r_1 + r_2)^2 - r_1 r_2 \sin^2 \gamma},$$

which, by addition, result in the equation,

$$R_1^2 + R_2^2 = (r_1 + r_2)^2 - 2r_1 r_2 \sin^2 \gamma.$$

$$\therefore \quad (R_1 + R_2)^2 - 2R_1 R_2 = (r_1 + r_2)^2 - 2r_1 r_2 \sin^2 \gamma.$$

$$\therefore \quad (R_1 + R_2)^2 = (r_1 + r_2)^2 - 2r_1 r_2 \sin^2 \gamma + 2R_1 R_2.$$

Multiplying (II$D$) by (III$D$), we find,

$$2R_1 R_2 = 2r_1 r_2 \sin^2 \gamma.$$

$$\therefore \quad R_1 + R_2 = r_1 + r_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54)$$

From which we conclude :

***15. The sum of the primary and secondary refractions is a constant, being equal to the sum of the elementary refractions for any combination, and all deviations of the axes of two combined congeneric cylinders.***

In the same manner, we obtain from the formulæ (VII$D$) and (VIII$D$),

$$R_1 - R_0 = r_1 - r_0, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55)$$

and therefore here also find,

***16. The sum of the principal positive and negative refractions is a constant, being equal to the sum of the positive and negative elementary refractions for any combination, and all deviations of the axes of two combined contra-generic cylinders.***

The total inherent refraction always remaining the same for any combination, the angle $\gamma$ merely performs the function of allotting the proportions of refraction $R_1$ and $R_2$, or $R_1$ and $R_0$, in the resultant principal planes.

By the equations (54) and (55), calculation may be greatly simplified. $R_1$ being determined for a specific value of $\gamma$, we may readily fix upon $R_2$ or $R_0$ by these equations.

This is demonstrated in the appended tables, although it has not been utilized in calculating ; on the contrary, a study of these led to the above deductions.

# IV. SPHERO-CYLINDRICAL EQUIVALENCE.

SINCE, for any combination of cylinders, the principal planes of refraction are at right angles to each other for all values of $\gamma$, there can be no reasonable doubts, under the provisions made at the opening of this demonstration, as to the equivalence of a sphero-cylindrical lens to one composed of combined cylinders. However, the use of such lenses being at present confined to the correction of errors of refraction in the human eye, it is evident, from the movements of the eye behind the fixed lens, that the visual axis cannot at all times coincide with the optical axis of the lens chosen, so that, in those practical attempts at substitution, which may at times prove to be unsatisfactory, the cause might seemingly be explained by the possibility of a difference becoming manifest for the more peripheral incident rays, although equally distant from the optical centre of either form of lens. In other words, the available field in the one may be greater or less than in the other, which, however, is likely to prove appreciable only in lenses of extreme curvature, and possibly in combinations of cylinders which widely differ in their individual refractions. This would remain to be shown.

To substitute a sphero-cylindrical lens for combined cylinders, the proposition is merely one demanding that the "focal interval" be the same, at the same distance from the principal plane, at the optical centre, for each of the compound lenses. The distances $F_1$ and $F_2$ being determined for any angular deviation $\gamma$ of the axes, in a combination of congeneric cylinders, for instance, the substitution is accomplished by making a sphero-cylindrical lens in which the focus of the spherical element is equal to $F_2$, and of the cylindrical element equal to $\frac{F_1 F_2}{F_2 - F_1}$, or, if expressed by refraction, $\frac{1}{F_1} - \frac{1}{F_2}$ sph. $= \frac{1}{F}$ cyl.

If the primary and secondary planes of the sphero-cylindrical lens are to coincide with those resulting from a combination of two definitely placed congeneric cylinders, the formula (I) and the articles 2 and 3 are to be referred to.

Comparing the sphero-cylindrical equivalent with the rotating cylinders, reference being had to Plate II, Fig. 2, a reduction of the angle $\gamma$ from $90°$ would be equivalent to a spherical element of the focus $F_2$, constantly decreasing from the focus $f_2$ to $\infty$, associated with a cylindrical element of the focus $F_c$, constantly increasing from the focus $\frac{f_1.f_2}{f_2 - f_1}$ to $\frac{f_1.f_2}{f_2 + f_1}$; or, in other words, a gradually decreasing potency of the spherical refraction $\frac{1}{F_2}$, from $\frac{1}{f_2}$ to $\frac{1}{\infty} = 0$, gives way to a proportionately increasing cylindrical refraction $\frac{1}{F_c}$, from $\frac{1}{f_1} - \frac{1}{f_2}$ to $\frac{1}{f_1} + \frac{1}{f_2}$. As an instance, if $f_1 = f_2 = f$, $\frac{1}{F_c}$ will increase from $\frac{1}{f_1} - \frac{1}{f_2} = 0$ to $\frac{2}{f}$, or twice the refraction of either cylinder. In this case, all successive values of cylindrical refraction will therefore be inherent between 0 and $\frac{2}{f}$.

Should a means be devised to suppress the spherical element for each successive value of $\gamma$, the remaining varying cylindrical element being thus rendered available for measuring corresponding degrees of astigmatism in the eye, the formulæ here advanced would prove of service in obtaining the graduations upon the rotating plates of such an instrument.

While cases of anomalous ocular refraction demanding a correction by combined cylinders are fortunately exceedingly rare, we may nevertheless be permitted to passingly allude to certain methods of procedure in such instances. We shall confine the subject to congeneric cylinders. In a case of astigmatism, for which the diagnosis has resulted in fixing upon two cylinders combined under the angle $\gamma$, the lenses are to be withdrawn from the trial frame and inserted in a graduated cell, so arranged as to facilitate their being rigidly fixed in any desired position for $\gamma$.

The positions of the principal planes of refraction are then estimated for this fixed combination, in the usual manner, without regard to the nature of the elements constituting it; the proportions of spherical and cylindrical refraction being revealed through neutralization by lenses from the trial set. The so determined lenses are then to be substituted in the trial frame, when rotation of the cylinder will lead to that position of it which is most acceptable to the patient. The spherical and cylindrical elements will probably then also bear of further modification, as a means of excluding any error which may have been caused by lack of absolute contact of the original cylinders in the cell. The formulæ may be resorted to as a further and more definite verification.

It having been shown that successive changes in the angle $\gamma$ are associated with corresponding changes of $F_1$ and $F_2$, the above substitution would indeed seem advisable, since the present appliances for grinding cylindro-cylindrical lenses are not constructed with sufficient precision to enable opticians to fix the relative positions of the cylinders beyond mere approximation.

As an illustration, let us select two congeneric cylinders of equal foci, say 20 inches, combined under the angle $\gamma = 60°$. Introducing these values in the formulæ (IV) and (V), we find,

$$F_1 = \frac{20}{1 + \cos 60°} = \frac{20}{1 + 0.5} = 13.33,$$

$$F_2 = \frac{20}{1 - \cos 60°} = \frac{20}{1 - 0.5} = 40.$$

We then obtain the cylindrical refraction $\frac{1}{F_c}$, for the desired sphero-cylindrical equivalent, from the equation,

$$\frac{1}{F_1} - \frac{1}{F_2} = \frac{1}{F_c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

Substituting herein the calculated values for $F_1$ and $F_2$ gives,

$$\frac{1}{13.33} - \frac{1}{40} = \frac{1}{F_c} = \frac{1}{20}.$$

$\frac{1}{F_2} = \frac{1}{40}$ being the spherical element, we therefore have the sphero-cylindrical equivalent,

$$\frac{1}{40} \text{ sph.} \supset \frac{1}{20} \text{ cyl.}$$

as an available substitute for the cylindro-cylindrical lens,

$$\frac{1}{20} \text{ cyl. axis } 0° \supset \frac{1}{20} \text{ cyl. axis } 60°,$$

without regard to a definite position of these lenses before the eye.

By way of comparison, allowing the optician to make an error of apparently so small an amount as 2°, in producing the same cylindro-cylindrical lens, we obtain, by introducing $\gamma = 62°$ in the same formulæ,

$$F_1 = \frac{20}{1 + \cos 62°} = \frac{20}{1 + 0.469} = \frac{20}{1.47} = 13.61,$$

$$F_2 = \frac{20}{1 - \cos 62°} = \frac{20}{1 - 0.47} = \frac{20}{0.53} = 37.73.$$

Substituting these values in the equation (56), we have,

$$\frac{1}{13.61} - \frac{1}{37.73} = \frac{1}{F_c} = \frac{1}{21.29},$$

from which we obtain the sphero-cylindrical lens,

$$\frac{1}{37.73} \text{ sph.} \supset \frac{1}{21.29} \text{ cyl.}$$

Had the optician been required to make a sphero-cylindrical lens $\frac{1}{40}$ sph. $\supset$ $\frac{1}{20}$ cyl., his execution of it presenting such discrepancies as $\frac{1}{37.73}$ sph. $\supset$ $\frac{1}{21.29}$ cyl., would certainly be rejected as being unsatisfactory, a notable difference of 2.27 inches focal distance being manifest in the spherical element.

On the other hand, instances are likely to occur for which it will be impossible, by the advanced method of neutralization, to accurately arrive at the sphero-cylindrical equivalent.

Since $\frac{1}{20}$ cyl. axis 0° $\supset$ $\frac{1}{20}$ cyl. axis 62° = $\frac{1}{37.73}$ sph. $\supset$ $\frac{1}{21.43}$ cyl., we should evidently be unable to satisfactorily neutralize such spherical and cylindrical elements by any of the lenses in the series of a trial set.

In those instances, therefore, where satisfactory neutralization of the principal planes of refraction cannot be attained for the combined cylinders, in the graduated cell, the cylindro-cylindrical lens will have to be chosen, again under the proviso, however, of a faultless mechanical execution. The sphero-cylindrical equivalent being, however, generally available, *we are to suspect error in our estimation of the refraction of an eye seeming to demand cylinders combined under acute or obtuse angles.* Having found an opportunity to apply the formulæ in practice, I take pleasure in citing the following case.

A cylindro-cylindrical lens $-\frac{1}{40}$ cyl. axis 0° $\supset$ $-\frac{1}{40}$ cyl. axis 70° had been prescribed for Mr. G. B. Owen, of New York, by his oculist in Philadelphia, in 1880–'1, the above correction having been worn continually since that time, while affording vision $= \frac{6}{6}$ for the left eye.

This case being known to me, I was anxious to make the substitution of the sphero-cylindrical equivalent, which I obtained as follows:

The lenses being congeneric concave cylinders of equal refraction, by the formulæ (IV) and (V), for $f = 40$ and $\gamma = 70°$, we have,

$$F_1 = \frac{40}{1 + 0.34202} = 29.806 = 30,$$

$$F_2 = \frac{40}{1 - 0.34202} = 60.79 = 60,$$

it being admissible to neglect the fractions for such focal distances.

By article 2, we find the position of the cylindrical axis equal $\frac{\gamma}{2} = 35°$, and consequently the sphero-cylindrical equivalent,

$$-\frac{1}{60} \text{ sph.} \supset -\frac{1}{60} \text{ cyl. axis } 35°.$$

This lens has been substituted with the knowledge and to the entire satisfaction of the patient.

It is therefore obvious that the meridian (125°) of greatest refraction in the eye had not been disclosed by the diagnosis.

The weak spherical element, in the substituted lens, while being an appreciable factor to the patient, might easily have been overlooked by the practitioner.

In similar cases, the advanced formulæ must prove of value in fixing upon the true state of the refraction.

# V. VERIFICATION OF THE FORMULÆ.

In the following tables, the Dioptric and Dioptral Formulæ have been applied to combinations of cylinders of the inch and metric systems, respectively, it being inadmissible to substitute the generally adopted inch-system equivalents for dioptres, in calculating, as the frequent repetitions of the former as factors in the dioptral formulæ would increase the neglected differences to an unwarrantable degree. For the purpose of obtaining reliable results, the calculations have been carried to the fifth decimal place under the radicals. The angles 30°, 45°, and 60° have been chosen so as to exhibit appreciable differences in the corresponding resultant refractions, which are thereby also brought within the lens-series of the inch and metric systems. The elementary foci and refractions have, in a measure, been arbitrarily selected, it being noticeable that the secondary refraction will generally be beyond the limits of neutralization for combinations of weaker cylinders, in which the axes deviate by less than 30°.

The *Approximates* given for refraction, in Table 1, will at times appear to conflict with the articles 15 and 16; this, however, is to be attributed to changes of proportion occasioned by the adopted substitutions.

To substantiate the resultant refractions given in the tables, through the experiment of neutralization, the cylindrical axes should first be accurately determined, when the cylinders are to be so united as to insure *absolute* contact of their plane surfaces.

Great care should also be taken to accurately and rigidly combine the cylinders under the specified angles, as the slightest variation will prove misleading. In the practical experiment, the observer's eye will generally fail to appreciate the neglect of fractions made necessary by the available lenses of an oculist's trial case.

## 1. TABLES IN VERIFICATION OF THE DIOPTRIC FORMULÆ.

### FOR COMBINED CONGENERIC CYLINDERS.

| Elementary Foci. | Axial Deviation. | Primary Focus. | Primary Refraction. | Secondary Focus. | Secondary Refraction. |
|---|---|---|---|---|---|
| $f_1 < f_2$. | $\gamma$ | $F_1$ | *(Approximate.)* | $F_2$ | *(Approximate.)* |
| $16 \supset 24$ | 30° | 10.2576 | 1/10 | 149.7422 | 1/160 |
| " " | 45° | 11.1555 | 1/11 | 68.8347 | 1/72 |
| " " | 60° | 12.5559 | 1/12 | 40.7773 | 1/40 |

### FOR COMBINED CONTRA-GENERIC CYLINDERS.

| Elementary Foci. | Axial Deviation. | Positive Focus. | Positive Refraction. | Negative Focus. | Negative Refraction. |
|---|---|---|---|---|---|
| $f_0 > f_1$. | $\gamma$ | $+F_1$ | *(Approximate.)* | $-F_0$ | *(Approximate.)* |
| $14 \supset 10$ | 35° | 16.9799 | +1/16 | 32.9799 | −1/32 |
| " " | 45° | 13.2046 | +1/13 | 21.2046 | −1/22 |
| " " | 60° | 11.2537 | +1/11 | 16.5870 | −1/16 |
| $f_0 < f_1$. | $\gamma$ | $+F_1$ | *(Approximate.)* | $-F_0$ | *(Approximate.)* |
| $14 \supset 20$ | 30° | 47.5527 | +1/48 | 23.5527 | −1/24 |
| " " | 45° | 30.4131 | +1/30 | 18.4131 | −1/18 |
| " " | 60° | 23.7316 | +1/24 | 15.7315 | −1/16 |

## 2. TABLES IN VERIFICATION OF THE DIOPTRAL FORMULÆ.

### FOR COMBINED CONGENERIC CYLINDERS.

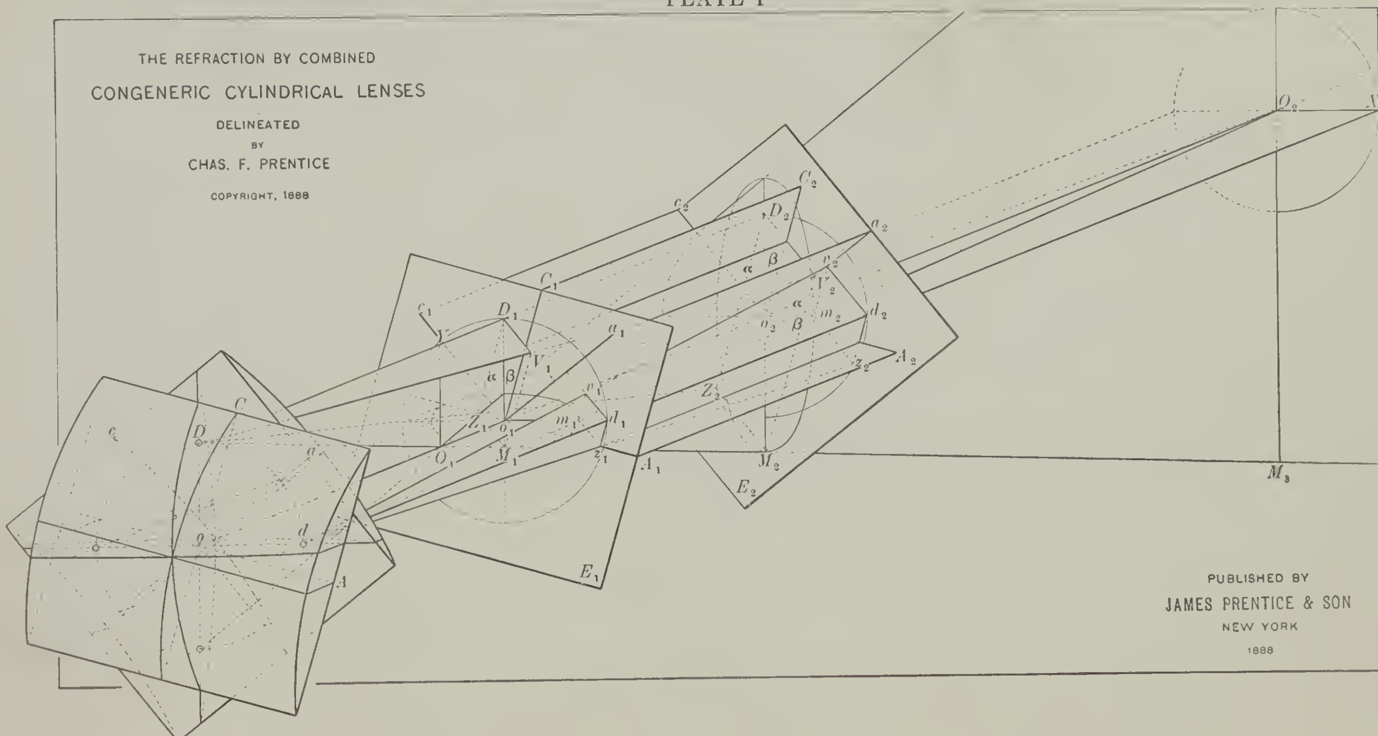
| Elementary Refractions. | Axial Deviat'n. | Primary Refraction. | | Secondary Refraction. | | $R_1 + R_2 = r_1 + r_2$ |
|---|---|---|---|---|---|---|
| $r_1 > r_2$ | $\gamma$ | $R_1$ | *(Approx.)* | $R_2$ | *(Approx.)* | |
| $2.5 \supset 1.5D.$ | 30° | $3.75D.$ | $3.75D.$* | $0.25D.$ | $0.25D.$ | $4D.$ |
| " " | 45° | 3.46 | 3.5 | 0.54 | 0.5 | 4 |
| " " | 60° | 3.09 | 3. | 0.91 | 1. | 4 |

### FOR COMBINED CONTRA-GENERIC CYLINDERS.

| Elementary Refractions. | Axial Deviat'n. | Positive Refraction. | | Negative Refraction. | | $R_1 - R_0 = r_1 - r_0$ |
|---|---|---|---|---|---|---|
| $r_1 > -r_0$ | $\gamma$ | $+R_1$ | *(Approx.)* | $-R_0$ | *(Approx.)* | |
| $+4 \supset -2.75D.$ | 30° | $2.397D.$ | $+2.5D.$ | $1.147D.$ | $-1.25D.$ | $+1.25D.$ |
| " " | 45° | 3.052 | +3. | 1.802 | −1.75 | +1.25 |
| " " | 60° | 3.564 | +3.5 | 2.314 | −2.25 | +1.25 |
| $r_1 < -r_0$ | $\gamma$ | $+R_1$ | *(Approx.)* | $-R_0$ | *(Approx.)* | $R_1 - R_0 = r_1 - r_0$ |
| $+2 \supset -2.75D.$ | 30° | $0.856D.$ | $+0.75D.$ | $1.606D.$ | $-1.5D.$ | $-0.75D.$ |
| " " | 45° | 1.325 | +1.25 | 2.075 | −2. | −0.75 |
| " " | 60° | 1.690 | +1.75 | 2.440 | −2.5 | −0.75 |

* If $4D.$ be written, then $R_1 + R_2 = 4.25D.$, which would be more refraction than is inherent in the combination, yet in neutralizing by $4D.$ the error will scarcely be detected.

## PLATE I




THE REFRACTION BY COMBINED

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DELINEATED

BY

CHAS. F. PRENTICE



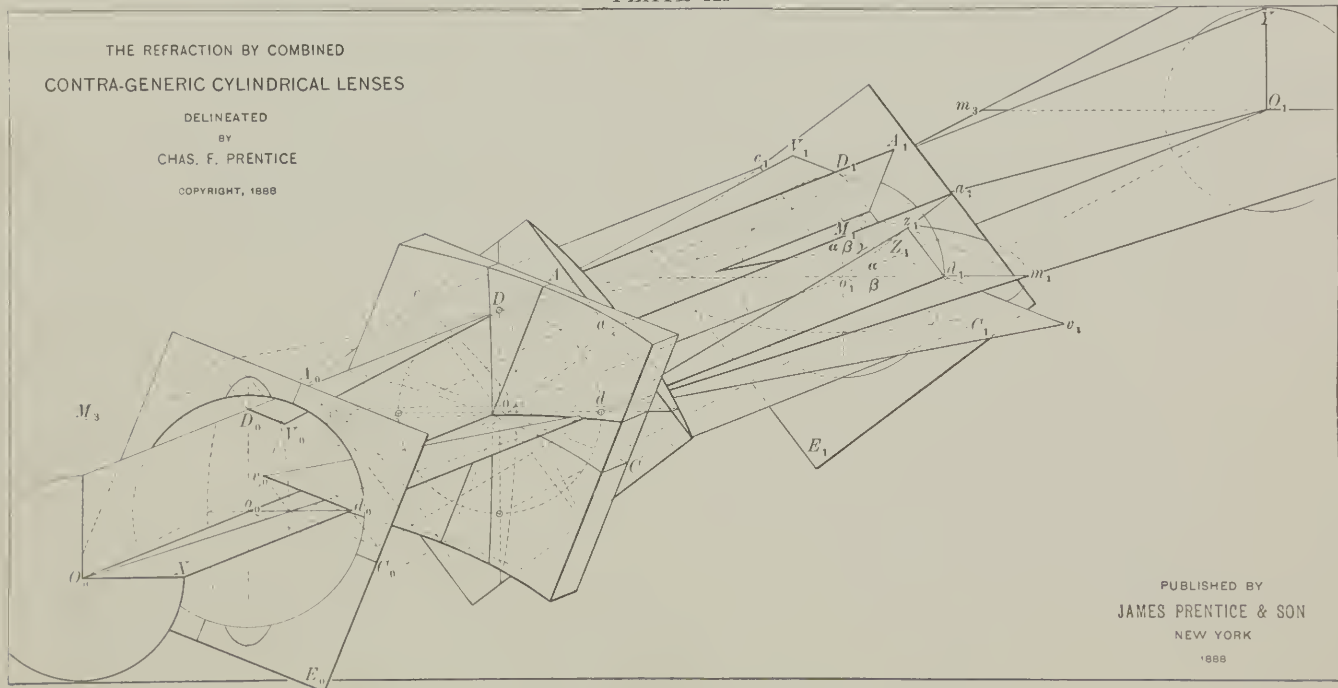
PUBLISHED BY

JAMES PRENTICE & SON

NEW YORK

1888

# PLATE III



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