ELEMENTS OF GEOMETRY

AFTER LEGENDRE,

WITH

A SELECTION OF GEOMETRICAL EXERCISES, AND
HINTS FOR THE SOLUTION OF THE SAME.

BY

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PREFACE.

The Geometry of Legendre has held its high place among works on Elementary Geometry for more than sixty years. During this period a number of modifications have been suggested by various editors and commentators in the methods of treating particular subjects, but the great body of his work has stood the test of time as the most successful modification of Euclid which has ever appeared. It forms the basis of all the later text-books on Elementary Geometry which have appeared in France, and of all the recent works designed to modernize the Euclidian Geometry for the schools and colleges of England, and of nearly all the geometrical instruction in America, for thirty or forty years.

In the preparation of the present edition a careful analysis has been made of such works as Planche (Cahiers de Géométrie pour servir de complement au traité de Legendre), Blanchet's Legendre, Bobillier, Amiot, and of the very complete work of Rouchè and De Comberousse, and the editor has adopted such additions and changes as he considered improvements on the original. These changes consist mainly in the discussion of parallels; in the treatment of tangencies (in Book II.); in the addition of some theorems and the omission of a few; in the substitution of the method of limits for the method of the *reductio ad absurdum* in the treatment of the measure of the circle and of the "three round bodies;" and in the fuller treatment of the plane and the triedral in Book V.

While this edition is made from a completely new translation of the French edition of Legendre of 1865, yet the translation has been carefully compared with that of Sir David Brewster, and his words adopted where they seemed fittest. Brewster's example has also been followed in making the "Doctrine of Proportion" introductory, instead of breaking the continuity of the text, by inserting it between two of the books.

The chief new feature of the edition is the addition, to each book, of exercises adapted to the order of the theorems of the book. These exercises are such as have been thoroughly tested in the instruction of classes in Elementary Geometry. They serve the purpose of increasing the pupil's knowledge of geometrical truths as well as the more important one of giving him thorough instruction in the methods of Geometrical Analysis. They are numerous, in order that the teacher may have a wide range of selection. They have been chosen mainly from the work of Planche, alluded to above, but some of them from the collections of Guilmin, Amiot, Ritt, Rouchè and De Comberousse, and from Potts' Euclid.

While most of the exercises relate to the methods of pure Geometry, a number of numerical problems have been added to some of the books. It is believed that these arithmetical applications will interest the pupil and help to fix his knowledge of the theorems on which they depend.

The feature of Hints to the Solutions, which serve as a guide to the study of Geometrical Analysis, will be appreciated both by teachers and pupils. For, while it is most desirable to engage the attention of beginners with geometrical exercises, help enough should be given to prevent them from being disheartened.

In conclusion, the editor cannot refrain from adverting to the excellence of the mechanical execution of the book and from expressing the hope that this will prove a practically useful edition of Elementary Geometry.

CHAS. S. VENABLE.

UNIVERSITY OF VIRGINIA,
January, 1875.

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ELEMENTS OF GEOMETRY.

INTRODUCTION.

[For the sake of those who have not studied the Doctrine of Proportion, it is requisite to prefix to the treatise of Legendre a brief outline of its fundamental principles. After studying Book I. and Book II. to Prop. XVI.. the beginner should study Sections II. and III. of this Introduction ; reserving, whenever it is thought advisable, the remaining sections for the review of the book.]

I. PRELIMINARY NOTIONS.

1. Our first notion of whole numbers arises from considering distinct and similar objects. The measure of magnitudes brings us to a necessary extension of this first notion.

2. When a magnitude is the sum of 2, 3, 4 · · · parts, equal to another magnitude of the same species, we say that the first is a *multiple* of the second, and that the second is an *aliquot part* of the first.

3. Two magnitudes are said to be *commensurable* with each other when they are both multiples of a third magnitude, which is called their *common measure.* When there is no third quantity of which they are both multiples, they are said to be *incommensurable* with each other.

4. To *measure* a magnitude, we seek a common measure between this magnitude and an arbitrary magnitude of the same species, which we call the *unit of measure,* or simply the *unit.*

If this common measure is the unit itself, and the given magnitude contains it, for example, three times, we say that the magnitude is measured by the number 3. If the common measure is an aliquot part of the unit; for example, if, when the unit is divided into five equal parts, the given magnitude is the sum of three of these parts; we say that this magnitude is three-fifths of the unit, and that it is measured by the fractional number $\frac{3}{5}$.

To sum up : *to measure a magnitude commensurable with the unit, is to seek how often this magnitude contains the unit or aliquot parts of the* *unit.* According as the magnitude is a multiple of the unit or a multiple of an aliquot part of the unit, *the number which expresses its measure is entire or fractional.* Conversely, every magnitude measured by an entire or fractional number is commensurable with the unit of measure.

5. Let us now consider a magnitude incommensurable with the chosen unit of measure. Here we shall find useful the following definition: *The limit of a variable number is a number which that variable number may approach in value as near as we please, but which it can never reach.*

Now, conceive the unit to be divided into any number, n, of parts equal to one another and less than the magnitude, G, to be measured. Taking 1, 2, 3, 4 . . . . of these parts we shall form a series of magnitudes,

$$A_1 \; A_2 \; A_3 \; A_4 \; . \; . \; . \; . \; A_k \; A_{k+1} \qquad (1)$$

measured respectively by the numbers

$$\frac{1}{n} \quad \frac{2}{n} \quad \frac{3}{n} \quad \frac{4}{n} \qquad \frac{k}{n} \quad \frac{k+1}{n}.$$

By going far enough in the series (1) we shall find two consecutive magnitudes, A_k and A_{k+1}, between which G lies. Now G differs from A_k or A_{k+1} by a quantity less than $A_{k+1} - A_k$, and this difference, being the nth part of the unit, can be made as small as we please by making n sufficiently great.

The magnitude G being then the common limit of the commensurable numbers A_k and A_{k+1}, the number which measures it is, by definition, the common limit of the numbers $\frac{k}{n}$ and $\frac{k+1}{n}$, which measure A_k and A_{k+1}; for, as this measure differs from $\frac{k}{n}$ or $\frac{k+1}{n}$ by less than $\frac{1}{n}$, we can make either of these numbers approach it as near as we please by taking n sufficiently great.

6. A number is called *commensurable* or *incommensurable* according as the magnitude of which it expresses the measure, is commensurable or incommensurable with the unit adopted. The commensurable numbers are entire numbers or fractions.

The result of operations to be performed on incommensurable numbers, is the limit of the results obtained by substituting for them commensurable numbers which approach them more and more nearly in value.

II. RATIOS.

NOTE.—In treating of Ratios and Proportion we shall first confine the discussion to commensurable quantities, and then extend the demonstrations to incommensurable quantities.

1. Ratio is the relation which one quantity bears to another in respect of magnitude, the comparison being made by considering what multiple, part, or parts, one is of the other.

Thus, in comparing 6 with 3 we observe that it has a certain magnitude with respect to 3, which it contains twice; again, in comparing it with 2 we see that it has a different *relative* magnitude, for it contains 2 three times; or, 6 is greater when compared with 2 than it is when compared with 3. The ratio of a to b is usually expressed by two points placed between them thus, $a : b$; and the former, a, is called the *antecedent* of the ratio, the latter, b, the *consequent*.

2. Since in the ratio $a : b$ the comparison is made in regard to quantuplicity, the ratio may evidently be expressed by what is necessary to *multiply* b *by to obtain* a. But this multiplier is the fraction $\frac{a}{b}$.

Then the ratio $a : b$ is measured by the fraction $\frac{a}{b}$, and for shortness we may say that the *ratio* of a to b *is equal* to $\frac{a}{b}$ *or is* $\frac{a}{b}$, or, in general,

A ratio is measured by the fraction which has for its numerator the antecedent of the ratio, and for its denominator the consequent of the ratio.

3. Hence, we may say that the ratio of a to b is equal to the ratio of c to d, when $\frac{a}{b} = \frac{c}{d}$.

4. If the terms of a ratio be multiplied or divided by the same quantity, the ratio is not altered.

For $$\frac{a}{b} = \frac{ma}{mb}.$$

REMARK.—ma and mb are called *equimultiples* of a and b.

5. In general, we can operate on ratios by the rules for operating on fractions.

III. PROPORTION.

Four quantities are said to be *proportionals* when the first is the same multiple, part, or parts of the second that the third is of the fourth; that is, when $\frac{a}{b} = \frac{c}{d}$, the four quantities a, b, c, d, are called proportionals. This is usually expressed by saying a is to b as c is to d, and is thus represented $a : b :: c : d$, or thus, $a : b = c : d$.

The terms a and d are called the *extremes*, and b and c the *means.*

Theorem I.

If four quantities are proportionals, the product of the extremes is equal to the product of the means.

Let a, b, c, d be the four quantities; then, since they are proportionals, $\frac{a}{b} = \frac{c}{d}$; and, by multiplying both sides of the equation by bd, we have $ad = bc$.

Note.—It is evident that since the product of two magnitudes has no meaning, the multipliers, at least, must be abstract numbers.

Corollary 1. Hence, if the first is to the second as the second is to the third, the product of the extremes is equal to the square of the mean. Thus, if $a : b = b : c$, then $ac = b^2$.

Cor. 2. Any three terms in a proportion being given, the fourth may be determined from the equation $ad = bc$.

$$\text{For } d = \frac{bc}{a},\ c = \frac{ad}{b},\ b = \frac{ad}{c},\ a = \frac{bc}{d}.$$

Hence, we have the Single Rule of Three in Arithmetic.

Theorem II.

If the product of two quantities be equal to the product of two others, the four are proportionals; the factors of either product being taken for the means and the factors of the other for the extremes.

Let $xy = ab$; then, dividing by ay, thus, $\frac{x}{a} = \frac{b}{y}$, or $x : a = b : y$.

Theorem III.

If $a : b = c : d$ and $c : d = e : f$, then also $a : b = e : f$.

Because $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$, therefore $\frac{a}{b} = \frac{e}{f}$, or $a : b = e : f$.

Theorem IV.

If four quantities be proportionals, they are proportionals when taken inversely.

If $a : b = c : d$, then $b : a = d : c$.

For $\frac{a}{b} = \frac{c}{d}$; divide unity by each of these equal quantities, thus,

$$\frac{b}{a} = \frac{d}{c},$$

or, $$b : a = d : c.$$

Theorem V.

If four quantities be proportionals, they are proportionals when taken alternately.

If $a : b = c : d$, then $a : c = b : d$.

For $\frac{a}{b} = \frac{c}{d}$; multiply by $\frac{b}{c}$, thus, $\frac{a}{c} = \frac{b}{d}$;

or, $$a : c = b : d.$$

Remark 1.—Theorems IV. and V. are immediate consequences of Theorem II. For we can change the order of the terms in any way which still renders the product of the extremes equal to the product of the means.

Remark 2.—Unless the four quantities are of the same kind, the alternation of the terms cannot take place; because this operation supposes the first to be some multiple, part, or parts of the third. One line may have to another line the same ratio as one weight has to another weight, but there is no relation with respect to magnitude between a line and a weight. In such cases, however, if the four quantities be *represented by numbers*, or by other quantities which are all of the same kind, the alternation may take place, and the conclusions drawn from it will be just.

Theorem VI.

When four quantities are proportionals, the first together with the second is to the second, as the third together with the fourth is to the fourth.

If $a : b = c : d$, then $a + b : b = c + d : d$.

For $\frac{a}{b} = \frac{c}{d}$; and, adding 1 to both sides,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1;$$

that is, $$\frac{a+b}{b} = \frac{c+d}{d}.$$

or, $$a + b : b = c + d : d.$$

This operation is called *componendo*, and the quantities are said to be in proportion by composition.

Theorem VII.

Also, the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth.

For $\frac{a}{b} = \frac{c}{d}$; subtracting 1 from both sides,

$$\frac{a}{b} - 1 = \frac{c}{d} - 1;$$

that is,

$$\frac{a-b}{b} = \frac{c-d}{d},$$

or,

$$a - b : b = c - d : d.$$

This operation is called *dividendo*, and the quantities are said to be proportionals by division.

Cor. Since $\frac{a}{b} = \frac{c}{d}$ gives $\frac{b}{a} = \frac{d}{c}$ (Theorem IV.), then Theorems VI. and VII. give

$$a + b : a = c + d : c$$

$$a - b : a = c - d : c,$$

or, by inversion,

$$a : a + b = c : c + d$$

$$a : a - b = c : c - d.$$

That is, *the first is to the sum of the first and second, as the third is to the sum of the third and fourth.* Again, *the first is to its excess above the second, as the third is to its excess above the fourth.* This last operation is called *convertendo.*

Theorem VIII.

When four quantities are proportionals, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.

If $a : b = c : d$, then $a + b : a - b = c + d : c - d$.

For (Theorem VI.)

$$\frac{a+b}{b} = \frac{c+d}{d},$$

and (Theorem VII.)

$$\frac{a-b}{b} = \frac{c-d}{d},$$

Therefore,

$$\frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d};$$

that is,

$$\frac{a+b}{a-b}=\frac{c+d}{c-d},$$

or, $$a+b : a-b = c+d : c-d.$$

Theorem IX.

When any number of quantities are proportionals, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let $$a : b = c : d = e : f;$$

then $$a : b = a+c+e : b+d+f.$$

Because $\frac{a}{b}=\frac{c}{d}$, $ad = bc$; because $\frac{a}{b}=\frac{e}{f}$, $af = be$; also, $ab = ba$,

hence, $$ab + ad + af = ba + bc + be,$$

that is, $$a(b+d+f) = b(a+c+e).$$

Hence (Theorem II.),

$$a : b = a+c+e : b+d+f;$$

and similarly when more quantities are taken.

Theorem X.

When four quantities are proportionals, if the first and second be multiplied or divided by any quantity, as also the third and fourth, the resulting quantities will be proportionals.

Let $a : b = c : d$; then $ma : mb = nc : nd$.

For $\frac{a}{b}=\frac{c}{d}$, therefore, $\frac{ma}{mb}=\frac{nc}{nd}$,

or, $$ma : mb = nc : nd.$$

Theorem XI.

If the first and third be multiplied or divided by any quantity, and also the second and fourth, the resulting quantities will be proportionals.

Let $a : b = c : d$, then $ma : nb = mc : nd$.

For $\frac{a}{b}=\frac{c}{d}$, therefore, $\frac{ma}{b}=\frac{mc}{d}$ and $\frac{ma}{nb}=\frac{mc}{nd}$,

or, $$ma : nb = mc : nd.$$

Theorem XII.

In two ranks of proportionals, if the corresponding terms be multiplied together the products will be proportionals.

If $a : b = c : d$,

and $e : f = g : h$;

then $ae : bf = cg : dh$.

For $\frac{a}{b} = \frac{c}{d}$ and $\frac{e}{f} = \frac{g}{h}$, therefore, $\frac{ae}{bf} = \frac{cg}{dh}$,

that is, $ae : bf = cg : dh$.

This is called *compounding* the proportions. The proposition is true if applied to *any number* of proportions.

Theorem XIII.

If four quantities be proportionals, the like powers or roots of these quantities will be proportionals.

If $a : b = c : d$, then $a^n : b^n = c^n : d^n$.

For $\frac{a}{b} = \frac{c}{d}$, therefore, $\frac{a^n}{b^n} = \frac{c^n}{d^n}$ when n may be whole or fractional;

that is, $a^n : b^n = c^n : d^n$.

Theorem XIV.

If three quantities, a, b, c, *be in continued proportion,*

that is, if $a : b = b : c$, then $a : c = a^2 : b^2$.

For $\frac{a}{b} = \frac{b}{c}$; multiply by $\frac{a}{b}$,

therefore, $\frac{a}{b} \times \frac{a}{b} = \frac{a}{b} \times \frac{b}{c}$,

that is, $\frac{a^2}{b^2} = \frac{a}{c}$,

or, $a : c :: a^2 : b^2$.

IV. INCOMMENSURABLE QUANTITIES.

These theorems are applicable to incommensurable quantities.

In the definition of Proportion, it is supposed that one quantity is some determinate multiple, part, or parts of another: or that the fraction formed by taking one of the quantities as a numerator and the other as a denominator, is a determinate fraction. This will be the case whenever the two quantities have any common measure whatever. Let x be a common measure of a and b, and let $a = mx$, $b = nx$, then, $\frac{a}{b} = \frac{mx}{nx} = \frac{m}{n}$, where m and n are whole numbers.

But, if the quantities are *incommensurable*, the value of $\frac{a}{b}$ cannot be exactly expressed by any fraction, $\frac{m}{n}$, whose numerator and denominator are whole numbers. Yet a fraction of this kind may be found which will express its value to *any required degree of accuracy*—that is, a fraction which may *approach the limit* $\frac{a}{b}$ *as nearly in value as we please.*

Suppose x to be a measure of b, and let $b = nx$; also let a be greater than mx, but less than $(m + 1)x$; then $\frac{a}{b}$ is greater than $\frac{m}{n}$, but less than $\frac{m+1}{n}$, or, the difference between $\frac{m}{n}$ and $\frac{a}{b}$ is less than $\frac{1}{n}$; and as x is diminished, since $nx = b$, n is increased and $\frac{1}{n}$ diminished; therefore, by diminishing x, the difference between $\frac{m}{n}$ and $\frac{a}{b}$ may be made less than any that can be assigned. In other words, the incommensurable ratio $\frac{a}{b}$ is the *limit* which a varying commensurable ratio $\frac{m}{n}$ may approach as nearly in value as we please, but never reach.

If c and d as well as a and b be incommensurable, and if, when $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, $\frac{c}{d}$ lie also between $\frac{m}{n}$ and $\frac{m+1}{n}$, however the magnitudes m and n are increased, then $\frac{a}{b}$ is equal to $\frac{c}{d}$. For, if they are not equal, they must have some assignable difference; and because each of them lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, this difference is less than $\frac{1}{n}$; but since n may, by the supposition, be increased without limit, $\frac{1}{n}$ may be diminished without limit, that is, it may become less than any assignable magnitude; therefore, $\frac{a}{b}$ and $\frac{c}{d}$ have no assignable difference; that is, $\frac{a}{b}=\frac{c}{d}$.

Hence, all the propositions respecting proportionals are true of the four magnitudes a, b, c, d, when incommensurable.

V. EUCLID'S DEFINITION OF PROPORTION.

It will be useful to compare the definition of proportion which has been given in this chapter with that which is given in the Fifth Book of Euclid.

The latter definition may be stated thus: *Four quantities are proportionals when, if any equimultiples be taken of the first and third, and, also, any equimultiples of the second and fourth, the multiple of the third is greater than, equal to, or less than the multiple of the fourth, according as the multiple of the first is greater than, equal to, or less than the multiple of the second.*

We will first show that the property involved in this definition follows from the algebraical definition.

For, suppose $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$, therefore, $\frac{pa}{qb}=\frac{pc}{qd}$. Hence, pc is greater than, equal to, or less than qd, according as pa is greater than, equal to, or less than qb.

Next, we may deduce the algebraical definition of proportion from Euclid's. Let a, b, c, d be four quantities, such that pc is greater than, equal to, or less than qd, according as pa is greater than, equal to, or less than qb, then shall $\frac{a}{b}=\frac{c}{d}$. First, suppose c and d are *commensurable;* then we can take p and q such that $pc=qd$; hence, by hypothesis, $pa=qb$.

Thus $$\frac{pa}{qb}=1=\frac{pc}{qd},$$

and $$\frac{a}{b}=\frac{c}{d}.$$

If, however, a, b, c, d be incommensurable, the above equalities cannot be obtained; but we can always make pa approach as near as we please to qb by giving proper values to p and q: *i. e.*, we can make pa differ from qb by a quantity less than b, or make pa lie between qb and $(q + 1)b$. Then, also, will pc lie between qd and $(q + 1)d$; *i. e.*, both $\frac{a}{b}$ and $\frac{c}{d}$ lie between $\frac{q}{p}$ and $\frac{q+1}{p}$. Also, p and q may be increased without limit; therefore, $\frac{a}{b} = \frac{c}{d}$, or a, b, c, d are proportionals according to the algebraical definition.

It will be seen that Euclid's definition of proportion includes both commensurable and incommensurable quantities.

PLANE GEOMETRY.

BOOK I.

FUNDAMENTAL PRINCIPLES.

DEFINITIONS.

1. GEOMETRY is the science which has for its object the measurement of extension.

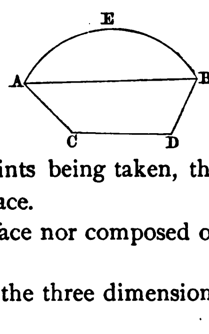
Extension has three dimensions, length, breadth, and height.

2. A *line* is length without breadth. The extremities of a line are called *points;* a point, therefore, has no extension.

3. A *straight line* is the shortest distance from one point to another.

4. Every line which is neither a straight line, nor composed of straight lines, is a *curve line.*

Thus, AB is a straight line. ACDB a *broken* line, or a line composed of straight lines, and AEB is a curve line.



5. A *surface* is that which has length and breadth without height or thickness.

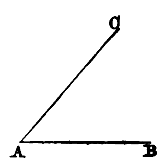
6. A *plane* is a surface in which any two points being taken, the straight line joining them lies wholly in the surface.

7. Every surface, which is neither a plane surface nor composed of plane surfaces, is a *curved* surface.

8. A *solid*, or *body*, is that which combines all the three dimensions of extension.

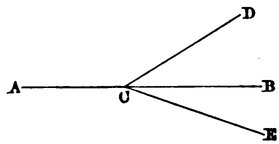
Note.—The word *volume* is often used to designate a *solid.*

9. When two straight lines, AB, AC, meet each other, the quantity, greater or less, by which they are separated from each other, in regard to their position, is called an *angle.* The point of meeting or *intersection*, A, is the *vertex* of the angle, and the lines AB, AC are its *sides.*



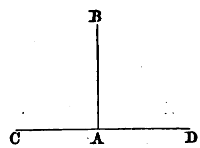
The angle is sometimes designated simply by the letter at the vertex, sometimes by three letters, BAC or CAB, care being taken to place the letter at the vertex in the middle.

Angles, like all other quantities, are susceptible of addition, subtraction, multiplication, and division: thus, the angle DCE is the sum of the two angles DCB, BCE; and the angle DCB is the difference of the two angles DCE, BCE.

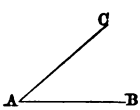


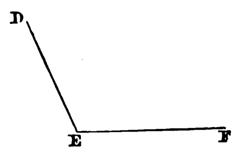
NOTE.—Two angles, DCB and BCE, which have the same vertex, C, and a common side, CB, are called *adjacent* angles.

10. When a straight line, AB, meets another straight line, CD, so as to make the adjacent angles, BAC, BAD, equal to one another, each of these angles is called a *right angle*, and the line AB is said to be *perpendicular* to CD.

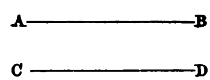


11. Every angle, BAC, less than a right angle, is an *acute angle*; and every angle, DEF, greater than a right angle, is an *obtuse angle*.

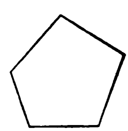




12. *Parallel* straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.



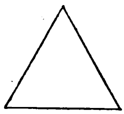
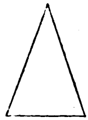
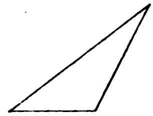
13. A *plane figure* is a plane terminated on all sides by lines. If the lines are straight, the space which they enclose is called a *rectilineal figure*, or *polygon*. The lines themselves are called the sides; and taken together they form the contour, or *perimeter*, of the polygon.



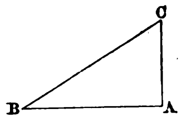
14. The polygon of three sides, the simplest of all, is called a *triangle;* the polygon of four sides is called a *quadrilateral;* that of five sides, a *pentagon;* that of six, a *hexagon*, etc.

15. An *equilateral triangle* is one which has three equal sides; an

isosceles triangle is that which has two sides equal; and a *scalene triangle* is one which has three unequal sides.

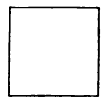




16. A *right-angled* triangle is one which has a right angle. The side opposite the right angle is called the *hypothenuse:* thus, ABC is a right-angled triangle, with the right angle at A; the side, BC, being the hypothenuse.



17. Among the quadrilaterals we distinguish:

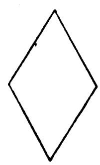
The *square*, which has its sides equal, and its angles right angles. (*See* Prop. xxx., Book I.)



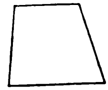
The *rectangle*, which has its angles right angles, but not all its sides equal. (*See* the same Prop.)

The *parallelogram*, or *rhomboid*, which has its opposite sides parallel.

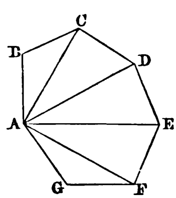
The *lozenge*, or *rhombus*, which has all its sides equal, but its angles are not right angles.



Lastly, the *trapezoid*, of which only two sides are parallel.



18. A *diagonal* of a polygon is a line which joins two vertices, not adjacent to one another. Thus, AC, AD, AE, AF, are diagonals.



19. An *equilateral* polygon is one which has all its sides equal; an *equiangular* polygon, one which has all its angles equal.

20. Two polygons are *mutually equilateral,* when they have their sides equal, each to each, and placed in the same order; that is to say, when following their perimeters in the same direction, the first side of the one is equal to the first side of the other, the second of the one to the second of the other, the third to the third, and so on.

In the same way with respect to their angles, two polygons are said to be *mutually equiangular.*

In both cases, the equal sides, or the equal angles, are called *homologous* sides or angles.

N. B.—In the four first books, it is only plane figures, or figures traced on a plane surface, which will come under consideration.

EXPLANATION OF TERMS AND SIGNS.

An *axiom* is a self-evident proposition.

A *theorem* is a truth which becomes evident by means of a course of reasoning, called a *demonstration.*

A *problem* is a question proposed, which requires a *solution.*

A *lemma* is a subsidiary truth employed for the demonstration of a theorem or the solution of a problem.

The common name, *proposition,* is applied indifferently to theorems, problems, and lemmas.

A *corollary* is an obvious consequence which flows from one or more propositions.

A *scholium* is a remark on one or more preceding propositions, tending to point out their connection, their use, their restriction, or their extension.

An *hypothesis* is a supposition made either in the enunciation of a proposition, or in the course of a demonstration.

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. The whole is greater than its part.

3. The whole is equal to the sum of all its parts.

4. From one point to another only one straight line can be drawn.

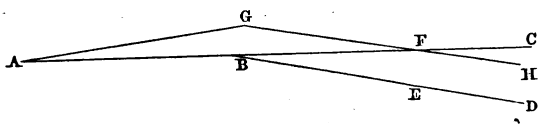
5. Two magnitudes, lines, surfaces, or solids, etc., which coincide throughout their whole extent, are equal to one another.

PROPOSITION I.

THEOREM.

Two straight lines which have two points, A *and* B, *common, coincide throughout their whole extent.*

In the first place, the two lines coincide between A and B; for, otherwise, there would be two straight lines from A to B, which is impossible (Ax. 4). Suppose, however, that they separate from each other at B, the one becoming BC, the other, BE. Turn the line



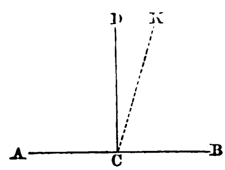
ABE about the point A until one of the points, E, of the line falls on one of the points, F, of the line ABC. In this movement the point B will fall on some point above, as G, and the line ABED will take the position AGFH. Whence, it would follow, that from the point A to the point F there could be two straight lines, which is impossible (Ax. 4). Hence, the lines cannot separate, and, therefore, they coincide throughout their whole extent.

PROPOSITION II.

THEOREM.

Through a point, C, *on a straight line,* AB, *only one perpendicular,* CD, *can be drawn to that line.*

For, through the point C draw any other straight line, CK. Then we shall have the angle ACK greater than ACD (Ax. 2). But ACD is equal to BCD (Def. 10). Therefore, ACK is greater than BCD; but BCD is greater than BCK (Ax. 2); much more, then, is ACK, greater than BCK. Hence, the straight line CK makes with AB angles which are not equal, and is, therefore, not perpendicular to this line (Def. 10). Therefore, etc.

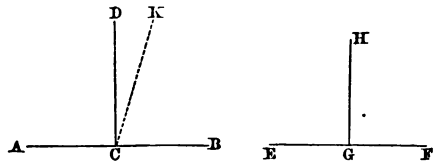


PROPOSITION III.

THEOREM.

All right angles are equal to one another.

Let the straight line CD be perpendicular to AB, and GH to EF, then shall the angles ACD, EGH, be equal to one another. Take the four lines, CA, CB, GE, GF, all equal to one another; the line AB is, then, equal to the line EF. Place the line EF on AB so that the point E shall



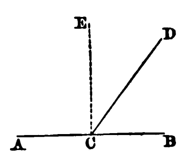
fall on A and the point F on B. These two lines thus placed shall coincide with one another; for, otherwise, there would be two straight lines from A to B, which is impossible (Ax. 4); then G, the middle point of EF, falls on C, the middle point of AB. The side GE being thus applied to CA, the side GH will fall along CD; for, suppose, if possible, that it falls along a line, CK, different from CD; then we should have two perpendiculars, CD and CK, through the same point, C, to the same straight line, which is impossible (Prop. II.); then, the side GH cannot fall along a line different from CD, and, therefore, it falls along CD, and, therefore, the angle EGH is equal to the angle ACD. Therefore, all right angles are equal.

PROPOSITION IV.

THEOREM.

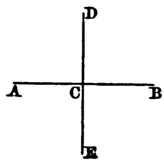
Every straight line, CD, *which meets another straight line,* AB, *makes with it two adjacent angles,* ACD, BCD, *whose sum is equal to two right angles.*

At the point C, erect on AB the perpendicular CE. The angle ACD is the sum of the angles ACE, ECD; therefore, ACD + BCD is the sum of the three angles ACE, ECD, BCD; but the first of these three angles is a right angle; and the other two together make up the right angle BCE; therefore, the sum of the two angles ACD, BCD, is equal to two right angles.

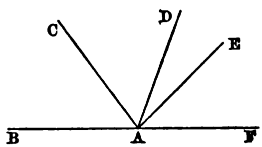


COR. I. If one of the angles, ACD, BCD, be a right angle, the other must be a right angle also.

Cor. 2. If the line DE is prependicular to AB, then, reciprocally, AB will be perpendicular to DE. For, since DE is perpendicular to AB, the angle ACD must be equal to its adjacent angle DCB, and both of them must be right angles. But, since ACD is a right angle, its adjacent ACE must also be a right angle; therefore, the angle ACE = ACD; hence, AB is perpendicular to DE.



Cor. 3. All the successive angles, BAC, CAD, DAE, EAF, formed on the same side of the straight line BF, taken together, are equal to two right angles; for their sum is equal to that of the two adjacent angles BAC, CAF.

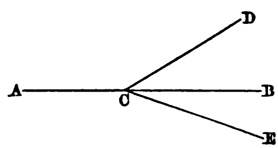


PROPOSITION V.

Theorem.

If two adjacent angles, ACD, DCB, *having the same vertex,* C, *are together equal to two right angles, the two exterior sides,* AC, CB, *shall be in the same straight line.*

For, if CB is not the prolongation of AC, let CE be that prolongation; then the line ACE being straight, the angles ACD, DCE are together equal to two right angles (Prop. IV.). But, by hypothesis, the angles ACD, DCB, are, together, also equal to two right angles; therefore, ACD + DCB must be equal to ACD + DCE; from these equals take away the common angle ACD, and there remains the part DCB, equal to the whole, DCE, which is impossible; therefore, CB is the prolongation of AC.

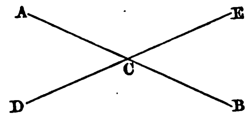


PROPOSITION VI.

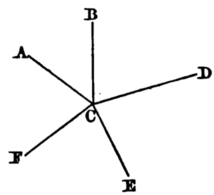
Theorem.

Whenever two straight lines, AB, DE, *cut one another, the opposite or vertical angles are equal.*

For, since DE is a straight line, the sum of the angles ACD, ACE, is equal to two right angles (Prop. IV.); and since AB is a straight line, the sum of the angles ACE, BCE, is also equal to two right angles; therefore, the sum ACD + ACE is equal to the sum ACE + BCE. From each of these take away the same angle, ACE, and there remains the angle ACD, equal to its opposite or vertical angle, BCE. It may be shown in the same manner that the angle ACE is equal to its opposite angle BCD.



SCHOLIUM.—The four angles formed about a point by two straight lines which cut each other, are, together, equal to four right angles; for the angles ACE, BCE, taken together, are equal to two right angles, and the two others, ACD, BCD, are likewise equal to the same.



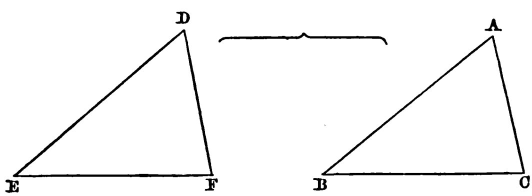
In general, when any number of straight lines, CA, CB, etc., meet in a point, C, the sum of all the successive angles, ACB, BCD, DCE, ECF, FCA, is equal to four right angles; for, if four right angles were formed at the point C by means of two lines perpendicular to each other, the same space would be occupied by the four right angles which is filled by the successive angles ACB, BCD, etc.

PROPOSITION VII.

THEOREM.

Two triangles are equal, when an angle and the two sides which contain it in the one, are equal to an angle and the two sides which contain it in the other, each to each.

Let the angle A be equal to the angle D, the side AB equal to the side DE, and the side AC equal to DF; then shall the triangle ABC be equal to the triangle DEF. For these triangles may be placed



the one on the other, so that they shall perfectly coincide. First, if the side DE be placed on its equal, AB, so as to coincide with it, the point D will fall on A and the point E on B; and since the angle D is equal to the angle A, when the side DE is placed on AB, the side DF will fall along AC. Besides, DF is equal to AC; therefore, the point F will fall on C, and the third side, EF, will exactly cover the third side, BC (Ax. 4); therefore, the triangle DEF is equal to the triangle ABC (Ax. 5).

COR. When in two triangles these three things are equal, namely: the angle A = D, the side AB = DE, the side AC = DF, the three others are equal also; namely: the angle B = E, the angle C = F, and the side BC = EF.

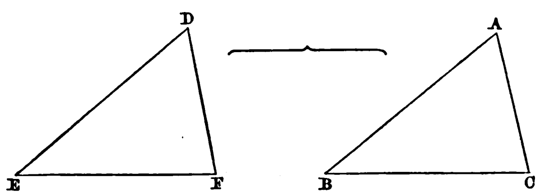
PROPOSITION VIII.

THEOREM.

Two triangles are equal, if two angles and the interjacent side of the one are equal to two angles and the interjacent side of the other, each to each.

Let the side BC be equal to the side EF, the angle B equal to the angle E, and the angle C to the angle F; then shall the triangle DEF be equal to the triangle ABC.

For, to apply the one to the other, let EF be placed on its equal, BC; the point E will fall on B, and the point F on C. And since the angle E is equal to the angle B, the side ED will fall along BA;



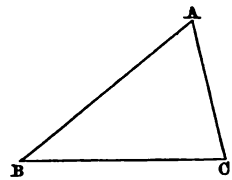
and the point D will be found somewhere in the line BA. In like manner, since the angle F is equal to the angle C, the line FD will fall along CA, and the point D will be found somewhere in the side CA. Hence, the point D, which is found at the same time in the two lines BA and CA, must fall at their intersection, A; therefore, the two triangles ABC, DEF, coincide with each other and are perfectly equal.

COR. Whenever in two triangles these three things are equal, namely: BC = EF, B = E, C = F; the other three are equal also; namely: AB = DE, AC = DF, and A = D.

PROPOSITION IX.

THEOREM.

In every triangle, any side is less than the sum of the other two.



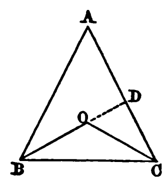
For, the straight line BC, for example, is the shortest distance from B to C (Def. 3); therefore, BC is less than AB + AC.

COR. The difference, BC − AC, of any two sides is less than the third side, AB. For, since $BC < AB + AC$, take AC from both, then $BC - AC < AB$.

PROPOSITION X.

THEOREM.

If from any point, O, *within the triangle,* ABC, *two straight lines,* OB, OC, *be drawn to the extremities of either side,* BC, *the sum of these two straight lines will be less than that of the two other sides,* AB, AC.



Let BO be produced to meet the side AC in D; the straight line OC is shorter than OD + DC (Prop. IX.); add BO to each, and we have

$$BO + OC < BO + OD + DC,$$

or,

$$BO + OC < BD + DC.$$

In like manner $BD < BA + AD$; add DC to each and we have

$$BD + DC < BA + AC.$$

But we have just found

$$BO + OC < BD + DC.$$

Therefore, still more is

$$BO + OC < BA + AC.$$

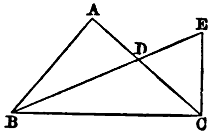
SCHOLIUM.—*When two triangles,* ABC *and* EBC, *which have a common side,* BC, *cut each other, the sum of the two sides which do not intersect is less than the sum of the sides which do intersect.*

For, in the triangle ABD we have AB < AD + DB, and in the triangle ECD,

$$EC < DC + ED.$$

Adding these two inequalities, member to member, we obtain

$$AB + EC < AC + EB.$$



PROPOSITION XI.

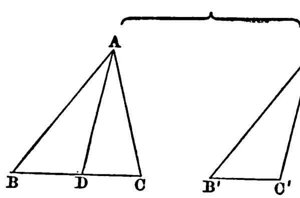
THEOREM.

If two sides, AB, AC, *of a triangle,* ABC, *be equal to the two sides,* A′B′, A′C′, *of another triangle,* A′B′C′, *each to each; and if the angle,* BAC, *contained by the former, is greater than the angle,* B′A′C′, *contained by the latter, the third side,* BC, *of the first triangle shall be greater than the third side,* B′C′, *of the second.*

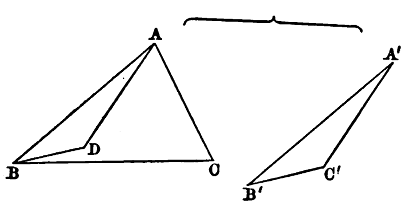
For, place the triangle A′B′C′ on the triangle ABC so that A′B′ coincides with its equal, AB, and the side A′C′, which makes, with A′B′, an angle less than BAC, falls along AD within the angle BAC. Then BD is equal to B′C′ (Prop. VII.), and we have only to show that BC is greater than BD.

Now, there may be three cases, according as the point D falls on BC, or within the triangle ABC, or without it.

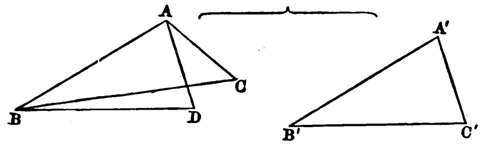
In the first case the truth of the theorem is evident, since BD is a part of BC.



In the second case we have AC + BC > AD + BD (Prop. X.). Whence, taking AC from one side and its equal, AD, from the other, we have BC > BD.



In the third case (Prop. X., Schol.), AD + BC > AC + BD;



whence, taking from the one side AD and from the other its equal, AC, we have BC > BD.

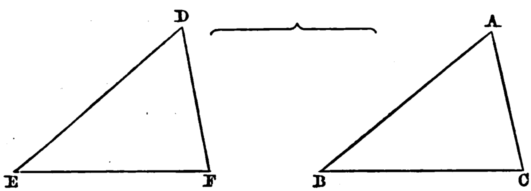
PROPOSITION XII.

THEOREM.

Two triangles are equal, when the three sides of the one are equal to the three sides of the other, each to each.

Let the side AB = DE, AC = DF, BC = EF; then shall the angles A = D, B = E, C = F.

For, if the angle A were greater than the angle D, as the sides AB, AC, are respectively equal to the sides DE, DF, it would follow (by the last Proposition) that the side BC must be greater than EF; and



if the angle A were less than D, it would follow that the side BC must be less than EF; but BC is equal to EF; therefore the angle A can be neither greater nor less than the angle D; it is, therefore, equal to it. In the same manner it may be shown that the angle B = E, and that the angle C = F.

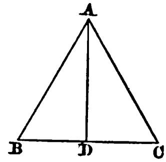
SCHOLIUM.—It may be observed that the equal angles are opposite to the equal sides: thus, the equal angles A and D are opposite to the equal sides BC, EF.

PROPOSITION XIII.

THEOREM.

In an isosceles triangle, the angles opposite to the equal sides are equal.

Let the side AB = AC; then shall the angle C be equal to B. Join the vertex A, and D, the middle point of the base, BC; the two triangles ABD, ADC, have the three sides of the one equal to the three sides of the other, each to each; namely, AD common, AB = AC by hypothesis, and BD = DC by construction; therefore (by the last Proposition) the angle B is equal to the angle C.



COR. Hence, every equilateral triangle is also equiangular.

SCHOLIUM.—From the equality of the triangles ABD, ACD, it follows, also, that the angle BAD is equal to DAC, and BDA to ADC, hence, the latter two are right angles; therefore, *a straight line drawn from the vertex of an isosceles triangle to the middle of its base, is perpendicular to that base, and divides the angle at the vertex into two equal parts.*

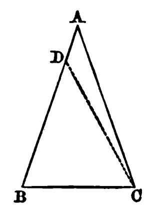
In a triangle which is not isosceles any side is taken indifferently as the base, and then the *vertex* is the vertex of the opposite angle. In an isosceles triangle, the base is that side which is not equal to one of the others.

PROPOSITION XIV.

THEOREM.

Conversely, if two angles of a triangle be equal, the sides opposite them shall be equal, and the triangle shall be isosceles.

Let the angle ABC be equal to ACB; then the side AC shall be equal to the side AB. For, if these sides be not equal, let AB be the greater; from AB cut off BD = AC, and join DC. The angle DBC is (by hypothesis) equal to ACB; and the two sides DB, BC are equal to the two sides AC, CB; therefore, the triangle DBC (Prop. VII.) must be equal to the triangle ACB. But the part cannot be equal to the whole (Ax. 2); hence, the sides AB, AC cannot be unequal; therefore, the triangle ABC is isosceles.

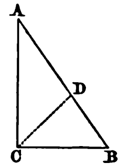


PROPOSITION XV.

THEOREM.

Of two sides of a triangle, that one is the greater which is opposite to the greater angle. And conversely, of two angles of a triangle, that one is the greater which is opposite to the greater side.

First.—Let the angle C be greater than B; then shall the side AB, opposite to C, be greater than the side AC, opposite to B. Make the angle BCD = B; then in the triangle BDC we shall have BD = DC (Prop. XIII.). But AD + DC > AC (Prop. IX.), and AD + DC = AD + DB = AB; therefore, AB is greater than AC.



Secondly.—Suppose the side AB > AC; then shall the angle C, opposite to AB, be greater than the angle B, opposite to AC. For if C were less than B, then by what has just been shown we must have AB < AC, which is contrary to the hypothesis; if we had C = B it would follow (Prop. XIII.) that AB = AC, which is also contrary to the hypothesis; therefore, the angle C must be greater than the angle B.

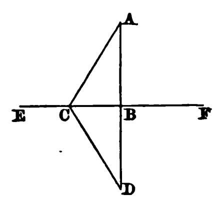
PROPOSITION XVI.

THEOREM.

From a point, A, *without a straight line,* EF, *only one perpendicular can be drawn to that line.*

For, suppose it is possible to draw two perpendiculars, AB and AC; produce one of them, AB, till BD = AB, and join DC.

The triangle CBD is equal to ABC: for the angles CBD and CBA are right angles, the side CB is common, and the side BD = AB; therefore, the triangles are equal (Prop. VII.), and hence, the angle BCD = BCA; but the angle BCA is a right angle, by hypothesis; therefore, the angle BCD is also a right angle. But, if the adjacent angles BCA, BCD, are together equal to two right angles, the line ACD must be straight (Prop. V.); whence it follows that from the point A to the point D, two straight lines, ABD and ACD, can be drawn, which is impossible (Ax. 4); therefore, it is equally impossible that two perpendiculars can be drawn from the same point to the same straight line.



PROPOSITION XVII.

THEOREM.

If, from a point, A, *without a straight line,* DE, *a perpendicular,* AB, *be drawn to that line, and oblique lines,* AE, AC, AD, *etc., be drawn to different points of the same line:*

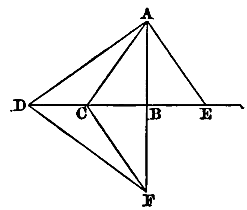
First.—The perpendicular AB *shall be shorter than any oblique line:*

Secondly.—Two oblique lines, AC, AE, *which meet* DE *on different sides of the perpendicular, at equal distances,* BC, BE, *from it, shall be equal.*

Thirdly.—Of any two oblique lines, AC *and* AD, *or* AE *and* AD, *that which meets* DE *farther from the perpendicular shall be the longer.*

Produce the perpendicular AB till BF is equal to AB, and draw FC, FD.

First.—The triangle BCF is equal to the triangle BCA, for the right angle CBF = CBA, the side BC common, and the side BF = BA; therefore, the third side, CF, is equal to the third side, AC. Now, the straight line AF is shorter than AC + CF, a broken line; hence, AB, the half of AF, is shorter than AC, the half of AC + CF, therefore, the perpendicular is shorter than any oblique line.



Secondly.—If we suppose BE = BC, then, as AB is common and the angle ABE = ABC, it follows that the triangle ABE = ABC (Prop. VII.); hence, the sides AE, AC are equal; therefore, two oblique lines, meeting DE at equal distances from the perpendicular, are equal.

Thirdly.—In the triangle DFA the sum of the lines AC, CF, is less (Prop. X.) than the sum of the sides AD, DF; therefore, AC, the half of the line AC + CF, is shorter than AD, the half of AD + DF; hence, the oblique line which meets DE farther from the perpendicular, is the longer.

Cor. 1. The perpendicular measures the true distance of a point from a line, since it is shorter than any other distance.

Cor. 2. From the same point three equal straight lines cannot be drawn to the same straight line; for, if there could, we should have two equal oblique lines on the same side of the perpendicular, which is impossible.

Cor. 3. Any two equal oblique lines, AC and AE, must cut DE at equal distances from the perpendicular.

For, if BC be greater or less than BE, then by the Proposition it would follow that AC would be greater or less than AE; but both of these results are contrary to the hypothesis; therefore, BC = BE.

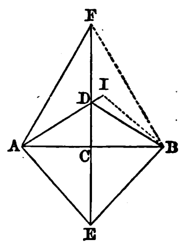
Cor. 4. In like manner, we may prove (the converse of Thirdly) that of two unequal oblique lines, AD and AE, the longer cuts the line DE farther from the perpendicular.

Cor. 5. Since AB is less than any oblique line, AC, the angle ACB is less than the right angle ABC (Prop. XV.). *Therefore, when two lines,* AC *and* DE, *intersect, the perpendicular drawn from any point of one on the other falls on the side of the acute angle.* Hence, in every right angled triangle two of the angles are acute.

PROPOSITION XVIII.

Theorem.

If from C, *the middle of the straight line* AB, *a perpendicular,* EF, *be drawn to this line: then, first, every point in the perpendicular is equally distant from the two extremities* A, *and* B, *of the line; secondly, every point without the perpendicular is unequally distant from these extremities.*



First.—Since we suppose AC = CB (Hyp.), the two oblique lines AD, DB, are equally distant from the perpendicular; and, therefore, equal. Therefore, every point in the perpendicular is equally distant from the extremities A and B.

Secondly.—Let I be a point without the perpendicular. Join IA, IB; one of these lines will cut the perpendicular in D; from D draw DB; we shall have DB = DA. But the straight line IB is less than ID + DB, and ID + DB = ID + DA = IA. Therefore, IB $<$ IA. Therefore, every point out of the perpendicular is unequally distant from the extremities A and B.

Cor. 1. It results from the above that the points on the perpendicular drawn to a line at its middle point, are the only points in the plane of the figure which possess the property of equidistance from the extremities of the line.

In Plane Geometry the name *Geometric Locus* is given to a line containing all the points in the plane *which fulfil a given geometrical condition,* or, as it is expressed, possess a *particular geometric property.*

We can now express the double theorem above, thus: *The perpendicular, drawn to a straight line at its middle point, is the geometric locus of the points which are equidistant from the extremities of the line.*

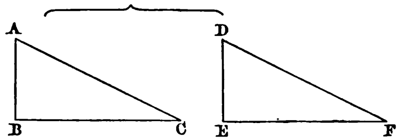
Cor. 2. Two points determine a straight line. Therefore, when a straight line has two points equally distant from the extremities of another straight line it is perpendicular to that line at its middle point.

PROPOSITION XIX.

THEOREM.

Two right angled triangles are equal, when the hypothenuse and the side of the one are equal to the hypothenuse and the side of the other, each to each.

Let the hypothenuse AC = DF, and the side AB = DE. Then shall the triangle ABC be equal to the triangle DEF. This equality will be manifest if the remaining sides, BC and EF, are equal. Now



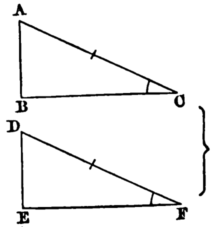
conceive the two triangles to be placed so that the equal sides AB and DE coincide. Then, since the right angle B = E, the sides BC and EF will form one and the same straight line (Prop. V.). Hence, AC and DF being two equal oblique lines, the distances BC and EF from the perpendicular must be equal (Prop. XVII. Cor. 3,). Therefore, the two triangles are equal.

PROPOSITION XX.

THEOREM.

Two right angled triangles are equal, when they have the hypothenuse and an acute angle of the one respectively equal to the hypothenuse and an acute angle of the other.

In the triangles ABC and DEF, right angled at B and E, let AC = DF and C = F. Apply DEF on ABC so that the angle F shall coincide with the angle C, DF will then coincide with AC, the point D falling on A, and FE will fall along CB. Therefore, the side DE, perpendicular to FE, must coincide with AB, perpendicular to CB, otherwise there would be two perpendiculars drawn from the same point to the same straight line. Hence, the point E, being in AB and BC,



must fall on B; and the two triangles coincide throughout their whole extent, and are therefore equal (Ax. 5).

SCHOLIUM. Two right angled triangles are equal when they have two sides of the one equal to two sides of the other, or one side and an acute angle of one equal to one side and an acute angle of the other.

DEFINITIONS.

To *bisect* means in Geometry to *divide into two equal parts.*

The line which divides an angle into two equal parts, is called the *bisectrix* of that angle.

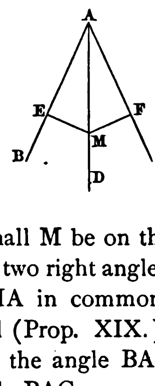
PROPOSITION XXI.

THEOREM.

1. *Any point,* M, *on the bisectrix,* AD, *of an angle,* BAC, *is equally distant from the two sides of that angle.*

2. Conversely, *Every point,* M, *in the interior of the angle,* BAC, *which is at equal distances from the sides,* AB *and* AC, *shall lie on* AD, *the bisectrix of that angle.*

First.—The perpendicular ME, drawn from M to AB, and the perpendicular MF, drawn from M to AC, measure the distances from M to these lines respectively. If M be on the bisectrix AD, then shall ME = MF. For the two right angled triangles MAE and MAF, have the hypothenuse MA in common, and the acute angle MAE = MAF, since AD bisects the angle BAC. Hence, the triangles are equal (Prop. XX.). Therefore, ME = MF.



Secondly.—Conversely, suppose ME = MF, then shall M be on the bisectrix AD of the angle BAC. For, join MA; the two right angled triangles MAF and MAE have the hypothenuse MA in common, and ME = MF by hypothesis; hence they are equal (Prop. XIX.). Therefore, the angle MAE = MAF, and MA divides the angle BAC in half, hence, M lies on the bisectrix AD of the angle BAC.

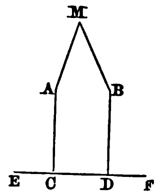
It follows that any point within the angle BAC not on the bisectrix, must be unequally distant from the sides AB and AC.

COR. *The bisectrix of an angle is the geometric locus of all points situated within the angle which are equidistant from its sides.*

PROPOSITION XXII.

THEOREM.

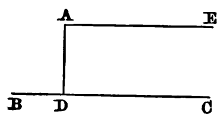
Two straight lines, AC *and* BD, *which are perpendicular to the same straight line,* EF, *are parallel.*



For, if they could meet there would be two perpendiculars drawn from their point of intersection, M, to the same straight line, EF, which is impossible (Prop. XVI.).

COR. 1. The square and rectangle are parallelograms.

COR. 2. *Through a point,* A, *without a straight line,* BC, *one parallel can always be drawn to that line.*



From the point A draw AD perpendicular to BC, and draw also AE perpendicular to AD. The two straight lines AE and BC, being both perpendicular to AD, are parallel.

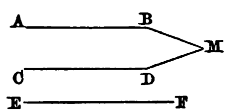
AXIOM 6.

Two intersecting straight lines cannot be parallel to the same straight line.

Hence it follows:

I. *Through a point without a straight line only one parallel can be drawn to that line.*

II. *Two straight lines,* A *and* B, *which are parallel to a third,* C, *are parallel to each other.*



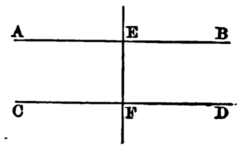
For if A and B could meet we would have two intersecting straight lines parallel to the same straight line.

PROPOSITION XXIII.

THEOREM.

When two straight lines, AB *and* CD, *are parallel, every straight line,* EF, *perpendicular to the one,* AB, *shall be perpendicular to the other,* CD.

For, EF, perpendicular to AB, must meet CD (Ax. 6). Through the point of intersection, F, conceive a perpendicular to be drawn to EF. This perpendicular will be parallel to AB (Prop. XXII.) and must coincide with CD, since from F only one parallel can be drawn to AB. Therefore, EF is perpendicular to CD.

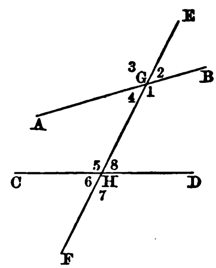


This theorem is often enunciated more briefly, thus:

Two parallels have their perpendiculars common.

DEFINITIONS.

When two straight lines, AB and CD, are cut by a third, EF, the eight angles at their points of intersection, G and H, are named as follows:



1. The four angles, 1, 4, 5, 8, which lie within the lines AB and CD, are called *interior angles.*

2. The four others, 2, 3, 6, 7, which lie without, are called *exterior angles.*

3. Two interior angles on opposite sides of the secant line and not adjacent, as 1 and 5 or 4 and 8, are called *alternate interior angles*, or simply, *alternate angles.*

4. Two exterior angles on opposite sides of the secant line and not adjacent, as 2 and 6 or 3 and 7, are called *alternate exterior angles.*

5. Two angles on the same side of the secant, one exterior and the other interior, and not adjacent, are called *corresponding angles.* Such are 1 and 7, or 4 and 6, or 2 and 8, or 3 and 5.

6. Finally, the two angles 1 and 8 or 4 and 5 are called *interior angles on the same side;* and the angles 2 and 7 or 3 and 6 are *exterior angles on the same side.*

7. Two angles which together are equal to two right angles are said to be *supplements* of each other, or *supplementary.* Hence, two angles which are supplements of the same angle are equal to each other.

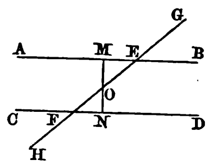
PROPOSITION XXIV.

THEOREM.

If two parallel lines, AB *and* CD, *are cut by a third line,* GH :

1. *The alternate interior angles are equal.*
2. *The alternate exterior angles are equal.*
3. *The corresponding angles are equal.*
4. *The sum of the interior angles on the same side is equal to two right angles.*
5. *The sum of the exterior angles on the same side is equal to two right angles.*

First.—Through the middle point, O, of the secant line, EF, draw MN, a common perpendicular, to the parallels. OM will fall in the acute angle FEA and ON in the acute angle EFD. The two right angled triangles MOE and ONF have their hypothenuses, OE and OF, equal by construction, and the opposite or vertical angles, MOE and FON, equal. These triangles are therefore equal, and hence the alternate interior angles MEO and OFN are equal.



Also, the alternate interior angles BEF and EFC are equal, for these angles are the supplements respectively of the equal angles MEO, OFN.

Secondly.—The alternate exterior angles GEB and CFH are equal, for they are the opposite or vertical angles respectively of the angles MEO, OFN. So, also, HFD = GEA.

Thirdly.—The corresponding angles GEB and EFD are equal. For GEB = AEF, and AEF = EFD. Therefore, GEB = EFD. So, also, AEG = CFE, etc.

Fourthly.—The sum of the interior angles on the same side, FEB and EFD, is equal to two right angles. For we have BEF + AEF = two right angles; but AEF = EFD; therefore, BEF + EFD = two right angles. So, likewise, AEF + CFE = two right angles.

Fifthly.—The sum of the exterior angles on the same side, GEB + HFD, is equal to two right angles. For HFD + EFD = two rights, and EFD = corresponding angle GEB. Therefore, HFD + GEB = two rights.

SCHOLIUM. When the secant line is perpendicular to one of the parallels the eight angles formed are all right angles. When it is oblique to the parallels, there are formed four equal acute angles and four equal obtuse angles, and each acute angle is the supplement of each obtuse angle.

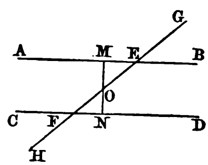
PROPOSITION XXV.

Theorem.

Conversely, *When two straight lines,* AB *and* CD, *are cut by a third,* GH, *so as to make :*

1. *The alternate interior angles equal ; or,*
2. *The alternate exterior angles equal ; or,*
3. *The corresponding angles equal ; or,*
4. *The sum of the interior angles on the same side equal to two right angles ; or,*
5. *The sum of the exterior angles on the same side equal to two right angles—Then these two straight lines are parallel.*

First.—Let the angle AEF be equal to its alternate angle EFD. Conceive a line drawn through F parallel to AB. This parallel must make, with GH (Prop. XXIV.), an angle equal to AEF, and, therefore, equal to EFD. Hence, this parallel coincides with FD, and, therefore, CD is parallel to AB.



Second.—If the alternate exterior angles GEB and CFH are equal, then their opposite or vertical angles are equal, that is, AEF = EFD ; and, therefore (1), AB is parallel to CD.

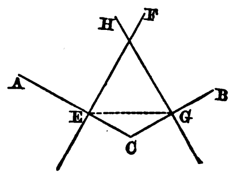
Third.—Let the corresponding angles GEB and EFD be equal. Then, since GEB is equal to AEF we have AEF = EFD. Therefore (1) AB is parallel to CD.

Fourth.—If the sum of the interior angles on the same side, BEF and EFD, is equal to two right angles, then, since BEF + AEF = two right angles, we must have AEF = EFD. Therefore (1), AB is parallel to CD.

Fifth.—Let GEB + HFD = two right angles; then, since EFD + HFD = two right angles, we must have GEB = EFD. Therefore (3) AB is parallel to CD.

Cor. 1. From the Propositions XXIV. and XXV., it follows that if two straight lines are cut by a third so that the angles formed do not fulfil the conditions which we have just enunciated, *the two lines are not parallel.* In particular, *If a straight line meet two straight* *lines so as to make the two interior angles on the same side of it less than two right angles, these straight lines being continually produced will meet on the side on which the angles are whose sum is less than two right angles.*

NOTE.—This is Euclid's axiom, on which he bases the theory of parallels.

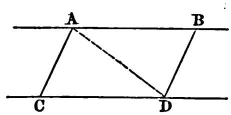


COR. 2. *Two straight lines,* EF, GH, *respectively perpendicular to two straight lines which intersect each other, must also intersect.* For, drawing the line EG, we see that each of the interior angles, FEG, HGE, is less than a right angle; therefore, the sum of these angles is less than two right angles. Hence, EF and GH must meet.

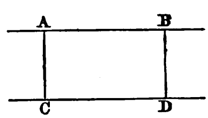
PROPOSITION XXVI.

THEOREM.

Two parallels, AC, BD, *intercepted between two parallels,* AB, CD, *are equal.*



For, join AD. The angles BAD, ADC are equal, being alternate angles with reference to the parallels AB, CD (Prop. XXIV.); also the angles ADB, DAC are equal, being alternate angles with reference to the parallels AC, BD. Hence, the two triangles ABD, ACD have a common side, AD, and two adjacent angles in each equal; hence, these triangles are equal; therefore, the side BD, opposite the angle BAD, is equal to the side AC, opposite to the angle ADC.



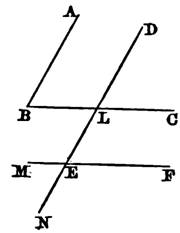
COR. If the two lines AC and BD are perpendicular to AB, and, hence, also to CD, they measure the distances of the points A and B of the straight line AB from the straight line CD. But AC and BD are equal, being parallels intercepted between parallels. Therefore, since A and B are any two points on AB, it follows that *two parallels are everywhere equally distant.*

PROPOSITION XXVII.

THEOREM.

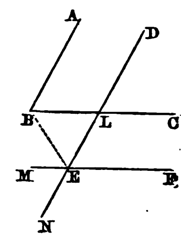
Two angles which have their sides parallel, each to each, are equal or supplementary.

1. Let ABC, DEF be two angles whose sides are parallel and *lie in the same direction.* Then will these angles be equal. For, the angles DLC, DEF are corresponding angles, and therefore equal. But for the same reason DLC = ABC; therefore, ABC = DEF.



2. Let ABC, MEN, be two angles whose sides are parallel, but *lie in opposite directions.* Then will these angles be equal. For MEN = DEF, and DEF = ABC. Therefore, MEN = ABC.

3. Lastly, Let the angles be ABC, DEM, whose sides are parallel, but two of these sides, BA and ED, lie in the same direction, and the two others, BC and EM, in contrary directions. Then will these angles be supplements of each other. For DEM is the supplement of DEF, and DEF = ABC. Therefore, DEM is the supplement of ABC.



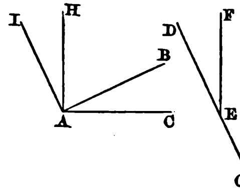
Scholium. Two parallel sides of two angles are said to lie in the *same direction* when they are both on the *same side* of the line which joins the vertices of the two angles. They are said to lie in *opposite directions* when they are on *opposite sides* of this line.

PROPOSITION XXVIII.

Theorem.

If two angles have their sides perpendicular, each to each, these angles will be either equal or supplementary.

1. Let BAC, DEF be two angles whose sides are perpendicular, each to each. Through the point A draw the straight line AI, perpendicular to AB, and the straight line AH, perpendicular to AC; the lines AI, AH, will be respectively parallel to the lines DE, EF, and lie in the same direction; hence, the angle IAH is equal to DEF.



But we have	BAC + HAB = one right angle.
and	IAH + HAB = one right angle,
Therefore,	BAC = IAH,
or,	BAC = DEF, the equal of IAH.

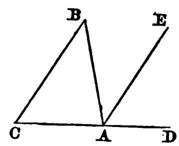
2. If we consider the angle formed by the straight line EF and the prolongation of DE, we see that the angle FEG is the supplement of the angle BAC.

PROPOSITION XXIX.

THEOREM.

In every triangle the sum of the three angles is equal to two right angles.

Let ABC be any triangle. Produce the side CA towards D; and at the point A draw AE, parallel to BC. Since AE and CB are parallel, and CAD cuts them, the angle DAE is equal to its corresponding angle ACB; also, since AB cuts the parallels, the alternate angles ABC and BAE are equal. Hence, the sum of the angles of the triangle ABC is equal to the sum of the three angles CAB, BAE, EAD, formed about the point A on the same side of the straight line CD; therefore (Prop. IV.), the sum of the three angles of the triangle is equal to two right angles.



COR. 1. *Any angle of a triangle is the supplement of the sum of the other two,* and thus, two angles of a triangle being given, or merely their sum, the third is found by subtracting this sum from two right angles.

COR. 2. If two angles of one triangle are respectively equal to two angles of another triangle, the third angle of the first will be equal to the third angle of the second, and the two triangles will be mutually equiangular.

COR. 3. Two triangles which have a side and two angles of the one respectively equal to a side and two angles of the other, are equal, whether these angles be adjacent to the side or not.

COR. 4. At least two of the angles of every triangle are acute.

COR. 5. In every right angled triangle the sum of the two acute angles is equal to one right angle.

COR. 6. In an equilateral triangle each angle is the third part of two right angles or two-thirds of one right angle; so that if the right angle is expressed by unity the angle of an equilateral triangle will be expressed by $\frac{2}{3}$.

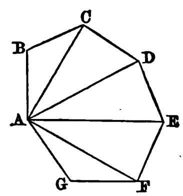
COR. 7. The angle BAD is equal to the sum of the two angles B and C. Thus, *every exterior angle of a triangle,* that is, every angle formed by one side and the prolongation of another, *is equal to the sum of the two interior angles which are not adjacent to it.*

PROPOSITION XXX.

THEOREM.

The sum of all the interior angles of a polygon is equal to as many times two right angles as there are units in the number of sides, diminished by two.

Let ABCD, etc., be the proposed polygon. If, from the vertex of any one angle, A, diagonals, AC, AD, AE, etc., be drawn, it is plain that the polygon will be divided into five triangles, if it has seven sides; into six triangles, if it has eight sides; and, in general, into as many triangles as the polygon has sides, less two; for these triangles may be considered as having the point A for a common vertex, and for bases, the several sides of the polygon, excepting the two sides which form the angle A. It is evident, also, that the sum of all the angles of these triangles does not differ from the sum of the angles of the polygon; hence, this last sum is equal to as many times two right angles as there are triangles in the figure; that is, as there are units in the number of the sides of the polygon, less two.



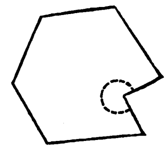
COR. 1. The sum of the angles of a quadrilateral is equal to two right angles multiplied by $4 - 2$, which makes four right angles.

Therefore, if all the angles of a quadrilateral are equal, each one of them will be a right angle. And this justifies Definition 17, where it was supposed that the four angles of a quadrilateral are right angles, in the case of the rectangle and the square.

COR. 2. The sum of the angles of a pentagon is equal to two right angles multiplied by $5 - 2$, which makes six right angles. Therefore, when the pentagon is equiangular, that is to say, when its angles are equal, the one to the other, each one of them is equal to the fifth part of six right angles, or to $\frac{6}{5}$ of one right angle.

COR. 3. The sum of the angles of a hexagon is equal to $2 \times (6-2)$ or eight right angles; therefore, in the equiangular hexagon each angle is $\frac{8}{6}$ or $\frac{4}{3}$ of a right angle.

COR. 4. If we designate the number of the sides of the polygon by n, the sum of its angles (the right angle being unity) will be expressed by $2(n - 2)$ or, $2n - 4$.

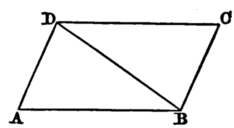


SCHOLIUM. When this proposition is applied to polygons which have *re-entrant* angles, each re-entrant angle must be regarded as greater than two right angles. But to avoid all ambiguity we shall henceforth limit our reasoning to polygons with *salient* angles, which may be otherwise named *convex polygons.* Every convex polygon is such that a straight line, drawn at pleasure, cannot meet the perimeter of the polygon in more than two points.

PROPOSITION XXXI.

THEOREM.

The opposite sides of a parallelogram are equal and the opposite angles are also equal.



1. Two opposite sides, AB and CD, for example, are equal to each other; for, by definition, they are two parallels intercepted between two parallels.

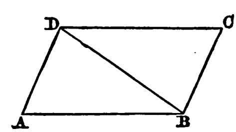
2. Any two opposite angles, ADC, ABC, are equal, for they are formed by parallel sides lying in opposite directions (Prop. XXVII.). And the same is true of opposite angles BAD, BCD.

COR. A diagonal divides a parallelogram into two equal triangles.

PROPOSITION XXXII.

THEOREM.

If the opposite sides of a quadrilateral, ABCD, *be equal, so that* AB = CD *and* AD = BC, *the equal sides shall be parallel, and the figure shall be a parallelogram.*



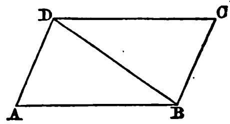
Draw the diagonal BD. The two triangles ABD, BDC, have the three sides of the one equal to the three sides of the other, each to each; therefore, they are equal; hence the angle ADB, opposite the side AB, is equal to the angle DBC, opposite to the side CD; therefore (Prop. XXV.) the side AD is parallel to BC. For a like reason, AB is parallel to CD; therefore, the quadrilateral ABCD is a parallelogram.

COR. The rhombus is a parallelogram.

PROPOSITION XXXIII.

THEOREM.

If two opposite sides, AB, CD, *of a quadrilateral are equal and parallel, the two other sides are also equal and parallel, and the figure,* ABCD, *is a parallelogram.*

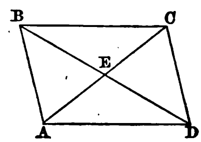


Draw the diagonal BD. Since AB is parallel to CD, the alternate angles, ABD, BDC, are equal (Prop. XXIV.). Moreover, the side AB = DC, and the side DB is common; therefore, the triangle ABD is equal to the triangle DBC (Prop. VII.); hence, the side AD = BC, and the angle ADB = DBC, and, consequently, AD is parallel to BC; therefore, the figure ABCD is a parallelogram.

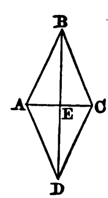
PROPOSITION XXXIV.

THEOREM.

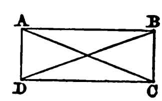
The two diagonals, AC, DB, *of a parallelogram divide each other mutually into two equal parts, that is, mutually bisect each other.*



For, comparing the triangles ADE, CEB, we find the side AD = CB, the angle ADE = CBE (Prop. XXIV.), and the angle DAE = ECB; hence, these triangles are equal (Prop. VIII.); therefore, AE, the side opposite the angle ADE, is equal to EC, the side opposite the angle EBC; hence, also, DE = EB.



SCHOLIUM 1. In the case of the rhombus, the diagonals are at right angles to each other. For, the sides AB and AD being equal, and, also, CB and CD, it follows (Prop. XVIII., Cor. 2) that AC is perpendicular to DB.



SCHOLIUM 2. In the case of the rectangle the diagonals are equal. For the two right angled triangles, ACD and BCD, have the side DC common, and the side AD = BC; hence, they are equal, and the hypothenuse AC = the hypothenuse DB.

SCHOLIUM 3. In the case of the square the diagonals are at right angles to each other, and are also equal.

EXERCISES ON BOOK I.

These exercises are theorems to be demonstrated, illustrative of the methods of demonstration used in Book I. These methods are (1) the *reductio ad absurdum* (applied generally to prove the converse of theorems), (2) superposition, (3) the comparison of equal triangles, and (4) the comparison of angles by parallels, or (5) by the use of Prop. XXIX. and its consequences. [*See* Hints to Solutions, p. 321.]

No auxiliary lines will be needed in these demonstrations except those indicated in a few of the propositions.

1. The distance of any point, M, of a straight line, AB, from the middle point, O, of this straight line, is equal to the half difference of the distances of this point M, from the extremities, A and B, of the line. If the point M is taken on the prolongation of the line AB, the distance MO is the half sum of MA and MB.

2. ACB being an angle, CO its bisectrix, and CM a straight line drawn at pleasure from the vertex so as to fall within the angle, the angle MCO is equal to the half difference of the angles MCA, MCB. If the straight line CM is exterior to the angle ACB, the angle MCO is the half sum of the angles MCA, MCB.

3. If through a point, O, on a straight line, AB, we draw two straight lines, OC and OD, on different sides of AB, so that the angle AOC = BOD, the two straight lines OC and OD are one and the same straight line.

4. If four straight lines, OA, OB, OC, OD, are drawn through the same point, O, so that the opposite angles, DOA and BOC, are equal to one another, and also the angles AOB, COD, then the sides OA and OC are in the same straight line, and also the sides OB and OD.

5. If two straight lines, AB and CD, intersect each other in O, and two straight lines, OM and ON, bisect the opposite angles, AOC, BOD, these lines, OM and ON, form one and the same line; and if OM and OP bisect two adjacent angles, AOC and AOD, these two lines, OM and OP, are perpendicular to each other.

6. Two quadrilaterals which have two angles of the one equal to two angles of the other and contained by three sides, equal, each to each, and similarly placed, are equal.

7. Two quadrilaterals which have two sides of the one equal to two sides of the other, and adjacent to three angles, equal, each to each, and similarly placed, shall be equal.

8. Definition : A median of a triangle is a line joining a vertex with the middle point of the opposite side.

THEOREM. A median of a triangle is less than the half sum of the two sides which contain the angle from whose vertex it is drawn. (Auxiliary Construction : Produce the median beyond the middle point of the side on which it rests, until the part produced is equal to the median, and join the extremity of the produced line with either extremity of the side.)

9. The sum of the medians of a triangle is less than the sum of the three sides of the triangle, and greater than half this sum.

10. Any convex polygonal line, ACDEFB (one which has no re-entrant angles), is less than the line which envelops it, and terminates in the same line, AB. (Auxiliary Construction : Produce AC, CD, DE, EF, until they meet the enveloping line.)

11. Any convex polygonal line is less than the line which envelops it entirely.

12. The sum of the three lines OA, OB, OC, from a point, O, within the triangle ABC, to the three vertices, is less than the sum of the three sides of the triangle, and greater than half this sum.

13. Two quadrilaterals which have one angle in each equal, and the four sides of the one equal to the four sides of the other, each to each, and similarly placed, are equal.

14. Two quadrilaterals are equal when they have one diagonal and four sides of the one equal to one diagonal and four sides of the other, each to each.

15. Two quadrilaterals are equal when they have the two diagonals and three sides of the one equal to the two diagonals and three sides of the other, each to each.

16. If, on the sides of a regular (equiangular and equilateral) polygon, ABCDEFGH, we take points a, b, c, d, e, f, g, h, so that $Aa = Bb = Cc$, etc., and join these points, we form a polygon $abcdefgh$, also regular.

17. If in the same polygon lines be drawn through the vertices A, B, C, D, etc., making equal angles with the sides AB, BC, CD,

etc., they will form a polygon which is also regular. Also show the same if the lines be drawn through the points *a*, *b*, *c*, *d*, etc.

18. If from two given points two straight lines be drawn to the same point in a given right line, and equally inclined to this line, the sum of these two lines will be less than the sum of the two lines drawn from the two given points to any other point of the given line. (Auxiliary Construction: Draw a perpendicular from one of the given points to the straight line, prolong it till the part below the line is equal to the part above, and join its extremity with the other given point.)

19. Definition: Two points, A and A′, are said to be symmetrical with regard to an indefinite straight line, *xy*, when this line *xy* is perpendicular to the line AA′ at its middle point.

THEOREM. If A and A′ and B and B′ are symmetrical with regard to *xy*, then the two symmetrical straight lines AB, A′B′, are equal. Also, the angle CAB of two straight lines, AB and AC, is equal to the angle C′A′B′ of the two symmetrical lines A′B′, A′C′.

20. If, through the vertex, A, of a triangle, ABC, the line *xy* be drawn perpendicular to the bisectrix of the angle A, and M be any point of *xy*, then the perimeter of the triangle BMC is greater than that of the given triangle ABC.

The preceding theorems can be demonstrated without using the theory of parallels.

21. If, through the middle point of a straight line terminating in two parallel lines, a second straight line be drawn, also terminating in the parallels, this second straight line will also be bisected at this point.

22. If two equal straight lines, AB and CD, terminating in two parallels, AC and BD, cut one another in O, we shall have AO=OC and OB = OD.

23. Conversely, if two straight lines, AB and CD, contained between two parallels, cut each other in O, so that AO = OC, then shall AB = CD.

24. If, through each of the vertices of a given triangle, a line be drawn parallel to the opposite side, a new triangle will be formed, equal to four times the given triangle. And each side of the new triangle is double the corresponding side of the given triangle.

25. (Corollary of 24.) The straight line which joins the middle

points of two sides of a triangle is parallel to the third side and equal to the half of it. And, conversely, if, through the middle point, D, of the side, AB, of a triangle, ABC, a straight line, DE, be drawn parallel to the side BC, the line DE, will meet AC at its middle point, and will be equal to the half of BC.

26. The locus of the middle points of any number of straight lines which extend from a given point to a given straight line is a straight line parallel to the given line.

27. The perpendiculars erected to the sides of a triangle, ABC, at their middle points meet in a common point. [Method of proof by means of loci: First, the perpendiculars to AB and BC at their middle points, must meet (Prop. XXV., Cor.) in some point, O. But any point on the perpendicular to AB is equidistant from A and B, and any point on the perpendicular to BC is equidistant from B and C, hence, their intersection, O, is equidistant from A and C, and must, therefore, be on the perpendicular to AC at its middle point, since that is the locus of all points equidistant from A and C. Therefore, O is the common point of intersection of the three perpendiculars.]

28. The three altitudes of a triangle (the three perpendiculars let fall from the vertices on the opposite sides) meet in a common point. (An easy Corollary of 27, by constructing the triangle which is four times the given triangle, 24.)

29. The three lines which bisect the angles of a triangle meet in a common point. (Proof by method of loci, as in 27.)

30. The bisectrix of the angle A, and the bisectrices of the two exterior angles at B and C, of a triangle, ABC, meet in a common point.

31. The three medians of a triangle (the lines which join the three vertices to the middle points of the opposite sides) meet in a common point, which divides each median in the ratio of two to one. (Auxiliary lines—a line joining the feet of two of the medians, and a second line joining the middle points of the parts of these two medians between their point of intersection and the vertices from which they are drawn. Then use 25.)

32. The two bisectrices of the angles at the base of a triangle form, with this base, a second triangle, the angle at the vertex of which is equal to one right angle plus the half of the angle at the vertex of the given triangle. Also, find the value of the angle at the vertex of

the second triangle, when the given triangle is equilateral—when it is isosceles, and the angle at the vertex is the half of one of the angles at the base.

33. In any right angled triangle, ABC, right angled at A, the median AO is equal to one-half the hypothenuse BC.

34. Conversely, if the median AO of the triangle ABC is equal to one-half the side BC, the triangle ABC is right angled at A.

35. If, in a triangle, ABC, right angled at A, the hypothenuse, BC, is double the side AB of the right angle, the angle C, opposite to AB, is one-third of a right angle. (Auxiliary construction : Produce BA, beyond A to D, until the part produced, AD, is equal to AB, and join DC.)

36. Conversely, if, in a right angled triangle, one of the acute angles is one-third of a right angle, the side opposite to this acute angle is one-half the hypothenuse.

37. If through the three vertices of a triangle, ABC, three bisectrices of the exterior angles be drawn, the three partial triangles and the whole triangle thus formed will be mutually equiangular.

38. Show, also, that each one of the angles of the given triangle ABC will be the supplement of the double of the opposite angle, in the large triangle of the preceding construction.

39. The angle made by the median, AO, of a triangle right angled at A, with the altitude, AD, is equal to the difference of the two acute angles.

40. If the sides of a convex polygon be extended in the same direction (that is, if, beginning at any vertex, we extend the sides in order at all the vertices in succession, making the complete circuit of the figure in the same direction), the sum of the exterior angles thus formed is equal to four right angles.

41. If the sides of an equiangular and equilateral pentagon be produced to meet, the angles formed by these lines are equal, and their sum is equal to two right angles.

42. If the sides of an equilateral and equiangular hexagon be produced to meet, the angles formed by these lines are together equal to four right angles.

43. If an exterior square be described on each side of an equi-

angular and equilateral hexagon, and if, then, the consecutive vertices of the adjacent squares be joined, the figure produced will be an equiangular and equilateral dodecagon.

44. What number of sides has the polygon, the sum of whose angles is twenty-six right angles?

45. What equiangular polygon has each angle equal to $\frac{5}{3}$ of a right angle?

46. Find angle of each equiangular polygon up to twenty sides.

47. The entire plane space about a point can be filled without leaving vacant intervals, by equal equilateral triangles; also by equal squares; also by equal regular hexagons.

48. These are the only three regular polygons which, by taking any number of the same polygons, serve to fill up plane space without leaving vacant intervals.

49. The space can be filled by using regular octagons and squares together; also by using regular dodecagons and equilateral triangles together.

50. If, in a quadrilateral, the opposite angles are equal, the figure is a parallelogram.

51. The four bisectrices of the angles of a quadrilateral form a second quadrilateral, whose opposite angles are supplementary. Show, also, if the first quadrilateral is a parallelogram, the second will be a rectangle whose diagonals are parallel to the sides of the given parallelogram, and equal to the difference between two adjacent sides—and, finally, if the first figure is a rectangle, the second will be a square.

52. Definition: The centre of a figure is a point which bisects all lines drawn through it terminating in the perimeter of the figure.

THEOREM. The intersection of the diagonals of a parallelogram is the centre of the parallelogram, and every line passing through this centre divides the parallelogram in half. Conversely, a quadrilateral, the intersection of whose diagonals is the centre of the figure, is a parallelogram.

53. The lines which join the middle points of the sides of a quadrilateral successively, form a parallelogram equal to one-half of the quadrilateral, whose perimeter is equal to the sum of the diagonals of the quadrilateral. Also show when this parallelogram is a rhom-

bus; when it is a rectangle; and when it is a square. (Auxiliary lines: First, the diagonals of the quadrilateral to prove the parallelogram; and, Second, lines joining the points where the diagonals of the quadrilateral meet the sides of the parallelogram, to prove the parallelogram one-fourth of the quadrilateral, using (24).

54. The lines which join the middle points of the opposite sides of a quadrilateral, bisect each other. (Corollary of 53.)

55. If, through the extremities of each diagonal of a quadrilateral, parallels be drawn to the other diagonal, a parallelogram will be formed equivalent to double the quadrilateral.

56. The diagonals of a parallelogram inscribed in a given parallelogram (that is, which has its vertices on the sides of a given parallelogram), intersect in the centre of this parallelogram.

57. Parallelograms can always be inscribed in a rectangle whose sides are respectively parallel to the diagonals of the rectangle, and therefore at any vertex the sides make equal angles with the side of the rectangle, and the perimeter of each parallelogram is equal to the sum of the diagonals of the rectangle. (Auxiliary Construction: Prolong the line drawn through any point of one side parallel to a diagonal, until it meets the opposite side produced.)

58. (A billiard ball, striking the cushion of the table, rebounds, making the angle of reflexion equal to the angle of incidence.) Show that a ball, sent parallel to one of the diagonals of the table, will, after striking the four cushions, return to the point from which it set out.

59. If from any point on the base of an isosceles triangle, perpendiculars be let fall on the two other sides, the sum of these two perpendiculars is equal to the perpendicular let fall from the vertex of one of the angles at the base on the opposite side. (Auxiliary lines: Through one of the extremities of the base draw a line parallel to the other side of the triangle, and prolong the perpendicular on that side till it meets this parallel.)

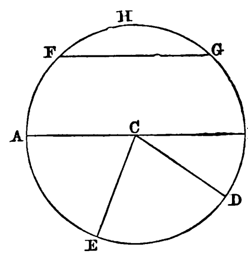
60. The sum of the three perpendiculars, drawn from any point within an equilateral triangle on the three sides, is equal to the altitude of the triangle. (Auxiliary line: Through the point draw a parallel to the base. The theorem is then an easy corollary of 59.)

BOOK II.

THE CIRCLE, AND THE MEASUREMENT OF ANGLES.

DEFINITIONS.

1. The *circumference of a circle* is a curve line, all the points of which are equally distant from a point within, called the *centre*.



The *circle* is the space terminated by this curved line.

N. B.—Sometimes, in common language, the circle is confounded with its circumference; but it will be always easy to recur to the correct expression, by recollecting that the circle is a surface, which has length and breadth, while the circumference is but a line.

2. Every straight line, CA, CE, CD, etc., drawn from the centre to the circumference, is called a *radius* or *semi-diameter;* every line, as AB, which passes through the centre, and which is terminated on both sides, at the circumference, is called a *diameter.*

From the definition of a circle, it follows, that all the radii are equal; that all the diameters are equal also, and each double of the radius.

3. A portion of the circumference, such as FHG, is called an *arc.*

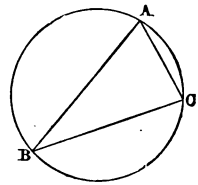
The *chord* or *subtense* of the arc is the straight line FG which joins its two extremities.

4. A *segment* is the surface or portion of a circle comprised between the arc and its chord.

N. B.—To the same chord, FG, correspond always two arcs, FHG, FEG, and, consequently, also two segments; but the smaller one is always meant unless the contrary is expressed.

5. A *sector* is the part of the circle included between an arc, DE, and the two radii, CD, CE, drawn to the extremities of that arc.

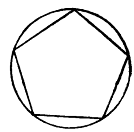
6. A *straight line* is said to be *inscribed in a circle*, when its extremities are in the circumference, as AB.



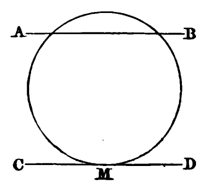
An *inscribed angle* is one, such as BAC, whose vertex is in the circumference, and which is formed by two chords.

An *inscribed triangle* is one which, like BAC, has its three vertices in the circumference.

And, in general, an *inscribed figure* is one of which all the angles have their vertices in the circumference. The circle is said at the same time to be *circumscribed* about this figure.



7. A *secant* is a line which meets the circumference in two points : AB is a secant.

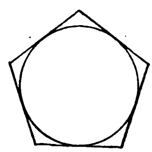


8. A *tangent* is a line which has only one point in common with the circumference : CD is a tangent.

The point, M, is called the *point of contact.*

9. In like manner, two circumferences are *tangent* to each other when they have only one point in common.

10. A polygon is *circumscribed about a circle* when all its sides are tangents to the circumference : in the same case we say that the circle is *inscribed* in the polygon.

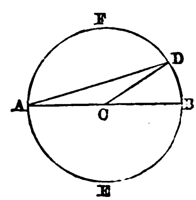


PROPOSITION I.

THEOREM.

Every diameter, AB, *divides the circle and its circumference into two equal parts.*

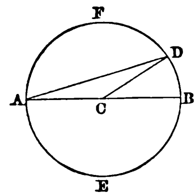
For, if we apply the figure AEB to AFB, the common base, AB, retaining its position, the curve line AEB must fall exactly on the curve line AFB, otherwise, there would be, in the one or the other, points unequally distant from the centre, which is contrary to the definition of a circle.



PROPOSITION II.

THEOREM.

Every chord is less than the diameter.



For, if the radii AC, CD be drawn to the extremities of the chord AD, we shall have the straight line AD < AC + CD, or AD < AB.

COR. Hence, the greatest straight line which can be inscribed in a circle is equal to its diameter.

PROPOSITION III.

THEOREM.

A straight line cannot meet a circumference in more than two points.

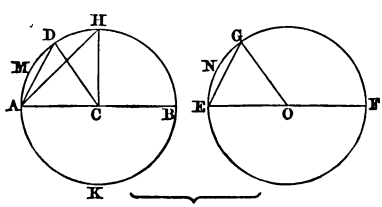
For, if it met it in three, those three points would be equally distant from the centre; there would, therefore, be three equal straight lines drawn from the same point to the same straight line, which is impossible (Book I., Prop. XVII. Cor. 2).

PROPOSITION IV.

THEOREM.

In the same circle, or in equal circles, equal arcs are subtended by equal chords; and, conversely, equal chords subtend equal arcs.

If the radii AC and EO are equal, and the arc AMD equal to the arc ENG, then the chord AD will be equal to the chord EG. For, since the diameters AB, EF, are equal, the semicircle AMDB may be applied exactly to the semicircle ENGF, and the curve line AMDB will coincide entirely with the curve line ENGF. But the part AMD is equal to the part ENG by hypothesis; therefore, the point D will fall on the point G; therefore, the chord AD is equal to the chord EG.

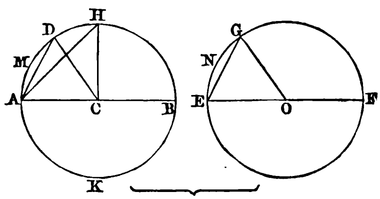


Conversely, supposing the radius AC = EO, if the chord AD = EG, then will the arc AMD be equal to the arc ENG. For, drawing the radii CD, OG, the two triangles ACD, EOG, having all their sides equal, each to each, viz.: AC = EO, CD = OG, and AD = EG, are themselves equal (Book I., Prop. XII.); therefore, the angle ACD = EOG. But, placing the semicircle ADB on its equal, EGF, since the angle ACD = EOG, it is evident that the radius CD will fall on the radius OG, and the point D on the point G; therefore, the arc AMD is equal to the arc ENG.

PROPOSITION V.

Theorem.

In the same circle, or in equal circles, a greater arc is subtended by a greater chord; and, conversely, the greater chord subtends the greater arc; the arcs being always supposed to be less than a semi-circumference.



Let the arc AH be greater than the arc AD, and draw the chords AD, AH, and the radii CD, CH: the two sides AC, CH, of the triangle ACH are equal to the two sides AC, CD, of the triangle ACD; the angle ACH is greater than ACD; therefore (Book I., Prop. XI.), the third side, AH, is greater than the third side, AD; therefore, the chord which subtends the greater arc is the greater.

Conversely, if we suppose the chord AH greater than AD, we conclude from the same triangles that the angle ACH is greater than ACD, and, therefore, that the arc AH is greater than AD.

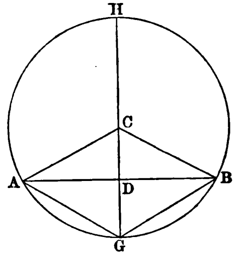
Scholium. The arcs here treated are less than the semicircumference. If they were greater, the reverse property would exist in them; as the arc increased, the chord would diminish, and conversely; thus, the arc AKBD being greater than AKBH, the chord AD of the first is less than the chord AH of the second.

PROPOSITION VI.

Theorem.

The diameter, GH, *perpendicular to a chord,* AB, *divides this chord and also the arcs* AGB, AHB, *which it subtends, into two equal parts.*

Let C be the centre of the circle, and D the point in which the diameter HG meets the chord AB. The radii CA, CB, considered with regard to the perpendicular CD, are two equal oblique lines; hence, they lie equally distant from that perpendicular (Book I., Prob. XVII., Cor. 3); hence AD=DB, or the chord AB is bisected at D.



Again, as we have shown that CG is perpendicular to AB at its middle point, every point of the perpendicular must be equidistant from A and B. Now, G is one of those points; therefore the chords GA and GB are equal; hence, the arcs GA and GB are also equal: or the arc, AGB, is bisected at the point G. In like manner we may show that the arc AHB is bisected at the point H.

Scholium. The centre, C, the middle point, D, of the chord AB, and the middle points, G and H, of the arcs subtended by this chord, are four points situated in the same line, perpendicular to the chord. Now, two points determine a straight line; therefore, every straight line which passes through two of the points mentioned, must necessarily pass through the third, and will be perpendicular to the chord.

It follows, also, that *the perpendicular, drawn to a chord at its middle point, will pass through the centre and through the middle points of the arcs subtended by that chord.* Also, *the geometric locus of the middle points of a system of parallel chords is the diameter perpendicular to those chords.*

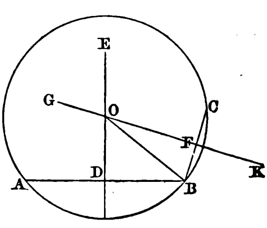
PROPOSITION VII.

Theorem.

Through three given points, A, B, C, *not in a straight line, one circumference may always be made to pass, and but one.*

We are to prove that there is one point, and only one, equally distant from the three points, A, B, C.

Draw AB, BC, and bisect those straight lines by the perpendiculars DE, FG. These lines, DE and FG, being perpendicular respectively to two lines, AB and BC, which meet, will meet each other in a point, O (Book I., Prop. XXV., Cor.).



And, moreover, since this point, O, lies in the perpendicular DE, it is equally distant from the two points A and B (Book I., Prop. XVIII.); and since the same point, O, lies in the perpendicular FG, it is also equally distant from the two points B, C; therefore the three distances, OA, OB, OC, are equal; therefore the circumference described from the centre O, and with the radius OB, will pass through the three given points, A, B, C.

We have now shown that one circumference may always be made to pass through three given points, not in a straight line; it may also be proved that only one circumference can be made to pass through these points.

For, if there were a second circumference passing through them, its centre, being equally distant from A, B, and C, must be at the same time in the two lines DE, FG. Now, two straight lines cannot cut each other in more than one point; therefore, there is only one circumference which can pass through three given points.

Scholium. This theorem may be enunciated thus: *Three points not in the same straight line determine a circumference.*

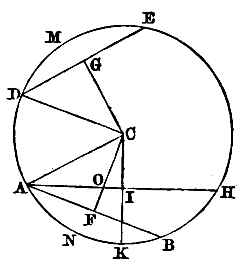
Cor. Two circumferences cannot meet in more than two points; for if they have three points common, they must have the same centre, and form one and the same circumference.

PROPOSITION VIII.

Theorem.

Two equal chords of a circle are equally distant from the centre; and of two unequal chords, the less is the farther from the centre.

First.—Let the chord AB = DE; bisect those chords by the perpendiculars CF, CG, and draw the radii CA, CD.



In the right angled triangles CAF, CDG, the hypothenuses, CA, CD, are equal; and the side AF, the half of AB, is equal to the side DG, the half of DE; hence, the triangles are equal (Book I., Prop. XIX.), and CF is equal to CG: therefore, the two equal chords, AB, DE, are equally distant from the centre.

Second.—Let the chord AH be greater than DE. The arc AKH will be greater than the arc DME (Prop. V.); on the arc AKH take the part ANB = DME, draw the chord AB, and let fall CF, perpendicular to this chord, and CI, perpendicular to AH. It is evident that CF is greater than CO, and CO greater than CI (Book I., Prop. XVII.); much more, therefore, is CF > CI.

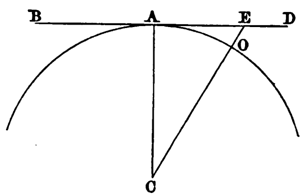
But CF = CG, since the chords AB, DE, are equal; hence, we have CG > CI: therefore, of two unequal chords, the less is the farther from the centre.

PROPOSITION IX.

Theorem.

The perpendicular, BD, *drawn to the radius,* CA, *at its extremity,* A, *is a tangent to the circumference.*

For, every oblique line, CE, is longer than the perpendicular, CA (Book II., Prop. XVII.); therefore the point E is outside of the circle; therefore the line BD has no point but A common to it and the circumference; hence, BD is a tangent (Def. 8).



Reciprocally. The radius, CA, *drawn to the point of contact of the tangent,* BD, *is perpendicular to the tangent.*

For all the points of the tangent, BD, except the point A, are exterior to the circumference; hence, the radius CA is the shortest line which can be drawn from the centre to the tangent, and is, therefore, perpendicular to it.

Cor. 1. *Through a point,* A, *of a circumference, one tangent can always be drawn to this circumference, and but one.*

Cor. 2. *Every tangent is parallel to the chords which are bisected by the diameter through the point of contact, and two tangents at the extremities of the same diameter are parallel.*

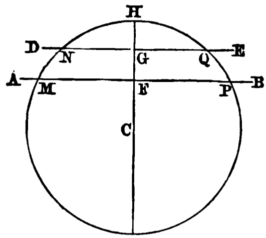
Cor. 3. *All the straight lines at a given distance from a given point, are tangent to a circumference of which the given point is the centre and the given distance is the radius.*

PROPOSITION X.

Theorem.

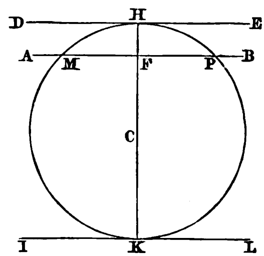
Two parallels, AB, DE, *intercept equal arcs,* MN, PQ, *on the circumference.*

There may be three cases.



First.—If the two parallels are secants, draw the radius CH perpendicular to the chord MP; it will be, at the same time, perpendicular to its parallel NQ (Book I., Prop. XXIII.); therefore the point H will be at once the middle of the arc MHP, and of the arc NHQ (Prop. VI.); therefore we shall have the arc MH = HP, and the arc NH = HQ: from this results MH − NH = HP − HQ, in other words, MN = PQ.

Second.—When, of the two parallels, AB, DE, one is a secant, and the other a tangent; draw the radius CH at the point of contact, H. This radius will be perpendicular to the tangent DE (Prop. IX.), and also to its parallel, MP. But since CH is perpendicular to the chord MP, the point H is the middle of the arc MHP; therefore the arcs MH, HP, included between the parallels AB, DE, are equal.



Third.—If the two parallels, DE, IL, are tangents, the one at H, the other at K, then H and K must be the two extremities of a diameter; and therefore the arc HMK = HPK: each one of these arcs being a semi-circumference.

DEFINITIONS.

Two points which are situated on the same perpendicular to a straight line, and at equal distances from the foot of the perpendicular, are said to be *symmetrical points*, with reference to the line. The line is called an *axis of symmetry*, with reference to the points.

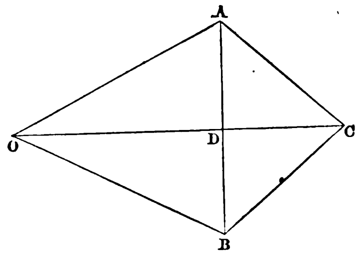
PROPOSITION XI.

THEOREM.

If two circumferences have a common point, A, *situated without the line,* CO, *which joins the centres* C *and* O, *these two circumferences have another common point,* B, *which is symmetrical with* A *with reference to this line of centres* CO.

Draw AD perpendicular to CO, and produce it until DB = AD.

Join OA, OB, also CA, CB. Since DB = AD the oblique line OB = OA. Hence, the circumference described from the centre O with the radius OA, must pass through the point B. In like manner CB = CA, and the circumference described from the centre C with radius CA, must pass through B. The point B is then a second point common to the two circumferences.



COR. 1. AB will be a common chord of the two circumferences. Therefore, *when two circumferences cut each other, the line which joins their centres is perpendicular to their common chord at its middle point.*

COR. 2. *If two circumferences are tangent to each other, the point of contact is situated on the line of the centres.* For otherwise the circumferences would have a second common point, and consequently would cut each other.

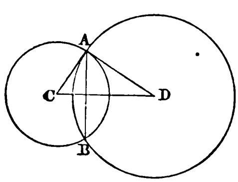
SCHOLIUM. Two circumferences can have *two* points in common, that is, cut each other; or have *one* point common, that is, be tangent to one another, externally or internally; or, finally, have *no* point common, that is, be entirely exterior to one another, or one interior to the other. Their possible relative positions are therefore *five* in number.

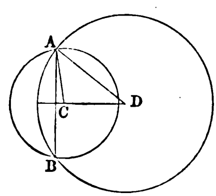
PROPOSITION XII.

THEOREM.

If two circumferences cut each other, the distance between their centres is less than the sum of their radii, and greater than the difference.

For, join the centres, C and D, to one of the points of intersection, A. We thus form a triangle in which the line of centres, CD, and



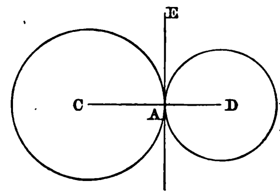


the radii DA and CA are the three sides. Hence it follows that the side CD $<$ CA + DA, and also CD $>$ DA − CA (Book I., Prop. IX., and Cor.).

PROPOSITION XIII.

THEOREM.

If two circumferences touch each other externally, the distance, CD, *between their centres, is equal to the sum of their radii,* CA, AD.

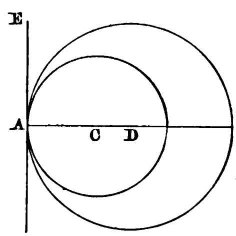


Let A be the point of contact. It must be situated on the line joining the centres (Prop. XI., Cor. 2). We have, therefore, CD = CA + AD.

PROPOSITION XIV.

THEOREM.

If one circumference touches another internally, the distance, CD, *between their centres is equal to the difference of their radii,* CA, AD.



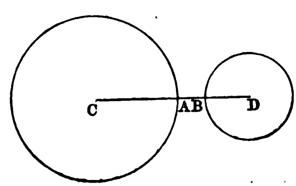
The point of contact, A, is on the line of centres (Prop. XI., Cor. 2). We have, therefore, DC = DA − CA.

SCHOLIUM. If two circumferences are tangent externally or internally, they have a common tangent at the point of contact.

PROPOSITION XV.

THEOREM.

If two circumferences are wholly exterior to each other, the distance between their centres is greater than the sum of the radii.



For, draw the line of centres, CD, cutting the circumferences in A and B respectively. We then have

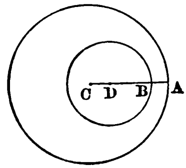
CD = CA + AB + DB.

Therefore, CD $>$ CA + DB.

PROPOSITION XVI.

THEOREM.

If two circumferences have no point in common, and one lies within the other, the distance between the centres is less than the difference of their radii.



Let the line CD, which joins the centres, cut the two circumferences in the points A and B respectively. We then have

$$CA = CD + DB + BA;$$

hence, $CA > CD + DB$,

and, therefore, $CD < CA - DB$.

SCHOLIUM. If the distance between the centres of two circles is nothing, that is, if they are described from the same centre, they are said to be *concentric*.

COR. 1. The reciprocals of the preceding theorems with regard to the positions of two circumferences are true. They may be enunciated as follows :

1. *If the distance between the centres is less than the sum of the radii, and greater than the difference, the two circumferences cut each other.*

2. *If the distance between the centres is equal to the sum of the radii, the two circumferences will touch each other externally.*

3. *If the distance between the centres is equal to the difference of the radii, one circumference will touch the other internally.*

4. *If the distance between the centres is greater than the sum of the radii, the circumferences have no point in common, and are exterior to each other.*

5. *If the distance between the centres is less than the difference of the radii, the circumferences have no point in common, and one lies within the other.*

COR. 2. All circles which have their centres on the right line CD (*See* Fig., Prop. XIV.), and which pass through the point A, are tangent to each other; they have only the point A common. And, if, through the point A, the straight line AE be drawn perpendicular to CD, it will be a common tangent to all the circles.

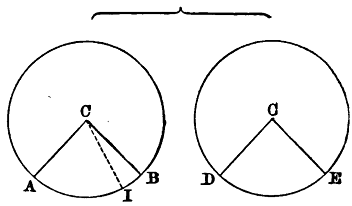
PROPOSITION XVII.

THEOREM.

In the same circle, or in equal circles, equal angles, ACB, DCE, *having their vertices at the centre, intercept equal arcs,* AB, DE, *on the circumference.*

Conversely, if the arcs AB, DE, *are equal, the angles* ACB, DCE, *will also be equal.*

First.—Since the angles ACB and DCE may be placed the one upon the other, and since their sides are equal, it is plain that the point A will fall on D, and the point B on E. But, in that case, the arc AB must also fall on the arc DE; for if they did not exactly coincide, there would be, in the one or the other, points unequally distant from the centre; which is impossible; hence the arc AB = DE.



Second.—If we suppose AB = DE, the angle ACB will be equal to DCE; for, if these angles are not equal, suppose ACB to be the greater, and let ACI be taken, equal to DCE. From what has just been shown, we shall have AI = DE: but, by hypothesis, the arc AB = DE; therefore we should have AI = AB, or a part equal to the whole, which is absurd: therefore the angle ACB = DCE.

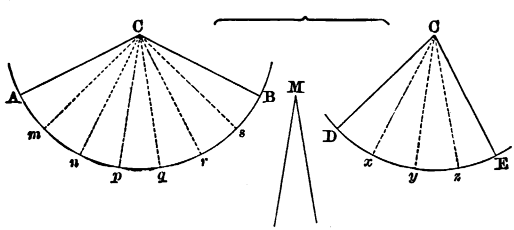
PROPOSITION XVIII.

THEOREM.

In the same circle, or in equal circles, if two angles at the centre, ACB, DCE, *are to each other as two whole numbers, the intercepted arcs,* AB, DE, *will be to each other as the same numbers, and we shall have*

the angle ACB : the angle DCE : : the arc AB : the arc DE.

Suppose, for example, that the angles ACB, DCE, are to each other as 7 is to 4; or, which is the same thing, suppose that the angle



M, which may serve as a common measure, is contained seven times in the angle ACB, and four times in DCE.

The seven partial angles AC*m*, *m*C*n*, *n*C*p*, etc., into which ACB is divided, being each equal to any one of the four partial angles into which DCE is divided; each of the partial arcs Am, mn, np, etc., will be equal to each of the partial arcs Ax, xy, etc. (Prop. XVII.); therefore, the whole arc AB will be to the whole arc DE, as 7 is to 4.

But the same reasoning would evidently hold good if, in place of 7 and 4, we had any other numbers whatever; hence, if the ratio of the angles ACB, DCE, can be expressed in whole numbers, the arcs AB, DE, will be to each other as the angles ACB, DCE.

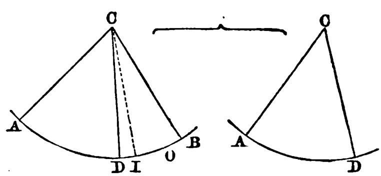
Scholium. Conversely, if the arcs AB, DE, are to each other as two whole numbers, the angles ACB, DCE, will be to each other as the same whole numbers, and we shall have ACB : DCE : : AB : DE; for the partial arcs Am, mn, etc., Dx, xy, etc., being equal, the partial angles ACm, mCn, etc., DCx, xCy, etc., will also be equal.

PROPOSITION XIX.

Theorem.

Whatever be the ratio of the two angles ACB, ACD, *these two angles will always be to each other as the arcs,* AB, AD, *intercepted between their sides and described from their vertices, as centres, with equal radii.*

Let the less angle be placed on the greater. If the proposition is not true, the angle ACB will be to the angle ACD as the arc AB is to an arc greater or less than AD. Suppose this arc to be greater, and let it be represented by AO; we shall thus have:



The angle ACB : angle ACD : : arc AB : arc AO.

Next, conceive the arc AB is divided into equal parts, each of which is less than DO; there will be at least one point of division between D and O: let I be that point; and join CI. The arcs AB, AI, will be to each other as two whole numbers, and we will have, by the preceding theorem:

The angle ACB : angle ACI : : arc AB : arc AI.

Comparing these two proportions with each other, and observing that the antecedents are the same, we conclude that the consequents are proportional, and thus we find

The angle ACD : angle ACI : : arc AO : arc AI.

But the arc AO is greater than the arc AI; hence, if this proportion be true, the angle ACD must be greater than the angle ACI; on the contrary, however, it is less; hence, the angle ACB cannot be to the angle ACD as the arc AB is to an arc greater than AD.

By a process of reasoning entirely similar, it may be shown that the fourth term of the proportion cannot be less than AD; hence it is AD itself, and therefore we have,

Angle ACB : angle ACD : : arc AB : arc AD.

Cor. Since the angle at the centre of a circle, and the arc intercepted between its sides, have such a connection that when one of them increases or diminishes in any ratio whatever, the other increases or diminishes in the same ratio, we are authorized to establish the one of those magnitudes as the measure of the other, *provided we take as unit-angle the angle at the centre which intercepts on the circumference the arc chosen as unit-arc;* and we shall henceforth assume in this sense, that *every angle at the centre has for its measure the arc intercepted between its sides.* It is only required that, in the comparison of angles with each other, the arcs which serve to measure them be described with equal radii, as all the foregoing propositions imply.

Scholium 1. It appears most natural to measure a quantity by a quantity of the same species; and upon this principle it would be convenient to refer all angles to the right angle; which, being made the unit of measure, an acute angle would be expressed by some number between 0 and 1; an obtuse angle by some number between 1 and 2. It has been found, however, more simple to measure angles by arcs of a circle, on account of the facility with which arcs can be made equal to given arcs, and for various other reasons. It is easy, also, to obtain the direct measurement in *right angle* units through this indirect measurement by arcs, since, on comparing the arc which serves as a measure to any given angle, with the fourth part of the circumference, we find the ratio of the given angle to a right angle, or its direct measure.

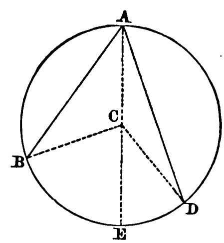
Scholium 2. In order to facilitate the comparison of angles by means of arcs, the circumference is divided into 360 equal parts, called *degrees;* each degree into 60 equal parts, called *minutes;* and each minute into 60 equal parts called *seconds.* Arcs are valued in degrees, minutes, and seconds, of the circumference. Thus, we say an arc of 36 degrees, 15 minutes, 21 seconds, which we write, $36° \ 15' \ 21''$. And we call an angle of $36° \ 15' \ 21''$, the angle which intercepts between its sides on any circumference described from its vertex or centre an arc of $36° \ 15' \ 21''$.

SCHOLIUM 3. By repeating almost literally the demonstrations of the preceding theorems, we can show that *two sectors*, ACB, ACD, *taken in the same circle, or in equal circles, are to each other as the arcs* AB, AD, *the bases of these sectors.*

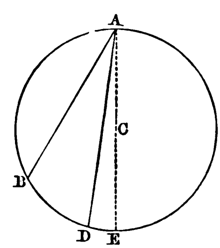
PROPOSITION XX.

THEOREM.

The inscribed angle BAD *is measured by half the arc*, BD, *included between its sides.*

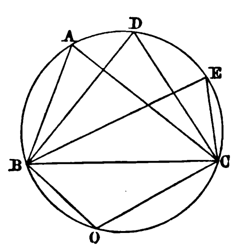


Let us first suppose that the centre of the circle lies within the angle BAD. Draw the diameter AE, and the radii CB, CD. The angle BCE, being exterior to the triangle ABC, is equal to the sum of the two interior angles CAB, ABC (Book I., Prop. XXIX., Cor. 7); but the triangle BAC being isosceles, the angle CAB = ABC; hence, the angle BCE is double of BAC. Since the angle BCE lies at the centre, it is measured by the arc BE; hence, BAC will be measured by the half of BE. For a like reason, the angle CAD will be measured by the half of ED; hence, BAC + CAD, or BAD, will be measured by the half of BE + ED, or of BD.



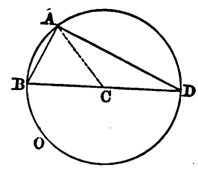
Suppose, in the second place, that the centre, C, lies without the angle BAD, then, drawing the diameter AE, the angle BAE will be measured by the half of BE; the angle DAE by the half of DE; hence, their difference, BAD, will be measured by the half of BE minus the half of ED, or by the half of BD.

Therefore, every inscribed angle is measured by half the arc included between its sides.



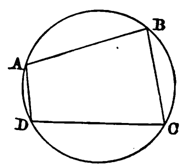
COR. 1. All the angles, BAC, BDC, etc., inscribed in the same segment, are equal; for they are measured by the half of the same arc, BOC.

COR. 2. Every angle, BAD, inscribed in a semi-circle, is a right angle; for it is measured by the half of the semi-circumference BOD, or the fourth part of the circumference. The same thing may be shown in another way by a reference to the Theorem 33, Exercises on Book I.



COR. 3. Every angle, BAC (*See* Fig., Cor. 1), inscribed in a segment greater than a semi-circle is an acute angle, for it is measured by the half of the arc BOC, less than a semi-circumference. And every angle, BOC, inscribed in a segment less than a semi-circle is an obtuse angle; for it is measured by the half of the arc BAC, greater than a semi-circumference.

COR. 4. The opposite angles, A and C, of an inscribed quadrilateral, ABCD, are together equal to two right angles; for the angle BAD is measured by half the arc BCD; and the angle BCD is measured by half the arc BAD; hence, the two angles BAD, BCD, taken together, are measured by half the circumference; hence, their sum is equal to two right angles.



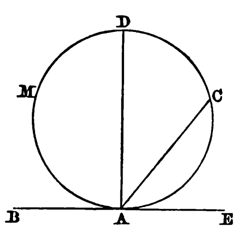
This corollary may also be conveniently expressed as follows: Every angle, BAC (*See* Fig., Cor. 1), inscribed in one of the segments determined by the chord BC, is the supplement of any angle, BOC, inscribed in the other segment determined by the same chord, BC.

PROPOSITION XXI.

THEOREM.

The angle, BAC, *formed by a tangent and a chord, is measured by half the arc,* AMDC, *included between its sides.*

From A, the point of contact, draw the diameter AD. The angle BAD is right (Prop. IX.), and is measured by half the semi-circumference, AMD; the angle DAC is measured by the half of DC; hence, BAD + DAC, or BAC, is measured by the half of AMD plus the half of DC, or by half the whole arc, AMDC. In like manner it may be shown that the angle CAE is measured by half the arc, AC, included between its sides.

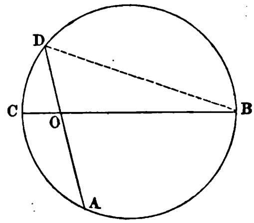


PROPOSITION XXII.

THEOREM.

The angle, AOB, *whose vertex*, O, *is within the circumference, is measured by half the arc*, AB, *included between its sides, plus half the arc*, DC, *included between their prolongations.*

For, join DB. The angle, AOB, exterior to the triangle, ODB, is equal to the sum of the two interior angles, D and B (Book I., Prop. XXIX., Cor. 7). But D is measured by half of the arc AB, and B by half the arc CD (Prop. XX.). Hence, the angle AOB is measured by the half of AB, plus the half of CD.

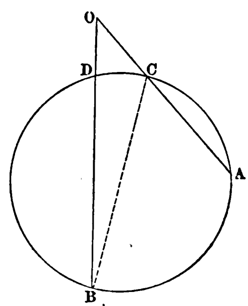


PROPOSITION XXIII.

THEOREM.

The angle, AOB, *whose vertex is without the circumference, and whose sides are secants, is measured by half the greater arc*, AB, *intercepted between its sides, minus half the less arc*, CD.

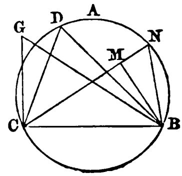
Join CB. The angle ACB, exterior to the triangle OCB, is equal to the sum of the interior angles, O and B. Then the angle O is equal to the difference of the angles ACB and B. But the angle ACB is measured by half of AB, and the angle B by half of CD. Hence, the angle O is measured by the half of AB minus the half of CD.



SCHOLIUM. This proposition is still true when one of the sides of the angle is tangent to the circumference, also when both of them are tangents; and the demonstration is the same.

COR. The arc, BAC, of the circumference, in which the angle CDB is inscribed, is the *locus* of the vertices of the angles equal to CDB, whose sides pass through C and B, and which lie on the same side of CB with D.

For, first, every angle inscribed in BAC has the same measure as CDB, that is, half the arc CB; second, every angle, CMB, whose sides pass through C and B, and whose vertex, M, lies within the segment, has a greater measure than half the arc CB (Prop. XXII.); and, third, every angle CGB, whose sides pass through C and B, and whose vertex lies without the arc BAC, has a less measure than half the arc CB (Prop. XXIII.).

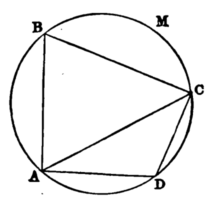


PROPOSITION XXIV.

THEOREM.

If two opposite angles, ADC, ABC, *of a quadrilateral, are together equal to two right angles, this quadrilateral may be inscribed in a circle.*

Describe a circumference through the three points A, D, C, and join AC. The angle ADC will be measured by half the arc AMC; hence, the angle ABC, being the supplement of ADC, is equal to any one of the angles inscribed in the segment AMC. Hence, its vertex, B, must be on the arc AMC (Prop. XXIII., Cor.). Therefore, the circumference passing through A, D, and C, passes also through B.



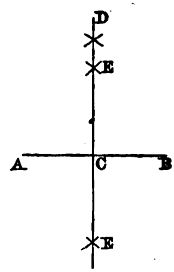
COR. If the sum of two opposite angles of a quadrilateral is greater or less than two right angles, the quadrilateral cannot be inscribed in a circle.

PROBLEMS RELATING TO THE FIRST TWO BOOKS.

PROBLEM I.

To divide the given straight line AB *into two equal parts.*

From the points A and B, as centres, with a radius greater than the half of AB, describe two arcs cutting each other in D. The point D will be equally distant from the points A and B. Find, in like manner, above or beneath the line AB, a second point, E, equally distant from the points A and B; through the two points D, E, draw the line DE: it will bisect the line AB at the point C.

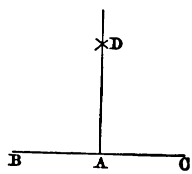


For the two points D and E, being each equally distant from the extremities A and B, must both be in the perpendicular raised from the middle of AB (Book I., Prop. XVIII., Cor. 2). But through two given points only one straight line can pass; hence, the line DE will itself be that perpendicular which divides the line AB into two equal parts at the point C.

SCHOLIUM. The problem, *On a given line*, AB, *as diameter, describe a circle*, depends immediately on the above. For the first step is to find the *middle point* of the line AB, which is the centre of the circle.

PROBLEM II.

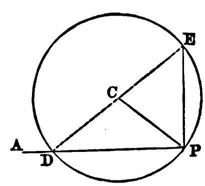
At a given point, A, *in the line* BC, *to erect a perpendicular to this line.*



Take two points, B and C, at equal distances from A; then, from the points B and C, as centres, with a radius greater than BA, describe two arcs cutting each other in D; draw AD, and it will be the perpendicular required.

For the point D, being equally distant from B and from C, belongs to the perpendicular erected at the middle of BC; therefore, AD is that perpendicular.

SCHOLIUM. If the point, P, were the extremity of the line, and if the line could not be produced beyond it, then a different construction must be employed.

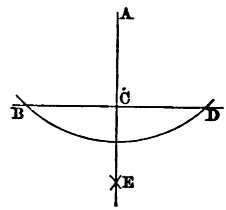


Thus, from any point, C, taken without the line, with a radius equal to the distance CP, describe a circumference, and from D, where it cuts AP, draw the diameter DE, and join EP; it will be the perpendicular required (Prop. XX., Cor. 2).

PROBLEM III.

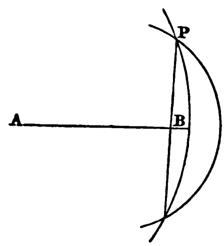
From a given point, A, *without the straight line* BD, *to let fall a perpendicular to this line.*

From the point A, as a centre, and with a radius sufficiently great, describe an arc, cutting the line BD at the two points, B and D; then mark a point, E, equally distant from the points B and D, and draw AE; it will be the perpendicular required.



For the two points A and E are each equally distant from the points B and D; therefore, the line AE is perpendicular to BD, at its middle point.

SCHOLIUM. If the point P were opposite the extremity of the line AB, or nearly so, and if AB could not be produced beyond this extremity, the following construction may be employed.

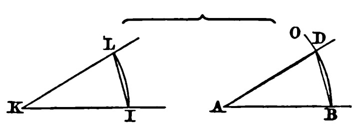


From any point, A, of AB, with a radius equal to its distance from P, describe an arc; then, from a second point in AB, with its distance from P as a radius, describe another arc. Join P with the other point of intersection of these arcs, and this line will be the perpendicular required (Prop. XI., Cor. 1).

PROBLEM IV.

At the point A *in the line* AB, *to make an angle equal to the given angle,* K.

From the vertex, K, as a centre, and with any radius, describe the arc IL, terminating in the two sides of the angle; from the point A, as a centre, and with a radius, AB, equal to KI, describe the indefinite arc BO; then take a radius equal to the chord LI; with which, from the point B, as a centre, describe an arc cutting the indefinite arc BO in D; draw AD, and the angle DAB will be equal to the given angle K.

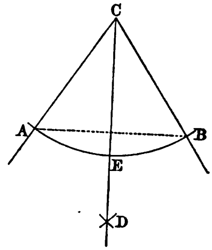


For the two arcs, BD, LI, have equal radii and equal chords; therefore, they are equal (Prop. IV.); therefore, the angles BAD, IKL, measured by them, are equal.

PROBLEM V.

To divide a given arc, or a given angle, into two equal parts.

First.—To divide the arc AB into two equal parts. From the points A and B, as centres, and with the same radius, describe two arcs cutting each other in D; through the point D and the centre C draw CD, it will bisect the arc AB in the point E.



For, the two points C and D are each equally distant from the extremities A and B of the chord AB; therefore, the line CD is perpendicular to the middle of this chord; hence, it divides the arc AB into two equal parts at the point E (Prop. VI., part 2).

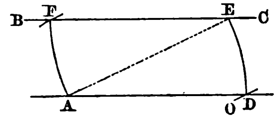
Second.—To divide the angle ACB into two equal parts, we first describe the arc AB, from the vertex C, as a centre; and bisect this arc as above. It is plain that the line CD will divide the angle ACB into two equal parts.

Scholium. By the same construction we may divide each of the halves, AE, EB, into two equal parts. Thus, by successive subdivisions, we may divide a given angle, or a given arc, into four equal parts, into eight, into sixteen, and so on.

PROBLEM VI.

Through a given point, A, *to draw a parallel to the given line* BC.

From the point A, as a centre, and with a radius sufficiently great, describe the indefinite arc EO; from the point E, as a centre, with the same radius, describe the arc AF; take ED = AF, and draw AD; this will be the parallel required.

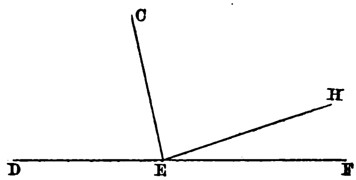


For, joining AE, we see that the alternate angles AEF, EAD, measured by equal arcs, are equal; therefore, the lines AD, EF, are parallel (Book I., Prop. XXV.).

PROBLEM VII.

Two angles, A *and* B, *of a triangle, being given, to find the third.*

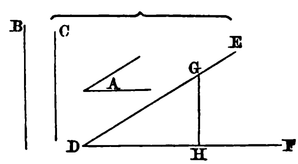
Draw the indefinite line DEF. At any point, as E, make the angle DEC = A (Prob. IV.), and the angle CEH = B; the remaining angle, HEF, will be the third angle required; for, these three angles taken together are equal to two right angles.



PROBLEM VIII.

Two sides, B *and* C, *of a triangle, and the angle* A, *which they contain, being given, to describe the triangle.*

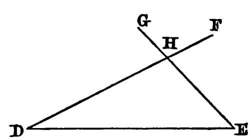
Having drawn the indefinite line DE, make, at the point D, the angle EDF, equal to the given angle A (Prob. IV.); then take DG = B, DH = C, and draw GH: DGH will be the required triangle (Book I., Prop. VII.)



PROBLEM IX.

One side and two angles of a triangle being given, to describe the triangle.

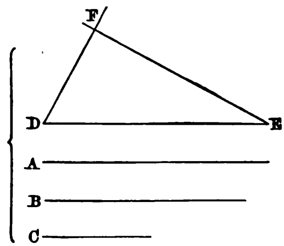
The two given angles will be either both adjacent to the given side, or the one adjacent, and the other opposite: in the latter case, find the third angle (Prob. VII.), and the two adjacent angles will thus be known; draw the straight line DE, equal to the given side; at the point D make an angle, EDF, equal to one of the adjacent angles, and, at E, the angle DEG, equal to the other: the two lines DF, EG, will cut each other in H, and DEH will be the triangle required.



PROBLEM X.

The three sides, A, B, C, *of a triangle being given, to describe the triangle.*

Draw DE, equal to the side A. From the point E, as a centre, with a radius equal to the second side, B, describe an arc; from D, as a centre, and with a radius equal to the third side, C, describe another arc cutting the former in F; draw DF, EF, and DEF will be the triangle required.



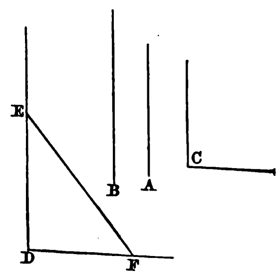
SCHOLIUM. If one of the sides were greater than the sum of the other two, the arcs would not cut each other; and the solution will be possible only when the sum of two sides, however taken, is greater than the third side.

PROBLEM XI.

Two sides, A *and* B, *of a triangle, and the angle* C, *opposite to the side* B, *being given, to describe the triangle.*

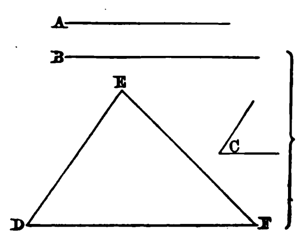
There are two cases.

First.—When the angle, C, is right or obtuse, make the angle, EDF equal to the angle C; take DE = A; from the point E, as a centre, with a radius equal to the given side B, describe an arc, cutting DF in F; draw EF; then DEF will be the triangle required.

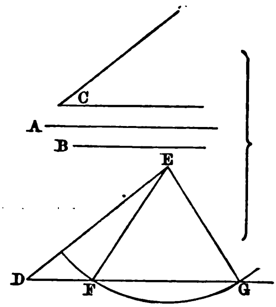


In this first case, the side, B, must be greater than A, for the angle C, being right or obtuse, is the greatest angle of the triangle; and the side opposite to it must, therefore, also be the greatest.

Second.—If the angle, C, is acute, and B greater than A, the same construction will again apply, and DEF will be the triangle required.



But, if the angle, C, is acute, and the side B less than A, then the arc described from the centre, E, with the radius EF = B, will cut the side DF in two points, F and G, lying on the same side of D; therefore, there will be two triangles, DEF, DEG, either of which will satisfy the problem.

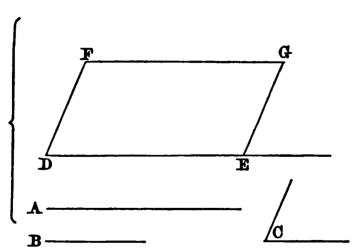


SCHOLIUM. The problem would be impossible in all cases, if the side B were less than the perpendicular let fall from E on the line DF.

PROBLEM XII.

The adjacent sides, A *and* B, *of a parallelogram, and the angle,* C, *which they contain, being given, to describe the parallelogram.*

Draw the line DE = A; at the point D, make the angle FDE = C; take DF = B; describe two arcs, one from F, as a centre, with a radius FG = DE, the other from E, as a centre, with a radius EG = DF; to the point G, in which these two arcs cut each other, draw FG, EG; DEGF will be the parallelogram required.



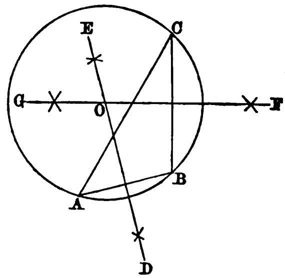
For, by construction, the opposite sides are equal; hence, the figure described is a parallelogram (Book I., Prop. XXXII.), and it is formed with the given sides and the given angle.

COR. If the given angle is right, the figure will be a rectangle; if, further, the sides are equal, it will be a square.

PROBLEM XIII.

To describe a circumference through three given points, A, B, C.

Join AB, BC, or suppose them to be joined. The centre of the required circle must be in a perpendicular to AB at its middle point, and also in a perpendicular to BC at its middle point.

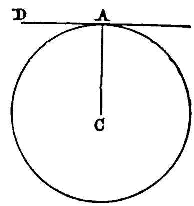


Therefore, bisect AB and BC by the perpendiculars DE, FG (Prob. I.); the point O, where these perpendiculars meet, will be the centre sought. The centre being known, we can describe the circumference with one of the equal lines, OA, OB, OC.

SCHOLIUM. The above construction serves for describing a circumference in which a given triangle, ABC, shall be inscribed. Also, *To find the centre of a given circle or arc*, we take upon it three points, A, B, C, and apply the above solution.

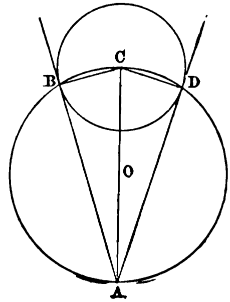
PROBLEM XIV.

Through a given point, to draw a tangent to a given circle.



If the given point, A, is in the circumference, draw the radius CA, and draw AD perpendicular to CA; AD will be the tangent required (Prop. IX.).

If the point A lies without the circle: suppose the problem solved, and AB to be a tangent drawn from A to the circle, and B the point of contact. Since the tangent is perpendicular to the radius at its extremity, CBA is a right angle, and hence the point B must lie on the circumference, which has AC for its diameter. Hence, to solve the problem, we have the following construction: Join A, and the centre C, by the straight line AC. On AC, as diameter, describe a circumference intersecting the given circumference in B: join AB; this will be the tangent required. Since the circumference on AC cuts the given circle in two points, there will be two tangents from A.



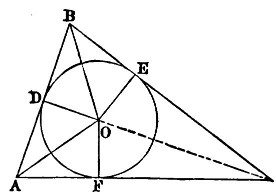
SCHOLIUM. *Two tangents,* AB, AD, *can be drawn to a given circle from a point,* A, *without. These are equal, and the straight line which joins the point* A *with the centre of the circle, bisects the angle* BAD *of the tangents, and also the angle* BCD *of the two radii to the points of contact.*

For the right angled triangles CBA, CDA, have the hypothenuse CA common, and the side CB = CD ; hence, they are equal ; hence, AD = AB, angle CAD = CAB, and also angle ACB = ACD.

PROBLEM XV.

To inscribe a circle in a given triangle, ABC.

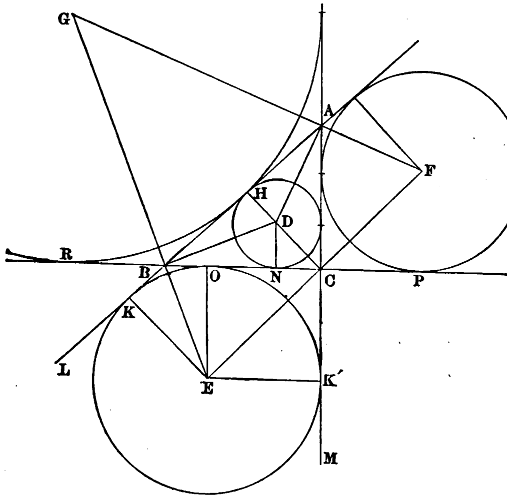
Suppose the problem solved. Since AB and AC are tangents to the inscribed circle, the centre, O, must lie on the bisectrix, AO, of the angle A, of the triangle (Prob. XIV., Schol.). So, also, it must lie on the bisectrix, BO, of the angle B.



Hence the construction: Bisect the angles A and B by the lines AO and BO (Prob. V.). These lines will meet (Book I., Prop. XXV., Cor. 1) in a point, O, equally distant from the three sides AB, AC, BC (Book I., Prop. XXI.). If, then, from O, perpendiculars, OD, OF, OE, be drawn to the sides of the triangle, these perpendiculars will be equal, and the circumference described from the point O, as centre, with OD as radius, will be tangent to the three sides (Prop. IX.).

SCHOLIUM 1. The point O, being equally distant from the sides BC, AC, lies also on the bisectrix of the angle C ; hence, *the three bisectrices of the angles of a triangle meet in the same point.*

SCHOLIUM 2. We see, in like manner, that the bisectrices of the exterior angles of a triangle intersect each other in the points E, F, G, each one of which lies on one of the bisectrices of the interior angles. Each one of these points, then, is equally distant from the sides of the triangle, and each is the centre of a circle which is tangent to one of the sides of the triangle, and to the prolongation of the two other sides. These three circles are called the *escribed circles* of the triangle.

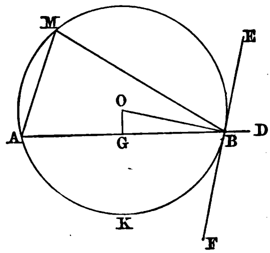


There are, then, in general, four circumferences tangent to three given straight lines which intersect so as to form a triangle.

PROBLEM XVI.

On a given straight line, AB, *to describe a segment of a circle which shall contain a given angle; that is, a segment all the angles inscribed in which shall be equal to a given angle.*

At the point B make the angle ABF, equal to the given angle (Prob. IV.). Then, if a circle be described to touch BF in B, and to pass through A, the segment AMB of that circle will be the segment required.



For, the angle ABF, formed by a tangent and a chord, will be measured by half the arc AB (Prop. XXI.), and any angle inscribed in AMB will also be measured by the half of AB (Prop. XXIII., Cor.). To find the centre of this circle draw BO perpendicular to EB, and GO perpendicular to AB at its middle point. The point of intersection, O, of these perpendiculars is the centre of the circle required (Props. IX. and VI.).

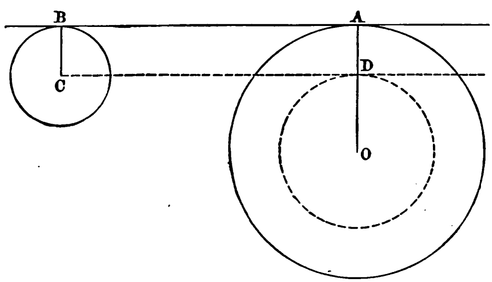
With the centre O and radius OA or OB, describe a circle; it will touch BE at B and pass through A, and, therefore, the segment AMB contains an angle equal to the angle ABF, that is, to the given angle.

PROBLEM XVII.

To draw a common tangent to two given circles.

Let the centres of the circles be C and O.

First.—With centre O and radius OD, equal to the difference of the radii OA and CB of the given circles, describe a circle. From the centre C, draw a tangent, CD, to this circle, touching it in D. Join

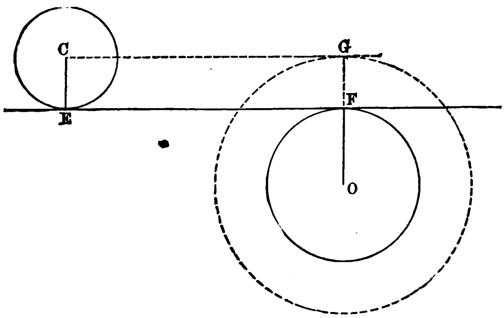


OD, and let it be produced through D to meet the given circumference whose centre is O, in the point A. Through C draw CB, parallel to OA, and join AB. AB will be tangent to both circles.

For, DA is, by construction, equal and parallel to CB; hence, AB is parallel to CD. But, CD is perpendicular to OA, since it touches the circle at D; hence, ABDC is a rectangle, and AB is perpendicular to the radii OA and CB, and is therefore tangent to both circles.

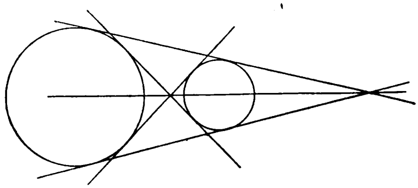
The construction shows that there are two exterior common tangents, since two tangents can be drawn from C to the circumference whose radius is OD. The problem is always possible unless one of the circumferences is interior to the other.

Second.—With centre O, and radius OG, equal to the sum of the radii of the given circles, describe a circle. From the centre C, draw a tangent to this circle, touching it at G. The remainder of the construction is the same as in the first case; EF will be tangent to both circles.

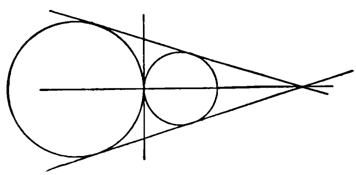


There are two solutions, also, in this case, the two tangents being called interior tangents. The problem is impossible in this case if the two circles cut each other.

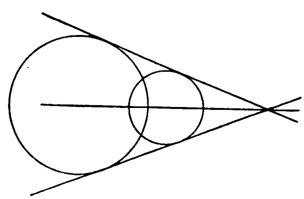
SCHOLIUM I. Two circles which are wholly outside one another have four common tangents, two exterior and two interior.



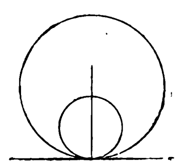
Two circles which touch externally have three common tangents, two exterior and one interior.



Two circles which intersect one another have two common tangents which are exterior.



Two circles which touch internally have one common tangent.



Where one of the circles is wholly inside the other, they have no common tangent.

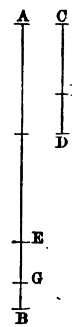
SCHOLIUM 2. It is evident (Prob. XIV., Schol.) *that the two exterior tangents are equal, and, also, the two interior tangents are equal, and the two interior and the two exterior intersect respectively on the line of the centres of the circles, and this line bisects their angles.* If the circles are equal, the exterior tangents are parallel.

PROBLEM XVIII.

To find the numerical ratio of two given straight lines, AB, CD, *provided that these two lines have a common measure.*

From the greater line, AB, cut off a part equal to the less, CD, as many times as possible: for example, twice, with the remainder, BE.

From the line CD cut off a part equal to the remainder, BE, as many times as possible, once, for example, with the remainder, DF.



From the first remainder BE, cut off a part equal to the second remainder, DF, as many times as possible, with the remainder, BG.

From the second remainder, DF, cut off a part equal to the third, BG, as many times as possible.

Continue thus until you have a remainder which is contained exactly a certain number of times in the preceding one.

Then this last remainder will be the *common measure* of the proposed lines; and regarding it as unity we shall easily find the values of the preceding remainders; and at last, those of the two proposed lines, and, hence, their ratio in numbers.

For example, if we find GB to be contained exactly twice in FD, BG will be the common measure of the two proposed lines. If we make BG = 1, we shall have FD = 2; but EB contains FD once *plus* GB; therefore, EB = 3; CD contains EB once *plus* FD; therefore, CD = 5; and, lastly, AB contains CD twice *plus* EB; therefore, AB = 13; therefore, the ratio of the two lines AB, CD, is that of 13 to 5. If the line, CD, was taken for unity, the line AB would be $\frac{13}{5}$; and if the line AB was taken for unity, the line CD would be $\frac{5}{13}$.

SCHOLIUM. The method which has just been explained is the same as that prescribed by arithmetic for finding the common divisor of two numbers; it needs, therefore, no other demonstration.

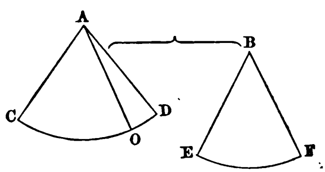
It is possible that, how far soever we continue the operation, we may never find a remainder which is contained an exact number of times in the preceding. In that case the two lines have no common measure, and are *incommensurable.* An instance of this will be seen hereafter in the ratio of the diagonal to the side of the square. In such cases, then, the exact ratio in numbers cannot be found.

The Ratio of Incommensurables is the limiting term of a series of ratios expressed in numbers. (See Introduction.)

PROBLEM XIX.

Two angles, A *and* B, *being given, to find their common measure, if they have one, and, by means of it, their ratio in numbers.*

Describe, with equal radii, the arcs CD, EF, to serve as measures for the angles; proceed then in the comparison of the arcs CD, EF, as in the preceding problem; for an arc may be cut off from an arc of the same radius, as a straight line from a straight line.



We shall thus arrive at the common measure of the arcs CD, EF, if they have one, and thence to their ratio in numbers. This ratio will be the same as that of the given angles (Prop. XVII.); and if DO is the common measure of the arcs, DAO will be that of the angles.

SCHOLIUM. It may happen, also, that the arcs compared have no common measure. In that case, the remark with regard to Incommensurables in the Scholium of the previous proposition, applies.

GEOMETRICAL ANALYSIS.

The words Analysis and Synthesis are used in Geometry in a special sense. Synthesis is a mode of reasoning which begins with some established truth or something given, and ends with some new result, with something required either to be done or to be proved. Synthesis leads from principles to consequences.

Analysis, or the method of resolution, is the reverse of Synthesis, or a method of reasoning from consequences to principles. The course of consecutive deduction is the same in both.

The synthetic method is pursued by Legendre as by Euclid in his elements of Geometry. They commence with certain assumed principles and definitions, and proceed to the solution of problems and the demonstration of theorems by successive inferences from them. The student has only to follow the reasoning by which the successive truths are established, without regard to the method of the discovery of these truths.

In Geometrical Analysis, we begin with assuming the truth of some theorem, or the solution of some problem; that is, assuming that what is required to be done has been effected; and we deduce from this assumption consequences which we can compare with known results, and thus test the truth of our assumption.

As Leslie has expressed it: "Analysis presents the medium of invention, while Synthesis naturally directs the course of instruction."

It is impossible to indicate any general and certain method for the demonstration of new theorems, or for the solution of problems by Analysis. Yet certain steps may be given which will render the investigation of new propositions easy and natural. These do not constitute so much a direct method of solution as a convenient way of searching for a suggestion. We give these steps separately for theorems and problems, remarking that the Geometrical Analysis is more extensively useful in discovering the solution of problems than for investigating the truth of theorems.

ANALYSIS OF THEOREMS.*

Assume that the Theorem is true.

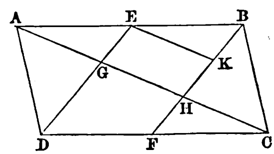
Construct the figure and examine any consequences that result from this assumption as a truth temporarily admitted, by the aid of other known truths respecting the figure. If any one of these consequences is known to be false, we have arrived at a *reductio ad absurdum*, which proves that the theorem is false. If a consequence can be deduced which coincides with some result already established, we start from this consequence, and endeavor, by retracing our steps, to give a synthetical demonstration of the theorem. This retracing our steps synthetically is essential, because a proposition may be false and yet furnish consequences that are true.

These directions are necessarily vague, however, as no certain rule can be given by which we can combine our assumption with truths already established. Nothing but experience, ingenuity, and a ready recollection of these truths will here avail the student. As an example to illustrate the steps indicated, we give the following

THEOREM.

If two opposite sides of a parallelogram be bisected, and two lines be drawn from the points of bisection to the opposite angles, these two lines trisect the diagonal.

Let ABCD be a parallelogram of which the diagonal is AC. Let AB be bisected in E, and DC in F; also let DE, FB, be joined, cutting the diagonal in G, H. Then is AC trisected in G and H.



Assume the theorem to be true; that is, AG = GH = HC, and draw EK parallel to GH.

Now (Book I., Prop. XXXIII.), ED and FB are parallel, and therefore EK = GH, being parallels intercepted between parallels. And, hence, if the theorem. is true, EK = AG. Therefore, the two triangles AGE and EKB are equal, as a consequence of an assumption, as they have AE=EB, angle EAG=BEK by reason of the parallels, and AG = EK by assumption.

Now, let us see whether these triangles are equal, from the known relations of the parts of the figure.

We have AE=EB by construction, EAG=BEK, and AEG=EBK, by reason of the parallels; hence, these two triangles are equal (Book I., Prop. VIII.), and so the consequence deduced from our assumption agrees with previously established results.

We now retrace our steps and give the *Synthesis*. Draw EK parallel to AC. The two triangles AGE and EKB are equal, having AE = EB, and the angles adjacent to these sides equal. Therefore, AG = EK. But EK = GH, being opposite sides of a parallelogram. Hence, AG = GH. Similarly, by drawing through F a line parallel to GH, and meeting DE, we can prove GH = HC. Therefore, AG = GH = HC, and the diagonal AC is trisected in G and H.

EXERCISES ON BOOK II.

THEOREMS.

1. A line from a point, A, without a circle, to the centre, O, meets the circumference in the points B and C. Show that AB is the shortest line, and AC the longest line, which can be drawn from A to the circumference. Prove, also, the same for a point, A, within the circle.

2. Show that the shortest distance between two circumferences is measured on the line which joins the centres: taking, first, the circles exterior to each other; and, second, one interior to the other.

3. The chord through a point, A, in a circle, which is perpendicular to the radius through that point, is shorter than any other chord which can be drawn through A.

4. If two equal chords cut each other, the parts of the one are equal to the parts of the other respectively.

5. Conversely, if two chords which intersect each other have one part equal in each, the two chords are also equal.

6. If two tangents to a circle intersect each other, the parts of these tangents, from the point of intersection to the points of contact, are equal; and, also, the bisectrix of the angle of the two tangents passes through the centre of the circle.

7. The sum of two of the opposite sides of a circumscribed quadrilateral is equal to the sum of the two other opposite sides.

8. Prove the converse of Theorem 7. That is, if the sum of two opposite sides of a quadrilateral is equal to the sum of the two other opposite sides, then a circle tangent to three sides will be tangent to the fourth.

9. The square and rhombus are the only parallelograms in which a circle can be inscribed.

10. If three circles are tangent to each other externally, the tangents drawn through the three points of contact meet in one point.

11. If a circle be inscribed in a triangle, the distance from the vertex of any angle to the points of contact of its sides is equal to the semi-perimeter, minus the side lying opposite to this angle.

12. Suppose a circle, O, tangent to the two sides, AB and AC, of

an angle, BAC, at the points B and C; then draw a tangent, DE, to this circle, terminating in the two sides of the angle. Show that the perimeter of the triangle ADE is constant, whatever point of the arc BC we take as the point of contact of the tangent DE.

13. Show that if in the above figure we join D and E with the centre, O, the angle DOE is constant.

14. If through one of the points of intersection of two circumferences a line be drawn parallel to the line of centres, the sum of the two chords intercepted on this parallel is double the distance of the centres.

The preceding theorems can be demonstrated without using the theorems of Book II. which relate to the measure of angles.

15. If two chords intersect on the circumference, the angle contained by one of them and the prolongation of the other (called an *exscribed angle*), is measured by half the sum of the arcs of the chords.

16. If two triangles have their angles equal, and are inscribed in the same circle, they are equal.

17. Three circumferences which pass respectively through the three vertices of a triangle and cut each other on the sides, all meet in the same point.

18. The circle which passes through the vertex of a triangle and through the adjacent feet of two perpendiculars, from the vertices on the opposite sides, passes also through the point of intersection of the perpendiculars.

19. The angles of the triangle formed by joining the three feet of the altitudes of a triangle, are bisected by these altitudes. (Use one of the circles described in the previous theorem, and the circle described on one side as a diameter, and compare inscribed angles.)

20. If two chords, AB, CD, intersect each other in a circle, the sum of the arcs, AC + BD, which they intercept on the circumference is equal to the sum of the arcs intercepted by two diameters parallel to these chords.

21. If the three vertices of an equilateral triangle, ABC, be joined to any point, P, of the circumscribed circle, by lines PA, PB, PC, then the line of junction, PA, which crosses the triangle, is equal to the sum of the other two. (Auxiliary Construction: On PA take PD = PB, and join BD, in order to obtain triangles for comparison.)

22. If, from the middle point, A, of an an arc, BC, two chords, AD, AE, be drawn, which cut the chord BC in the points F and G, the four points D, E, F, and G belong to the same circumference (that is, DEFG is an inscriptible quadrilateral).

23. Through the point of contact of two circles, tangent externally or internally, two straight lines are drawn which cut the two circumferences in four other points; then, if the two points on the first circle be joined, and the two points on the second also joined, these chords of junction will be parallel.

24. If, from any two points in the circumference of a circle, there be drawn two straight lines to a point in a tangent to the circle, they will make the greatest angle when drawn to the point of contact.

Geometric Loci to be Determined.

1. Find the locus of all the points which are at a given distance from a given point.

2. Find the locus of all the points which are at a given distance from a given straight line.

3. Find the locus of all the points at a given distance from a given circumference.

4. The locus of all the points, any one of which is equidistant from two straight lines (parallel or intersecting).

5. The locus of all the points, any one of which is equidistant from the circumferences of two equal circles.

6. The locus of the vertices of right angled triangles having the same hypothenuse.

7. The locus of all the points, the sum of whose distances from two intersecting straight lines is equal to a given line, is the perimeter of a rectangle of which the two given lines are diagonals (*See* Book I., Exercise 59).

8. Find the locus of the middle points of chords of a circle parallel to a given straight line.

9. Find the locus of the middle points of chords equal to a given line.

10. Find the locus of the middle points of chords of a circle which pass through a given point. (Let the point be taken within the circle, on the circumference, and exterior to it.)

11. Find the locus of the extremities of lines of given length drawn from every point of a given straight line parallel to a given line and all in the same direction.

12. The same, when the parallels are drawn from every point of a given circumference.

13. Two straight lines being given, from all the points of the first perpendiculars are let fall on the second, and these perpendiculars are prolonged until the parts prolonged are equal to the perpendiculars; find the locus of the extremities.

14. All the points of a straight line are joined with a given point, and all these lines of junction are prolonged till the prolongations are equal to the lines themselves; find the locus of their extremities.

15. The same problem, taking a circle instead of a straight line.

16. Find the locus of all the points from which tangents of a given length, equal to a given line, can be drawn to a given circle.

17. Find the locus of all the points of intersection of pairs of tangents which make with each other an angle equal to a given angle.

18. The locus of the vertices of triangles having the same base and their angles at the vertex equal to a given angle, is the arc of a segment of a circle containing the given angle, and constructed on the base of the triangle as chord (*See* Prob. XVI.).

19. The locus of the centres of the inscribed circles of these same triangles is the arc of a segment constructed on the base, and containing an angle equal to one right angle plus one half the given angle at the vertex.

20. Find the locus of the points of intersection of the altitudes of these same triangles.

21. Find the locus of centres of circles of given radius which divide a given circumference in half.

22. Find the locus of centres of circles which pass through two given points, A and B.

23. Find the locus of the centres of circles which are tangent to a given straight line at a given point.

24. Find the locus of the centres of circles tangent to a given circle at a given point.

25. Find the locus of centres of circles of given radius tangent to a given straight line.

26. Find the locus of centres of circles of given radius tangent to a given circle.

27. Find the locus of the centres of circles tangent to two parallel straight lines.

28. Find the locus of the centres of circles tangent to two intersecting straight lines.

29. An equilateral triangle being given, find the locus of all the points, the distance of any one of which from one of the vertices of the triangle is equal to the sum of its distances from the two other vertices (*See* Theorem, Exercise 21).

Solution of Problems by the Intersection of Loci.

The method of intersection of loci is one that admits of very frequent application in the solution of problems. When a point has to be found which satisfies *one condition*, it gives, as we see above, a *locus*. These problems are sometimes called *local* problems. When a point has to be found which satisfies *two* conditions, the problem is, if it is possible, determinate, and the point is the intersection of the loci, which are obtained by imposing each condition separately. Thus, when a point is to be found at a given distance from two given points, we find it at the intersection of two circles described from the given points as centres, with the given distance as radii. These circles being the loci of points at the given distance from the given points respectively. Again, if a point is to be found equally distant from two given points, and, also, equally distant from two given straight lines, this point must be at the intersection of the two loci which arise from the solution of the local problems.

The only loci used in Elementary Plane Geometry are the straight line and the circle.

Problems I., II., X., XIII., XIV., etc., are instances of the method of intersections of loci.

The determinate problems which result from the thirty loci found in the preceding exercises and in Book I., Prop. XVIII., and which are obtained by the combination of these loci, two and two, are 435 in number, and are very easy to make.

Determinate Problems.

The solution of a problem in Elementary Geometry consists,

1. In indicating how the ruler and compasses are to be used in making the construction required.

2. In proving that the construction so given is correct.

3. In showing when there is more than one solution, and in discussing the limitations which in some cases exist, within which alone the solution is possible.

Solution of Problems by Geometrical Analysis.

As has been stated in connection with the demonstration of Theorems by Analysis, Geometrical Analysis is not so much a method, as a way of searching for a suggestion. We assume that the problem is solved, draw the figure, and deduce consequences from this assumption, combined with truths and results already established. If a consequence can be deduced which contradicts some truth already established, this amounts to a demonstration that our assumption is inadmissible; that is, the problem cannot be solved. If a consequence can be deduced which coincides with some established truth, we then endeavor, starting from this consequence, to retrace our steps and give a synthetical solution of the problem.

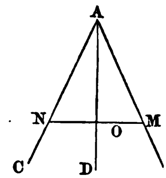
We cannot prescribe any certain rule by which we are to combine our assumption with previously established truths, so as to discover the relation which gives a clue to the construction. This must depend on the student's familiarity with the truths previously established, and his ingenuity in applying them. It will be seen, however, that, in general, we proceed by successive substitutions; that is, we bring the proposed problem to depend on another, and this on a third, etc., until we arrive at a problem whose solution is known.

For example: Problem VI. is reduced to the problem previously solved, of the construction of an angle equal to a given angle; and all the problems with reference to perpendiculars are reduced to finding points equidistant from two given points; and the problem of drawing a common tangent to two circles, is reduced to the drawing a tangent to a circle from a point without. We add one or two more instances of the solution of problems by analysis.

(1.) *It is required to draw a line which shall pass through a given point, and make equal angles with two given intersecting lines.*

Let O be the given point, and AB, AC, the given lines.

Analysis.—Suppose the problem solved, and the line MON were the line required; and the angle M = angle N. Then MAN would be an isosceles triangle of which MON is the base. And, therefore, MON would be perpendicular to the bisectrix of the angle MAN. Hence, beginning with this last step, we have the following

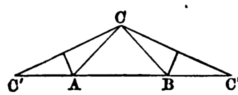


Solution.—Bisect the angle MAN by the line AD, and through O draw a perpendicular to AD, and prolong it to meet the sides AB, and AC, in M and N. MON will be the line required, since it is plain it must make equal angles with AB and AC.

(2.) *Required to construct a triangle, given the angles, and the perimeter.*

Let C′C″ be a line equal in length to the perimeter.

Suppose the problem solved, and BAC to be the triangle required. It follows, then, that AC′ = AC and BC″ = BC, and if lines be drawn connecting C with C′ and with C″, then these triangles CAC′ and CBC″ are isosceles, and the angle C′ = ½A, and C″ = ½B. We know, then, the side C′C″, and the angles C′ and C″, and can construct the triangle CC′C″, and then use it to construct BAC. Hence, the following



Solution.—Draw a straight line, C′C″, equal to the perimeter. At C′ and C″, make the angles C′ and C″, equal to one-half the given angles A and B respectively. We get, thus, the triangle C′C″C. Erect perpendiculars to the middle points of C″C and C′C, and produce them till they meet the line C′C″ in B and A respectively. Join CA, CB, then ACB is the triangle required.

Exercises in the Solution of Determinate Problems.

1. Erect a perpendicular to a given line at its extremity, without prolonging the line (by a construction differing from that given in the text).

2. Through a given point without a straight line, draw a second straight line which shall make with the first a given angle.

3. Draw a tangent to a circle, parallel to a given straight line.

4. Draw, in a circle, a chord of given length, passing through a given point.

5. Draw, in a circle, a chord of given length, and parallel to a given straight line.

6. Between two parallels draw a straight line of given length passing through a given point.

7. Between two intersecting straight lines draw a straight line of given length, and parallel to a given straight line.

8. With a given radius describe a circle which shall pass through two given points.

9. With a given radius describe a circle which shall pass through a given point, and be tangent to a given straight line.

10. With a given radius describe a circle which shall pass through a given point, and be tangent to a given circle.

11. With a given radius describe a circle tangent to two given straight lines.

12. With a given radius describe a circle tangent to a given straight line and given circle.

13. With a given radius describe a circle tangent to two given circles.

14. Describe a circle which shall cut three equal chords of given length from three given straight lines.

15. With a given radius describe a circle which shall be at the same given distance from three given points not in the same straight line.

16. Inscribe a circle in a given rhombus.

17. Find a point on a given straight line, at equal distances from two given points.

18. Trisect a right angle.

19. Trisect a given straight line.

20. From the vertices of a triangle as centres, describe three circles which shall touch each other, two and two.

21. Find how many circles may be constructed equal to a given circle, touching it, and tangent to each other, two and two.

CONSTRUCTION OF TRIANGLES, ETC.

22. Construct an isosceles triangle, given its base and adjacent angle.

23. Construct an isosceles triangle, given its base and the radius of the inscribed circle.

24. Construct a right angled triangle, given the hypothenuse and one of the acute angles.

25. Construct a right angled triangle, given the hypothenuse and the perpendicular let fall from the right angle on the hypothenuse.

26. Construct a right angled triangle, given the hypothenuse and the radius of the inscribed circle.

27. Construct a triangle, given one side, an adjacent angle, and the sum of the two other sides.

28. Construct a triangle, given one side, an adjacent angle, and the difference of the two other sides.

29. Construct a right angled triangle, given the radius of the inscribed circle, and the radius of the circumscribed circle.

30. Construct a right angled triangle, given the radius of the inscribed circle, and one of the acute angles.

31. Construct a right angled triangle, given the radius of the inscribed circle, and one of the sides containing the right angle.

32. Construct a right angled triangle, given the median and altitude drawn from the vertex of the right angle.

33. Construct a triangle, given two sides and one altitude (two problems).

34. Construct a triangle, given two sides and one median line (two problems).

35. Construct a triangle, given the angles and the radius of the circumscribed circle.

36. Construct a triangle, given the three medians.

37. Construct a triangle, given the middle points of the three sides.

38. Construct a triangle, given two vertices; and, first, the point of intersection of the medians; second, the point of intersection of the three altitudes; third, the point of intersection of the three bisectrices of the angles.

39. Construct a triangle, given one vertex and the feet of two altitudes.

40. Construct a triangle, given the feet of the three altitudes.

41. Construct a triangle, given the centres of the three escribed circles.

42. Construct a triangle, given the radius of the inscribed circle, the radius of an escribed circle, and one angle.

43. Construct a triangle, given the angles and one altitude.

44. Construct a square, given its diagonal.

45. Construct a rhombus, given its two diagonals.

46. Construct a rectangle, given the diagonal and one side.

47. Construct a square, given the sum of its diagonal and side.

48. Construct a square, given the difference of its diagonal and side.

49. Construct a trapezoid, given its four sides, and it being stated, also, which two are parallel.

50. Construct a pentagon, given the middle points of its sides.

51. Construct a rectangle, given its perimeter and its diagonal.

52. Construct a rhombus, given one side and the sum of its diagonals.

53. Construct a rhombus, given one of its angles and the radius of the inscribed circle.

BOOK III.

THE PROPORTIONS OF FIGURES.

DEFINITIONS.

1. Those figures whose surfaces are equal, we shall call *equivalent figures.*

Two figures may be equivalent, although very dissimilar: for example, a circle may be equivalent to a square, a triangle to a rectangle, etc.

The denomination of *equal* figures we shall reserve for such as when applied the one to the other, coincide in all their points: of this kind are two circles the radii of which are equal, two triangles of which the three sides are equal, each to each, etc.

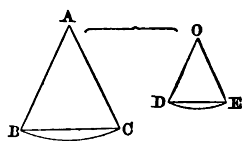
2. Two figures are *similar* when they have their corresponding angles equal, each to each, and their *homologous sides* proportional.

By *homologous sides* are meant those which have a corresponding position in the two figures, or which are adjacent to equal angles. These angles themselves are called *homologous angles.*

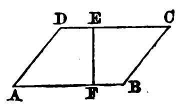
Two equal figures are always similar; but two similar figures may be very unequal.

3. In different circles *similar arcs, similar sectors, similar segments* are those which correspond to equal angles at the centre.

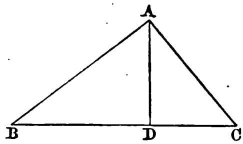
Thus, if the angle A is equal to the angle O, the arc BC will be similar to the arc DE, the sector ABC to the sector ODE, etc.



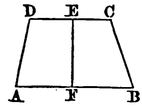
4. The *altitude* of a parallelogram is the perpendicular, EF, which measures the distance between the two opposite sides, AB, CD, taken as bases.



5. The altitude of a triangle is the perpendicular, AD, let fall from the vertex of an angle, A, on the opposite side, BC, taken as a base.



6. The altitude of a trapezoid is the perpendicular, EF, drawn between its two parallel sides, AB, CD.



7. The *area* and the surface of a figure are terms which are nearly synonymous. Area designates more particularly the superficial magnitude of a figure, in so far as it is measured, or compared, with other surfaces.

N. B.—In order to understand this and the following books, it will be necessary to bear in mind the theory of proportions, for which we refer to the introduction. [*See* Introduction.] We shall make only one observation, which is very important for fixing the true import of propositions, and for dissipating all obscurity, whether in their enunciation or their demonstration.

If we have the proportion A : B :: C : D, it is known that the product of the extremes, $A \times D$, is equal to the product of the means, $B \times C$.

This truth is incontestible for numbers: we add that it is equally so for magnitudes of any kind whatever, provided they be expressed, or we may imagine them expressed, in numbers; and this may always be supposed. For example, if A, B, C, D are lines, we may conceive that one of these four lines, or a fifth, if requisite, serves for a common measure for them all, and that it is taken for unity; then A, B, C, D will each represent a certain number of units, entire or fractional, commensurable or incommensurable, and the proportion between the lines A, B, C, D becomes a numerical proportion.

The product of the lines A and D, which is also named their *rectangle*, is then nothing else than the number of linear units contained in A, multiplied by the number of linear units contained in B; and it is easily conceived that this product can be, and must be, equal to that which results similarly from the lines B and C.

The magnitudes A and B may be of one kind, for example, lines; and the magnitudes C and D of another kind, for example, surfaces; in this case it will be necessary always to regard these magnitudes as numbers: A and B will be expressed in linear units, C and D in superficial units, and the product, $A \times D$, will be a number, as will be the product $B \times C$.

In general, in every operation connected with proportions, the terms of these proportions must be regarded as so many numbers, each of the kind appropriate to its nature, and there will be no difficulty in conceiving these operations and the consequences which result from them.

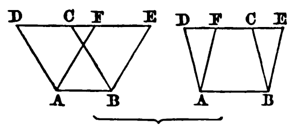
We must call attention, also, to the circumstance that several of our demonstrations are founded on some of the simplest rules of algebra, which are themselves based upon familiar axioms : thus, if we have $A = B + C$, and if we multiply each member by the same quantity, M, we conclude $A \times M = B \times M + C \times M$; likewise, if we have $A = B + C$, and $D = E - C$, and if we add together these equations, cancelling $+C$ and $-C$, which destroy each other, we shall have $A + D = B + E$, and so in other cases. All this is sufficiently evident in itself; but in case of difficulty, it will be well to consult the books on algebra, and thus to combine the study of the two sciences.

PROPOSITION I.

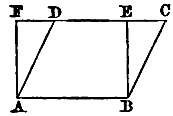
Theorem.

Parallelograms which have equal bases and equal altitudes, are equivalent.

Let AB be the common base of the two parallelograms, ABCD, ABEF. Since they are supposed to have the same altitude, the upper bases, DC, FE, will both lie in one straight line parallel to AB. Now, from the nature of parallelograms, we have $AD = BC$, and $AF = BE$; for the same reason we have $DC = AB$, and $FE = AB$; therefore, $DC = FE$; hence, if DC and FE be taken away from the same line, DE, the remainders, CE and DF, will be equal. It follows that the triangles DAF, CBE are mutually equilateral, and consequently equal (Book I., Prop. XII.).



But if, from the quadrilateral ABED we take away the triangle, ADF, there will remain the parallelogram ABEF; and if, from the same quadrilateral, ABED, we take away the triangle CBE, there will remain the parallelogram ABCD; therefore, the two parallelograms ABCD, ABEF, which have the same base and the same altitude, are equivalent.

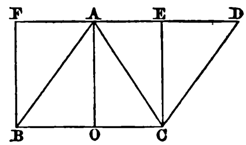


COR. Every parallelogram, ABCD, is equivalent to the rectangle, ABEF, which has the same base and the same altitude.

PROPOSITION II.

THEOREM.

Every triangle, ABC, *is the half of the parallelogram which has the same base and the same altitude.*



Draw through the vertices A and C of the triangle parallels, AD and CD, to the opposite sides. We form, thus, a parallelogram, ABCD, of same base, BC, and same altitude with the triangle. ABC is the half of ABCD.

For the triangles ABC, ACD are equal (Book I., Prop. XXXI., Cor.).

COR. 1. Therefore, a triangle, ABC, is the half of the rectangle, BCEF, which has the same base, BC, and the same altitude, AO; for the rectangle, BCEF, is equivalent to the parallelogram, ABCD.

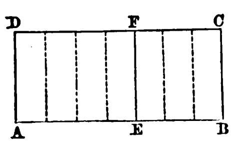
COR. 2. All triangles which have equal bases and equal altitudes are equivalent.

PROPOSITION III.

THEOREM.

Two rectangles of the same altitude are to each other as their bases.

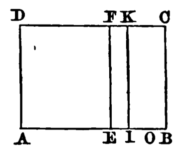
Let ABCD, AEFD, be two rectangles which have AD for their common altitude: they will be to each other as their bases, AB, AE.



Suppose, first, that the bases, AB, AE, are commensurable, and are to each other, for example, as the numbers 7 and 4. If we divide AB into seven equal parts, AE will contain four of these parts; at each point of division, erect a perpendicular to the base, and seven partial rectangles will thus be formed, which will be equal to each other, because all have the same base and the same altitude.

The rectangle ABCD will contain seven partial rectangles, while AEFD will contain four; therefore, the rectangle ABCD is to the rectangle AEFD as 7 is to 4, or as AB is to AE. The same reasoning may be applied to any ratio as well as to that of 7 to 4; hence, whatever be that ratio, provided its terms be commmensurable, we shall have

ABCD : AEFD : : AB : AE.



Suppose, in the second place, that the bases, AB, AE, are incommensurable: it is to be shown that still we shall have

ABCD : AEFD : : AB : AE.

For, if this proportion is not true, the three first terms remaining the same, the fourth term will be greater or less than AE.

Suppose it to be greater, and that we have

ABCD : AEFD : : AB : AO.

Divide the line AB into equal parts less than EO. There will be at least one point, I, of division between E and O; from this point draw IK, perpendicular to AI; the bases AB, AI will be commensurable, and thus, from what is proved above, we shall have

ABCD : AIKD : : AB : AI.

But, by hypothesis, we have

ABCD : AEFD : : AB : AO.

In these two proportions the antecedents are equal; hence, the consequents are proportional, and we find

AIKD : AEFD : : AI : AO.

But AO is greater than AI; hence, if this proportion be correct, the rectangle AEFD must be greater than AIKD.; on the contrary, however, it is less; hence, the proportion is impossible; and, therefore, ABCD cannot be to AEFD as AB is to a line greater than AE.

Exactly in the same manner it may be shown that the fourth term of the proportion cannot be less than AE; therefore, it is equal to AE.

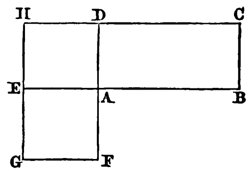
Therefore, whatever be the ratio of the bases, two rectangles, ABCD, AEFD, of the same altitude, are to each other as their bases, AB, AE.

PROPOSITION IV.

THEOREM.

Any two rectangles, ABCD, AEGF, *are to each other as the products of their bases multiplied by their altitudes, so that*

ABCD : AEGF : : AB × AD : AE × AF.



Having placed the two rectangles so that the angles at A are opposite angles at the vertex, produce the sides GE, CD, till they meet in H. The two rectangles ABCD, AEHD, having the same altitude, AD, are to each other as their bases AB, AE : in like manner the two rectangles AEHD, AEGF, having the same altitude, AE, are to each other as their bases, AD, AF ; thus we have the two proportions

ABCD : AEHD : : AB : AE,
AEHD : AEGF : : AD : AF.

Multiplying the corresponding terms of these proportions together, and observing that the mean term, AEHD, may be omitted, since it is a multiplier of both the antecedent and the consequent, we shall have

ABCD : AEGF : : AB × AD : AE × AF.

SCHOLIUM 1. Hence, the product of the base by the altitude may be assumed as the measure of a rectangle, provided we understand by this product the product of two numbers, one of which is the number of linear units contained in the base, and the other the number of linear units contained in the altitude.

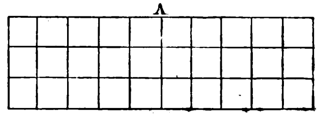
This measure, however, is not absolute, but only relative ; it supposes that the area of any other rectangle is computed in a similar manner by measuring its sides by the same linear unit : we thus obtain a second product, and the ratio of the two products is equal to that of the rectangles, agreeably to the proposition just demonstrated.

For example, if the base of the rectangle A, contains three units, and its altitude ten, that rectangle will be represented by the number 3 × 10, or 30, a number which, when thus isolated, signifies nothing ; but if we have a second rectangle, B, whose base contains twelve units, and whose altitude contains seven, the second rectangle will be represented by the number 7 × 12, or 84 : whence we shall infer that the two rectangles, A and B, are to each other as 30 is to 84 ;

therefore, if we agree to take the rectangle A, for the unit of measure of surfaces, the rectangle B, would, in that case, have $\frac{84}{30}$ for its absolute measure; that is to say, it would be equal to $\frac{84}{30}$ of superficial units.

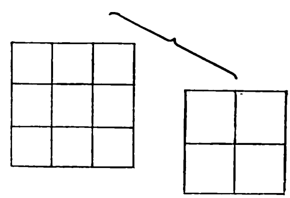
It is more common and more simple to take the square for the unit of surface, and the square is chosen whose side is the unit of length; then the product of the base and altitude expresses the number of unit-squares in the rectangle.

A



For example, the number 30, by which we have measured the rectangle A, represents 30 superficial units, or 30 of these squares whose side is equal to unity. This is evident in the accompanying figure.

In geometry, the product of two lines is frequently spoken of as synonymous with their *rectangle*, and this expression has even passed into arithmetic, to designate the product of two unequal numbers, as the expression *square* is used to express the product of a number multiplied by itself.



The squares of the numbers 1, 2, 3, etc., are 1, 4, 9, etc. We see, also, that the square made on a double line is four times as great as the square made on a single one; on a triple line it is nine times as great, and so on.

SCHOLIUM 2. The measure of the rectangle may be presented in a more general manner. Thus, if R, R′ be two rectangles, B, H, and B′, H′, their bases and altitudes respectively, then

$$\frac{R}{R'} = \frac{B \times H}{B' \times H'};$$

and if we take R′ for the unit of measure of R, we have

$$R = \frac{B \times H}{B' \times H'}.$$

Thus, the *measure of a rectangle, when we take another rectangle for the unit, is equal to the product of its base and altitude, divided by the product of the base and altitude of the unit rectangle.*

When we take as this unit rectangle the square whose side is equal to the unit of length, so that $B' = H' = 1$, then the denominator,

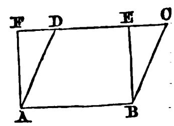
$B' \times H' = 1$, and $R = B \times H$. In this case, then, *the measure of the rectangle* is the *product of its base by its altitude.*

PROPOSITION V.

THEOREM.

The area of any parallelogram is equal to the product of its base by its altitude.

For the parallelogram ABCD is equivalent to the rectangle, ABEF, which has the same base, AB, and the same altitude, BE (Prop. I.); but this rectangle has for its measure $AB \times BE$ (Prop. IV.); therefore, $AB \times BE$ is equal to the area of the parallelogram ABCD.



COR. Parallelograms of the same base are to each other as their altitudes, and parallelograms of the same altitude are to each other as their bases: for, A, B, C being any three magnitudes, we have, generally,

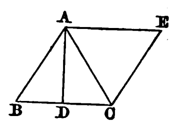
$$A \times C : B \times C :: A : B.$$

PROPOSITION VI.

THEOREM.

The area of a triangle is equal to the product of its base by half its altitude.

For the triangle, ABC, is the half of the parallelogram, ABCE, which has the same base, BC, and the same altitude, AD (Prop. II.): but the surface of the parallelogram $= BC \times AD$ (Prop. V.); therefore, that of the triangle $= \frac{1}{2} BC \times AD$, or $BC \times \frac{1}{2} AD$.



COR. Two triangles of the same altitude are to each other as their bases, and two triangles of the same base are to each other as their altitudes.

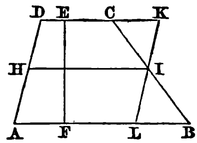
PROPOSITION VII.

THEOREM.

The area of the trapezoid, ABCD, *is equal to its altitude,* EF, *multiplied by half the sum of its parallel bases,* AB, CD.

Through I, the middle point of the side CB, draw KL, parallel to the opposite side, AD; and produce DC till it meets KL.

In the triangles IBL, ICK, we have the side IB=IC, by construction; the angle LIB=CIK; and since CK and BL are parallel, the angle IBL=ICK (Book I., Prop. XXIV.); hence, the triangles are equal (Book I., Prop. VII.); therefore, the trapezoid ABCD is equivalent to the parallelogram ADKL, and is measured by EF × AL.



But we have AL=DK; and since the triangles IBL and KCI are equal, the side BL=CK; hence, AB+CD=AL+DK=2AL, and accordingly, AL is half the sum of the bases AB, CD; hence, the area of the trapezoid ABCD is equal to the altitude, EF, multiplied by half the sum of the bases, AB, CD, which is expressed thus:

$$\text{ABCD} = \text{EF} \times \left(\frac{\text{AB} + \text{CD}}{2}\right).$$

Scholium. If through I, the middle point of BC, IH be drawn, parallel to the base AB, the point H will also be the middle of AD; for, since the figure AHIL is a parallelogram, as also DHIK, their opposite sides being parallel, we have then AH = IL and DH = IK; but, since the triangles BIL, CIK are equal, we already have IL = IK; therefore, AH = DH.

It may be observed that the line $\text{HI} = \text{AL} = \dfrac{\text{AB} + \text{CD}}{2}$; therefore the area of the trapezoid may also be expressed by EF × HI; it is therefore equal to the altitude of the trapezoid multiplied by the line which joins the middle points of its inclined sides.

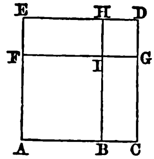
PROPOSITION VIII.

Theorem.

If a line, AC, *is divided into two parts,* AB, BC, *the square described on the whole line* AC *is equivalent to the square described on one part,* AB, *plus the square described on the other part,* BC, *plus twice the rectangle contained by the two parts* AB, BC; *which is expressed thus:*

$$\overline{\text{AC}}^2, \textit{ or } (\text{AB} + \text{BC})^2 = \overline{\text{AB}}^2 + \overline{\text{BC}}^2 + 2\text{AB} \times \text{BC}.$$

Construct the square ACDE; take AF = AB; draw FG parallel to AC, and BH parallel to AE. The square ACDE is divided into four parts: the first, ABIF, is the square described on AB, since we made AF=AB; the second, IGDH, is the square described on BC; for, since we have AC = AE, and AB = AF, the difference AC — AB is equal to the difference AE — AF, which gives BC = EF; but IG = BC, and DG = EF, since the lines are parallel; therefore, HIGD is equal to the square described on BC, and these two squares being taken away from the whole square, there remain the two rectangles BCGI, EFIH, each of which is measured by AB × BC, hence, the proposition is true.



SCHOLIUM. This property is equivalent to that which is demonstrated in algebra in obtaining the square of a binomial, which is expressed thus,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

PROPOSITION IX.

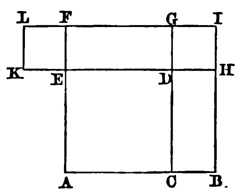
THEOREM.

If the line AC *is the difference of the two lines* AB, BC, *the square described on* AC *is equivalent to the square on* AB, *plus the square on* BC, *minus twice the rectangle contained by* AB *and* BC; *that is to say, we will have*

$$\overline{AC}^2, \text{ or } (AB - BC)^2 = \overline{AB}^2 + \overline{BC}^2 - 2AB \times BC.$$

Construct the square ABIF; take AE = AC; draw CG parallel to BI, HK parallel to AB, and complete the square EFLK.

The two rectangles CBIG, GLKD are each measured by AB × BC: if we take these rectangles from the whole figure, ABILKEA, which is equivalent to $\overline{AB}^2 + \overline{BC}^2$, there will evidently remain the square ACDE; hence, our theorem is true.



SCHOLIUM. This proposition is equivalent to the algebraic formula,

$$(a - b)^2 = a^2 + b^2 - 2ab.$$

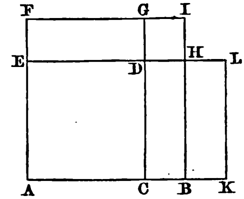
PROPOSITION X.

THEOREM.

The rectangle contained by the sum and the difference of two lines, is equal to the difference of the squares of these lines; and thus we have

$$(AB + BC) \times (AB - BC) = \overline{AB}^2 - \overline{BC}^2.$$

On AB and AC construct the squares ABIF, ACDE; produce AB till BK = BC, and complete the rectangle AKLE. The base, AK, of the rectangle is the sum of the two lines AB, BC; its altitude, AE, is the difference of the same lines; therefore, the rectangle



$$AKLE = (AB + BC) \times (AB - BC).$$

But this same rectangle is composed of the two parts ABHE + BHLK; and the part BHLK is equal to the rectangle EDGF, for BH = DE, and BK = EF; therefore, AKLE = ABHE + EDGF. These two parts make up the square ABIF minus the square DHIG, which latter is the square described on BC; therefore, finally,

$$(AB + BC) \times (AB - BC) = \overline{AB}^2 - \overline{BC}^2.$$

SCHOLIUM. This proposition is equivalent to the algebraic formula

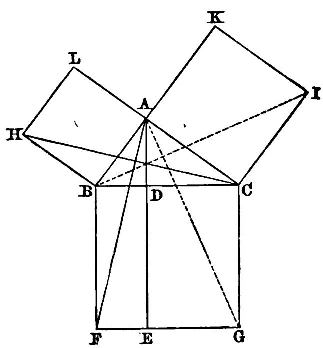
$$(a + b)(a - b) = a^2 - b^2.$$

PROPOSITION XI.

THEOREM.

The square described on the hypothenuse of a right angled triangle is equivalent to the sum of the squares described on the other two sides.

Let the triangle ABC be right angled at A. Having formed squares on the three sides, let fall, from A, on the hypothenuse, the perpendicular AD, which prolong to E; and draw the diagonals AF, CH. The angle ABF is made up of the angle ABC, together with the right angle CBF; the angle CBH is made up of the same angle, ABC, together with the right angle ABH; hence, the angle ABF = HBC. But AB=BH, being sides of the same square, and BF = BC, for the same reason; therefore, the triangles ABF, HBC have two sides and the included angle in each equal; therefore, they are themselves equal (Book I., Prop. VII.)



The triangle ABF is the half of the rectangle BDEF (or, for the sake of shortness, BE), which has the same base, BF, and the same altitude, BD (Prop. II.). The triangle HBC is likewise the half of the square AH; for, the angles BAC and BAL being both right, AC and AL form one and the same straight line, parallel to HB; therefore, the triangle HBC and the square AH, which have the common base BH, have also the common altitude AB; therefore, the triangle is the half of the square.

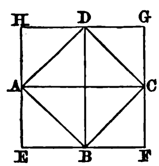
It has already been proven that the triangle ABF is equal to the triangle HBC; hence, the rectangle BDEF, which is double of the triangle ABF, is equivalent to the square AH, which is double of the triangle HBC. In the same manner it may be proved that the rectangle CDEG is equivalent to the square AI; but the two rectangles BDEF, CDEG, taken together, make up the square BCGF; therefore the square BCGF, described on the hypothenuse, is equal to the sum of the squares ABHL, ACIK, described on the other two sides; in other words,

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2.$$

COR. 1. Hence, the square of one of the sides of the right angle is equivalent to the square of the hypothenuse diminished the square of the other side, which is expressed thus:

$$\overline{AB}^2 = \overline{BC}^2 - \overline{AC}^2.$$

COR. 2. Let ABCD be a square, and AC its diagonal; the triangle ABC being right angled and isosceles, we will have



$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = 2\overline{AB}^2;$$

therefore, *the square described on the diagonal* AC *is double the square described on the side* AB.

This property may be made evident by drawing parallels to BD through the points A and C, and parallels to AC through the points B and D; we will thus form a new square, EFGH, which will be the square of AC. Now, EFGH evidently contains eight triangles, each equal to ABE; and ABCD contains four of these; hence, the square EFGH is double ABCD.

Since $$\overline{AC}^2 : \overline{AB}^2 :: 2 : 1,$$

by extracting the square root, we shall have

$$AC : AB :: \sqrt{2} : 1;$$

therefore, *the diagonal of a square is incommensurable with its side.* A property which will be explained more fully in another place.

Cor. 3. It has been shown in the proposition that the square AH is equivalent to the rectangle BDEF (*See* the figure for Prop. XI.); but, by reason of the common altitude BF, the square BCGF is to the rectangle BDEF as the base BC is to the base BD; therefore,

$$\overline{BC}^2 : \overline{AB}^2 :: BC : BD.$$

Hence, *the square of the hypothenuse is to the square of one of the sides of the right angle as the hypothenuse is to the segment adjacent to that side.* We call *segment* here the part of the hypothenuse cut off by the perpendicular let fall from the right angle; thus, BD is the segment adjacent to the side AB, and DC is the segment adjacent to the side AC.

We would have also,

$$\overline{BC}^2 : \overline{AC}^2 :: BC : CD.$$

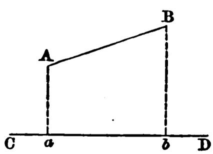
Cor. 4. The rectangles BDEF, DCGE, having also the same altitude, are to each other as their bases, BD, CD. But these rectangles are equivalent to the squares $\overline{AB}^2$, $\overline{AC}^2$; hence,

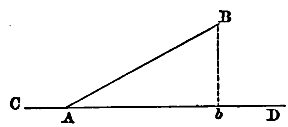
$$\overline{AB}^2 : \overline{AC}^2 :: BD : DC.$$

Therefore, *the squares of the two sides of the right angle are to each other as the segments of the hypothenuse adjacent to these sides.*

DEFINITION.

The projection of a straight line, AB, on another, CD, is the distance, *ab*, between the feet of the perpendiculars let fall from the





points A and B on CD. If one of the points of AB, as A, is on CD, then A*b* is the projection.

PROPOSITION XII.

Theorem.

In any triangle, the square of the side opposite to an acute angle is less than the sum of the squares of the other two sides, by twice the rectangle contained by one of these sides, and the projection of the other on it.

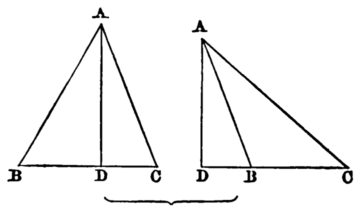
Let C be an acute angle of the triangle ABC; draw AD perpendicular to BC. Then shall

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 - 2BC \times CD.$$

There are two cases.

First.—When the perpendicular falls within the triangle ABC, we have BD = BC − CD, and consequently (Prop. IX.),

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2BC \times CD.$$



Adding $\overline{AD}^2$ to each, and observing that the right angled triangles ABD, ADC give

$$\overline{AD}^2 + \overline{BD}^2 = \overline{AB}^2,$$

and $$\overline{AD}^2 + \overline{DC}^2 = \overline{AC}^2,$$

we have $$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2BC \times CD.$$

Second.—When the perpendicular AD falls without the triangle ABC we have BD = CD − BC, and, consequently (Prop. IX.),

$$\overline{BD}^2 = \overline{CD}^2 + \overline{BC}^2 - 2CD \times BC.$$

Adding $\overline{AD}^2$ to both we find, as before,

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2BC \times CD$$

PROPOSITION XIII.

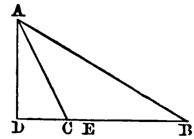
Theorem.

In any triangle having an obtuse angle, the square of the side opposite the obtuse angle is greater than the sum of the squares of the other two sides, by twice the rectangle contained by either of these sides and the projection of the other on it.

Let AB be the side opposite the obtuse angle C, of the triangle ABC; draw the perpendicular AD from A to BC. We shall have

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 + 2BC \times CD.$$

The perpendicular cannot fall within the triangle; for if it fell at any point, such as E, the triangle ACE would have both the right angle E and the obtuse angle C, which is impossible (Book I., Prop.



XXIX.); hence, the perpendicular falls without, and we have BD = BC + CD. From this we have (Prop. VIII.),

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 + 2BC \times CD.$$

Adding $\overline{AD}^2$ to each member of this equation, and reducing, as in the preceding theorem, we shall get

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2BC \times CD.$$

SCHOLIUM. The right angled triangle is the only triangle in which the sum of the squares of the two sides is equal to the square of the third side; for, if the angle included by these sides is acute, the sum of their squares will be greater than the square of the opposite side; and if it is obtuse, the sum will be less.

PROPOSITION XIV.

THEOREM.

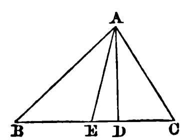
The sum of the squares of two sides of a triangle is equivalent to twice the square of the median to the third side together with twice the square of half the third side.

Let ABC be the triangle, AE the median to the side BC (that is, the line which joins the vertex A to the middle point of BC). Then we shall have

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{AE}^2 + 2\overline{BE}^2.$$

Let fall the perpendicular AD on the base BC. The triangle AEC gives (by Theorem XII.)

$$\overline{AC}^2 = \overline{AE}^2 + \overline{EC}^2 - 2EC \times ED.$$



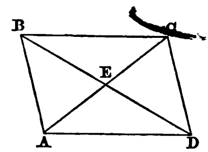
The triangle ABE gives (by Theorem XIII.)

$$\overline{AB}^2 = \overline{AE}^2 + \overline{EB}^2 + 2EB \times ED.$$

Hence, by adding, and observing that EB = EC, we shall have

$$\overline{AB}^2 + \overline{AC}^2 = 2\overline{AE}^2 + 2\overline{EB}^2.$$

COR. 1. Hence, *in every parallelogram, the sum of the squares of the sides is equal to the sum of the squares of the diagonals.*



For the diagonals, AC, BD, bisect each other (Book I., Prop. XXXIV.); consequently, the triangle ABC gives

$$\overline{AB}^2 + \overline{BC}^2 = 2\overline{AE}^2 + 2\overline{BE}^2.$$

The triangle ADC gives likewise,

$$\overline{AD}^2 + \overline{DC}^2 = 2\overline{AE}^2 + 2\overline{DE}^2.$$

Adding these two equations, member to member, and observing that BE = DE, we shall have

$$\overline{AB}^2 + \overline{AD}^2 + \overline{DC}^2 + \overline{BC}^2 = 4\overline{AE}^2 + 4\overline{DE}^2.$$

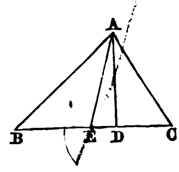
But $4\overline{AE}^2$ is the square of 2AE, or of AC; $4\overline{DE}^2$ is the square of BD; therefore, the sum of the squares of the sides is equal to the sum of the squares of the diagonals.

COR. 2. If, in the figure of the proposition, we suppose the points A and C to remain fixed, and the point B to change its position so that $AB^2 + BC^2$ remains constant, then, since $\overline{AB}^2 + \overline{BC}^2 = 2\overline{AE}^2 + 2\overline{BE}^2$, it follows that BE remains constant. Hence, the point B must lie on a circle of which E is the centre, and BE the radius.

PROPOSITION XV.

THEOREM.

The difference of the squares of two sides of a triangle is equal to twice the rectangle contained by the third side and the projection of its median on the third side.



Let ABC be the given triangle.

We have, as in the last proposition,

$$\overline{AB}^2 = \overline{AE}^2 + \overline{EB}^2 + 2EB \times ED,$$

$$\overline{AC}^2 = \overline{AE}^2 + \overline{EC}^2 - 2EC \times ED.$$

Subtract the second from the first, observing that EB = EC, we have

$$\overline{AB}^2 - \overline{AC}^2 = 4EB \times ED; \text{ or,}$$

$$\overline{AB}^2 - \overline{AC}^2 = 2BC \times DE.$$

COR. If the points B and C remain fixed, while the vertex A moves so that the difference, $AB^2 - AC^2$, remains constant, the equation above shows that the projection DE, of the median from A, remains constant. Hence, A must always be in a straight line perpendicular to BC.

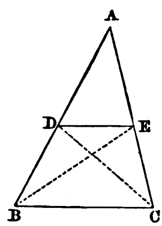
PROPOSITION XVI.

THEOREM.

The line DE, *drawn parallel to the base of a triangle,* ABC, *divides the sides* AB, AC, *proportionally; so that we have*

AD : DB : : AE : EC.

Join BE and DC. The two triangles BDE, DEC, have the same base, DE; they have also the same altitude, since the vertices, B and C, are situated in a line parallel to the base; hence, these triangles are equivalent (Prop. II.). The triangles ADE, CDE, whose common vertex is D, have also the same altitude, and are to each other as their bases, AE, EC; therefore, ADE : DEC : : AE : EC.



In the same way, ADE : BDE : : AD : DB.

But the triangle BDE = DEC; therefore, because of the common ratio in these two proportions, we shall have

AB : DB : : AE : EC.

COR. 1. Hence, by composition, we have

AD + DB : AD : : AE + EC : AE; or, AB : AD : : AC : AE,

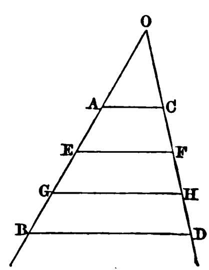
and also AB : BD : : AC : CE.

COR. 2. *If between two straight lines,* AB, CD, *any number of parallels be drawn, as* AC, EF, GH, BD, *etc., those straight lines will be cut proportionally, and we shall have*

AE : CF : : EG : FH : : GB : HD.

For let O be the point where the straight lines AB, CD, meet. In the triangle OEF, the line AC being drawn parallel to the base EF, we shall have

OE : OF : : AE : CF.



In the triangle OGH, we shall likewise have

OE : OF : : EG : FH.

And by reason of the common ratio, OE : OF, those two proportions give

AE : CF : : EG : FH.

It may be proved, in the same manner, that

EG : FH : : GB : HD,

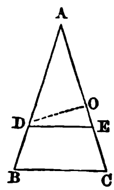
and so on; hence, the lines AB, CD, are cut proportionally by the parallels EF, GH, etc.

PROPOSITION XVII.

THEOREM.

Conversely, if the sides AB, AC, *are cut proportionally by the line* DE, *this line will be parallel to the third side,* BC.

For, if DE be not parallel to BC, suppose that DO is; then, by the corollary of the preceding theorem, we shall have



AB : AD : : AC : AO (1).

But by hypothesis,

AD : BD : : AE : EC, and also
AB : AD : : AC : AE (3).

Comparing (1) and (3), we see that AO must be equal to AE, which is impossible; hence, the parallel to BC, drawn through the point D, cannot differ from DE; hence, DE is that parallel.

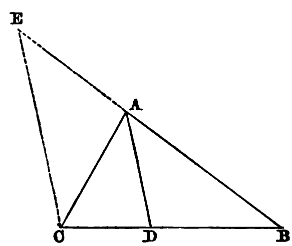
PROPOSITION XVIII.

THEOREM.

The bisectrix of the angle BAC, *of a triangle, divides the opposite side* BC *into two segments,* BD, DC, *proportional to the adjacent sides* AB, AC; *so that we shall have*

BD : DC : : AB : AC.

Through the point C draw CE parallel to AD, till it meets BA produced. In the triangle BCE, the line AD is parallel to the base CE: hence, we have the proportion (Prop. XVI.),

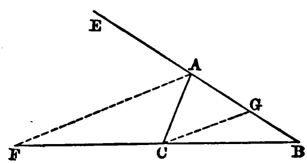


BD : DC : : AB : AE.

But the triangle ACE is isosceles; for, since AD and CE are parallel, we have the angle ACE = DAC, and the angle AEC = BAD (Book I., Prop. XXIV.): but, by hypothesis, DAC = BAD; hence, the angle ACE = AEC, and consequently AE=AC (Book I., Prop. XIV.); substituting then AC in the place of AE in the above proportion, we shall have

BD : DC : : AB : AC.

SCHOLIUM. *The bisectrix*, AF, *of the exterior angle* CAE *intercepts on the base produced two segments*, BF, CF, *also proportional to the sides* AB, AC.



Draw CG parallel to AF; in the triangle BAF we have

BF : FC : : BA : AG.

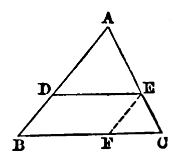
We can, then, in the same manner as above, show that the triangle AGC is isosceles, and that AG = AC.

Therefore, BF : FC : : AB : AC.

PROPOSITION XIX.

THEOREM.

A straight line, DE, *drawn parallel to one of the sides*, BC, *of the triangle* ABC, *cuts off a triangle*, ADE, *similar to* ABC.



For, first, the two triangles ADE, ABC, have their angles respectively equal. For the angle A is common, and the angles ADE=ABC, by reason of the parallels DE, BC, as also AED = ACB.

Secondly, the homologous sides are proportional; for DE being parallel to BC, we have (Prop. XVI.)

AD : AB : : AE : AC,

and, drawing EF parallel to AB, we have

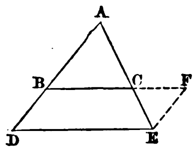
AE : AC : : BF : BC.

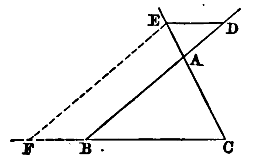
And since the parallels DE, BF, intercepted between parallels, are equal.

AE : AC : : DE : BC.

Therefore, the triangles ADE and ABC, having their angles equal, and their homologous sides proportional, are similar (Def. 2).

SCHOLIUM. The proposition holds true when the parallel DE is below BC, or above A, the demonstration being the same in these cases.



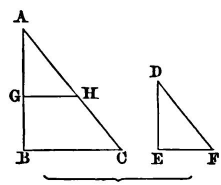


PROPOSITION XX.

THEOREM.

Two equiangular triangles are similar.

Let ABC, DEF, be two triangles which have their angles equal, each to each, namely, A = D, B = E, C = F. Then the triangles will be similar. For, on AB, homologous to DE, take AG = DE, and draw GH parallel to BC. The triangle AGH thus formed is similar to ABC (Prop. XIX.). But the triangles DEF and AGH have the angle A = D by hypothesis; the side AG = DE by construction, and the angle AGH = E, since each is equal to the angle B. Hence, these triangles are equal, and therefore DEF is also similar to ABC.



COR. For the similarity of two triangles it is enough that they have two angles equal each to each; since the third angle will then be equal in both, and the two triangles will be equiangular. Two right angled triangles are similar when they have one acute angle equal.

SCHOLIUM. Observe that in similar triangles the homologous sides are opposite to the equal angles.

PROPOSITION XXI.

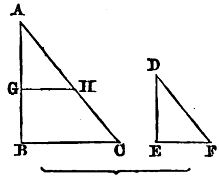
THEOREM.

Two triangles which have their homologous sides proportional, are similar.

Let ABC, DEF, be the two triangles, and suppose,

$$AB : DE :: AC : DF, \text{ and}$$
$$AB : DE :: BC : EF.$$

Then will the two triangles be similar. For on AB take AG=DE, and draw GH parallel to BC. The triangle AGH is similar to ABC (Prop. XIX.), and their homologous sides give the proportion,



AB : AG : : AC : AH ; or, since AG=DE,
AB : DE : : AC : AH.

But we have by hypothesis,

AB : DE : : AC : DF.

Hence, AH = DF. In like manner we can prove GH = EF. Hence, the triangle AGH is equal to the triangle DEF, and therefore DEF is similar to ABC.

SCHOLIUM 1. It results from the last two propositions, that *in triangles equality among the angles is a consequence of proportionality of the sides, and conversely.* This fundamental property is not true of other polygons. For example, a square and a rectangle have their angles equal, but their homologous sides are not proportional. And a square and rhombus have their sides proportional, but their angles are not equal.

SCHOLIUM 2. The two preceding propositions, which are in reality only one, together with that concerning the square of the hypothenuse, are the most important and fruitful propositions of geometry; they are sufficient, almost, of themselves for application to all cases, and for the solution of every problem. The reason is, that all figures may be divided into triangles, and any triangle into two right angled triangles. Thus, the general properties of triangles involve, by implication, those of all figures.

PROPOSITION XXII.

THEOREM.

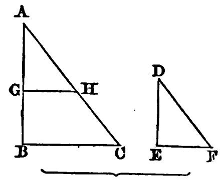
Two triangles which have an equal angle included between proportional sides, are similar.

Let the angle A = D, and suppose we have

AB : DE : : AC : DF ;

then shall the triangle ABC be similar to DEF.

Take AG = DE, and draw GH parallel to BC; the triangle AGH will be similar to the triangle ABC (Prop. XIX.); we shall have, therefore,



AB : AG : : AC : AH;

but, by hypothesis,

AB : DE : : AC : DF,

and, by construction, AG = DE; hence, AH = DF. The two triangles AGH, DEF, have, therefore, an equal angle included between equal sides; hence, they are equal; therefore, DEF is also similar to ABC.

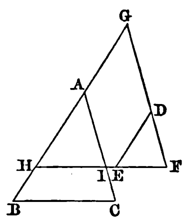
Cor. Two right angled triangles are similar, when they have any two sides of the one proportional to the homologous sides of the other.

PROPOSITION XXIII.

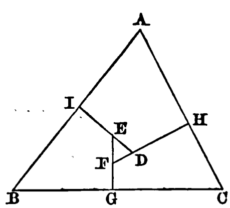
Theorem.

Two triangles which have their homologous sides parallel, or perpendicular to each other, are similar.

For, *First*, if the side AB is parallel to DE, and BC to EF, the angle ABC will be equal to DEF (Book I., Prop. XXVII.); if, further, AC is parallel to DF, the angle ACB will be equal to DFE, and, also, BAC to EDF; hence, the triangles ABC, DEF, are equiangular; consequently they are similar.



Second. If the side DE is perpendicular to AB, and the side DF to AC, two angles, I and H, of the quadrilateral AIDH will be right; and, since the four angles together are equal to four right angles (Book I., Prop. XXX.), the remaining two, IAH, IDH, are equal to two right angles. But the two angles EDF, IDH, are also equal to two right angles; therefore, the angle EDF is equal to IAH or BAC. In like manner, if the third side EF is perpendicular to the third side BC, it may be shown that the angle DFE = C, and DEF = B; hence, the two triangles ABC, DEF, which have the sides of the one perpendicular to the corresponding sides of the other, are similar.



SCHOLIUM 1. In the case of the sides being parallel, the homologous sides are the parallel sides, and in the case of the sides being perpendicular, the homologous sides are the perpendicular sides. Thus, in the latter case, DE is homologous to AB, DF to AC, and EF to BC.

SCHOLIUM 2. The case of the sides being perpendicular might furish a relative situation of the two triangles different from that which is supposed in the last figure, but we can always suppose within the triangle ABC, a triangle DEF to be constructed, whose sides should be parallel to those of the triangle compared with ABC, and then the demonstration would be the same with that given in the case of the last figure.

Both cases of the proposition are sometimes proved thus: By (Book I., Props. XXVII. and XXVIII.) two angles which have their sides parallel or perpendicular are either equal or supplementary. Now, no two pair of these corresponding angles of the triangles can be supplementary, for then the sum of the angles in the two triangles would be greater than four right angles. Hence, at least two of the angles in one triangle must be equal to two of the angles in the other, and, therefore, the third angle must be equal; therefore, the triangles, being equiangular, are similar.

PROPOSITION XXIV.

THEOREM.

The lines AF, AG, *etc., drawn at pleasure from the vertex of a triangle, divide proportionally the base* BC *and its parallel* DE, *so that we have*

DI : BF : : IK : FG : : KL : GH, etc.

For, since DI is parallel to BF the triangles ADI and ABF are equiangular, and we have the proportion

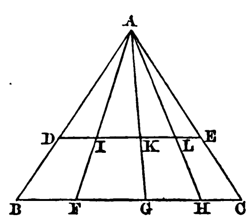
DI : BF : : AI : AF;

and, since IK is parallel to FG, we have

AI : AF : : IK : FG;

hence, the ratio AI : AF being common, we will have

DI : BF : : IK : FG.



In the same manner we shall find

IK : FG : : KL : GH, etc.;

therefore, the line DE is divided at the points I, K, L, in the same proportion as the base BC is at the points F, G, H.

Cor. Therefore, if BC were divided into equal parts at the points F, G, H, the parallel DE would also be divided into equal parts at the points I, K, L.

PROPOSITION XXV.

Theorem.

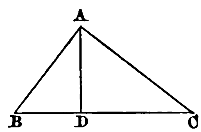
If from the right angle, A, *of a right angled triangle, the perpendicular* AD *be let fall on the hypothenuse,*

First.—The two partial triangles, ABD, ADC, *will be similar to each other, and to the whole triangle,* ABC ;

Second.—Either side, AB *or* AC, *will be a mean proportional between the hypothenuse* BC *and the adjacent segment* BD *or* DC ;

Third.—The perpendicular AD *will be a mean proportional between the two segments* BD, DC.

First.—The triangles BAD and BAC have the common angle B; also the right angle BDA is equal to the right angle BAC ; therefore, the third angle, BAD, of the one is equal to the third angle, C, of the other : hence, those two triangles are equiangular and similar. In the same manner it may be shown that the triangle DAC is similar to the triangle BAC ; hence, all the triangles are similar.



Second.—Since the triangles BAD and BAC are similar, their homologous sides are proportional. Now, BD in the small triangle, and BA in the large one, are homologous, because they lie opposite to equal angles, BAD, BCA ; the hypothenuse BA, of the small triangle, is homologous with the hypothenuse BC of the large triangle ; hence, the proportion

$$BD : BA :: BA : BC.$$

By the same reasoning we should find

$$DC : AC :: AC : BC.$$

Hence, each of the sides AB, AC, is a mean proportional between the hypothenuse and the segment adjacent to this side.

Third.—Since the triangles ABD, ADC, are similar, by comparing their homologous sides, we have

$$BD : AD :: AD : DC ;$$

hence, the perpendicular AD is a mean proportional between the segments BD, DC, of the hypothenuse.

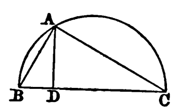
Scholium. Since BD : AB : : AB : BC, the product of the extremes will be equal to that of the means, or $\overline{AB}^2 = BD \times BC$. In the same way we have $\overline{AC}^2 = DC \times BC$: therefore,

$$\overline{AB}^2 + \overline{AC}^2 = BD \times BC + DC \times BC;$$

the second member is the same thing as $(BD + DC) \times BC$, and it reduces to $BC \times BC$, or $\overline{BC}^2$; therefore we have $\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$; therefore, the square described on the hypothenuse BC is equal to the sum of the squares formed on the other two sides, AB, AC. Thus we again arrive at the property of the square of the hypothenuse by a path very different from that which was previously pursued; from which it is seen that, strictly speaking, the proposition of the square of the hypothenuse is a consequence of the proportionality of the sides in equiangular triangles.

Thus, the most important propositions of geometry are reduced, so to speak, to this single one, that equiangular triangles have their homologous sides proportional.

Cor. If, from a point, A, of the circumference, we draw the two chords, AB, AC, to the extremities of the diameter BC, the triangle BAC will be right angled at A (Book II., Prop. XX., Cor. 2); hence, first, *the perpendicular* AD *is a mean proportional between the two segments*, BD, DC, *of the diameter*, or, what amounts to the same, the square $\overline{AD}^2$ is equal to the rectangle $BD \times DC$.



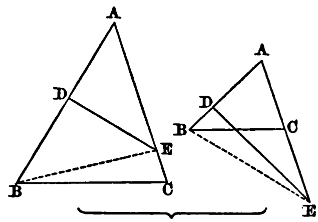
Hence, also, in the second place, *the chord* AB *is a mean proportional between the diameter* BC *and the adjacent segment* BD, or, what is the same thing, $\overline{AB}^2 = BD \times BC$. In like manner, we have $\overline{AC}^2 = CD \times BC$; therefore, $\overline{AB}^2 : \overline{AC}^2 : : BD : DC$. Also, comparing $\overline{AB}^2$ and $\overline{BC}^2$, we have $\overline{AB}^2 : \overline{BC}^2 : : BD : BC$; we would likewise have $\overline{AC}^2 : \overline{BC}^2 : : DC : BC$. These proportions between the squares of the sides, compared with each other or with the square of the hypothenuse, have been already given in Corollaries 3 and 4 of Proposition XI.

PROPOSITION XXVI.

THEOREM.

Two triangles, which have one angle in each equal, are to each other as the rectangles of the sides which contain the equal angle. Thus, the triangle ABC *is to the triangle* ADE *as the rectangle* AB × AC *is to the rectangle* AD × AE.

Draw BE. The two triangles ABE, ADE, whose common vertex is E, have the same altitude, and are to each other as their bases, AB, AD (Prop. VI., Cor.) ; therefore,



ABE : ADE : : AB : AD.

In like manner we have

ABC : ABE : : AC : AE.

Multiplying these two proportions together, term for term, and omitting the common term ABE, we shall have

ABC : ADE : : AB × AC : AD × AE.

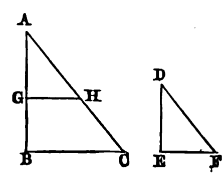
COR. Hence, the two triangles would be equivalent, if the rectangle AB × AC were equal to the rectangle AD × AE, or if we had AB : AD : : AE : AC, which would be the case if a line drawn from D to C were parallel to BE.

PROPOSITION XXVII.

THEOREM.

Two similar triangles are to each other as the squares of their homologous sides.

Let the angle A = D and the angle B = E. Then, first, by reason of the equal angles A and D, we shall have, according to the last proposition,



ABC : DEF : : AB × AC : DE × DF.

Moreover, because of the similarity of the triangles, we have

AB : DE : : AC : DF.

If we multiply this proportion, term by term, by the identical proportion

$$AC : DF :: AC : DF,$$

we shall have

$$AB \times AC : DE \times DF :: \overline{AC}^2 : \overline{DF}^2.$$

Therefore,

$$ABC : DEF :: \overline{AC}^2 : \overline{DF}^2.$$

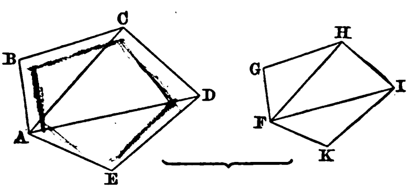
Therefore, two similar triangles, ABC, DEF, are to each other as the squares of the homologous sides AC, DF, or as the squares of any other two homologous sides.

PROPOSITION XXVIII.

THEOREM.

Two similar polygons are composed of the same number of triangles, similar, each to each, and similarly situated.

From any angle, A, in the polygon ABCDE, draw the diagonals AC, AD, to the other angles. From the corresponding angle, F, in



the other polygon, FGHIK, draw diagonals FH, FI to the other angles.

Since the polygons are similar, the angle ABC is equal to its homologous angle FGH (Def. 2); and the sides AB, BC must also be proportional to the sides FG, GH; that is,

$$AB : FG :: BC : GH.$$

Wherefore, the triangles ABC, FGH have each an equal angle contained between proportional sides; hence, they are similar (Prop. XXII.); therefore, the angle BCA is equal to GHF. Take away these equal angles from the equal angles BCD, GHI, there remains ACD = FHI; but since the triangles ABC, FGH are similar, we have

$$AC : FH :: BC : GH;$$

and since the polygons are similar (Def. 2),

BC : GH : : CD : HI;

therefore,

AC : FH : : CD : HI.

But the angle ACD we already know is equal to FHI; hence, the triangles ACD, FHI have an equal angle contained by proportional sides, and are consequently similar.

In the same manner might all the remaining triangles be shown to be similar, whatever were the number of sides of the proposed polygons; therefore, two similar polygons are composed of the same number of triangles, similar and similarly situated.

SCHOLIUM. The converse proposition is equally true:

If two polygons are composed of the same number of triangles, similar and similarly situated, those two polygons will be similar.

For the similarity of the respective triangles will give the angles ABC = FGH, BCA = GHF, ACD = FHI; therefore BCD = GHI; likewise, CDE = HIK, etc. Further, we shall have

AB : FG : : BC : GH : : AC : FH : : CD : HI, etc.;

hence, the two polygons have their angles equal and their sides proportional; therefore, they are similar.

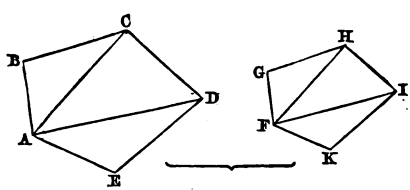
PROPOSITION XXIX.

THEOREM.

The contours or perimeters of similar polygons are to each other as their homologous sides, and the surfaces are to each other as the squares of these sides.

First.—Since, by the nature of similar figures we have

AB : FG : : BC : GH : : CD : HI, etc.,



we conclude from this series of equal ratios that the sum of the antecedents, AB + BC + CD, etc. (the perimeter of the first polygon), is to the sum of the consequents, FG + GH + HI, etc. (the perimeter of the second polygon), as one of the antecedents is to its consequent, or as the side AB is to its homologous side FG.

Second.—Since the triangles ABC, FGH are similar, we shall have (Prop. XXVII.)

$$ABC : FGH :: \overline{AC}^2 : \overline{FH}^2;$$

in the same manner, the similar triangles ACD, FHI give

$$ACD : FHI :: \overline{AC}^2 : \overline{FH}^2;$$

therefore, by reason of the common ratio $\overline{AC}^2 : \overline{FH}^2$ we have

$$ABC : FGH :: ACD : FHI.$$

By the same mode of reasoning we should find

$$ACD : FHI :: ADE : FIK;$$

and so on, if there were more triangles.

From this series of equal ratios we conclude: That the sum of the antecedents, ABC + ACD + ADE, or the polygon ABCDE, is to the sum of the consequents, FGH + FHI + FIK, or the polygon FGHIK, as one antecedent, ABC, is to its consequent, FGH, or as $\overline{AB}^2$ is to $\overline{FG}^2$; therefore, the surfaces of similar polygons are to each other as the squares of their homologous sides.

COR. If three similar figures be constructed on the three sides of a right angled triangle, the figure constructed on the hypothenuse will be equal to the sum of the other two: for the three figures are proportional to the squares of their homologous sides; but the square of the hypothenuse is equal to the sum of the squares of the other two sides; therefore, etc.

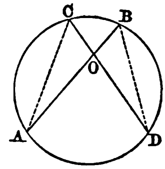
PROPOSITION XXX.

THEOREM.

The segments of two chords, which intersect each other in a circle, are reciprocally proportional; that is to say, we will have

$$AO : DO :: CO : OB.$$

Join AC and BD. In the triangles ACO, BOD, the angles at O are equal, as being opposite angles at the vertex; the angle A is equal to the angle D, because both are inscribed in the same segment (Book II., Prop. XX., Cor. 1); for the same reason, the angle C=B: the triangles are therefore similar, and their homologous sides give the proportion



AO : DO : : CO : OB.

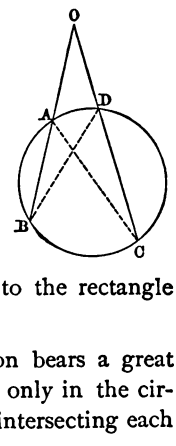
Cor. From this follows AO × OB = DO × CO; therefore, the rectangle of the two segments of one of the chords is equal to the rectangle of the two segments of the other.

PROPOSITION XXXI.

Theorem.

If, from the same point, O, *without a circle, the secants* OB, OC, *be drawn, terminating in the concave arc, the whole secants will be reciprocally proportional to their external segments; that is to say, we shall have*

OB : OC : : OD : OA.



For, joining AC, BD, the triangles OAC, OBD, have the angle O common; and the angle B = C (Book II. Prop. XX., Cor. 1); these triangles are, therefore, similar, and their homologous sides give the proportion

OB : OC : : OD : OA.

Cor. Therefore, the rectangle OA × OB is equal to the rectangle OC × OD.

Scholium. It may be observed that this proposition bears a great analogy to the preceding, and that it differs from it only in the circumstance that the two chords, AB, CD, instead of intersecting each other within the circle, intersect outside of it.

The following proposition may also be regarded as a particular case of this one.

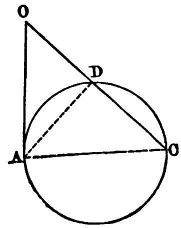
PROPOSITION XXXII.

Theorem.

If, from the same point, O, *without a circle, a tangent,* OA, *and a secant,* OC, *be drawn, the tangent will be a mean proportional between the secant and its external segment; so that we shall have*

OC : OA : : OA : OD ; *or, what is the same,* $\overline{OA}^2 = OC \times OD$.

For, joining AD and AC, the triangles OAD, OAC, have the angle O common ; further, the angle OAD, formed by a tangent and a chord (Book II., Prop. XXI.), has for its measure the half of the arc AD, and the angle C has the same measure ; hence, the angle OAD = C ; therefore the two triangles are similar, and we have the proportion,



$$OC : OA :: OA : OD,$$

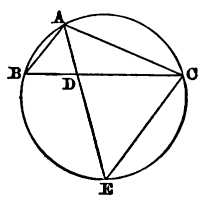
which gives $\overline{OA}^2 = OC \times OD$.

PROPOSITION XXXIII.

THEOREM.

In a triangle, ABC, *if the angle* A *be bisected by the line* AD, *terminating in the opposite side, the rectangle of the sides* AB, AC, *will be equal to the rectangle of the segments* BD, DC, *together with the square of the bisecting line* AD.

Describe a circumference to pass through the three points A, B, C ; prolong AD till it meets the circumference, and join CE.



The triangle BAD is similar to the triangle EAC ; for, by hypothesis, the angle BAD = EAC ; also, the angle B = E, since they are both measured by half of the arc AC ; hence, these triangles are similar, and the homologous sides give the proportion BA : AE : : AD : AC ; hence, BA × AC = AE × AD ; but AE = AD + DE, and, multiplying both members by AD, we have $AE \times AD = \overline{AD}^2 + AD \times DE$; but AD × DE = BD × DC (Prop. XXX.) ; therefore, finally,

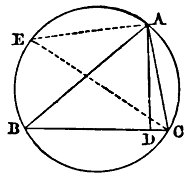
$$BA \times AC = \overline{AD}^2 + BD \times DC.$$

PROPOSITION XXXIV.

THEOREM.

In every triangle, ABC, *the rectangle of the two sides,* AB, AC, *is equal to the rectangle contained by the diameter,* CE, *of the circumscribed circle and the perpendicular,* AD, *let fall on the third side,* BC.

For, joining AE, the triangles ABD, AEC, are right angled, the one at D, the other at A; also, the angle B=E; these triangles are, therefore, similar, and they give the proportion



$$AB : CE :: AD : AC;$$

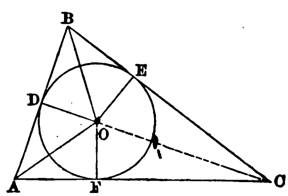
and hence, $AB \times AC = CE \times AD.$

COR. If these equal quantities be multiplied by the same quantity, BC, we shall have $AB \times AC \times BC = CE \times AD \times BC$. Now, $AD \times BC$ is double of the surface of the triangle (Prop. VI.); therefore, *the product of the three sides of a triangle is equal to its surface multiplied by twice the diameter of the circumscribed circle.*

The product of three lines is sometimes called a solid, for a reason which will be seen hereafter. Its value is easily conceived by imagining that the lines are reduced to numbers, and multiplying together the numbers in question.

SCHOLIUM. It may also be demonstrated that *the area of a triangle is equal to its perimeter multiplied by half the radius of the inscribed circle.*

For, the triangles AOB, BOC, AOC, which have a common vertex at O, have for a common altitude the radius of the inscribed circle; hence, the sum of these triangles will be equal to the sum of the bases AB, BC, AC, multiplied by the half of the radius OD; hence, the surface of the triangle ABC is equal to its perimeter multiplied by the half of the radius of the inscribed circle.



PROPOSITION XXXV.

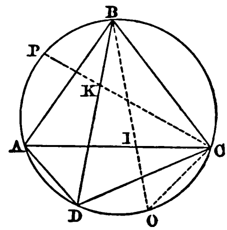
THEOREM.

In every quadrilateral, ABCD, *inscribed in a circle, the rectangle of the two diagonals,* AC, BD, *is equal to the sum of the rectangles of the opposite sides, so that we have*

$$AC \times BD = AB \times CD + AD \times BC.$$

Take the arc CO = AD; and draw BO, meeting the diagonal AC in I. The angle ABD = CBI, since the one is measured by the half of AD, and the other by the half of CO, equal to AD; the angle

ADB = BCI, because they are both inscribed in the same segment AOB; hence, the triangle ABD is similar to the triangle IBC, and we have the proportion



$$AD : CI :: BD : BC;$$

from which results

$$AD \times BC = CI \times BD.$$

Again, the triangle ABI is similar to the triangle BDC; for the arc AD being equal to CO, if we add OD to each, we shall have the arc AO = DC; hence, the angle ABI = DBC; also, the angle BAI = BDC, because they are inscribed in the same segment; hence, the triangles ABI, DBC are similar, and their homologous sides give the proportion

$$AB : BD :: AI : CD;$$

from which results

$$AB \times CD = AI \times BD.$$

Adding the two results found, and observing that

$$AI \times BD + CI \times BD = (AI + CI) \times BD = AC \times BD,$$

we shall have

$$AD \times BC + AB \times CD = AC \times BD.$$

SCHOLIUM. Another theorem concerning the inscribed quadrilateral may be demonstrated in the same manner.

The similarity of the triangles ABD and BIC, gives the proportion

$$BD : BC :: AB : BI,$$

hence,

$$BI \times BD = BC \times AB.$$

If CO be drawn, the triangle ICO, similar to ABI, will be similar to BDC, and will give the proportion

$$BD : CO :: DC : OI;$$

hence,

$$OI \times BD = CO \times DC,$$

or, because CO = AD,

$$OI \times BD = AD \times DC.$$

Adding together the two results, and observing that

$$BI \times BD + OI \times BD \text{ is equal to } BO \times BD,$$

we shall have

$$BO \times BD = AB \times BC + AD \times DC.$$

If we had taken BP = AD, and had drawn CKP, a similar course of reasoning would have given us

$$CP \times CA = AB \times AD + BC \times CD.$$

But the arc BP being equal to CO, if BC be added to each of them it will follow that CBP = BCO; hence, the chord CP is equal to the chord BO, and, consequently, the rectangles BO × BD and CP × CA are to each other as BD is to CA; therefore,

$$BD : CA :: AB \times BC + AD \times DC : AD \times AB + BC \times CD.$$

Therefore, *the two diagonals of an inscribed quadrilateral are to each other as the sums of the rectangles of the pairs of sides which terminate at the extremities of each diagonal.*

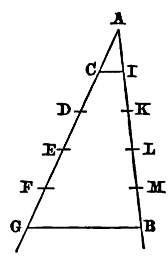
These two theorems may be used for finding the diagonals when the sides are known.

PROBLEMS RELATING TO BOOK III.

PROBLEM I.

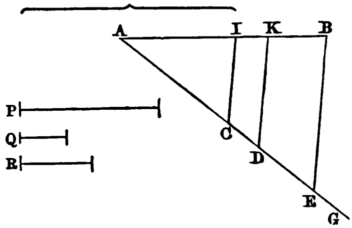
To divide a given straight line into any number of equal parts, or into parts proportional to given lines.

First.—Let it be proposed to divide the line AB into five equal parts. Through the extremity A, draw the indefinite straight line AG, and taking AC of any magnitude, apply it five times on AG. Join the last point of division, G, and the extremity B, by the line GB; then draw CI parallel to GB: then will AI be the fifth part of AB, and thus, by applying AI five times on AB, the line AB will be divided into five equal parts.



For, since CI is parallel to GB, the sides AG, AB are cut proportionally in C and I (Prop. XVI.). But AC is the fifth part of AG; therefore, AI is the fifth part of AB.

Second.—Let it be proposed to divide the line AB into parts proportional to the given lines P, Q, R. Through the extremity A draw the indefinite line AG; take AC = P, CD = Q, DE = R; join the extremities E and B, and through the points C, D, draw CI, DK, parallel to EB; the line AB will be divided into parts AI, IK, KB, proportional to the given lines P, Q, R.

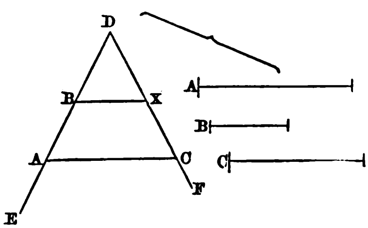


For, on account of the parallels CI, DK, EB, the parts AI, IK, KB, are proportional to the parts AC, CD, DE (Prop. XVI., Cor. 2); and, by construction, these are equal to the given lines P, Q, R.

PROBLEM II.

To find a fourth proportional to three given lines, A, B, C.

Draw the two indefinite lines DE, DF, making any angle with each other. On DE take DA = A and DB = B; on DF take DC = C, draw AC, and through the point B draw BX, parallel to AC; DX will be the fourth proportional required: for, since BX is parallel to AC, we have the proportion



DA : DB : : DC : DX;

now, the first three terms of this proportion are equal to the three given lines; therefore, DX is the fourth proportional required.

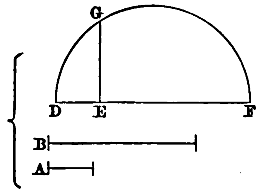
Cor. In the same manner may be found a third proportional to two given lines A, B, for it will be the same as the fourth proportional to the three lines A, B, B.

PROBLEM III.

To find a mean proportional between two given lines, A *and* B.

On the indefinite line DF take DE = A, and EF = B; on the whole line DF, as a diameter, describe the semicircle DGF; at the point E erect, on the diameter, the perpendicular EG, meeting the circumference in G; EG will be the mean proportional sought.

For, the perpendicular GE, let fall from a point of the circumference on the diameter, is a mean proportional between the two segments of the diameter DE, EF (Prop. XXV., Cor.); now, these segments are equal to the given lines A and B.

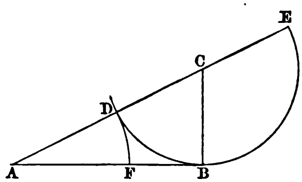


PROBLEM IV.

To divide the given line AB *into extreme and mean ratio, that is, into two parts, so that the greater part shall be a mean proportional between the whole line and the other part.*

At the extremity, B, of the line AB erect the perpendicular BC, equal to the half of AB; from the point C, as a centre, and with the radius CB, describe a semicircle; draw AC, cutting the circumference in D; and take AF = AD: the line AB will be divided at the point F in the manner required, that is to say, we shall have

$$AB : AF :: AF : FB.$$



For AB, being perpendicular to the extremity of the radius CB, is a tangent; and if AC be produced till it again meets the circumference in E, we shall have (Prop. XXXII.)

$$AE : AB :: AB : AD;$$

hence, by division,

$$AE - AB : AB :: AB - AD : AD.$$

But, since the radius BC is the half of AB, the diameter DE is equal to AB, and, consequently, $AE - AB = AD = AF$; also, because $AF = AD$, we have $AB - AD = FB$; therefore,

$$AF : AB :: FB : AD, \text{ or } AF;$$

therefore, by inversion,

$$AB : AF :: AF : FB.$$

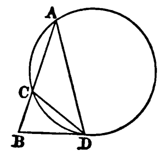
Scholium.. This division of the line AB is called division in mean and extreme ratio. It may further be observed that the secant AE is divided in mean and extreme ratio at the point D; for, since AB=DE, we have

$$AE : DE :: DE : AD.$$

PROBLEM V.

To describe an isosceles triangle in which each of the angles at the base shall be double the angle at the vertex.

Consider the problem solved so that each of the angles at B and D is double of the angle at A. Bisect the angle at D by the straight line DC. Then the angle ADC is equal to the angle A, and BCD = ADC + A is equal to twice the angle A. But, by hypothesis, the angle ABD is equal to double the angle A. Therefore, BD=DC=CA, and, since the angle BDC is equal to the angle at A, the straight line BD will touch the circle described around the triangle ACD in D (Book II., converse of Prop. XXI.). Therefore, $AB \times BC = \overline{BD}^2 = \overline{AC}^2$. Hence, AB : AC : : AC : BC, and AB is divided at C in extreme and mean ratio (Prob. IV.).



Hence, to describe the required isosceles triangle, take two equal lines and the greater part obtained by dividing one of them in mean and extreme ratio, and with these construct the triangle.

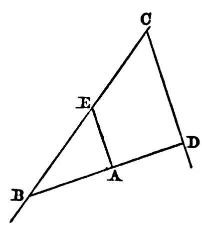
PROBLEM VI.

Through a given point, A, *in the given angle* BCD, *to draw the line* BD, *so that the parts* AB, AD, *comprised between the point* A *and the two sides of the angle, shall be equal.*

Through the point A draw AE parallel to CD, take BE = CE, and through the points B and A draw BAD, which will be the line required. For, AE being parallel to CD, we have

$$BE : EC :: BA : AD;$$

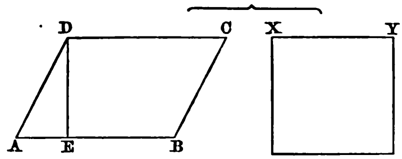
or BE = EC; therefore, BA = AD.



PROBLEM VII.

To construct a square equivalent to a given parallelogram, or to a given triangle.

First.—Let ABCD be the given parallelogram, AB its base, and DE its altitude; between AB and DE find a mean proportional, XY

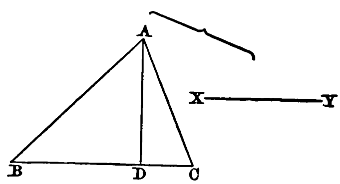


(Prob. III.); the square constructed on XY will be equivalent to the parallelogram ABCD. For we have, by construction

$$AB : XY :: XY : DE;$$

therefore, $\overline{XY}^2 = AB \times DE$; now $AB \times DE$ is the measure of the parallelogram, and $\overline{XY}^2$ that of the square, therefore, they are equivalent.

Second.—Let ABC be the given triangle, BC its base, AD its altitude; find a mean proportional between BC and the half of AD, and let XY be that mean; the square constructed on XY will be equivalent to the triangle ABC.



For, since we have

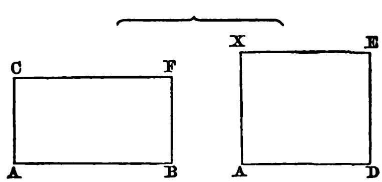
$$BC : XY :: XY : \tfrac{1}{2}AD,$$

it follows that $\overline{XY}^2 = BC \times \tfrac{1}{2}AD$; hence, the square constructed on XY is equivalent to the triangle ABC.

PROBLEM VIII.

On a given line, AD, *to construct a rectangle*, ADEX, *that shall be equivalent to the given rectangle* ABFC.

Find a fourth proportional to the three lines AD, AB, AC, and let AX be that fourth proportional; the rectangle constructed on AD and AX will be equivalent to the rectangle ABFC.



For, since we have

$$AD : AB :: AC : AX,$$

there results from this $AD \times AX = AB \times AC$; therefore, the rectangle ADEX is equivalent to the rectangle ABFC.

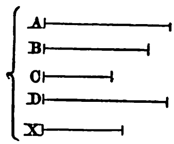
PROBLEM IX.

To find two lines whose ratio shall be the same as the ratio of the rectangle of the two given lines A *and* B *to the rectangle of the two given lines* C *and* D.

Let X be a fourth proportional to the three lines B, C, D; then will the two lines A and X have the same ratio as the two rectangles $A \times B$, $C \times D$.

For, since we have $B : C :: D : X$, it follows that $C \times D = B \times X$; therefore,

$$A \times B : C \times D :: A \times B : B \times X :: A : X.$$

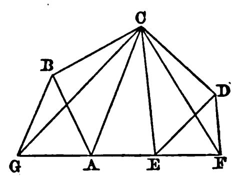


COR. Hence, to obtain the ratio of the squares constructed on the given lines A and C, find a third proportional, X, to the lines A and C, so as to have $A : C :: C : X$, and you will have $A^2 : C^2 :: A : X$.

PROBLEM X.

To find a triangle that shall be equivalent to a given polygon.

Let ABCDE be the given polygon. Draw first the diagonal CE, cutting off the triangle CDE; through the point D draw DF, parallel to CE, and meeting AE prolonged; join CF, and the polygon ABCDE will be equivalent to the polygon ABCF, which has one side less.



For, the triangles CDE, CFE, have the common base CE; they have also the same altitude, since their vertices D, F, are situated in a line, DF, parallel to the base; therefore, these triangles are equivalent. Add to each the figure ABCE, and there will result the polygon ABCDE, equivalent to the polygon ABCF.

In like manner we may cut off the angle B by substituting for the triangle ABC the equivalent triangle AGC, and thus the pentagon ABCDE will be changed into an equivalent triangle, GCF.

The same process can be applied to any other polygon, for by diminishing the number of its sides successively one by one, we shall arrive finally at the equivalent triangle.

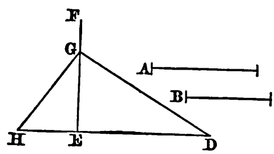
SCHOLIUM. We have already seen that every triangle may be changed into an equivalent square (Prob. VI.); and thus a square may always be found equivalent to a given rectilineal figure, an operation which is called *squaring* the rectilineal figure, or finding its *quadrature*.

PROBLEM XI.

To construct a square which shall be equal to the sum, or to the difference of two given squares.

Let A and B be the sides of the given squares.

First.—If it is required to find a square equivalent to the sum of these squares, draw the two indefinite lines ED, EF, at right angles to each other; take ED=A and EG=B; draw DG, and DG will be the side of the required square.



For, the triangle DEG being right angled, the square made on DG is equal to the sum of the squares made on ED and EG.

Second.—If it is required to find a square equal to the difference of the given squares, form the same right angle FEH; take GE, equal to the shorter of the sides A and B; from the point G, as a centre, with a radius GH, equal to the other side, describe an arc cutting EH in H; the square described on EH will be equal to the difference of the squares described on the lines A and B.

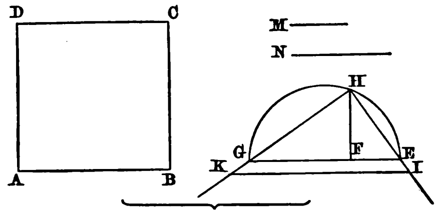
For, the triangle GEH is right angled, its hypothenuse GH = A, and the side GE = B; hence, the square constructed on EH, etc.

SCHOLIUM. Thus may be found a square equal to the sum of any number of squares; for the construction which reduces two of them to one, will reduce three of them to two, and these two to one, and so for the others. It would be the same, if any of the squares were to be subtracted from the sum of the others.

PROBLEM XII.

To construct a square which shall be to the given square ABCD, *as the line* M *is to the line* N.

On the indefinite line EG, take EF = M, and FG = N ; on EG, as a diameter, describe a semicircle, and at the point F erect on the diameter the perpendicular FH. From the point H draw the chords



HG, HE, and prolong these chords indefinitely ; on the first, take HK, equal to the side AB of the given square, and through the point K draw KI, parallel to EG ; HI will be the side of the square sought.

For, by reason of the parallels KI, GE, we have

$$HI : HK :: HE : HG;$$

hence,

$$\overline{HI}^2 : \overline{HK}^2 :: \overline{HE}^2 : \overline{HG}^2 ;$$

but, in the right angled triangle EHG (Prop. XI., Cor. 4), the square of HE is to the square of HG as the segment EF is to the segment FG, or as M is to N ; therefore,

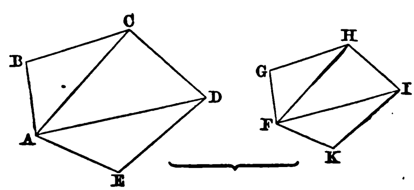
$$\overline{HI}^2 : \overline{HK}^2 :: M : N.$$

But HK = AB ; therefore, the square constructed on HI is to the square constructed on AB as M is to N.

PROBLEM XIII.

On the side FG, *homologous to* AB, *to describe a polygon similar to the given polygon* ABCDE.

In the given polygon draw the diagonals AC, AD ; at the point F make the angle GFH = BAC, and at the point G, the angle



FGH = ABC; the lines FH, GH, will cut each other in H, and FGH will be a triangle similar to ABC; likewise, on FH, homologous to AC, construct the triangle FIH, similar to ADC, and on FI, homologous to AD, construct the triangle FIK, similar to ADE. The polygon FGHIK will be the required polygon, similar to ABCDE.

For these two polygons are composed of the same number of triangles, similar to each other and similarly situated (Prop. XXVIII.).

PROBLEM XIV.

Two similar figures being given, to construct a similar figure which shall be equal to their sum, or to their difference.

Let A and B be two homologous sides of the given figures; find a square equal to the sum, or to the difference of the squares constructed on A and B; let X be the side of this square, and X will be, in the required figure, the side homologous to A and B in the given figures. The figure itself may then be constructed by the preceding problem.

For, the similar figures are as the squares of their homologous sides; now, the square of the side X is equal to the sum, or to the difference of the squares constructed on the homologous sides A and B; therefore, the figure constructed on the side X is equal to the sum or to the difference of the similar figures constructed on the sides A and B.

PROBLEM XV.

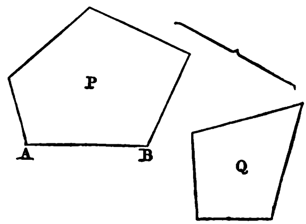
To construct a figure similar to a given figure, and which shall bear to it the given ratio of M *to* N.

Let A be a side of the given figure, X the homologous side in the figure sought. The square of X must be to the square of A as M is to N (Prop. XXIX.). X will be found, therefore, by Problem XII; X being known, the rest will be accomplished by Problem XIII.

PROBLEM XVI.

To construct a figure similar to the figure P, *and equivalent to the figure* Q.

Find M, the side of the square equivalent to the figure P, and N, the side of the square equivalent to the figure Q. Then, let X be a fourth proportional to the three given lines, M, N, AB; on the side X, homologous to AB, describe a figure similar to the figure P; it will also be equivalent to the figure Q.



For, calling Y the figure constructed on the side X, we have

$$P : Y :: \overline{AB}^2 : X^2;$$

but, by construction,

$$AB : X :: M : N, \text{ or } \overline{AB}^2 : X^2 :: M^2 : N^2;$$

therefore,

$$P : Y :: M^2 : N^2.$$

But, by construction, also, $M^2 = P$ and $N^2 = Q$; therefore,

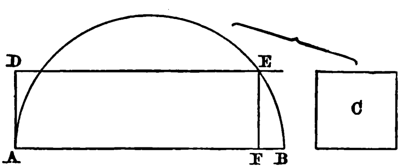
$$P : Y :: P : Q;$$

hence, $Y = Q$; hence, the figure Y is similar to the figure P, and equivalent to the figure Q.

PROBLEM XVII.

To construct a rectangle equivalent to a given square, C, and which shall have the sum of its adjacent sides equal to a given line, AB.

On AB, as a diameter, describe a semicircle; draw the line DE parallel to the diameter, at a distance, AD, from it equal to the side of the given square C; from the point E, where the parallel cuts the circumference, draw EF perpendicular to the diameter; AF and FB will be the sides of the rectangle required.



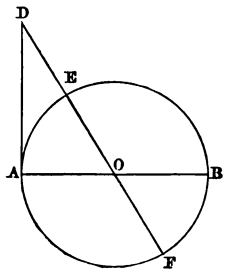
For, their sum is equal to AB; and their rectangle $AF \times FB$ is equal to the square of EF (Prop. XXV., Cor.), or the square of AD; hence, that rectangle is equivalent to the given square C.

SCHOLIUM. To render the problem possible, the distance AD must not exceed the radius; that is to say, the side of the square C must not exceed the half of the line AB.

PROBLEM XVIII.

To construct a rectangle equivalent to a square, C, *and the difference of the adjacent sides of which shall be equal to a given line,* AB.

On the given line AB, as a diameter, describe a circle; at the extremity of the diameter draw the tangent AD, equal to the side of the given square C; through the point D and the centre O draw the secant DF; DE and DF will be the adjacent sides of the required rectangle.

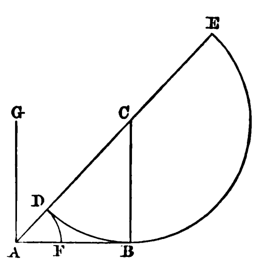


For, first, the difference of these sides is equal to the diameter EF or AB; second, the rectangle DE × DF is equal to $\overline{AD}^2$ (Prop. XXXII.); therefore, this rectangle will be equivalent to the given square C.

PROBLEM XIX.

To find the common measure, if there is one, between the diagonal and the side of a square.

Let ABCG be any square whatever, and AC its diagonal. We must first apply CB on CA as often as it can be contained there (Book II., Prob. XVIII.); and for that purpose let the semicircle DBE be described from the centre, C, with the radius CB. It is evident that CB is contained once in AC with the remainder AD; the result of the first operation is, therefore, the quotient 1 with the remainder AD, which must be compared with BC, or its equal, AB.



We may take AF = AD, and actually apply AF on AB; we should find it to be contained twice with a remainder; but, as that remainder and those which succeed it go on diminishing, and would soon elude our comparison by their smallness, this would be but an imperfect mechanical method from which we could draw no conclusion for determining whether the lines AC, CB, have, or have not a common measure. There is a very simple way, however, by which we may avoid these decreasing lines, and operate only on lines which remain always of the same magnitude.

Thus, the angle ABC being right, AB is a tangent, and AE a secant drawn from the same point, so that we have (Prop. XXXII.),

$$AD : AB :: AB : AE.$$

Accordingly, in the second operation, in which AD is to be compared with AB, we may, instead of the ratio of AD to AB, take that of AB to AE; now, AB, or its equal, CD, is contained twice in AE, with the remainder AD; therefore, the result of the second operation is the quotient 2 with the remainder AD, which must be compared with AB.

Thus, the third operation again consists in comparing AD with AB, and may be reduced in the same manner to a comparison of AB, or its equal, CD, with AE, and we shall again have 2 for a quotient and AD for a remainder.

Hence, it is evident the process will never terminate, and, therefore, there is no common measure between the diagonal and the side of a square; a truth which was already known by arithmetic (since these two lines are to each other :: $\sqrt{2} : 1$) (Prop. XI.), but which acquires a greater degree of clearness by the geometrical investigation.

Scholium. It is hence impossible, also, to find in numbers the exact ratio of the diagonal to the side of the square; but an approximation may be made to it as near as we please, by means of the continued fraction, which is equal to that ratio. The first operation gave us 1 for a quotient; the second, and all the others *ad infinitum* give 2; thus, the fraction in the case is

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}}}, \text{ etc., } \textit{ad infinitum.}$$

For example, if we calculate this fraction as far as the fourth term inclusively, we find that its value is $1\frac{12}{29}$ or $\frac{41}{29}$; so that the approximate ratio of the diagonal to the side of the square is :: 41 : 29. A closer approximation to the ratio might be found by calculating a greater number of terms.

EXERCISES ON BOOK III.

Theorems.

1. If, through any point in the diagonal of a parallelogram, lines be drawn parallel to the sides, the two parallelograms so formed through which the diagonal does not pass are equivalent to one another.

2. The area of a trapezoid is equal to one of the non-parallel sides multiplied by the half sum of the perpendiculars let fall on this side from the extremities of the opposite side.

3. The square constructed on a line equal to the sum of three lines is equal to the sum of the squares on these lines, together with twice the sum of the rectangles on these lines, taken two and two.

4. If, on the two sides AB, AC, of a triangle, ABC, parallelograms ABEF, ACGH, be constructed, and then the point A be joined with the intersection D of the sides EF and GH, the sum of these two parallelograms is equal to the parallelogram one of whose sides is BC, and the other equal and parallel to AD.

From this theorem deduce Proposition XI.

5. Demonstrate Proposition XII. independently of Proposition XI., by the construction of the squares on the sides, and completing the construction after the manner of XI.

6. Demonstrate Proposition XIII. after the same manner.

7. If, in figure, Proposition XI., the lines FH, GI, LK be joined, each of the triangles so formed is equal to the triangle ABC.

8. The irregular hexagon formed by joining the exterior corners of the squares as above, is equal to the area of the square described on the hypothenuse of a right angled triangle, one of whose sides is equal to the hypothenuse of the original triangle and the other is equal to the sum of its sides.

9. If a straight line be drawn from one of the acute angles of a right angled triangle bisecting the opposite side, the square upon that line is less than the square upon the hypothenuse by three times the square upon half the bisected side.

10. If, from the middle point of a straight line as centre, with any radius, a circle be described, the sum of the squares of the distances of any point of this circle from the two extremities of the straight line

is constant. Consider the case in which the radius of the circle is equal to half of the given straight line.

11. If, from the middle point of the line joining the centres of two given circles as a centre, with any radius, a circle be described, the sum of the squares of the tangents drawn from any point of this circle to the two first circles is constant.

12. If, at any point of a straight line a perpendicular be drawn to this line the difference of the squares of the distances of any point of this perpendicular from the two extremities of the given line is constant.

13. If we draw a perpendicular to the line joining the centres of two circles, the difference of the squares of the tangents drawn from any point of this perpendicular to the two circles is constant.

14. Hence, show (and without having recourse to the Prop. XXXII. Book III.) that when two circles intersect each other the tangents drawn to the two circles from any point of the line of intersection produced are equal.

15. If through a point in the interior of a circle two chords be drawn at right angles to one another, the sum of the squares of the four segments (parts) of the chords is equal to the square of the diameter.

16. In any quadrilateral the sum of the squares of the sides is equal to the sum of the squares of the diagonals *plus* four times the square of the straight line which joins the middle points of these diagonals.

17. The sum of the squares on the diagonals of a quadrilateral is double the sum of the squares on the sides of the parallelogram formed by joining the middle points of its sides.

18. Hence, show that this sum is also double the sum of the squares on the lines which join the middle points of the opposite sides of the quadrilateral.

19. O is the point of intersection of the diagonals of a square, ABCD, and P any other point whatever. Prove that

$$AP^2 + BP^2 + CP^2 + DP^2 = 4OA^2 + 4OP^2.$$

20. In a trapezoid the sum of the squares of the diagonals is equal to the sum of the squares of the opposite non-parallel sides *plus* twice the rectangle of the opposite parallel sides.

21. The sum of the squares of the three sides of a triangle is three times the sum of the squares of the three straight lines which join the vertices to the point of intersection of the medians.

22. Four times the sum of the squares of the medians of a triangle is equal to three times the sum of the squares on the sides of the triangle.

23. If G be the point of intersection of the medians of a triangle, ABC, and M any point whatever in the plane of the triangle, then

$$MA^2 + MB^2 + MC^2 = AG^2 + BG^2 + CG^2 + 3MG^2.$$

24. If, from a point, O, in the plane of a triangle, ABC, perpendiculars, OD, OE, OF, be let fall on the three sides, the sum of the squares on the three non-adjacent segments, CE, BD, AF, is equal to the sum of the squares of the three other segments, AE, CD, BF, and we shall have

$$CE^2 + BD^2 + AF^2 = AE^2 + CD^2 + BF^2.$$

25. Conversely, if points, D, E, F, be taken on the sides of a triangle, ABC, so that $CE^2 + BD^2 + AF^2 = AE^2 + CD^2 + BF^2$, the perpendiculars to the sides erected at D, E, F, respectively, meet in the same point.

26. Show that Proposition 24 is true for any polygon.

27. Deduce from Proposition 25, first, that the perpendiculars drawn to the sides of a triangle, at the middle points, meet in the same point. Second, that the three altitudes of a triangle meet in a common point. Third, that the perpendiculars drawn to the sides of a triangle at the points of contact of the escribed circles, meet in a common point.

N. B.—All the preceding theorems can be demonstrated without using Proposition XX., Book III.

28. First. If any point in the plane of a polygon be joined with the vertices of this polygon and all these lines of junction be produced so that the prolongations are proportional to the lines themselves, the polygon which is formed by joining the extremities of these lines will be similar to the given polygon. Second. Prove the same, substituting the condition that the lines of junction all be revolved through the same angle.

29. If two circles are tangent externally, the portion of the exterior

common tangent comprehended between the points of contact is a mean proportional between the diameters of the two circles.

30. If two triangles have an angle in one equal to an angle in the other, and a second angle in one supplementary to an angle in the other, the sides of the triangles respectively opposite to these angles are proportional.

31. In every triangle the distance of the centre of the circumscribed circle from one of the sides is equal to half the straight line which joins the opposite vertex to the point of intersection of the altitudes.

32. In any triangle, ABC, the middle points of the sides, a, b, c, the feet of the altitudes, α, β, γ, the middle points, p, q, r, of the distances of the vertices from the point of intersections of the altitudes are nine points situated on the same circumference. The centre of this circle is the middle point of the straight line which joins the centre, O, of the circumscribed circle of the triangle with the point of intersection of the altitudes, and its radius is equal to half the radius of this circle.

NOTE.—This circle is known in ancient and modern geometrical analysis as the *nine points* circle.

33. If, from the three vertices of a triangle and from the intersection of its medians perpendiculars be let fall on any line whatever, the last perpendicular is one-third of the sum of the three first.

34. The area of a triangle contained by the medians of a given triangle is three-fourths of the area of the given triangle.

35. The straight lines drawn through the point of contact of two circles which touch externally or internally are cut proportionally by the circles.

36. If in two circles two radii be drawn parallel and in the same direction, the line which joins the extremities of the two radii passes through the point of intersection of the common exterior tangents; if the parallel radii are drawn in opposite directions the straight line which joins their extremities passes through the intersection of the common interior tangents.

37. The point of intersection of the three altitudes of a triangle, that of the three medians, and the centre of the circumscribed circle are in the same straight line, and the second divides the distance between the first two in the ratio of 1 to 2.

38. If between the two sides of an angle any number of parallel

lines be drawn, all the middle points of these parallels are on a straight line which passes through the vertex of the angle.

39. The intersections of the diagonals of the trapezoids which we obtain in considering these parallels two and two respectively, are in the same straight line.

40. The three common chords of three circles which intersect each other, two and two, meet each other in one point.

41. If P be a given point within a circle on the radius AC, and a point Q be taken without on the prolongation of the same radius, so that CP : CA : : CA : CQ; then, if from any point, M, in the circumference the straight lines MP and MQ be drawn, they will bear to each other everywhere a constant ratio, and we shall have MP : MQ : : AP : AQ.

42. Prove the converse of Proposition XXXI.

Geometric Loci.

1. Find the locus of all the points such that if any one of these points be joined with two given points the triangle thus formed has a given area.

2. A straight line being given, find the locus of all the points such that the perpendicular let fall from any one of them on the line shall be a mean proportional between the segments of the line between the foot of the perpendicular and the two extremities.

3. The locus of all the points, the sum of the squares of the distances of any one of which, from two fixed points, is equal to a given square, is a circle.

4. Find the locus of all the points, the sum of the squares of the distances of any one of which from *three* given points is equal to a given square; from four points; from any number of points.

5. Find the locus of all the points, the sum of the squares of the tangents from one of which to *two* given circles shall be equal to a given square; the same for *three* given circles; for *any* number of given circles.

6. The locus of all the points, the difference of the squares of the distances of any one of which from two given points shall be equal to a given square, is a straight line perpendicular to the line which joins the two given points.

7. The locus of all the points, the difference of the squares of the tangents of any one of which to two given circles shall be equal to a given square, is a straight line. Consider the case when the given difference is *nothing*, that is, when the tangents are equal.

Definition: This last locus is called the *radical axis* of the two circles; that is, the radical axis of two circles is the line the tangents from any point of which to the two circles are equal.

8. The radical axes of three circles are parallel or meet in the same point.

Definition: The point of meeting of the radical axes of three circles is called the *radical centre.*

9. The radical axis of two circles is the locus of the centres of the circles which cut the two given circles at right angles.

10. Find the locus of all the points such that the sum of the squares of the distances of any one of them from the two sides of a right angle shall be equal to a given square.

11. Find the locus of all the points the sum of the squares of the distances of any point of which from the four sides of a rectangle shall be equal to a given square.

12. A straight line and a point being given, the point is joined with all the points of the straight line, and all the lines of junction are divided into segments proportional to the given lines, find the locus of the points of division.

13. The same problem, replacing the straight line by a circle.

14. The locus of all the points, the ratio of the distances of any one of which from two given straight lines is a given number, is a straight line.

15. The locus of points, the ratio of the distances of any one of which to two fixed points is a given number, is a circle.

16. The locus of points from any one of which two given circles subtend the same angle between the tangents is a circle.

17. Any number of parallel lines cut two given straight lines. Find the locus of the middle point of these parallels.

18. Find the locus of the intersection of the diagonals of the trapezoids formed in the figure of the preceding problem.

PROBLEMS.

1. Deduce from the properties of chords which intersect in a circle (Prop. XXX.) a means of constructing a fourth proportional to three given lines; also, a third proportional to two given lines.

2. Through a given point within a given angle draw a straight line which shall be divided at this point in the ratio of $m : n$. Consider the case in which the point is exterior to the angle.

3. Through a given point on one of the sides of a triangle draw a straight line which shall divide the triangle into two equivalent parts; also, into parts proportional to the lines m and n.

4. Construct a square which shall be a mean proportional between two given parallelograms.

5. Construct a rectangle equivalent to a given rectangle and the sum of two adjacent sides equal to a given line.

6. Find in the interior of a triangle a point which, if we join to the three vertices of the triangle, the triangles thus obtained shall be equivalent.

7. Divide a triangle into two equal parts by a straight line parallel to a given straight line.

8. Construct a right angled triangle, given the hypothenuse and knowing that one of the sides of the right angle is equal to the segment of the hypothenuse not adjacent to it.

9. Construct a triangle, given the three altitudes.

10. Construct a triangle, given one median and the angles.

11. Construct a triangle, given one altitude and the angles.

12. Construct a right angled triangle, given one of the sides containing the right angle and the non-adjacent segment of the hypothenuse formed by the perpendicular let fall from the right angle on the hypothenuse.

13. Describe a circle which shall pass through two given points and be tangent to a given straight line.

14. Describe a circle which shall pass through two given points and be tangent to a given circle.

15. Describe a circle which shall pass through a given point and be tangent to two given straight lines.

16. Describe a circle which shall pass through a given point and shall touch a given straight line and a given circle.

17. Describe a circle which shall pass through a given point and be tangent to two given circles.

18. Describe a circle which shall be tangent to two given straight lines and to a given circle.

19. Draw a chord through a given point in the interior of a circle which shall be divided in a given ratio at the given point.

20. Through a given point exterior to a circle draw a secant such that the exterior part shall be four-fifths of the whole secant.

21. Construct a triangle, given the base, the line which joins the vertex to the middle point of base, and the ratio of the two other sides.

22. Construct a triangle, given two sides and the bisectrix of the angle which they contain.

23. Construct a triangle similar to a given triangle and the vertices of which shall rest on three given concentric circles.

24. Inscribe a square in a given triangle.

25. Find a point in the interior of a triangle so that the lines drawn from it to the three vertices of the triangle shall divide it into three equal triangles.

26. Find the radical axis of two circles which do not intersect or touch one another.

27. Three circumferences being given, find upon one of them a point so that the tangents drawn from this point to the other two circumferences shall be equal.

28. Three circles being given, find on one of them a point so that the difference of the squares of the tangents drawn from this point to the two other circumferences shall be equal to a given square.

29. Given two points and a circumference, find on this circumference a point, C, the distances of which from the two points, A and B, shall be in the ratio of two given lines.

30. Given a circumference and a triangle, ABC, find on the circumference a point, X, so that the sum of the squares of the distances of this point from the three vertices, A, B, C, shall be equivalent to a given square.

NUMERICAL PROBLEMS.

1. Find the area of a triangle whose base is 548 yards and perpendicular 265 meters.

2. Find the area of a trapezoid whose two parallel sides are 48.2 meters, 30.5 meters, and altitude 27.45 meters.

(Compute the same in feet, the meter being 39.37 inches, nearly.)

3. Compute the area of a parallelogram whose base is 145.6 meters, and altitude 72.48 meters.

4. Two sides of a triangle are respectively 15 and 12 feet, and the altitude corresponding to the third side is 9 feet. Find the third side and the distances of the other two sides from the opposite vertices.

5. The sides about the right angle of a R. A. T. are 3 meters and 4 meters respectively. Determine, to within a centimeter,—

First.—The hypothenuse and corresponding altitude.

Second.—The projections of the given sides on the hypothenuse.

Third.—The radii of the inscribed and circumscribed circles.

Fourth.—The portions of the sides fixed by the points of contact of the inscribed circles.

Fifth.—The segments of each side determined by the bisectrix of the opposite angle.

Sixth.—The lengths of the three bisectrices.

Seventh.—The three medians.

6. Two circles whose radii are respectively 1.2 meters and 3 decimeters cut each other at right angles (that is, their tangents at the point of intersection are perpendicular to each other). Compute,

First.—The length of the common chord.

Second.—The length of the part of the line of centres intercepted between the centres of the two circles.

7. The sides of a triangle are 3 meters, 5 meters, and 6 meters, respectively. Compute to within a centimeter,

First.—The segments cut off on each side by the bisectrix of the opposite angle.

Second.—The segments determined by the points of contact of the inscribed circle.

Third.—The lengths of the three bisectrices.

Fourth.—The segments of each side determined by the corresponding altitudes.

Fifth.—The three altitudes.

Sixth.—The three medians.

Seventh.—The radius of the circumscribed circle.

Eighth.—The radius of the inscribed circle.

8. Given in the straight line AC, AB = 4 inches, BC = 5 inches, and in the line A′C′, parallel to AC, A′B′ = 1.24 inches, and B′C′ = 3.10 inches: discover whether AA′, BB′, and CC′, meet in the same point.

9. Given the side of an equilateral triangle equal to 10 feet: find its area.

10. Given the area of an equilateral triangle equal to 36 square feet: find its side to within .001 of a foot.

11. Given one of the equal sides of an isosceles triangle equal to 10 feet, and one of the equal angles equal to one-third of a right angle: find the area of the triangle.

12. Given the sum of the squares of the distances of a point, P, from two points, A and B (12 inches apart), equal to 200 square inches: find the radius of the circle which is the locus of P.

13. The length of a tangent, AB, to a circle (whose radius is 290 feet) from the point of contact, A, to the point B is 100 feet. If this tangent is divided into four equal parts, find the lengths of the perpendiculars erected to it at the three points of division, and terminating in the circumference.

14. With the same data as in Problem 13, determine the external portions of the secants from the three points of division which pass through the centre of the circle.

BOOK IV.

OF REGULAR POLYGONS, AND THE MEASUREMENT OF THE CIRCLE.

DEFINITION.

A polygon which is at once equiangular and equilateral, is called a *regular polygon.*

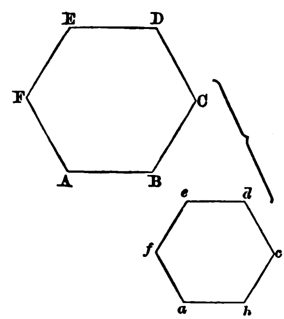
Regular polygons may have any number of sides. The equilateral triangle is one of three sides; and the square, one of four.

PROPOSITION I.

THEOREM.

Two regular polygons of the same number of sides are similar figures.

For example, let ABCDEF, *abcdef*, be two regular hexagons. The sum of all the angles is the same in each figure, being equal to eight right angles (Book I., Prop. XXX.). The angle A is the sixth part of that sum; so is the angle a; hence, the angles A and a are equal; and for the same reason, the same is true of the angles B and b, C and c, and so on.



Again, since from the nature of the polygons, the sides AB, BC, CD, etc., are equal, and likewise the sides ab, bc, cd, etc., it is plain that AB : ab : : BC : bc : : CD : cd; hence, the two figures in question have their angles equal and their homologous sides proportional; therefore they are similar (Book III., Def. 2).

COR. The perimeters of two regular polygons having the same number of sides are to each other as their homologous sides, and their surfaces are as the squares of these same sides (Book III., Prop. XXIX.).

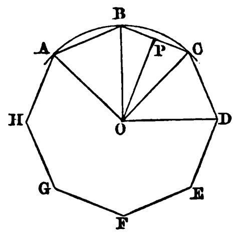
SCHOLIUM. The angle of a regular polygon is determined by the number of its sides (Book I., Prop. XXX.).

PROPOSITION II.

THEOREM.

A circle may be circumscribed about any regular polygon; and a circle may be inscribed in any regular polygon.

Let ABCDE, etc., be a regular polygon; describe a circle through the three points A, B, C; O being the centre, and OP the perpendicular let fall from it on the middle of the side BC; join AO and OD.



If the quadrilateral OPCD be placed on the quadrilateral OPBA, they will coincide, for the side OP is common; the angle OPC = OPB, being right; hence, the side PC will fall along its equal PB, and the point C will fall on B.

Besides, from the nature of the polygon, the angle PCD = PBA, hence, CD will fall along BA, and since CD = BA, the point D will fall on A, and the two quadrilaterals will entirely coincide. The distance OD is therefore equal to AO; and consequently the circle which passes through the three points A, B, C, will pass also through the point D: by similar reasoning it may be shown that the circle which passes through the three vertices B, C, D, will pass through the vertex E, and so of all the rest; hence, the circle which passes through the three points A, B, C, passes through the vertices of all the angles of the polygon, which is therefore inscribed in this circle.

Again, in reference to this circle, all the sides AB, BC, CD, etc., are equal chords; they are therefore equally distant from the centre (Book II., Prop. VIII.); hence, if from the point O, as a centre, with the radius OP, a circle be described, it will touch the side BC and all the other sides of the polygon, each at its middle point, and the circle will be inscribed in the polygon, or the polygon circumscribed about the circle.

SCHOLIUM. The point O, the common centre of the inscribed and circumscribed circles, may be regarded also as the centre of the polygon, and the angle AOB, formed by the two radii drawn to the extremities of the same side, AB, is called the *angle at the centre.*

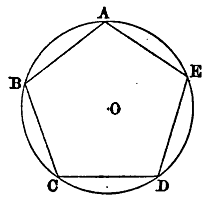
Since all the chords AB, BC, etc., are equal, it is plain that all the angles at the centre are equal; and therefore the value of each is found by dividing four right angles by the number of the sides of the polygon.

PROPOSITION III.

THEOREM.

Every equilateral polygon inscribed in a circle is regular.

Let ABCDE be an inscribed equilateral polygon. Then, since the chords AB, BC, etc., are equal, the arcs AB, BC, etc., are equal; hence, the arcs ABC, BCD, etc., are also equal; and therefore the angles A, B, C, etc., inscribed in equal arcs, are equal. Hence the polygon ABCDE is regular.



SCHOLIUM. In order to inscribe a regular polygon of a certain number of sides in a given circle, we have to divide the circumference into as many equal parts as the polygon has sides, and join the points of division. This division of the circumference can, however, be effected geometrically in only a limited number of cases.

PROPOSITION IV.

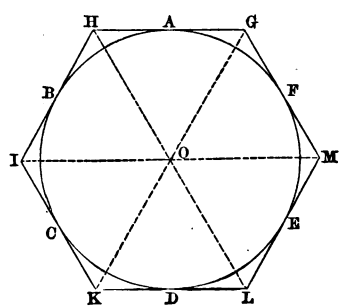
THEOREM.

Every equiangular polygon circumscribed about a circle is regular.

Let GHIKLM be a circumscribed polygon, and let the angles

$$G = H = I = K = L = M.$$

Since the centre O is equally distant from the sides GH, HI, etc., the bisectrices of the angles G, H, I, etc., must all pass through this centre. And since H=G, the angle OHG = OGH; therefore, OG=OH. We show, in the same



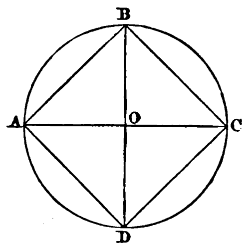
manner, that OG=OM, OH=OC, etc. Hence, the sides GH, HI, etc., are chords of the same circle, and equidistant from the centre. They are, therefore, equal, and the polygon GHIKLM is regular.

COR. The arcs AB, BC, CD, etc., are all equal. Therefore, to circumscribe a regular polygon of a certain number of sides about a given circle, we have to divide the circumference into as many equal parts as the polygon has sides, and draw tangents to the circle at the points of division.

PROPOSITION V.

PROBLEM.

To inscribe a square in a given circle.



Draw two diameters, AC, BD, cutting each other at right angles; join the extremities A, B, C, D; the figure ABCD will be the inscribed square; for the angles AOB, BOC, etc., being equal, the chords AB, BC, etc., will be equal, and the angles ABC, BCD, etc., being inscribed in semicircles, are right.

SCHOLIUM. The triangle BOC being right angled and isosceles, we have (Book III., Prop. XI.) BC : BO :: $\sqrt{2}$: 1; hence, *the side of the inscribed square is to the radius as the square root of 2 is to unity.*

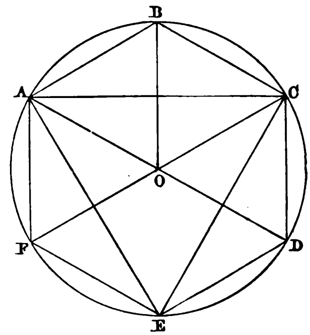
PROPOSITION VI.

PROBLEM.

In a given circle, to inscribe a regular hexagon and an equilateral triangle.

Suppose the problem solved; and that AB is a side of the inscribed hexagon: the radii AO, OB, being drawn, the triangle AOB will be equilateral.

For the angle AOB is the sixth part of four right angles; hence, taking the right angle for unity, we shall have AOB = $\frac{4}{6}$ = $\frac{2}{3}$; and the two other angles, ABO, BAO, of the same triangle, are together equal to $2 - \frac{2}{3}$ or $\frac{4}{3}$, and being equal, each one of them $= \frac{2}{3}$; hence, the triangle ABO is equilateral; therefore, the side of the inscribed hexagon is equal to the radius.



Hence, to inscribe a regular hexagon in a given circle, the radius must be applied six times to the circumference; which will bring us back to the point from which we set out.

The hexagon ABCDEF being inscribed, the equilateral triangle ACE may be formed by joining the vertices of the alternate angles.

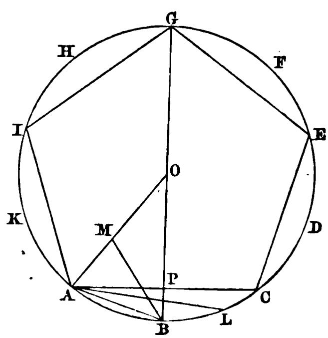
SCHOLIUM. The figure ABCO is a parallelogram, and even a rhombus, since AB=BC=CO=AO; hence (Book III., Prop. XIV. Cor. 1), the sum of the squares of the diagonals, $\overline{AC}^2 + \overline{BO}^2$, is equal to the sum of the squares of the sides, which is $4\overline{AB}^2$ or $4\overline{BO}^2$; and taking away $\overline{BO}^2$ from both, there will remain $\overline{AC}^2 = 3\overline{BO}^2$; hence, $\overline{AC}^2 : \overline{BO}^2 :: 3 : 1$, or $AC : BO :: \sqrt{3} : 1$; therefore, *the side of the inscribed equilateral triangle is to the radius as the square root of 3 is to unity.*

PROPOSITION VII.

PROBLEM.

In a given circle, to inscribe a regular decagon, then a pentagon, and also a regular polygon of fifteen sides.

First.—Suppose the problem solved, and that AB is a side of the inscribed decagon. Draw the radii AO, BO. Taking the right angle equal to unity, we shall have the angle AOB at the centre $= \frac{4}{10} = \frac{2}{5}$; and the two other angles, OAB and OBA, of the same triangle are together equal to $2 - \frac{2}{5} = \frac{8}{5}$, and being mutually equal, each of them must be equal to $\frac{4}{5}$. Hence, each of the angles OAB and OBA is



double the angle AOB. Therefore, to find AB we must construct on the radius OA, as one of the equal sides, an isosceles triangle, with the angle at the base double the angle at the vertex (Book III., Prob. V.). To do this, we divide OA in extreme and mean ratio at the point M (Book III., Prob. IV.), and take AB equal to the greater part OM ; AB is then the side of the regular inscribed decagon.

Second. By joining the alternate corners of the regular decagon, the regular inscribed pentagon ACEGI will be formed.

Third.—AB being still the side of the decagon, let AL be the side of the hexagon; the arc BL will then, with reference to the whole circumference, be $\frac{1}{6} - \frac{1}{10} = \frac{1}{15}$; hence, the chord BL will be the side of the regular polygon of fifteen sides or pentadecagon. It is evident, also, that the arc CL is one-third of CB.

SCHOLIUM. A regular polygon being inscribed, if we divide the arcs subtended by its sides into two equal parts, and draw the chords of the semi-arcs, these chords will form a new regular polygon of double the number of sides of the first; thus it is seen that the square may serve for inscribing successively regular polygons of 8, 16, 32, etc., sides. In like manner, the hexagon may be used for inscribing regular polygons of 12, 24, 48, etc., sides; the decagon, for inscribing polygons of 20, 40, 80, etc., sides; the regular polygon of 15 sides, for inscribing polygons of 30, 60, 120, etc., sides (1).

PROPOSITION VIII.

PROBLEM.

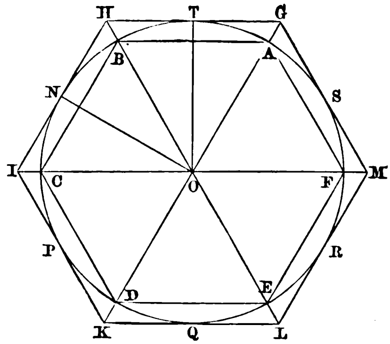
A regular inscribed polygon, ABCDEF, *being given, to circumscribe a similar polygon about the same circle.*

There are two ways of readily solving this problem :

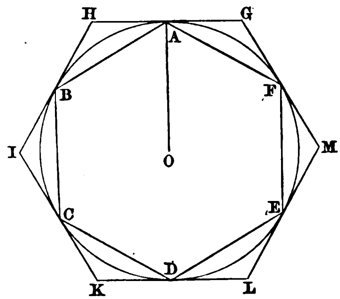
First.—Draw the radii OT, ON, etc., perpendicular to the sides

(1) It was for a long time believed that these polygons were the only ones which could be inscribed in a given circle by the process of elementary geometry, or, what is equivalent to the same thing, by the resolution of equations of the first and second degree; but it has been proved by Gauss, in his work entitled *Disquisitiones Arithmeticæ*, that by the same means may be inscribed a regular polygon of seventeen sides, and in general one of $2^n + 1$ sides, provided that $2^n + 1$ is a prime number.

AB, BC, etc., and then draw tangents to the circle at the points T, N, etc., we thus have a regular circumscribed polygon (Prop. IV., Cor.) of the same number of sides with ABCDEF, and, therefore, similar to it (Prop. I.).



Second.—Draw tangents to the circle at the corners of the inscribed polygon A, B, C, D, E, F, we thus form a regular circumscribed polygon (Prop. IV., Cor.) equal in all respects to the one constructed by the first method.



COR. 1. Conversely, if the circumscribed polygon GHIK, etc., were given, and we were required to deduce from it the inscribed polygon ABC, etc., it would only be necessary to draw from the vertices G, H, I, etc., of the given polygon, lines OG, OH, etc., meeting the circumference at the points A, B, C, etc. Then join AB, BC, etc., and thus prove the regular inscribed polygon. An easier solution would be simply to join the points of contact of the circumscribed polygon. This would likewise form a regular inscribed polygon similar to the circumscribed.

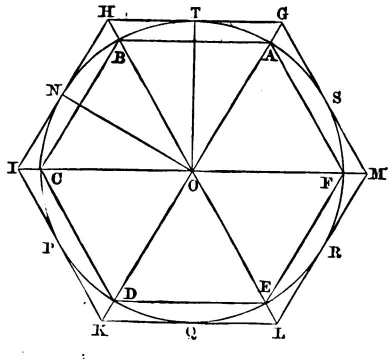
COR. 2. Therefore, we can circumscribe about a circle all the regular polygons which may be inscribed within it, and conversely.

PROPOSITION IX.

THEOREM.

The area of a regular polygon is equal to its perimeter multiplied by the half of the radius of the inscribed circle.

Let, for example, GHIK, etc., be a regular polygon. The triangle GOH will be measured by GH × $\frac{1}{2}$OT, and the triangle OHI by HI × $\frac{1}{2}$ON ; but ON = OT ; hence, the two triangles taken together will be measured by (GH + HI) × $\frac{1}{2}$OT. By continuing the same operation for the other triangles, it will be seen that the sum of them all, or the whole polygon, is measured by the sum of the bases GH, HI, IK, etc., or the perimeter of the polygon, multiplied by $\frac{1}{2}$OT, the half the radius of the inscribed circle.



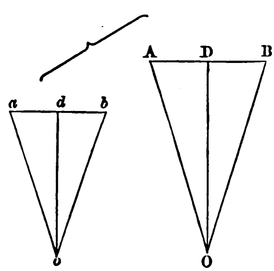
SCHOLIUM. The radius of the inscribed circle, OT, is nothing else than the perpendicular let fall from the centre on one of the sides ; it is called the *apothem* of the polygon.

PROPOSITION X.

THEOREM.

The perimeters of regular polygons having the same number of sides are to each other as the radii of the circumscribed circles, and, also, as the radii of the inscribed circles ; their surfaces are to each other as the squares of these same radii.

Let AB be a side of the one polygon, O its centre, and, consequently, OA the radius of the circumscribed circle, and OD, perpendicular to AB, the radius of the inscribed circle. In like manner, let *ab* be a side of the other polygon, *o* its centre, *oa* and *od* the radii of the circumscribed and inscribed circles.



The perimeters of the two polygons are to each other as the sides AB and *ab* ; but the angles A and *a* are equal, as being each a half of the angle of the polygon ; so, also, are the angles B and *b* ; therefore, the triangles ABO, *abo*, are similar, as are also the right angled triangles ADO, *ado* ; therefore AB : *ab* : : AO : *ao* : : DO : *do* ; hence, the perimeters of the polygons are to each other as the radii AO, *ao*, of the circumscribed circles, and, also, as the radii DO, *do*, of the inscribed circles.

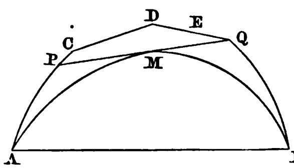
The surfaces of those polygons are to each other as the squares of the homologous sides AB, *ab*; they are, therefore, to each other, also, as the squares of the radii of the circumscribed circles, AO, *ao*, or as the squares of the radii of the inscribed circles, OD, *od*.

PROPOSITION XI.

LEMMA.

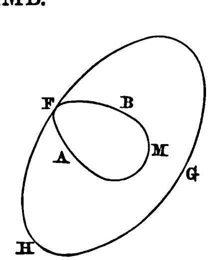
Any curved, or polygonal line, which envelops the convex line AMB *from one extremity to the other, is longer than the enveloped line* AMB.

We have already said that by convex line we understand a line, polygonal or curve, or partly curve and partly polygonal, such that a straight line cannot cut it in more than two points. The arcs of a circle are essentially convex; but the present proposition extends in its application to any line which fulfils the required condition.



This being premised, if the line AMB is not shorter than any of those which embrace it, there will be among these latter one line shorter than all the others, which will be shorter than AMB, or, at most, equal to AMB. Let ACDEB be this enveloping line; anywhere between those two lines, draw the straight line PQ, not meeting the line AMB, or, at least, only touching it. The straight line PQ is shorter than PCDEQ; hence, if, instead of the part PCDEQ, we substitute the straight line PQ, we will have the enveloping line APQB shorter than APDQB. But, by hypothesis, this must be the shortest of all; therefore, that hypothesis is false; hence, all the enveloping lines are longer than AMB.

SCHOLIUM. In the same manner it can be shown that a convex line AMB, returning into itself, is shorter than any line enveloping it on all sides, whether the embracing line FHG touches AMB in one or several points, or surrounds without touching it.



MEASURE OF THE CIRCLE.

LIMITS.

DEFINITIONS.

1. A variable magnitude is one which takes successively different values according to some law.

Examples: The angle of regular polygons take different values according to the number of sides of the polygon. The perimeter of a polygon, inscribed in a given circle, varies when the number of sides changes, as also the perimeter of a polygon circumscribed about a given circle.

2. If the successive values of a variable magnitude approach nearer and nearer to a constant magnitude of the same kind, so that the difference between it and the constant magnitude may be made less than any magnitude of the kind that can be assigned, the Constant is called the *limit* of the Variable, and the Variable is said to *converge* to the Constant.

The limit of a variable, therefore, is that constant quantity of the same kind to which the variable may be brought as near as we please, but which it can never reach.

Examples: Arithmetic and Geometry furnish numerous examples of *limits* which variables thus approach.

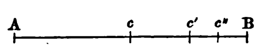
The angle of a polygon of m sides has for its value

$$\frac{2m - 4}{m} = 2 - \frac{4}{m}$$

(the right angle being unity).

Now, if we suppose the number of the sides of the polygon to increase the value of the angle increases, and since we can take m as great as we please, $\frac{4}{m}$ may become as small as we please. We conclude, therefore, that the successive values of the angle of the regular polygons will have two right angles for the *limit*.

In the same manner, if we take the middle point, c, of a straight line, AB, and then the middle point, c', of cB, and so on in succession, the lines Ac, Ac', Ac'', will have AB for a limit.



It is well to observe that a magnitude may vary without having a limit.

Fundamental Principle of Limits.

If the corresponding successive values of two variables are always equal, and each one converges to a limit, then shall the two limits be equal.

In fact, two magnitudes always equal only present a single value, and it seems useless to demonstrate that one variable value cannot converge at the same time to two constant magnitudes which differ from each other. But this important principle can perhaps be made clearer by the following demonstration.

Suppose the two constant limits A and B of the variables, differ by a quantity D. Then, as the variable which has A for a limit may be made to differ from it by a quantity less than $\frac{1}{2}$D, and the second variable may be made to approach B to within less than $\frac{1}{2}$D, the variables would be unequal, which is contrary to the hypothesis. Therefore, the limits A and B cannot be unequal.

Consequences of the principle:

First.—The limit of the sum of two variables is equal to the sum of the limits of these variables.

Second.—The limit of the product of two variables is equal to the product of their limits.

Third.—The limit of the product of a variable by a constant is equal to the product of the limit of the variable by the constant.

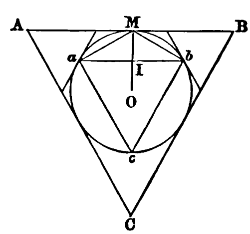
Fourth.—The limit of the ratio of two variables is equal to the quotient of their limits.

PROPOSITION XII.

Theorem.

1. *The circumference is the common limit to which the perimeters of similar regular inscribed and circumscribed polygons converge when we continue to double the number of their sides.*

2. *The area of the circle is the limit to which the areas of these same polygons converge.*



First.—Let *abc* be a regular inscribed polygon, and ABC the regular similar circumscribed polygon.

The length of the circumference is comprised between the perimeters of these polygons; and if the number of their sides be doubled, it is evident that the perimeter of the inscribed polygon will continually increase, while the perimeter of the circumscribed polygon diminishes. They then approach nearer and nearer to the circumference of the circle as we continue to double the number of their sides; and to prove that they approach it as near as we please, we shall show that their difference can become less than any assigned magnitude. Let P and p be the perimeters of the polygons ABC, abc, then,

$$P : p :: OM : OI,$$

whence, $$P - p : P :: OM - OI \text{ or } IM : OM.$$

From this we find $$P - p = \frac{P \times IM}{OM}.$$

But, IM is shorter than Mb; Mb is shorter than the arc which it subtends; and the subtended arcs can decrease without limit, for they follow the terms of the progression, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \cdots\cdot$; besides, P diminishes continually and OM is constant; hence, $P - p$ converges to zero. Therefore, P and p converge to the same limit, the circumference of the circle.

Second.—Let S and s be the areas of the same polygons. We can show as above that S and s continue to approach the area of the circle as the number of the sides of the polygons is doubled. To prove that the circle is the limit of these areas we must show that $S - s$ can be made smaller than any assigned magnitude.

But we have

$$S : s :: \overline{OM}^2 : \overline{OI}^2,$$

whence,

$$S - s : S :: OM^2 - OI^2 \text{ or } \overline{Ib}^2 : \overline{OM}^2.$$

We deduce from that

$$S - s = \frac{S \times \overline{Ib}^2}{OM^2},$$

and it is evident that the difference converges to zero, for S diminishes as the number of sides increases; Ib less than Mb can become as small as we please, and OM is constant. Hence, $S - s$ converges to zero, and, therefore, S and s both converge to the circle as a common limit.

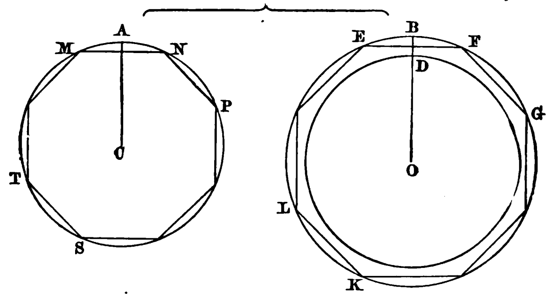
REMARK.—The apothems of the successive inscribed polygons have the radius of the circle as their limit.

PROPOSITION XIII.

THEOREM.

1. *The circumferences of two circles are to each other as their radii.*
2. *The areas of two circles are to each other as the squares of their radii.*

First.—Inscribe in the two circles whose radii are OB and CA two regular similar polygons. Let P and P′ be the perimeters of these



polygons; designate the radii OB and CA by R and R′ and the circumferences by C and C′. Then, from Prop. X.,

$$\frac{P}{P'} = \frac{R}{R'}.$$

This proportion is true whatever be the number of sides of the polygons. But when the number of these sides is increased the perimeters P and P′ converge to the limits C and C′, and their ratio to the limit $\frac{C}{C'}$.

Therefore, $$\frac{C}{C'} = \frac{R}{R'}, \quad (1).$$

Second.—Let A and A′ be the areas of the same circles, S and S′ the areas of two regular similar inscribed polygons. We shall have (Prop. X.)

$$\frac{S}{S'} = \frac{R^2}{R'^2}.$$

This proportion is true whatever be the number of the sides of these polygons. But when the number of sides is increased without limit, S and S′ converge to the limits A and A′ and their ratio to $\frac{A}{A'}$.

Therefore, $$\frac{A}{A'} = \frac{R^2}{R'^2}.$$

COR. 1. Since $$\frac{C}{C'} = \frac{R}{R'},$$

we have $$\frac{C}{R} = \frac{C'}{R'}, \text{ or } \frac{C}{2R} = \frac{C'}{2R'}.$$

Hence, the ratio of a circumference to its diameter is the same for all circumferences; or, in other words,

The ratio of the circumference to the diameter is a constant number.

This number, which is usually represented by π, is incommensurable.* It can be expressed then only approximately in figures; but it can be computed (as we shall show) with any degree of approximation we may wish. Its value in decimals and that of its reciprocal are

$$\pi = 3.14159265358979323846 \cdots$$

$$\frac{1}{\pi} = 0.31830988618379067153.$$

COR. 2. From $\frac{C}{2R} = \pi$ we have

$$C = 2\pi R, \text{ also } R = \frac{C}{2\pi}.$$

We see,

First.—To compute the length of a circumference when the radius is given we multiply twice the length of the radius by the number π.

Second.—To compute the radius of a given circumference we divide the semi-circumference by the number π.

PROPOSITION XIV.

THEOREM.

1. *Similar arcs,* AB, DE, *are to each other as their radii,* AC, OD.

2. *Similar sectors,* BCA, DOE, *are to each other as the squares of their radii.*

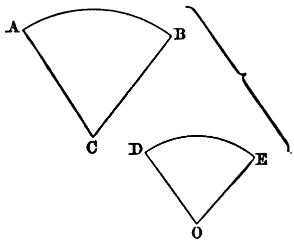
---

* NOTE.—Lambert demonstrated first that π is incommensurable. Legendre proved later that the same is true of the square of π.

First.—We have

Arc BA : circumference AC : : C : 4 right angles,

Arc DE : circumference OD : : O : 4 right angles.



Also, angle C = O.

Hence, arc BA : arc DE : : cir. AC : cir. OD : : AC : OD.

Second.—In like manner we have

Sect. ACB : area AC : : C : 4 right angles,

Sect. DOE : area DO : : O : 4 right angles.

Hence,

Sect. ACB : sect. DOE : : area AC : area DO : : $\overline{AC}^2 : \overline{DO}^2$.

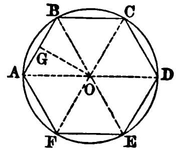
Cor. Similar segments are to each other as the squares of the radii of their respective circles.

PROPOSITION XV.

THEOREM.

The area of a circle is equal to the product of its circumference by half the radius.

The area of the circle is the limit of the areas of regular inscribed polygons, the number of whose sides increase without limit. Let S, C, and R be the area, circumference, and radius of the given circle, and s, p, a, the area, perimeter, and apothem of a regular polygon inscribed in this circle. We have (Prop. IX.) $s = p \times \frac{1}{2}a$.



When we continually double the number of sides of the polygon, s converges to the limit S, p to the limit C, and a to the limit R. We have then $S = C \times \frac{1}{2}R$.

Cor. 1. We have seen that $C = 2\pi R$,

therefore, $$S = 2\pi R \times \frac{R}{2} = \pi R^2 \text{ or } \frac{S}{R^2} = \pi.$$

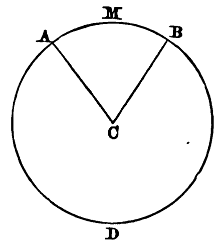
Hence, the number π, the ratio of the circumference of the circle to the diameter, is also the ratio of the area of the circle to the square on the radius.

And $$S = \pi R^2 \text{ gives } R^2 = \frac{S}{\pi}, \text{ or } R = \sqrt{\frac{S}{\pi}}.$$

Hence,

First.—To compute the area of a circle of given radius, we multiply the square of the radius by π.

Second.—To compute the radius of a circle of given area, we divide the area by π, and take the square root of the result.



COR. 2. The area of a sector is equal to the arc of this sector multiplied by half the radius.

For the sector ACB : the whole circle : : the arc AMB : the whole circumference ABD ;

or, $: : \text{AMB} \times \frac{1}{2}\text{AC} : \text{ABD} \times \frac{1}{2}\text{AC}$.

But the whole circle $= \text{ABD} \times \frac{1}{2}\text{AC}$.

Hence, the sector $\text{ACB} = \text{AMB} \times \frac{1}{2}\text{AC}$.

SCHOLIUM. We have seen, already, that the problem of the quadrature of the circle consists in finding a square equal in area to a circle whose radius is given. This problem is solved when we know the ratio of the area of the circle to the square on the radius, or, what is the same thing, the ratio of the circumference to the radius or the diameter.

As we have said, this can only be found approximately, but the approximation may be carried as far as we please. So this question, which occupied much of the attention of Geometers when the methods of approximation were less known, is now classed among the idle questions with which those only concern themselves who possess the most elementary ideas of geometry.

This approximation is made by the method of series, a method purely algebraic. "Before the discovery of these series an approximation was made to the value of π by calculating the lengths of the perimeters of an inscribed and circumscribed regular polygon of the same number of sides. Thus Archimedes (250 B.C.) found $\pi = \frac{22}{7}$, a result which is too great by nearly $\frac{1}{800}$ of the diameter. Peter Metius (1571–1635), by a similar process, obtained $\frac{355}{113}$ for an approximate value of π, a remarkable result, which is accurate to 5 places of decimals. Vieta (1540–1663) extended this approximation by the same method to 10 places of decimals, which number was increased to 35 by Ludolph Van Ceulen ; a labor of vast extent when the means are considered, and of which he was so proud that he directed, after the example of Archimedes, that the numbers should be engraved upon his tomb.

De Lagny (1660–1734), by a process not given, but most probably by a very convergent series, found the value of π to 127 places of decimals, and some notion of the prodigious accuracy of this determination may be formed from the following hypothesis :

If the diameter of the Universe be 100000000000 times the distance of the sun from the Earth, and if a distance which is 100000000000 times this diameter, be divided into parts each of which is 1000000000000th part of an inch ; then, if a circle be described whose diameter is 100000000000 times that distance, repeated 100000000000 times as often as each of those parts of an inch is contained in it : then the error in the circumference of this circle, as calculated from this approximation, will be less than 1000000000000th part of the 1000000000000th part of an inch."—(*Peacock's Calculus*, Art. 37.)

It may be added that some enthusiastic computer has carried the approximation to 600 decimal places.

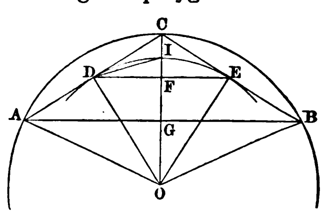
We shall proceed to give two of the simplest elementary methods of obtaining an approximation for the value of π.

PROPOSITION XVI.

PROBLEM.

Given the radius r *and the apothem* a *of a regular polygon, find the radius* r′ *and the apothem* a′ *of the regular polygon which has the same perimeter and double the number of sides.*

Let AB be the side, and O the centre of the given polygon. Drawing the radius OGC perpendicular to AB, we shall have OC $= r$, OG $= a$.



Draw CA and CB, and join the middle points, D and E, of these two chords. The straight line DE, parallel to AB, and equal to the half of it, will be the side of the regular polygon which has the same perimeter as the first, and twice its number of sides. Besides, the angle DOE, being the half of the angle at the centre, AOB, of the first polygon, the point O will be the centre of the new polygon, and we shall have

$$OD = r' \quad OF = a'.$$

But the point F is the middle of CG; then,

$$OF = \tfrac{1}{2}(OG + OC) \text{ or } a' = \tfrac{1}{2}(a + r) \quad (1).$$

Moreover, the right angled triangle ODC gives

$$OD^2 = OC \times OF, \text{ or } r' = \sqrt{r \times a'} \quad (2).$$

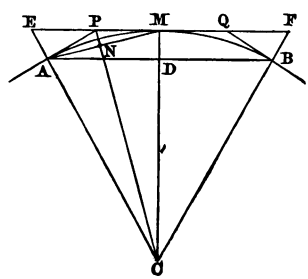
The relations (1) and (2) solve the problem proposed.

SCHOLIUM. We see from the figure that OF is greater than OG, and that OD is less than OC. Then, *when we pass from a regular polygon to a regular isoperimetrical polygon of double the number of sides, the apothem increases and the radius diminishes*, so that the difference between the radius and apothem diminishes continually. The triangle AOG shows that this difference, OA — OG, is less than AG, that is, less than the half of the side of the corresponding polygon. But the perimeter of the polygon remaining constant the length of each side converges to zero when we double their number. Hence, the excess of the radius over the apothem can become as small as we please.

PROPOSITION XVII.

PROBLEM.

Given the perimeters, p, P, *of two similar regular polygons inscribed and circumscribed about the same circle, find the perimeters* p′, P′, *of the regular inscribed and circumscribed polygons of a double number of sides.*



Let AB, EF, be the sides of the polygons whose perimeters are p, P, and m, the number of these sides. Draw the chord AM, and at the points A and B construct the tangents AP, BQ; finally draw the straight line PC; AM and PQ will be the sides of the inscribed and circumscribed polygons of $2m$ sides, the perimeters of which are p', P′.

We have then

$$\frac{P}{p} = \frac{CE}{CA \text{ or } CM}.$$

And as CP is the bisectrix of the angle ECM we have, also,

$$\frac{\text{PE}}{\text{PM}} = \frac{\text{CE}}{\text{CM}}.$$

Then, by cause of the common ratio,

$$\frac{\text{P}}{p} = \frac{\text{PE}}{\text{PM}}.$$

Whence, $$\frac{\text{P} + p}{2p} = \frac{\text{EM}}{2\text{PM}} \text{ or PQ}.$$

But the straight lines EM, PQ, are contained $2m$ times in the perimeters P, P′.

Hence, $$\frac{\text{P} + p}{2p} = \frac{\text{P}}{\text{P}'}, \text{ or } \text{P}' = \frac{2\text{P}p}{\text{P} + p}.$$

To find p' we remark that the two triangles PMN, MAD, are equiangular, and, therefore,

$$\frac{\text{AM}}{\text{AD}} = \frac{\text{PM}}{\text{MN}},$$

but the straight lines AM, AD, are contained $2m$ times in p' and p, and the straight lines PM, MN, $4m$ times in P′ and p'. Therefore,

$$\frac{p'}{p} = \frac{\text{P}'}{p'}, \text{ or } p' = \sqrt{\text{P}'p}.$$

PROPOSITION XVIII.

Problem.

To find the approximate ratio of the circumference to the diameter, that is, the approximate value of π.

We can now use the results of Props. XVI. and XVII. to find an approximate value of π. *First*, Prop. XVI. gives the

Method of Isoperimeters.

Let us take the circumference $\text{C} = 2$; the formula $\pi = \frac{\text{C}}{2\text{R}}$ gives, then, $\pi = \frac{1}{\text{R}}$ or $\frac{1}{\pi} = \text{R}$. That is, the number $\frac{1}{\pi}$ is the radius of the circumference which is equal to 2. Hence, the apothem, a, and radius, r, of every regular polygon whose perimeter is equal to 2, are approximate values of $\frac{1}{\pi}$, because the inscribed and circumscribed circumferences of such a polygon being the one less and the other greater than 2, their radii, a and r, must contain between them the radius, R, of the circumference, equal to 2, that is, the number $\frac{1}{\pi}$ must lie between a and r in value.

Now, take a square as our first polygon, with perimeter equal to 2, its side is equal to $\frac{1}{2}$, its apothem $a_1 = \frac{1}{4}$, and its radius $r_1 = \frac{1}{4}\sqrt{2}$. From these we can find successively the apothems and radii of the regular polygons of this same perimeter 2, and eight sides, using the formulæ $a_2 = \frac{1}{2}(a_1 + r_1)$; $r_2 = \sqrt{a_2 \times r_1}$, etc., and proceeding in the same manner we find the apothems and radii of isoperimetrical polygons of 16, 32, etc., sides.

In this series,

$$a_1, r_1, a_2, r_2, a_3, r_3, a_4, r_4, \text{ etc.},$$

of which each term, beginning with the third, a_2, is alternately the arithmetical and geometrical mean between the two which precede it, the odd terms, a_1, a_2, a_3, etc., go on increasing but are always less that R or $\frac{1}{\pi}$, while the even terms, r_1, r_2, r_3, etc., continually decrease but are always greater than R or $\frac{1}{\pi}$. Moreover, since the difference, $r_1 - a_1$, $r_2 - a_2$, etc. (Prop. XVI.), become less than any assignable quantity by carrying the process far enough, it follows that the terms of the series have $\frac{1}{\pi}$ for their limit.

Now, $a_1 = \frac{1}{4}$ is the arithmetical mean between 0 and $\frac{1}{2}$, and $r_1 = \frac{1}{4}\sqrt{2}$ is the geometric mean between $\frac{1}{2}$ and $\frac{1}{4}$.

Therefore, we can give the following theorem, called Schwab's rule, having been first published by that mathematician in 1813:

The number $\frac{1}{\pi}$ is the limit of the terms of a series of numbers of which the first term is 0, *the second* $\frac{1}{2}$, *and each term beginning with the third is alternately the arithmetical and the geometrical mean between the two preceding terms.*

Computing these means until we find two consecutive terms which agree to $m + 1$ places of decimals, either of these will be an approximate value of $\frac{1}{\pi}$ true to within a decimal place of the $m+1$th order.

Below is such a table computed to a polygon of 8192 sides.

TABLE.

NUMBER OF SIDES OF POLYGON.	APOTHEM.	RADIUS.
4	$a_1 = .2500000$	$r_1 = .3535534$
8	$a_2 = .3017767$	$r_2 = .3266407$
16	$a_3 = .3142087$	$r_3 = .3203644$
32	$a_4 = .3172866$	$r_4 = .3188218$
64	$a_5 = .3180541$	$r_5 = .3184376$
128	$a_6 = .3182460$	$r_6 = .3183418$
256	$a_7 = .3182939$	$r_7 = .3183179$
512	$a_8 = .3183059$	$r_8 = .3183119$
1024	$a_9 = .3183089$	$r_9 = .3183104$
2048	$a_{10} = .3183096$	$r_{10} = .3183100$
4096	$a_{11} = .3183098$	$r_{11} = .3183099$
8192	$a_{12} = .3183099$	$r_{12} = .3183099$

$$\therefore \pi = \frac{1}{.3183099} = 3.14159,$$ true to five decimal places.

Second.—Using the results of Prop. XVII., we have for determining the approximate value of π, the

METHOD OF PERIMETERS.

For $R = \frac{1}{2}$ the formula $\pi = \frac{C}{2R}$ becomes $\pi = C$; the number π is equal to the circumference whose diameter is equal to 1. Therefore, the perimeter of every polygon inscribed in this circumference is an approximate value of π less than π. And the perimeter of every polygon circumscribed about this circumference is an approximate value of π greater than π.

Now, beginning with the perimeters of the inscribed and circumscribed squares, we have $p = 2\sqrt{2}$, $P=4$, and if we use the formulæ $P' = \frac{2Pp}{P+p}$ and $p' = \sqrt{P' \times p}$ of Prop. XVII., we can compute a series of values of perimeters of inscribed and circumscribed polygons nearer and nearer in value to one another and to their limit π.

For the octagons we would get

$$P' = 3.3137085 \qquad p' = 3.0614675,$$

and continuing the process to P_{11} and p_{11} of the circumscribed and inscribed polygons of 8192 sides, we would get

$$P_{11} = 3.1415928 \qquad p_{11} = 3.1415926;$$

hence, π, which lies between these two numbers, may be taken equal to 3.1415927 to within a decimal of the 7th order.

The method of perimeters thus given is a very laborious one, but a very simple one in principle. Archimedes used a method of perimeters beginning with the hexagon and continuing the process to polygons of 96 sides, and found the value $\pi = \frac{22}{7}$ before given, which is sufficient approximation for ordinary application.

COR. The formulæ $P' = \frac{2Pp}{P+p}$ and $p' = \sqrt{P'p}$ may be written

$$\frac{1}{P'} = \frac{1}{2}\left(\frac{1}{P} + \frac{1}{p}\right); \quad \frac{1}{p^1} = \sqrt{\frac{1}{P'} \cdot \frac{1}{p}},$$

and since $\frac{1}{\pi}$ is the limit of $\frac{1}{P'}$, and of $\frac{1}{p}$ when $R = \frac{1}{2}$, we arrive again at Schwab's Theorem, given under the method of Isoperimeters.

For $\frac{1}{P}, \frac{1}{p}, \frac{1}{P'}, \frac{1}{p'} \dots$ are, beginning with the third term, alternately the arithmetical and geometrical means between the two preceding terms. And beginning with the circumscribed and inscribed squares, we have for $R = \frac{1}{2}$, $\frac{1}{P} = \frac{1}{4}$, and $\frac{1}{p} = \frac{\sqrt{2}}{4}$, and we have Schwab's Theorem in the same words as before given.

PROPOSITION XIX.

THEOREM.

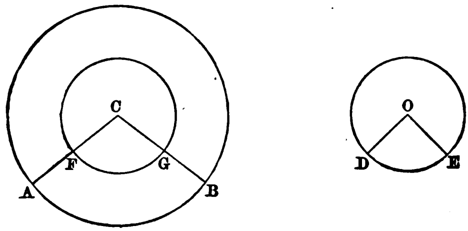
Two angles at the centres of unequal circles are to each other as their intercepted arcs divided by the radii of the circles.

Let the angles C and O be at the centres of the circles AC, DO, respectively. Then shall we have

$$\text{angle C} : \text{angle O} :: \frac{AB}{AC} : \frac{DE}{DO}.$$

Take radius CF = OD and describe the circle CF, then, since FG is included between radii AC and CB,

$$\text{angle C} : \text{angle O} :: \text{FG} : \text{DE};$$



or, since CF = DO,

$$\text{angle C} : \text{angle O} :: \frac{\text{FG}}{\text{CF}} : \frac{\text{DE}}{\text{DO}}.$$

But, by reason of similar arcs,

$$\frac{\text{FG}}{\text{CF}} = \frac{\text{AB}}{\text{AC}}.$$

Hence, $\text{angle C} : \text{angle O} :: \frac{\text{AB}}{\text{AC}} : \frac{\text{DE}}{\text{DO}}$; which was to be proved.

Cor. The $\frac{\text{arc}}{\text{radius}}$ may be taken as the measure of the angle. The unit of this measure is the arc equal in length to the radius. This measure of angles is called *Circular* measure; or, better, *Radial* measure. To express the unit in degrees, minutes, and seconds, we have semi-circumference $= \pi R = 180°$.

$$\therefore \quad R = \frac{180°}{\pi} = \frac{180°}{3.1416} = 57° \; 17' \; 44''.8 = 3437'.8 = 206264''.8.$$

These values of the radius in degrees, minutes, and seconds, are of the greatest importance in numerical problems connecting the measure of the circle and of angles.

EXERCISES ON BOOK IV.

THEOREMS.

1. An inscribed equiangular polygon is regular if the number of its sides be odd.

2. A circumscribed equilateral polygon is regular if the number of sides be odd.

3. The diagonals of a regular hexagon divide each other in the ratio of two to one.

4. The regular inscribed hexagon is double the equilateral triangle inscribed in the same circle, and one-half of the circumscribed equilateral triangle.

5. The regular inscribed hexagon is three-fourths of the regular hexagon circumscribed about the same circle.

6. The side of the circumscribed equilateral triangle is double the side of the inscribed equilateral triangle, and the altitude is three times the radius of the circle.

7. The area of the regular inscribed dodecagon is equal to three times the square of the radius.

8. The square of the side of the regular inscribed decagon, together with the square of the radius, is equal to the square of the side of the regular inscribed pentagon.

9. The diagonals of a regular pentagon divide each other in extreme and mean ratio.

10. If we join the first, fourth, seventh, etc., vertices of a regular inscribed decagon, each one of the joining chords is equal to the radius of the circle plus the side of the decagon. And, applying this chord to the circumference, after going around three times the extremity will fall at the point of starting. And thus a regular reentrant decagon will be formed (called a star decagon).

11. If the first, fifth, ninth, etc., vertices be joined, we form the star pentagon, or we effect the same by joining the alternate vertices of the above star decagon.

12. If a circumference be divided into five equal parts, and the points of division, A, B, C, D, E, be joined by the lines AC, CE, EB, BD, DA, these lines will form by their intersection a regular pentagon.

13. If, from any point within a regular polygon of n sides, we let fall perpendiculars on the sides, the sum of these perpendiculars will be equal to n times the apothem of the polygon.

14. If, on the side of a square, we take distances from the vertices equal to one-half of the diagonal, and join the points thus taken on adjacent sides, we form a regular octagon.

15. If two circles cut each other at right angles (that is, if their tangents at the point of intersection be perpendicular to each other), and if the distance between the centres be double one of the radii, then the common chord, is the side of a regular hexagon in the greater circle, and of an equilateral triangle in the other.

16. If we describe two equal semi-circumferences on the diameter of a given semicircle, and then inscribe a circle in the space between these three semicircles, touching the three semi-circumferences, the diameter of this circle will be one-third of the diameter of the first circle.

17. If, on the sides of a triangle, ABC, right angled at C, we describe three semicircles, AMCNB, AOC, and BPC (these last two exterior to the triangle), then the curvilinear spaces, cut off by the semi-circumference on the hypothenuse from the other two semicircles, are together equal to the area of the triangle (these spaces are called the Lunules of Hippocrates).

18. Show, by the consideration of the perimeters of the regular inscribed hexagon and circumscribed square, that the value of π is comprised between the numbers 3 and 4.

19. Show that the semi-circumference of the circle is nearly equal to the sum of the side of the inscribed equilateral triangle and the side of the inscribed square.

20. If we divide the diameter, AB, of a circle into two parts, AC, CB, and describe semi-circumferences on AC and CB, on different sides of AB: 1st, the curve line composed of these two semi-circumferences divides the circle into two parts proportional to AC and CB; and 2d, this line is also equal to the semi-circumference on AB.

21. If we divide the diameter, AB, of a circle into any number of equal parts, as, for example, five, so that the points of division fall at C, D, E, and F, and if upon AC, AD, AE, and AF, we describe semi-circumferences on one side of AB, and semi-circum-

ferences on BC, BD, BE, and BF, all falling on the other side of AB, then the curve lines made up of these semi-circumferences, two and two respectively, shall divide the circle into five equal parts.

22. A circular ring (that is, the space included between two concentric circles) is equal to the circle which has for its diameter the chord of the greater circle which is tangent to the smaller.

23. If we describe a semi-circumference on CA, the radius of a given circle whose centre is C, and, dividing this radius into any number of equal parts, for example, into four, erect perpendiculars at the points of division, meeting the semi-circumference on AC in the points M, N, and O: then the circles described with the centre C and radii AM, AN, and AO, respectively, will divide the circle of radius CA into four equal parts.

24. If we make a circumference roll along a fixed circumference of double radius, and within it, then, a point on the rolling circumference will describe a diameter of the fixed circle.

PROBLEMS.

1. In an equilateral triangle inscribe three equal circles touching one another, and each touching two sides of the triangle.

2. In a given circle inscribe three equal circles touching one another and the given circle.

3. In a given square inscribe four equal circles tangent each to two of the others and to one side of the square.

4. In a given circle inscribe four equal circles, tangent each to two of the others and to the given circle.

5. Compute the side of a regular decagon, the radius of the circle being R; also the side of the regular pentagon.

6. Determine by a single and the same construction the side of the regular decagon, and the side of the regular pentagon inscribed in a given circle.

7. Compute the area of the regular inscribed hexagon, octagon, dodecagon, and equilateral triangle, the radius of the circle being 2.4 meters. Make the computation also in feet (the meter being 39.37 inches nearly).

8. Three equilateral triangles whose sides are respectively 3 feet, 5

feet, and 2 feet, being given, find the equilateral triangle equal to their sum.

9. Radius of circle being 3.20 meters, compute to within .001 meter the area of inscribed and circumscribed equilateral triangle, and area of regular inscribed and circumscribed hexagon (compute the same in feet).

10. Area of regular dodecagon being 3888 square meters, find the area of regular decagon inscribed in the same circle.

11. Construct a circle equivalent to the sector of a given circle whose arc is $32^\circ\ 24'$.

12. The fore wheels of a carriage have a radius of 24 centimeters, the hind wheels a radius of 40 centimeters; how many revolutions does each make in five kilometers.

13. The area of a regular hexagon being 6400 square feet, find the area of the circle circumscribed about it.

14. Find the area of a segment whose arc is 60°, the radius of the circle being 512 meters.

15. Find the area of a circle in which the area of the sector whose arc is $36^\circ\ 40'$, is 1200 square meters.

16. A circle, a square, and an equilateral triangle, all have the same perimeter, equal to one meter. Compare their areas.

17. The radius of the circumscribed circle being unity, compute the side and apothem of each of the following regular polygons:

First, Equilateral triangle; Second, Square; Third, Octagon; Fourth, Pentagon; Fifth, Decagon; Sixth, Dodecagon; Seventh, Polygon of 20 sides.

18. Three equal circumferences, with the radius 6 inches, touch each other. Compute the area inclosed between them.

19. Four equal circles are inscribed in a square, each touching two of the others, the side of the square being four inches. Compute that part of the surface of the square which is exterior to the circles.

20. If an arc of 45° on one circumference is equal to an arc of 60° on another circle, what is the ratio of the areas of the circles.

21. Find the number of degrees, minutes, and seconds in the angle $\frac{1}{10}$ (the unit of measure being the angle corresponding to the arc equal to the radius).

22. Express $22\frac{1}{2}°$ in radial measure.

23. Express angle .7854 in degrees, minutes, and seconds.

24. Find the radial measure of an angle whose arc is π feet, the radius of the circle being π yards.

25. What is the radius of a circle the arc of $8''.9$ on which is 3956 miles long.

26. What is the length of an arc of $16'\ 2''$ on a circumference whose radius is 91684792 miles.

27. The circumference of a circle is 300 feet, find its area.

28. The area of a circle is 1000 square meters, find its radius.

29. Find the radius of a circle which is equal in area to the sum of the areas of three circles whose radii are 4, 6, and 6.93 feet, respectively.

GEOMETRY IN SPACE.

BOOK V.

PLANES AND SOLID ANGLES.

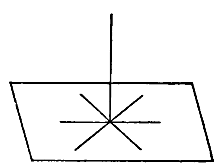
I. Determination of Planes in Space, etc.

DEFINITIONS.

1. We have seen (Def. 6, Book I.) that a *plane* is a surface in which any two points being taken, the straight line joining them lies wholly in the surface.

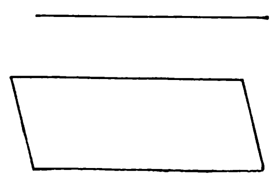
Note.—This surface is indefinite in extent, yet, to represent it we assign limits to it—that is, we represent a plane by a figure traced in it—but the plane must be conceived to extend indefinitely beyond the sides of the figure. This figure is usually a parallelogram in the Theorems of this Book.

2. A straight line is *perpendicular* to a plane when it is perpendicular to all the straight lines which pass through its foot in the plane. Conversely, the plane is perpendicular to the line. The *foot* of the perpendicular is the point in which it meets or pierces the plane.

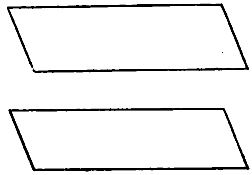


3. A straight line is *oblique* to a plane when it meets the plane without being perpendicular to it.

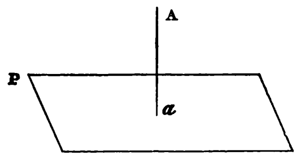
4. A straight line is *parallel* to a plane when it cannot meet that plane, how far soever both be produced. Conversely, this plane is parallel to the line.



5. Two planes are parallel to each other when they cannot meet, how far soever both be produced.



6. The *projection* of a point, A, on a plane, P, is the foot, *a*, of the perpendicular let fall from the point on the plane. The perpendicular A*a* is called the *projecting line* of the point, and the plane is called the *plane of projection.*



7. The projection of a line on a plane is the line which contains the feet of the perpendiculars let fall from all the points of the line on the plane.

PROPOSITION I.

THEOREM.

A straight line cannot be partly in a plane and partly out of it.

For, by Definition 1, when a straight line has two points common with a plane, it lies wholly in that plane.

COR. A straight line can meet a plane in one point only.

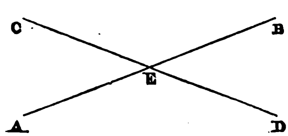
SCHOLIUM. To discover whether a surface is a plane, we must apply a straight line in different directions to the surface, and observe if it touches the surface throughout its whole length.

PROPOSITION II.

THEOREM.

Two planes, P *and* Q, *which have three points,* A, B, *and* C, *not in the same straight line, in common, coincide throughout their whole extent.*

Join any two of the points, as A and B. Let D be any point of the plane P, on the opposite side of AB from C. Join CD. The lines CD and AB, being in the same plane, P, must meet each other in some point, E. But, since the point C, the straight line AB, and its point, E, are in the plane Q, by hypothesis, the straight line CE, and its point, D, must be in this plane (Prop. I.). Hence, the point D is

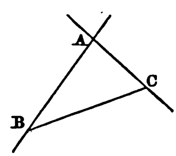


common to both planes. And, as we may make the same contruction for each of the points, A, B, and C, and the line joining the other two, it follows that every point of the plane P is common to the plane Q; therefore the two planes coincide.

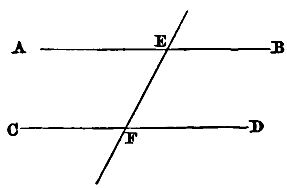
COR. 1. *Through three points, not in the same straight line, one plane may be made to pass, and but one.* In other words, *Three points not in the same straight line determine the position of a plane.*

COR. 2. *A point,* C, *and a straight line,* AB, *determine the position of a plane.*

COR. 3. *Two straight lines,* AB *and* AC, *which intersect each other, determine the position of a plane.*



For the plane P of the three points A, B, and C contains the lines AB and AC, since each of these lines will have two points in that plane; and conversely, the plane of AB and AC cannot be different from the plane P, since it is the plane of the three points A, B, and C.



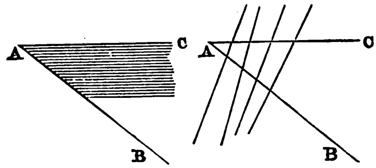
COR. 4. Hence also two parallels, AB and CD, determine the position of a plane; for, drawing the secant EF, the plane of the two straight lines, AE, EF, is that of the parallels, AB, CD.

COR. 5. Hence, *through a point we can draw in space only one parallel to a given straight line.*

For a line drawn through this point, parallel to the given line, must be in the plane determined by the point and the given line, and in a plane only one parallel can be drawn to a given line through a given point.

SCHOLIUM 1. In Geometry in Space, the name Geometric Locus is given to a line or a surface containing *all* the points in space which fulfil *a given condition as to position*, or, as it is expressed, possess a particular geometrical property. Thus it follows from this Proposition that: *The Geometric Locus of a line which passes through a fixed point and meets a fixed line is the plane determined by the point and fixed line.*

SCHOLIUM 2. The plane of an angle, BAC, may be generated by a straight line moving along the side AB, and remaining constantly parallel to the side AC; or, more generally, by a straight line which moves, resting in any manner whatever on the two lines AB and AC.



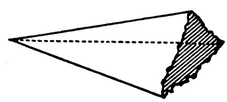
PROPOSITION III

THEOREM.

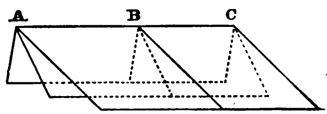
If two planes cut each other, their common intersection will be a straight line.

For, if among the points common to the two planes, there be three which are not in the same straight line, then the planes passing each through these three points must form only one and the same plane, which contradicts the hypothesis.

COR. 1. Three planes may meet in a point. This point is where the line common to two of the planes pierces the third plane.



COR. 2. Through three points, A, B, and C, in the same straight line, any number of planes may be drawn. Hence, at each point of a straight line, AB, any number of perpendiculars may be drawn to that line. For we can draw one in each of the planes which contains AB.

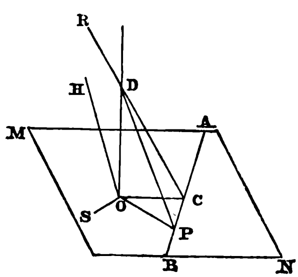


PROPOSITION IV.

THEOREM.

At any point, O, *of á plane,* MN, *one perpendicular can always be drawn to that plane, and but one.*

First.—From the point O draw a perpendicular, OC, to any line AB of the plane, and at C erect another perpendicular, CR, to this line. Finally, in the plane OCR, determined by CO and CR, let the perpendicular OD be drawn to the line OC. Then will OD be perpendicular to any line, OP, drawn



through its foot in the plane MN—that is, it will be perpendicular to the plane (Def. 2). For, join DP, then the right angled triangles DCP, OCP, and DCO, give (Prop. XI., Book III.) the equations

$$DP^2 = DC^2 + CP^2$$
$$OC^2 + CP^2 = OP^2$$
$$DC^2 = OC^2 + OD^2$$

Adding and striking out OC^2, DC^2, and CP^2, common to both sides, and we have $DP^2 = OP^2 + OD^2$. Hence, the angle DOP is a right angle, and the line DO is perpendicular to the plane MN.

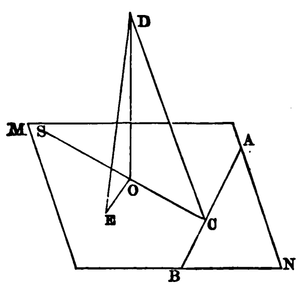
Secondly.—Any other line, OH, drawn through O, will be oblique to the plane MN. For the plane determined by OD and OH cuts the plane MN in the line OS, and since DOS is a right angle, HOS is less than a right angle. Therefore, OD is the only perpendicular which can be drawn to the plane MN, at the point O.

PROPOSITION V.

THEOREM.

From a point, D, *without a plane,* MN, *one perpendicular can be drawn to that plane, and but one.*

First.—From the point D draw a perpendicular, DC, to any line, AB, of the plane. Then, at C, draw in the plane MN another perpendicular, CS, to the line AB; and finally, from D let fall the perpendicular DO, on the line CS. We can then show, as in the preceding proposition, that DO will be perpendicular to MN.



Secondly.—It is impossible to draw another perpendicular to the plane from the point D. For, let DE be that perpendicular, and join OE. Then the triangle DOE would have two right angles, DOE and DEO, which is impossible.

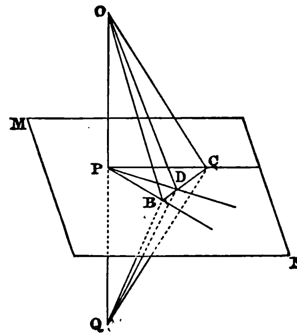
PROPOSITION VI.

THEOREM.

If a straight line, OP, *is perpendicular to two others,* PB, PC, *which intersect each other at its foot* P *in the plane* MN, *it will be perpendicular to that plane.*

Draw any other line, PD, in the plane, then we have to show that OP will be perpendicular to this line.

For, draw the line BDC, cutting the lines PB, PD, and PC, in B, D, and C, respectively. Prolong OP until PQ is equal to OP, and join OB, OD, OC, and QB, QD, QC. The triangles QBC and OBC are equal, because they have BC common; the oblique line OB = QB, and OC = QC (Book I., Prop. XVII.). Hence, the angle OCB = angle QCB. Hence, the triangles OCD and QCD, having the side DC common, and OC = QC, and the angles OCB and QCB equal, are equal, and therefore OD = QD. Hence, the straight line PD, having two of its points, P and D, equally distant from the extremities O and Q of the line OQ, is perpendicular to it at the point P. Hence the line OP is perpendicular to every line drawn through its foot in the plane MN, and is therefore perpendicular to the plane (Def. 2).



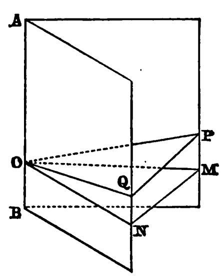
SCHOLIUM. Thus it is evident, not only that a straight line may be perpendicular to all the straight lines which pass through its foot in a plane, but that it always must be so whenever it is perpendicular to two straight lines drawn through its foot in the plane; which shows the propriety of our second definition.

PROPOSITION VII.

THEOREM.

At a point, O, *of a straight line,* AB, *a plane can always be drawn perpendicular to this line, and but one.*

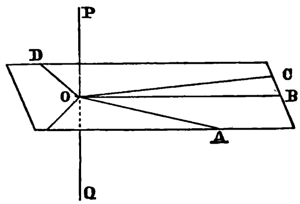
First.—At the point O erect two perpendiculars, OM and ON, to the line AB, in any two planes containing AB. Then that line is perpendicular to the plane MON, determined by these lines, OM and ON (Prop. VI.), and conversely, the plane MON is perpendicular to AB.



Secondly.—Any other plane drawn through O will be oblique to AB. For, let OP and OQ be the intersections of this plane with the planes AOM and AON. We have the angle AOP < the right angle AOM, and the angle AOQ < the right angle AON. Hence, AB is oblique to the plane POQ, and conversely, the plane POQ is oblique to that line. Therefore only one plane can be drawn through O perpendicular to AB.

COR. *The locus of the perpendiculars* OA, OB, OC, OD, *etc., drawn to the same point,* O, *of a straight line,* PQ, *is a plane perpendicular to that line.*

For, if three of these perpendiculars, as OA, OB, and OC, are not in the same plane, there would be three planes, AOB, AOC, BOC, perpendicular to the line PQ, at the same point, O, which is impossible.

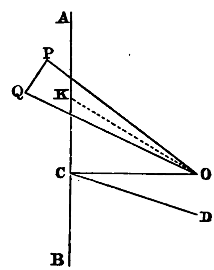


PROPOSITION VIII.

THEOREM.

From a point, O, *without a straight line,* AB, *one plane can be drawn perpendicular to that line, and but one.*

First.—From the point O let fall the perpendicular OC on AB; then at the point C erect another perpendicular, CD, to this line. Then the line AB is perpendicular to the plane OCD, determined by the lines OC and CD, and conversely, the plane OCD is perpendicular to AB.



Secondly.—Suppose it possible to draw through O another plane perpendicular to this line; and let OPQ be that plane. Join O and the point, K, in which the line AB meets the plane OPQ. Then OK will be perpendicular to AB, and we shall have two perpendiculars, OC and OK, from the same point to the same straight line, which is impossible. Hence, OCD is the only plane which can be drawn through O perpendicular to AB.

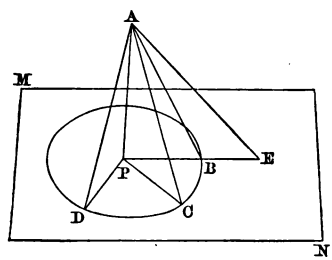
PROPOSITION IX.

THEOREM.

If from any point, A, *without a plane,* MN, *we let fall a perpendicular on that plane, and draw different oblique lines to meet the plane, then,*

1. *The perpendicular is shorter than any oblique line.*
2. *Oblique lines which meet the plane at points equally distant from the perpendicular are equal.*
3. *Of oblique lines unequally distant from the perpendicular the more distant is the longer.*

First. We have AP $<$ AB, because in a plane APB the perpendicular to the straight line PB is shorter than any oblique line drawn to the same.



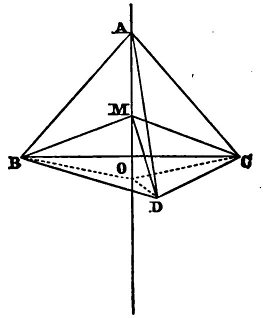
Secondly. The angles APB, APC, APD being right, if we suppose the distances PB, PC, PD equal to each other, the triangles APB, APC, and APD will each have an equal angle contained by equal sides, hence they are equal. Therefore the hypothenuses, or the oblique lines AB, AC, AD, will be equal to each other.

Thirdly. If the distance PE is greater than PD or its equal, PB, it is plain that the oblique line AE will be greater than AB (Prop. XVII., Book I.), or its equal AD.

COR. 1. The perpendicular AP, being the shortest of all the lines which join the point A to the plane MN, measures the true distance from the point to the plane.

COR. 2. All the oblique lines which terminate in the circumference BCD, described from P, the foot of the perpendicular, as a centre, are equal; therefore a point, A, out of a plane being given, the point P, at which the perpendicular let fall from A would meet the plane, may be found by marking upon that plane three points, B, C, D, equally distant from the point A, and then finding the centre of the circle which passes through these points; this centre will be P, the point sought.

COR. 3. Every straight line which has two of its points, A and M, equally distant from the three vertices, B, C, D, of a triangle, BCD, is perpendicular to the plane of this triangle; and the foot of the perpendicular, O, will be the centre of the circle circumscribing the triangle BCD.

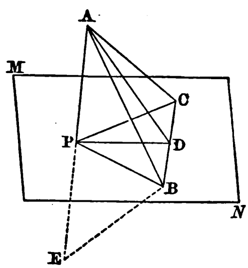


PROPOSITION X.

THEOREM.

If from the foot, P, *of a perpendicular,* PA, *to the plane* MN, *a perpendicular,* PD, *be drawn to a line,* BC, *of the plane, any line,* DA, *which joins the foot,* D, *of this second perpendicular to a point,* A, *of the first, will itself be perpendicular to the line* BC *of the plane.*

Take DB = DC, and join PB, PC, AB, AC. Since DB = DC, PB = PC, and hence the oblique line AB = AC (Prop. IX.). Therefore, the line AD has two of its points, A and D, equally distant from the extremities B and C. Hence, AD is perpendicular to BC at its middle point (Book I., Prop. XVIII., Cor. 2). This proposition is called the *Theorem of the three perpendiculars.*



COR. It is evident that BC is perpendicular to the plane APD, since it is at once perpendicular to the two straight lines AD and PD.

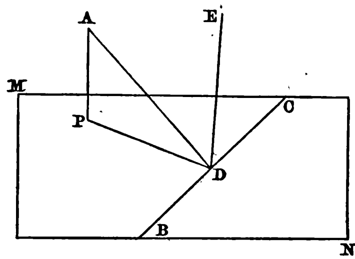
SCHOLIUM. The two straight lines AE, BC, afford an instance of two lines which cannot meet, and still are not parallel. BC lies in the plane MN, and AE pierces that plane in the point P. The two lines are, therefore, not situated in the same plane, and hence cannot meet. We may say that two straight lines drawn arbitrarily in space do not generally meet. For none of the lines drawn in the plane MN will meet AE, except those which pass through the point P. Both the lines AE and BC are perpendicular to the line PD. Hence, *in space two lines may be perpendicular to the same straight line and still not be parallel.*

PROPOSITION XI.

THEOREM.

If the line AP *is perpendicular to the plane* MN, *every line,* DE, *parallel to* AP *will be perpendicular to the same plane.*

The plane of the parallels AP, DE must meet the plane MN, since AP meets that plane, and its intersection with MN will be PD; and ED must meet PD, and therefore also must meet the plane MN; in the plane draw BC perpendicular to PD, and join AD. By the corollary of the preceding Theorem, BC is perpendicular to the plane APDE; therefore the angle BDE is a right angle: but the angle EDP is a right angle also, since AP is perpendicular to PD, and DE is parallel to AP; therefore, the line DE is perpendicular to the two straight lines DP, DB; hence it is perpendicular to their plane, MN (Prop. VI.).



COR. 1. Conversely, if the straight lines AP, DE are perpendicular to the same plane, they will be parallel; for, if they be not so, draw through the point D a line parallel to AP; this parallel will be perpendicular to the plane MN; and therefore, at the same point, D, more than one perpendicular might be drawn to the same plane, which is impossible (Prop. IV.).

COR. 2. Two lines, A and B, parallel to a third line, C, are parallel to each other; for, conceive a plane perpendicular to the line C; the lines A and B, parallel to this perpendicular, will be perpendicular to the same plane; therefore, by the preceding corollary, they will be parallel to each other.

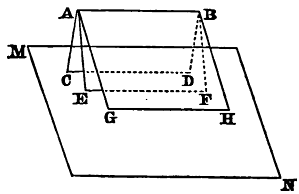
It is understood that the lines are not in the same plane, for if so, the proposition would be already known (Book I., Axiom 6).

PROPOSITION XII.

THEOREM.

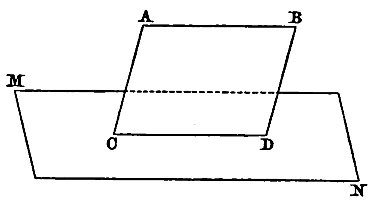
All the planes, AD, AF, AH, *etc., which contain a straight line,* AB, *parallel to a plane,* MN, *intersect this plane in lines,* CD, EF, GH, *parallel to* AB, *and to each other.*

The straight line AB, being in the same plane with each one of the intersections, CD, EF, GH, etc., is parallel to each. For, if it met any one of these lines lying in the plane MN, it would meet that plane, which is impossible by the hypothesis (Def. 4).



Also, the lines CD, EF, GH, etc., being parallel to AB, are parallel to each other (Prop. XI., Cor. 2).

COR. 1. If a line, AB, is parallel to a plane, MN, through any point, C, of this plane, a line may be drawn parallel to AB in the plane. For, if through the point C and the line AB we pass a plane, the intersection, CD, of this plane with MN will be parallel to AB. Conversely, when a line, AB, is parallel to a plane, MN, a parallel to AB through any point, C, of MN lies in that plane; otherwise there would be two parallels through the same point, C, to the same straight line, AB.



COR. 2. *A straight line,* AB, *parallel to two planes,* MN *and* PQ, *which intersect each other, is parallel to their line of intersection.*

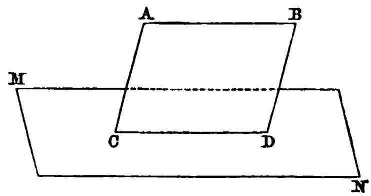
For, the parallel to AB through any point, C, of their intersection, must lie in both planes.

PROPOSITION XIII.

THEOREM.

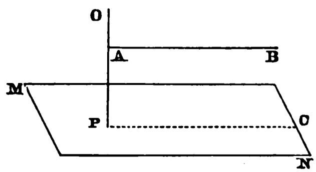
If a straight line, AB, *is parallel to a straight line,* CD, *drawn in the plane* MN, *it will be parallel to that plane.*

For, if the line AB, which is in the plane ABCD, could meet the plane MN, this could only be in some point of CD, the common intersection of the two planes; but AB cannot meet CD, since it is parallel to it; hence it will not meet the plane MN. Therefore (Def. 4) it is parallel to that plane.



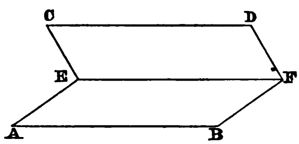
Cor. 1. *A straight line,* AB, *and a plane,* MN, *perpendicular to the same straight line,* OP, *are parallel.*

For the plane BAP intersects the plane MN in a line, PC, perpendicular to OP, and therefore parallel to AB. Hence, AB is parallel to MN.



Cor. 2. Through a point, A, without a plane, MN, any number of lines may be drawn parallel to that plane.

Cor. 3. If two intersecting planes, AF and CF, contain two parallels, AB and CD, their common intersection, EF, will be parallel to these lines. For, the straight line AB, being parallel to CD, must be parallel to the plane CF, and being parallel to this plane it must be parallel to EF, and similarly, CD must also be parallel to EF.



Cor. 4. *The parallels intercepted between a plane,* MN, *and a straight line,* AB, *parallel to it, are equal.*

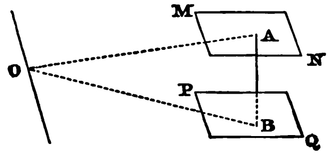
Thus, AC and BD (see diagram above) are equal, for the figure ABCD is a parallelogram.

PROPOSITION XIV.

Theorem.

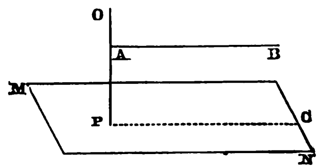
Two planes, MN, PQ, *perpendicular to the same straight line,* AB, *are parallel to each other.*

For, if they meet, let O be one point of their common intersection. Then, from O we would have two planes perpendicular to the straight line AB, which is impossible (Prop. VIII.). Hence the planes MN and PQ cannot meet. Therefore they are parallel (Def. 5).



COR. *The geometric locus of the straight lines drawn parallel to the plane* MN *through the same point,* A, *is a plane, parallel to the plane* MN.

For these lines all lie in a plane perpendicular to the line AP (Prop. VII., Cor.), which is perpendicular to the plane MN (Prop. XIII., Cor. 1).

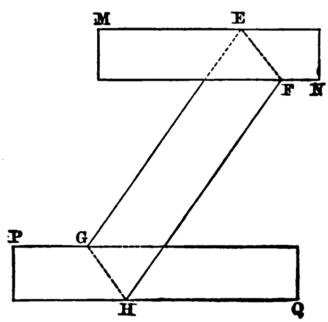


PROPOSITION XV.

THEOREM.

The intersections, EF, GH, *of two parallel planes,* MN, PQ, *with a third plane,* FG, *are parallel.*

For, if the lines EF, GH, which lie in the same plane, were not parallel, they would meet each other when produced; and therefore, the planes MN, PQ, in which these lines lie, would also meet, which is impossible (Def. 5). Hence, EF and GH are parallel.

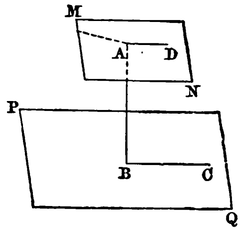


PROPOSITION XVI.

THEOREM.

Any straight line, AB, *perpendicular to a plane,* MN, *is also perpendicular to a plane,* PQ, *parallel to* MN.

Any plane, ABCD, containing AB, meets the planes MN and PQ in two parallel lines, AD, BC (Prop. XV.). But AB is perpendicular to AD by hypothesis (Def. 2). Hence, it is also perpendicular to its parallel, BC. Therefore, AB is perpendicular to any line, BC, drawn through its foot in the plane PQ, and is therefore perpendicular to that plane (Def. 2).



COR. 1. *Two planes,* A *and* B, *parallel to a third plane,* C, *are parallel to each other.*

For if a straight line be drawn perpendicular to the plane C, it will be also perpendicular to the planes A and B, which are therefore parallel to each other (Prop. XIV.).

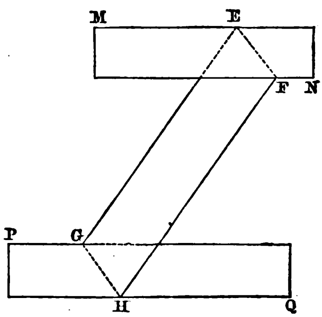
COR. 2. *Through a given point only one plane may be drawn parallel to a given plane.*

PROPOSITION XVII.

THEOREM.

The parallels, EG, FH, *included between two parallel planes,* MN, PQ, *are equal.*

Through the parallels EG, FH, pass the plane EGHF, meeting the parallel planes in EF, GH. The intersections EF, GH are parallel to each other (Prop. XV.), as are also EG, FH (by hypothesis); hence, the figure EGHF is a parallelogram; therefore, EG = FH.

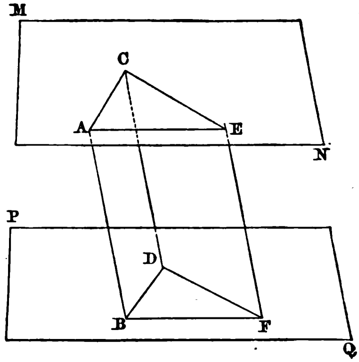


COR. Hence it follows that *two parallel planes are everywhere equidistant,* for if EG and FH are perpendicular to the two planes MN, PQ, they will be parallel to each other (Prop. XI., Cor. 1); hence they are equal.

PROPOSITION XVIII.

THEOREM.

If two angles, CAE, DBF, *not situated in the same plane, have their sides parallel and lying in the same direction, these angles will be equal, and their planes will be parallel.*



Take AC = BD, AE = BF, and join CE, DF, AB, CD, EF. Since AC is equal and parallel to BD, the figure ABDC is a parallelogram (Book I., Prop. XXXIII.). Therefore CD is equal and parallel to AB. For a similar reason EF is equal and parallel to AB; hence also CD is equal and parallel to EF; hence the figure CEFD is a parallelogram, and the side CE is equal and parallel to DF; therefore the triangles CAE and DBF are equal (Book I., Prop. XII.). Therefore the angle CAE = DBF.

Again, the planes CAE and DBF are parallel. For these planes contain the parallels CA, DB, and the parallels AE, BF, and if they could intersect each other, their common line of intersection would be parallel both to AC and AE (Prop. XIII., Cor. 3), which is absurd (Book I., Ax. 6, p. 37). Hence these planes cannot meet; they are, therefore, parallel.

COR. If two parallel planes, MN, PQ, are met by two other planes, CABD, EABF, the angles CAE, DBF formed by the intersections of the parallel planes will be equal. For (Prop. XV.) the intersection AC is parallel to BD, and AE to BF, therefore the angle CAE = DBF.

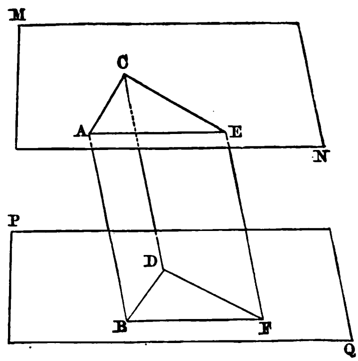
SCHOLIUM 1. We see, moreover, here a method of drawing a plane through a given point parallel to a given plane. Draw through the point two straight lines parallel to any two intersecting lines lying in the plane, and pass a plane through the two parallels thus drawn.

SCHOLIUM 2. We measure the *inclination or angle of two lines which cannot meet in space*, by the angle which is formed by drawing through the same point parallels to these two straight lines. And by this Proposition this angle is the same whatever point we take. We thus speak of lines at right angles to one another which do not meet. For example, in figure, Prop. X., the lines AP and BC are perpendicular to each other without meeting; for AP is perpendicular to a line through P, parallel to BC.

PROPOSITION XIX.

THEOREM.

If three straight lines AB, CD, EF, *not situated in the same plane, are equal and parallel, the triangles* ACE, BDF, *formed respectively by joining the extremities of these straight lines, will be equal, and their planes will be parallel.*

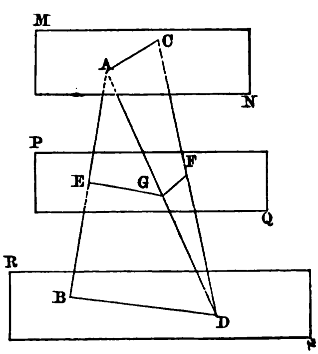


For, since AB is equal and parallel to CD, the figure ABCD is a parallelogram; therefore, the side AC is equal and parallel to BD. For a similar reason, the sides AE, BF, are equal and parallel, as well as also CE, DF; therefore the two triangles ACE, BDF are equal: and, as in the last proposition, their planes are parallel.

PROPOSITION XX.

THEOREM.

Two straight lines, AB, CD, *which are cut by three parallel planes, are cut proportionally.*



Suppose the line AB to meet the parallel planes MN, PQ, RS at the points A, E, B, and the line CD to meet the same planes at the points C, F, D; then shall we have

AE : EB : : CF : FD.

Draw AD, meeting the plane PQ in G, and join AC, EG, GF, BD; the intersections EG, BD, of the parallel planes PQ, RS, by the plane ABD, are parallel (Prop. XV.); therefore, AE : EB : : AG : GD (Book

III., Prop. XVI.); likewise, the intersections AC, GF being parallel, we have AG : GD : : CF : FD; therefore, by reason of the common ratio, AG : GD, we have

AE : EB : : CF : FD.

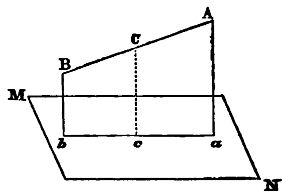
SCHOLIUM. All lines drawn from A to the plane RS, are cut proportionally by the parallel plane PQ.

PROPOSITION XXI.

THEOREM.

The projection of a straight line, AB, *on a plane,* MN, *is a straight line.*

For all the perpendiculars, Aa, Bb, Cc, let fall from the different points of the line AB on the plane MN are parallel (Prop. XI., Cor. 1), and they lie in the plane ABa, and their feet must lie in the straight line, ab, in which this plane meets the plane MN. Hence, the straight line ab is the projection of AB on the plane MN.



SCHOLIUM. (1.) When the straight line meets the plane, the point of intersection is evidently one point of its projection on the plane.

(2.) When the straight line is parallel to a plane, it is parallel to its projection on that plane.

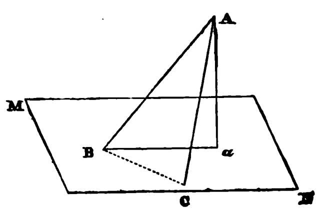
(3.) When a straight line is perpendicular to a plane, its projection on that plane is a point.

PROPOSITION XXII.

THEOREM.

The acute angle, ABa, *which a straight line,* AB, *makes with its projection,* aB, *on the plane* MN, *is smaller than the angle,* ABC, *which* AB *makes with any other line,* BC, *in the plane* MN.

For, take BC = Ba, and join AC. The two triangles ABa and ABC have the side AB common, Ba = BC, and the third side, AC, of the one greater than the third side, Aa, of the other (Prop. IX.). Therefore, the angle ABa, opposite Aa, is less than the angle ABC, opposite to AC (Book I., Prop. XI.).



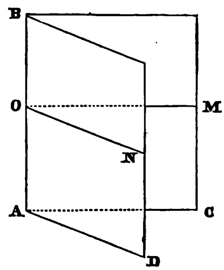
COR. The obtuse angle which a line makes with the prolongation of its projection on a plane, is greater than the angle which it makes with any other line in the plane.

SCHOLIUM. The angle which a straight line makes with its projection on the plane, is called *the angle of the straight line and the plane.* It is the complement of that which the straight line makes with the perpendicular let fall from any one of its points on the plane.

II. DIEDRAL AND SOLID ANGLES.

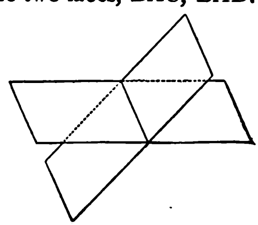
DEFINITIONS.

1. A *diedral angle*, or *diedral*, is the mutual inclination or opening of two planes, BAC, BAD, which meet each other. The planes, BAC, BAD, are called the *faces* of the diedral, and their intersection, AB, is called the *edge* of the diedral. A diedral is denoted by four letters, those of the edge being placed in the middle. Thus we say, Diedral CBAD.

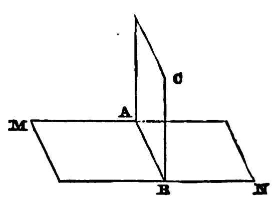


2. *The rectilineal angle* of a diedral, CBAD, is the angle, MON, formed by two lines, OM and ON, perpendicular to the edge, AB, at the same point, O, and lying respectively in the two faces, BAC, BAD.

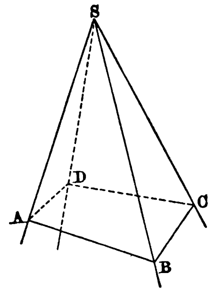
3. Two planes which intersect each other form four diedrals.



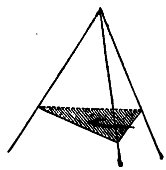
4. A plane, CAB, is *perpendicular* to another plane, MN, when it makes with it two adjacent diedrals, CABM, CABN, equal to each other. The equal diedrals are then called *right* diedrals.



5. *A solid or polyedral angle* is the angular space inclosed between several planes which meet at the same point. Thus, the solid angle S is formed by the planes ASB, BSC, CSD, DSA. The point S is called the vertex of the solid angle. The straight lines SA, SB, SC, SD are called its edges.



6. The plane angles or faces ASB, BSC, CSD, DSA, and the diedrals ASBC, BSCD, BASD, ASDC, which the faces make with each other, are called the *parts* of the solid angle. The number of these parts is always double the number of the edges.

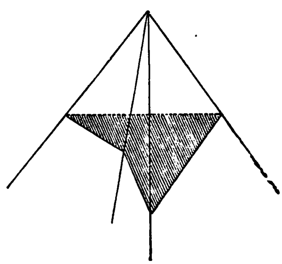


7. The simplest solid angle is the *triedral*, formed by three planes. Its parts are three plane angles and three diedral angles.

8. A rectangular triedral is one which has one of its *diedrals* right, a bi-rectangular triedral has two of its diedrals right, and a *tri-rectangular* triedral has all three of its diedrals right.

9. An *iso-edral* triedral has two of its faces or plane angles equal. An *iso-angular* triedral has two of its diedrals equal.

10. A solid angle is *equiedral* when all its faces are equal, and *equiangular* when all its diedrals are equal.

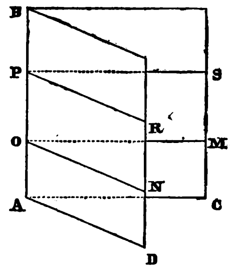


11. A solid angle is said to be *convex* when the plane of no one of its faces produced can ever cut it. If this condition is not fulfilled, the solid angle is *concave*, or has re-entrant angles. A *triedral* is always convex.

PROPOSITION XXIII.

THEOREM.

Any two rectilinear angles, MON, SPR, *of a diedral,* CBAD, *are equal.*



For, the straight lines OM and PS, being perpendicular to the edge AB in the plane BAC, are parallel. In like manner ON and PR are parallel. Hence, the angle MON = angle SPR (Prop. XVIII.). Therefore, the rectilineal angle of any diedral, CABD, is invariable, whatever be the position of its vertex on the edge, AB.

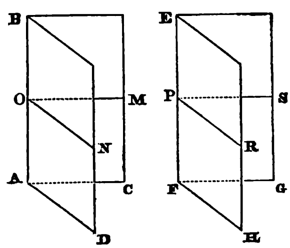
SCHOLIUM. The plane of the rectilineal angle of a diedral is perpendicular to its edge.

PROPOSITION XXIV.

THEOREM.

Two equal diedrals, CABD *and* GFEH, *have their rectilineal angles,* MON *and* SPR, *equal; and conversely.*

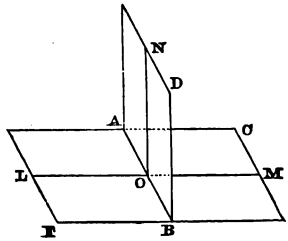
Place GFEH on CABD so that they shall coincide, and so that the point P shall fall on the point O. The sides PS and PR will coincide with OM and ON respectively, because we can only draw one perpendicular to the line AB at the point O in each of the planes BAC, BAD. Hence, the angle MON = angle SPR.



Conversely, Let MON = SPR, then shall the diedrals be equal. For, place the diedral GFEH on CABD so that the angle SPR coincides with MON. Then the edge EF, perpendicular to the plane SPR, will coincide with the edge AB, perpendicular to the plane MON, since only one perpendicular can be drawn from the point O to this plane. Thus, the plane EFG of the angle EPS will coincide with the plane BAC of the angle BOM (Prop. II., Cor. 3). For the same reason the planes FEH and BAD will coincide. Hence

diedral GFEH = diedral CABD.

COR. 1. *A right diedral angle,* CABD, *has a right rectilineal angle,* MON, *and conversely.*



For, since the diedral CBAD = diedral FABD, angle MON = angle LON. Hence, MON is a right angle (Def. 10, Book I.). Conversely, if MON is a right angle, MON = LON, and therefore CABD = FABD. Hence, CABD is a right diedral.

COR. 2. *Through a straight line*, AB, *lying in a plane*, FC, *we can always pass one plane*, AD, *perpendicular to that plane, and but one.*

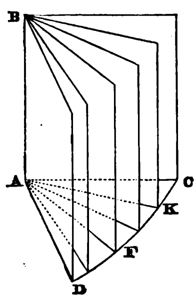
COR. 3. *All right diedrals are equal.*

PROPOSITION XXV.

THEOREM.

Two diedrals, CBAD, CBAF, *are to each other as their rectilineal angles*, CAD, CAF.

First.—Suppose the two rectilineal angles CAD, CAF to be commensurable, and that their greatest common measure, CAK, is contained five times in the first and three times in the second. So that



CAD : CAF : : 5 : 3. (1.)

The planes determined by the edge, AB, and the lines of division, will divide the diedral CBAD into five, and the diedral CBAF into three diedrals, each equal to CBAK (Prop. XXIII.). Hence

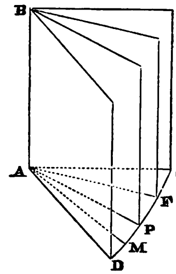
Diedral CBAD : Diedral CBAF : : 5 : 3. (2.)

And by reason of the common ratios in proportions (1) and (2) we shall have

diedral CBAD : diedral CBAF : : CAD : CAF.

Secondly.—Suppose the angles CAD and CAF to be incommensurable, then we shall still have CBAD : CBAF : : CAD : CAF. (1.)
For, if this proportion be not true let the fourth term be CAM instead of CAF. So that CBAD : CBAF : : CAD : CAM. (2.)

Divide the angle CAD into equal parts smaller than FAM, so that at least one line of division, AP, falls between AF and AM. The



angles CAD and CAP being commensurable, it follows that

CBAD : CBAP : : CAD : CAP. (3.)

Now, by reason of the common antecedents in hese two proportions (2) and (3), we shall have

CBAF : CBAP : : CAM : CAP,

which proportion is false,

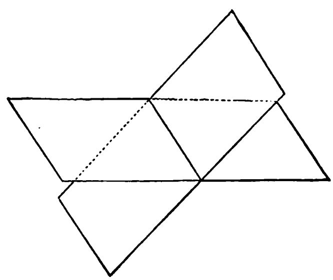
since CBAF is less than CBAP,

while CAM is greater than CAP.

Hence, the fourth term of proportion (1) cannot differ from CAF.

COR. 1. We can assume the rectilineal angle of a diedral as its measure. For the ratio of a diedral to a right diedral is equal to the ratio of its rectilineal angle to a right angle. This is usually expressed as follows: *The angle between two planes is the angle formed by two lines drawn, one in each plane, perpendicular to their common intersection at the same point.*

COR. 2. *When two planes cut each other,*



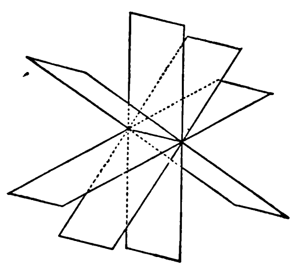
1. *The sum of the four diedrals thus formed is equal to four right diedrals.*

2. *The adjacent diedrals are supplementary.*

3. *The opposite diedrals are equal.*

For, the intersections of these planes with a third plane perpendicular to their common edge, form four rectilineal angles, which measure the four diedral angles respectively.

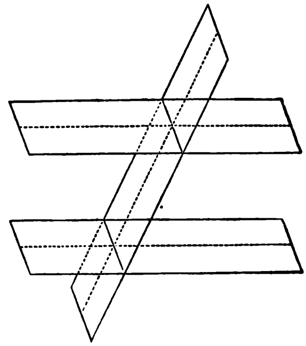
COR. 3. *When two planes cut each other,*



1. *The plane bisectors of the opposite diedrals coincide.*

2. *The plane bisectors of the adjacent diedrals are at right angles to each other.* (The same demonstration.)

COR. 4. In general, the angles formed by parallel planes and a secant plane, have the same properties as the angles formed by two parallel lines and a secant line given in (Prop. XXIV., Book I.). The reciprocals of these properties (Book I., Prop. XXV.) are only true for planes when the diedrals considered have parallel edges. The properties of plane angles demonstrated in Prop. XXVII., Book I., are also true for diedrals.

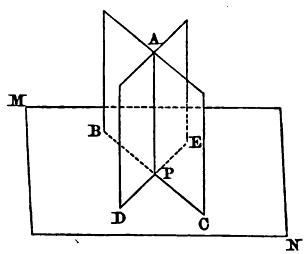


PROPOSITION XXVI.

THEOREM.

If the line AP *be perpendicular to the plane* MN, *any plane,* APB, *passed through* AP, *will be perpendicular to the plane* MN.

Let BC be the intersection of the planes AB, MN; in the plane MN draw DE, perpendicular to BP; then the line AP, being perpendicular to the plane MN, will be perpendicular to each of the two straight lines BC, DE (Def. 2); but the angle APD, formed by the two perpendiculars PA, PD to the common intersection, BP, measures the angle of the two planes (Prop. XXV., Cor. 1). Therefore, since this angle is right, the two planes are perpendicular to each other.



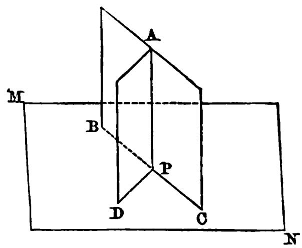
SCHOLIUM 1. This theorem can be enunciated thus: *A plane,* MN, *perpendicular to a straight line situated in a second plane, is perpendicular to that plane.*

SCHOLIUM 2. When three straight lines, such as AP, BP, DP, are perpendicular to each other, each of those lines is perpendicular to the plane of the other two, and the three planes are perpendicular to each other.

PROPOSITION XXVII.

THEOREM.

If the plane AB *is perpendicular to the plane* MN, *any line,* PA, *drawn in the plane* AB, *perpendicular to the common intersection,* PB, *will be perpendicular to the plane* MN.



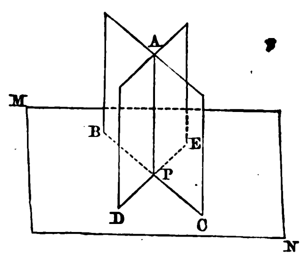
For, in the plane MN, draw the line PD perpendicular to PB; then, because the planes are perpendicular, angle APB is right; therefore, the line AP is perpendicular to the two straight lines PB, PD; therefore, it is perpendicular to their plane MN.

COR. *If the plane* AB *is perpendicular to the plane* MN, *and, at a point,* P, *of the common intersection, we erect a perpendicular to the plane* MN, *that perpendicular will be in the plane* AB; for if not, then in the plane AB we could draw a line, AP, perpendicular to the common intersection, BP, which would be at the same time perpendicular to the plane MN. Therefore, at the same point, P, there would be two perpendiculars to the plane MN, which is impossible (Prop. IV.).

PROPOSITION XXVIII.

THEOREM.

If two planes, AB, AD, *are perpendicular to a third plane,* MN, *their common intersection,* AP, *will be perpendicular to that third plane.*



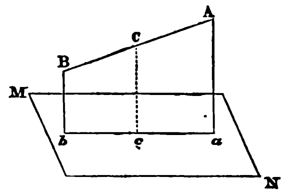
For at the point P we erect a perpendicular to the plane MN; that perpendicular must be at once in the plane AB and in the plane AD (Prop. XXVII., Cor.); therefore, it is their common intersection, AP.

PROPOSITION XXIX.

THEOREM.

Through a straight line, AB, *oblique to the plane* MN, *one plane can always be drawn perpendicular to* MN, *and but one.*

For if from any point, C, of the line AB, a perpendicular, C*c*, be let fall on the plane MN, the plane determined by AB and C*c* will be perpendicular to that plane. Moreover, it is the only plane through AB, perpendicular to MN; because if we can draw two planes through AB, perpendicular to MN, then AB, itself, must be perpendicular to that plane (Prop. XXVIII.), which is contrary to the hypothesis.

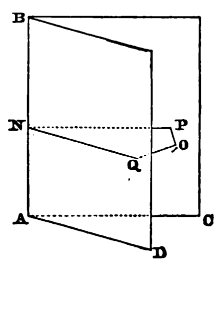


PROPOSITION XXX.

THEOREM.

The perpendiculars, OP, OQ, *let fall on the faces,* AC, AD, *of a diedral,* CABD, *from an interior point,* O, *form an angle,* POQ, *whose plane is perpendicular to the edge,* AB, *and which is the supplement of the diedral.*

The planes AC, AD, being perpendicular to the straight lines OP, OQ, are both perpendicular to the plane, POQ, of these lines (Prop. XXVI.). Hence, their common intersection, AB, is perpendicular to this plane (Prop. XXVIII.), and, conversely, the plane POQ is perpendicular to AB at N; therefore, the diedral CABD is measured by PNQ (Prop. XXV. Cor. 1). But in the quadrilateral OPNQ, OPN and OQP are right angles. Hence, the angles POQ and PNQ are supplementary—that is, POQ = 2 right angles − PNQ.

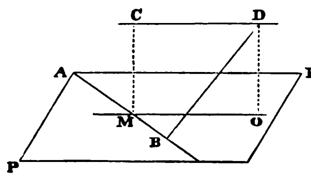


PROPOSITION XXXI.

THEOREM.

The shortest distance between two straight lines, AB *and* CD, *not situated in the same plane, is the straight line which meets them both at right angles.*

First.—Such a line can always be drawn. Through any point, A, of the line AB, draw AE parallel to CD. Then the plane P, determined by AB and AE, is parallel to CD (Prop. XIII.). Now, from any point, D, of CD, let fall the perpendicular DO, on the plane P. The intersection, OM, of the plane CDO with P, is parallel to CD (Prop. XII.). Finally, through the point of intersection, M, of OM and AB, draw MC perpendicular to MO. This line will meet CD. It is, moreover, perpendicular to the plane P, to the line AB, and to the line CD, parallel to MO. We have thus a common perpendicular, MC, to the lines CD and AB.



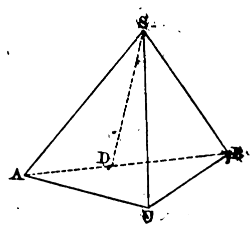
Secondly.—MC is shorter than any other line, DB, joining AB and CD. For DO, perpendicular to P, is shorter than the oblique line DB (Prop. IX.), and MC = DO. Hence, MC is shorter than DB.

PROPOSITION XXXII.

THEOREM.

In a triedral, the sum of any two of the plane angles is greater than the third.

This proposition requires a demonstration only when the plane angle which is compared to the sum of the other two is greater than either of them. Suppose, then, the triedral S to be formed by three plane angles, ASB, ASC, BSC, whereof the angle ASB is the greatest; we are to show that ASB < ASC + BSC.



In the plane ASB make the angle BSD = BSC; draw the straight line ADB at pleasure; and, having taken SC = SD, join AC, BC.

The two sides BS, SD are equal to the two sides BS, SC; the angle BSD = BSC; therefore, the triangles BSD, BSC are equal; therefore BD = BC. But AB < AC + BC; taking BD from one side, and from the other its equal BC, there will remain AD < AC. The two sides AS, SD are equal to the two sides AS, SC; the third side AD is less than the third side AC; therefore (Book I., Prop. XI.), the angle ASD < ASC. Adding BSD = BSC, we shall have ASD + BSD or ASB < ASC + BSC.

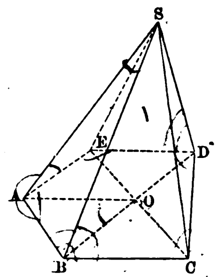
COR. *In a triedral, one of the plane angles is always greater than the difference of the other two.* For, since ASC + BSC > ASB, we have ASC > ASB − BSC.

PROPOSITION XXXIII.

THEOREM.

The sum of the plane angles which form a solid angle is always less than four right angles.

Cut the solid angle S by any plane, ABCDE; from a point, O, in that plane, draw to the several angles the lines OA, OB, OC, OD, OE.



The sum of the angles of the triangles ASB, BSC etc., formed about the vertex S, is equivalent to the sum of the angles of an equal number of triangles, AOB, BOC, etc., formed about the vertex O. But at the point B, the angles ABO, OBC, taken together, make the angle ABC less than the sum of the angles ABS, SBC (Prop. XXXII); in the same manner, at the point C, we have BCO + OCD < BCS + SCD; and so with all the angles of the polygon ABCDE. Whence, it follows, that the sum of the angles at the bases of the triangles whose vertex is in O, is less than the sum of the angles at the bases of the triangles whose vertex is in S; hence, to make up the deficiency, the sum of the angles formed about the point O is greater than the sum of the angles about the point S. But the sum of the angles about the point O is equal to four right angles (Book I., Prop. VI., Scholium); therefore, the sum of the plane angles which form the solid angle S is less than four right angles.

SCHOLIUM. This demonstration supposes that the solid angle is convex; if it were otherwise, the sum of the plane angles would be unlimited, and might be of any magnitude.

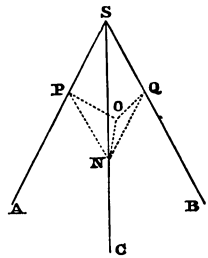
PROPOSITION XXXIV.

THEOREM.

An iso-edral triedral is also iso-angular, and conversely.

Let ASC = BSC; from a point, N, of the edge SC, let the planes

NPO, NQO be drawn perpendicular to the edges SA, SB; they will be perpendicular to the plane ASB (Prop. XXVI.). Their intersection, NO, being perpendicular to this plane (Prop. XXVIII.), the angles NOP and NOQ are right (Def. 2). The two right angled triangles NPS and NQS have the angles NSP = NSQ by hypothesis, and the hypothenuse SN common. They are, therefore, equal; hence NP = NQ.



Hence, the two right angled triangles NOP and NOQ have the hypothenuse NP = hypothenuse NQ, and NO common. They are, therefore, equal, and angle NPO = NQO. But these angles measure the diedrals CSAB and CSBA (Prop. XXV., Cor. 1). Hence CSAB = CSBA, and the triedral is iso-edral.

Conversely. Let the diedral CSAB = diedral CSBA. Then the triangles NPO and NQO have the angle NPO = angle NQO by hypothesis, and the side NO common. They are, therefore, equal, and NP = NQ. Then the triangles NSP, NSQ, having the hypothenuse NS common and NP = NQ, are equal, and hence NSP = NSQ, or ASC = CSB, and the triedrals are iso-angular.

Cor. 1. *An equiedral triedral is equiangular, and conversely.*

Cor. 2. If two diedrals, BSAC, ASBC, of a triedral are right, the plane angles opposite to them are also right. For the faces ASC, BSC, being in that case perpendicular to ASB, their common intersection, SC, is also perpendicular to that plane, and hence CSB and CSA are right angles (Def. 2).

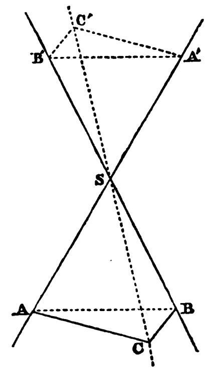
Cor. 3. *The three plane angles of a tri-rectangular triedral are right.*

PROPOSITION XXXV.

Theorem.

Two triedrals may have all the parts of the one equal to all the parts of the other, each to each, and yet not coincide when placed the one upon the other.

Prolong the three edges of any triedral, SABC, in such manner as to form the triedral SA′B′C′ opposite to SABC, at the vertex. Then, in these two triedrals,



ASB = A′SB′, BSC = B′SC′, CSA = C′SA′,

since they are opposite or vertical plane angles. Also

diedral CSAB = C′SA′B′
diedral ASBC = A′SB′C′
diedral BSCA = B′SC′A′,

because they are diedrals opposite at the same edge. Thus the two triedrals have the six parts of the one equal to the six parts of the other, each to each. Nevertheless they cannot coincide. For, if we apply A′SB′ to ASB so that SA′ falls along SA, and SB′ along SB, the edges SC′, SC will be on different sides of ASB. If we let SA′ fall along SB, and SB′ along SA, the equal parts do not then correspond. Hence, the triedrals S-ABC and S-A′B′C′ do not admit of superposition.

Two triedrals thus related, that is, having all their parts equal, but arranged in different order, are called *symmetrical triedrals*, or *triedrals equal by symmetry*.

SCHOLIUM. *An iso-edral triedral has no symmetrical triedral.*

For, if the angle ASB = angle BSC, we can effect the superposition of the triedrals S-ABC, S-A′B′C′, by placing A′SB′ on its equal, ASB, so that SA′ falls along SB and SB′ on SA. Hence, *two iso-edral triedrals having all the parts of the one equal to all the parts of the other, each to each, coincide when applied one to the other.*

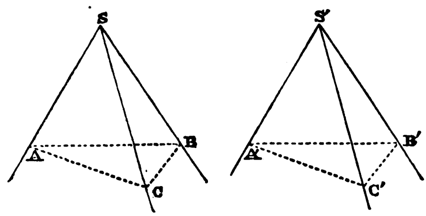
PROPOSITION XXXVI.

THEOREM.

Two triedrals are either equal or symmetrical when a diedral and the two plane faces which contain it in the one are equal to a diedral and the two plane faces which contain it in the other, each to each.

If the parts are arranged in the same order, we demonstrate the equality of the two triedrals by coincidence, as in Prop. VII., Book I., Plane Geometry.

If the parts are not placed alike in the two figures, we cause the second triedral to coincide with the triedral which is opposite to the first at the vertex, and thence conclude that the two given triedrals are symmetrical.



PROPOSITION XXXVII.

THEOREM.

Two triedrals are either equal or symmetrical, if two diedrals and the interjacent plane face of the one are equal to two diedrals and the interjacent plane face of the other, each to each.

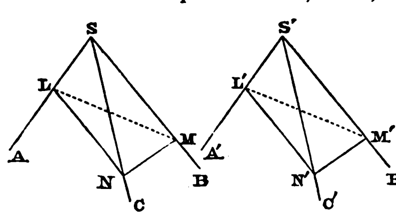
The demonstration is similar to that in Proposition VIII., Book I. We cause thus the second triedral to coincide either with the first or with its symmetrical triedral.

PROPOSITION XXXVIII.

THEOREM.

Two triedrals are either equal or symmetrical when the plane angles of the one are equal to the plane angles of the other, each to each.

Let the angle ASB = A'S'B', BSC = B'S'C', CSA = C'S'A'. Take SL = S'L', and at the point L erect in the planes ASB, ASC, respectively, two perpendiculars, LM, LN, to the edge SA, and join MN. Similarly, in the planes A'S'B', A'S'C', draw two perpendiculars, L'M', L'N', to the edge S'A', and join M'N'.



The triangles SLM and S'L'M' are equal, having SL = S'L' by construction, angle ASB = A'S'B' by hypothesis, and SLM = S'L'M', being right angles. Hence

$$SM = S'M', \text{ and } LM = L'M'.$$

Similarly, triangle SLN = triangle S′L′N′, and hence

$$SN = S'N', \ LN = L'N'.$$

Therefore, the triangles MSN and M′S′N′ are equal (Prop. VII., Book I.), and hence MN = M′N′.

Hence the two triangles LMN and L′M′N′ are equal (Book I., Prop. XII.), and therefore the angle MLN = M′L′N′. But these angles measure the diedrals BSAC, B′S′A′C′, respectively (Prop. XXV., Cor. 1). Hence, BSAC = B′S′A′C′. Hence (Prop. XXXVI.) the two triedrals are either equal or symmetrical.

SCHOLIUM. This demonstration fails when two plane angles, ASB and ASC, are right (as will be seen from the construction). But, in that case, the edge SA is perpendicular to the plane BSC, and the edge S′A′ to the plane B′S′C′. Hence, when we place B′S′C′ on BSC, the perpendicular S′A′ will coincide with SA (Prop. IV.), and thus the two triedrals will coincide.

PROPOSITION XXXIX.

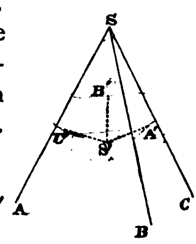
THEOREM.

The perpendiculars, S′C′, S′A′, S′B′, *drawn from an interior point,* S′, *of a triedral,* S, *on its three faces,* ASB, ASC, BSC, *form a second triedral,* S′, *the plane angles and diedrals of which are respectively the supplements of the diedrals and plane angles of the triedral* S.

First, we have (Prop. XXX.)

angle B′S′C′ = 2 right angles — diedral BSAC
angle C′S′A′ = 2 right angles — diedral CSBA
angle A′S′B′ = 2 right angles — diedral ASCB.

Moreover, the three planes, B′S′C′, C′S′A′, A′S′B′ are perpendicular, respectively, to the edges, SA, SB, SC (Prop. XXX.); hence, conversely, the triedral S has the same properties with regard to the triedral S′ that S′ has with regard to S.



Hence, we have

angle ASB = 2 right angles — diedral A′S′C′B′
angle BSC = 2 right angles — diedral B′S′A′C′
angle CSA = 2 right angles — diedral C′S′B′A′.

Scholium 1. These triedrals, S and S′, are said to be *supplementary* to each other, or the two taken together are called *supplementary* triedrals.

Scholium 2. If perpendiculars be drawn, at S′, to the planes ASB, ASC, BSC, so that they shall fall on the same sides of these planes with the edges SC, SB, SA, respectively, opposite to these planes, the triedral thus formed will be also *supplementary* to S′-ABC, since it will fulfil all the conditions of S′-A′B′C′.

PROPOSITION XL.

Theorem.

Two triedrals are either equal or symmetrical when the three diedrals of the one are equal to the three diedrals of the other, each to each.

If we construct the triedrals supplementary to these two, these new triedrals will (Prop. XXXIX.) have the plane angles of the one equal to the plane angles of the other, and hence (Prop. XXXVIII.) the diedrals of the one will be equal to the diedrals of the other, each to each. Therefore (Prop. XXXIX.), the given triedrals will have their plane angles equal, each to each. And hence, since all the parts of the one are equal to all the parts of the other, each to each, these two triedrals will either be equal or symmetrical.

PROPOSITION XLI.

Theorem.

The sum of the three diedrals of a triedral is always greater than two right angles and less than six.

Since each diedral of a triedral is equal to two right angles, *minus* the opposite plane angle of the supplementary triedral, the sum of the three diedrals is equal to six right angles, *minus* the sum of the plane angles of the supplementary triedral. But this sum (Prop. XXXIII.) is less than four right angles. Therefore, the difference above is greater than two right angles.

The second part is evident, since each diedral is less than two right angles.

EXERCISES ON BOOK V.

THEOREMS.

1. If we draw a plane perpendicular to a straight line at its middle point, 1st. Any point on this plane will be equally distant from the extremities of the line. 2d. Any point not on this plane will be unequally distant from these extremities.

2. If any angle, BAC, revolve about its side AB, returning to its first position, every point of AC will describe a circumference, the plane of which is perpendicular to AB.

3. Two planes which are parallel to the same straight line, are either parallel to each other or their intersection is parallel to this line.

4. Two planes which are parallel to two planes which intersect each other, will also intersect, and the line of intersection of the first planes will be parallel to the line of intersection of the second.

5. If from the projections, P and Q, of the same point, O, upon two planes which intersect, perpendiculars be drawn to the line of intersection, they will meet it at the same point.

6. Conversely, if the perpendiculars from two points, P and Q, on the common intersection of the two planes in which they lie, meet that intersection at the same point, then P and Q are the projections of the same point, O, in space on these two planes.

7. Two planes perpendicular to the same plane, P, and containing two lines, AB, A′B′, parallel to each other, are parallel. Show also that this is not true if the lines AB and A′B′ are perpendicular to the plane P.

8. The projections of two parallel lines on the same plane are parallel. Show also that the converse of this is not true.

9. The angles which two parallel lines oblique to a plane make with that plane, are equal.

10. If two planes, P and Q, intersect each other, a straight line in P, perpendicular to their common intersection, makes a greater angle with the plane Q than any other straight line drawn in P.

NOTE.—This line through any point in P, perpendicular to the common intersection, is called the *line of greatest inclination* to the plane Q.

11. A straight line and plane which are perpendicular to the same plane are parallel.

12. Any point on the plane bisector of a diedral angle is equally distant from the faces of the diedral, and any point within the diedral not on this plane bisector is unequally distant from the faces of the diedral.

13. The perpendiculars let fall from the same point on planes which have a common intersection are all in the same plane. Show also that this is true of perpendiculars let fall from the same point on planes whose intersections are parallel.

14. If, through one of the diagonals of a parallelogram, we pass any plane, the perpendiculars let fall from the extremities of the other diagonal on this plane will be equal.

15. In any triedral, the three planes perpendicular to the three faces, and containing the bisectrices of the plane angles, meet in the same straight line.

16. The three planes which bisect the diedral angles of a triedral meet in the same straight line.

17. The three planes containing the three edges of a triedral and the bisectrices of the opposite plane angles meet in the same line.

18. The three planes drawn through the three edges of a triedral, perpendicular to the opposite faces, meet in the same line.

19. Show that if, in the place of the triedral, we have three planes whose intersections two and two are parallel, the last four theorems are equally true.

20. If through the middle point of the perpendicular to two straight lines, not situated in the same plane, we draw a plane parallel to these two lines, this plane will bisect every line which joins the two lines.

21. When a straight line is parallel to a plane, the shortest distance from this line to any line of the plane not parallel to the first line is constant.

22. If three straight lines, A, B, C, not situated in the same plane, lie two and two in the same planes, these lines either all meet in the same point or are all parallel.

23. Every section of a rectangular triedral, by a plane perpendicular to one of its edges, is a right angled triangle.

24. In any triedral the greater plane face lies opposite the greater diedral, and conversely.

Geometric Loci.

1. Find the locus of all the points at a given distance from a given plane.

2. Find the locus of all the points, any one of which is equally distant from two given points.

3. Find the locus of all the points in space, any one of which is equally distant from two given straight lines which lie in the same plane.

4. Find the locus of all the points, any one of which is equidistant from two given planes.

5. Find the locus of all the points, any one of which is equidistant from three given points.

6. Find the locus of all the points, any one of which is at equal distances from three given straight lines situated in the same plane.

7. Find the locus of all the points, any one of which is equally distant from three given planes.

8. Find the locus of all the points, any one of which is equally distant from the three edges of a given triedral.

9. Find the locus of the points, any one of which is equidistant from two given points, and also equidistant from two given straight lines which lie in the same plane.

10. Find the locus of the points, any one of which is equidistant from two given points and also from two given planes.

11. Find the locus of the points, any one of which is equidistant from two given straight lines in the same plane, and also from two given planes.

Note.—In problems 9, 10, and 11, we determine a locus by the intersection of two loci, as in a plane we solve a determinate problem by the intersection of two loci.

12. Find the locus of two points, the difference of the squares of the distances of each one of which from two given points is constant.

13. Find the locus of the points in space, any one of which is equally distant from all points of the circumference of a circle.

14. Find the locus of all the points in a given plane, which are at a given distance from a given point, A, without the plane.

15. Find the locus of the points in a plane such that the sum of

the squares of each one of them from two given points, A and B, without the plane, is constant.

16. Find the locus of the points in a given plane, the difference of the squares of the distances of each one of which from two given points, A and B, without the plane, is constant.

17. Find the locus of the feet of the perpendiculars drawn from a given point, A, without a given plane, to the different straight lines drawn through a given point, B, in the plane.

PROBLEMS.

1. Through a given point draw a straight line parallel to two given planes.

2. Through a given point drawn a plane parallel to two given straight lines.

3. Through a given point draw a plane perpendicular to two given planes.

4. Through a given point draw a straight line which shall meet two given straight lines not situated in the same plane.

5. Draw a line parallel to a given straight line which shall meet two given straight lines not situated in the same plane.

6. Find a point equidistant from four given points not in the same plane. Discuss the problem if the four points are all in the same plane.

7. Find upon a straight line a point such that the difference of the squares of its distances from two given points is a given square.

8. Through a given straight line draw a plane which shall be parallel to a given straight line.

9. In a given plane and through a given point in this plane, draw a straight line perpendicular to a straight line in space.

10. Given a plane, P, and two points, A and B, situated on the same side of the plane, find a point, M, in P, such that the sum of the distances AM, BM shall be the least possible.

11. Given a plane, P, and a triangle, ABC, find the point of the plane equidistant from the three points A, B, C.

12. Cut a quadriedral angle by a plane, so as to make the section a parallelogram.

BOOK VI.

POLYEDRONS.

DEFINITIONS I.

1. The name *solid polyedron*, or simply *polyedron*, is given to every solid terminated by planes or plane faces. (These planes will themselves be terminated, it is evident, by straight lines.)

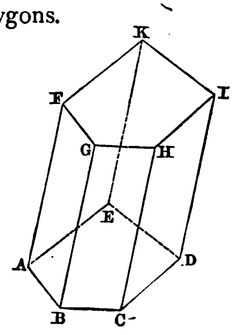
The polyedron which has four faces, is named a *tetraedron;* that which has six, a *hexaedron;* that which has eight, an *octaedron;* that which has twelve, a *dodecaedron;* that which has twenty, an *icosaedron,* and so on.

The tetraedron is the simplest of all polyedrons; because at least three planes are required to form a solid angle, and these three planes leave an opening which requires at least a fourth plane to close it.

2. The common intersection of two adjacent faces of a polyedron is called the *side* or *edge* of the polyedron.

3. A *regular polyedron* is one whose faces are all equal regular polygons, and all whose solid angles are equal to each other. There are five such polyedrons. (*See Appendix to Book VI.*)

4. The *prism* is a solid bounded by several parallelograms, terminated at both ends by equal and parallel polygons.



To construct this solid let ABCDE be any polygon; then if in a plane parallel to ABC, the lines FG, GH, HI, etc., be drawn, equal and parallel to the sides AB, BC, CD, etc., thus forming the polygon FGHIK, equal to ABCDE; if, in the next place, the vertices of the angles in one plane be joined with the homologous vertices in the other, by the straight lines AF, BG, CH, etc., the faces ABGF, BCHG, etc., will be parallelograms, and the solid ABCDEFGHIK, thus formed, will be a prism.

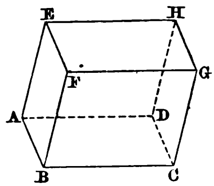
5. The equal and parallel polygons ABCDE, FGHIK are called the *bases of the prism;* the parallelograms, taken together, constitute

the *lateral* or *convex surface of the prism;* the equal straight lines AF, BG, CH, etc., are called the *sides of the prism.*

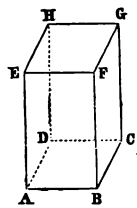
6. The *altitude of a prism* is the distance between its two bases, or the perpendicular let fall from a point in the upper base on the plane of the lower base.

7. A *prism* is *right* when the sides, AF, BG, etc., are perpendicular to the planes of the bases; and then each of them is equal to the altitude of the prism, and the faces are rectangles. In any other case the prism is *oblique*, and the altitude less than the side.

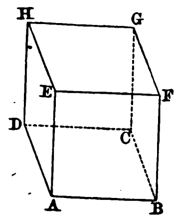
8. A *prism* is *triangular*, *quadrangular*, *pentagonal*, *hexagonal*, etc., according as its base is a triangle, a quadrilateral, a pentagon, a hexagon, etc.



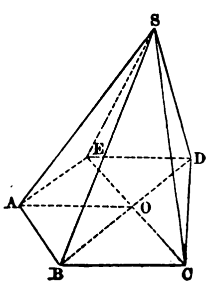
9. The prism whose base is a parallelogram has all its faces parallelograms. It is called a *parallelopipedon.* Parallelopipedons are *right* or *oblique.* The faces of the right parallelopipedon are rectangles.



10. Among right parallelopipedons we distinguish the *rectangular parallelopipedon,* whose bases are rectangles as well as its faces.



11. Among rectangular parallelopipedons we distinguish the *regular hexaedron* or *cube,* bounded by six equal squares. The lengths of three edges, AB, AD, AE, of a rectangular parallelopipedon, which meet in the same vertex, are called the *dimensions* of the parallelopipedon. The dimensions of a cube are equal.

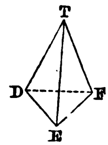


12. A *pyramid* is the solid bounded by several triangular planes, proceeding from the same point, S, and terminating in different sides of the same polygon, ABCDE.

The polygon ABCDE is called the *base* of the pyramid; the point S its vertex, and the triangles ASB, BSC, etc., taken together, form the *convex* or *lateral surface* of the pyramid.

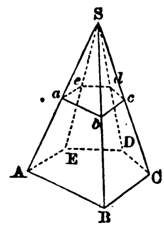
13. The *altitude* of a pyramid is the perpendicular let fall from the vertex upon the plane of the base (produced, if necessary).

14. A pyramid is *triangular*, *quadrangular*, etc., according as its base is a triangle, quadrilateral, etc. Among these we note, especially, the *triangular pyramid* or *tetraedron* (mentioned in Def. 1.), a figure with *four* triangular faces, *four* vertices, and *six* edges.



15. A pyramid is *regular* when its base is a regular polygon, and when, at the same time, a perpendicular let fall from the vertex on the plane of the base passes through the centre of the base; this line is called the axis of the pyramid.

16. The *frustum* of a pyramid or *truncated* pyramid, is the part of the pyramid which remains when any part towards the vertex is cut off by a plane parallel to the base. Thus ABCDE, *abcde*, is a frustum of a pyramid, S, ABCDE.



17. The *diagonal* of a polyedron is the straight line which joins the vertices of two solid angles which are not adjacent.

18. By the *vertices* of a polyedron, we mean the points situated at the vertices of its different solid angles.

NOTE.—The only polyedrons we intend at present to treat of, are polyedrons with *salient* angles, or *convex* polyedrons. They are such that their surface cannot be intersected by a straight line in more than two points. In polyedrons of this kind, the plane of any face, when produced, can in no case cut the solid; the polyedron, therefore, cannot be in part above the plane of any face and part below it. It must be wholly on the same side of this plane.

19. By the *volume* or *solidity* of a polyedron, we mean its magnitude or extent.

PROPOSITION I.

THEOREM.

Two polyedrons having the same number of vertices, and these vertices being the same points, will coincide.

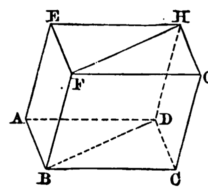
For, suppose one polyedron to be already constructed; if a second is to be formed, having the same vertices, and in the same number, the planes of the latter cannot all pass through the same points with those of the former, else the two polyedrons will not differ from each other; but if they do not all pass through the same points with the planes of the first, some of the new planes must cut the first polyedron, one or more of whose vertices will therefore lie above these planes, and one or more below, which cannot be the case in a convex polyedron: hence, if two polyedrons have the same vertices, and in the same number, they must necessarily coincide.

PROPOSITION II.

THEOREM.

In every parallelopipedon the opposite faces are equal and parallel.

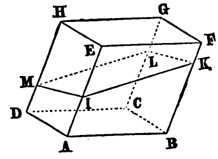
By the definition of this solid, the bases, ABCD, EFGH, are equal parallelograms, and their sides are parallel; it remains then to show that the same is true for any two opposite lateral faces, such as AEHD, BFGC. Now, AD is equal and parallel to BC, since the figure ABCD is a parallelogram. For a like reason AE is equal and parallel to BF; hence the angle DAE is equal to the angle CBF (Book V., Prop. XVIII.), and the plane DAE parallel to CBF; hence also the parallelogram DAEH is equal to the parallelogram CBFG. In the same way it may be shown that the opposite parallelograms, ABFE, DCGH, are equal and parallel.



COR. 1. Since the parallelopipedon is a solid bounded by six planes, of which those lying opposite to each other are equal and parallel, it follows that *any two opposite faces may be taken for the bases of the parallelopipedon.*

COR. 2. *If a plane cutting a parallelopipedon meets two opposite faces, the section will be a parallelogram.*

For the opposite sides of the section IKLM are parallel, being the intersections of the same plane by two parallel planes; hence, IKLM is a parallelogram.

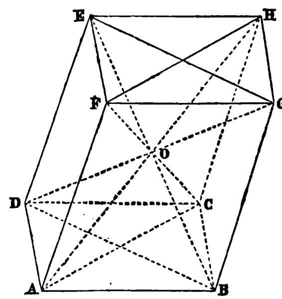


SCHOLIUM.—If three straight lines, AB, AE, AD, passing through the same point, A, and making given known angles with each other, are known, a parallelopipedon may be constructed on them. For this purpose a plane must be passed through the extremity of each line, parallel to the plane of the other two; that is, through the point B, a plane parallel to DAE, through the point D, a plane parallel to BAE, and through the point E, a plane parallel to BAD. The mutual intersections of these planes will form the parallelopipedon required.

PROPOSITION III.

THEOREM.

In every parallelopipedon the opposite solid angles are symmetrical; and the diagonals drawn through the vertices of these angles bisect each other.



First.—Compare the solid angle A with its opposite one, H; the angle DAF, equal to DEF, is also equal to CHG; the angle BAF = BGF = CHE; and the angle BAD = BCD = GHE. Hence, the three plane angles which form the solid angle A are respectively equal to the three plane angles which form the solid angle H; moreover, it is easy to see that their arrangement in one is different from that in the other; hence, the two solid angles A and H are symmetrical (Book V., Prop. XXXV.).

Secondly.—Let two diagonals, FC, AH, be drawn respectively through opposite vertices; since AF is equal and parallel to CH, the figure AFHC is a parallelogram; hence, the diagonals FC, AH will mutually bisect each other. It may be shown in the same manner that the diagonal FC, and another, EB, also bisect each other; hence, the four diagonals mutually bisect each other in a common point.

SCHOLIUM 1. The opposite solid angles of a rectangular parallelopipedon, being equiedral triedrals, are not symmetrical but equal.

SCHOLIUM 2. The point, O, of intersection of the four diagonals of a parallelopipedon may be regarded as the *centre* of the parallelopipedon. For, any line, MN, passing through O, and terminating in the surface of the parallelopipedon, is bisected at the point O. For the plane determined by this line and the diagonal AH will cut the opposite faces, ABGF and DCHE, in parallel lines, AM and HN; the two triangles AOM, HON are therefore equal, and OM = ON. (The figure is easily constructed.)

PROPOSITION IV.

THEOREM.

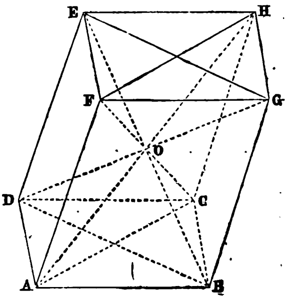
In any parallelopipedon the sum of the squares of the four diagonals is equal to the sum of the squares of the twelve edges.

The parallelograms

ACHF, BDEG, ABCD

give (Book III., Prop. XIV., Cor.),

1. $AH^2 + CF^2 = 2AF^2 + 2AC^2$
2. $BE^2 + DG^2 = 2DE^2 + 2BD^2$
3. $AC^2 + BD^2 = 2AB^2 + 2AD^2$;

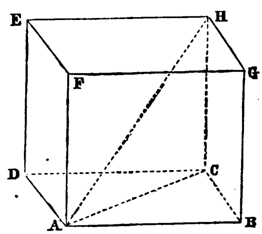


adding these equations, after multiplying the third by two, noting that DE = AF, and cancelling terms common to both sides, we have

$$AH^2 + BE^2 + CF^2 + DG^2 = 4AB^2 + 4AD^2 + 4AF^2,$$

that is, the sum of the squares of the twelve edges, since they are equal to each other four and four.

COR. 1. *In a rectangular parallelopipedon the square of a diagonal is equal to the sum of the squares of three edges which meet in the same vertex.*



For, since each of the parallelograms, as ACHF, which determine the diagonals is in that case a rectangle, the four diagonals are all equal, that is, AH = BE = CF = DG.

Hence, $$4AH^2 = 4AB^2 + 4AD^2 + 4AF^2,$$

or $$AH^2 = AB^2 + AD^2 + AF^2.$$

COR. 2. *The diagonal of a cube is to its edge as $\sqrt{3}$ is to 1.* For, since, in the cube, AB = AD = AF, we have

$$AH^2 = 3AF^2, \text{ whence } AH = AF\sqrt{3}.$$

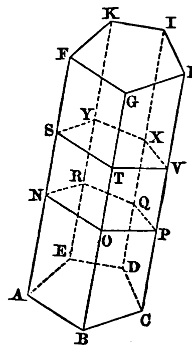
$$\therefore AH : AF :: \sqrt{3} : 1.$$

PROPOSITION V.

THEOREM.

In every prism, ABCI, *the sections,* NOPQR, STVXY, *formed by parallel planes, are equal polygons.*

For the sides NO, ST are parallel, being the intersections of two parallel planes by a third plane, ABGF. These same sides, NO, ST, are included between the parallels NS, OT, which are sides of the prism; hence, NO is equal to ST. For like reasons, the sides, OP, PQ, QR, etc., of the section NOPQR are respectively equal to the sides, TV, VX, XY, etc., of the section STVXY. And since the equal sides are at the same time parallel, it follows that the angles, NOP, OPQ, etc., of the first section, are respectively equal to the angles, STV, TVX, etc., of the second (Book V., Prop. XVIII.). Hence, the two sections, NOPQR, STVXY, are equal polygons.



COR. *Every section of a prism made by a plane parallel to its base, is equal to that base.*

SCHOLIUM 1. A section of a prism made by a plane perpendicular to its lateral edges is called the *right section.*

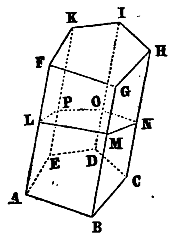
SCHOLIUM 2. A *truncated prism* is a part of a prism cut off by a plane *not parallel* to the base.

PROPOSITION VI.

THEOREM.

The convex surface of a prism is equal to the product of the perimeter of its right section by its lateral edge.

Let LMNOP be the right section of the prism AH. The sides,



LM, MN, NO, etc., of this section are the altitudes of the parallelograms, ABGF, BGHC, etc., which form the convex surface of the prism. And these parallelograms have then for bases the equal lateral edges, AF, BG, CH, etc., of the prism. The sum of their measures will then be

$$AF \times LM + BG \times MN \cdot \cdot \cdot \cdot \cdot + EK \times PL,$$

or, since, $AF = BG = \cdot \cdot \cdot \cdot \cdot EK,$

$$\text{convex surface} = AF(LM + MN + NO + OP + PL)$$
$$= AF \times \text{perimeter of right section.}$$

Cor. *The convex surface of a right prism is equal to the product of the perimeter of its base by its altitude.*

Scholium 1. To obtain *the whole surface of a prism we add to the convex surface found above, twice the area of the base.*

Scholium 2. The whole surface of a rectangular parallelopipedon whose three edges from the same vertex are a, b, and c, is $2(ab + ac + bc)$. The whole surface of a cube whose edge is c, is $6c^2$.

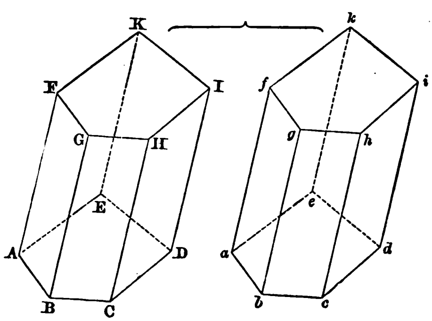
Scholium 3. If p = perimeter of the base, r = the apothem of the same, and h the altitude of a right prism with regular base, its whole surface $= ph + 2p \times \frac{r}{2}$, or $p(h + r)$ (Book IV., Prop. IX.).

PROPOSITION VII.

Theorem.

Two prisms are equal when a solid angle in each is contained by three planes which are respectively equal and similarly placed.

Let the base ABCDE be equal to the base *abcde*, the parallelogram ABGF equal to the parallelogram *abgf*, and the parallelogram BCHG equal to the parallelogram *bchg*, then will the prism ABCI be equal to the prism *abci*.



For, lay the base ABCDE

upon its equal, *abcde*; these two bases will coincide. But the three plane angles which form the solid angle B are respectively equal to the three plane angles which form the solid angle *b*, namely, ABC = *abc*, ABG = *abg*, and GBC = *gbc*; they are also similarly situated; hence the solid angles B and *b* are equal (Book V., Prop. XXXVIII.), and therefore the side BG will fall on its equal, *bg*. It is likewise evident that, by reason of the equal parallelograms, ABGF, *abgf*, the side GF will fall on its equal, *gf*, and in the same manner, GH on *gh*; hence, the upper base, FGHIK, will exactly coincide with its equal, *fghik*, and the two solids will be identical, since they have the same vertices.

PROPOSITION VIII.

THEOREM.

Two right prisms which have equal bases and the same altitude are equal.

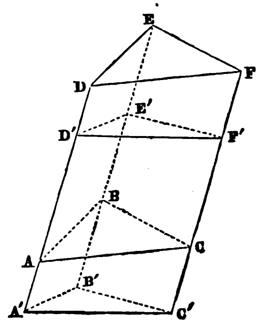
For, if we make the lower bases coincide, the lateral edges at the coinciding vertices will coincide (Book V., Prop. IV.); and since these are equal to the given altitude, the upper bases of the prisms will also coincide (Book V., Prop. VII.). Therefore, the prisms are equal.

SCHOLIUM. The preceding demonstration applies to the case of two right truncated prisms of the same base, whose corresponding lateral edges are equal. Hence, *two right truncated prisms which have equal bases and equal corresponding lateral edges are equal.*

PROPOSITION IX.

THEOREM.

Every oblique triangular prism is equivalent to the right triangular prism which has for its base the right section of the oblique prism, and for its altitude the lateral edge of the oblique prism.



Let ABCDEF or AF be the oblique prism. Through the point E′, of the edge BE, draw a right section, D′EF′. Prolong the edge BE, below the base ABC, till BB′ = EE′, and through B′ draw a plane parallel to the right section. The intersection of this plane with the prolongations of the faces of the prism will form a triangle, A′B′C′, equal to the triangle D′E′F′, and A′B′C′D′E′F′ will be a right prism, having for its base the right section of the oblique prism, AH, and for its altitude the lateral edge, BG. For since BB′ = EE′, B′E′ is equal to BE.

The prism AF = prism A′F′, because the part ABCD′E′F′ is common to both, and the truncated right prism A′B′C′ABC is equal to the truncated right prism D′E′F′DEF (Prop. VIII., Scholium).

SCHOLIUM. This proposition is equally true of any polygonal oblique prism.

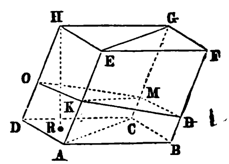
PROPOSITION X.

THEOREM.

The two triangular prisms into which a parallelopipedon is divided by a plane passing through its opposite diagonal edges are equivalent.

First.—If the parallelopipedon is a right one, the proposition is evident (Prop. VIII.), since the two prisms will be right prisms with equal bases and altitudes.

Secondly.—Let AG be an oblique parallelopipedon. The plane AEGC, containing the opposite edges, AE and GC, divides this parallelopipedon into two triangular prisms, ABCEFG, ACDEGH, which we are to prove equivalent. Draw the right section of the parallelopipedon AG. This section, OKLM, is a parallelogram (Prop. II., Cor. 2), and the two triangles, KLM and KMO, into which it is divided by the diagonal, KM, are respectively the *right sections* of the prisms ABCEFG, ACDEGH.



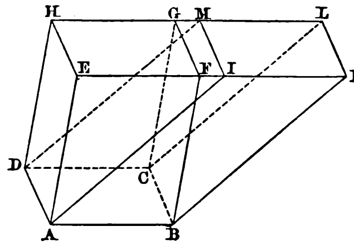
The triangular prism ABCEFG is equivalent to the right prism which has KLM for its base and AE for its altitude (Prop. IX.), the triangular prism ACDEGH is equivalent to the right prism which has KMO for its base and AE for its altitude. But these two right prisms are equivalent (Prop. VIII.). Therefore, the two triangular prisms ABCEFG, ACDEGH are equivalent, and each one of them is one half of the parallelopipedon AG.

PROPOSITION XI.

THEOREM.

If two parallelopipedons, AG, AL, *have a common base,* ABCD, *and their upper bases,* EFGH, IKLM, *in the same plane, and between the same parallels,* EK, HL, *they will be equivalent.*

There may be three cases, according as EI is greater than, equal to, or less than EF; but the demonstration is the same for all. In the first place, then, we shall show that the triangular prism AEIDHM is equal to the triangular prism BFKCGL.



Since AE is parallel to BF, and HE to GF, the angle AEI = BFK, HEI = GFK, and HEA = GFB. Of these six angles the first three form the solid angle E, the other three the solid angle F; hence, since the plane angles are respectively equal, and similarly arranged, it follows that the solid angles, E and F, are equal. Now, if the prism AEM is placed on the prism BFL, the base AEI being laid on the base BFK will coincide with it, because they are equal; and, since the solid angle E is equal to the solid angle F, the side EH will fall along its equal, FG: nothing more is required to prove that the two prisms will coincide throughout their whole extent (Prop. VII.); for the base AEI and the edge EH determine the prism AEM, as the base BFK and the edge FG determine the prism BFL (Def. 5); hence, the prisms are equal.

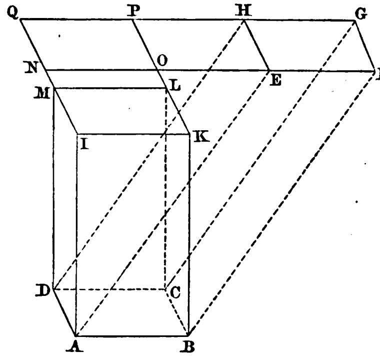
But, if the prism AEM is taken away from the solid AL there will remain the parallelopipedon AIL; and if the prism BFL is taken away from the same solid, AL, there will remain the parallelopipedon AEG; hence, the two parallelopipedons, AIL, AEG, are equivalent.

PROPOSITION XII.

THEOREM.

Two parallelopipedons having the same base and the same altitude are equivalent.

Let ABCD be the common base of the two parallelopipedons AG, AL; since they have the same altitude, their upper bases, EFGH, IKLM, will be in the same plane. Also, the sides EF and AB will be equal and parallel, as well as IK and AB; hence, EF is equal and parallel to IK; for a like reason, GF is equal and parallel to LK. Let the sides EF, HG be produced, likewise, LK, IM, till, by their intersections they form the parallelogram NOPQ; this parallelogram will evidently be equal to each of the bases, EFGH, IKLM. Now, if a third parallelopipedon be conceived, having for its lower base the same, ABCD, and NOPQ for its upper, this third parallelopipedon will be equivalent to the parallelopipedon AG (Prop. XI.), since, with a common lower base, their upper bases lie in the same plane, and between the parallels GQ, FN.



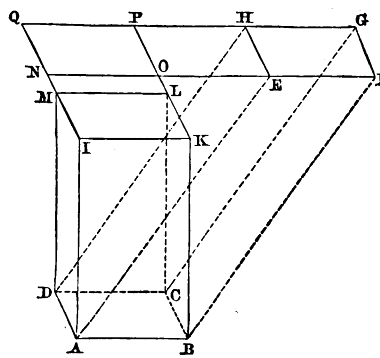
For the same reason, this third parallelopipedon will also be equivalent to the parallelopipedon AL; hence the two parallelopipedons AG, AL, which have the same base and same altitude, are equivalent.

PROPOSITION XIII.

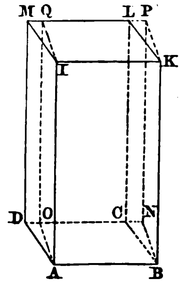
Theorem.

Any parallelopipedon may be changed into an equivalent rectangular parallelopipedon, having the same altitude and an equivalent base.

Let AG be the proposed parallelopipedon. From the points A, B, C, D, draw AI, BK, CL, DM, perpendicular to the plane of the base, thus forming the parallelopipedon AL, equivalent to the parallelopipedon AG, and having its lateral faces, AK, BL, etc., rectangles. Hence, if the base, ABCD, is a rectangle, AL will be the rectangular parallelopipedon, equivalent to the proposed parallelopipedon AG.



But, if ABCD is not a rectangle, draw AO and BN, perpendicular to CD, and OQ and NP, perpendicular to the base; you will then have the solid ABNOIKPQ, which will be a rectangular parallelopipedon : for, by construction, the bases ABNO and IKPQ are rectangles ; so also are the lateral faces, since the edges, AI, OQ, etc., are perpendicular to the plane of the base; hence, the solid AP is a rectangular parallelopipedon. But the two parallelopipedons, AP, AL, may be considered as having the same base, ABKI, and the same altitude, AO ; hence, they are equivalent ; hence, the parallelopipedon AG, which was first changed into an equivalent parallelopipedon, AL, is again changed into an equivalent rectangular parallelopipedon, AP, having the same altitude, AI, and a base, ABNO, equivalent to the base ABCD.

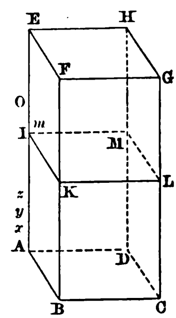


PROPOSITION XIV.

THEOREM.

Two rectangular parallelopipedons, AG, AL, *which have the same base,* ABCD, *are to each other as their altitudes,* AE, AI.

First.—Suppose that the altitudes, AE, AI, are to each other as two whole numbers, for example, as 15 is to 8. Divide AE into fifteen equal parts, whereof AI will contain eight, and through the points of division, x, y, z, etc., draw planes parallel to the base. These planes will cut the solid AG into fifteen partial parallelopipedons, all equal to each other, because they have equal bases and equal altitudes ; equal bases because every section, as MIKL, of a prism, made parallel to its base, ABCD, is equal to this base (Prop. V., Cor.) ; equal altitudes because these altitudes are the equal divisions, Ax, xy, yz, etc. But, of the fifteen equal parallelopipedons, eight are contained in AL ; hence, the solid AG is to the solid AL as 15 is to 8, or, generally, as the altitude AE is to the altitude AI.



Secondly.—If the ratio of AE to AI cannot be expressed in numbers, it can be shown, nevertheless, that

solid AG : *solid* AL : : AE : AI.

For, if this proportion is not correct, suppose

solid AG : *solid* AL : : AE : AO.

Divide AE into equal parts, such that each shall be less than OI; there will be at least one point of division, m, between O and I.

Let P be the parallelopipedon having for base ABCD, and for altitude Am; since the altitudes, AE, Am, are to each other as two whole numbers, we shall have

sol. AG : P : : AE : Am.

But, by hypothesis, *sol.* AG : *sol.* AL : : AE : AO;

hence, *sol.* AL : P : : AO : Am.

But AO is greater than Am; hence, for the proportion to be correct, the solid AL must be greater than P. Now, on the contrary, it is less; hence, it is impossible that the fourth term of the proportion,

sol. AG : *sol.* AL : : AE : x,

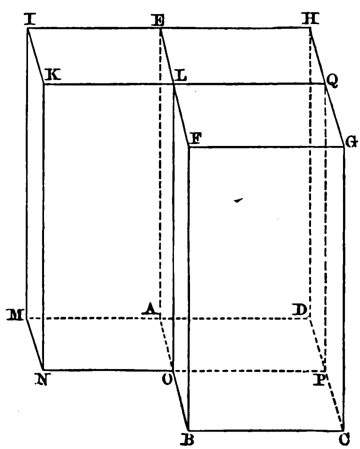
can be a greater line than AI. By like reasoning it may be shown that the fourth term cannot be less than AI; hence it is equal to AI; therefore rectangular parallelopipedons having the same base, are to each other as their altitudes.

PROPOSITION XV.

THEOREM.

Two rectangular parallelopipedons, AG, AK, *having the same altitude,* AE, *are to each other as their bases,* ABCD, AMNO.

Having placed the two solids by the side of each other, as the figure represents, produce the plane ONKL until it meets the plane DCGH in PQ; we will thus have a third parallelopipedon, AQ, which may be compared with each of the parallelopipedons, AG, AK. The two solids, AG, AQ, having the same base, AEHD, are to each other as their altitudes, AB, AO; in like manner the two solids, AQ, AK, having the same base, AOLE, are to each other as their altitudes, AD, AM. Thus we have the two proportions



sol. AG : *sol.* AQ : : AB : AO,

sol. AQ : *sol.* AK : : AD : AM.

Multiplying together the corresponding terms of these two proportions, and omitting in the result the common multiplier, *Sol.* AQ,

we have *sol.* AG : *sol.* AK : : AB × AD : AO × AM.

But AB × AD represents the base ABCD, and AO × AM represents the base AMNO ; hence, two rectangular parallelopipedons having the same altitude are to each other as their bases.

PROPOSITION XVI.

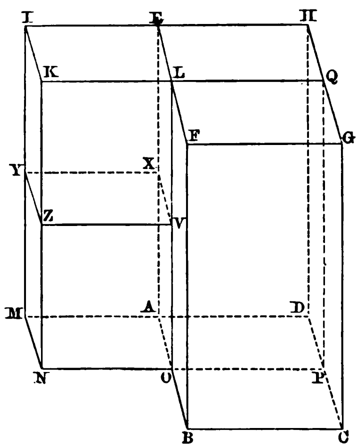
Theorem.

Any two rectangular parallelopipedons are to each other as the products of their bases by their altitudes, or as the products of their three dimensions.

For, having placed the two solids, AG, AZ, so that their surfaces have the common angle BAE, produce the planes necessary to complete the third parallelopipedon, AK, having the same altitude as the parallelopipedon AG. By the last proportion we shall have

sol. AG : *sol.* AK : : ABCD : AMNO

But the two parallelopipedons AK, AZ, having the same base, AMNO, are to each other as their altitudes, AE, AX ; thus we have



sol. AK : *sol.* AZ : : AE : AX.

Multiplying together the corresponding terms of these two proportions, and omitting in the result the common multiplier, *Sol.* AK, we obtain

sol. AG : *sol.* AZ : : ABCD × AE : AMNO × AX.

In the place of the bases, ABCD, AMNO, put AB × AD and AO × AM ; it will give

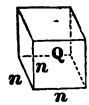
sol. AG : *sol.* AZ : : AB × AD × AE : AO × AM × AX.

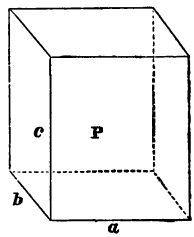
Hence, any two rectangular parallelopipedons are to each other, etc.

PROPOSITION XVII.

THEOREM.

The volume of a rectangular parallelopipedon, P, *is equal to the product of its base,* $a \times b$, *by its height,* c, *or equal to the product of its three dimensions,* $a \times b \times c$.





For, comparing the parallelopipedon P with the cube Q, constructed on the edge n, we have

$$P : Q :: a \times b \times c : n \times n \times n,$$

or $$\frac{P}{Q} = \frac{a}{n} \times \frac{b}{n} \times \frac{c}{n}.$$

Now, if we suppose the cube Q to be the unit of measure of the parallelopipedon, and n the linear unit on which it is constructed, then the ratio $\frac{P}{Q}$ will express the measure of the volume of the parallelopipedon P, and the ratios $\frac{a}{n}, \frac{b}{n}, \frac{c}{n}$ are its three dimensions considered as abstract numbers. Hence, volume $P = a \times b \times c$.

SCHOLIUM 1. Let $a = 3$ inches, $b = 2$ inches, $c = 5$ inches, and let the cubic inch be the unit of solid measure of volumes.

Then the ratio $$\frac{P}{\text{cubic inch}} = 3 \times 2 \times 5 = 30,$$

or $$\text{Volume } P = 30 \times \text{cubic inch} = 30 \text{ cubic inches}.$$

In order, then, to comprehend the nature of this measurement, it is necessary to reflect that the product of the three dimensions of a parallelopipedon is a number which signifies nothing of itself, and would be different if a different linear unit had been assumed. But if the three dimensions of another parallelopipedon are valued according to the same linear unit, and multiplied together, the two products will be to each other as the solids, and will serve to express their relative magnitude.

SCHOLIUM 2. The three dimensions of a cube being equal to each other, if the side is 1, the volume will be $1 \times 1 \times 1 = 1$; if the side is 2, the volume will be $2 \times 2 \times 2 = 2^3 = 8$; if the side is c, the volume will be $c \times c \times c = c^3$. Hence it is that in arithmetic the *cube* of a number is the name given to the third power of the number.

SCHOLIUM 3. The side, x, of a cube equivalent to a given volume, V, is equal to the cube root of V. For $x^3 = V$; and hence $x = \sqrt[3]{V}$. The problem *of the duplication* of the *cube*, famous among the ancient Greek Geometers, consists in determining the side of a cube which shall be double a given cube. Now, if x and c be the sides of the two cubes, then $x^3 = 2c^3$, or $x = c\sqrt[3]{2}$, that is, the side of the required cube would have to be to the side of the given cube as the *cube root of 2 is to unity.* Now, the square root of 2 is easily found by a geometrical construction. But the cube root of 2 cannot be so found, that is, by the simple operations of elementary Geometry, which employ no other lines than straight lines and circumferences. The problem admits of solution, however, by other methods no less rigorous than those of elementary Geometry.

PROPOSITION XVIII.

THEOREM.

The volume of a parallelopipedon, and generally of any prism, is equal to the product of its base by its altitude.

First.—Any parallelopipedon is equivalent to a rectangular parallelopipedon having the same altitude, and an equivalent base (Prop. XIII.). Now, the volume of the latter is equal to its base multiplied by its altitude; hence the volume of the former is likewise equal to the product of its base by its altitude.

Secondly.—Any triangular prism is half the parallelopipedon so constructed as to have the same altitude and a base twice as great (Prop. X.). But the volume of the latter is equal to its base multiplied by its altitude; hence, that of a triangular prism is equal to the product of its base (half that of the parallelopipedon) multiplied by its altitude.

Thirdly.—Any prism may be divided into triangular prisms having for their bases the different triangles which form the polygon which serves as its base, and for their common altitude the altitude of the prism. But the volume of each triangular prism is equal to its base multiplied by its altitude; and, since the altitude is the same for all, it follows that the sum of all the partial prisms will be equal to the sum of all the triangles which constitute their bases, multiplied by the common altitude.

Hence, the volume of any polygonal prism is equal to the product of its base by its altitude.

COR. 1. *The volume of any prism is equal to the product of its right section by its lateral edge* (Prop. IX.).

COR. 2. *First.—Any two prisms are equivalent when they have equal bases and equal altitudes, or when the products of their bases by their altitudes respectively are equal.*

Secondly.—Two prisms of the same altitude are to each other as their bases.

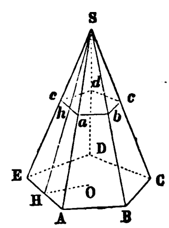
Thirdly.—Two prisms of equivalent bases are to each other as their altitudes.

PYRAMIDS.

PROPOSITION XIX.

THEOREM.

The convex surface of a regular pyramid S-ABCDE *is equal to the perimeter of the base,* ABCDE, *multiplied by half the slant height,* SH.



For, the convex surface, S, is composed of five isosceles triangles, each equal to SAE $= \text{AE} \times \frac{\text{SH}}{2}$. Hence, convex surface S $= 5\text{AE} \times \frac{\text{SH}}{2}$. But 5AE $=$ (AB + BC + CD . . . .) $=$ perimeter, ABCDE. Therefore, S $=$ perimeter, ABCDE $\times \frac{\text{SH}}{2}$.

SCHOLIUM. The area of the base, ABCDE, being equal to the product of its perimeter by half the apothem, OH (Book IV., Prop. IX.), the whole surface of the pyramid is

$$\text{perimeter ABCDE} \times \frac{\text{SH}}{2} + \text{perimeter ABCDE} \times \frac{\text{OH}}{2},$$

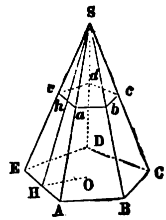
or

$$\text{perimeter ABCDE} \times \frac{\text{SH} + \text{OH}}{2}.$$

PROPOSITION XX.

THEOREM.

The convex surface of the frustum, AD′, *of a regular pyramid, is equal to the sum of the perimeters of its bases,* ABCDE, *abcde, multiplied by half the slant height,* H*h*, *of the frustum.*



For this convex surface, S, is composed of five trapezoids, each equal to AE*ae*, and therefore each measured by $\frac{AE + ae}{2} \times Hh$. Therefore,

$$S = 5\left(\frac{AE + ae}{2}\right) \times Hh = (5AE + 5ae) \times \frac{Hh}{2}.$$

But $5AE$ = perimeter ABCDE, and $5ae$ = perimeter *abcde*. Hence, S = (perimeter ABCDE + perimeter *abcde*) $\times \frac{Hh}{2}$.

SCHOLIUM. To get the whole surface of the frustum, we add to the convex surface the areas of the two bases, ABCDE, *abcde*.

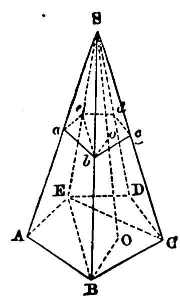
PROPOSITION XXI

THEOREM.

If a pyramid, S-ABCDE, *is cut by a plane, abd, parallel to its base,*

1. *The edges,* SA, SB, SC, *etc., and the altitude,* SO, *will be divided proportionally at a, b, c, o.*

2. *The section abcde will be a polygon similar to the base,* ABCDE.



First.—Conceive a plane to pass through the vertex, S, parallel to the planes ABCDE, *abcde*. Then all the edges, SA, SB, etc., and the altitude, SO, being cut by three parallel planes in the points S, A, *a*, B, *b*, O, *o*, etc., will be divided proportionally (Book V., Prop. XX., Schol.), and we shall have

SA : S*a* : : SB : S*b* : : SO : S*o*.

Secondly.—Since *ab* is parallel to AB, *bc* to BC, *cd* to CD, etc., the angle *abc* = ABC, *bcd* = BCD, and so on. Also by reason of the similar triangles SAB, S*ab*, we have

AB : *ab* : : SB : S*b*,

and by reason of the similar triangles SBC, S*bc*, we have

SB : S*b* : : BC : *bc*;

hence AB : *ab* : : BC : *bc*.

In like manner we may prove

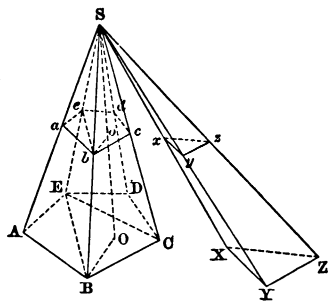
BC : *bc* : : CD : *cd*, and so on.

Hence, the polygons ABCDE, *abcde* have their angles respectively equal, and their homologous sides proportional, hence they are similar.

PROPOSITION XXII.

THEOREM.

If two pyramids, S-ABCDE, S-XYZ, *which have the same altitude and the bases in the same plane, be cut by the same plane parallel to the plane of the bases, the sections, abcde, xyz, will be to each other as the bases,* ABCDE, XYZ.



For the polygons ABCDE, *abcde,* being similar (Prop. XXI.), are to each other as the squares of their homologous sides AB, *ab* (Book III., Prop., XXIX.).

But $\quad AB : ab :: SA : Sa.$

Hence

$$ABCDE : abcde :: SA^2 : Sa^2.$$

For the same reason

$$XYZ : xyz :: SX^2 : Sx^2.$$

But since *abc* and *xyz* are in one plane, we have likewise

$$SA : Sa :: SX : Sx$$ (Book V., Prop. XX., Schol.);

hence

$$ABCDE : abcde :: XYZ : xyz,$$

or

$$abcde : xyz :: ABCDE : XYZ.$$

COR. Hence

If the bases, ABCDE, XYZ, *are equivalent, the sections made at equal altitudes are also equivalent.*

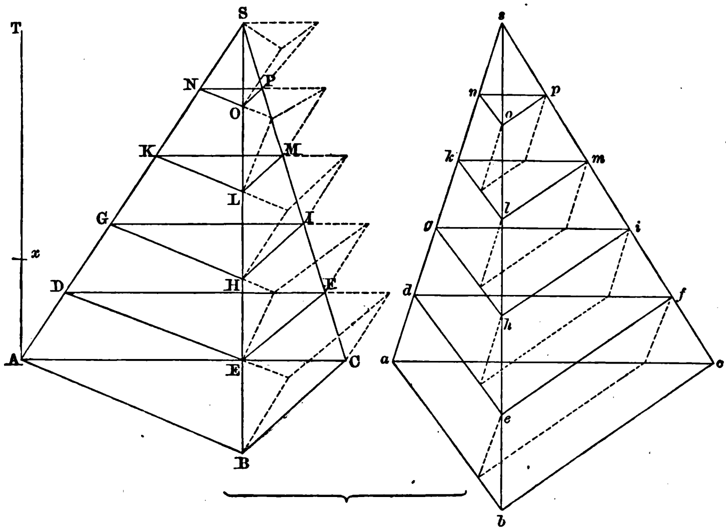
PROPOSITION XXIII.

THEOREM.

Two ~~triangular~~ pyramids which have equivalent bases and equal altitudes, are equivalent.

Let S-ABC, *s-abc*, be the two pyramids. Let their equivalent bases, ABC, *abc*, be situated in the same plane, and let AT be their common altitude. If these pyramids are not equivalent, let *s-abc* be the smaller, and suppose A*x* to be the altitude of a prism which, constructed in the base ABC, is equal to their difference.

Divide the common altitude, AT, into equal parts smaller than Ax, and let k be one of these parts ; through the points of division pass planes parallel to the plane of the bases : the sections made in the two pyramids by each of these planes will be equivalent (Prop. XXII., Cor.), namely, DEF to def, GHI to ghi, etc. This being granted,



upon the triangles ABC, DEF, GHI, etc., taken as bases, construct exterior prisms, having for edges the parts AD, DG, GK, etc., of the edge SA ; in like manner upon the triangles def, ghi, klm, etc., taken as bases, construct in the second pyramid interior prisms, having for edges the corresponding parts of sa ; all these partial prisms will have the common altitude k.

Now, the sum of the exterior prisms of the pyramid S-ABC is greater than that pyramid, and the sum of the interior prisms of the pyramid s-abc, is less than that pyramid ; hence, the difference between the sum of all the exterior prisms, and the sum of all the interior ones, should be greater than the difference between the two pyramids.

But, beginning with the bases, ABC, abc, the second exterior prism, DEFG, is equivalent to the first interior prism, $defa$, because their bases, DEF, def, are equivalent, and they have the same altitude, k ; for the same reasons, the third exterior prism, GHIK, and the second interior prism, $ghid$, are equivalent ; the fourth exterior and third interior ; and so on to the last of each series. Hence, all the exterior prisms of the pyramid S-ABC, excepting the first, ABCD, have equivalent corresponding ones in the interior prisms of the pyramid *s-abc* ; therefore, the prism ABCD is the difference between the sum of the exterior prisms of the pyramid S-ABC, and the sum of the interior prisms of the pyramid *s-abc*. But, the difference between these two sets of prisms has already been proved greater than the difference between the two pyramids, which latter difference we suppose to be ABCx ; hence, the prism ABCD, must be greater than the prism ABCx ; but in reality it is less ; for they have the same base, ABC, and the altitude, k, of the first, is less than the altitude, Ax, of the second.

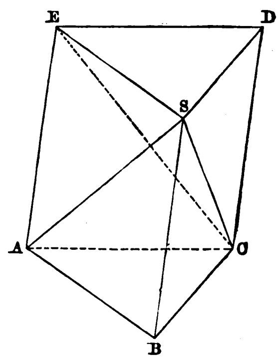
Hence, the supposed inequality between the two pyramids cannot exist. Hence, the two pyramids, S-ABC, *s-abc*, having equivalent bases and equal altitudes, are equivalent.

PROPOSITION XXIV.

THEOREM.

Every triangular pyramid is the third part of the triangular prism having the same base and the same altitude.

Let S-ABC be a triangular pyramid, ABCDES a triangular prism, having the same base and the same altitude; then will the pyramid be equal to a third of the prism. Cut from the prism the pyramid S-ABC by the plane SAC; there will remain the solid S-ACDE, which may be considered as a quadrangular pyramid whose vertex is S, and whose base is the parallelogram ACDE : draw the diagonal CE, and pass the plane SCE, which will divide the quadrangular pyramid into two triangular pyramids, S-ACE, S-DCE. These two triangular pyramids have for their common altitude the perpendicular let fall from S on the plane ACDE; they have equal bases, the triangles ACE, DCE being halves of the same parallelogram ; hence, the two pyramids, S-ACE, S-DCE, are equivalent (Prop. XXIII.). But, the pyramid S-DCE, and the pyramid S-ABC, have equal bases, ABC, DES ; they have also the same altitude, for this altitude is the distance of the parallel planes, ABC, DES. Hence, the two pyramids, S-ABC,



S-DCE, are equivalent. Now, the pyramid S-DCE has already been proved equivalent to the pyramid S-ACE ; hence, the three pyramids, S-ABC, S-DCE, S-ACE, which compose the prism ABCD, are equivalent. Hence, the pyramid S-ABC is the third of the prism ABCD, which has the same base and the same altitude.

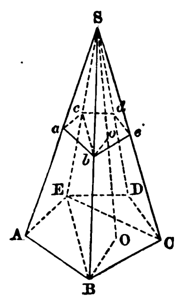
COR. *The volume of a triangular pyramid is equal to one-third of the product of its base by its altitude.*

PROPOSITION XXV.

THEOREM.

Every pyramid, S-ABCDE, *is measured by the third part of the product of its base,* ABCDE, *by its altitude,* SO.

For, by passing the planes SEB, SEC through the diagonals EB, EC, the polygonal pyramid S-ABCDE will be divided into several triangular pyramids, having the same altitude, SO. But, by the preceding theorem, each of these pyramids is measured by multiplying its base, ABE, BCE, or CDE, by the third part of its altitude, SO ; hence, the sum of these triangular pyramids, or the polygonal pyramid S-ABCDE, will be measured by the sum of the triangles ABE, BCE, CDE, or the polygon ABCDE, multiplied by one-third of SO ; *hence, every pyramid is measured by the third of the product of its base by its altitude.*



COR. 1. *Every pyramid is the third part of the prism having the same base and the same altitude.*

COR. 2. *First.—Any two pyramids are to each other as the products of their bases by their altitudes.*

Secondly.—Two pyramids having the same altitude are to each other as their bases.

Thirdly.—Two pyramids having equivalent bases are to each other as their altitudes.

SCHOLIUM 1. In order to find the volume of a polyedron, we divide it into pyramids, compute the volumes of these pyramids, and add together the numbers thus obtained.

In order to divide the polyedron into pyramids, we can assume any point whatever in space and join it to all the vertices of the polyedron. The bases of the different pyramids thus formed will be the faces of the polyedron, and the altitudes of the pyramids the perpendiculars let fall from the point taken upon the planes of these faces. The volume of the polyedron will be the arithmetical or the algebraic sum of the volumes of the pyramids, according as their common vertex (the point taken) is within or without the polyedron.

Sometimes the division of the polyedron into pyramids is effected by taking one of the vertices, and from that drawing diagonals to all the other vertices not adjacent to this.

SCHOLIUM 2. When a point can be found in the interior of a polyedron, equidistant from all its faces, the pyramids composing the polyedron, which have this point for a common vertex, will have as a common altitude the perpendicular let fall from this point on any one of the faces, and *the volume of the polyedron will have for its measure the third of the product of its surface by this perpendicular.*

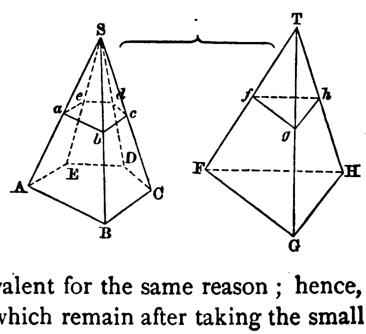
As any polygonal pyramid may be divided into triangular pyramids (tetraedrons), it is evident that any polyedron may be divided also into tetraedrons.

PROPOSITION XXVI.

THEOREM.

If a polygonal pyramid, S-ABCDE, *and a triangular pyramid,* T-FGH, *having equivalent bases lying in the same plane and the same altitude, be cut by a plane, abd, parallel to the plane of the bases, the frustums,* ABD-*abd,* FGH-*fgh, thus cut off, will be equivalent.*

For the plane *abd,* produced, forms in the triangular pyramid a section, *fgh,* situated at the same height above the common plane of the bases; and therefore, since the base ABCDE is equivalent to FGH, the section *abcde* will be equivalent to the section *fgh* (Prop. XXII., Cor.).



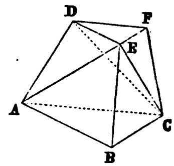
Hence, the pyramids S-*abcde* and T-*fgh* are equivalent, for their altitude is the same and their bases are equivalent. The whole pyramids S-ABCDE, T-FGH are equivalent for the same reason; hence, the frustums ABD-*abd*, FGH-*fgh*, which remain after taking the small pyramids from the wholes respectively, are equivalent.

PROPOSITION XXVII.

THEOREM.

Every frustum of a pyramid is equal to the sum of three pyramids, having for their common altitude the altitude of the frustum, and for bases the lower base of the frustum, the upper base, and a mean proportional between the two bases.

From the preceding theorem it follows that if the proposition can be proved in the single case of the frustum of a triangular pyramid, it will be true of any other frustum. Let ABC-DEF be the frustum of a triangular pyramid. The planes AEC, DEC divide it into three triangular pyramids, E-ABC, E-DCF, E-DCA. The first, E-ABC, has for its base the lower base, ABC, of the frustum; its altitude is likewise that of the frustum, since its vertex E lies in the plane of the upper base, EDF.



If we take the point C for its vertex, the second pyramid, E-DCF, has for its base DEF, the upper base of the frustum; its altitude is also the altitude of the frustum, since its vertex, C, lies in the lower base, ABC.

Thus we know two of the pyramids which compose the frustum. It remains to consider the third pyramid, E-DCA. To measure this, we compare it with the second, E-DCF. These two pyramids having the same altitude (considered with reference to the common vertex, E), are to each other as their bases, CDA, CDF.

But $$CDA : CDF :: AC : DF,$$
since the triangles have the same altitude. Hence
$$E\text{-}DCA : E\text{-}DCF :: AC : DF.$$

But since the bases of the frustum, ABC and DEF, are similar,
$$AC : DF :: \sqrt{ABC} : \sqrt{DEF} \text{ (Book III., Prop. XXVII.)}.$$
Therefore
$$E\text{-}DCA : E\text{-}DCF :: \sqrt{ABC} : \sqrt{DEF},$$
or
$$E\text{-}DCA = E\text{-}DCF \frac{\sqrt{ABC}}{\sqrt{DEF}}.$$
Now
$$E\text{-}DCF = DEF \times \frac{1}{3} \text{ of the altitude of frustum.}$$

Hence

$$\text{E-DCA} = \sqrt{\text{ABC}} \cdot \sqrt{\text{DEF}} \cdot \times \frac{1}{3} \text{ of the altitude of the frustum.}$$

Therefore, the third pyramid, E-DCA, is equivalent to a pyramid having for its base a mean proportional between the two bases of the frustum, and for its altitude the altitude of the frustum.

Hence, the frustum of a pyramid is equivalent to three pyramids whose common altitude is the altitude of the frustum, and whose bases are respectively the lower and upper bases of the frustum, and a mean proportional between these two bases.

SCHOLIUM. Let V be the volume of a frustum of a pyramid, B and B′ its bases, and H its altitude. The volumes of the three pyramids being

$$B \times \frac{H}{3},\ B' \times \frac{H}{3},\ \sqrt{BB'} \times \frac{H}{3};$$

we have

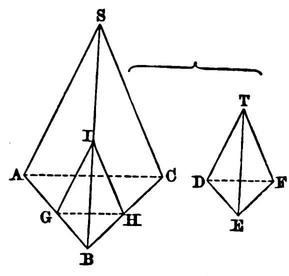
$$V = (B + B' + \sqrt{BB'}) \times \frac{H}{3}.$$

SIMILAR POLYEDRONS.

DEFINITIONS II.

1. Two tetraedrons, S-ABC, T-DEF, are *similar* when they have all their homologous edges proportional, that is, when SA : TD : : SB : TE : : SC : TF : : AB : DE, etc.

2. Two polyedrons are *similar* when they are composed of the same number of similar tetraedrons, similar each to each, and similarly situated.

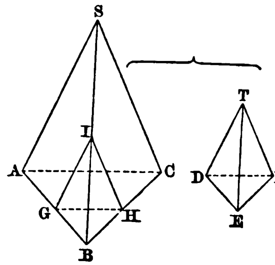


PROPOSITION XXVIII.

THEOREM.

In two similar tetraedrons, S-ABC, T-DEF, *the homologous faces are similar, and the homologous triedrals are equal.*

Any two homologous faces, SAB, TDE, are similar, because their sides are, by definition, proportional. And since the plane angles of a triedral in one tetraedron are therefore equal respectively to the plane angles of the homologous triedral in the other, these triedrals are equal (Book V., Prop. XXXVIII.).



PROPOSITION XXIX.

THEOREM.

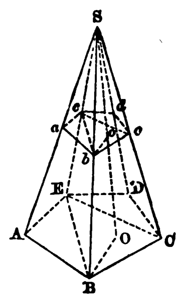
Conversely, *Two tetraedrons are similar, 1st, when their homologous faces are similar; 2d, when their homologous triedrals are equal.*

First.—When the homologous faces are similar, the homologous edges are proportional, and therefore the tetraedrons similar.

Secondly.—When the homologous triedrals are equal, the plane angles which form them are respectively equal, and therefore the homologous faces have the angles of the one equal to the angles of the other, and are therefore similar. Hence, the tetraedrons are similar.

COR. 1. *Two similar tetraedrons have their six homologous diedrals equal each to each, and conversely.*

COR. 2. *Every section, abcde, parallel to the base of a pyramid,* S-ABCDE, *determines another pyramid,* S-*abcde*, *similar to the first.*



For the planes SEC, SEB, divide the two pyramids into tetraedrons, S-ABE and S-*abe*, S-BCE and S-*bce*, S-CDE and S-*cde*, similar each to each, because their faces are similar. These tetraedrons are also similarly situated. Hence, by definition, the two pyramids are similar.

PROPOSITION XXX.

THEOREM.

In two similar polyedrons:

1. *The homologous faces are similar, each to each, and their inclinations are the same.*

2. *The homologous solid angles are equal.*

First.—The two polyedrons being composed of the same number of tetraedrons, similar each to each and similarly situated, their surfaces are also composed of the same number of triangles, similar each to each and similarly grouped. Moreover, the inclination of two adjacent triangles of the first surface is equal to the inclination of the two homologous triangles of the second; for these inclinations are either the homologous diedrals of two similar tetraedrons, or they are the sums of a like number of homologous diedrals: whence it results that two similar polyedrons are contained by the same number of faces, similar each to each, and equally inclined to each other.

Secondly.—The homologous solid angles are equal; for all their plane angles and diedrals are equal each to each and similarly grouped.

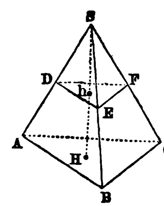
COR.—*The edges, the diagonals, and in general all the homologous lines of two similar polyedrons, are proportional.*

PROPOSITION XXXI.

THEOREM.

Similar tetraedrons are to each other as the cubes of their homologous edges.

We can always place the two tetraedrons so that they shall have a triedral, S, in common (Prop. XXIX, Cor. 2.). Then, since the bases, ABC, DEF, are similar, we have,



$$ABC : DEF :: AB^2 : DE^2. \qquad (1)$$

And since the angles SAB and SDE are equal, as also SBC and SEF, the plane DEF is parallel to the plane ABC. Therefore,

$$SH : Sh :: SA : SD :: AB : DE,$$

or, $$\tfrac{1}{3}SH : \tfrac{1}{3}Sh :: AB : DE. \qquad (2)$$

Multiplying the proportions (1) and (2), term by term, we have,

$$ABC \times \tfrac{1}{3}SH : DEF \times \tfrac{1}{3}Sh :: AB^3 : DE^3.$$

That is,

$$S\text{-}ABC : S\text{-}DEF :: AB^3 : DE^3.$$

PROPOSITION XXXII.

THEOREM.

In two similar polyedrons:

1. *The surfaces are to each other as the squares of their homologous edges.*
2. *The volumes are to each other as the cubes of these edges.*

First.—The areas of similar polygons being proportional to the squares of their homologous sides, the homologous faces of the two polyedrons form a series of equal ratios; hence, the sums of these faces, that is, the surfaces of the two polyedrons, are to each other as the squares of these same sides or edges.

Secondly.—Similar tetraedrons being proportional to the cubes of their homologous edges, the tetraedrons of which the two polyedrons are composed form a series of equal ratios; hence the sums of the antecedents and of the consequents, that is, the volumes of the two polyedrons, are to each other as the cubes of these same edges.

COR.—*Two similar pyramids are to each other as the cubes of their homologous edges, as also two similar prisms.*

EXERCISES ON BOOK VI.

THEOREMS.

1 Show that two tetraedrons are equal,

First.—When they have an equal diedral contained by two plane faces equal each to each, and similarly situated.

Second.—When they have an equal face adjacent to three diedrals, equal each to each, and similarly situated.

Third.—When they have three faces equal each to each, and similarly situated.

Fourth.—When they have one edge equal, and the plane angles of three faces equal, and similarly situated.

Fifth.—When they have an edge and five diedrals equal each to each, and similarly situated.

2. Two triangular prisms are equal when they have their lateral faces equal and arranged in the same manner.

3. The volume of a triangular prism is equal to the product of one of its lateral faces by half the distance from this face to the opposite edge.

4. Every plane which contains the line joining the middle points of two opposite faces of a parallelopipedon will divide that parallelopipedon into two equal parts.

5. Show that the formula

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

is verified by the geometrical construction when a and b are two parts of a given line.

6. Two tetraedrons which have a solid angle equal, are to each other as the product of the three edges which meet in the vertices of these equal solid angles.

7. The plane bisector of a diedral angle of a tetraedron divides the opposite edge into two parts proportional to the faces adjacent to this diedral.

8. If a plane be drawn, containing one edge of a tetraedron and the middle point of the opposite edge, it will divide the tetraedron into two equivalent tetraedrons.

9. Every plane which passes through the middle points of two opposite edges of a tetraedron, divides this body into two equivalent parts.

10. The six planes drawn perpendicular to the six edges of a tetraedron at their middle points, meet in a common point which is equidistant from the four vertices of the tetraedron.

11. The six plane bisectors of the diedral angles of a tetraedron meet in a common point equidistant from the four faces of the tetraedron.

12. The four straight lines which join the vertices of a tetraedron with the intersections of the medians of the opposite faces meet in a common point which divides each line of junction in the ratio of 3 to 1. (That is, the part of the line towards the vertex will be to the part towards the face as 3 to 1.)

13. The four perpendiculars erected to the faces of a tetraedron at the centres of their circumscribed circles meet in a common point.

14. The three straight lines which join the middle points of the opposite edges of a tetraedron meet in a common point which bisects these lines.

15. The different points of intersection mentioned in Exercises 9 to 14 inclusive, all become one and the same point when the tetraedron is regular.

16. When a tetraedron has one of its triedrals trirectangular, then the square of the area of the face opposite to this right triedral is equal to the sum of the squares of the other three faces.

17. The distance of the centre of a parallelopipedon from any plane is equal to one-eighth of the sum of the distances of its eight vertices from the same plane.

18. If a point is at a constant distance from the centre of a parallelopipedon, the sum of the squares of its distances from the vertices is constant.

19. The altitude of a regular tetraedron is equal to the sum of the perpendiculars let fall from any point taken in the interior of the tetraedron on its four faces.

20. *First.*—The volume of a truncated triangular prism is equivalent to three pyramids, having the base of the prism for a common base, and the three points in which the three edges pierce the inclined cutting plane as vertices.

Second.—It is also equivalent to its base multiplied by one-third of the distance of this base to the point of intersection of the medians of the inclined upper base.

Third.—It is also equal to its right section multiplied by one-third of the sum of its three edges.

21. The volume of a truncated parallelopipedon is equal to the product of its right section by the arithmetical mean of its four lateral edges.

22. If planes be drawn through the vertices of a tetraedron parallel to the opposite faces, the tetraedron formed by these planes is not similar to the first.

23. The squares of the volumes of two similar polyedrons are proportional to the cubes of their homologous faces.

PROBLEMS.

1. Find a point in the interior of a tetraedron, such that being joined to the four vertices, the tetraedron is divided into four equivalent tetraedrons.

2. Draw a plane parallel to the base of a pyramid cutting off a small pyramid which shall be $\frac{1}{8}$ of the given pyramid — $\frac{1}{27}$ — $\frac{1}{64}$.

3. In a tetraedron, S-ABC, through E, the middle point of the edge SB, let the plane DEF be passed parallel to the base ABC; the plane EGH, parallel to the face ASC, and the plane EDH; the pyramid, S-ABC, is thus divided into two equivalent triangular prisms, and into two tetraedrons of the same base and altitude. Divide these two pyramids in the same manner, and deduce the volume of the pyramid as the limit of the sum of the series of successive prisms thus obtained.

4. Cut a cube by a plane so as to make the intersection a regular hexagon.

5. Compute the altitude of a prism, knowing the volume, v, and the base, b; the same of a pyramid.

6. Given in a frustum of a pyramid the lower base, B, the height, h, and the volume, v. Compute the other base.

7. Compute the surface and volume of a regular tetraedron, the edge being given.

8. Find the entire surface of a regular pyramid, the slant height being 12 feet, and each side of the hexagonal base being 3 feet.

9. Find the convex surface of the frustum of a pentagonal regular pyramid whose slant height is 40 feet, each side of the lower base 8 feet, and each side of the upper base 5 feet.

10. Find the surface and volume of a frustum of a pyramid whose bases are squares, each side of the lower base being 12 feet, each side of the upper base 6 feet, and the height 4 feet.

11. Find the surface and volume of a block of marble in the shape of a rectangular parallelopipedon whose three dimensions are 4 feet 6 inches, 2 feet 3 inches, 3 feet 9 inches.

12. Find the whole surface of a triangular prism whose altitude is 15 feet, and its bases equilateral triangles, whose sides are 4 feet.

13. Find the volume of a regular hexagonal pyramid whose slant height is 18 feet, and each side of the base 5 feet.

14. Find the volume of the frustum of a pyramid, given the altitude 12 feet, and the bases regular dodecagons, the radii of whose circumscribing circles are 3.6 and .8 feet respectively.

15. Compute the three dimensions of a rectangular parallelopipedon, knowing that they are proportional to the numbers $\frac{2}{3}$, $\frac{4}{5}$, and $\frac{3}{4}$, the volume of the parallelopipedon being 2 cubic yards.

16. Compute the volume of a rectangular parallelopipedon, of which the surface is 5 square yards, and the three dimensions proportional to the numbers 4, 6, 9.

17. The height of a pyramid is 4.5 meters, and its base is a square whose side is 1.2 meters. Compute the corresponding dimensions of a similar pyramid, the volume of which is 7.29 cubic meters.

18. The base of a regular pyramid is a hexagon, each of whose sides is 3 feet in length, and its convex surface is ten times the area of its base. Find its height.

19. Find the volume of a right truncated triangular prism whose base is an equilateral triangle, the side of which is 6 feet, and the three lateral edges of which are 10, 12, and 15 feet. (See Exercises, Theorem 20.)

20. The greatest pyramid of Egypt is 150 yards high, and its base is a square whose side is 250 yards. Find its volume and its convex surface.

21. The edge, SA, of a pyramid, S-ABCD, being four feet six inches long, find the parts into which it is divided by a plane parallel to the base which divides the convex surface, first, into two equivalent parts; second, into two parts proportional to the numbers 3 and 5.

22. A regular pyramid has a hexagon for its base whose side is 15 feet, and its faces make with the base an angle equal to two-thirds of a right angle. Find the volume.

23. A right prism has for its base a regular hexagon. Find its altitude, knowing that its volume is 3 cubic feet and its convex surface 12 square feet.

APPENDIX TO BOOK VI.

THE REGULAR POLYEDRONS.

PROPOSITION I.

THEOREM.

There can only be five regular polyedrons.

For *regular polyedrons* were defined as those all of whose faces are equal regular polygons, and all whose solid angles are equal. These conditions cannot be fulfilled except in a small number of cases.

First.—If the faces are equilateral triangles, polyedrons may be formed of them, having solid angles contained by three of those triangles, by four, or by five: hence arise three regular bodies, the *tetraedron*, the *octaedron*, and the *icosaedron*. No other can be formed with equilateral triangles, for six angles of these triangles are equal to four right angles, and cannot form a solid angle (Book V., Prop. XXXIII.).

Secondly.—If the faces are squares, their angles may be arranged by threes; hence results the *hexaedron*, or *cube*.

Four angles of a square are equal to four right angles, and cannot form a solid angle.

Thirdly.—In fine, if the faces are regular pentagons, their angles may likewise be arranged in threes; and the regular *dodecaedron* will result.

We can go no farther; for three angles of a regular hexagon are equal to four right angles; three of a heptagon are greater.

Hence, there can be only five regular polyedrons; three formed with equilateral triangles, one with squares, and one with pentagons.

SCHOLIUM. It will be proved in the following proposition that these five polyedrons actually exist, and that all their dimensions may be determined when one of their faces is known.

PROPOSITION II.

PROBLEM.

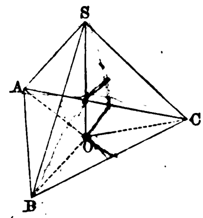
One of the faces of a regular polyedron being given, or only its side, to construct the polyedron.

This problem embraces five, which will be solved in succession.

1. CONSTRUCTION OF THE TETRAEDRON.

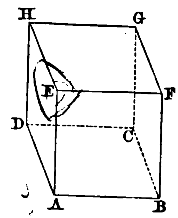
Let ABC be the equilateral triangle which is to be one face of the tetraedron. At the point O, the centre of this triangle, erect

OS perpendicular to the plane ABC; terminate this perpendicular in S, so that AS = AB; join SB, SC, and the pyramid S-ABC will be the required tetraedron.



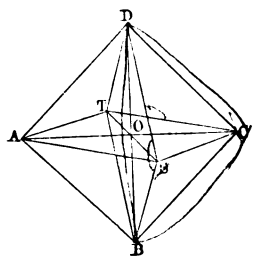
For, by reason of the equal distances OA, OB, OC, the oblique lines SA, SB, SC are equal (Prop. IX., Book V.). One of them, SA = AB; hence, the four faces of the pyramid S-ABC are triangles equal to the given triangle ABC. And the solid angles of this pyramid, are all equal, because each of them is formed by three equal plane angles; hence this pyramid is a regular tetraedron.

2. Construction of the Hexaedron.



Let ABCD be a given square; on the base, ABCD, construct a right prism whose altitude, AE, shall be equial to the side AB. It is evident that the faces of this prism are equal squares, and that its solid angles are equal, since they are each formed by three right angles; hence, this prism is a regular hexaedron, or cube.

3. Construction of the Octaedron.



Let AB be the given edge: on AB describe the square ABCD; at the point O, the centre of this square, erect TS perpendicular to its plane, and terminating on both sides in T and S, so that OT = OS = AO; then join SA, SB, TA, etc.; you will have a solid, SABCDT, composed of two quadrangular pyramids, S-ABCD, T-ABCD, united together by their common base, ABCD; this solid will be the required octaedron.

For the triangle AOS is right angled at O, as is the triangle AOD; the sides AO, OS, OD are equal; hence, those triangles are equal; hence, AS = AD. It may be shown in like manner that all the other right angled triangles, AOT, BOS, COT, etc., are equal to the triangle AOD; hence all the sides, AB, AS, AT, etc., are equal, and consequently the solid SABCDT is bounded by eight equilateral triangles whose sides are equal to AB. Moreover, the solid angles of this polyedron are all equal; for instance, the angle S is equal to the angle B.

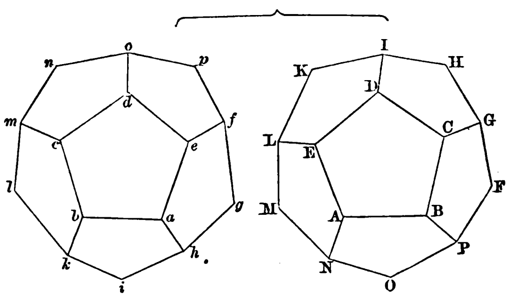
For it is evident that the triangle SAC is equal to the triangle DAC, and therefore the triangle ASC is right; hence the figure SATC is a square, equal to the square ABCD. But, comparing the pyramid

B-ASCT with the pyramid S-ABCD, the base of the first may be placed on the base, ABCD, of the second; then the point O being a common centre, the altitude OB of the first will coincide with the altitude OS of the second, and the two pyramids will exactly apply to each other in all points; hence the solid angle S is equal to the solid angle B; hence the solid SABCDT is a regular octaedron.

SCHOLIUM. If three equal straight lines, AC, BD, ST, are perpendicular to each other, and bisect each other, the extremities of these straight lines will be the vertices of a regular octaedron.

4. CONSTRUCTION OF THE DODECAEDRON.

Let ABCDE be a given regular pentagon; let ABP, CBP be two plane angles equal to the angle ABC; with these plane angles form the solid angle B. The mutual inclination of two of these planes,* we will call K. In like manner, at the points C, D, E, A, form solid angles equal to solid angle B, and similarly situated: the plane CBP will be the same as the plane BCG, since they are both inclined by the same quantity, K, to the plane ABCD. Hence, the pentagon BCGFP, equal to the pentagon ABCDE, may be described in the



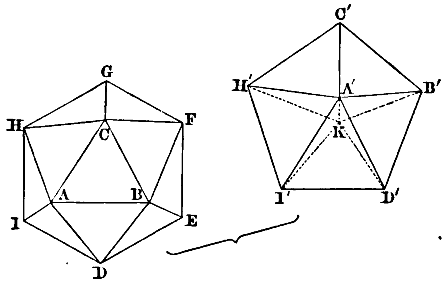
plane PBCG. If the same thing is done in each of the other planes, CDI, DEL, etc., we shall have a convex surface, PFGH, etc., composed of six equal regular pentagons, and each inclined to its adjacent plane by the same quantity, K. Let *pfgh*, etc., be a second surface equal to PFGH, etc., then may these two surfaces be united so as to form a single continuous convex surface. For, the angle *opf*, for example, may be joined to the two angles OPB, BPF, to make a solid angle, P, equal to the angle B; and in this junction no change will take place in the inclination of the planes BPF, BPO, that inclination being already such as is required to form the solid angle B. But, whilst the solid angle P is formed, the side *pf* will fall along its equal PF, and at the point F will be found the three planes, PFG, *pfe*, *efg*, united and forming a solid angle equal to each of the solid angles already formed;

* K is determined by methods not given in these elements.

and this junction will take place without changing either the state of the angle P, or that of the surface *efgh*, etc. ; for the planes PFG, *efp*, already united at P, have the proper inclination K, as have the planes *efg*, *efp*. Continuing the comparison thus, step by step, we see that the two surfaces will mutually adjust themselves to each other so as to form a single continuous convex surface ; which will be that of a regular dodecaedron, since it is composed of twelve equal regular pentagons, and has all its solid angles equal.

5. Construction of the Icosaedron.

Let ABC be one of its faces : we must first form a solid angle with five planes equal to ABC, and each equally inclined to its adjacent plane. To do this, on the side B′C′ equal to BC, construct the regular pentagon B′C′H′I′D′ ; at the centre of this pentagon erect a perpendicular to its plane, terminating in A′, so that B′A′ = B′C′ ; join A′C′, A′H′, A′I′, A′D′ ; the solid angle A′, formed by the five planes B′A′C′, C′A′H′, etc., will be the solid angle required. For, the oblique lines A′B′, A′C′, etc., are equal, and one of them, A′B′, is equal to the side B′C′ ; hence, all the triangles, B′A′C′, C′A′H′, etc., are equal to each other, and to the given triangle, ABC.



It is further manifest that the planes B′A′C′, C′A′H′, etc., are each equally inclined to their adjacent planes ; for, the solid angles B′, C′, etc., are all equal, being each formed by two angles of equilateral triangles, and one of a regular pentagon. Let the inclination of the two planes in which are the equal angles be called K ; the angle K will, at the same time, be the inclination of each of the planes composing the solid angle A′ to its adjacent plane.

This being granted, if at each of the points A, B, C, a solid angle be formed equal to the angle A′, we will have a convex surface, DEFG, etc., composed of ten equilateral triangles, each one of which will be inclined to its adjacent triangle by the quantity K ; and the angles D, E, F, etc., of its contour will alternately combine three angles and two angles of equilateral triangles. Conceive a second surface equal to the surface DEFG, etc. ; these two surfaces may be mutually adapted to each other, if each triple angle of one is joined to a double angle of the other ; and, since the planes of these angles have already to each other the inclination, K, requisite to form a quintuple solid angle equal to the angle A, there will be nothing changed, by this junction, in the state of each particular surface, and the two together will form a single continuous surface, composed of twenty equilateral triangles. This surface will be that of the regular icosaedron, since all its solid angles are also equal.

SCHOLIUM 1. The following table of the numbers of the different elements of the regular polyedrons enables us to notice particularly some of the properties of each:

TABLE OF PARTS OF REGULAR POLYEDRONS.

	No. Faces.	No. Sides of each Face.	No. Vertices.	No. Plane Angles of Each Solid Angle.	No. Edges.
Regular Tetraedron	4	3	4	3	6
Regular Hexaedron	6	4	8	3	12
Regular Octaedron	8	3	6	4	12
Regular Dodecaedron	12	5	20	3	30
Regular Icosaedron	20	3	12	5	30

1. In the regular tetraedron the number of vertices is the same as the number of faces.

2. The number of vertices of the regular hexaedron is equal to the number of faces of the octaedron, and reciprocally.

3. The number of the vertices of the dodecaedron is equal to the number of faces of the icosaedron, and reciprocally.

4. The number of edges of the hexaedron and octaedron is the same.

5. The number of edges of the dodecaedron and icosaedron is the same.

The following properties may also be observed in considering these bodies:

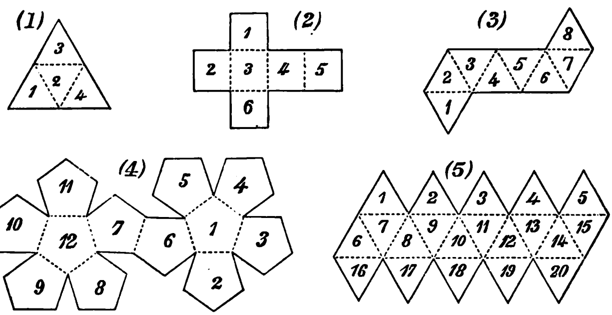
6. In a tetraedron the edges are opposite two and two, and each vertex is opposite to a face.

7. In the hexaedron and octaedron the vertices are opposite two and two. The same is true of the edges and faces.

8. In the dodecaedron the faces are opposite two and two, but neither the edges nor the vertices; but each edge is opposite to a vertex.

9. In the regular icosaedron the vertices are opposite two and two, but neither the edges nor the faces; but each edge is opposite to a face.

SCHOLIUM 2. The figures below represent the developments of the surfaces of the five regular polyedrons in their order. We can form these bodies thus: first, draw these figures on card-board; then cut them out, and also cut half through the board along the dotted lines. We can then bring the faces together in the required shape.



EXERCISES ON APPENDIX TO BOOK VI.

Theorems.

1. The polyedron which has for its vertices the centres of the four faces of a regular tetraedron is also a regular tetraedron.

2. The polyedron which has for its vertices the centres of the six faces of a regular tetraedron, is a regular octaedron.

3. Conversely, the polyedron which has for vertices the centres of the eight faces of a regular octaedron, is a regular hexaedron.

4. The polyedron which has for its vertices the centres of the faces of a regular dodecaedron, is a regular icosaedron.

5. Conversely, the polyedron which has for vertices the centres of the faces of a regular icosaedron, is a regular dodecaedron.

6. The polyedron which has for its vertices the middle points of the six edges of a regular tetraedron, is a regular octaedron.

7. Show that the volume of the regular tetraedron formed as indicated in 1, is $\frac{1}{27}$ of the volume of the first tetraedron.

Problems.

1. Compute the volume of a regular octaedron whose edge is given.

2. Compute the volume of a regular octaedron, the vertices of which are the middle points of the six edges of a regular tetraedron whose edge is given.

3. Compute the volume of a regular octaedron, the vertices of which are the centres of the six faces of a regular hexaedron whose edge is given.

4. Compute the volume of a regular hexaedron, whose vertices are the centres of the eight faces of a regular octaedron whose edge is given.

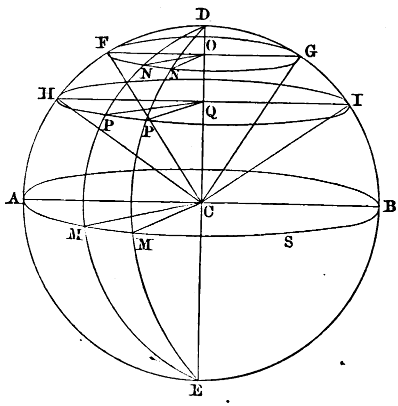
BOOK VII.

THE SPHERE.

DEFINITIONS.

1. The *sphere* is a solid terminated by a curved surface, all the points of which are equally distant from a point within called the centre.

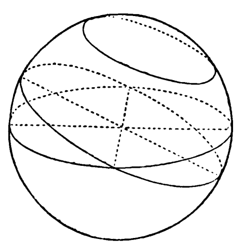
The sphere may be conceived as generated by the revolution of a semicircle, DAE, about its diameter, DE ; for the surface described in this movement by the semi-circumference, DAE, will have all its points equally distant from the centre, C. This surface, generated by DAE, is called the *surface of the sphere*, or the *spherical surface.* And for the sake of shortness, this surface is often called the sphere, as the circumference of a circle is often called the circle.



2. The *radius* of a sphere is a straight line drawn from the centre to any point of the surface ; the *diameter* or *axis* is a line passing through the centre and terminated on both sides by the surface.

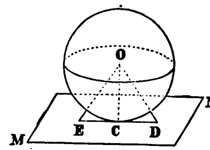
All the radii of a sphere are equal ; all the diameters are equal, and each double of the radius.

3. All the sections of a sphere by planes passing through its centre are obviously equal circles, which have the centre for their common centre, and for radii the radii of the sphere. It will be shown (Prop. II.), that all other sections of a sphere by planes, will be circles of smaller radii than the radii of the



sphere. This granted, a *great circle* is a section which passes through the centre; a *small circle* one which does not pass through the centre.

4. A *plane* is *tangent* to a sphere when their surfaces have but one point in common. This point is called the *point of contact* or *of tangency.*



5. The *pole of a circle* of a sphere is a point in the surface equally distant from all the points in the circumference of this circle. It will be shown (Prop. VIII.) that every circle of a sphere, great or small, has two poles.

6. The *angle of two arcs of great circles* on the sphere, is the angle of the planes of those arcs. Thus, the angle ADM (Fig. Def. 1), of the arcs AD and MD, is the angle of the planes ACD and MCD.

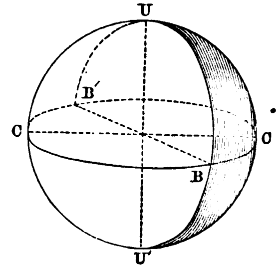
7. A *spherical triangle* is a portion of the surface of a sphere bounded by three arcs of great circles.

These arcs, named the sides of the triangle, are always supposed to be each less than a semi-circumference. ABC (Fig. Def. 12) is a spherical triangle of which AB, AC, and BC, are the sides.

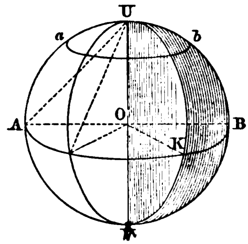
8. A spherical triangle takes the name of *scalene, isosceles, equilateral,* in the same cases as a rectilineal triangle.

9. A *spherical polygon* is a portion of the surface of a sphere terminated by several arcs of great circles. MN (Fig. Def. 12) is a spherical polygon. The *spherical triangle* is the simplest of the spherical polygons.

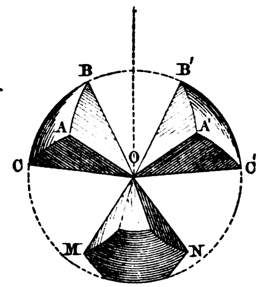
10. A *lune* is that portion of the surface of a sphere which is included between two great semi-circumferences meeting in the extremities of a common diameter.



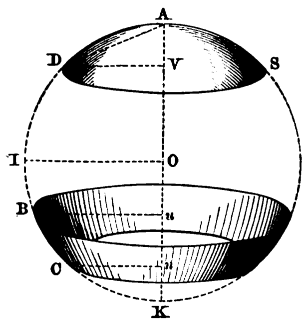
11. A *spherical wedge* or *ungula* is that portion of the solid sphere which is included between two great semicircles meeting in a common diameter. The *ungula* has a *lune* for its exterior spherical surface or *base*.



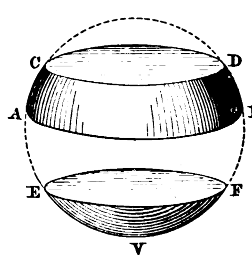
12. A *spherical pyramid* is a portion of the solid sphere included between the planes of a solid angle whose vertex is the centre, and which terminate in the surface of the sphere. The *base* of the pyramid is the spherical polygon intercepted by these planes.



13. A *zone* is the portion of the surface of the sphere included between two parallel planes which form its bases. One of these planes may be tangent to the sphere; in which case the zone has only a single base.



14. A *spherical segment* is the portion of the solid sphere included between two parallel planes which form its *bases*. One of these planes may be tangent to the sphere; in which case the segment has only a single base. The segment has a zone for the curved or spherical part of its surface.



15. The *altitude of a zone* or *of a segment* is the distance between

the two parallel planes which form the bases of the zone or segment, or the distance from the point of contact of the parallel tangent plane to the base, when they have only a single base.

16. Whilst the semicircle, DAE (Fig. Def. 1), revolving round its diameter, DE, describes the sphere, any circular sector, as DCF or FCH, describes a solid which is named a *spherical sector*.

17. A polyedron is said to be *inscribed* in a sphere when the vertices of all its solid angles are on the surface of the sphere.

18. A polyedron is said to be *circumscribed* about a sphere when all its faces are tangent planes to the spherical surface.

PROPOSITION I.

THEOREM.

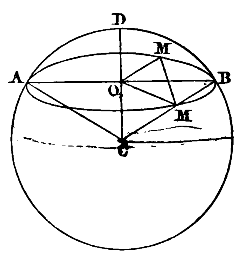
A straight line cannot meet the surface of a sphere in more than two points.

The demonstration is exactly the same as that of Proposition III., Book II.

PROPOSITION II.

THEOREM.

Every section of a sphere, made by a plane, is a circle.

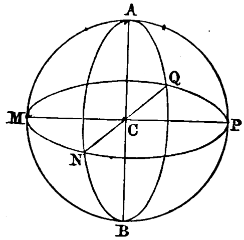


Let AMB be a section made by a plane in the sphere whose centre is C. From the point C, draw CO, perpendicular to the plane AMB; and different lines, CM, CM, to different points of the curve AMB, which terminates the section.

The oblique lines CM, CM, CB are equal, since they are radii of the sphere; they are therefore equally distant from CO (Book V., Prop. IX.); hence all the lines OM, OM, OB are equal; hence the section AMB is a circle, of which the point O is the centre.

COR. 1. If the section passes through the centre of the sphere, its radius will be the radius of the sphere; and all great circles are equal, as we have seen before (Def. 3). All other sections, since they have for diameters chords of the great circle, are smaller than the great circle.

COR. 2. *Two great circles always bisect each other.* For their common intersection, passing through the centre, is a diameter.



COR. 3. *Every great circle divides the sphere and its surface into two equal parts;* for if the two hemispheres were separated and afterwards placed in the common base, with their convexities turned to the same side, the two surfaces would coincide, no point of the one being nearer the centre than any point of the other.

COR. 4. *The centre of a small circle and that of the sphere, are in the same straight line perpendicular to the plane of the small circle.*

COR. 5. *One arc of a great circle, and but one, may be made to pass through two given points in the surface of a sphere, which are not the extremities of a diameter;* for the two given points and the centre of the sphere make three points, which determine the position of a plane. If, however, the two given points were at the extremities of a diameter, then these two points and the centre of the sphere could be in a straight line, and an infinite number of great circles might be passed through the given points. Three points on the surface of a sphere are necessary to determine a small circle.

COR. 6. Any great semicircle whatever of the sphere revolving about its diameter would generate the sphere.

COR. 7. Small circles are the less the further they lie from the centre of the sphere; for the greater CO is, the less is the chord AB, the diameter of the small circle AMB.

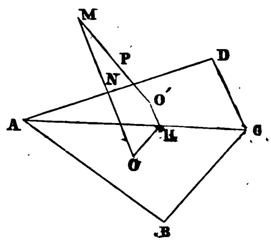
Or we may enunciate this corollary thus: *Two small circles equally distant from the centre of the sphere are equal; and of two small circles unequally distant from the centre of the sphere the greater is that one nearer to the centre.*

PROPOSITION III.

THEOREM.

Through four points, A, B, C, *and* D, *not in the same plane, the surface of one sphere may be made to pass, and but one.*

First.—Join AB, AC, AD, BC, CD, forming the triangles ABC, ADC, whose planes meet in AC. Erect a perpendicular, ON, to the plane of the triangle ABC, at O, the centre of its circumscribed circle. Erect a perpendicular, O′P, to the plane of ADC, at O′, the centre of its circumscribed circle. Draw perpendiculars from O and O′ to AC. They will meet AC at its middle point H. Hence, the planes NOH and PO′H are perpendicular to AC, at H (Book V., Prop. X., Cor.), and must, therefore, coincide (Book V., Prop. VII.). Hence, ON and O′P lie in the same plane, and they must meet, since they are perpendicular to lines OH and O′H, which meet (Book I., Prop. XXV., Cor. 2). The point M of their intersection is equally distant from A, B, and C, since it is on ON (Book V., Prop. IX., Cor. 3), and from A, C, and D, since it is on O′P. Hence, the four distances MA, MB, MC, and MD are equal. Therefore, the surface of a sphere with centre M and radius OA, will pass through the points A, B, C, D.



Secondly.—Only one sphere can be made to pass through these four points; for its centre, being equally distant from A, B, C, D, must be at the same time on the lines OM and O′P, and these lines have but one point of intersection. Hence, it must have the same centre and radius as the sphere already found, and hence be the same sphere.

SCHOLIUM. This theorem may be enunciated thus: *Four points not in the same plane determine the surface of a sphere.*

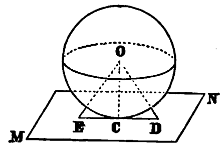
PROPOSITION IV.

THEOREM.

Every plane perpendicular to a radius at its extremity is tangent to the sphere. Conversely, *Every plane tangent to a sphere is perpendicular to the radius at the point of contact.*

Let MN be a plane perpendicular to the radius OC, at its extremity

C. Any point, E, being taken in this plane, and OE and CE joined, the angle OCE will be right, and thus the distance OE will be greater than OC. Hence, the point E is outside the sphere; and as the same is the case with every other point in the plane MN, it follows that this plane has only the single point C in common with the surface of the sphere; hence it is tangent to that surface (Def. 4).



Secondly.—The converse is true, because every point of the tangent plane MN, except the point C, being exterior to the sphere, the radius OC is the shortest line which can be drawn from the centre, O, to the plane MN. It is therefore perpendicular to this plane (Book V., Prop. IX.).

COR. 1. *Through a given point on the surface of a sphere, only one tangent plane can be drawn to this surface.*

COR. 2. *Every tangent plane is parallel to the planes of the small circles whose diameters are bisected by the radius through the point of contact.* And two tangent planes at the extremities of the same diameter are parallel.

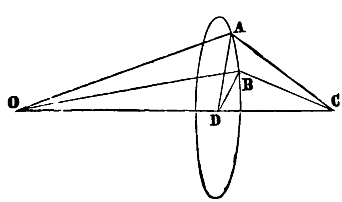
SCHOLIUM. A tangent plane to the sphere contains all the tangents drawn through the point of contact to the sections of the sphere which pass through this point. Two of these tangents determine the tangent plane.

PROPOSITION V.

THEOREM.

If two spheres have a common point situated without the line joining their centres, these two spheres will cut each other in a circle, the centre of which is on the line of their centres, and the plane of which is perpendicular to this line.

Suppose the two spheres have the centres O and C, and both surfaces passing through the point A. From the point A let fall the perpendicular AD on the straight line OC; and through AD draw a plane perpendicular to the line OC. In this plane, from the point D as a centre, describe a circle with a



radius equal to DA; every point upon the circumference of this circle will be one of the points of intersection of the two spheres. For, taking any point, B, of the circumference, and joining BD, OA, OB, CA, CB, the lines OA and OB are equal, since the right angled triangles ODA and OBA are equal. Hence, the point B is on the surface of the sphere of which O is the centre and OA the radius. The two equal triangles ADC, BDC give also AC = BC; hence B is on the sphere of which C is the centre and CA the radius, and hence B is common to both spheres. Hence, the common intersection of the two spheres is the circle DA, etc., etc.

COR. 1. *If two spheres are tangent to each other, the point of contact is situated on the line joining their centres.* For, otherwise, the spheres would have other common points, and consequently would cut each other.

COR. 2. The surface of a sphere cannot be passed through four points situated in the same plane, unless these four points are on the same circumference, and then any number of spheres may be passed through them, whose common intersection will be this circumference.

SCHOLIUM. Two spheres can have a *common circle*—that is, cut each other; or have *one* point in common—that is, be tangent to each other externally or internally; or, finally, be entirely exterior to each other, or one interior to the other. Their possible relative positions are thus *five* in number. Propositions XII., XIII., XIV., XV., XVI., and the Corollaries of XVI., of Book II., with regard to the different relative positions of two circles, apply equally to two spheres.

Spheres having the same centre are named *concentric spheres.*

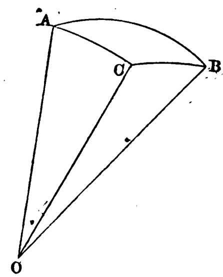
PROPOSITION VI.

THEOREM.

In every spherical triangle, ABC, *any side is less than the sum of the other two.*

Let O be the centre of a sphere; draw the radii OA, OB, OC.

Conceive the planes AOB, AOC, COB drawn; these planes will form a triedral angle at the centre, O, and the angles AOB, AOC, COB will be measured by AB, AC, BC, the sides of the spherical triangle ABC. But each of the three plane angles which form the triedral angle is less than the sum of the other two. (Book V., Prop. XXXII.); hence any side of the triangle ABC is less than the sum of the other two.



COR. 1. Each side of a spherical triangle is greater than the difference of the other two.

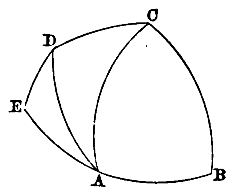
COR. 2. *Any side,* AB, *of a spherical polygon is less than the sum of the other sides,* AE + ED + DC + CB.

Join AD, AC, then we have

$$AD < AE + ED;$$

$$AC < AD + DC;$$

$$AB < AC + BC.$$



Adding these inequalities, and cancelling the equal terms on the two sides of the results, we have

$$AB < AE + ED + DC + CB.$$

SCHOLIUM. The demonstration of this theorem shows that each property of a triedral relative to its plane angles or its diedrals, pertains also to the sides and angles of a spherical triangle. For the plane angles of the triedral, constructed as in the proposition, are measured by the sides of the spherical triangle, and its diedrals are, by definition, equal to the angles of the triangle. There exists then a perfect analogy between the spherical triangle and the triedral which has for its vertex the centre of the sphere, and for its edges the radii to the vertices of the triangle.

The same remark applies to any spherical polygon, and its corresponding solid angle with vertex at the centre of the sphere. Thus, from Cor. 2, we can draw the conclusion *that any one plane angle of a solid angle is less than the sum of all the others.*

PROPOSITION VII.

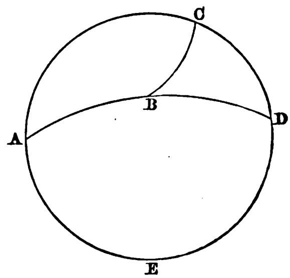
THEOREM.

The sum of the sides of every convex spherical polygon is less than the circumference of a great circle.

For, joining the vertices of the polygon with the centre of the sphere we form a solid angle whose plane angles are measured respectively by the sides of the polygon. But the sum of these plane angles is always less than four right angles (Book V., Prop. XXXIII.). Hence, the sum of the sides of the polygon is less than the measure of four right angles or the circumference of a great circle.

SCHOLIUM. The direct demonstration is also easy.

Consider the triangle ABC. Prolong the sides AB, AC till they meet again at D. The arcs ADB, ACD are semi-circumferences (Prop. II., Cor. 2). But in the triangle BCD we have,



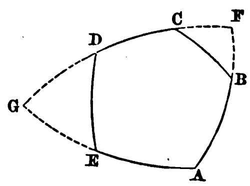
$$BC < BD + CD \text{ (Prop. VI.)}.$$

Adding AB + AC to each, we have,

$$AB + AC + BC < ABD + ACD,$$

that is to say, less than a circumference of a great circle.

Operating in the same manner on a polygon, replacing one side by the prolongations of the two sides adjacent to it, we see that if the theorem is true for any one convex polygon, it is true for the convex polygon which has one more side. Hence, it is true of all convex polygons.

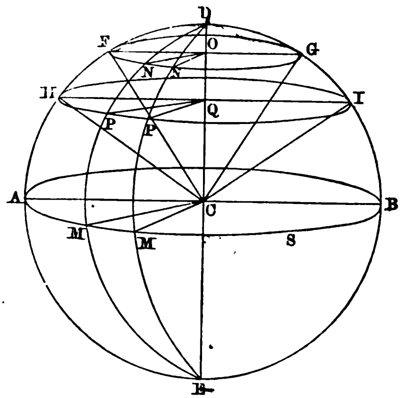


PROPOSITION VIII.

THEOREM.

If the diameter DE *be drawn perpendicular to the plane of the great circle* AMB, *its extremities,* D *and* E, *will be the poles of the circle* AMB *and of all the small circles, as* FNG, *etc., parallel to it.*

For DC, being perpendicular to the plane AMB, is perpendicular to all the straight lines, CA, CM, CB, etc., drawn through its foot in this plane: hence, all the arcs DA, DM, DB, etc., are quarters of the circumference; the same is the case with the arcs EA, EM, EB, etc.; hence, the points D and E are each equally distant from all the points of the circumference AMB; hence, they are the poles of this circumference (Def. 5).



Again, the radius DC, perpendicular to the plane AMB, is perpendicular to its parallel, FNG; hence, it passes through the centre, O, of the circle FNG (Prop. II., Cor. 4); hence, if the oblique lines DF, DN, DG be drawn, these oblique lines will be equally distant from the foot of the perpendicular DO, and will be equal (Book V., Prop. IX.). But the chords being equal, the arcs are equal; hence all the arcs DF, DN, DG, etc., are equal to each other; hence the point D is the pole of the small circle FNG, and for like reasons the point E is the other pole.

COR. 1. *Every arc,* DM, *drawn from a point in the arc of a great circle,* AMB, *to its pole is a quarter of the circumference, and is called, for the sake of shortness, a quadrant, and this quadrant makes at the same time a right angle with the arc* AM. For, the line DC being perpendicular to the plane AMC, every plane, DMC, which contains the line DC, is perpendicular to the plane AMC (Book V., Prop. XXVI.). Hence, the angle of these planes, or (Def. 6) the angle AMD, is a right angle.

COR. 2. One of the poles, D, of a given arc, AM, is on an arc, MD, perpendicular to AM, and at a quadrant's distance from M, measured on this arc; or, rather, the pole, D, of the arc AM is the point of intersection of two arcs, AD and MD, perpendicular to AM.

COR. 3. Conversely, If the distance of the point D to each of the points A and M is equal to a quadrant, then the point D will be the pole of the arc AM, and at the same time the angles DAM and AMD will be right.

For, let C be the centre of the sphere, and draw the radii CA, CD, CM; since the angles ACD, MCD are right, the line CD is perpendicular to the plane of the two straight lines CA and CM (Book V., Prop. VI.). Hence, the point D is the pole of the arc AM, and consequently the angles DAM, AMD are right.

SCHOLIUM I.—The properties of poles enable us to describe arcs of a circle on the surface of a sphere with the same facility as on a plane surface. It is evident, for instance, that by turning the arc DF, or any other line extending to the same distance, round the point D, its extremity F will describe the small circle FNG; and by turning the quadrant DFA round the point D, its extremity A will describe the arc of the great circle AM.

The poles are sometimes called centres, and the arcs used to describe the circles *polar distances*. The polar distances for the description of arcs of great circles are *quadrants*. Knowing the quadrant's length, we can thus easily solve the following

PROBLEMS.

1. *Given two points,* A *and* M, *of the arc,* AM, *of a great circle, produce that arc.*

Determine the pole, D, of AM, by the intersection of two arcs described from the points A and M, as centres, with a polar distance equal to a quadrant; then from the pole, D, with the quadrant as a polar distance, describe AM and its prolongation as required.

2. *At a given point,* M, *of an arc,* AM, *of a great circle, draw an arc of a great circle perpendicular to* AM.

Find the pole, D, of AM, as above, and with D as a centre and a quadrant as polar distance, prolong AM to S, until MS is a quadrant, then, with S as a centre, and SM as a polar distance, describe MD. It will be the perpendicular required.

3. *From a given point,* P, *draw an arc perpendicular to the given arc* AM.

Determine the pole, D, of AM, and prolong AM; from P as a centre, with a quadrant as polar distance, describe an arc cutting AM prolonged in S. Then from S as a centre, with the same polar distance, an arc through P, cutting AM in M. It will be the perpendicular, PM, required. The arc, MP, prolonged through D, will also be perpendicular to the arc AMB, prolonged on the other side of the sphere.

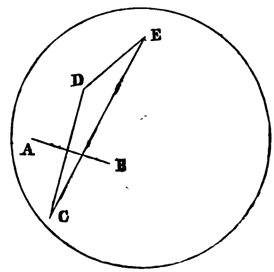
SCHOLIUM 2. In order to determine the quadrant necessary for the above graphical constructions on a sphere, we must determine the radius. This can be done by the following graphical construction in a plane.

PROPOSITION IX.

PROBLEM.

Determine the radius of a solid sphere or globe.

Take two points, A and B, on the surface of the globe, and with the same opening of the dividers describe two arcs cutting each other in C; then, from A and B, with another opening of the dividers, describe two other arcs cutting each other in D; then again, in like manner, two arcs cutting each other in E. The three points, C, D, and E, thus determined, lie on the circumference of a great circle of the sphere. For these points, being each equally distant from the points A and B, must be on a plane perpendicular to the straight line which joins these points at its middle point (Book V., Exercises; Theorem 1), and this plane must pass through the centre of the sphere, as that is also equidistant from A and B. Hence, the three points C, D, E are on a plane passing through the centre of the sphere, and therefore on the circumference of a great circle. Take off between the points of the dividers the distances CD, DE, CE, and with these construct the plane triangle CDE (Book II., Prop. X.). Then construct the circumference which circumscribes this triangle. This will be a great circle of the sphere, and its radius the radius of the sphere.



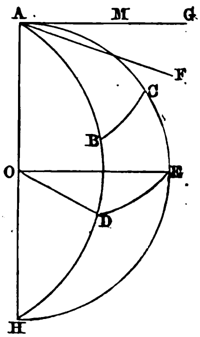
SCHOLIUM. We can find the poles of an arc of a small circle when we know three points, A, B, C. They are the intersections of two great circles, one perpendicular to the arc of a great circle which joins the two points A and B at its middle point, and the other drawn perpendicular to the arc of the great circle which joins B and C at its middle point.

PROPOSITION X.

THEOREM.

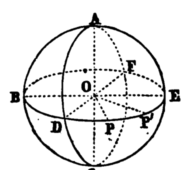
The angle, BAC, *formed by two arcs of great circles,* AB, AC, *is equal to the angle,* FAG, *formed by the tangents of these arcs at the point* A; *and is measured by the arc* DE, *described from the point* A *as a pole contained between the sides* AB, AC, *produced, if necessary.*

For the tangent AF, drawn in the plane of the arc AB, is perpendicular to the radius AO; and the tangent AG, drawn in the plane of the arc AC, is perpendicular to the same radius, AO. Hence, the angle FAG is equal to the diedral angle of the planes OAB, OAC (Book V., Prop. XXV., Cor. 1), which is that of the arcs AB, AC (Def. 6), and is called BAC.

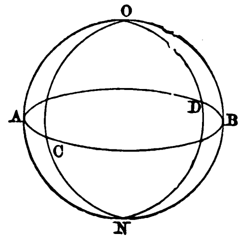


In like manner, if the arc AD and AE are both quadrants, the lines OD, OE will be perpendicular to AO, and the angle DOE will still be equal to the angle of the planes AOD, AOE; hence, the arc DE is the measure of the angle contained in these planes, or the measure of the angle CAB.

COR. 1. *The angle,* BAD, *of two arcs of great circles has also for its measure the arc of the great circle which joins the corresponding poles,* P *and* P′, *of the arcs* AB *and* AD. For, if we take, on the arc BD of which A is the pole, BP and DP′ on the same side of B, each equal to a quadrant, the point P will be the pole of the arc AB, and P′ the pole of the arc AD, and the arc PP′ equal to the arc BD, each being equal to a quadrant diminished by DP.



COR. 2. The angles of spherical triangles may be compared together by means of the arcs of great circles described from their vertices as poles, and included between their sides: thus it is easy to make an angle equal to a given angle.



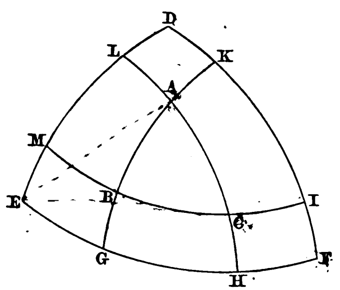
SCHOLIUM. The angles opposite at the vertex, such as ACO and BCN, are equal, for either of them is always the angle formed by the two planes ACB, OCN.

It is farther evident, that in the intersection of two arcs, ACB, OCN, the two adjacent angles, ACO, OCB, taken together, are always equal to two right angles.

PROPOSITION XI.

THEOREM.

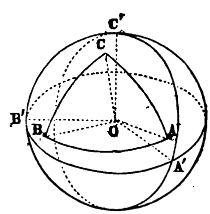
If from the vertices, A, B, C, *of the triangle* ABC, *as poles, the arcs* EF, FD, DE *be described, forming the triangle* DEF, *then will the three points* D, E, F *be reciprocally the poles of the sides* BC, AC, AB.



For, the point A being the pole of the arc EF, the distance AE is a quadrant; the point C being the pole of the arc DE, the distance CE is also a quadrant; hence, the point E is at the distance of the length of a quadrant from each of the points A and C; hence it is the pole of the arc AC (Prop. VIII.). It may be shown by the same method that D is the pole of the arc BC, and F that of the arc AB.

COR. Hence the triangle ABC may be described by means of DEF, as was DEF by means of ABC. Each triangle is said to be the *polar triangle* of the other, and triangles thus associated are called *polar triangles.*

SCHOLIUM. The two triedrals, OABC, OA′B′C′, which correspond to two polar triangles, ABC, A′B′C′, are *supplementary* triedrals (Book V., Prop. XXXIX., Scholium 2). For, by the construction for determining the point C′, we see that the edge OC′ is perpendicular to the plane AOB (Prop. VIII.), and it lies on the same side of this plane as OC, and we can reason in the same manner with regard to other edges, OB′, OA′. Since these triedrals, OABC, OA′B′C′,



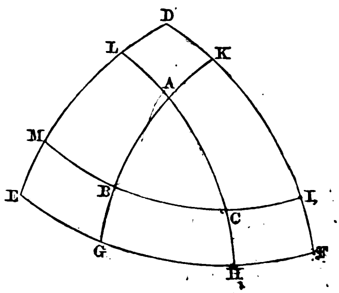
are supplementary, it follows that each angle of one of the triangles ABC, A′B′C′ is the *supplement* of the opposite side of the other triangle. But this property, by virtue of which polar triangles are also called *supplementary triangles*, is important enough for a direct demonstration, which we give in the next theorem.

PROPOSITION XII.

THEOREM.

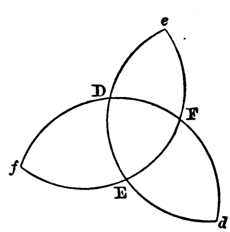
Each angle in one of the two polar triangles, ABC, DEF, *will be measured by the semi-circumference, less the opposite side in the other triangle.*

Produce, if necessary, the sides AB, AC until they meet EF in G and H; since the point A is the pole of the arc GH, the angle A will be measured by the arc GH (Prop. X.). But the arc EH is a quadrant, and likewise GF, E being the pole of AH, and F the pole of AG; hence, EH + GF is equal to a semi-circumference. Now EH + GF is the same as EF + GH; hence, the arc GH, which measures the angle A, is equal to a semi-circumference minus the side EF; in like manner the angle B will be measured by ½ *circ.* — DF, and the angle C by ½ *circ.* — DE.



This property must be reciprocal in the two triangles, since each of them is described in the same manner by means of the other. Thus we shall find the angles, D, E, F, of the triangle DEF to be measured respectively by ½ *circ.* — BC, ½ *circ.* — AC, ½ *circ.* — AB. The angle D, for example, is measured by the arc MI; but MI + BC = MC + BI = ½ *circ.*; hence the arc MI, measure of the angle D, = ½ *circ.* — BC, and so of the others.

SCHOLIUM.—It must be observed that besides the triangle DEF three others might be formed by the intersection of the three arcs DE, EF, DF. But the present proposition refers only to the central triangle, which is distinguished from the three others by the fact that the two angles A and D lie on the same side of BC, the two, B and E, on the same

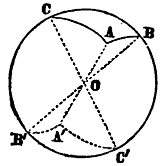


side of AC, and the two, C and F, on the same side of AB (see diagram of Proposition).

PROPOSITION XIII.

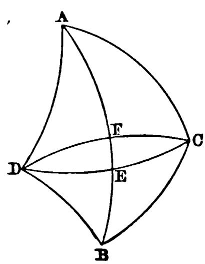
Theorem.

Two spherical triangles may be formed on the surface of a sphere which have all the parts of the one equal to all the parts of the other, each to each, and yet so arranged that the triangles will not admit of superposition.



Let ABC be a spherical triangle. Construct the triedral OABC with its vertex at the centre, and prolong its edges, AO, BO, CO, through O till they meet the surface of the sphere in the points A′, B′, C′, respectively. The triedral OA′B′C′ thus formed will be symmetrical with OABC (Book V., Prop. XXXV.). And since the plane faces and diedrals of OA′B′C′ are equal, each to each, to the plane faces and diedrals of OABC, the sides and angles of the triangle A′B′C′ are equal, each to each, to the sides and angles of the triangle ABC. But these triangles will evidently not admit of superposition, for it would be impossible to apply them to each other exactly, since the parts are not arranged alike.

Scholium 1. The triangles ABC and A′B′C′ are called *symmetrical triangles*. This designation cannot apply to isosceles spherical triangles. For two isosceles triangles which have the parts of the one equal to the parts of the other, each to each, will exactly coincide when applied to one another.



Scholium 2. This symmetrical of a spherical triangle may also be formed by taking two of the vertices, A and B, of the given triangle as poles, and describing two arcs of small circles through C, and then joining the other point of intersection, D, of these arcs with A and B, by arcs AD and BD of great circles. ADB will thus be the symmetrical of ABC.

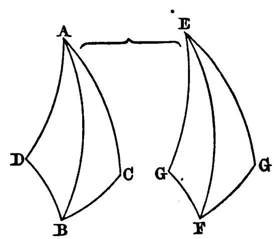
PROPOSITION XIV.

THEOREM.

Two spherical triangles on the same sphere or on equal spheres are equal in all their parts :

1. *When they have each an equal angle included between equal sides.*
2. *When two angles and the included side of the one are respectively equal to two angles and the included side of the other.*

This theorem is a consequence of the analogous properties of triedral angles (Book VI., Props. XXXVI., XXXVII.). It can also be demonstrated directly. For, if the parts of the two triangles are arranged alike, one of them may be placed on the other, as is done in the like case of rectilineal triangles (Book I., Props. VII. and VIII.).



If the parts are arranged in inverse order in the two, then the first triangle may be placed on the symmetrical of the second so as to coincide with it.

PROPOSITION XV.

THEOREM.

Two triangles on the same sphere or on equal spheres are equal in all their parts :

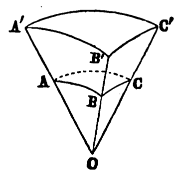
1. *When they have their sides equal, each to each.*
2. *When they have their angles equal, each to each.*

This theorem is a consequence of the analogous properties of triedral angles (Book V., Props. XXXVIII. and XL.).

Or, the second part may be proved from the first by the consideration of the polar triangles of the two given triangles.

SCHOLIUM. The second part of this proposition is not applicable to rectilineal triangles, in which only *proportionality* among the sides is the result of *equality* among the angles. But the difference in this respect between rectilineal and spherical triangles is easily accounted for. In the present proposition, as in propositions which treat of the comparison of triangles, it is expressly stated that the triangles are drawn on the same sphere, or on equal spheres. Now, similar arcs are proportional to their radii ; hence, on equal spheres, two triangles cannot be similar without being equal. Therefore, it is not strange that equality of angles should produce equality of sides.

The case would be different if the triangles were drawn on unequal spheres; then, the angles being equal, the triangles would be similar, and the homologous sides would be to each other as the radii of the spheres. These similar triangles would have *equal* triedrals at the centres of the two spheres.

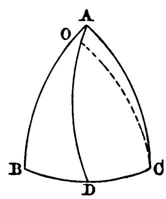


PROPOSITION XVI.

THEOREM.

In every isosceles spherical triangle, the angles opposite the equal sides are equal; and, conversely, if two angles of a spherical triangle are equal, the triangle is isosceles.

This theorem is a consequence of the analogous property of iso-edral and iso-angular triedrals (Book V., Prop. XXXIV.). Or it may be demonstrated directly after the manner of the same propositions with regard to rectilineal triangles (Book I., Props. XIII., XIV.).



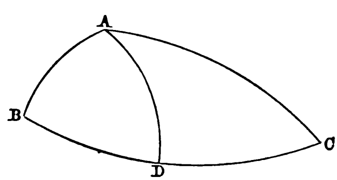
SCHOLIUM. The last demonstration proves the angle BAD = DAC and the angle BDA = ADC. Hence, the last two are right angles. Hence, *The arc of a great circle drawn from the vertex of a spherical isosceles triangle to the middle of the base is perpendicular to the base and bisects the vertical angle.*

PROPOSITION XVII.

THEOREM.

In any spherical triangle the greater side is opposite the greater angle, and, conversely, the greater angle is opposite to the greater side.

This theorem is demonstrated in the same manner as the similar theorem for rectilinear triangles (Book I., Prop. XV.). It is also a consequence of the analogous property of a triedral (Book V., Exercises, Theorem 24).



PROPOSITION XVIII.

THEOREM.

The sum of the angles of a spherical triangle is less than six right angles and greater than two right angles.

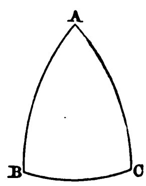
This theorem is a consequence of Proposition XLI. of Book V. But we will give, also, a direct demonstration.

First, Each angle of a spherical triangle is less than two right angles (*see the following scholium*); hence, the sum of the three angles is less than six right angles.

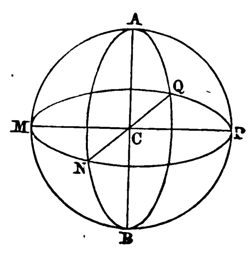
Secondly, The measure of each angle of a spherical triangle is equal to a semi-circumference *minus* the corresponding side of the polar triangle (Prop. XII.); hence, the sum of all the three is measured by three semi-circumferences *minus* the sum of the sides of the polar triangle. But this last sum is less than a circumference (Prop. VII.). Therefore, taking it from three semi-circumferences, the remainder will be greater than a semi-circumference, which is the measure of two right angles; hence, the sum of the three angles of a spherical triangle is greater than two right angles.

COR. 1. The sum of the angles of a spherical triangle is not constant like that of all the angles of a rectilineal triangle; it varies between two and six right angles, without reaching either limit. Thus two given angles do not serve to determine the third.

COR. 2. A spherical triangle may have two or three angles right, or two ~~or three obtuse.~~



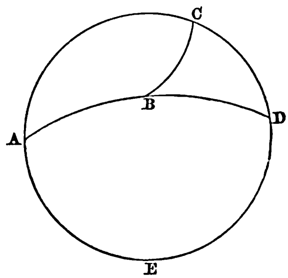
If the triangle ABC is *bi-rectangular,* that is to say, if it has two right angles, B and C, the vertex, A, will be the pole of the base, BC (Prop. VIII., Cor. 1), and the sides AB, AC, will be quadrants. If, in addition, the angle A is right, the triangle ABC will be *tri-rectangular;* its angles will be right and its sides quadrants.



The tri-rectangular triangle is contained eight times in the surface of the sphere, as is seen by the figure, supposing the arc MN to be a quadrant.

The tri-rectangular triangle is evidently its own polar triangle.

SCHOLIUM. In all the preceding observations we have supposed, in conformity with Def. 6, that our spherical triangles have each of their sides always less than a semi-circumference; whence it follows that their angles are always less than two right angles; for if the side AB is less than a semi-circumference, as also AC, both those arcs must be produced in order to meet at D. Now, the two angles ABC, CBD, taken together, are equal to two right angles; hence, the angle ABC alone is less than two right angles.



We may observe, however, that there are spherical triangles in which certain of the sides are greater than a semi-circumference, and certain of the angles greater than two right angles. Thus, if the side AC be prolonged so as to form a whole circumference, ACE, the part which remains after subtracting the triangle ABC from the hemisphere is a new triangle, also designated by ABC, whose sides are AB, BC, AEDC. It is evident, then, that the side AEDC is greater than the semi-circumference AED; and at the same time the angle B, opposite to it, exceeds two right angles by the quantity CBD.

Finally, we have excluded from the definition those triangles whose sides and angles are so large, because their solution, or the determination of their parts, is always reducible to that of the triangles included in the definition. Indeed, it is plain enough that if the sides and angles of the triangle ABC are known, it will be easy to find the angles and sides of the triangle of the same name which is the difference between a hemisphere and the former triangle.

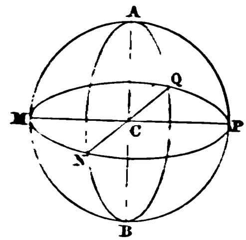
PROPOSITION XIX.

THEOREM.

The lune, AMBNA, *is to the surface of the sphere as the angle,* MAN, *of this lune is to four right angles, (or as the arc,* MN, *which measures that angle, is to the circumference).*

Suppose, in the first place, that the ratio of the arc MN and the circumference MNPQ is commensurable; is, for example, as 5 is to 48.

The circumference MNPQ being divided into 48 equal parts, MN will contain 5 of them. And if the pole A were joined with the several points of division, by as many quadrants, we will have 48 triangles in the hemisphere AMNPQ, all equal, because all their parts are equal. Hence, the whole sphere must contain 96 of these partial triangles, and the lune AMBNA will contain 10 of them; hence, the lune is to the sphere as 10 is to 96, or as 5 is to 48, in other words, as the arc MN is to the circumference.



If the arc MN is not commensurable with the circumference, we can still prove, by a mode of reasoning frequently exemplified already (Books II., III., V., VI.), that in this case also the lune is to the sphere as the arc MN is to the circumference.

COR. 1. Two lunes are to each other as their respective angles.

COR. 2. It has been already shown that the whole surface of the sphere is equal to eight tri-rectangular triangles (Prop. XVIII., Cor. 2); hence, if the area of one such triangle is taken for unity, and T be taken as the symbol of this unit, the surface of the sphere will be represented by 8T.. This granted, the surface of the lune whose angle is A will be expressed by $2A \times T$ (the angle A being estimated by taking the right angle as unity); for we have Lune : 8T : : A : 4; hence, Lune $= 2A \times T$. Here, then, we have two different unities; one for angles, being the *right angle*, the other for surfaces, being the *tri-rectangular spherical triangle*, or that triangle whose angles are all right and whose sides are quadrants. It is in this sense that we say a lune has for its measure the double of its angle. Thus, for instance, if the angle of the lune is 30°, or $\frac{1}{3}$ of a right angle, the lune is $\frac{2}{3}$, that is, $\frac{2}{3}$ of the tri-rectangular triangle.

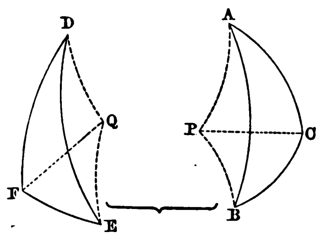
SCHOLIUM. The spherical ungula bounded by the planes AMB, ANB, is to the whole solid sphere as the angle A is to four right angles. For, the lunes being equal, the spherical ungulas will also be equal; hence, two spherical ungulas are to each other as the angles formed by the planes which enclose them.

PROPOSITION XX.

THEOREM.

Two symmetrical spherical triangles are equal in surface.

Let ABC, DEF be two symmetrical spherical triangles. Since the sides of ABC are equal, each to each, to those of DEF, the rectilinear triangles formed by the chords of these sides are equal. Hence, the circles which pass through the points A, B, C, and through the points D, E, F, are equal, and the corresponding poles, P and Q, of these circles respectively are situated in the same manner with regard to the given triangles, and the polar distances of the two circumferences are the same. Hence, joining these poles to the vertices of the triangles respectively by the arcs of great circles, PA, PB, PC, QD, QE, QF, these six arcs are all equal. Therefore, the triangles PAB and QDE have their three sides equal, each to each; they are, moreover, isosceles, hence they are superposable and equal. In like manner, triangle PAC = QDF, and PBC = QEF. Hence, DQF + FQE − DQE = APC + CPB − APB, or DFE = ABC.



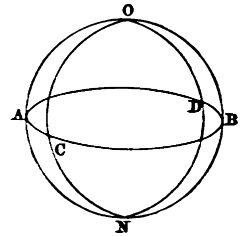
SCHOLIUM. The poles P and Q might lie inside the triangles ABC, DEF; then it would be necessary to add together the three triangles DQF, FQE, DQE, to form the triangle DEF, and in like manner it would be necessary to add the three triangles APC, CPB, APB, to form the triangle ABC. In all other respects the demonstration and the result would still be the same.

PROPOSITION XXI.

THEOREM.

If the circumferences of two great circles, AOB, COD, *intersect each other on the surface of the hemisphere,* AOCBD, *the sum of the opposite triangles,* AOC, BOD, *thus formed, will be equal to the lune whose angle is* BOD.

For, producing the arcs OB, OD on the other hemisphere until they meet at N, the arc OBN will be a semi-circumference, and AOB one also: taking OB from both, we shall have BN = AO. For a like reason, we have DN = CO, and BD = AC; hence, the two triangles AOC, BND have their three sides respectively equal; besides, they are symmetrical; hence they are equal in surface (Prop. XX.), and the sum of the triangles AOC, BOD is equivalent to the lune, OBNDO, whose angle is BOD.



SCHOLIUM. The symmetricality of the triangles AOC, BND could be shown by constructing their corresponding triedrals at the centre of the sphere, which are symmetrical.

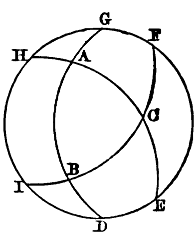
It is evident, likewise, that the two spherical pyramids which have for bases the triangles AOC, BOD, taken together, are equivalent to the spherical ungula whose angle is BOD.

PROPOSITION XXII.

THEOREM.

The surface of any spherical triangle is to the surface of the tri-rectangular triangle, as the excess of the sum of its three angles above two right angles is to one right angle.

Let ABC be the proposed triangle; produce its sides until they meet the great circle DEFG, drawn at pleasure without the triangle. By the preceding theorem, the two triangles ADE, AGH, taken together, are equivalent to the lune whose angle is A, and whose measure is $2A \times T$ (Prop. XIX., Cor. 2); thus we have $ADE + AGH = 2A \times T$; for a like reason, $BGF + BID = 2B \times T$, and $CIH + CFE = 2C \times T$. But the sum of these six triangles exceeds the hemisphere by twice the triangle ABC, and the hemisphere is represented by $4T$; hence, twice the triangle $ABC = 2A \times T + 2B \times T + 2C \times T - 4T$, and consequently $\frac{ABC}{T} = \frac{A + B + C - 2}{1}$; which was to be proved.



COR. 1. Making $T = 1$, we have $ABC = A + B + C - 2$. Hence we sometimes say *the spherical triangle is measured by the excess of the sum of its angles above two right angles.* As many right angles as there are in this measure, just so many tri-rectangular triangles, or eighths of the sphere, which are the units of surface (Prop. XIX., Cor. 2), will be contained in the proposed triangle.

For example, if the angles of a given triangle are respectively 110°, 80°, and 20°, then the three angles are equal to $2\frac{1}{3}$ right angles, and the proposed triangle will be represented by $2\frac{1}{3} - 2$, or $\frac{1}{3}$; hence, it will be equal to $\frac{1}{3}$ of the tri-rectangular triangle, or to one-twenty-fourth of the surface of the sphere.

Cor. 2. The spherical triangle ABC is equivalent to the lune whose angle is $\frac{A + B + C}{2} - 1$; likewise the spherical pyramid whose base is ABC is equivalent to a spherical ungula whose angle is

$$\frac{A + B + C}{2} - 1.$$

SCHOLIUM. While the spherical triangle ABC is compared with the tri-rectangular triangle, the spherical pyramid, whose base is ABC, is compared with the tri-rectangular pyramid, and from these comparisons the same proportions result. The solid angle at the vertex of the pyramid is compared likewise with the solid angle at the vertex of the tri-rectangular pyramid, and this comparison is established by the coincidence of the parts. Now, if the bases of the pyramids coincide, it is evident that the pyramids themselves will coincide, as also the solid angles at their vertices. From this several consequences result.

First.—Two triangular spherical pyramids are to each other as their bases; and since a polygonal pyramid may be divided into several triangular pyramids, it follows that any two spherical pyramids are to each other as the polygons which form their bases.

Secondly.—The solid angles at the vertices of these same pyramids are likewise in proportion to the bases; hence, to compare any two solid angles, their vertices must be placed at the centre of two equal spheres, and these solid angles will be to each other as the spherical polygons intercepted between their planes or faces.

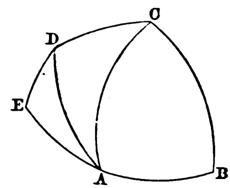
The vertical angle of the tri-rectangular pyramid is formed by three planes perpendicular to each other; this angle, which may be called a *right solid angle*, will serve as a very fit unit of measure for all other solid angles. This granted, the same number that gives the area of a spherical polygon will give the measure of the corresponding solid angle. For example, if the area of the spherical polygon is $\frac{3}{4}$, that is, if it is $\frac{3}{4}$ of the tri-rectangular triangle, the corresponding solid angle will also be $\frac{3}{4}$ of the right solid angle.

PROPOSITION XXIII.

THEOREM.

The surface of a spherical polygon is equal to the sum of its angles minus the product of two right angles by the number of sides in the polygon less two; the right angle being the unit of measure of angles, and the tri-rectangular triangle being the unit of measure of surfaces.

Through one of the vertices, A, let diagonals, AC, AD, be drawn to all the other vertices; the polygon ABCDE will be divided into as many triangles, *minus* two, as it has sides. But the surface of each triangle is measured by the sum of its angles *minus* two right angles; and it is evident that the sum of all the angles of all the triangles is equal to the sum of the angles of the polygon; hence the surface of the polygon is equal to the sum of its angles diminished by twice as many right angles as it has sides, *minus* two.



SCHOLIUM. Let s be the sum of the angles of a spherical polygon, n the number of its sides; the right angle being considered unity, the surface of the polygon will be measured by

$$s - 2(n - 2), \text{ or } s - 2n + 4.$$

For example, let the angles of a spherical pentagon be 110°, 120°, 150°, 85°, and 81°, respectively. Then s $= 6\frac{1}{15}$ right angles, $2n = 10$ right angles, and $s - 2n + 4 = \frac{1}{15}$ of a right angle. Hence, the polygon is $\frac{1}{15}$ of the tri-rectangular triangle, or $\frac{1}{120}$ of the whole surface of the sphere.

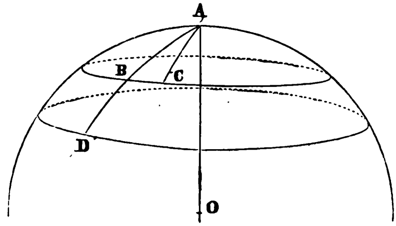
PROPOSITION XXIV.

LEMMA.

1. *If the arc* AB *of a great circle be equal to the arc* AC, *then the shortest path from* A *to* B, *on the surface of the sphere, will be equal to the shortest path from* A *to* C.

2. *If the arc* AD *is greater than the arc* AC, *the shortest path from* A *to* D, *on the surface of the sphere, will be greater than the shortest path from* A *to* C.

First.—If we cause the arc AB to revolve about the diameter of the sphere which passes through the point A as an axis; then AB, in one of its positions, will coincide with AC, and the shortest path from A to B will coincide with the shortest path from A to C.



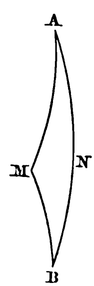
Secondly.—If we cause AD and AC both to revolve about the diameter through A, the points D and C will generate two circles whose centres are on the diameter, and whose planes are perpendicular to the diameter, and every line on the surface of the sphere which joins A to D will cut the circumference described by the point C. Therefore, the shortest path from A to D will be equal to the shortest path from A to C, *plus* a certain line. Therefore, etc., etc.

PROPOSITION XXV.

Theorem.

The shortest path from one point to another on the surface of a sphere is the arc of a great circle, less than a semi-circumference, which joins the two points.

Let AB be the arc of the great circle joining the two points. If AB is not the shortest path from A to B; then suppose that there is without AB a point, M, on the shortest path. Draw the arcs of great circles AM, BM, and take AN = AM. Then (Prop. XXIV.) the shortest path from A to N is equal to the shortest path from A to M, and hence, if our supposition be true as to M, the shortest path from M to B must be less than the shortest path from N to B.



But (Prop. VI.) AB < AM + MB, and taking AN from one side and its equal AM from the other, there remains NB < MB, or (Prop. XXIV.) the shortest path from N to B less than the shortest path from M to B. Hence, our supposition of a point outside of AB, lying on a shorter path, has led us to an absurd contradiction. Hence, no point of the shortest distance from A to B can lie without the arc AB of the great circle joining these points. Therefore, etc.

EXERCISES ON BOOK VII.

THEOREMS.

1. If a point, C, without a sphere, be joined with the centre, O, of the sphere, by the line CAOB, meeting the surface in the points C and B, then CA is the shortest line and CB the longest line which can be drawn from C to the surface of the sphere.

2. C is at equal distances from all points of any circle described from A and B as poles.

3. If, from the centre, O, of a sphere, we draw OAC perpendicular to the line EF without the sphere, meeting the sphere in A, and EF in C, then AC is the shortest distance from the line EF to the sphere.

4. If, from the centre, O, of a sphere, we draw OAC perpendicular to a plane, MN, without the sphere, meeting the surface of the sphere in A and the plane in C, then will AC be the shortest distance from the plane to the sphere.

5. The smallest circle whose plane passes through a given point within a sphere, is that one whose plane is perpendicular to the radius through the given point.

6. If, from any point on a sphere as a pole, with a polar distance equal to one-third of a quadrant, we describe a circle on the sphere, the radius of this circle will be one-half the radius of the sphere.

7. If we describe a circle with a polar distance equal to one-fifth of a quadrant, the diameter of the circle will be the greater part of the radius of the sphere when that radius is divided in extreme and mean ratio.

8. *First.*—If the sides of a spherical polygon A'B'CD'E' be described from the vertices of a polygon ABCDE, as poles, then will the vertices of A'B'C'D'E' be poles of the sides of ABCDE.

Secondly.—In two polar spherical polygons, each angle of one of them is the supplement of that side of the other of which its vertex is the pole.

9. If the arc of a great circle bisect the angle of two other arcs,

then every point on this bisecting arc is equally distant from the two sides of the angle, and every point within the angle not on this bisecting arc is unequally distant from these two sides.

10. Two spherical triangles are equal in surface when their polars have equal perimeters, and conversely.

11. The sum of two angles of a spherical triangle is always less than the third angle increased by two right angles.

12. Show that if through any point in the surface of a sphere three chords be drawn at right angles to each other, the sum of the squares of these chords is equal to the square of the diameter.

13. If through a point within a sphere two chords be drawn, the product of the parts of one chord is equal to the product of the parts of the other.

14. If from a point without a sphere two secants be drawn terminating in the concave surface, the product of one secant by its external part is equal to the product of the other by its external part.

15. If from a point without a sphere a secant and tangent be drawn, the product of the secant by its external part is equal to the square of the tangent.

16. Tangents drawn to two intersecting spheres from any point of their plane of intersection are equal.

17. A sphere may be inscribed in, and one circumscribed about, any one of the regular polyedrons, and these spheres will be concentric.

18. The surfaces of two regular polyedrons of the same number of faces, are to each other as the squares of the radii of the inscribed spheres; and also as the squares of the radii of the circumscribed spheres; and their volumes are to each other as the cubes of these radii.

19. The volumes of polyedrons circumscribed about the same sphere, are to each other as their surfaces.

20. The tangent planes common to two spheres meet the line of centres in the same points.

Geometric Loci.

1. The locus of points, the sum of the squares of the distances of each of which from two fixed points is constant, is a sphere.

2. The locus of points, such that the distances of each one of them from two fixed points are in a constant ratio, is a sphere.

3. Find the locus of the centres of sections made in a sphere by planes which pass through a given point.

4. Find the locus of the centres of sections made in a given sphere by planes which contain a given straight line.

5. Find the locus of the points of contact of tangents drawn from the same point to a given sphere.

6. Find the locus of points of contact of planes tangent to a sphere and parallel to a given straight line.

7. Find the locus of the poles of great circles making a given angle with a given great circle.

8. Find the locus of all the points on the surface of a sphere, each one of which is equidistant from two given points on the surface.

9. Find the locus of all the points on the surface of a sphere, each one of which is equidistant from two great circles of the sphere.

10. The locus of the vertices of a spherical triangle, whose vertical angle is equal to the sum of the other two, is a small circle of the sphere: find its pole and polar distance.

11. Find the locus of the points from which a given straight line appears under the same angle.

12. Find the locus of the centres of spheres of a given radius tangent to a given plane.

13. Find the locus of centres of spheres of a given radius tangent externally to a given sphere.

14. What is the locus of the centres of spheres tangent to a given plane at a given point? to a given sphere at a given point?

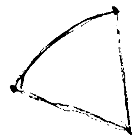
15. Find the locus of the centres of spheres of a given radius and passing through two given points.

16. Find the locus of the centres of spheres which have three points in common.

17. Find the locus of the centres of spheres of given radius which are tangent to two given intersecting planes.

18. Find the locus of centres of spheres tangent to three given planes.

19. Find the locus of the centres of spheres of given radius tangent to two given spheres.



PROBLEMS.

1. Bisect a given arc of a great circle.

2. Bisect the angle contained by two given arcs of great circles.

3. Circumscribe a circle about a given spherical triangle.

4. Inscribe a circle in a given spherical triangle.

Scholium : The pole of the circle inscribed in a spherical triangle is also the pole of the circle circumscribed about the polar triangle ; and the radii of these circles are complements of each other.

5. At a point on an arc of a great circle draw a second arc making a given angle with the first.

6. From a point without an arc of a great circle draw a second arc, making a given angle with the first.

7. Construct a spherical triangle, given an angle and two adjacent sides.

Application : Given the latitudes and difference of longitudes of two places on the earth, construct the distance between them, *i. e.*, the arc of great circle which joins them (regarding the earth as a sphere).

8. Construct a spherical triangle, given one side and the adjacent angles.

9. Construct a spherical triangle, given three sides.

Application : Given the latitudes of two places on the earth and the distance between them, construct the difference of longitude.

10. Construct a spherical triangle, knowing the three angles.

NOTE.—The above problems may be solved graphically on a globe with a pair of spherical dividers or some simple substitute for these. In the absence of a globe the solutions can be indicated merely.

11. Compute the area of the spherical triangle in terms of the tri-rectangular triangle, given its angles 61°, 109°, and 127°.

12. Compute the area of the spherical triangle in terms of the tri-rectangular triangle, its angles being 52° 36′, 72° 15′, 87° 40′.

13. Compute the angles of a spherical triangle, knowing that they are to each other as the numbers 4, 6, and 7, the area of the triangle being one-fourth of the tri-rectangular triangle.

14. Given the radius of the inscribed sphere in each case, compute the edges and volume of the regular tetraedron, hexaedron, and octaedron.

15. Given the side of a cube, find the diameter of the circumscribing sphere by a plane construction.

16. A tangent to the earth's surface (regarded as a sphere) from the top of a vertical pole ten feet high touches the earth at a distance of (about) four miles from the pole. Find the radius of the earth in miles.

Indicate the solutions of the following problems:

17. Construct a sphere of given radius which shall pass through three given points.

18. Construct a sphere of given radius which shall pass through two given points and be tangent to a given plane.

19. Construct a sphere of given radius which shall pass through a given point and be tangent to two given planes which intersect each other.

20. Construct a sphere of given radius which shall be tangent to three given planes.

21. Construct a sphere of given radius which shall be tangent to a given sphere and pass through two given points.

22. Construct a sphere of given radius which shall pass through a given point and be tangent to two given spheres.

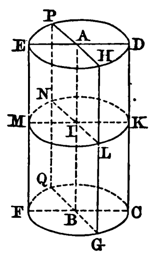
23. Construct a sphere of given radius which shall pass through a given point and be tangent to a given plane and a given sphere.

BOOK VIII.

MEASURES OF THE THREE ROUND BODIES.

DEFINITIONS.

1. A *cylinder of revolution* is the solid generated by the revolution of a rectangle, ABCD, conceived to turn about the immovable side AB.



In this movement the sides AD, BC, remaining always perpendicular to AB, describe the equal circles DHP, CGQ, which are called the *bases of the cylinder*, and the side CD describes its *convex surface.*

The immovable line AB is styled the *axis of the cylinder.*

Every section, KLM, made in the cylinder perpendicular to the axis, is a circle equal to each of the bases: for, whilst the rectangle ABCD turns about AB, the line IK, perpendicular to AB, describes a circular plane equal to the base, and this plane is nothing else than the section made perpendicular to the axis at the point I.

Every section, PQGH, made through the axis, is a rectangle, double the generating rectangle ABCD.

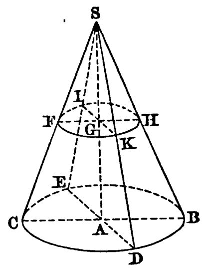
NOTE.—The cylinder of revolution is often called a *right cylinder with circular base.*

2. A *cone of revolution* is the solid generated by the revolution of the right-angled triangle SAB, conceived to turn about the immovable side SA.

In this movement the side AB describes a circle, BDCE, which is called the *base of the cone*, and the hypothenuse, SB, describes its *convex surface.*

The point S is called the *vertex of the cone*, SA the *axis*, or the *altitude*, and SB the *side*, or *slant height.*

Every section, HKFI, made perpendicular to the axis, is a circle; every section, SDE, made through the axis, is an isosceles triangle, double the generating triangle SAB.



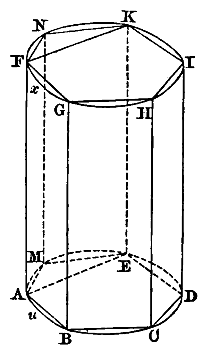
NOTE.—The cone of revolution is often called a *right cone with circular base.*

3. If from the cone SCDB we cut off the cone SFKH, by a section parallel to the base, the remaining solid, CBHF, is called a *truncated cone,* or *frustum of a cone.*

It may be conceived as described by the revolution of the trapezoid ABHG, whose angles, A and G, are right, about the side AG. The immovable line AG is called the *axis,* or the *altitude of the frustum,* the circles BDC, HFK are its *bases,* and BH is its *side.*

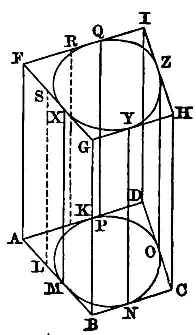
4. Two cylinders, or two cones, are *similar* when they are generated by similar rectangles, or by similar triangles respectively, (*i. e.*) when their axes are to each other as the diameters of their bases.

5. If, in the circle ACD, which forms the base of a cylinder, a polygon, ABCDE, be inscribed, and on the base, ABCDE, a right prism be erected, equal in altitude to the cylinder, the prism is said to be *inscribed in the cylinder,* or the cylinder to be *circumscribed about the prism.* It is evident that the edges, AF, BG, CH, etc., of the prism, being perpendicular to the plane of the base, are included in the convex surface of the cylinder; hence, the prism and the cylinder touch each other along these edges.



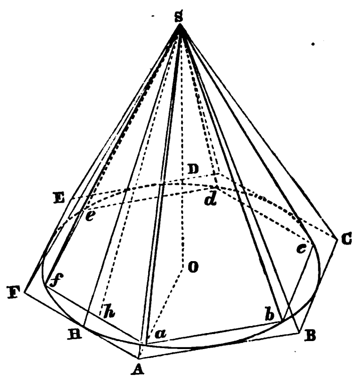
6. Similarly, if ABCD is a polygon circumscribed about the base of a cylinder, and on the base, ABCD, a right prism be erected, equal in altitude to the cylinder, the prism is said to be *circumscribed about the cylinder,* or the cylinder *inscribed in the prism.*

Let M, N, etc., be the points of contact of the sides AB, BC, etc., and at the points M, N, etc., let MX, NY, etc., be drawn perpendicular to the plane of the base; it is evident that these perpendiculars will be at the same time in the sur-



face of the cylinder, and in that of the circumscribed prism ; hence, they will be their lines of contact.

7. If, in the circle which forms the base of a cone, a polygon, *abcdef*, be inscribed, and the vertex S of the cone be joined to the vertices of the polygon, the pyramid thus formed is said to be *inscribed in the cone*, and the cone to be circumscribed about the pyramid. The edges of the pyramid are on the convex surface of the cone, and hence are common to both the pyramid and cone.



8. Similarly, if ABCDEF is a polygon circumscribed about the base of a cone, and if the vertices of this polygon be joined to S, the vertex of the cone—a pyramid is thus formed which is said to be *circumscribed about the cone*, and the cone is said to be *inscribed in the pyramid.*

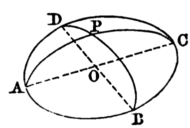
N. B.—The cylinder of revolution, the cone of revolution, and the sphere, are the *three round bodies* treated of in "The Elements." The treatment of cylinders and cones, other than those of revolution, is usually left for the higher branches of mathematics.

PROPOSITION I.*

LEMMA.

A plane surface, OABCD, *is less than every other surface*, PABCD, *terminating in the same perimeter*, ABCD.

This proposition is almost evident enough to be ranked as an axiom ; for the plane may be regarded among surfaces as the straight line is among lines : the straight line is the shortest between two given points, as the plane is the least of all surfaces having the same perimeter. Yet, since it is expedient to reduce the axioms to the least number possible, we give a demonstration that will remove all doubt concerning this truth.



A surface being extended in length and breadth, one surface can-

* Propositions I. and II. are preliminary Lemmas concerning surfaces.

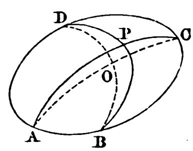
not be conceived as greater than another unless the dimensions of the first in some direction exceed those of the second ; and if it should happen that the dimensions of one surface are in all directions less than the dimensions of another surface, the first surface would evidently be the least of the two. Now, in whatever way we pass the plane BPD, cutting the plane surface along BD, and the other surface along BPD, the straight line BD will always be less than BPD ; hence, the plane surface OABCD is less than the surface PABCD, which envelops it.

PROPOSITION II.

Lemma.

Every convex surface, OABCD, *is less than any other surface enveloping it whilst resting on the same perimeter,* ABCD.

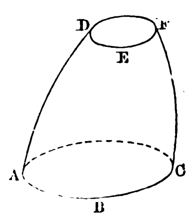
We repeat here that by *convex surface* is understood a surface which cannot be cut by a straight line in more than two points : nevertheless, it is possible that a straight line may apply itself exactly, in a certain direction, to a convex surface ; examples of this are seen in the surfaces of the cone and the cylinder. We will also observe that the name *convex surface* is not limited to curved surfaces alone ; it includes *polyedral* surfaces, or surfaces composed of several faces, and also surfaces partly curved and partly polyedral.



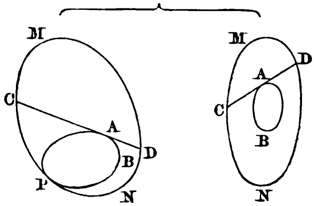
This granted, we demonstrate this Lemma after the manner of Proposition I. For, if we make sections of the surfaces, in any direction, the sections of the enveloped convex surface OABCD will always be less than the corresponding section of the enveloping surface PABCD (Book IV., Prop. XI.). Hence, the surface OABCD is less than the enveloping surface PABCD.

Scholium. By an entirely similar mode of reasoning it may be proved :

First.—That if a convex surface terminated by two perimeters, ABC, DEF, is enveloped by any other surface terminated by the same perimeters, the enveloped surface will be the smaller of the two.



Secondly.—That if a convex surface, AB, is enveloped on all sides by another surface, MN, whether they have any points, lines, or planes in common, or have no point in common, the enveloped surface will always be less than the enveloping surface.

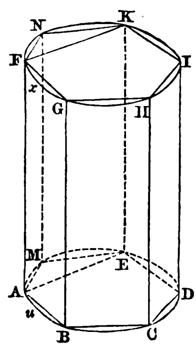


PROPOSITION III.

THEOREM.

The convex surface of a cylinder of revolution is greater than the convex surface of any inscribed prism, and less than the convex surface of any circumscribed prism.

For the convex surface of the cylinder and that of the inscribed prism may be considered as having the same length, since every section made in either, parallel to AF, is equal to AF; and if these surfaces be cut in order to obtain the breadth of them by planes parallel to the base, or perpendicular to the edge, the one section will be equal to the circumference of the base, the other to the perimeter of the polygon, ABCDE, which is less than that circumference.



Hence, with equal length the cylindrical surface has greater breadth than the surface of the prism. Hence, the former is greater than the latter. By a similar demonstration the convex surface of the cylinder might be shown to be less than that of any circumscribed prism, ABCDFGHI (See Fig. Def. 6).

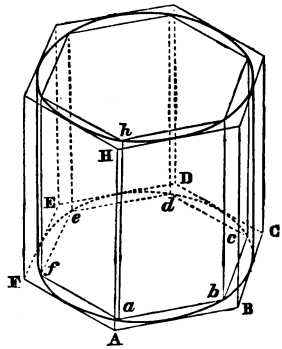
PROPOSITION IV.

THEOREM.

1. *The convex surface of a cylinder of revolution is the common limit of the convex surfaces of the inscribed and circumscribed regular prisms when the number of sides of the bases of these prisms is increased indefinitely.*

2. *The volume of the cylinder is at the same time the common limit of the volumes of these two prisms.*

First.—The convex surface of the cylinder is comprised between the convex surface of the inscribed prism and that of the circumscribed prism.



The convex surface of inscribed prism = perimeter of its base *abcdef* × altitude *ah*, and the convex surface of circumscribed prism = the perimeter of its base ABCDEF × altitude AH. And the difference of the two is

perimeter ABCDEF × AH —*perimeter abcdef* × *ah*.

Now, the altitudes are equal, and equal to that of the cylinder, and as the number of sides of two bases is increased without limit, the difference of the perimeters,

(ABCDEF — *abcdef*,)

may be made as small as we please (Book IV., Prop. XII.).

Hence, the difference of the convex surfaces of the prisms may be made as small as we please, and still more can the difference between the convex surface of the cylinder and either of them be made as small as we please. Therefore, the convex surface of the cylinder is the common limit of the two convex surfaces of the prisms when the number of sides of the bases is increased indefinitely. (See Definition of Limit. Book IV., page 161.)

Secondly.—The volume of the cylinder is comprised between the volumes of the two prisms. But the volume of inscribed prism = area *abcde* × altitude *ah*. And volume of circumscribed prism = area ABCDE × altitude AH. And the difference between these volumes is

(area ABCDE — area *abcde*) × altitude AH.

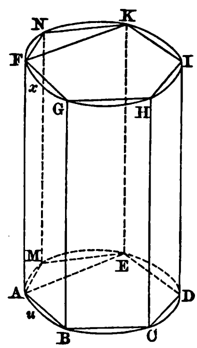
Now, by increasing the number of sides of the bases indefinitely, the difference between the areas of the bases can be made as small as we please. Hence, the difference between the volumes of the inscribed and circumscribed prism will become as small as we please, and hence, still more may the difference between the volume of the cylinder and either of them be made less than any assignable quantity. Therefore, the volume of the cylinder is the common limit of the volumes of the two prisms, when the number of sides of the bases is increased without limit.

PROPOSITION V.

THEOREM.

The convex surface of a cylinder of revolution is equal to the product of the circumference of its base by its altitude.

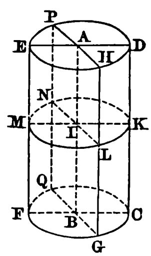
Let S = convex surface of the cylinder, circ. OA = circumference of its base, and AF its altitude. Let s = convex surface of the inscribed prism, p = the perimeter of its base. Then $s = p \times$ AF (Book VI., Prop. VI.), whatever be the number of sides of its base. But when the number of sides is increased indefinitely, s converges to the limit S (Prop. IV.), and p to its limit circ. OA. Therefore, we have at the limit (see Fundamental Principle of Limits, p. 162),



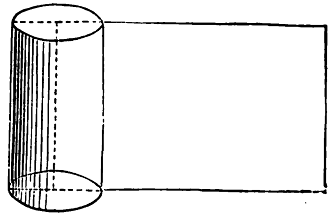
$$S = \text{circ. OA} \times \text{AF}.$$

Hence, the convex surface of the cylinder is equal to the product of the circumference of its base by its altitude.

COR. If a line EF revolve around a line AB to which it is parallel, it will generate a surface which is measured by the line EF, multiplied by the circumference MLKN, described by its middle point M. For the surface described is the convex surface of the cylinder, whose altitude is EF, and the circumference of whose base is equal to that of any parallel section MNKL.



SCHOLIUM I. This theorem may also be established thus: Beginning at any particular generating line, we may roll the convex surface out on a plane; this development is a rectangle which has for its altitude the altitude of the cylinder, and for its base the circumference of the base of the cylinder. Hence, it is equal to the product of these two dimensions.



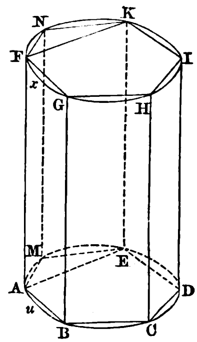
SCHOLIUM 2. If R = the radius of the base of the cylinder, and H the altitude, then the convex surface $S = 2\pi R \times H$; and to get the whole surface, we add to this the sum of the areas of the two bases, or $2\pi R^2$. Thus the whole surface $= 2\pi RH + 2\pi R^2 = 2\pi R(H + R)$.

PROPOSITION VI.

THEOREM.

The volume of a cylinder of revolution is equal to the product of its base by its altitude.

Let V, area OA, and AF, be the volume of the cylinder, the area of its base, and its altitude; let v, b, be the volume and area of the base of a regular prism inscribed in the cylinder, we have



$$v = b \times AF \text{ (Book VI., Prop. XVIII.)},$$

whatever be the number of sides of the base b of the prism. But when the number of these sides is increased indefinitely, V is the limit of v and circ. OA of polygon b. Hence, passing to the limit,

$$V = B \times AF.$$

COR. 1. *First.—Cylinders of the same altitude are to each other as their bases.*

Second.—Cylinders of the same base are to each other as their altitudes.

Third.—Any two cylinders are to each other as the products of their bases and altitudes.

COR. 2. *Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases.*

For, the bases being as the squares of the diameters, in the case of similar cylinders are to each other as the squares of the altitudes (Def. 3). Hence, the cylinders being to each other as the products of the bases by their altitudes, are to each other, also, as the cubes of their altitudes, and also (Def. 3) as the cubes of the diameters of their bases.

SCHOLIUM. Let R be the radius of a cylinder's base, and H the altitude; then area OA $= \pi R^2$. Hence, the volume

$$V = \pi R^2 \times H = \pi R^2 H,$$

also $$\pi R^2 H = 2\pi RH \times \tfrac{1}{2}R.$$

But $2\pi RH$ is the convex surface of the cylinder (Prop. V., Scholium 2). Hence, *The volume of a cylinder of revolution is equal to its convex surface multiplied by one half of its radius.*

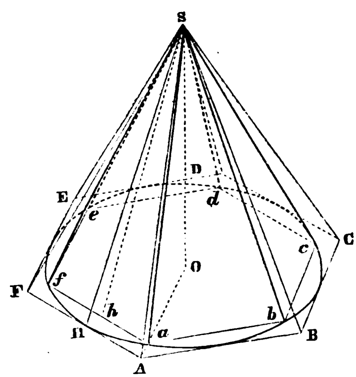
PROPOSITION VII.

THEOREM.

1. *The convex surface of a cone of revolution is the common limit of the convex surfaces of the inscribed and circumscribed regular pyramids when the number of sides of their bases is increased without limit.*

2. *The volume of the cone is at the same time the common limit of the volumes of the two pyramids.*

First.—We can show, as in the case of the convex surface of the cylinder (Prop. III.), that the convex surface of the cone is greater than the convex surface of the inscribed pyramid, and less than that of the circumscribed pyramid, and is therefore comprised between these two convex surfaces.



Now the convex surface of inscribed pyramid = perimeter of base $abcdef \times \frac{1}{2}$ slant height Sh (Book VI., Prop. XIX.).

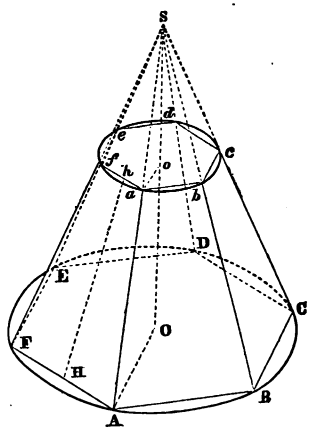
And the convex surface of circumscribed pyramid = perimeter of base ABCDEF $\times \frac{1}{2}$ slant height SH.

Hence, their difference = perimeter ABCDEF $\times \frac{1}{2}$SH − perimeter $abcdef \times \frac{1}{2}Sh$. But when the number of sides of the bases is increased indefinitely, the perimeter $abcdef$ converges to the perimeter ABCDEF (Book IV., Prop. XII.), and the slant height Sh converges to the slant height SH, or the side of the cone, as its limit. Hence, the difference between the convex surfaces of the pyramids may be made as small as we please. Therefore, still more can the difference between the convex surface of the cone and either of these be made as small as we please. Hence, it is the common limit of these two convex surfaces.

Secondly.—The volume of the cone is comprised between the volumes of the two pyramids, and, by exactly the same reasoning employed in regard to the volume of the cylinder (Prop. IV.), it becomes evident that the volume of the cone is the common limit of the volumes of the inscribed and circumscribed regular pyramids when the number of the sides of the bases is increased without limit.

COR. *The convex surface of the frustum of a cone is the limit of the convex surfaces of the inscribed and circumscribed frustums of regular pyramids; and also the volume of the frustum is the limit of the volumes of these pyramidal frustums, when the number of sides of the bases is increased indefinitely.*

For the convex surface of the frustum of the cone ABCDEF, *abcdef*, is the difference of the convex surfaces of the two cones, S-ABCDEF and S-*abcdef*. And the convex surface of the regular inscribed frustum of a pyramid is the difference of the two pyramids, S-ABCDEF and S-*abcdef*. But the convex surfaces of the cones are the limit of the convex surfaces of these pyramids, when the number of sides of the bases is increased, etc. Hence, their difference, or the convex surface of the frustum of the cone, is equal to the limit of the difference between the convex surfaces of the two inscribed pyramids, or the regular inscribed frustum (see Fundamental Principle of Limits, Consequence First, page 162), and similarly for the convex surface of the regular circumscribed frustum.



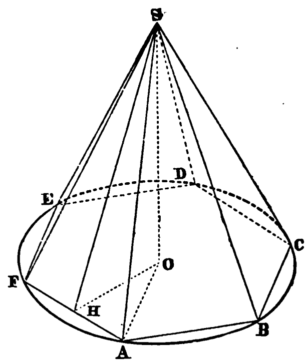
The same reasoning applies to the volume of the frustum of the cone.

PROPOSITION VIII.

THEOREM.

The convex surface of a cone of revolution is equal to the circumference of its base, multiplied by half of its side.

Let S = convex surface of the cone, circ. OA = the circumference of its base, and SA its side; and s = convex surface, p the perimeter of the polygon which forms its base, and SH = the slant height of the inscribed pyramid. Then

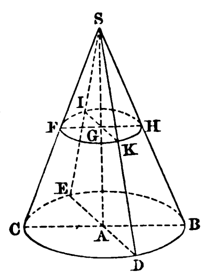


$s = p \times \frac{1}{2}$SH (Book VI., Prop. XIX.),

whatever be the number of sides of the base of the pyramid. Now, when this number of sides is increased indefinitely, the convex surface s converges to the limit S (Prop. VII.), p to the limit circ. OA (Book IV., Prop. XII.), and SH to the limit SA. Therefore,

$$S = \text{circ. OA} \times \tfrac{1}{2}\text{SA}.$$

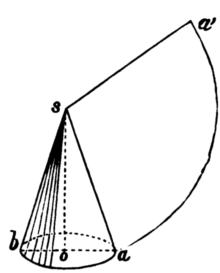
COR. The convex surface of the cone is also *equal to the product of its side by the circumference of the circular section of the cone, made by a plane parallel to the base through the middle point of its side.*



For the radius of this section is one-half the radius of the base, and hence the circumference of this section is one-half the circumference of the base (Book IV., Prop. XIII.). Hence,

$$S = \text{circ. CA} \times \tfrac{1}{2}\text{SC} = \tfrac{1}{2}\text{circ. CA} \times \text{SC} = \text{circ. GF} \times \text{SC}.$$

SCHOLIUM 1. Another mode of demonstrating this theorem is as follows:



Since all the points of the circumference of the base are equally distant from the vertex of the cone, if we unroll the convex surface on a plane, beginning at the side Sa, the surface rolled out will be a circular sector, Saa', the radius of which is the side Sa, and the arc, aa', of which is equal to the circumference Oa. Hence, the convex surface of the cone, which is this sector, is equal to $aa' \times \frac{1}{2}Sa$, or equal to

$$\text{circumference } Oa \times \tfrac{1}{2}Sa.$$

SCHOLIUM 2. If R = the radius of the base of the cone, and L its side, then the convex surface

$$S = 2\pi R \times \tfrac{1}{2}L = \pi RL.$$

And to obtain the whole surface, we must add to this the area of the base $= \pi R^2$. Thus:

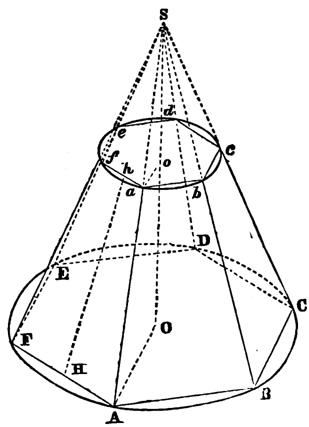
$$\text{The whole surface} = \pi R^2 + \pi RL = \pi R(R + L).$$

PROPOSITION IX.

THEOREM.

The convex surface of the frustum of a cone of revolution is equal to its side multiplied by half the sum of the circumferences of its two bases.

Let S, Aa, circ. OA, circ. oa, be the convex surface, the side, and the circumferences of the bases of the frustum of the cone; s, Hh, p, p', the convex surface, slant height, and perimeters of the bases of the inscribed frustum of the regular pyramid. Then



$s = \frac{1}{2}(p + p')\text{H}h$(Book VI., Prop. XX.),

whatever be the number of sides of the bases of the frustum. But, when the number of sides is increased indefinitely, s converges to the limit S (Prop. VII., Cor.), Hh to the limit Aa, p to the limit circ. OA, and p' to circ. oa (Book IV., Prop. XII.).

Hence, passing to the limits,

$$S = \tfrac{1}{2}(\text{circ. OA} + \text{circ. } oa) \times \text{A}a.$$

SCHOLIUM. If R and R′ are the radii of the bases of the frustum of the cone, and L its side, we have

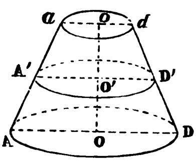
$$\text{circ. OA} = 2\pi R,$$

and $\text{circ. } oa = 2\pi R'$;

consequently $S = \pi(R + R')L$,

And to get the whole surface we must add to S the sum of the areas of the two bases $\pi R^2 + \pi R'^2$.

COR. If, through the middle point A′, of the side A*a*, we draw a plane parallel to the bases; the radius of the circular section determined by this plane is parallel to the radii, AO, *ao*, of the bases, and equal to their half sum (Book III., Prop. VII.). It follows, then, that the circumference A′O′ is equal to half the sum of the circumferences AO, *ao* (Book IV., Prop. XIII.). Hence, it may also be asserted, that the *surface of the frustum of a cone is equal to its side multiplied by the circumference of a section at equal distances from the two bases.*



SCHOLIUM. *If any line lying in the plane with another line, and wholly on one side of it, revolve about this line as an axis, the surface generated is always equal to the revolving line multiplied by the circumference described by its middle point.* For this line describes either the convex surface of a cylinder, of a cone, or of the frustum of a cone, and in each case the measure of the surface has been shown to be as above stated (Prop. V. Cor., Prop. VIII. Cor., and Cor. above).

PROPOSITION X.

THEOREM.

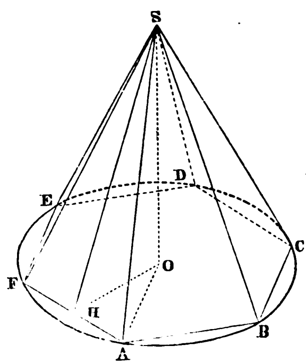
The volume of a cone of revolution is equal to the product of its base by one-third of its altitude.

Let V, area OA, and SO, be the volume of the cone, the area of its base, and its altitude; let v and b be the volume and area of the base of a regular pyramid inscribed in this cone. Then

$$v = b \times \tfrac{1}{3}SO \text{ (Book VI., Prop. XXV.)}$$

whatever be the number of sides of the base. But, when the number of these sides is increased indefinitely, v converges to V as its limit (Prop. VII.), and b to the limit area OA. Hence,

$$V = \text{area OA} \times \tfrac{1}{3}SO,$$



or, the volume of a cone is equal to the product of its base by one-third of its altitude.

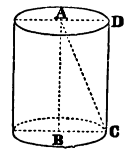
COR. 1. *First.—A cone is the third of a cylinder having the same base and the same altitude.*

Second.—Cones of equal altitudes are to each other as their bases.

Third.—Cones of equal bases are to each other as their altitudes.

Fourth.—Similar cones are to each other as the cubes of the diameters of their bases, or as the cubes of their altitudes.

COR. 2. When a rectangle, ABCD, revolves about one of its sides, AB, the triangle ABC generates a cone whose volume is one-third of the volume of the cylinder generated by ABCD. Hence, the triangle ADC must at the same time generate a volume which is two-thirds of the cylinder generated by ABCD.



SCHOLIUM. If R be the radius of the cone's base, H the altitude, and V the volume of the cone, then

$$V = \pi R^2 \times \tfrac{1}{3}H, \text{ or } \tfrac{1}{3}\pi R^2 H.$$

PROPOSITION XI.

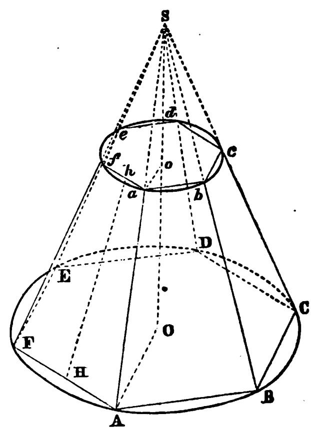
THEOREM.

The volume of the frustum of a cone is equivalent to the volumes of three cones whose common altitude is the altitude of the frustum, and whose bases are the lower base of the frustum, the upper base of the frustum, and a mean proportional between them.

Let V, area OA, area *oa*, and O*o*, be the volume, the area of the lower base, the area of the upper base, and the altitude of the frustum of the cone, and *v*, B, B′, the volume, the lower base, and the upper base of the frustum of the regular pyramid inscribed in the frustum of the cone. We have (Bock VI., Prop. XXVII., Scholium),

$$v = \tfrac{1}{3}Oo \times (B + B' + \sqrt{BB'}),$$

whatever be the number of sides of the bases of the frustum. But, when the number of these sides is increased indefinitely,



v converges to the limit V, a to the limit area OA, and b to the limit area oa, while Oo remains the same. Hence,

$$V = \tfrac{1}{3}Oo \times (\text{area } OA + \text{area } oa + \sqrt{\text{area } OA \times \text{area } oa}),$$

or

$$V = \text{area } OA \times \tfrac{1}{3}Oo + \text{area } oa \times \tfrac{1}{3}Oo + \sqrt{\text{area } OA \times \text{area } oa} \times \tfrac{1}{3}Oo.$$

Or, which is the same thing, the volume of the frustum of the cone is equivalent to the volumes of three cones, etc.

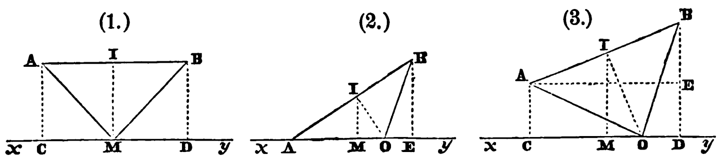
SCHOLIUM. If R is the radius of the lower base, and r the radius of the upper base, then area OA $= \pi R^2$, area $oa = \pi r^2$, and hence

$$V = \tfrac{1}{3}\pi H(R^2 + r^2 + Rr).$$

PROPOSITION XII.

THEOREM.

The area of the surface generated by the base, AB, *of an isosceles triangle,* OAB, *which revolves about a fixed axis, xy, lying in its plane, and passing through its vertex (without cutting the triangle), is equal to the circumference which has for its radius the altitude* OI *of the triangle multiplied by the projection of the base,* AB, *on the axis, xy.*



The base, AB, of the triangle, may be parallel to the axis (Fig. 1), or may meet it in A (Fig. 2), or may be inclined to it without meeting it (Fig. 3).

In all these relative positions (Prop. IX., Scholium) the measure of the surface is

circ. IM $\times$ AB.

Hence, when AB is parallel to xy, the theorem is true, since IM is the altitude of the triangle, and AB is equal to its projection CD on xy.

For the other positions of AB, we have to show that

circ. IM $\times$ AB is equal to circ. OI $\times$ CD.

Draw AE (Fig. 3) parallel to xy till it meets BD in E.

Then the triangles ABE and OMI, having their sides respectively perpendicular to each other are similar. Hence, their homologous sides give the proportion

$$MI : AE :: OI : AB.$$

Hence, $MI \times AB = OI \times AE = OI \times CD.$ (Fig. 3.)

Therefore,

$$2\pi \times MI \times AB = 2\pi \times OI \times AE = 2\pi \times OI \times CD.$$

That is,

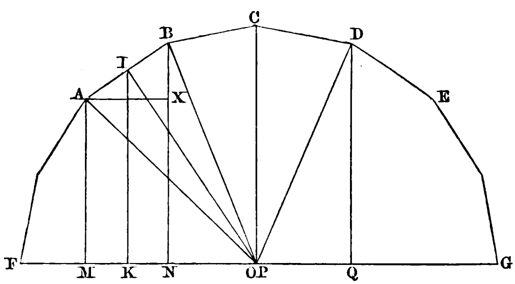
$$\text{circ. } MI \times AB = \text{circ. } OI \times AE = \text{circ. } OI \times CD.$$

Hence, the surface generated by AB is equal to circumference OI multiplied by its projection on axis xy.

PROPOSITION XIII.

THEOREM.

The area of the surface generated by a portion of a regular broken line, ABCD, *revolving about an axis,* FG, *which passes through the centre of its inscribed circle, is equal to the projection,* MQ, *of the contour* ABCD *on* FG, *multiplied by the circumference of the circle inscribed in* ABCD.



For, since the broken line is regular, all the triangles, AOB, BOC, etc., are equal and isosceles. Now, by the last theorem, the surface described by AB, the base of the triangle AOB (which we shall call surf. AB), is equal to MN × circumference OI.

So that we have,

$$\text{surf. } AB = MN \times \text{circ. } OI$$

similarly, $\text{surf. } BC = NO \times \text{circ. } OI$

and $\text{surf. } CD = OQ \times \text{circ. } OI$

Hence, adding these we get

$$\text{surf. } ABCD = (MN + NP + PQ) \text{ circ. } OI = MQ \times \text{circ. } OI.$$

Hence, it is equal to the altitude multiplied by the circumference of the inscribed circle.

Cor. If the regular broken line is the half perimeter of a regular polygon of an even number of sides, and if the axis, FG, passes through two opposite vertices, F and G, the whole surface described by the revolution of the semi-polygon FACG will be equal to its axis, FG, multiplied by the circumference of the inscribed circle. This axis will, at the same time, be the diameter of the circumscribed circle.

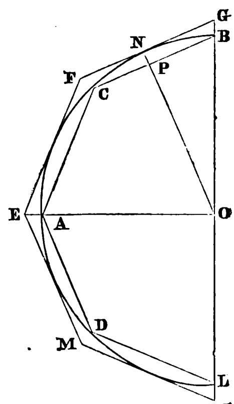
PROPOSITION XIV.

Theorem.

1. *The surface of a sphere is the common limit of the surfaces described by the revolution of the semi-perimeters of the regular inscribed and circumscribed polygons of a great circle of the sphere about an axis which joins two opposite vertices, when the number of sides of the polygon is increased without limit.*

2. *The surface of a zone, of one base, is the common limit of the surfaces described by the regular broken lines inscribed in and circumscribed about the arc of the great circle which generates the zone, when the number of sides of these broken lines is increased without limit.*

First.—The sphere is greater than the surface a, described by the revolution of semi-perimeter of the inscribed polygon, because it envelops it (Prop. II., Scholium). For the same reason, it is less than the surface A, described by the revolution of the regular circumscribed polygon. Hence, the sphere is comprised between these two surfaces. But (Prop. XIII., Scholium),



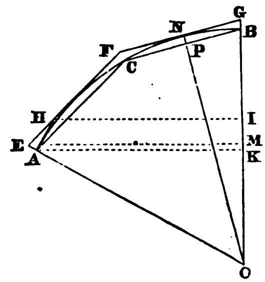
$$A = GK \times \text{circ. } ON, \text{ and } a = LB \times \text{circ. } OP.$$

Hence,

$$A - a = GK \times \text{circ. } ON - LB \times \text{circ. } OP.$$

Now, when the number of sides of the polygons is increased without limit, GK converges to the limit LB, and circ. OP to the limit circ. ON. Hence, the difference GK × circ. OA − LB × circ. OP, or $A - a$, may be made as small as we please. Hence, still more can the difference between the surface of the sphere and either one of these surfaces be made as small as we please, and therefore the surface of the sphere is the common limit of the two when the number of the sides of the polygons is increased indefinitely.

Secondly.—The surface of the zone of one base is comprised between the surfaces, S and *s*, described by the regular broken lines EFG, ACB, revolving about GO.



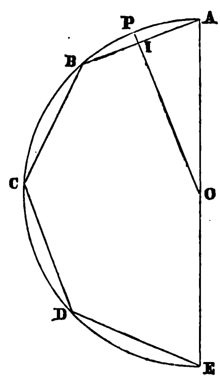
For, the zone is greater than *s* (Prop. II., Scholium), since it envelops it. It is also smaller than the surface S. For, draw the tangent AH, the surfaces described by EHFG and AHFG have a common part generated by HFG; but the surface described by EH is greater than that described by AH, since the measure of the first is

$\pi EH \times \frac{1}{2}(HI + EM)$, and of the second is $\pi AH \times \frac{1}{2}(HI + AK)$ (Prop. IX.), and $EM > AK$, and the oblique line $HE > HA$, which is perpendicular to OA. Hence, the surface described by EFG is greater than the surface described by AHFG, which is greater than the zone, since it envelops it. Hence, surface EHFG is greater than the zone. The surface of the zone being then comprised between S and *s*, we can show exactly as in the case of the whole sphere, that it is the common limit of these two surfaces when the number of sides of the polygons is increased without limit.

PROPOSITION XV.

THEOREM.

The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.



Let S be the surface of the sphere, and OA the radius of the generating great circle; let *s* be the surface generated by the semi-perimeter of the regular inscribed polygon, and OI the radius of the inscribed circle of this polygon.

Then $s = AE \times \text{circ. } OI$ (Prop. XIII., Schol.); but when the number of sides of the polygon is increased without limit, *s* converges to the limit S (Prop. XIV.), and OI to the limit OA, while the diameter AE remains fixed.

Hence, passing to the limit,

$$S = AE \times \text{circ. } OA,$$

or, the surface of the sphere is equal to the product of the diameter by the circumference of a great circle.

COR. 1. *The surface of the sphere is four times the area of a great circle.*

For, let R = the radius of the sphere.

Then $S = \text{circ. } R \times 2R = 2\pi R \times 2R = 4\pi R^2 = 4 \times$ area of great circle, since the area of a great circle $= \pi R^2$.

Cor. 2. Placing $2R = D$, we have $4R^2 = D^2$.

Hence, also $S = \pi D^2$, or *the surface of the sphere is equal to the area of the circle whose radius is the diameter of the sphere.*

COR. 3. *The surfaces of two spheres are to each other as the squares of their radii or the squares of their diameters, or as the squares of the circumferences of their great circles.*

For, let R, R', D, D', C, C', be the radii, diameters, and circumferences of the great circles, respectively, of two spheres whose surfaces are S and S'.

Then

$$S : S' :: 4\pi R^2 : 4\pi R'^2 :: R^2 : R'^2 :: D^2 : D'^2 :: C^2 : C'^2.$$

SCHOLIUM. The surface of the sphere being thus determined and compared with plane surfaces, it will be easy to find the absolute value of the various lunes and spherical triangles whose ratio to the surface of the whole sphere has been determined in Book VII., Props. XIX., XXII.

First, the lune whose angle is A, estimated in right angles as units, has been found to be $2A \times$ the tri-rectangular triangle; that is, $2A \times \frac{1}{8}$ of the surface of the sphere.

Hence, if the radius of the sphere $= R$, the lune whose angle is $A = 2A \times \frac{1}{8} \times 4\pi R^2 = A \times \pi R^2$. If the angle of the lune is given in degrees; for example, if it be α°, then

$$A = \frac{\alpha}{90}, \text{ and the lune } = \frac{\alpha}{90} \times \pi R^2;$$

or, since the surface of the lune is to the surface of the sphere as the arc which measures the angle of the lune is to the circumference of a great circle, *the lune = diameter of the sphere $\times$ the arc which measures its angle.*

The spherical triangle whose angles, estimated in right angle units, are A, B, C, has been found to be $(A + B + C - 2) \times \frac{1}{8}$ of the surface of the sphere.

Hence, if radius of sphere $= R$,

Area of this spherical triangle $= (A + B + C - 2) \dfrac{\pi R^2}{2}$; or, if α, β, γ, be the angles of the triangle in degrees; then

$$A = \frac{\alpha}{90}, \ B = \frac{\beta}{90}, \ C = \frac{\gamma}{90}.$$

$$\text{Hence, area of triangle} = \frac{(\alpha + \beta + \gamma - 180)}{180} \pi R^2.$$

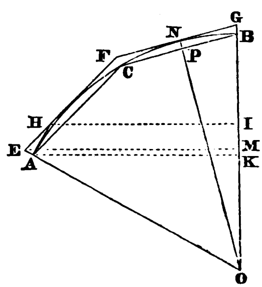
The measurement of spherical polygons follows immediately from that of triangles.

PROPOSITION XVI.

Theorem.

The surface of a zone is equal to the product of its altitude by the circumference of a great circle of its sphere.

First.—We will consider the zone of one base. Let S be the surface of the zone generated by the revolution of the arc AB of the great circle about the axis OB, and s the surface generated by the revolution of the regular broken line inscribed in the arc AB about the same axis. We have,



$$s = \text{circ. OP} \times \text{BM (Prop. XIII.)},$$

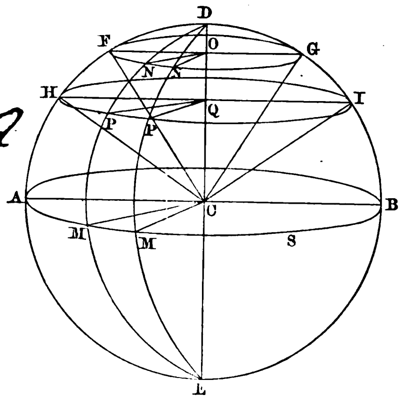
whatever be the number of sides of the regular broken line.

But, when this number of sides is increased indefinitely, s converges to the limit S (Prop. XIV.), and circ. OP to circ. ON, BM to BK. Hence, passing to the limits,

$$S = \text{circ. ON} \times \text{BK},$$

or, every spherical zone with one base is measured by its altitude multiplied by the circumference of a great circle.

Secondly.—The zone with two bases, described by the arc HF, has also for its measure the circumference of a great circle multiplied by its altitude. For it is the difference of two zones of a single base described by DH and DF. It has, therefore, for its measure



$$(DQ - DO) \times \text{circ. } CA,$$

or

$$OQ \times \text{circ. } CA.$$

Hence, any spherical zone is measured by its altitude multiplied by the circumference of a great circle.

COR. *First.—Two zones taken in the same sphere, or in equal spheres, are to each other as their altitudes;*

Second.—Any zone is to the surface of the sphere as the altitude of the zone is to the diameter of the sphere.

SCHOLIUM. If R = radius of the sphere, and H the altitude of the zone, then

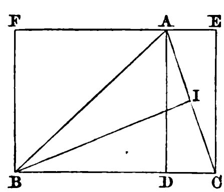
$$\text{the surface of the zone, } S = 2\pi R \times H.$$

PROPOSITION XVII.

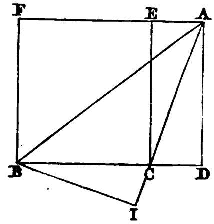
THEOREM.

If the triangle BAC, *and the rectangle* BCEF, *having the same base and the same altitude, turn simultaneously about the common base* BC, *the solid described by the revolution of the triangle will be a third of the cylinder described by the revolution of the rectangle.*

Let fall the perpendicular AD upon the axis; the cone described by the triangle ABD is the third part of the cylinder described by the rectangle AFBD (Prop. X., Cor. 1); also the cone described by the triangle ADC, is the third part of the cylinder described by the rectangle ADCE; hence, the sum of the two cones, or the solid described by ABC, is the third part of the two cylinders taken together, or of the cylinder described by the rectangle BCEF.



If the perpendicular AD falls without the triangle, the solid described by ABC will be the difference of the cones described by ABD and ACD; but, at the same time, the cylinder described by BCEF will be the difference of the cylinders described by AFBD, AECD. Hence, the solid described by the revolution of the triangle will still be the third of the cylinder described by the revolution of the rectangle of the same base and same altitude.



SCHOLIUM. The surface of the circle whose radius is AD is $\pi \times AD^2$; hence

$$\pi \times AD^2 \times BC,$$

is the measure of the cylinder described by BCEF, and

$$\tfrac{1}{3}\pi \times AD^2 \times BC, \text{ or } \tfrac{1}{3}\pi \times AD \times AD \times BC, \qquad (1)$$

is that of the solid described by the triangle ABC.

Moreover, if we let fall the perpendicular BI on AC, we have

$$AD \times BC = AC \times BI.$$

Hence, substituting AC × BI for AD × BC in (1), we have,

$$\text{The volume described by triangle } ABC = \tfrac{1}{3}\pi\, AD \times AC \times BI.$$

But $\pi \times AD \times AC$ is the surface generated by the line AC (Prop. VIII.).

Hence, $\text{vol. } ABC = \text{surf. } AC \times \tfrac{1}{3}BI.$

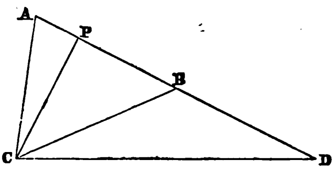
That is, *the volume generated by a triangle revolving around one of its sides is equal to the convex surface of the cone described by either one of the other sides multiplied by one-third of the distance of this side from the opposite vertex.*

PROPOSITION XVIII.

THEOREM.

The volume generated by the revolution of a triangle about an axis situated in its plane and passing through its vertex (without cutting the triangle), is equal to the surface described by the side opposite to this vertex multiplied by the distance of this side from the vertex.

First.—Let ACB be the triangle which turns about the axis CD, passing through its vertex, C. Prolong AB to meet CD in D. Then



vol. ABC = vol. ACD − vol. BCD.

But (Prop. XVII., Scholium),

vol. ACD = surf. AD × $\frac{1}{3}$CP,

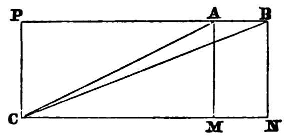
and

vol. BCD = surface BD × $\frac{1}{3}$CP.

Hence, vol. ABC = (surf. AD − surf. BD) × $\frac{1}{3}$CP,

or vol. ABC = surf. AB × $\frac{1}{3}$CP.

Secondly.—When the base, AB, of the triangle ACB is parallel to the axis through the vertex, C, the theorem is still true. For the



vol. ACB = vol. BCP − vol. ACP.

Now, vol. BCP = $\frac{2}{3}$ of cylinder described by rectangle CNBP (Prop. X., Cor. 2), or = surf. BP × $\frac{1}{3}$CP (Prop. VI., Scholium). Also vol. ACP = surf. AP × $\frac{1}{3}$CP. Hence,

vol. ACB = (surf. BP − surf. AP) × $\frac{1}{3}$CP, or = surf. AB × $\frac{1}{3}$CP.

If the perpendicular CP falls within the triangle, then

vol. ACB = vol. BCP + vol. ACP,

and we have the same result.

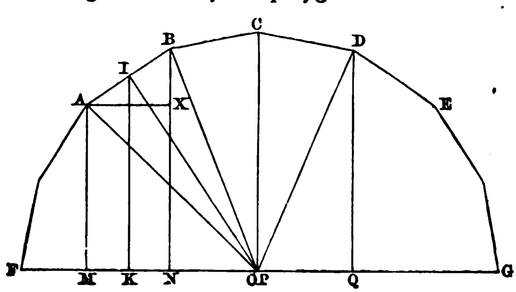
Hence, etc.

PROPOSITION XIX.

Theorem.

ABCD *being a portion of a regular broken line, if we conceive the polygonal sector* AOD, *lying on one side of the diameter* FG *in its plane, to make a revolution about this diameter, the volume described will be measured by the surface generated by the perimeter* ABCD, *multiplied by one-third of the apothem* OI.

For the volume generated by the polygonal sector AOD is the sum



of the volumes generated by the equal isosceles triangles AOB, BOC, COD.

But, vol. AOB = surf. AB × $\frac{1}{3}$OI.
vol. BOC = surf. BC × $\frac{1}{3}$OI.
vol. COD = surf. CD × $\frac{1}{3}$OI.

Hence, adding these,

vol. AOD = (surf. AB + surf. BC + surf. CD) × $\frac{1}{3}$OI
= surf. ABCD × $\frac{1}{3}$OI.

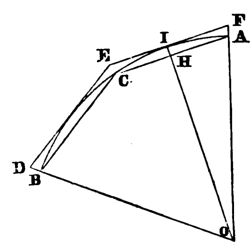
Cor.—The volume described by the semi-polygon, if the whole polygon has an even number of sides, and if the axis passes through two opposite vertices, has for its measure surface FABCDEG × $\frac{1}{3}$OI.

PROPOSITION XX.

Theorem.

The volume of a spherical sector is the common limit of the volumes generated by the revolution of similar regular inscribed and circumscribed polygonal sectors of the corresponding circular sector about the axis of the sector, when the number of sides of the polygonal lines is increased without limit.

Let AOB be the circular sector, which, by its revolution around AO, generates the spherical sector.



Inscribe in and circumscribe about the arc AB two similar regular broken lines, BCA, DEF. The volume of the sector is evidently comprised between V and v, the volumes generated respectively by the revolution of the polygonal sectors DEFO, BCAO, about the same axis, AO.

But, V = surf. DEF × $\frac{1}{3}$OI.
and, v = surf. ABC × $\frac{1}{3}$OH.
Hence, V − v = $\frac{1}{3}$(surf. DEF × OI − surf. ABC × OH).

And, when the number of sides of these inscribed and circumscribed polygonal lines is increased indefinitely, we know that OH approaches OI as its limit, and we have shown (Prop. XIV.) that surf. DEF and surf. ABC approach the same limit. Hence, the difference, V − v, may be made as small as we please; and, therefore, the volume of the sector, which is always comprised between V and v, is their common limit.

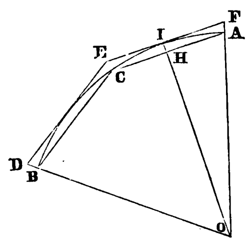
Cor. The whole sphere is the common limit of the volumes generated by the revolution of similar regular inscribed and circumscribed semi-polygons of its semi-great circle, about a diameter passing through two opposite vertices.

PROPOSITION XXI.

Theorem.

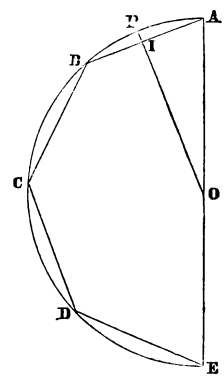
Every spherical sector is measured by the zone which forms its base multiplied by one-third of the radius; and the whole sphere has for its measure the product of its surface by one-third of its radius.

First.—Let V be the volume of the spherical sector, S the surface of the zone which forms its base, OA the radius of the sphere, v the volume generated by the regular inscribed polygonal sector, s the surface described by its regular broken line, and OH its apothem.



Then $v = s \times \frac{1}{3}\text{OH}$, whatever be the number of sides of the regular broken line of the polygonal sector (Prop. XIX.). But when we increase the number of these sides indefinitely, v converges to the limit V (Prop. XX.), s converges to the limit S (Prop. XIV.), and OH to the limit OA.

Hence, $\text{V} = \text{S} \times \frac{1}{3}\text{OA}$,



or, the volume of the sector is equal to the surface of its zone multiplied by one-third of the radius.

Secondly.—The reasoning for the sector applies without change to the whole sphere, or we may proceed thus: a circular sector may increase till it becomes a semicircle, in which case the spherical sector generated by its revolution is the whole sphere. Hence, the volume of the sphere is equal to its surface multiplied by a third of the radius.

Scholium 1. V being the volume of a sphere, R the radius, and D the diameter, we have,

$$V = \text{surface of sphere} \times \tfrac{1}{3}R = 4\pi R^2 \times \tfrac{1}{3}R = \tfrac{4}{3}\pi R^3.$$

And since $R = \frac{1}{2}D$, $R^3 = \frac{1}{8}D^3$. Hence, also, $V = \frac{4}{3}\pi \times \frac{1}{8}D^3 = \frac{1}{6}\pi D^3$.

SCHOLIUM 2. V being the volume of a spherical sector, H the altitude of its zone, and R the radius of the sphere, then

$$V = \text{surface of zone} \times \tfrac{1}{3}R = 2\pi RH \times \tfrac{1}{3}R, \text{ or, } v = \tfrac{2}{3}\pi R^2 H.$$

COR. 1. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

COR. 2. The volume of a spherical wedge is equal to the lune which forms its base multiplied by one-third of the radius.

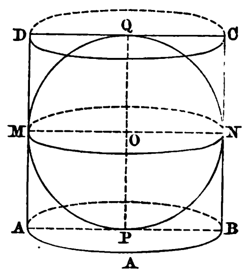
PROPOSITION XXII.

THEOREM.

1. *The surface of a sphere is to the whole surface of the circumscribed cylinder (including its bases) as 2 is to 3.*

2. *The volumes of these two bodies are to each other in the same ratio.*

First.—Let MNPQ be a great circle of the sphere, ABCD the circumscribed square; if the semicircle PMQ, and the half square PADQ, are made to revolve at the same time about the diameter PQ, the semicircle will generate the sphere, while the half square will generate the cylinder circumscribed about the sphere.



The altitude, AD, of the cylinder is equal to the diameter, PQ; the base of the cylinder is equal to the great circle; its diameter, AB, being equal to MN; hence the convex surface of the cylinder (Prop. V.) is equal to the circumference of the great circle multiplied by its diameter. This measure is the same as that of the surface of the sphere (Prop. XV.); whence it follows that *the surface of the sphere is equal to the convex surface of the circumscribed cylinder.*

But the surface of the sphere is equal to four great circles (Prop. XV., Cor. 1); hence the convex surface of the circumscribed cylinder is also equal to four great circles; adding to this the two bases, each equal to a great circle, the whole surface of the circumscribed cylinder will be equal to six great circles; hence the surface of the sphere is to the whole surface of the circumscribed cylinder as 4 is to 6, or as 2 is to 3.

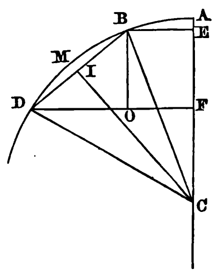
Secondly.—The volume of the circumscribed cylinder is equal to its convex surface multiplied by one-half the radius of its base (Prop. VI., Scholium), that is, equal to four great circles $\times \frac{1}{2}$ radius of the sphere, while the volume of the sphere is equal to four great circles $\times \frac{1}{3}$ of the radius. Hence,

volume of sphere : volume of cylinder : : $\frac{1}{3} : \frac{1}{2}$, or as 2 : 3.

PROPOSITION XXIII.

THEOREM.

The volume generated by the circular segment BMD, *revolving around a diameter*, AC, *exterior to it, is equivalent to one-sixth of the cylinder which has for its radius the chord*, BD, *of the segment, and for its altitude the projection*, EF, *of this chord on the diameter*, AC; *that is, it has for its measure* $\frac{1}{6}\pi BD^2 \times EF$.



Let fall upon the axis the perpendiculars BE, DF; from the centre, C, draw CI perpendicular to the chord BD, and draw the radii CB, CD.

We have (Props. XXI. and XVIII.),

spherical sector DCB = zone BMD $\times \frac{1}{3}$CB = $\frac{2}{3}\pi CB^2 \times EF$;

also volume generated by triangle DCB = surf. BD $\times \frac{1}{3}$ CI = $\frac{2}{3}\pi CI^2 \times EF$.

Hence,

vol. BMD = spherical sector DCB − vol. DCB = $\frac{2}{3}\pi \times (CB^2 - CI^2) \times EF$.

But, in the triangle CBI,

$$CB^2 - CI^2 = BI^2 = \tfrac{1}{4}BD^2.$$

Hence,

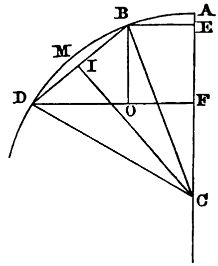
$$\text{vol. BMD} = \tfrac{2}{3}\pi \times \tfrac{1}{4}BD^2 \times EF = \tfrac{1}{6}\pi BD^2 \times EF,$$

which was to be proved.

PROPOSITION XXIV.

THEOREM.

Every segment of a sphere included between two parallel bases is measured by the half sum of its bases multiplied by its altitude plus the volume of the sphere whose diameter is this same altitude.



Let BE, DF, be the radii of the bases of the segment, EF its altitude, the segment being generated by the revolution of the circular space BMDFE about the axis FE. The solid described by the segment BMD is equal to

$$\tfrac{1}{6}\pi BD^2 \times EF \text{ (Prop. XXIII.)};$$

the frustum of a cone described by BDFE is equal to $\tfrac{1}{3}\pi EF \times (BE^2 + DF^2 + BE . DF)$ (Prop. XI.);

hence, the segment of the sphere which is the sum of these two solids is equal to $\tfrac{1}{6}\pi EF \times (2BE^2 + 2DF^2 + 2BE . DF + BD^2)$. (1)

But, drawing BO parallel to EF, we shall have,

$DO = DF - BE$, $DO^2 = DF^2 - 2DF . BE + BE^2$ (Book III., Prop. IX.), and consequently,

$$BD^2 = BO^2 + DO^2 = EF^2 + DF^2 - 2DF . BE. + BE^2.$$

Putting this value in place of BD^2 in the expression (1) for the value of the segment, and striking out the parts which destroy each other, we shall obtain for the volume of the segment

$$\tfrac{1}{6}\pi EF \times (3BE^2 + 3DF^2 + EF^2),$$

an expression which may be decomposed into two parts; the one,

$$\tfrac{1}{6}\pi EF \times (3BE^2 + 3DF^2), \text{ or } EF \times \left(\frac{\pi BE^2 + \pi DF^2}{2}\right),$$

is the half sum of the bases multiplied by the altitude; the other, $\tfrac{1}{6}\pi EF^3$, represents the sphere of which EF is diameter (Prop. XXI., Scholium 1); hence every segment of sphere, etc.

Cor. If either of the bases is nothing, the segment in question becomes a segment with a single base. Hence, *Any spherical segment with a single base is equivalent to half the cylinder having the same base and the same altitude, plus the sphere of which this altitude is the diameter.*

General Scholium.

It is well to give a recapitulation of the expressions which have already been found in the scholiums to the different propositions on the measures of the three round bodies.

1. Let R = radius of the cylinder, H its altitude.

Its convex surface $= 2\pi RH$.
Its whole surface $= 2\pi R(R + H)$.
Its volume $= \pi R^2H$.

2. Let R = the radius of the cone's base, L its side, and H its altitude.

Its convex surface $= \pi RL$.
Its whole surface $= \pi R(L + R)$.
Its volume $= \frac{1}{3}\pi R^2H$.

3. Let R and R′ be the radii of the bases of the frustum of the cone, L its side, and H its altitude.

The convex surface of the frustum $= \frac{1}{2}\pi L(R + R')$.
The volume of the frustum $= \frac{1}{3}\pi H(R^2 + R'^2 + RR')$.

4. Let R = the radius of the sphere and D its diameter.

The surface of the sphere $= 4\pi R^2 = \pi D^2$.
The volume of the sphere $= \frac{4}{3}\pi R^3 = \frac{1}{6}\pi D^3$.

5. Let R be the radius of the sphere, H the altitude of a zone.

The surface of the zone $= 2\pi RH$.
The volume of the spherical sector which has this zone for its base $= \frac{2}{3}\pi R^2H$.

6. Let P and Q be the two bases of a spherical segment, H its altitude.

$$\text{The volume of the segment} = \frac{P + Q}{2}.\ H + \frac{1}{6}\pi H^3.$$

If the spherical segment have but one base,

Its volume $= \frac{1}{2}PH + \frac{1}{6}\pi H^3$.

7. If A, B, C be the values of the angles of a spherical triangle, expressed in right angle units, and R the radius of the sphere,

$$\text{The area of the spherical triangle} = \frac{\pi R^2}{2}(A + B + C - 2),$$

or, if α°, β°, γ° be the angles of a spherical triangle in degrees,

$$\text{The area of the triangle} = \frac{\pi r^2}{180}(\alpha + \beta + \gamma - 180).$$

EXERCISES ON BOOK VIII.

THEOREMS.

1. If, through a point within or without a cylinder of revolution and the axis of the cylinder, a plane be passed cutting the surface in a straight line, the perpendicular drawn from the point to this straight line will be the shortest distance from the point to the cylinder.

2. The volumes of two cylinders whose convex surfaces are equivalent, are to each other as the radii of the bases divided by the altitudes.

3. The convex surfaces of two cylinders whose volume is the same, are to each other as the altitudes divided by the radii of the bases.

4. The convex surfaces of two similar cylinders are to each other as the squares of their altitudes, or the squares of the diameters of their bases.

5. The convex surface of a cone whose angle at the vertex is two-thirds of a right angle, is double the area of the base, and one-half of the area of a circle whose radius is equal to the side of the cone.

6. The area of a zone with one base is equal to the area of the circle whose radius is the chord of the arc which generates the zone.

7. If the semi-circumference of a circle be divided into three equal parts, and revolved about the diameter through its two extremities, the zone generated by the middle one of the three arcs will be equivalent to the sum of the two zones generated by the other two parts of the semicircle. And the sector which has for its base the zone generated by the middle arc will be equal to the sum of the two sectors which have the other two zones for bases.

8. If the half of a regular polygon of an even number of sides be inscribed in a semicircle, and a similar half polygon be circumscribed about the same semicircle, the surface of the sphere generated by the revolution of the semicircle about its diameter, will be a mean proportional between the surfaces generated by the perimeters of the two semi-polygons about the same diameter.

9. The volume of a cone which has for its base the great circle of a sphere, and for its altitude a diameter of the sphere, is one-half the volume of the sphere.

10. The volume of a spherical shell (bounded by the surfaces of two concentric spheres) is equal to four times that of the frustum of a cone the radii of whose bases are the diameters of the two spheres and whose altitude is the difference of these radii.

11. The volume of the solid generated by a triangle revolving about a straight line which passes through its vertex (without cutting the triangle), is equivalent to the area of the triangle multiplied by the circumference described by the point of intersection of its medians.

12. If, in each one of a number of concentric semicircles we draw a chord of the same given length parallel to a common axis, passing through the centre, and then revolve the circular segments thus obtained about the common axis, the volumes generated by them will all be equivalent.

13. If, through the points of division of the semicircle (when divided as in Exercise 7), planes be drawn perpendicular to the axis of revolution, the middle segment thus determined will be equivalent to $\frac{11}{8}$ of the sum of the two extreme segments.

14. The volume generated by the revolution of a regular hexagon about one of its sides, a, has for its measure $\frac{9\pi a^3}{2}$.

15. The surface of the circumscribed cylinder of a sphere is a mean proportional between the surface of the sphere and that of the circumscribed equilateral cone. And the same relation exists between the volumes of these bodies.

(NOTE.—An equilateral cone is one generated by the revolution of one-half of an equilateral triangle about its altitude.)

16. If the surface of a sphere be represented by the number 16, then will the convex surface of the inscribed cylinder whose section through the axis is a square, its whole surface, the convex surface of the inscribed equilateral cone, and its whole surface, be represented respectively by the numbers 8, 12, 6, 9.

17. The surface of the sphere being 4, the convex surface of the circumscribed equilateral cone, its whole surface, the convex surface and whole surface of the circumscribed cylinder, will be respectively 6, 9, 4, 6.

18. If the volume of the sphere be 32, the volumes of the inscribed equilateral cone, of the inscribed cylinder (whose section is a square),

of the circumscribed cylinder, and of the circumscribed equilateral cone, will be respectively 9, $12\sqrt{2}$, 48, 72.

19. The volume generated by the revolution of a regular semi-hexagon about one of its diameters is equivalent to that of a cylinder whose altitude and the radius of whose base are equal to the radius of the circumscribed circle of the hexagon.

20. If a parallelogram be revolved successively about two adjacent sides, the two volumes generated will be reciprocally proportional to these sides.

LOCI.

1. What is the locus of all the points which are at a given distance from a given straight line.

2. What is the locus of all the points which are at a given distance from the surface of a given cylinder of revolution.

3. What is the locus of all the straight lines passing through a given point, and making a given angle with a given plane.

4. What is the locus of all the points which are at a distance a from a point A, and at a distance b from a point B.

5. What is the locus of all the points which are at a distance a from a point, A, and a distance b from a plane, MN.

6. State what the locus would be of a point which is at a distance a from a straight line, A, and at a distance b from a plane, MN.

7. What is the locus of all the points, M, from which, if lines be drawn to three given points, A, B, C, the angles AMB and AMC would both be right angles.

8. Find the locus of all the points, the distances from each one of which from two given spheres of equal radii are equal.

9. Three points, A, B, C, being given, find the locus of all the points, such that the sum of the squares of the distances from each one of them to the two points A and B shall be equal to a given square, and the sum of the squares of their distances from the two points A and C is equal to a second given square.

PROBLEMS.

1. With a given altitude construct a cylinder whose convex surface is equal to the sum of the areas of the two bases.

2. Compute the radius of the base and the altitude of a cylinder, given the convex surface and the volume.

3. Find the convex surface of a frustum of a cone, the radii of whose bases are 5.4 feet and 3.6 feet, and whose altitude is 2.4 feet.

4. Find the volume of a frustum of a cone, the radii of whose bases are 5.4 feet, and 3.6 feet, and whose side is 3 feet.

5. Compute the volume of a cone, the radius of whose base is 8 feet, and whose side is 10 feet.

6. Compute the convex surface of a cone, whose altitude is 16 feet, and the radius of whose base is 12 feet.

7. Compute the volume and whole surface of a cylinder, the radius of whose base is 16 feet 9 inches, and whose altitude is 20 feet.

8. The radius of the base of a cone is 9 feet, and its altitude is 12 feet. If this cone is rolled out completely on a plane (see Prop. VIII., Schol. 1), find the radius and angle of the circular sector which is the developed convex surface of the cone.

9. The side of a cone being 1.8 yards, find the parts into which it is divided by a plane parallel to the base of the cone which divides the convex surface, *First*, Into two equivalent parts, *Secondly*, Into two parts proportional to the numbers 3 and 5.

10. Find the surface of the sphere whose radius is 20 inches.

11. The surface of a sphere being 1000 square feet, find its volume.

12. The meter is one ten-millionth part of a quadrant of the earth's great circle (earth being regarded as a sphere). Find the surface of the earth in square kilometers. *Ans.*, $\frac{1,600,000,000}{\pi}$.

13. The diameter of the earth (regarded as a sphere) is 7912.5 miles, find its surface in square miles.

14. Find the volume of a sphere whose radius is 6 feet.

15. The length of one second on the arc of a great circle of a sphere being one foot, find the volume of the sphere.

16. The radii of two spheres are respectively 6 feet and 8 feet. Find a sphere whose surface shall be equivalent to the sum of the surfaces of these two spheres.

17. Find the volume of a spherical shell, the internal radius being 6 inches and the external radius 9 inches.

18. Find the radius of a sphere whose volume is 12 cubic feet.

19. The diameters of the earth, the moon, and the sun are to each other as the numbers 1, $\frac{3}{11}$, and 108. Compare the surfaces and volumes of these bodies.

20. Find the surface of a zone whose altitude is 4 feet, the radius of the sphere being 10 feet.

21. Find the surface of a zone of one base, the radius of the sphere being 20 feet, and the radius of the base of the zone being 12 feet.

22. The angles of a triangle on the earth's surface (regarded as a sphere) are 87°, 72°, and 21° 0′ 1″.5, find the area of the triangle in square miles, the diameter of the earth being 7912 miles.

23. Find the volume of a spherical segment of one base, whose altitude is $\frac{1}{10}$ of a foot, the radius of the sphere being one foot.

24. Find the volume of a spherical sector generated by the revolution of a circular sector whose arc is 30° about one of its radii, the radius of the sphere being 2.4 feet.

25. Given the volume generated by the revolution of an equilateral triangle about one of its sides to be 27 cubic feet, find the side of the triangle.

26. Compute the volume generated by the revolution of an equilateral triangle whose side is 2 inches, about a perpendicular to its base produced, which is at a distance of 2 inches from its nearest vertex. *Ans.*, 33.648068 cubic inches.

27. The side AB of a parallelogram, ABCD, is 10 inches, the side BC 4 inches, and the angle ABC = $\frac{1}{3}$ of a right angle. Find the volume generated by this parallelogram, *First*, When it revolves about AB. *Secondly*, When it revolves about BC.

28. If we join the middle points of two of the sides of a triangle and then revolve it about the third side, what will be the ratio of the volumes generated by the two parts of the triangle?

29. Given the radius of a sphere, find the sides respectively of its inscribed regular tetraedron, cube, and octaedron.

HINTS TO SOLUTIONS OF EXERCISES.

BOOK I.

1. MO is the difference of AO and MA, and also MO is the difference of MB and OB; hence, result. For second part, MO is the sum of MB and OB, and MO is the difference of MA and OA, hence, the conclusion.

2. The same reasoning applies in this case.

3. By *reductio ad absurdum* from Prop. VI.

4. By *reductio ad absurdum* from Prop. VI.

5. By Props. V and VI.

6. By superposition (or by drawing diagonals and making comparison of equal triangles).

7. By superposition or by comparison of triangles, Prop. VII.

8. Construction given. Compare equal triangles and use Prop. IX.

9. Get three inequalities, as in 8, and add.

10. Construction given. Use Prop. IX. Sum the inequalities thus obtained and cancel equals on the two sides of resulting inequality.

11. Same method as in 10.

12. Use Prop. X. for first and Prop. IX. for last, getting in each case three inequalities, and adding them respectively.

13. By comparison of triangles, Prop. VII.

14. By comparison of triangles, Prop. VII.

15. By comparison of triangles, Props. XII. and VII.

16. By comparison of equal triangles.

17. By similar method.

18. Construction given. Assume any other point and join it to

two given points and extremity of produced perpendicular. Then use Prop. IX.

19. Draw AA′ and BB′ and CC′, meeting *xy* in points M, N, and O. Now conceive the part of figure above *xy* revolved around *xy*, the points A, B, and C will fall on A′, B′, C′, and thus proposition is proved. This revolution is merely a convenient mode of superposition.

20. Corollary of 18.

21. By comparison of triangles, Props. XXIV. and VII.

22. Let fall perpendiculars from A and C on the parallel BD, and compare the two triangles thus formed. Then use Prop. XIV.

23. Use Props. XIII. and XXIV.

24. Compare the triangles formed by the construction with the original triangle.

25. An easy corollary of 24. The converse by the *reductio ad absurdum.*

26. Follows directly from 25.

30. The bisectrix of the exterior angle at B is the locus of all points within this angle, equidistant from BC and AB, produced. The bisectrix of the exterior angle, C, is the locus of points within this angle, equidistant from BC and AC, produced. These bisectrices must meet (Prop. XXV., Scholium, etc.).

32. Use Prop. XXIX.

33. Draw from O a line, OM, parallel to side AB, and then use 25 (converse), and compare the triangles AOM and COM.

34. Use Prop. XIII. in each one of the isosceles triangles, AOC and AOB, and then converse of Prop. XXIX., Cor. 5.

35. Triangle BCD, formed by given construction, is an equilateral triangle, and perpendicular CA bisects angle A, etc.

36. At C make angle ACD = BCA, and complete triangle ACD. Then BCD is equilateral, etc.

37. Let the bisectrices of the exterior angles B and C meet in O; of exterior angles A and C meet in N; of exterior angles A and B in M. Then use Prop. XXIX. to prove semiexterior angle A in triangle ABM equal to angle O in triangle OCB. Using, also, triangles NCA and OMN.

38. Follows directly from 37, or equality of angle O and semi-exterior angle A.

39. Angle DAO = a right angle minus angle AOD, and AOD = 2C = C + C = C + a right angle diminished by B, whence result.

40. Apply Prop. IV. at each vertex and then Prop. XXX. for the sum of interior angles.

41. Apply Corollary of Prop. XXX. and Prop. XXIX.

42. Same method.

43. Angle of equiangular hexagon is $\frac{4}{3}$ of a right angle. Hence, the triangles formed by sides of squares and lines joining the corners are equilateral, etc.

44. Let number of sides be x; then sum of angles (Prop. XXX., Cor. 4) $= 2x - 4 = 26$, whence x.

45. Each angle $= \dfrac{2x-4}{x} = \dfrac{5}{3}$, whence x.

46. Apply Prop. XXX., Cor. 4, as above.

47 and 48. Compare the angles of these polygons and of others with four right angles.

50. The sides are parallel, by XXIV.

51. Let quadrilateral be ABCD. Let also bisectrices of angles A and B meet in M; those of angles C and D meet in N; first, angle M = two right angles $- \frac{1}{2}(A + B)$, angle N = two right angles $- \frac{1}{2}(C + D)$, etc. Hence M + N = two right angles. Second, if ABCD is a parallelogram, angle M = two rights minus the half of two rights, etc. (Prop. XXV.) Third. A rectangle with its diagonals at right angles to one another, hence a square. Prop. XXXIV., Scholium 3.

52. See (21). Then draw diagonals and divide the surfaces to be compared into triangles, which compare by VII.

53. See Exercises 25 and 24. This parallelogram is a rhombus when the given quadrilateral is a rectangle; and a rectangle when the quadrilateral is a rhombus; a square when the quadrilateral is a square.

55. The figure is a parallelogram (Def.). Compare the areas by means of equal triangles.

56. See 21.

57. Let ABCD be the rectangle. Take a point, M, on the side AB, and draw a line parallel to the diagonal AC, meeting the side BC in the point N. Draw NP parallel to the diagonal BD, meeting DC in P, and draw PQ parallel to AC, meeting AD in Q, and, lastly, join QM. Then shall MNPQ be a parallelogram. Prolong MN to meet DC, produced, in R. Then compare the triangles NCR and NCP, and by means of these and the parallelogram CRMA, CP = AM. Then compare triangles AMQ, NCP, etc.

58. Let H be position of ball in rectangle ABCD, and HM parallel to diagonal BD. HMNPQS the path of the ball, making angles HMA = DMN, MND = PNC, NPC = BPQ, BQP = AQS. Then, since HM is parallel to diagonal BD, and HMA = DMN, DNM = ACD, and MN is parallel to AC. Similarly, NP is parallel to DB, and PQ to AC, and QS to BD, and QS must meet AD in point M (57). It is easy to prove it must pass through H, and then the length of path is shown by (57).

59. Either perpendicular prolonged as in auxiliary construction is equal to the perpendicular from angle at base on opposite side. Now compare the triangle formed by this construction, of which the prolongation is one side, with the triangle cut off by the second perpendicular from the given triangle, and the demonstration is easy.

60. The three altitudes of the equilateral triangle are all equal, and the equilateral triangle cut off by the parallel to the base (auxiliary construction) has the property of the isosceles triangle demonstrated in (59). And to these two perpendiculars (constructed as in 59) we have to add the perpendicular from point on base.

BOOK II.

1. Draw another line from A, meeting circumference in M and N, join OM and ON, and form inequalities by (Book I., Prop. IX.), in the triangles AOM and AON. The demonstration is then easy. Proceed in the same manner for point A within the circumference.

2. Let O and C be centres, and let OC meet the circumferences in M and N respectively. Take any other point, P, on one circumference, and Q on the other, and compare PQ and MN. From P draw

PC to centre of other circle, meeting circumference in S. Now, PQ is longer than PS, and by using inequality (Book I., Prop. IX.) of sides of triangle PCO, we show MN less than PS. (This for circles exterior to one another.) The second is easy after the above hints.

3. Analysis: If the chord perpendicular to the radius is the shortest, it must be farthest from the centre (Prop. VIII). Hence, draw any other chord through A, and compare its distance from centre with the distance of this perpendicular chord from centre.

4. Join centre and point of intersection of chords, and let fall perpendiculars on chords from centre. If the theorem be true, the parts between feet of perpendiculars and point of intersection of chords must be equal.

5. Converse is easy.

6. Prob. XIV., Scholium.

7. From Scholium of Prob. XIV., the tangents from each vertex of quadrilateral are equal. Then sum the equalities thus obtained for each vertex.

8. Construct a circumference tangent to the three sides AB, BC, AD, of the quadrilateral ABCD, which fulfils the conditions $AB + CD = AD + BC$. Assume that it is not tangent to the fourth side, DC. Through D draw a tangent DC′ to the circle, and then the *reductio ad absurdum* is easy.

9. A necessary corollary from 8 and the definitions of these figures.

10. Join the centres of the three circles, and inscribe a circle in this triangle. It is easy to show that this circle is tangent to the sides of the triangle at the points of contact of the three circles (Scholium, Prob. XIV.), and hence the three tangents are radii, etc.

11. Use Scholium, Prob. XIV., to get an equality at each vertex, and from these equations the proof is easy.

12. Let DE touch the circumference at the point M, then by Scholium, Prop. XIV., $DM = DB$, and $EM = EC$, etc., whatever be the position of M on the arc BC.

13. By same Scholium, angle $ODE = \frac{1}{2}BDE = \frac{1}{2}(A + AED)$, (Book I., Prop. XXIX., Cor.) and similarly for OED, and then $DOE = 2$ right angles minus $(ODE + OED)$, etc.

14. This follows at once if we let fall perpendiculars from the centres on this parallel.

15. The angle is the supplement of the inscribed angle between the chords, and hence, Prop. XX., its measure as asserted in Exercise.

16. Props. XX. and IV.

17. Let triangle be ABC; draw any circumference through A, cutting AB in H, and AC in K. Construct a second circumference through points B and H, cutting BC in M, and the first circumference in point I. We are to show that C, K, I, and M are on the same circumference. Draw IH, IK, and IM, and prove (Prop. XX., Cor. 4) that angle MIK + angle C = two right angles, and then by Prop. XXIV., etc.

18. This follows immediately from Prop. XXIV.

19. Construct circle through A, and feet M and N of perpendiculars from B on AC, and from C on AB. Let O be foot of perpendicular from A on BC. Construct also a circle on AB as diameter, this circle will pass through M and O (Prop. XX., Cor. 2). Then (Prop. XX.) angle OMB = OAB, and BMN = OAB, etc.

20. Use Prop. XXVII., Book I., and Prop. XXII., Book II.

21. Analysis: If theorem be true, and we take PD = PB, then must CP = DA, and hence, moreover, the triangles CPB and DBA must be equal (Prop. XX., and Book I., Prop. VII.), and, therefore, BD must be equal to BP. Hence, if our theorem be true, BD = BP, or the triangle BDP is equilateral. Hence, we simply prove this fact (by Prop. XX.), and then, by synthesis, retrace the steps of our analysis to make clear the truth of the theorem.

22. Analysis: If the four points D, E, F, and G, are on the same circumference, then the sum of the opposite angles D and G of the quadrilateral is equal to two right angles. Hence, we endeavor to show by Props. XX. and XXII., that D + G = 2 right angles.

23. Draw the common tangent. Then, by Props. XX. and XXI., and by Prop. XXV., Book I., by means of this tangent, show the chords to be parallel.

24. Corollary of Theorem, Exercise 18, Book I.

GEOMETRIC LOCI.

3. Two circumferences concentric with the given one, and at the given distance from it (distance measured on radii, or radii produced).

4. The locus is a line parallel to the two lines, or, in second case, two lines at right angles to each other, the bisectrices of the adjacent angles of the intersecting lines (Book I., Prop. XXI.).

5. Any point of this locus must be also equidistant from the centres of the circles. Hence, apply Cor. 1, Prop. XVIII., Book I.

6. Prop. XX., Cor. 2.

7. Let AB, AC, be the two given intersecting lines. At A erect perpendiculars AP, AR, to AB and AC, each equal to the given line. At their extremities draw lines PN and RM, parallel to AB and AC respectively, cutting AC and AB in N and M. Then, by Theorem, Exercise 59, Book I., any point on MN possesses the property required for the locus. Prolong NA and MA to equal distances on the other side of A, etc.

8. Prop. VI., Scholium.

9. Prop. IX., Cor. 3.

10. Let A be the given point, C the centre. Join AC, and join middle point of chords through A with centre. Then apply 6.

12. A circumference with centre at the given distance from the given centre, the radius of which is equal to the given radius.

16. Circumference concentric with given circumference. Prob. XIV., Scholium.

19 and 20. These problems the same as 18, as we can establish a constant relation between the angle at the vertex of a triangle and the angle at the centre of inscribed circle and the angle at the intersection of the altitudes from extremities of given base.

21 and 22. See Prop. VI.

23. Prop. IX.

24. Props. XIII. and XIV.

25. See 2.

26. See 3.

27 and 28. See 4.

DETERMINATE PROBLEMS.

2. At any point, O, of given line, AB, draw a line making with OA an angle equal to given angle, etc.

3. Analysis: The radius perpendicular to the tangent must be perpendicular to the given straight line. Hence, construction (two solutions).

4. Analysis: All chords of given length must be at the same distance from the centre, and must therefore be tangent to the circle described with the given distance and concentric with the given circle. Hence, construct to find distance of chord from centre, and then construction is easy (two solutions). Prob. XIV., Scholium.

5. Same analysis as 4, and then use 3.

6. Suppose problem solved. Any line parallel to the required line and intercepted between the parallels is equal to it. Hence, from any point on one line as centre and radius of the given length describe a circle cutting the other line. Join the centre with the point of intersection. The remainder of the construction is easy (two solutions).

7. Analysis: If the line were drawn it would form with one of the lines, a parallel to it, and a parallel to the required line through the point of intersection of the two given lines of the required length, a parallelogram. Hence, first draw through the point of intersection of the two given lines a line parallel to the given straight line and of the required length, etc.

8. The two loci, the intersection of which determine the centre, are, first, the line perpendicular to the chord joining the two points; and second, the circumference described from one of the points as centre with given radius (two solutions).

9. The two loci in this case are, first, the circumference described from the given point as centre with the given radius, and, second, the lines parallel to tangent at a distance equal to given radius from it (four solutions).

10. Loci are, first, the same as first in 9, and, second, a circle concentric with given circle, the radius of which is equal to the given radius plus the radius of given circle (two solutions).

11. Analysis: If problem is solved, the centre must be on one of the bisectrices of angle between lines. Use 7 to draw a line equal to

the given radius between one of the given lines and this bisectrix (four solutions). If the two given lines are parallel the problem is only possible when the given radius is equal to half the distance between the lines.

12. The loci the intersection of which gives the solution are apparent from the previous problems (a circle and two straight lines). (Four solutions possible, in general.)

13. These loci also apparent (two circles). (Two solutions possible, in general.)

14. Analysis: If the problem be solved the centre of required circle is equally distant from the three sides. Hence, it is the centre of the inscribed circle of triangle. The remainder of the construction is easy.

15. Analysis: If centre be found it must be equally distant from the three given points A, B, and C. Hence, find it as in Prob. XIII., etc.

17. The locus of points any one of which is equidistant from the two given points we have from Book I., Prop. XVIII., Cor. 1.

18. The angle of an equilateral triangle is $\frac{2}{3}$ of a right angle. This gives a very simple construction (Probs. X. and V.).

19. Use the theorem illustrating geometrical analysis (page 86).

20. Suppose problem solved. Then the points of contact are (Theorem 10, Exercises) the points of contact of inscribed circle and sides of triangle. Hence construction.

21. Six equal circles can be constructed fulfilling the conditions.

Construction of Triangles, etc.

23. Analysis: Conceive the problem solved and the figure drawn, then the semi-base and the radius of the inscribed circle are sides of a right angled triangle, one of whose acute angles is half the adjacent angle. Hence, the construction is apparent.

25. The two loci the intersection of which determines the position of the vertex, are a circle (Prop. XX., Cor. 2) and two straight lines parallel to the hypothenuse (Geometric Loci, Exercise 2).

26. Let BAC, right angled at A, be the triangle required, and let the inscribed circle be drawn. Then (Prop. XIV., Scholium) if

r = radius, $2r$ + BC = AB + AC. Hence, one side, the angle opposite and sum of two other sides is given, to construct the triangle.

Suppose this problem solved and let BAC be the triangle required. Prolong BA to O, so that AO = AC, and then join OC. Then BO = BA + AC, and the angle BAC = twice the angle O. Hence, O = $\frac{1}{2}$BAC. We know enough in triangle BOC to construct it, and beginning with this we have the solution.

27. Consider the problem solved, and ABC the required triangle, angle B, side BC, and BA + AC being given. If BA be prolonged until BD = BA + AC and DC be joined, the triangle CAD is isosceles, and the triangle DBC can be constructed from the data. Hence, begin the solution by constructing a triangle with BC, angle B, and BD = BA + AC for the given parts. It is easy to complete the solution.

28. Consider the problem solved, and find, by proceeding in a manner similar to the above, a triangle which can be constructed from the data, and whose construction makes the solution of the problem apparent.

29. Same problem as 26.

30. Construct acute angle. Draw circle tangent to the two sides of it of the given radius by Exercise 11. Then use Exercise 3, to complete the triangle.

31. The triangle having the centre of the inscribed circle as vertex, and the given side as base, can be constructed from the data, and its construction leads to an easy solution of the problem.

32. The triangle of the median, and altitude, and part of base intercepted between their extremities, can be constructed from the data. The construction of this triangle renders the solution apparent.

33. First. If the altitude corresponds to one of the given sides as base, then the two loci, the intersection of which determines the position of the vertex, are the straight line parallel to base at a distance from it equal to the given altitude and a circle determined by the other given side (two solutions). Second. If the altitude corresponds to the unknown side, then the foot of the altitude is determined by the intersection of two loci (circles). (See Geometric Loci, Exercises 1 and 6.) This point once determined, the solution is easy (two solutions).

34. First. When the median corresponds to one of the given sides, the loci which determine the vertex are apparent. Second. When the median corresponds to the unknown side, the loci are described from the extremity and middle point of one of the sides as centres, with the median, and half the other side as radii respectively.

35. Describe the circumscribed circle, and use Prop. XXI. for the construction of arcs measuring the given angles.

36. Construct a parallelogram ABCD, two adjacent sides, AB and BC, of which are equal to 2M and 2M′ (M, M′, and M″, being the given medians), and one of the diagonals, AC, is equal to 2M″. Draw the other diagonal, BD, and divide it into three equal parts (19). Join A with the points of division E and F. AEF is the required triangle. (Give the analysis of this solution.)

37. See Exercises 24 and 25, Book I.

38. The construction of a triangle in each case is possible from the data, which renders the solution apparent.

40. Construct the triangle having for its vertices the three given points, and then base the construction for the completion of the solution on (Theorem 19 of Exercises).

41. Suppose the problem solved, and the triangle ABC constructed. Construct also the triangle OO′O″, of the centres of escribed circles. Now, lines joining O, O′, and O″, respectively with A, B, and C, bisect the angles A, B, and C, and are perpendicular to O′O″, OO″, and OO′, respectively. Hence, we begin by constructing OO′O″, and base the rest of the construction on (Theorem 19, Exercises).

42. First. Let both circles be within the sides of the given angle. Suppose the problem solved and the figure drawn. The third side of the required triangle is evidently a common interior tangent to the two circumferences. Second. Let the escribed circle be within the sides of one of the unknown angles. Then, by supposing the problem solved, and constructing the figure, we see that the third side must be a common exterior tangent to the two circles. Hence, use Prob. IV., and then Prob. 11. (Exercises), and then Prob. XVII. Constructing in the second case the escribed circle within the supplement of the given angle.

43. Draw a straight line, and take a point on it as the extremity of the side (supposed to be along the line), to which the given alti-

tude corresponds. Then the locus of the vertex is apparent. Draw this locus, and then use Prob. IV., beginning at the assumed point.

47. Suppose the problem solved, and that from the line BE = the sum of diagonal and side, we have found the part DE equal to the side, and constructed the isosceles triangle BDC, right angled at C, as one-half the required square. Join C with E, then the triangle CDE must be isosceles. Hence, the angle E must be known (Book I., Prop. XXIX., Cor. 7). The angle B is half a right angle. Therefore, the triangle BEC can be constructed from the data, Prob. IX., and this triangle constructed, the solution is evident.

48. After solving 47, this problem presents no difficulty. See also 28.

49. Suppose the problem solved, and through one extremity of the shorter of the parallel sides, draw a line parallel to the inclined side through the other extremity. This auxiliary line enables us easily to bring back the solution to the construction of a triangle.

50. Join the given middle points. We thus construct a pentagon. The diagonals of the required pentagon must be parallel to the sides of this first pentagon, and double them in length (Book I., Exercise 25). Suppose the problem solved, and draw two of these diagonals from the same vertex. The sides of the triangle formed by joining the middle points of these diagonals with the given middle point of the side of required pentagon opposite their vertex, are parallel to two non-adjacent sides of the auxiliary pentagon (and equal to them), and to the side of the required pentagon. This triangle can be constructed from the data of the problem, and its construction solves it.

51. This is to construct a right angled triangle, given the hypothenuse and sum of the other two sides. (The solution is indicated in Hints to Solution of Problem 26.)

52. The same as the above.

53. This is the same as problem 25.

BOOK III.

1. Calling these parallelograms (A) and (B) respectively, then (A) and two triangles make up one-half of the given parallelogram. Also (B) and two triangles which are respectively equal to the first two make up the other half. Hence, etc.

2. Through each extremity of one inclined side, draw a parallel as well as a perpendicular to the opposite inclined side. We construct thus two parallelograms, each measured by this opposite inclined side multiplied by one of the perpendiculars, and we have to show that the trapezoid is equal to the half sum of these two parallelograms.

3. Construct the figure and demonstrate the theorem after the manner of Prop. VIII.

4. Construct the figure and prolong DA until it divides the parallelogram on BC, which call BOPC, into two parallelograms, by meeting the side OP in the point M. Now prolong the sides OB and PC until they meet respectively the sides EF and GH. We thus form two parallelograms having AD for a common side, by means of which we can compare (using Prop. I.) the parallelogram BM with ABEF, and also CM with ACGH, . . . and prove their equivalence, etc.

To prove Theorem XI. from this we have only to show that the diagonal through A of the rectangle formed by prolonging IK and HL, is the prolongation of AD, and equal to the hypothenuse of the triangle.

5. Take A as the acute angle, and then, by completing the construction as in XI., and pursuing precisely the same method of comparing the rectangles by means of equal triangles, we find the square on BC equivalent to the sum of two rectangles, one of which is less than the square on AB by a certain rectangle (say M), and the other of which is less than the square on AC by a second rectangle (N). Then we can easily show (M) and (N) to be equivalent, and also to be measured respectively as stated in the theorem.

6. Take A as the obtuse angle, and by the same mode we find the square on BC equivalent to the sum of two rectangles, one of which is greater than the square on AB by a rectangle (M), and the other greater than the square on AC by a rectangle (N), and then it is easy to show that (M) and (N) are equal, and each measured as stated in Theorem XIII.

7. The triangle KAL (*See* Fig., Prop. XI.) can be easily proved to

be equal to ABC. To compare either of the others, as ICG, with ABC, conceive this triangle to revolve around the point C until CG becomes the prolongation of CB, then CI will coincide with CA, and the triangle ICG (in its new position) and the triangle ABC have equal bases and the same vertex; hence, etc.

8. This is easily shown by remembering (7), and adding up the three squares and four triangles which make up the hexagon.

9. This follows at once by applying XI. to the original triangle and to the new right angled triangle formed by drawing the median named.

10. An obvious corollary of Theorem XIV. For when we join a point of circumference to the given points the median of this triangle is the constant radius, and half the base is also constant.

11. A corollary of (10). Since the sum of the squares of the two lines joining any point of the third circle with the centres of the two given circles is constant, and these two lines with the tangents and radii to the points of contact form right angled triangles, hence, by Prop. XI. the sum of squares of tangents must also be constant.

12. An immediate corollary of Prop. XV.

13. Construct the figure and then by the same process as in (10) show this to be a corollary of (12).

14. Call the lines joining the point taken on line of intersection with the centres of the circles m and n, and the radii of the circles R and r, respectively. Then the difference of the squares of the tangents is $m^2 - n^2 - (r^2 - R^2)$ (13), but by (12) $m^2 - n^2 = R^2 - r^2$. Hence, etc.

15. Let AB and DE be the chords intersecting at right angles in O, and let C be the centre of the circle. Join AD and BE and the diameter ECS, and join DS and BS. Then use Prop. XI. in triangles AOD and BOE, and remember that, also, EDS, EBS are right angles, and DS, therefore, parallel to AB, and hence, AD = BS, etc.

16. Construct the figure ABCD and join middle points M and N of the diagonals, then join the middle point, M, of the diagonal, BD, to the vertices A and C. Then apply Prop. XIV. to the triangles DAB, DCB, and AMC, and combine the results.

17. Because each side of the parallelogram formed by joining the middle points of sides is one-half the diagonal to which it is parallel (Exercise 53, Book I.).

18. Follows from (17) by corollary of XIV.

19. Form the triangles PAC and PBD and join PO. Then apply XIV. and combine results.

20. First, show that the line joining the middle points of diagonals of a trapezoid is equal to half the difference of the parallel sides, and then the theorem is an easy corollary of 16.

21. Use Prop. XIV. for each median, remembering (Exercise 31, Book I.), and also that the square on $\frac{1}{3}$ of a line is equal to $\frac{1}{9}$ of the square on the line, and the square on $\frac{2}{3}$ of a line is equal to $\frac{4}{9}$ of the square on the line, and combine results.

22. Use Prop. XIV. for each median, and combine results.

23. Construct the figure, join M with A, B, C, and G, etc., and then use XIV. and combine.

24. Construct the figure, join point O with vertices, apply Prop. XI., and combine results.

25. Prove this by a *reductio ad absurdum*, granting (24).

26. Proceed as in (24) and combine the results by addition, as in that exercise.

27. Simply show that in each case

$$CE^2 + BD^2 + AF^2 = AE^2 + CD^2 + BF^2,$$

when E, D, F are, first, the feet of the perpendiculars to middle points of sides. Second, the feet of the three altitudes. Third, the points of contact of the three escribed circles. (In this last case, we must first show from the property that the two tangents from a point without a circle are equal, that the distance from the vertex, C, of a triangle to the point of contact, S, of escribed circle and side CB, is equal to half the perimeter minus the side AC, etc., for the other points of contact.)

28. Easy corollary of Prop. XX., and converse.

29. Construct the figure and also the circle within the greater circle whose radius is equal to the difference of the radii of the two given circles. Then the square of the part of the common tangent named is equal to the square of the distance between the centres plus the square of the difference of the radii (Prob. XVII., Book II. and Prop. XI.). But the distance between the centres is equal to the sum of the radii. Therefore, etc.

30. Let ABC and EBF be the two triangles, having angles at B supplementary and angles C and F equal. Place them so that EB shall fall along AB. (EB being the shorter.) Then FB and BC will form one and the same straight line. Through E draw a parallel to AC, meeting BC in M. Then EM will be equal to EF, and the similar triangles BAC and BEM give an easy demonstration of the theorem (Prop. XX.).

31. Draw through the vertices of the triangle parallels to the opposite sides. We thus construct a triangle equal to four times the given triangle, whose sides are double the homologous sides of the first (Book I., Exercise 24) and similar to it (Prop. XIX.). Now, the distance from vertex to intersections of altitudes of the first triangle is the distance of centre of circumscribed circle to the side of second triangle. Therefore, this line is double the distance of centre of circumscribed circle of the first triangle from the homologous sides, since these lines are homologous, and therefore bear the same ratio to one another as the homologous sides.

32. Draw the altitudes Aα, Bβ, Cγ, and let M be their point of intersection. Since triangle MBα is right angled at α the angle $q\alpha$M $=$ qMα. Similarly the angle $r\alpha$M $=$ rMα. Hence, angle $q\alpha r$ $=$ BMC; also, we have by reason of parallels, angle qpr $=$ BAC.

Therefore, $qar + qpr =$ BAC $+$ BMC $=$ two right angles. Therefore, the circle through pqr passes through α. Similarly, it may be shown to pass through β and γ.

Again, qarM is a parallelogram. Hence, angle $qar = q$Mr; therefore $qar + qpr =$ two right angles. Hence, the circle through p, q, r passes through a, and; also, by similar proof, through b and c.

Second.—The lines $a\alpha$, $b\beta$, $c\gamma$, are chords of the circle, and perpendiculars erected at the middle points of these pass through the centre. Now, any one of these perpendiculars, as that one at middle point of $a\alpha$, must (by property of trapezoid) meet OM at its middle point, N (O being centre of circumscribed circle). Hence, N is the centre, and the radius Np $=$ one-half OA (Book I., Exercise 24).

33. Let ABC be the triangle, G the point of intersection of medians, and M the middle point of BC. Draw perpendiculars from A, B, C, G, and I on the given line, then apply the property of the trapezoid (Prop. VII., Scholium).

34. Construct triangle ABC and draw the medians AM, BN, CO, meeting in G. Prolong AM until MO $=$ MG and join CO. The area of triangle COG, contained by sides equal to two-thirds of each

median is equal to $\frac{4}{9}$ of the area of this triangle of the medians (Props. XXI. and XXVII.). Now, the triangle AGC = triangle CGO (Prop. II., Cor.), and triangle AGC is one-third of the triangle ABC, having same base and one-third of the altitude. Therefore, etc.

35. Use similar triangles.

36. Construct the figure. Then, by similar triangles, in first, the distances from point where secant cuts line of centres to the centres respectively, are proportional to the radii. Then these quantities being in proportion, are in proportion by division. Thus, three terms of the proportion are fixed, and, consequently, the fourth term (the distance from one of the centres to the point of intersection) is fixed, and the same which we get by proceeding with the common tangent, etc., and similarly for the interior tangents.

37. Draw in triangle ABC the median AD, and DO perpendicular to middle point of BC, to the centre, O, of the circumscribed circle. Through A, B, and C draw parallels to the opposite sides of the triangle, forming thus, A′B′C′, similar to ABC. Through A, middle point of B′C′, draw AO, perpendicular to B′C′, and prolong it to O′, the centre of circumscribed circle of A′B′C′. O′ is the intersection of altitudes of the triangle ABC (*See* 32).

Join OO′ meeting AD in G. The triangles O′GA and OGD are similar, but (32) O′A = 2OD; hence, AG = 2GD, and G is the point of intersection of medians (Book I., Exercise 31). Moreover, O′G = 2OG.

39. The line in each case is parallel to the parallels.

40. Let the circles have the centres A, B, and C. Let the common chord, MN, of the circles A and B meet the common chord PR of the circles B and C in the point O. We wish to prove that TS, the common chord of circles A and C, will pass through this point. Join SO, and suppose it meets circle A in some point, T′, instead of T, and the circumference C in T″. Then, using Prop. XXX., we prove the theorem by easy *reductio ad absurdum.*

41. Join CM, then the triangles CMP and CMQ are similar (Prop. XXII.). The proportions resulting from this similarity serve to demonstrate the theorem.

42. This converse is, If four points be situated on two straight lines, meeting in any point, O, so that OA × OB = OD × OC, then the

four points are on the circumference of the same circle. This is proved readily by the *reductio ad absurdum.*

GEOMETRIC LOCI.

1. The locus is a line parallel to the line joining the two given points.

2. See Prob. III.

3. Prop. XIV. and Exercise 10, Theorems.

4. Prop. XIV., etc. Combining first two points and then this result with the third, etc.

5. See Exercise 11, Theorems, for *two* circles. Then combine this result with third circle, etc.

6. See Exercise 12, Theorems.

7. See Exercise 13, Theorems, and also 14.

8. Let A, B, C, be the centres of the three circles, R, R′, R″ their radii. Suppose the radical axis of circle A and circle B, and the radical axis of circles B and C meet in a point O. Then

First.—$\overline{OA}^2 - \overline{OB}^2 = R^2 - R'^2$. *Second.*—$\overline{BO}^2 - \overline{CO}^2 = R'^2 - R''^2$, whence, by addition, $\overline{OA}^2 - \overline{CO}^2 = R^2 - R''^2$. Hence, point O is on radical axis of circles A and C. (This proof includes the case proved in Exercise 40, Theorems.)

9. Let C and O be the centres of the given circles, and A and B their points of intersection with a circle of centre P which cuts each of these circles at right angles. The tangents drawn to the given circles at A and B must pass through centre P of the orthogonal circle (being radii) and are therefore equal. Hence P is on the radical axis. N. B.—Two circles cut orthogonally when their radii, at point of intersection, are at right angles.

10. Let AOB be the angle and P one of the points, then let $PS^2 + PR^2 = \text{constant} = b^2$. Draw OP; OS = PR. $\therefore PS^2 + PR^2 = OP^2 = b^2$. Hence, OP, constant and locus, is a circle.

11. Also a circle by extending (10), or included in (4).

12. Construct the figure, and by reason of the proportionality of two sides about a common angle, we have a number of pairs of similar triangles. Hence, from equality of angles, the required locus is a line parallel to the given line.

13. Let O be the point. A, B, etc., points on circumference C. a, b, c, etc., points taken on OA, OB, and OC. Then, since by hypothesis $\frac{Oa}{OA} = \frac{Ob}{OB} = \frac{Oc}{OC}$, the triangles O$ac$ and OAC are similar. Also OBC and Obc. Hence, it is easy to prove $ca = cb$, and the locus is a circle with centre c.

14. Let OA and OB be the two given lines, and M be a point so that if MP and MN be its distances from OA and OB respectively, $\frac{MP}{MN} =$ constant. Join O and M and we can readily show that any other point on OM has this same property, etc.

15. Let A and B be the given points, and M a point so that $\frac{MA}{MB} = \text{constant} = \frac{a}{b}$. Draw AB and determine a point C on it, so that $\frac{AC}{CB} = \frac{a}{b}$, and also a point D on AB produced so that $\frac{AD}{BD} = \frac{a}{b}$, and then, by Prop. XVIII. and Scholium, we can easily show the locus to be a circle on the diameter DC.

16. The figure being constructed, and the half angles of the tangents being equal, it will be clear from the similar triangles formed by the tangents from the point M and the lines from M to centres O and O′, and the radii, that $\frac{MO}{MO'} =$ constant. Hence, the locus is a circle, by (15).

17 and 18. See Exercises 38 and 39, Theorems.

PROBLEMS.

1. Take a point on an indefinite straight line, and lay off on this line the lengths of two of the given lines, one on each side of the point, and at the point lay off, in any direction, the third line. Pass a circle through the three extremities of the lines thus laid off, etc.

2. Through the given point O draw a line parallel to one side, AC, of the angle BAC, meeting the other, AB, in D. Find a line which bears to AD the ratio of n to m, and lay it off from D towards B, as DH. Join HO, etc., using Prop. XVI.

3. *First.*—Let ABC be the triangle, and M the point on the side AC. Draw MB. If MC be greater than AM, the point of division

of the required line must fall between B and C. Draw BD. Then, $\frac{\text{MCD}}{\text{ACB}} = \frac{\text{MC} \times \text{CD}}{\text{AC} \times \text{CB}}$, (Prop. XXVI.), and this ratio is to be $\frac{1}{2}$.

$\therefore \frac{\text{MC} \times \text{CD}}{\text{AC} \times \text{CB}} = \frac{1}{2}$. And we can construct CD, Prob. II.

Second.—In view of the solution of first, second presents no difficulty.

4. Successive applications of Prob. III.

5. See Problem XVII.

6. The point of intersection of the medians. (See Hint to Theorem 34.)

7. The solution depends on the Prob. XII.

8. This requires the division of the hypothenuse into extreme and mean ratio, and then we must take the greater segment of the hypothenuse equal to the side. (Use in this problem Prop. XXV., Cor.)

Method of Similar Triangles.

The *method of similar triangles* in the solution of problems consists in drawing a figure similar to the required figure, deducing from the data of the problem all the elements necessary for the construction of the new figure, or some of them, and taking the others at will. The comparison of the known proportional lines in the two figures will enable us to construct those of the proposed problem.

9. Let x, x', x'' be the unknown sides AB, AC, BC of the required triangle ABC, and h, h', h'' the corresponding given altitudes. We must have

$$xh = x'h' = x''h'',$$

whence

$$\frac{x}{h'} = \frac{x'}{h} = \frac{x''}{\frac{hh'}{h''}}.$$

Construct, first, a fourth proportional a to the three lines h, h', h'', so that $a = \frac{hh'}{h''}$. Then construct a triangle A′B′C′ with the sides h, h', and a. A′B′C′ will be similar to ABC by reason of the proportion

$$\frac{x}{h'} = \frac{x'}{h} = \frac{x''}{a}.$$

Now, taking upon one of the altitudes, A′D′, of A′B′C′, a line A′D″ equal to the homologous given altitude, as h, and through D″ drawing B″C″ parallel to B′C′, the triangle A′B″C″ will be the triangle required.

10. Construct a triangle having its angles equal to the given angles. Draw a median (of course one particular median is given in the required triangle) and on it take a line equal to given median, etc.

11. Solution after same manner by the method of similar triangles.

12. Use Prop. XXII., Cor., and then construct hypothenuse as third proportional to the given segment and given side, etc.

13. Use Theorem XXXII. and Problem III. (first prolonging the line joining the two points to meet the given line). (Two solutions.)

14. Suppose the problem solved (A and B being the given points and O the centre of given circle). Suppose the common tangent to the two circles, MC, meets the line AB in M. Then $MC^2 = MA \times MB$. Also, if any other secant be drawn to the given circle from M, as MFD, then $MC^2 = MF \times MD$, and then $MF \times MD = MA \times MB$, and A, B, F, D are on the same circumference (Prop. XXXI, converse). Hence, the construction to find M is as follows: Pass any circle through A and B, cutting the given circle in any two points, F and D; and the chords AB and FD determine M. Hence the point of contact of required circle and given circle known. (Two solutions), etc.

15. Take the case of intersecting lines; the locus of the centre of the required circle is the bisectrix of the angle. Join the vertex of the angle with the given point. Now, if the problem were solved and any number of circles were drawn tangent to the two lines, the radii drawn to the points of intersection of one of these with this straight line will be parallel to the radii of the others. Hence, draw any circle touching the two lines, and draw the radii above spoken of; then, through given point, draw two lines parallel to these radii until they meet the bisectrix, and we have the required centres (two solutions). (A good illustration of the method of similar triangles.) This problem may be solved more simply, thus: Construct the bisectrix, let fall a perpendicular from the point M on the bisectrix, meeting it at O, and prolong this perpendicular till OM′ = OM, then M′ is a point of the circle, and the problem becomes the same as (13).

16. Given circle with centre O, a straight line CD, and a point A. Suppose the problem solved, and a perpendicular drawn from O to CD, meeting it at N, meeting the circumference in E and F, and draw FA, meeting the required circle also in M, and let P and H be the points of contact of required circle with the given circle and given line respectively, then N, E, P, and H are on one circumference, and FN × FE = FP × FH (Prop. XXXI). Also, FA × FM = FP × FH, therefore, FA × FM = FN × FE. Hence, we can find point M by constructing FM, a fourth proportional to FN, FE, and FA. M being determined, problem is brought back to (13).

17. Let A be the centre of the smaller circle, B the centre of the larger one, and M the given point. Draw a common tangent to the two circles, touching circle A at C and circle B at D, and meeting the line of centres, AB, in P. Join PM and divide it at N so that PM × PN shall be equal to PC × PD. Then pass a circle through M and N, touching either of the given circles (by 14). This will be the required circle. For, etc. (The exterior common tangent gives two solutions; so, also, in general the interior common tangent will give two, thus making four in all.)

18. Let AB, AC be the given lines and O the centre of the given circle. Suppose the problem solved and O′ to be the centre of the required circle, touching AB and AC and touching the given circle in M. The centre, O′, must be on the bisectrix, AD, of the angle BAC. From O let fall perpendicular OD on AD and prolong it until DO″ = DO. A circle passing through O and O″ and touching two lines parallel to AC and AB, and at a distance from them equal to the radius of given circle O, must be concentric with the required circle. Hence, construct a line parallel to AC and at a distance from it equal to radius of given circle. Then pass a circle through O, O″ touching this line (13). The centre of this circle is the centre of the required circle, etc.

19. Draw a diameter through the given point and a chord perpendicular to it. Suppose problem solved, and then use Prop. XXX. and also Prob. XII.

20. Draw any secant whatever. Then suppose the problem solved, and use Prop. XXXI. and also Prob. III.

21. Since the base is given and the ratio of the two sides, the locus of the vertex is a circle (*See* Exercise 15, Geometrical Loci); and

since median is given the locus of the vertex is a second circle described from middle point of base as centre. Hence, etc.

22. Suppose the problem solved; BAC the triangle required and AD the given bisectrix. Then (Prop. XVIII.) $\frac{BD}{DC} = \frac{AB}{AC}$ (1), and, also (Prop. XXXIII.), $AB \times AC = AD^2 + BD \times DC$. Whence, $BD \times DC = AB \times AC - AD^2$. We have then to construct, first, a mean proportional between AB and AC, and then a square equal to the difference of two squares; thus we get $BD \times DC = l^2$, or combining with (1) $BD^2 = l^2 \times \frac{AB}{AC}$. Then construct BD, etc. (Prob. II. and Prob. III.).

23. Let ABC be the given triangle, O′ the centre of the three circles. Suppose the problem solved and A′B′C′ the required triangle. Draw the radii O′A′, O′B′, O′C′. At the given point C suppose CAO made equal to C′A′O′ and ACO = A′C′O′. Then the quadrilaterals ABCO and A′B′C′O′ would be similar. And, hence, if we knew ABCO we could construct A′B′C′O′ and thus solve the problem. Now we can construct ABCO by determining O through the intersection of two loci. For $\frac{AO}{OC} = \frac{A'O'}{O'C'}$ (a known ratio), hence, locus of O, a circle (*See* Geometrical Loci, Exercise 15); and $\frac{OB}{OC} = \frac{O'B'}{O'C'}$ (a known ratio). Hence we know second locus of O, etc.

24. Suppose the problem solved. Construct the figure, and then, by similar triangles, we find x, the required side, $= \frac{ab}{a+b}$, or a fourth proportional to the base and altitude, and the sum of the two.

25. Same as 6.

26. Draw any circle whatever intersecting the two given circles and draw their chords of intersection (radical axes), prolonging them to their intersection. We thus determine the radical centre of the three and at once the radical axis of the first two.

27. Find radical axis of the two to which the equal tangents are to be drawn, etc.

28. Use (Exercise 6, Geometrical Loci) to get a locus whose intersections with the third circle give the points required.

29. Use the locus of a point whose distances from two fixed points bear a fixed ratio (*See* Geometrical Loci, Exercise 15).

30. Find the locus (*See* Geometrical Loci, Exercise 4) of the point, the sum of the squares of the distances of which from the three points A, B, and C is equal to a given square.

Numerical Problems.

The chief theorems used in the solution of the Numerical Exercises are the V., VI., VII., XI., XIV., XX., XXV., and XXXII. By a reference to these the solutions will be found very easy.

BOOK IV.

Theorems.

1. From the equality of the angles we can show the equality of the alternate arcs which the sides subtend. Hence, the alternate sides are equal, and, therefore, all the sides, if the number be odd.

2. From the equality of the sides, by a comparison of equal triangles, we show the alternate angles equal.

3. Let ABCDEF be the hexagon. Draw the diagonals AC, FB, and FC. FC is parallel to AB and equal to twice AB. Then proceed by comparison of similar triangles.

4. A construction of the figure and a comparison of equal triangles makes this clear.

5. The area of the inscribed is to that of the circumscribed as the square of the radius is to the square of the apothem.

6. This follows from the construction of the figure, and the last part from the property of the right angled triangle.

7. For the triangles having vertex at centre have for bases the radius and for altitudes one-half of the radius.

8. Take MN as side of pentagon and AB as side of decagon, O being centre of circle. Produce AB until AC = radius and draw OM, ON, OA, OB, and OC. Now, compare the triangles OMN and OAC. Then, from C draw a tangent, CP, to the circle, and join OP. Then use (Book III., Prop. XXXII.) and the property of the right angled triangle.

9. Let ABCDE be the pentagon. Draw the diagonals AC and

BE, meeting in O. Then examine the triangle BOC, and compare the triangles ABO and ABC.

10. Construct the decagon ABCDEFGHIK, draw the diameter AF and the chords BE and CF which subtend each $\frac{3}{10}$ of the circumference. Let these meet in a point M. Then the line DM, produced, must pass through O, the centre of the circle. Then prove the triangles FMO and CMD isosceles.

11. Presents no difficulty in the proof.

12. The lines form a regular star polygon, and by a comparison of the isosceles triangles formed by their intersections and also by examining the isosceles triangles formed by drawing also AB, BC, etc., we can readily prove the theorem.

13. Make the given point the vertex of n triangles, having the n sides for bases. Compare the expression for the area of the polygon thus obtained with that which we get by taking the centre of the polygon as the vertex of n equal triangles.

14. This construction gives us an equiangular octagon and we can easily find an expression for two of the sides and thus show the sides all equal.

15. Since the tangents at the point of intersection are at right angles, the radii are, also at right angles, and thus the right angled triangle has the hypothenuse double one of the sides. Whence, the angles at the centre of each circle corresponding to the common chord are known.

16. Join the centre of the small circle with the centre of the given semicircle and also with the centre of one of the semicircles described on the radius of the given circle. Then use Theorems XIII. and XIV., Book II., and Theorem XXXII., Book III.

17. The semicircle in the hypothenuse is equal to the sum of the other two semicircles, and also equal to the triangle, together with the two segments on the two sides.

18. π is the semicircumference to the radius 1, and is in value between these two semiperimeters.

20. The second is true because circumferences are proportional to their diameters. To prove the first, we compare the areas, remembering that the one is

$$\tfrac{1}{8}\pi\overline{AC}^2 + \tfrac{1}{8}\pi\overline{AB}^2 - \tfrac{1}{8}\pi\overline{BC}^2,$$

and the other is

$$\tfrac{1}{8}\pi\overline{BC}^2 + \tfrac{1}{8}\pi\overline{AB}^2 - \tfrac{1}{8}\pi\overline{AC}^2,$$

and also remembering that $a^2 - b^2 = (a + b)(a - b)$. Refer also to Exercise 1, Book I.

21. A corollary of the preceding or proved by comparing the areas after the same manner.

22. Because, if O be the centre and OB, OC, the radii, the area of the ring is $\pi\overline{OB}^2 - \pi\overline{OC}^2$.

23. We demonstrate this after the same manner as the solution of Problem XII., Book III. (using Book III., Prop. XXV., Cor.).

24. The angle at the centre of the fixed circle between the radius to point of contact of rolling circle and radius to the given point on the circumference of the same in its changed position, is an inscribed angle on the rolling circle, and, hence (Book IV., Prop. XIX.), the arc of fixed circle between the point of contact before rolling commenced and the point of contact in the new position is equal to the arc of rolling circle between given point in its changed position and point of contact. Therefore, etc.

PROBLEMS.

1. If the problem be solved the three circles fulfilling the conditions will be inscribed in the quadrilaterals formed by letting fall perpendiculars from the point of intersection of the bisectrices on the opposite sides; that is by the prolongation of the bisectrices. Hence, the construction is plain. (The distance of the centre of each small circle from the foot of the altitude on which it lies will be equal to one-half the side of the triangle).

2. If the problem were solved the three small circles would each touch one side of an equilateral triangle circumscribing the circle at its point of contact with the circle. Hence, circumscribe this triangle and divide it into three triangles by lines from the vertices to centres of circle, etc.

3. Draw the diagonals of the square, forming thus four equal triangles, etc. If a be the side, the radius of the inscribed circles $= \frac{a}{2}(\sqrt{2} - 1)$.

4. Conceive the problem solved and a square circumscribed about the given circle having its sides tangent at the points of contact. We

see thus that the solution of the problem can be brought back to the solution of the preceding one.

5. R being radius, let $x =$ side of decagon $\therefore$ (Book IV., Prop. VII.) $R : x :: x : R - x$, whence, a quadratic in x, and $x = R\frac{(\sqrt{5} - 1)}{2}$, and then Theorem 8 of exercises gives side of pentagon.

6. Theorem 8 of Exercises of Book IV., and the value of side of inscribed regular decagon when radius is R, $R\frac{(\sqrt{5} - 1)}{2}$ give the construction. Draw a diameter AOB, and a radius OC, perpendicular to it. Join C to M, the middle point of OA. Then, with M as a centre and MC as radius, describe an arc of a circle cutting the diameter AOB in point D on the other side of the centre; OD is the side of the decagon and CD the side of the pentagon.

7. Area of the octagon found from that of the square, by using Prop. XVI.

8. The triangles are proportional to the squares of their homologous sides, and any one antecedent is to its consequent as sum of antecedents to sum of consequents. Hence, if x be side of required triangle, $x^2 = 9 + 25 + 4$.

10. See Theorem 7, Exercises.

11. Find area of sector, and then, given area of circle, to find its radius.

14. From area of sector, which is readily found, subtract equilateral triangle whose side is the radius.

17. Compute side of octagon from that of square and of polygon of 20 sides from decagon.

18. The difference between the sum of three equal sectors and an equilateral triangle.

19. See Problem 3.

20. Areas of circles are proportional to the square of their circumferences.

21, 22, 23. (*See* Prop. XIX. Cor.)

29. See Problem 8.

BOOK V.

THEOREMS.

1. If the point be on the plane, join it with the middle point of the given line, and with its extremities. Then demonstrate, as in Prop. XVIII., Book I.

If the point be not on the plane, join it to the two extremities of the given line. The line joining the point to one extremity must pierce the plane ; join this point of piercing with the middle point of the given line, and with the other extremity ; we then have a figure the same as that used in the second part of Prop. XVIII., referred to.

2. An easy corollary of Prop. IX. (see Cor. 2).

3. See Cor. 2, Prop. XII.

4. Let P and Q be two planes parallel, respectively, to two planes, M and N, which intersect. If P be parallel to Q, then is also M parallel to Q, and then M and N, being both parallel to Q, would be parallel to each other, which is contrary to the hypothesis.

Then, if, through any point of intersection of P and Q, we draw a parallel to the intersection of M and N ; being parallel to M, it must lie in P, and being parallel to N, it must lie in Q, and hence, etc.

5. For they must both meet the line of intersection at the foot of the perpendicular from O on that line (Prop. X.).

6. For the perpendiculars to the two planes erected at P and Q must meet the perpendicular to the common intersection drawn as above, in the same point (Prop. X.).

7. For if the two planes meet, their line of intersection would be parallel to AB and A'B', and hence this intersection is impossible (Prop. XXIX.), unless AB and A'B' are perpendicular to P.

8. See 7, and Prop. XV.

9. See 8, and Prop. XVIII.

10. Draw in P, through the same point, A, the perpendicular AO, and any other line, AM, to the common intersection of P and Q. Project these lines on Q, in O*a* and M*a*, and then compare the angles at M and O in the triangles thus formed (see Prop. XXII.).

11. They cannot meet (see Props. XXVII. and XIII.).

12. Draw the faces of the diedral and the plane bisector. Then

from a point in the plane bisector draw perpendiculars to these faces, and from the feet of these perpendiculars draw perpendiculars to the edge of the diedral and join their common foot with the assumed point. Now compare the two right angled triangles thus formed. (The second part may be demonstrated after the same manner as secondly in Prop. XXI., Book I.).

13. *First.*—Each perpendicular is in a plane through the point perpendicular to the common intersection (Exercises 5 and 6, and Prop. V.).

Secondly.—Each perpendicular is in a plane, as above, perpendicular to each one of the common intersections (then Prop. V.).

14. Construct the figure; let fall the perpendiculars from extremities of diagonal on the plane, and prove them equal by a comparison of triangles.

15. Let SA, SB, SC be the edges of the triedral. Every point within the triedral equally distant from the edges SA, SB is on the plane containing the bisectrix of the angle ASB, and perpendicular to the plane ASB. Similarly for the plane perpendicular to the face ASC, and containing the bisectrix of the angle ASC. These two planes meet in a line, any point of which is equally distant from SB and SC; hence, their line of intersection must be, etc., etc.

16. Use Exercise 12, and reasoning similar to the above.

17. Take SA, SB, SC all equal, and join AB, AC, BC. The bisectrix of the angle ASB, of the isosceles triangle SAB, meets AB at its middle point, and so of the other bisectrices. Draw the medians of the triangle ABC; each one of these lies in one of the planes containing an edge and the bisectrix of the angle of the opposite face. Hence, the intersection of these medians is common to all the planes, hence, etc., etc.

18. Draw the planes. Let SM, SN, SO be their intersections with the opposite faces. Through A draw AB perpendicular to SM at M, and AC perpendicular to SN at N. Then draw BC, BN, and CM. Then CM and BN are perpendicular to AB and AC (Prop. X., Cor.), and their point of intersection is common to all three of the planes (Book I., Exercise 28). Hence, as the planes have already S in common, etc., etc.

19. In this case the proof varies but little for 15, and is the same for 16. In (15) the planes perpendicular to the faces contain, in the

new position, not the bisectrices of the angles of the two edges, but the lines parallel to the edges in each face and midway between them. A like change applies to 17. In the case of 18, the common line of intersection of the three planes is the perpendicular to the plane ABC, erected at the intersection of its altitudes AM, AN.

20. Every perpendicular from a point on either given line to the plane will be equal to one-half the perpendicular to the two given lines. Draw any line meeting the two given lines in points A and B. From the two points A and B let fall perpendiculars on the plane. Join the feet of these perpendiculars with the point in which AB meets the plane, and compare the triangles.

21. Because this shortest distance is in every case the distance from the line to the plane (as will be seen by constructing this shortest distance in one case (Prop. XXXI.).

22. A cannot be parallel to B and meet C (Prop. II., Cor. 5). A cannot meet B in one point and C in another, for then all three lines would be in the same plane. Hence, etc., etc.

23. Let SABC be the triedral, rectangular along the edge SA. If the plane is perpendicular to the edge SA, the truth of the theorem is evident. Let it be perpendicular next to SB, at any point, N, and let it meet SA in M, and SC in O. Construct the triangle MNO. It is easy to show that it is a right angled triangle (Prop. XXVIII., and Def. 2).

24. Let SABC be the triedral, having the diedral ASBC greater than the diedral ASCB. Cut off from ASBC a diedral, CSBD, equal to ASCB; let SD be the intersection of the plane SBD with the face ASC, then apply XXXIV. and XXXII., and make a demonstration similar to that in Book I., Prop. XV. The converse is proved by an easy *reductio ad absurdum*.

Loci.

1. Two planes parallel to the given plane.

2. See Theorem 1, Exercises.

3. If lines parallel, plane perpendicular to plane of lines, and containing line parallel to the two lines, and midway between them. If lines intersect, the locus is two planes perpendicular to each other, and to plane of the given lines. (See Book II., Exercises, Geometric Locus 4.)

4. Two perpendicular planes (see Exercises, Theorem 12).

5. See Prop. IX., Cor. 3.

6. Locus consisting of straight lines perpendicular to the plane of the three given lines (see Locus 3), or a single line if the three given lines meet in a point.

7. A locus consisting of straight lines (see Exercises, Theorems 12, 16, 19).

8. See Geometric Locus 3, and also Exercises, Theorem 15.

9. A straight line (See Loci 2 and 3).

10. A straight line (See Loci 2 and 4).

11. A straight line (See Loci 3 and 4).

12. A plane (See Geometric Locus 6, Exercises Book III.).

13. See Prop. IX., and Locus 5.

14. See Prop. IX. A circle.

15. All the points in question must be at the same distance from M, the middle point of AB (determined by the given sum of the squares of its distances from A and B), (Book III., Prop. XIV., Cor. 2). Hence, the locus (by Locus 14) is a circle.

16. A straight line. See Locus 12.

17. Let fall a perpendicular, AC, from A on the plane, C is one point of the locus. Draw BC, and any other line, BM, in the plane through B; let AM be perpendicular to BM at point M, and join MC. Then, Prop. X., Cor., MC is perpendicular to BM. Hence, the locus required is a circle on BC as diameter.

PROBLEMS.

NOTE.—For the most of these problems the pupil should be required simply to indicate the solution, or to give an analysis without making the drawing.

1. See Prop. XII., Cor. 2.

2. Draw through the given point two lines which determine the required plane, using Prop. XIII.

3. Use Prop. XXVIII.

4. Pass a plane through the point and one of the lines; find where the second line meets this plane, etc. (Discuss the Problem).

5. Let A and B be the two given straight lines, not situated in the

same plane, and C the straight line to which the required line meeting A and B is to be parallel. Pass a plane through B parallel to C (see Problem 8), and find where A meets it; then, etc.

6. Let the points be A, B, C, D. For the locus of points, any one of which is equidistant from A, B, and C, see Locus 5. Also see the same Locus 5, for points equidistant from B, C, and D. These lines are in the same plane and must meet (see Exercises, Theorem 6), and their point of intersection is equidistant from A, B, C, D. If the four points are in the same plane, the solution is impossible unless they are on the circumference of a circle, and then an infinite number of points satisfy the conditions (see Locus 13).

7. Use Locus 12.

8. At any point of the given straight line through which the required plane is to be drawn, determine a second line of this plane, using Prop. XIII.

9. Prolong the straight line in space until it meets the given plane. Through this point of meeting draw in the plane a line perpendicular to the given line. Then through the given point draw a line parallel to this line (see Prop. XVIII., Scholium 2).

10. The point, M, required will be such that the lines AM and BM shall make equal angles with the plane, P (see Book I., Exercises, 18).

11. Use Locus 5.

12. Let SO be the intersections of two diagonal planes through opposite edges of the quadriedral. Through O, any point of SO, draw AC and BD, so that they shall be bisected in O (see Book III., Problem VI.); these lines will be diagonals of the required parallelogram, etc.

BOOK VI.

Theorems.

1. These propositions can be easily demonstrated by a reference to Props. XXXVI., XXXVII., XXXVIII., XL., Book V., and by superposition.

2. The bases must be equal, also, and hence, etc.

3. See Prop. XVIII.

4. Construct the plane, then the comparison of the prisms (Prop. XVIII.) makes the truth of the theorem clear.

5. Construct a cube on the line $a + b$. Draw through the points of division, which separate the parts a and b of the edges, planes parallel to the faces. We will thus have one cube of the edge a, three rectangular parallelopipedons whose base is the square on a, and whose altitude is b, etc.

6. Let S-*abc*, and S-ABC be the two tetraedrons. Join A*b* and C*b*, thus forming a third tetraedron, S-A*b*C, which we can compare with either of the others, having the same vertex, A, with A-SBC, and the same vertex, *b*, with *b*-S*ac*. Then, tetraedrons having the same altitudes are to each other as their bases. But the bases SBC and S*b*C, corresponding to the vertex A, are to each other as the edges SB, S*b* (Book III., Prop. VI., Cor.), and the bases SAC, S*ac*, corresponding to the common vertex *b*, are to each other as the products

$$SA \times SC : Sa \times Sc \text{ (Book III., Prop. XXVI.)}.$$

A proper combination of the proportions demonstrates the theorem.

7. Let S-ABC be the tetraedron, and SAM (M being on BC) the plane bisector of the diedral BSAC. The tetraedrons S-AMC and S-AMB can be compared as having the common vertex S, and the common vertex M. These give two proportions (Prop. XXV., Cor. 2), the combination of which demonstrates the theorem, if we remember that the triangles AMC and AMB, having the same vertex, are to each other as their bases.

8. See Prop. XXIII.

9. Let S-ABC be the tetraedron, M and N the middle points of the opposite edges, SA and BC. Pass the plane through M and N, meeting AB in O, and SC in P. Join SO and SN. Then we see that the polyedron SMPNOB is composed of the quadrangular pyramid S-MONP, and the tetraedron S-ONB. Join PA and AN, then the polyedron AMPNOC is composed of the quadrangular pyramid A-MONP and the tetraedron P-ACN. Now, A-MONP and S-MONP are equal, since they have the same base and equal altitudes (SM and MA being equal). It remains then to show that the tetraedrons S-ONB andP-ACN are equal. Their bases, ONB and ACN, are to each other as BO : BA, and their altitudes are to each other as SC : PC. Therefore,

$$S\text{-}ONB : P\text{-}ACN :: BO \times SC : BA \times PC \text{ (Prop. XXV., Cor. 2)}.$$

Now, the distances of the vertices B and A from the cutting plane MONP, are equal respectively to the distances of C and S from the same plane, since BN = CN and AM = MS. But the first two are to each other as BO : AO, and the second two as PC : PS.

Hence BO : AO : : PC : PS,

whence BO : AB : : PC : SC, or BO × SC = AB × PC.

Therefore, S-ONP = P-ACN.

10. Use Problem 6 and Locus 2, Exercises, Book V.

11. Use Theorem 16, Exercises, Book V.

12. Let S-ABC be the tetraedron. The median lines SO and AO, which join the extremities S and A of the edge SA with the middle point, O, of the opposite edge, BC, are in the same plane. Divide SO and AO respectively, in the ratio of 2 to 1 at M and N. (See Exercise 31, Book I.). Draw SN and AM. These are two of the lines of which the theorem treats. They will meet in some point, P. Join MN, and compare the triangles MPN and APS, etc.

13. Use Problem 6, Exercises, Book V. (See also Theorem 10.).

14. Construct the figure, and any two of these lines will be seen to be the diagonals of a parallelogram.

15. Construct the figure, and the truth of the theorem becomes plain from the fact that all the faces are equilateral triangles, all the diedrals are equal, the feet of the perpendiculars from the vertices on the opposite faces are the centres of circumscribed circles, and intersections of medians, etc.

16. Let S be the vertex of the right solid angle, and ABC the opposite face, then shall

$$(ABC)^2 = (ASB)^2 + (ASC)^2 + (BSC)^2.$$

Let fall the perpendicular CN from C on AB, and the perpendicular SM from S on CN, join SN, MB, and MA. Then

ABC : ASB : : CN : SN, therefore : : SN : MN (Book III., Prop. XXV.).

But ASB : AMB : : SN : MN.

Hence $(ASB)^2 = AMB \times ABC$.

Similarly for BSC and ASC, etc.

17. Construct the figure, then any diagonal, AH, of the parallelo-

pipedon, the two perpendiculars Aa and Hh, from A and H on the given plane, and the projection, ah, of the diagonal on the plane, form a trapezoid. In this trapezoid the perpendicular Oo from the centre of the parallelopipedon on the plane, is parallel to Aa and Hh, and equal to half their sum (Book III., Prop. VII., Schol.), etc., so that we have $2Oo = (Aa + Hh)$, and so on for the remaining opposite corners.

18. Take a point, P, join it with O, the centre of the parallelopipedon, and with the extremities A and H of a diagonal. Then use Prop. XIV., Book III.

19. Exactly analogous to Theorem 13, Exercises, Book IV.

20. *First.*—Let ABC-LMN be the prism. Pass the plane MAC, cutting off the pyramid M-ABC, having M for a vertex and ABC for a base, leaving the quadrangular pyramid M-ACNL. Divide this by the plane MNA into the triangular pyramids M-ANC and M-ANL. It can be easily shown that M-ANC is equivalent to A-BNC or N-ABC, and that M-ANL is equivalent to B-ACL or L-ABC.

Second.—Let OO′ be the distance of the intersection of the medians, O, of the triangle LMN, from ABC. Then $OO' = \frac{1}{3}$ of the sum of the perpendiculars from L, M, and N, on the plane ABC. This can be proved after the manner of Theorem 17, above, using the property of medians proved in Theorem 31, Exercises, Book I.

Third.—Let EFG be the right section. Then the truncated prism ABC-LMN is decomposed into two, ABC-EFG and EFG-LMN. Then the measures of these two (by First) added together show the truth of the assertion in Third.

21. Draw diagonal planes resting on the diagonals of the base, and join the intersection of the diagonals of the base to the intersection of the diagonals of the upper base. We thus have four triangular prisms, which, added together, make twice the truncated parallelopipedon. Then apply Theorem 20, *Third.*

22. The intersections of these planes give us triangles similar to the faces respectively of the tetraedron. Hence, the diedrals of the new tetraedron will be equal. But they will nevertheless be not similar, as their triedrals are *symmetrical* and not equal.

23. Use Prop. XXXII.

PROBLEMS.

1. The point is the point of intersection of the four straight lines spoken of in the Theorem 12 of these Exercises.

2. Find the point in which the plane must cut the edge or altitude by remembering Prop. XXXII.

3. Calling b the area of the base of the pyramid S-ABC, and h its altitude, then the area of each one of the equivalent triangular prisms EHG-DMA, DEF-MCH, will have for its measure $\frac{b}{4} \times \frac{h}{2} = \frac{bh}{8}$. Hence, the two together will be $= \frac{bh}{4}$. Each of the triangular prisms, formed in the same manner, in the two tetraedrons S-DEF and B-EGH, will be each one equal to $\frac{b}{16} \times \frac{h}{4} = \frac{bh}{64}$, and as there are four of them, the four are equal together to $\frac{bh}{16}$, etc.

The volume of tetraedron will then be $bh(\frac{1}{4} + \frac{1}{16} + \frac{1}{64}$, etc., etc., without end). The limit of the series is $\frac{1}{3}$. Hence, volume $= \frac{bh}{3}$.

4. Referring to the figure of the cube in Def. 10, the plane cutting the edges AE, CG, EH, HG, AB, BC, at their middle points will make the intersection a regular hexagon.

5. See Props. XVIII. and XXV.

6. See Prop. XXVII., Scholium.

7. See Prop. XIX., Scholium, and Prop. XXV. If the edge is a, then the slant height $= \frac{1}{2}a\sqrt{3}$, the altitude $= \frac{a\sqrt{2}}{\sqrt{3}}$.

8. Prop. XIX., Scholium. Apothem of base $= \frac{1}{2}3\sqrt{3}$ feet.

9. Prop. XX.

10. Props. XX. and XXVII.

11. Prop. VI., Scholium 2, and Prop. XVII.

12. Prop. VI., Scholium 3.

13. Prop. XXV. (The altitude, the slant height, and the apothem of the base form a right angled triangle.)

14. Find the side of each decagon (Book IV., Prop. VII.), thence the apothem, and area. Then apply Prop. XXVII., Scholium.

15. If $\frac{2}{3}x$ be one of the edges, then the others are $\frac{3}{3}x$ and $\frac{4}{3}x$. Then apply Prop. XVII.

16. If $4x$ be one edge, then the two other edges are $6x$ and $9x$. Then apply Prop. VI., Scholium 2.

17. Compute the volume of the first pyramid, then apply Prop. XXXII.

18. Apply Prop. XIX.

19. The area of the base $= \frac{1}{4}(6)^2\sqrt{3}$. (Then use Theorem 20, Exercises.)

20. Props. XIX. and XXV.

21. Let x be the point of division. Then convex surface of whole pyramid : convex surface of pyramid cut off : : $(4.6)^2 : Sx^2$. But these convex surfaces are to each other, first, as 2 : 1 ; and secondly, as 3 : 5. In either case Sx is found by an easy computation or a simple construction (Book III., Problem III.).

22. The slant height of the pyramid and apothem of the hexagon are in the ratio of 2 to 1 (Book I., Exercise 36). The altitude and volume can then be easily found.

23. If x = side of hexagon, and y = the altitude of prism, then perimeter of the base $= 6x$, and area $= \frac{3}{2}x^2 \times \sqrt{3}$. Hence, with given convex surface and volume we can readily find x and y.

APPENDIX TO BOOK VI.

THEOREMS.

1. It has four vertices, and its edges are all equal. Hence its plane faces are all equal, and therefore its diedrals are all equal.

2, 3, 4, 5, 6. Similar reasoning to the above applies in these cases.

7. The centres of the faces are the intersections of the medians, and the line joining the intersections of the medians in two faces is one-third of the edge to which it is parallel. Therefore, etc., etc. (Prop. XXXII.)

PROBLEMS.

1. If the edge is $2a$, then the diagonal section is $4a^2$, and the altitude of each of the two square pyramids which form the octaedron is $a\sqrt{2}$. Then use Prop. XXV.

2. If the edge of the tetraedron is $4a$, that of the octaedron is $2a$, and its volume is found as in Problem 1.

3. If the edge of the hexaedron be $2a$, the edge of the octaedron is $a\sqrt{2}$.

4. If the edge of the octaedron is a, then the edge of the hexaedron is $\frac{a\sqrt{2}}{3}$.

BOOK VII.

THEOREMS.

1 See Exercises on Book II., Theorem 1.

2. See Book V., Prop. IX.

3. Any other line from O to EF is greater than OAC (Book I., Prop. XVII.), and these lines have an equal part, equal to the radius of the sphere, etc., etc.

4. The same reasoning applies as in 3, using Prop. IX., of Book V.

5. See Exercises, Book II., Theorem 3.

6. The angle at the centre of the sphere made by the radius of the sphere, perpendicular to the plane of the small circle, and the radius to the extremity of a diameter of the same, is one-third of a right angle. Then refer to Book I., Exercise 36.

7. The diameter will be the side of a regular decagon inscribed in a great circle. Then see Book IV., Prop. VII.

8. *First.*—This is demonstrated after the manner of Prop. XI.
Second.—This is demonstrated after the manner of Prop. XII.

9. Demonstrated after the manner of Prop. XXI., Book I. (assuming the perpendicular arc of the great circle to be the shortest distance from a given point to a given arc).

10. See Props. XXII. and XII.

11. See Props. VI. and XII.

12. See Book VI., Prop. IV., Cor. 1.

13. See Book III., Prop. XXX.

14. See Book III., Prop. XXXI.

15. See Book III., Prop. XXXII.

16. A corollary of 15.

17. A point equidistant from any four of the vertices is equidis-

tant from all of them. Whence follows the circumscribed sphere. The small circles of the sphere which circumscribe the faces of the polyedron thus inscribed in the sphere are all equal. Hence, their centres are equally distant from the centre of the sphere. Therefore, etc.

18. See Book VI., Prop. XXXII., and Book IV., Prop. X.

19. See Book VI., Prop. XXV., Scholium 2.

20. There are two points, one for the exterior tangent planes, and one for the interior tangent planes. See Book II., Problem XVII.

LOCI.

1. See Book III., Exercises, Geometric Locus 3.

2. See Book III., Exercises, Geometric Locus 15.

3. See Book II., Exercises, Geometric Locus 10.

(N. B.—The points of the surface found exterior to the given sphere do not form part of the locus.)

4. Let O be the centre of the sphere, and EF the given line, and M the foot of the perpendicular from O on one of the sections made by a plane, MEF. Let fall MC perpendicular on EF at C, and join OC. The locus is a sphere described on OC as diameter (see Book II., Exercises, Locus 10). The remark made in 3, applies also in this case.

5. Let P be the point and O the centre of the sphere. Pass a plane through SO. This intersects the sphere in a great circle. Draw two tangents from P to this circle. The chord joining the points of contact is the diameter of the locus required.

6. A great circle, the plane of which is perpendicular to the given line (see Book II., Exercises, Problem 3).

7. It is a circle of which the pole is the pole of the given arc. See Prop. X., Cor. 1.

8. See the Solution of Problem, Prop. IX., also Book V., Exercises, Geometric Locus 2.

9. See Exercises, Theorem 9. Also Book II., Exercises, Geometric Locus 4.

10. The circle has for its pole the middle of the base, and its polar distance is equal to one-half the base.

11. See Book II., Prop. XXIII., Cor.

12. See Book V., Exercises, Geometric Locus 1.

13. See Book II., Exercises, Geometric Locus 3.

14. See Prop. IV.

15. A circle whose centre is the middle point between the two given points.

16. See Prop. V., and Book V., Exercises, Locus 5.

17. See Book V., Exercises, Geometric Locus 4. And also Book II., Exercises, Problem 7. The locus is four lines parallel to the intersection of the planes.

18. See Book V., Exercises, Geometric Locus 7.

19. See Exercises, Locus 13.

PROBLEMS.

1. See Exercises, Geometric Locus 8, and also Book II., Problem I.

2. See Prop. XVI., Scholium, and also Book II., Problem V.

3. Use 1 to find the pole of the circle. See Book II., Problem XIII.

4. Use 2 to find the pole of the circle. See Book II., Problem XV.

SCHOLIUM. The arcs of the great circles bisecting the angles of one of the triangles will be perpendicular to and will bisect the corresponding sides in the polar triangle. Hence, the pole of the inscribed circle of the first is the pole of the circumscribed circle of the second. Again, if we join the angular points of the polar triangle to this common pole by arcs of great circles, these arcs will be perpendicular to the sides of the first triangle at the points of contact of the inscribed circle. Hence, etc.

5. The pole of the required arc must be on a small circle whose pole is the pole of the given arc, and whose polar distance is the arc which measures the given angle. See Geometric Locus 7, and also Prop. X., Cor. 1. It must also be on a great circle at a quadrant's

distance from the given point. Hence, the pole can be found, and the required arc described. (There will be two solutions.)

6. Discussion similar to the above. Show when there is one solution, when two, and when the solution is impossible.

7. See Book II., Problem VIII. Use 6.

Application: The complements of the latitudes form the two adjacent sides, and the difference of longitudes the included angle. The required distance is the third side of the triangle.

8. See Book II., Problem IX. Use 6.

9. See Book II., Problem X. (Discuss the problem as to when it is possible, etc.)

Application: The complements of the latitudes and the distance are the sides of the triangle, and the angle at the vertex opposite to the distance will be the difference of longitude.

10. Construct the polar triangle, first finding its sides by Prop. XII.

11. The area is equivalent to 1.3 times the area of the tri-rectangular triangle. See Prop. XXII., Cor. 1.

12. The excess over 180° is $32^\circ\ 31'$, or $1951'$. The right angle $= 90^\circ = 5400'$.

Hence the area is $\frac{1951}{5400}$ of the tri-rectangular triangle.

13. Let $4x =$ angle A (in right angle units), $6x =$ angle B, $7x =$ angle C. Then Prop. XXII., Cor. 1 will give us an equation to find x, etc.

14. If R = radius of the inscribed sphere, then, *First*, The edge of the tetraedron is equal to $4R\sqrt{\frac{3}{8}}$. (Book VI., Exercises, Theorem 12.) For the volume see Book VI., Exercises, Problem 7. *Secondly*, The side of cube $= 2R$. *Thirdly*, The radius of the inscribed sphere of the octaedron being the perpendicular from the centre on any face, is the perpendicular of a triangle of which half the edge is the hypothenuse, and $\frac{1}{3}$ of half the edge $\times \sqrt{3}$ the base. Hence, it is equal to the edge divided by $\sqrt{6}$. Therefore, the edge $= R\sqrt{6}$. For the volume, see Appendix to Book VI., Exercises, Problem 1.

15. The diameter is to the side of the cube as $\sqrt{3} : 1$ (Book VI., Prop. IV., Cor. 2). Then apply Scholium, Prop. VI., Book IV.

16. Theorem 15 (Exercises) will give (if we call the radius R in

miles) the equation $(4)^2 = \frac{10}{5280}\left(2R + \frac{10}{5280}\right)$, whence R can be found.

17. One locus of the centre of the sphere is a sphere described from any one of the given points with the given radius. The other locus is given by Locus 16. The intersections of these loci give the required centre. (There are two solutions.)

18. The intersections of the Loci 12 and 15 determine the centre of the required sphere, and show the number of solutions.

19. The locus of the centres of spheres of given radius tangent to two planes, is given in Exercises, Locus 17. The intersections of this with a sphere described from the given point as centre, with the given radius, determine the centre and the number of solutions, etc.

20. Locus 18 gives one locus of the centre of the required sphere, etc. If the sphere is to be in any particular triedral of the three planes, we can find the locus of centres of spheres tangent to three planes by Book V., Exercises, Theorem 16, and then apply to this line of centres and to one of the edges, Problem 7, Exercises, Book II.

21. The intersections of Loci 13 and 15 determine the centre of the required sphere, etc.

22. The intersections of two Loci determined in 13, and the intersection of this intersection by a sphere described from the given point as centre with the given radius, determine the centre of the required sphere, etc.

23. The intersections of Loci 12 and 13, and the intersections of this intersection by a sphere described from the given point as centre with the given radius, determine the centre of the required sphere.

BOOK VIII

Theorems.

1. The perpendicular in question will be perpendicular to the plane tangent to the cylinder along the line in which the plane through the given point and the axis cuts the surface of the cylinder. Hence, etc.

2. Let R, H, C, V, be respectively the radius of the base, altitude, convex surface, and volume of one cylinder, and R′, H′, C′, V′, those of the other.

Then, $V = \frac{CR}{2}$, $V' = \frac{C'R'}{2}$ (Prop. VI., Scholium), and since $C = C'$, and also $RH = R'H'$, the theorem easily follows.

3. $C = \frac{2V}{R}$, $C' = \frac{2V'}{R'}$. Then, since $V = V'$, and also $R^2H = R'^2H'$, the theorem is easily proved.

4. This follows from the fact that by definition 4, $R : R' :: H : H'$.

5. See Book I., Exercise 36, and then apply Prop. VIII.

6. Prop. XVI., Cor., and Book III., Prop. XXV., Cor.

7. See Props. XVI. and XXI., and refer to Book I., Exercise 36.

8. Compare the surfaces described by any two corresponding sides of each polygon, and the zone generated by the corresponding arc of the great circle (Props. XIII., XVI.), and then note the proportion which the homologous sides of the halves of the isosceles triangles with the common vertex at the centre resting on these corresponding sides give.

9. See figure, Prop. XXII. Construct in it the cone having the base and altitude of the cylinder. Then use this proposition, and also Prop. X., Cor. 1.

10. Let R, R′ be the radii of the exterior and interior spheres respectively. Then volume of shell $= \frac{4}{3}\pi(R^3 - R'^3)$. Then find the factors of $R^3 - R'^3$. Hence, etc.

11. Let ABC be the triangle, O the intersection of the medians, and xy the axis of revolution passing through A. Let fall the perpendiculars BM and CN on the axis xy, also the perpendicular OP from the point O on the same axis. Let $BM = b$ and $CN = c$, then (Book III., Exercises, Theorem 33), $OP = \frac{1}{3}(b + c)$. Now

vol. ABC = vol. BCNM − vol. ABM − vol. ACN $= \frac{1}{3}\pi \times MN(b^2 + c^2 + bc) - \frac{1}{3}\pi \times MA \times b^2 - \frac{1}{3}\pi \times NA \times c^2 = \frac{1}{3}\pi(b + c)(MA \times c + NA \times b) = \pi \times OP(MA \times c + NA \times b)$. (1)

But area of triangle ABC = area of trapezoid BCNM − area ABM − area ACN $= \frac{1}{2}(MA \times c + NA \times b)$. (2)

Then (1) and (2) give the required result.

12. This follows immediately from Prop. XXIII.

13. See Exercise 7, and Prop. XXIV.

14. The volume generated consists of a cylinder whose side is a, together with the two solids generated by the revolution of two triangles (one above and one below the rectangle which generates the cylinder) about an axis passing through their vertices.

15. For the side of the circumscribed equilateral cone and the diameter of its base, see Book IV., Prop. VI., Scholium. See also Prop. XXII.

16. For the altitude of the inscribed cylinder and the diameter of its base, see Book IV., Prop. V. For the side and diameter of the base of the inscribed equilateral cone, each, see Book IV. Prop. VI., Scholium. Then apply General Scholium, 1, 2, and 4.

17. Let R = radius of sphere. Then the side of equilateral cone and diameter of base (See Book IV., Prop. VI., Scholium), etc. Then apply General Scholium, 1, 2, and 4.

18. Find altitudes and radii of bases by reference to the Props. V. and VI. of Book IV., and then apply General Scholium, 1, 2, and 4.

19. See Prop. XIX., Cor.

20. Let a and b be the two adjacent sides, and h and h' the corresponding altitudes. Then the volume generated about a is $\pi h^2 \times a$, and that about b is $\pi h'^2 \times b$, but $ah = bh'$, etc.

Loci.

1. The convex surface of a cylinder of revolution having this line for an axis.

2. The convex surfaces of two cylinders of revolution having the same axis with the given cylinder.

3. See Book V., Prop. IX.

4. This locus is the intersection of two loci.

5. The circumference of a circle, or the circumferences of two circles the intersection of two loci.

6. The curve of intersection of a plane and a cylinder, or the curves of intersection of two planes and a cylinder.

7. A circumference the intersection of two spheres (See Book II., Prop. XXIII., Cor., and Prop. XX., Cor. 2).

8. A plane.

9. Circumference of a circle. See Book VII., Exercises, Locus 1.

PROBLEMS.

1. See General Scholium, 1. The radius of the base is equal to the given altitude.

2. Use General Scholium, 1.

3. The side is the hypothenuse of a right angled triangle, whose altitude is the given altitude of the frustum, and whose base is the difference of the radii of the bases of the frustum. This being found, apply General Scholium, 3.

4. Find the altitude from the triangle, the construction of which is given in Problem 3, and then apply General Scholium, 3.

5. Find the altitude from the data, and apply General Scholium, 2.

6. Find the side from the data, and apply General Scholium, 2.

7. General Scholium, 1.

8. Find the side of the cone from the data. This is the radius of the sector. Then apply Prop. XIX., Book IV., to find the angle.

9. Let SA $=$ 1.8 yards be the side of the given cone, and Sx the unknown side of the cone cut off. Then the two convex surfaces are to each other as $SA^2 : Sx^2$, whence by the data Sx can be found in both cases.

10. 11. 12. 13. 14. 15. Apply General Scholium, 4.

16. See Prop. XV., Cor. 3.

17. See these Exercises, Theorem 10.

18. Apply General Scholium, 4.

19. See Prop. XV., Cor. 3, and Prop. XXI., Cor. 1.

20. Apply General Scholium, 5.

21. Find the altitude of the zone from the data, and then apply General Scholium, 5.

22. Apply General Scholium, 7.

23. Find the radius of the base from the data, and then apply General Scholium, 6.

24. Find the altitude of the zone of the sector from the data, and then apply General Scholium, 5.

25. Let x be the required side. Then Prop. XVII. gives the expression for the given volume, whence x can be found.

26. Let ABC be the triangle, MN the axis, C the nearest vertex, BCN perpendicular to MN, so that CN $=$ 2 inches. Let AO be the altitude of the triangle. Then ON $= 3$ inches, and AO $= \sqrt{3}$, and AM, parallel and equal to ON, $= 3$ inches. Then

$$\text{vol. ABC} = \text{vol. ABNM} - \text{vol. ACNM}.$$

Apply General Scholium, 3.

27. See Exercises, Theorem 20.

First. $\pi a^2 h$, but $h = \frac{1}{2}b$. *Secondly.* $\pi b^2 h'$, but $h' = \frac{1}{2}a$. (See Book I., Exercise 36.)

28. Let ABC be the triangle, and DE the line parallel to BC bisecting the sides AB, AC at D and E and the altitude AM at the point N.

We are to compare the volumes described by the revolution of ADE and DECB about BC, or what is more simple, to compare vol. DECB with the vol. ABC. From D and E draw DO and EP perpendicular to BC, then

$$\text{vol. DECB} = \text{vol. DEPO} + \text{vol. DOB} + \text{vol. EPC},$$

which is easily computed and compared with vol. ABC.

29. Let the radius of the sphere $=$ R, then the side of the regular inscribed tetraedron $= \frac{4R}{3}\sqrt{\frac{3}{2}}$, the side of the inscribed cube $= \frac{2R}{\sqrt{3}}$, and the side of the inscribed regular octaedron is $R\sqrt{2}$. Hence the volumes can be easily computed. (See Exercises, Appendix to Book VI., Problems.)