RECOMMENDATIONS.

Cincinnati, August 14, 1844.

I have examined, in manuscript, a work on Algebra, by Prof. Horatio N. Robinson, entitled "An Universal Key to the Science of Algebra," now in course of publication in this city.

The plan of this work I believe to be *entirely original*, and I consider it well adapted to afford important assistance to those who are desirous of acquiring readiness and a certain tact in the use and discovery of methods by which they may unravel with facility the intricacies of Algebraic formulæ.

The examples of these methods presented in the book itself, and the assistance it affords in the discovery of others, will give occasion for intellectual exercise, in the highest degree pleasing and useful; and will doubtless so tend to smooth the path of the student in this science, as to render a study hitherto considered by many as altogether dry and laborious, a most delightful as well as profitable pursuit.

THOMAS J. MATTHEWS.

Cincinnati, August 24, 1844.

I have examined the manuscript of Professor H. N. Robinson's "Universal Key to the Science of Algebra," and believe it to be a work well calculated to aid the student in the acquisition of knowledge. Those who wish to acquire great *facility*, *accuracy*, and *practical skill*, in the management and solution of complex and difficult problems, may derive much advantage from the use of this work. Teachers, especially, may find it a valuable auxiliary in conducting their pupils through this science with that *spirit* and *activity* so necessary to secure success, and make the student a lover of the study.

G. R. HAND.

Having examined the above-named work in manuscript, I freely concur in the opinion of Mr. Hand.

O. WILSON.

RECOMMENDATIONS.

Cincinnati, August 27, 1844.

MR. ROBINSON,

Dear Sir:—Allow me to say that I am very glad you have resolved on the publication of your "Universal Key to Algebra." It cannot fail to be well received by those who desire to be introduced to the spirit and beauty of that science. There are but few works of character on that subject, from the elementary to the most elaborate, which I have not examined, and I do not hesitate to say, that, in none have I seen that elegance and conciseness of solution, and philosophical application of principles so clearly and strongly developed, as in your work. The *modus operandi* of your Algebraic solutions is a most important improvement, and must, when known, supersede the cumbrous and inelegant one now in common use. Giving your volume my cordial approbation,

I remain yours, respectfully,
JOHN W. PICKET, M. D. L. L. D.

Cincinnati, Sept. 28, 1844.

The undersigned, having examined a "Key to the Science of Algebra," by Professor H. N. Robinson, esteems it a work of much merit. Its ingenious and various methods of solving intricate problems, and reducing difficult equations, will render it a valuable acquisition to teachers, and all who wish to apply the machinery of Algebra to numerical calculations.

J. McL, EDWARDS.

A

UNIVERSAL KEY

TO THE SCIENCE OF

ALGEBRA:

IN WHICH SOME NEW MODES OF OPERATION ARE INTRODUCED

CORRESPONDING TO THE

CANCELLING SYSTEM IN NUMBERS;

WITH

ELEGANT AND CONCISE SOLUTIONS,

TO NEARLY ALL THE IMPORTANT AND DIFFICULT PROBLEMS

FOUND IN VARIOUS BOOKS IN USE.

CONTAINING, ALSO,

EXAMPLES IN EVERY SECTION FOR PRACTICE.

BY

HORATIO N. ROBINSON, A. M.

CINCINNATI:

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1844.

PREFACE.

WHEN a pupil is first initiated into Algebra, it is necessary that he should be carried through certain forms, and learn certain definite principles and rules of operation.

For instance, he must learn how to add, subtract, multiply, and divide all kinds of algebraical expressions, and it is not to be denied that some sections of all Algebras must be more tedious than interesting.

But when once an individual has learned the elementary operations, he is then prepared to be emancipated from technical rules, and can rely alone on general principles and good judgment. Indeed the student must pass by the stage of strict reliance on rules in Algebra, before that science can enlarge his understanding or elevate his mind, and the object of this little work is to guide him past that point, aid him in acquiring his independence, and to show him that there are often more ways than one to accomplish the same end, and that no one can be very skilful who is a strict adherent to set rules and forms. Rules may be materially varied, but principles are immutable; and while we speak lightly of rules, it is only in comparison *to magnify* more general and never-changing principles.

The more immediate object of this work, however, is to introduce a brief method of working common Algebraic problems, and contrast it with the common dead routine, too universally practised. This method has a strong analogy to the cancelling system, recently published in Talbot's Arithmetic, and is, in fact, part of the same system, the student being required to keep a steadfast eye on the relation of numbers, to save long and tedious operations.

It is not our purpose to propose any new problems. Should we do so, it might be said that we made them to correspond with our peculiarities of operation, and the same modes would not apply generally.

We shall therefore select such examples as are *curious*, or difficult of solution, and such as are considered important or interesting, and shall take as many such from all the different Algebras in use, as to render this a universal Key to the science, without being an elementary work.

As our books in general use, the works of professed Algebraists, do not solve many problems, and those only to illustrate a *rule*, which *rule* they *must then* literally follow, it must be that their humble imitators will as literally follow the same rules, and the whole soul-enlarging spirit of Algebra sink into literal formularies, and the acquisition of the science become a laborious task, rather than an intellectual pleasure. To avoid this tendency, and to awaken the true genius of analysis, is one object of this work, and better to attain this end, we shall solve almost every problem we introduce.

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CONTENTS.

KEY

TO THE

SCIENCE OF ALGEBRA.

EQUATIONS.—It is understood that the pupil already understands the formal rules of working an equation, and to these we will add the following observation:

An equation is the simplest possible idea. Certain things in weight, worth, or measure, are equal to certain other things.

An equation may be considered as a pair of scales, when exactly counterpoised, and, of course, we may *put on*, *take off*, *multiply* or *divide*, in short, do any thing to one side, if we do the same to the other side, the balance or *equality* is not destroyed.

Let this be the leading *Rule*, and a little practice will soon teach the best manner of reducing each and every case.

OBSERVATION 2.—When two quantities of the *same kind* are on both sides of an equation, we discourage the *formality* of transposition, and rather say *drop* from both sides. Or if the quantities are negative, or unlike, say add to both sides.

EXAMPLE.

$$x+2a+4=3a+6$$

We say *drop* $2a$ and 4 from both sides, and we have $x=a+2$.

Ex. 2. $x-3a+6=a+7$.

Drop 6, and add $3a$ to both sides, and $x=4a+1$.

Ex. 3. $x-9+a=7a-10$.

Here we may say drop -9, or add $+9$ to both sides. *Drop* a, and $x=6a-1$.

PROB. 1. What sum of money is that, whose third part, fourth part, and fifth part added together, amount to 94 dollars? (Day 79.)* Ans. $120.

SOLUTION. Let x=the money.

Then $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}=94$ Multiply by 60, the product of $3\cdot4\cdot5=60$.

And $20x+15x+12x=94\cdot60$.

$47x=94\cdot60$.

Divide the factor 94 by 47, and the quotient is 2.

And $x=2\cdot60=120$.

The peculiarity to which we would call attention is that 94 is not *actually* multiplied by 60, only indicated, whereby some labor is saved both in multiplication and division, the same as in the cancelling system in arithmetic.

PROB. 2. What number is that, of which if $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{2}{7}$ be added together the sum will be 73. (D. 30.) Ans. 84.

SOLUTION. $\frac{x}{3}+\frac{x}{4}+\frac{2x}{7}=73$.

Multiply by 12, and we have

$$4x+3x+\frac{24x}{7}=73\cdot12.$$

By 7, $28x+21x+24x=73\cdot12\cdot7$.

Adding the x, and $73x=73\cdot12\cdot7$.

Dividing by 73 $x=84$.

PROB. 3. The half, third, and fourth of a sum of money, is equal to $130; what is the sum?

Ans. 120.

*(Day 79,) means Day's Algebra, page 79, from whence we extract the Problem; but hereafter we shall not write the name in full, we shall write (D. 79.) R. 130,) C. 50,) &c., meaning Ryan's, Colburn's, Harney's, &c.

$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=130.$

Multiply by 12, and $6x+4x+3x=130.12.$
Or $13x=130\cdot12.$
Or $x=10\cdot12=120.$

PROB. 4. A teacher said that $\frac{1}{2}$, $\frac{1}{6}$, and $\frac{1}{12}$ of the number of his scholars amounted to 90. How many scholars had he

(H. 33.) Ans. 120.

$\frac{x}{2}+\frac{x}{6}+\frac{x}{12}=90$

Multiply by 12 and $6x+2x+x=90\cdot12.$
Or $x=10\cdot12=120.$

PROB. 5. One-seventh and $\frac{1}{6}$ of a certain number added together make 130. What is the number?

(H. 34) Ans. 420.

$\frac{x}{7}+\frac{x}{6}=130.$

Or $6x+7x=130\cdot42.$
$x=10\cdot42=420.$

Again, suppose $42x$ should represent the number—
Then $6x+7x$ or $13x=130.$
And $x=10.$

Therefore the number is 420, as before. Mr. H. has worked this last example in his book *the common way*, and in the operation he has the number 5460.

PROB. 6. $\frac{1}{2}$ $\frac{1}{3}$ and $\frac{1}{7}$ of a number are equal to 82. What is the number? (H. 35.) Ans. 34.

$\frac{x}{2}+\frac{x}{3}+\frac{x}{7}=82$

Multiply by 42 the product of $2\cdot3\cdot7.$
and $21x+14x+6x=82\cdot42.$
Or $41x=82.42.$
$x=2\cdot42=84.$

PROB. 7. A man sold a horse and a chaise for 800 dollars; $\frac{1}{2}$ of the price of the horse was equal to $\frac{1}{3}$ of the price of the chaise. What was the price of each?

(H. 35.) Ans. chaise, 480. Horse $320.

Let x=the price of the chaise,
Then $800-x$=the price of the horse.

Per ques. $400-\frac{x}{2}=\frac{x}{3}$
Or $6 \cdot 400-3x=2x$.
Or $6 \cdot 400=5x$.
Or $6 \cdot 80=x$. Or $x=480$.

N. B. It is of little consequence on which side of an equation the unknown term stands. Many of the changes usually made, can as well be dispensed with.

PROB. 8. Divide 48 into two such parts, that if the less be divided by 4, and the greater by 6, the sum of the quotients will be 9. (D. 77.) Ans. 12 the less.

Let x=the less; then $48-x$=the greater.

$\frac{x}{4}+\frac{48-x}{6}=9.$
Multiply by 4 and 6 or 24.
And $6x+4 \cdot 48-4x=9 \cdot 24$.
Or $2x+8 \cdot 24=9 \cdot 24$.
Therefore $2x=24$.
Or $x=12$.

In the above Equation *observe* that 4 times 48 is the same as $8 \cdot 24$; and 8 times 24 taken from both sides there will remain *one time* 24, or simply 24, on the right hand side.

PROB. 9. What number is that, a sixth part of which exceeds an eighth part of it, by 20? (D. 79.) Ans. 480.

$\frac{x}{6}-\frac{x}{8}=20 \quad 8x-6x=20 \cdot 48.$
$x=10 \cdot 48=480.$

PROB. 10. An estate is divided among four children, in such a manner that

The first has 200 dollars more than $\frac{1}{4}$ of the whole.
The 2d has 340 dollars more than $\frac{1}{5}$ of the whole.
The 3d has 300 dollars more than $\frac{1}{6}$ of the whole.
The 4th has 400 dollars more than $\frac{1}{8}$ of the whole.
What is the value of the estate?

(D. 79.) Ans. $4800

Let x=the estate,

$$\text{Then } 200+\frac{x}{4}=\text{1st}$$

$$340+\frac{x}{5}=\text{2d}$$

$$300+\frac{x}{6}=\text{3d}$$

$$400+\frac{x}{8}=\text{4th.}$$

$$\text{By addit'n } 1240+\frac{x}{4}+\frac{x}{5}+\frac{x}{6}+\frac{x}{8}=x.$$

Multiply both sides by 24 and we shall clear three fractional terms at once; and we shall have

$$24\cdot1240+6x+\frac{24x}{5}+4x+3x=24x.$$

By the *common* rule, and by *common practice*, we should *continue* clearing of fractions. But in many cases this is neither requisite nor expedient, by doing so in *all* cases we show only a blind adherence to rules.

Drop $13x$ from both sides and we have

$$24\cdot1240+\frac{24x}{5}=11x.$$

Multiply by 5,
And $5\cdot24\cdot1240+24x=55x.$
Or, $5\cdot24\cdot1240=31x.$

We can divide 124 by 31, therefore we can divide 1240 by 31, and the quotient is 40.

Therefore $5\cdot24\cdot40=x.$

Or, $4800=x.$

Prob. 11. In the composition of a quantity of gunpowder
The *nitre* was 10 lbs. more than $\frac{2}{3}$ of the whole.
The *sulphur* $4\frac{1}{2}$ lbs. less than $\frac{1}{6}$ of the whole.
The *charcoal* 2 lbs. less than $\frac{1}{7}$ of the *nitre*.
What was the amount of gunpowder?

(D. 81.) Ans. 69 lbs.

Let x=the whole,

Then $10+\frac{2x}{3}$=*nitre.*

$-4\frac{1}{2}+\frac{x}{6}$=*sulphur.*

$-2+\frac{10}{7}+\frac{2x}{21}$=*charcoal.*

Adding $3\frac{1}{2}+\frac{10}{7}+\frac{5x}{6}+\frac{2x}{21}=x;$

Multiplying by 6 gives $21+\frac{60}{7}+5x+\frac{4x}{7}=6x$

Or $21+\frac{60}{7}+\frac{4x}{7}=x.$ By dropping $5x$.

Multiply by 7, and drop $4x$ and we have

$7{\cdot}21+60=3x.$

Dividing by 3 gives $7{\cdot}7+20=x.$ Or, $x=69.$

PROB. 12. Four places are situated in the order of the letters A, B, C, D. The distance from A to D, is 34 miles. The distance from A to B is to the distance from C to D, as 2 to 3. And $\frac{1}{4}$ of the distance from A to B added to half the distance from C to D, is three times the distance from B to C. What are the respective distances?

(D. 82.) Ans. 12, 4 and 18.

Let $2x$=the distance from A to B.

Then the whole will be as represented below.

A $\quad 2x \quad$ B $\quad \frac{x}{6}+\frac{x}{2} \quad$ C $\quad 3x \quad$ D

And $5x+\frac{x}{6}+\frac{x}{2}=34.$ Or, $30x+x+3x=34{\cdot}6.$

Or, $x=6.$

PROB. 13. A merchant supported himself three years, for 50 pounds a year, and at the end of each year, added to that part of his stock which was not thus expended, a sum equal to one third of this part. At the end of the 3d year his original stock was doubled. What was the stock?

(D. 83.) Ans. 740 pounds.

N. B. Adding a third to *any thing* is equivalent to multiplying the same by $\frac{4}{3}$.
Let x=his stock.

Then x—50 multiplied by $\frac{4}{3}$ is the 2d year's stock.

Or $\frac{4x-200}{3}-50$=2d year's stock, less his support.

$\frac{16x-800}{9}-\frac{200}{3}-50$=3d do.

And $\frac{64x-3200}{27}-\frac{800}{9}-\frac{200}{3}=2x$

Multiplying by 27, we have

$$64x-3200-2400-1800=54x.$$

Or, $10x=7400$. $x=740$.

To illustrate the remark made in reducing the tenth problem, we extract the following equation from Day's Algebra, page 74, and we copy his work in reducing the same, figure for figure, word for word.

PROB. 14. Reduce the equation $\frac{3x}{4}+6=\frac{5x}{8}+7$.

Clearing of fractions, $24x+192=20x+224$.
Transposing and uniting terms, $4x=32$.
Dividing by 4, $x=8$.

We will now work the same equation, guided only by the *simple idea of equality*.

Observe that $\frac{3x}{4}$ is the same as $\frac{6x}{8}$. Now drop 6 from both sides, and also $\frac{5x}{8}$ from both sides, and we have $\frac{x}{8}=1$. or $x=8$.

Let us make another comparison, (not with the spirit of vanity,) but to show that the system of *Cancelling* and *abreviating*, can be carried to a great extent, and is far preferable to the usual method.

PROB. 15. It is required to divide the number 204 into two such parts, that $\frac{2}{5}$ of the less being subtracted from the greater, the remainder will be equal to $\frac{3}{7}$ of the greater subtracted from four times the less.

(Colb. sec. 6. Prob. 6.) Ans. **154 and 50.**

Let x=the greater part,
Then $204-x$=the less part.

$\frac{2}{5}$ of the less is $\frac{408-2x}{5}$.

By the conditions, $x-\frac{408-2x}{5}=816-4x-\frac{3x}{7}$.

Multiplying by 5, $5x-408+2x=4080-20x-\frac{15x}{7}$.

" 7, $35x-2856+14x=28560-140x-15x$.

Transposing and uniting terms, $204x=31416$.

$x=154$. $204-x=50$.

The above is a strict copy. Observe the heavy number 31416, all such numbers, we wish to avoid.

As $\frac{2}{5}$ of the less is spoken of, and $\frac{3}{7}$ of the greater, we can commence by taking $5x$ for the less or $7x$ for the greater.

We prefer to let $5x$=the less, then $204-5x$=the greater.

$\frac{2}{5}$ of $5x=2x$. Then $204-5x-2x=204-7x$.

Therefore by the conditions, $204-7x=20x-\frac{3\cdot204-15x}{7}$.

Transpose the $7x$ to the $20x$, and divide the whole by 3,

And $68=9x-\frac{204-5x}{7}$. Or, $7\cdot68=63x-204+5x$.

Or, $7\cdot68=68x-204$.

Divide by 68, and $7=x-3$. Or, $10=x$.

But $5x$= the less number; it is therefore 50 as before. Observe the highest number in this solution, is the given number 204.

Another Solution.

Let $7x$= the greater,
Then $204-7x$= the less.

Per question, $7x-\frac{2\cdot204-14x}{5}=4\cdot204-28x-3x$.

Transposing $31x$, we have $38x-\frac{2\cdot204-14x}{5}=4\cdot204$.

Or, $190x-2\cdot204+14x=20\cdot204$. Transposing $-2\cdot204$, Gives $204x=22\cdot204$. $x=22$. $7x=154$, the greater.

PROB. 16. Divide \$183 between two men, so that $\frac{4}{7}$ of what the first receives, shall be equal to $\frac{3}{10}$ of what the second receives. What will be the share of each?

(Col. sec. 5.) Ans. \$63 and 120.

Let $x=$1st. receives,

$183-x=$2d. receives.

$$\frac{4x}{7}=\frac{3\cdot183-3x}{10}$$

Or, $40x=21\cdot183-21x$. $61x=21\cdot183$.

Divide 183 by 61, and $x=21\cdot3=63$.

PROB. 17. A man has a lease for 99 years; and being asked how much of it was already expired, answered that two thirds of the time past, was equal to four-fifths of the time to come. Required the time past and the time to come.

(Col. sec. 5.) Ans. Time past, 54 years. 45 to come.

N. B. Keep the 99 as a factor.

PROB. 18. A man bought a horse and chaise for \$341. Now if $\frac{3}{8}$ of the price of the horse, be subtracted from twice the price of the chaise, the remainder will be the same as if $\frac{5}{7}$ of the price of the chaise be subtracted from three times the price of the horse. Required the price of each.

(Col. sec. 6.) Ans. Horse, \$152. Chaise, \$189.

The solution of this problem by the common routine of operation, would be long, tedious, and of course uninteresting. But by keeping the 341 as a factor, agreeable to our method, it is by no means difficult or tedious.

The most natural way, (and problaly the most easy *because it is the most natural*,) is to let $x=$the horse, and $341-x$ $=$the chaise; and in this condition we shall leave it for the reader to solve if he pleases.

A more brief and artificial method,is to let $7x=$the chaise. (We take this because we can take $\frac{5}{7}$ of it without a fraction.) Then $341-7x=$ the horse.

Per question, $14x-\frac{3\cdot341-21x}{8}=3\cdot341-21x-5x.$

$$\text{Or, } 40x-\frac{3\cdot341-21x}{8}=3\cdot341.$$

Multiplying by 8, adding the x^s, and transposing the $-3\cdot341$, we have $341x=27\cdot341$.

Or, $x=27$. Or, $7x=189$. Answer.

A similar solution may be obtained by taking $8x$ for the price of the horse.

Section II.

We shall devote this section simply to the reduction of *given* equations, taking such as we find in various books, not originating any to correspond with our peculiar manner of working.

1. Given $x-\frac{3x-3}{5}+4=\frac{20-x}{2}-\frac{6x-8}{7}+\frac{4x-4}{5}$

(D. 74.) (Trotten's Alge. 155.) Ans. 6.

Multiply by 10, and we have

$$10x-6x+6+40=100-5x-\frac{60x-80}{7}+8x-8.$$

Reduce and $x=46-\frac{60x-80}{7}$. Or $x=6$.

2. Reduce $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$. (D. 75.)

Let the reader *observe* that the first term of this equation may be written thus:

$\frac{6x}{9}+\frac{7}{9}$ and the last term thus $\frac{2x}{3}+\frac{4}{3}$.

Also observe that $\frac{4}{3}=\frac{12}{9}$. Hence, the equation may be expressed thus:

$\frac{6x}{9}+\frac{7}{9}+\frac{7x-13}{6x+3}=\frac{2x}{3}+\frac{12}{9}$.

By dropping equals from both sides, we have

$\frac{7x-13}{6x+3}=\frac{5}{9}$. Or $\frac{7x-13}{2x+1}=\frac{5}{3}$.

Clear of fractions and $21x-39=10x+5$.

$11x=44$. Or $x=4$.

3. Given $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}$ to find the value of x.

(R. 114.)

Observe that $\frac{7x}{21}=\frac{x}{3}$ drop these and transpose the minus term.

And $\frac{16}{21}=\frac{x+8}{4x-11}$. Or $64x-16\cdot11=21x+8\cdot21$.

$43x-22\cdot8=21\cdot8$.

Observe that 16 times 11, is the same as 22 times 8, halving one factor, and doubling the other.

Now transposing $22\cdot8$ and $43x=43\cdot8$. Or $x=8$.

4. Given $\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4}$ to find the value of x.

(R. 115.)

Observe that $\frac{9x}{36}=\frac{x}{4}$. Drop these, and reduce $\frac{20}{36}=\frac{5}{9}$.

Then $\frac{5}{9}=\frac{4x-12}{5x-4}$.

Or $25x-20=36x-108$. $88=11x$. $8=x$.

The following may be treated in the same manner:

5. Given $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$ to find x.

(R. 114.) Ans. $x=6$.

6. Given $\frac{20x+36}{25}+\frac{5x+20}{9x-16}=\frac{4x}{5}+\frac{86}{25}$ to find x.

(R. 115.) Ans. $x=4$.

From this last we perceive immediately that

$\frac{5x+20}{9x-16}=2$.

7. Given $\frac{21-3x}{3}-\frac{4x+6}{9}=6-\frac{5x+1}{4}$ to find x.

Ans. $x=3$.

Multiply by 3, and

$$21-3x-\frac{4x+6}{3}=18-\frac{15x+3}{4}.$$

Drop 18, and afterwards multiply by 3, or 4, or 12, according to fancy.

8. Given $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}$ to find x.

(R. 116.) Ans. 4.

9. Given $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}$ to find x.

(R. 116.) Ans. $x=7$.

N. B. This can be reduced *very* briefly.

10. Given $16x+5 : \frac{4x+14}{9x+31} :: 36x+10 : 1$ to find x.

(R. 115.) Ans. $x=5$.

If we multiply extremes and means we shall have a tedious operation.

We can do *any thing* to the antecedents and consequents of a proportion, provided we do the same thing to both. In this case, let us multiply the consequents by $(9x+31.)$ and we have

$$16x+5 : 4x+14 :: 36x+10 : 9x+31$$

Multiply the consequents of this last by 4,

And $16x+5 : 16x+56 :: 36x+10 : 36x+124$.

Take the difference between the antecedents and consequents in each couplet for the consequents of a new proportion, and we have

$$16x+5 : 1 :: 36x+10 : 114.$$

Change the means,

$$16x+5 : 36x+10 :: 1 : 114.$$

Double the antecedents, and

$$32x+10 : 36x+10 :: 102 : 114.$$

Or $32x+10 : 4x :: 102 : 12$.

Divide the consequents by 4 and

$$32x+10 : x :: 102 : 3.$$

Divide the last couplet by 3, and

$$32x+10 : x :: 34 : 1.$$

$$32x+10=34x. \quad \text{Hence } 10=2x. \quad \text{Or } x=5.$$

The above appears lengthy and tedious, to those who would count length by the number of times the same terms were written, but it contains very little mechanical labor, it is reason and observation almost entirely, the thoughts are only registered.

The following, though much easier, shonld be treated in the same manner.

12. Given $\frac{4x+3}{6x-43} : 1 :: 2x+19 : 3x-19$ to find x.

R. 115.) Ans. $x=8$.

The following is a very remarkable problem, taken from Ryan's Algebra, page 107; we shall copy his solution to the letter.

13. Given $\frac{x-5}{4} : x-5 :: \frac{2}{3} : \frac{3}{4}$ to find x.

Multiplying extremes and means, we have

$$\frac{3}{4}\left(\frac{x-5}{4}\right)=\frac{2}{3}(x-5)$$

Or $\frac{3x-15}{16}=\frac{2x-10}{3}$.

By clearing of fractions, $9x-45=32x-160$.

By transposition, $9x-32x=45-160$.

Collecting and changing signs, $23x=115$.

By division, $x=5$.

We take the equation $\frac{3}{4}\cdot\left(\frac{x-5}{4}\right)=\frac{2}{3}(x-5)$

Divide by $(x-5)$ and $\frac{3}{16}=\frac{2}{3}$ *an absurdity.*

But the equation is nevertheless *true*. Therefore to make it true we must take $x-5=0$. Or $x=5$, the answer.

Section III.

Equations of two unknown quantities.

1. Given $\left\{\begin{array}{l}\frac{x}{3}+\frac{y}{2}=8\\ \frac{x}{2}+\frac{y}{3}=7\end{array}\right\}$ to find x and y. (R. 149.)

Multiply both by 6, and we have

$$2x+3y=8\cdot6.$$
$$3x+2y=7\cdot6.$$

Add. $5x+5y=15\cdot6.$

Or, $x+y=3\cdot6.$ $2x=2y+6\cdot6.$

Subtract this last equation from the first, and we have

$$y=2\cdot6=12.$$

2. Given, $\frac{x}{6}+\frac{y}{4}=6.$ and $\frac{x}{4}+\frac{y}{6}=5\frac{2}{3}$ } to find the values of x and y. (R. 148.)

Sum $\frac{x+y}{6}+\frac{x+y}{4}=11\frac{2}{3}.$

As an *introduction* to substitution, (which we intend to use quite freely hereafter,) let us put $x+y=s$. This will be a general expression in all cases in which it may be required, in the succeeding part of this work.

Then, $\frac{s}{6}+\frac{s}{4}=11\frac{2}{3}.$ Multiply by 12, then

$2s+3s=4\cdot35.$ Or, $s=4\cdot7.$ That is, $x+y=28.$ (*a*)

But by the first equation	$2x+3y=72.$
Double equation (*a*)	$2x+2y=56.$
By subtraction	$y=16.$

This particular equation could have been reduced quicker by the common mode, but we wished to call attention to the fact that it could be reduced in this way.

3. Given. $\left\{\begin{array}{l}\frac{x}{8}+8y=194\\ \frac{y}{8}+8x=131\end{array}\right\}$ to find x and y.

(R. 148.) Ans. $x=16.$
$y=24.$

We copy the author's solution.

Multiplying the first equation by 8,

$$x+64y=1552. \qquad x=1552-64y.$$

Substituting this value for x, in the second equation, it becomes,

$$\frac{y}{8}+8(1552-64)y=131;$$

By reduction, $y+99328-4096y=1048$.

By transposition, $4095y=98280$.

By division, $y=24$.

We prefer the following solution. Add the two equations together, and we have

$$\frac{x+y}{8}+8(x+y)=325. \qquad \text{Let } x+y=s.$$

And afterwards multiply by 8, and $s+64s=325\cdot8$.

Divide by 65, and $s=5\cdot8$. That is $x+y=5\cdot8$.

But from the first equation, $x+64y=194\cdot8$.

Subtract $x+y=5\cdot8$.

The remainder is $63y=189\cdot8$.

And $y=3\cdot8=24$.

4. Given $\left\{\begin{array}{l}\frac{x}{7}+7y=99 \\ \frac{y}{6}+7x=51\end{array}\right\}$ to find x and y.

Ans. $x=7$.
(R. 114.) $y=14$.

5. Given $\left.\begin{array}{l}\frac{2x-y}{2}+14=18 \\ \text{And } \frac{2y+x}{3}+16=19\end{array}\right\}$ to find x and y.

(R. 145.) Ans. $x=5$. $y=2$.

Drop 14 from the first, and 16 from the last equation ***and then*** clear of fractions, and we have

$$2x-y=8. \qquad x+2y=9.$$

5. Given $x-\frac{2y-x}{23-x}=20-\frac{59-2x}{2}$ mn
And $y+\frac{y-3}{x-18}=30-\frac{73-3y}{3}$ } to find x and y. (R. 156.)

In clearing equations of fractions, such as the above, which contain both *simple* and *compound* denominators, it is generally preferable to multiply by the *simple quantities first*, and then *reduce*, as low as possible before we multiply by the *compound* quantities.

To commence with a large compound denominator, is too much like driving a wedge butt end foremost. However, no general direction will be best in *every possible case*.

Of the above, multiply the 1st by 2, and the 2d by 3, and reducing we have

$\frac{4y-2x}{23-x}=19$. And $\frac{3y-9}{x-18}=17$.

Which give $x=21$ and $y=20$.

6. Given $x+150 : y-50 = 3 : 2$.
And $x-50 : y+100 = 5 : 9$.

To find the values of x and y. (R. 158.)

Observe that 100 and 150 are multiplies of 50, of this circumstance advantage can be taken.

Multiplying and transposing terms we have

$2x-3y=-9\cdot50$. And $9x-5y=19\cdot50$.

Multiply the first by 5, and the 2d by 3, and

$10x-15y=-45\cdot50$. And $27x-15y=57\cdot50$.

Difference $17x=102\cdot50$.

Divide by 17 and $x=6\cdot50=300$.

7. Given $3x+6y+1=\frac{6x^2-24y^2+130}{2x-4y+3}$.

And $3x-\frac{151-16x}{4y-1}=\frac{9xy-110}{3y-4}$. to find the values of x and y.

(R. 158.)

Divide the numerator by the denominator of the fraction in the first equation, and we may obtain for a quotient $3x+6y+1$, and a remainder, or the equation becomes

$$3x+6y+1=3x+6y+1+\frac{-11x-14y+127}{2x-4y+3}$$

By dropping equals from both sides, the fraction must equal nothing; but when a fraction equals nothing, its numerator must equal nothing, and therefore

$$11x+14y=127.$$

Multiply the 2d Equation by $(3y-4)$ and

$$9xy-12x-\frac{(151-16x)(3y-4)}{4y-1}=9xy-110.$$

Drop $9xy$, and transpose the minus terms, as plus signs are more convenient, and we shall have

$$110=12x+\frac{(151-16x)\,(3y-4)}{4y-1}.$$

Clear of fractions. The multiplication must now be actually accomplished, and the reduction will be obvious. Giving $x=9$, and $y=2$.

8. Given, $16x+6y-1=\frac{128x^2-18y^2+217}{8x-3y+2}$,

And $\frac{10x+10y-35}{2x+2y+3}=5-\frac{54}{3x+2y-1}$. to find x and y.

The first equation can be reduced in the same manner as the first equation in the preceding example. In the 2d equation, multiply by $2x+2y+3$ *first*, and *not* by $3x+2y-1$. Making all possible reductions before multiplying by $3x+2y-1$.

Should we commence clearing this 2d equation of fractions, by multiplying by $3x+2y-1$, it would admit of little or no reductions, until it became *entirely* clear of fractions, and the operation would be tedious in the extreme. We make these remarks to show the importance of calling our reason and judgment into exercise, before commencing labor.

Section IV.

Equations of three or more unknown quantities.

It is well known that there must be as many *independent* equations as there are unknown terms to be found.

As a general rule, it is most expeditiuos to *exterminate first* those unknown terms, which appear in the least number of equations.

Eaxmple.

1. A Grocer had four kinds of wine marked A, B, C, and D. He mixed together 7 gallons of A, 5 gallons of B, and 8 gallons of C; and sold the mixture at $1,21 per gallon. He also mixed together 3 gallons of A, 10 of C, and 5 of D, and sold the mixture at $1,50 per gallon. At another time he mixed 8 gallons of A, 10 of B, 10 of C, and 7 of D, and sold the whole for $48,00. At another time he mixed 18 gallons of A, and 15 of D, and sold the mixture for $48,00. What was the value of each kind of wine? (Col. sec. 24.)

Per question,
$$7A+5B+8C=\$24,20.$$
$$3A+10C+5D=\$27,00.$$
$$8A+10B+10C+7D=\$48,00.$$
$$18A+15D=\$48,00.$$

A, B, C, and D representing the number of gallons, as well as the name of each kind of wine.

In the above equations, we observe that B appears only in the 1st and 3d. Our *first object* then, is to exterminate B in preference to any other quantity. The *exception* to this rule would be when the coefficients of some other letter in the several equations were alike.

Multiply the 1st equation by 2, and from the product subtract the third. Double the 2d equation, and divide the third by 3, and we shall have the three following equations:

$$6A+6C-7D=,40.$$
$$6A+20C+10D=54,00.$$
$$6A+0+5D=16,00.$$

Subtract the first from the 2d, and the 3d from the 2d, and we have the two following;

	$14C+17D=53{\cdot}60.$	
	$20C+5D=38,00.$	(a)
Difference	$-6C+12D=15,60.$	
Divide by 6	$-C+2D=2,60.$	(b)
Divide (a) by 5	$4C+D=7,60.$	
By addition	$3C+3D=10,20.$	
Divide by 3, and	$C+D=3,40.$	
Add (b)	$-C+2D=2,60.$	
	$3D=6.$	
	$D=2.$	
And then	$C=1,40.$	$A=1,00.$

2. There are three persons, A, B, and C, whose ages are as follows; if B's age be subtracted from A's, the difference will be C's age; if 5 times B's age and twice C's age, be added together, and from their sum A's age be subtracted, the remainder will be 147. The sum of all their ages is 96. What are their ages? (Col. sec. 24.)

$$A-B=C.$$
$$5B+2A-2B-A=147.$$
$$A+B+C=96.$$

Transpose C in the 1st equation, and add it to the 3d, and divide by 2, gives $A=48$.

Condense the 2d, giving A its value, divide by three, and we shall have $B=33$. Hence $C=15$.

3. Given $x+y+z=12$, And $x+2y-2z=10$, And $x+y-z=4$ } to find x, y and z. (Day 160.)

$$\left.\begin{array}{ll}\text{Given} & x+y+z=12\\ \text{And} & x+2y-2z=10\\ \text{And} & x+y-z=4\end{array}\right\}\text{ to find } x, y \text{ and } z.$$

Subtract the 3d from the first, and $2z=8$, or $z=4$.
Subtract the 3d from the 2d, and $y-z=6$. Or $y=10$.

4. Given $\left\{\begin{array}{l}xy=600\\ xz=300\\ yz=200\end{array}\right\}$ to find x, y and z. (Day 163.)

Multiply the 1st and 2d, and divide by the 3d,

And $x^2=\frac{600\cdot300}{200}=9\cdot100$. Or, $x=30$.

Another Method.

Divide the 1st by the 2d, and $\frac{y}{z}=2$. Or $y=2z$.

Put this value of y in the 3d equation, and $2z^2=200$, or $z^2=100$ or $z=10$.

5. A man, his wife, and a son's years made 96, of which the father and son's were equal the wife's and 15 years over, and the wife and son's were equal to the man's and two years over. What was the age of each? (R. 167.)

Let x, y and v=their respective ages.

Then 1st $x+y+v=96$.
And 2d $x+v=y+15$.
And 3d $y+v=x+2$.

Add the 2d and third together, observing that $x+y$ on both sides of the sum will cancel,

And $2v=17$. $v=8\frac{1}{2}$.

Add 1st and 2d, and $2x+17=111$.

6. Given $x+y-z=8$ $x+z-y=9$, and $y+z-x=10$, to find the values of x, y, and z.
(R. 163.) Ans. $x=8\frac{1}{2}$ $y=9$ $z=9\frac{1}{2}$.

7. Given $4x-4y-4z=24$
$6y-2x-2z=24$
$7z-y-x=24$ required x, y, and z.
(R. 162.)

Suppose $x+y+z=s$
Then $4x+4y+4z=4s$.
To this last add $4x-4y-4z=24$ the first given equa.
And $8x=4s+24$

Double the supposed equation and add the 2d to it, and
$8y=2s+24$ (b)
And $8z=s+24$ (c)

Add the last three and $8s=7s+3\cdot24$.
Or $s=3\cdot24$.

Put the value of s in equation (a) and we have

$$8x=13\cdot 24$$

Or $x=39$

By equation (b) $8y=7\cdot 24$. Or $y=21$. $z=12$.

$$8.\ \text{Given}\left\{\begin{array}{r} 2x+y-2z=40 \\ 4y-x+3z=35 \\ 3u+t=13 \\ y+u+t=15 \\ 3x-y+3t-u=49 \end{array}\right\}\quad \text{Ans.}\left\{\begin{array}{l} x=20 \\ y=10 \\ z=\ 5 \\ u=\ 4 \\ t=\ 1 \end{array}\right.$$

In solving this problem we will make an exception to our general rule, z exists only in two equations, yet we prefer to exterminate t before z, as the coefficients of t, are alike in two equations, and those two are the *smallest* equations.

SECTION V.

Involution and Powers.

$a.a.a.a$ means a, taken four times, as a factor, and it may be expressed more briefly thus, a^4, the 4, or the number of times any term is taken as a factor, is called an exponent.

a^2 multiplied by a^3 must be a, taken twice and three times as a factor, *in all five times;* or $a.^5$ That is, to multiply powers of the *same* letter add the exponents.

Multiply x^3 by x^4 Ans. x^7.

Multiply x by x^3 Ans. x^4.

When no exponent is expressed, one is always understood.

Multiply b^n by $b.^m$ Ans. b^{n+m}.

Multiply b by b^n. Ans. b^{1+n}.

If adding the exponents is a correct operation for the multiplication of powers, taking their difference, must be a correct operation for division.

Divide a^7 by a^2. Ans. a^5.

Reverse, Multiply a^5 by a^2. Ans. a^7.

Divide x^8 by x^5. Ans. x^3.

Divide x^m by x^n. Ans. x^{m-n}.
Multiply x^{m-n} by x^n. Ans. x^m.

These operations will be true, whatever be the values of m and n, *whether positive or negative, whole numbers or fractions.*

What is square of x^n?

It must be x^n, multiplied by x^n. But we multiply exponential quantities by adding the Exponents; that is $x^{n+n}=x^{2n}$.

In the same way we find x^n cubed is x^{3n}, and x^n, to the fourth power, is x^{4n}.

Therefore, to raise a quantity to any power multiply the exponents by the index of that *power.*

Inversely then, to extract the roots of any exponential quantities. Divide the exponents by the index of the root.

What is the square root of a^4. Ans. a^2.

Multiply a^2 by a^2, and we have a^4, hence we perceive that the principle must be correct.

What is the cube root of x^{3n}. Ans. x^n.

Cube x^n. It must be $x^n \cdot x^n \cdot x^n = x^{3n}$.

Hence x^n is the cube root of x^{3n}, and is obtained by dividing the exponent $3n$ by 3.

It is often the inquiry of young Algebraists, how we know? How it is arrived at? What is the propriety of expressing the square root, cube root, fourth root, &c., by the exponential fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c.

We answer this enquiry in the following manner:

What is the square root of a^2. Ans. a.

Or a^1. a and a^1 is the same thing, as much as a and $1a$ are the same.

Now how did we obtain the root of a^2, what *formal operation?* we divided the exponent by 2.

What then is the square root of a?

Its exponent is 1 understood, divide this by 2, and we have for the square root of a, $a^{\frac{1}{2}}$.

Reasoning in the same way, the cube root of a is $a^{\frac{1}{3}}$, the fourth root is $a^{\frac{1}{4}}$, the nth root is $a^{\frac{1}{n}}$, &c.

When exponential quantities have *expressed* coefficients we multiply the coefficients, and add the exponents thus multiply $2x^3$ by $3x^{\frac{1}{2}}$, and we have $6x^{\frac{7}{2}}$.

When the terms, the multiplier and multiplicand are not of the same kind or denomination, we multiply their coefficients, and write the literal parts as *factors*.

Thus, multiply $3a^3$ by $2x^6$ by $5y^2 = 30a^3x^6y^2$.

• Division being the reverse of multiplication, in all cases, requires no comment here. This not being intended as an elementary work, we make as few remarks as possible on the subject of this section; yet we have explained about all that is necessary. We have laid down the cardinal principles, and if the pupil takes *strong hold* of these, there will be no difficulty. Indeed we regard the trouble and perplexity (or a great part of it,) which most students experience among roots, powers, and exponential quantities, as occasioned by the many *artificial divisions* and cases, and the smoke and fog thrown round it, by *a redundancy of explanation.*

We wish to show in few words, that $a^0 = 1$, and $a^{-2} = \frac{1}{a^2}$ and $a^{-n} = \frac{1}{a^n}$, as we have found these expressions to be very troublesome and perplexing to students.

Take a^3 and divide it successively by a, and we have the following series:

$$a^3,\ a^2,\ a,\ 1,\ \frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3},\ \text{\&c.}$$

Divide again, and form another series, *rigidly adhering* to the principle that to divide any power of a, by a, and the *exponent becomes one less.*

Thus a^3, a^2, a^1, a^0, a^{-1}, a^{-2}, a^{-3}, &c.

Now a^3 in one series equals a^3 in the other.

That is, $a^3=a^3$, $a^2=a^2$, $a=a^1$, $a^0=1$, $a^{-1}=\frac{1}{a}$

$a^{-2}=\frac{1}{a^2}$, $a^{-3}=\frac{1}{a^2}$. In short $a^{-n}=\frac{1}{a^n}$.

Therefore the denominators of literal fractions may be thrown into the numerators by changing the exponental signs, thus

$\frac{a}{x^2}=ax^{-2}$, and $\frac{b^{-3}}{x}=\frac{x^{-1}}{b^{-3}}$ &c. &c.

Multiply $\frac{b^4}{a^{-2}}$ into $\frac{h^{-3}}{x}$ and $\frac{a^n}{y^{-3}}$. Ans. $\frac{a^{2+n}b^4y^3}{h^3x}$

Multiply $\frac{a^3-x^1}{a^3}$ by $\frac{a}{x^2-a^{-2}}$. Ans. $\frac{a^4-ax^4}{a^2x^2-1}$.

Divide $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$. Ans $a^{\frac{1}{2}-\frac{1}{3}}=a^{\frac{1}{6}}$.

Multiply $(a+b)^2$ by $(a+b)^{\frac{1}{2}}$. Ans. $(a+b)^{\frac{5}{2}}$.

Section VI.

As this work is not designed to teach that minutia which is requisite to draw out all the algebraic principles and modes of expression, but rather to call those principles into exercise, and handle them with skill and dexterity; we shall therefore take it for granted, that the student is already acquainted with the fact, that

$(a+b)^2=a^2+2ab+b^2$

And $(a-b)^2=a^2-2ab+b^2$.

That is, *the square of any binomial is equal to the square of the two parts, and twice the product of the parts.*

Also, that $(a+b)(a-b)=a^2-b^2$.

Or $(a^4+b^4)(a^4-b^4)=a^8-b^8$.

That is, the rectangle of the sum and difference of two quantities is equal to the difference of their squares.

By this last observation we readily perceive that a^4-b^4, or *any other binomial* having a *minus sign* between the terms can be resolved into factors, by taking half of the exponents, for new exponents, and the + and — sign between the terms, thus $a^4-b^4=(a^2+b^2)(a^2-b^2)$

But $a^2-b^2=(a+b)(a-b.)$

Therefore, $a^4-b^4=(a^2+b^2)\times(a+b)(a-b.)$

Also, $x-a=\left(x^{\frac{1}{2}}+a^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}}-a^{\frac{1}{2}}\right)$

And $x^n-a^{\frac{n}{2}}=\left(x^{\frac{n}{2}}+a^{\frac{n}{2}}\right)\left(x^{\frac{n}{2}}-a^{\frac{n}{2}}\right)$

Observe also, that

$$\frac{a^2-b^2}{a-b}=a+b.$$

And $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2.$

And $\dfrac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3.$

And $\dfrac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4.$

Hence in general terms, we may conclude that

$$\frac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}b+a^{n-3}b^2+a^{n-4}b^3,\ \&c. \quad (1)$$

The division running out when n is any whole number, *or when the exponents in the numerator are exact multiples of those in the denominator.*

From the above we learn,

1st. That the difference of any two equal powers of different quantities, is always divisible by the *difference of their roots.*

2d. The difference of any two equal powers of different quantities, is divisible by the *sum* of their roots, when the exponent of the power is an *even number.*

For $\dfrac{a^2-b^2}{a+b}=a-b.$

And $\dfrac{a^4-b^4}{a+b}=a^3-a^2b+ab^2-b^3.$

Hence we may conclude that in general

$$\frac{a^{2n}-b^{2n}}{a+b}=a^{2n-1}-a^{2n-2}b+a^{2n-3}b^2, \&c. : \ . \ . \ b^{2n-1}. \quad (2)$$

3d. The sum of any two equal powers of different quantities, is divisible by the *sum* of their roots, when the exponents of the power is an *odd number*.

For $\frac{a^3+b^3}{a+b}=a^2-ab+b^2$.

And $\frac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4$.

In general, $\frac{a^{2n+1}+b^{2n+1}}{a+b}=a^{2n}-a^{2n-1}b+ \ . \ . \ . \ b^{2n}. \quad (3)$

In the general formulæ (1) (2) (3) let it be observed that n may be any whole number whatever, odd or even.

From the preceding, we observe that

$$(a^3+a^2b+ab^2+b^3)(a-b)=a^4-b^4.$$

And $(a^4-a^3b+a^2b^2-b^3+b^4)(a-b)=a^5+b^5$, &c. &c.

That is, any quantity, the exponents of whose terms being regular, and in each term equaling the same constant number, can be brought into two terms, by multiplying by the difference or sum of the roots, according as the signs of the terms in the quantity, are *plus*, or alternately, plus and *minus*.

The mind and eye of the student cannot be too familiar with the *facts*, *form*, and symmetry of the several formulæ in this section; for on them depend very much our aptitude and quickness, to resolve into factors; to find multipliers which will change the form and simplify an expression, &c. &c.; all of which enables us to solve equations with skill and dexterity. Hence the study and full understanding of them we insist upon.

Section VII.

Solution of pure equations.

Pure equations are such as may have different powers or roots, but which, nevertheless, can be reduced without completing squares, (in the common algebraic acceptation of that term,) or without resort to the higher order of equations.

Observe well the relation of quantities, compare their exponents and coefficients, and be guided by your judgment.

To get clear of roots raise them to their respective powers. To reduce powers, extract their corresponding roots.

EXAMPLE.

1. Given $\sqrt{(1+x\sqrt{x^2+12})}=1+x$ to find x.

Square both sides, drop 1 from each side, and divide by x.

Then $\sqrt{x^2+12}=2+x$.

Square again and $x^2+12=4+4x+x^2$.

Therefore $x=2$, the answer.

2. Given $\dfrac{3x-1}{\sqrt{3x+1}}=1+\dfrac{\sqrt{3x-1}}{2}$ to find x. (R.229.)

Without *care* this would be a troublesome equation, and by ordinary management would come out a quadratic. But observe the numerator $3x-1$. *The sign minus is between the terms*, and we know that a^2-b^2 may be separated into factors. Now a^2 may equal $3x$ and $b^2=1$.

In short $3x-1=(\sqrt{3x}+1)(\sqrt{3x}-1)$

Or, $$\frac{3x-1}{\sqrt{3x+1}}=\sqrt{3x-1}$$

The equation then becomes $\sqrt{3x-1}=1+\dfrac{\sqrt{3x-1}}{2}$.

Let $\sqrt{3x-1}=y$. And we have $y=1+\dfrac{y}{2}$, or $y=2$.

Then $\sqrt{3x}=3$. Or $3x=9$. And $x=3$. Ans.

3. Given $\dfrac{5x-9}{\sqrt{5x}+3}-1=\dfrac{\sqrt{5x}-3}{2}$ to find x.

(R. 229.) Ans. $x=5$.

4. Given $\dfrac{ax-b^2}{\sqrt{ax}+b}=c+\dfrac{\sqrt{ax}-b}{c}$, to find x.

R (R. 230.) Ans. $x=\dfrac{1}{a}\left(b+\dfrac{c^2}{c-1}\right)^2$

The one immediately following is very simple, yet we have found many good students very much troubled with it, and for that reason we extract it.

5. $x^2+3x-7=x+2+\dfrac{18}{x}$, to find the value of x.

Ans. $x=3$ or -3.

6. Given $x^3+y^3=(x+y)\,xy$. And $x^2y+xy^2=4xy$. } to find x and y.

(R. 235.) Ans. $x=2$. $y=2$.

Agreeable to *Formala 3d, section* 6, we find that the first equation may be divided by $(x+y)$ and we shall have

$$x^2-xy+y^2=xy.$$
Hence, $x^2-2xy+y^2=0$
Or, $x-y=0$.

Divide the 2d equation by xy, and $x+y=4$. But $x-y=0$. Hence, $x=2$ and $y=2$.

7. Given $x^2y+xy^2=180$. And $x^3+y^3=189$. } to find the values of x and y.

Compare 180 with 189 in the mind. $180=20\cdot9$, and $189=21\cdot9$.

Multiply the first equation by 3, and add it to the 2d, and we have

$$x^3+3x^2y+3xy^2+y^3=81\cdot9=9^3$$
Cube root, $x+y=9$.
Divide the 1st by this, and $xy=20$.
Hence, $x=4$ or 5, $y=5$ or 4.

No person can be very skilful in Algebraic operations, who feels averse to substitution; for judicious substitution, stands in the same relation to common Algebraic notation, as Algebra stands to Arithmetic. We shall introduce it at first in a few cases where it may not be *very* necessary, more to familiarise the student with it, than any thing else.

8. Given $x+\sqrt{xy}+y=19.$ And $x^2+xy+y^2=133.$ } to find the values of x and y.

Let $x+y=s$, and $\sqrt{xy}=p$.

Hence, $x^2+2xy+y^2=s^2.$ $xy=p^2.$

Subtract this last from the preceding, and

$$x^2+xy+y^2=s^2-p^2.$$

Then the original, or given equations become

$$s+p=19, \text{ and } s^2-p^2=133,$$

Or, $(s+p)(s-p)=133.$

Hence, $s-p=7.$

Therefore, $s=13$, and $p=6.$

Or, $x+y=13$, and $xy=36.$

Hence, $x=9$ or 4, and $y=4$ or 9.

9. Given $x^2+xy=12.$ And $y^2+xy=24.$ } to find the values of x and y.

Let $x+y=s.$

Then $sx=12$, and $sy=36$.

By addition, $s(x+y)=s^2=36.$ Or, $s=\pm 6.$

Then $sx=6x=12$, or $x=\pm 2.$ m

10. Given $x^4+3x^2y^2+y^4=1296-4xy(x^2+xy+y^2,)$ and $x-y=4$, to find the values x and y.

(R. 235.)

We recognize the left hand side of the first equation, as a perfect binomial square, the root of which is x^2+y^2.

Let $x^2+y^2=s$, and $xy=p$.

Then $s^2=1296-4p(s+p)$, multiply and transpose, and $s^2+4ps+4p^2=1296$. Extracting the square root, gives $s+2p=36$.

But $s+2p=x^2+2xy+y^2=36$, or $x+y=\pm 6$.

But $x-y=4$. Therefore, $x=5$ or -1, $y=1$ or -5.

11 Given $x^8+x^4y^4+y^8=273,$ and $x^4+x^2y^2+y^4=21.$ } to find x and y.

Ans. $x=\pm 2\sqrt{1}.$

$y=\mp 1$, or $\pm\sqrt{-1}$.

The pupil is requested to work over the seventh problem by putting $x+y=s$, and $xy=p$.

The reader should *observe and* satisfy himself, that this notation must give

$$x^2+y^2=s^2-2p.$$
$$x^3+y^3=s^3-3sp.$$
$$x^4+y^4=s^4-4s^2p+2p^2.$$
$$x^5+y^5=s^5-5s^3p+5sp^2.$$

Fractional exponents are often very troublesome to the young algebraist, but such exponents can always be banished from *pure equations by substitution*, as the exponent in all *such* equations must be regular or multiples of each other, otherwise they would be complex equations.

EAXMPLE.

12. Given $x^{\frac{2}{3}}+y^{\frac{1}{3}}=6$, and $x^{\frac{4}{3}}+y^{\frac{2}{3}},=20$, to find the values of x and y.

Take that function of x which has the least exponent and put it equal to any single letter say P. Take also the function of any other term which contains its least *exponent*, and put it equal to any other simple letter, say Q. *And let this be a general rule.*

In the present example, Let $x^{\frac{2}{3}}=P$, $y^{\frac{1}{3}}=Q$. Then, by squaring, $x^{\frac{4}{3}}=P^2$, $y^{\frac{2}{3}}=Q^2$.

Now the given equations become

$$P+Q=6, \qquad (1)$$
$$\text{and } P^2+Q^2=20. \qquad (2)$$

By squaring equation (1), $P^2+2PQ+Q^2=36$.
Subtracting equation (2) we have $2PQ=16$.
Subtract this last from equation (2), and

$$P^2-2PQ+Q^2=4.$$

Therefore $\quad P-Q=2.$
But $\quad P+Q=6.$

$$2P=8 \text{ or } 4.$$
$$P=4 \text{ or } 2$$

That is $x^{\frac{2}{3}}=4$ or 2.

Square root, $x^{\frac{1}{3}}=2$ or $\sqrt{2}$.

Cubing, gives $x=8$ or $(2)^{\frac{3}{2}}$.

By a like process we find $y=32$ or 1024.

13. Given $xy^2+y=21$ and $x^2y^4+y^2=333$, to find the values of x and y.

By comparing exponents in the two equations, we perceive that if we put $P=xy^2$, and $Q=y$, we shall have

$$P+Q=21. \quad \text{And } P^2+Q^2=333.$$

Solved as the preceding gives $P=18$, $Q=3$.

That is $xy^2=18$, and $y=3$.

14. $\left.\begin{array}{l}\text{Given } x^{\frac{3}{2}}+x^{\frac{3}{4}}y^{\frac{3}{4}}+y^{\frac{3}{2}}=1009 \\ \text{And } x^3+x^{\frac{3}{2}}y^{\frac{3}{2}}+y^3=582193\end{array}\right\}$ to find x and y.

Let $x^{\frac{3}{4}}=P$. $y^{\frac{3}{4}}=Q$.

Then $x^{\frac{3}{2}}=P^2$. $y^{\frac{3}{2}}=Q^2$.

And $x^3=P^4$. $y^3=Q^4$.

Also let $a=1009$. And $b=582193$.

Now, using P and Q, the given equations become

$\left.\begin{array}{l}P^2+PQ+Q^2=a \\ P^4+P^2Q^2+Q^4=b\end{array}\right\}$ Equations having no fractional exponents.

Observe that these equations are exactly similar to those in Example 8.

Again, Assume $P^2+Q^2=s$. And $PQ=t$.

We then have $s+t=a=1009$.

And $s^2-t^2=b$.

By division $s-t=\frac{b}{a}=577$.

Therefore $s=793$, and $2t=432$.

We can now *go back* and find P and Q, afterwards x and y.

15. Given $x^{\frac{1}{4}}-y^{\frac{1}{4}}=3$, and $x^{\frac{1}{4}}+y^{\frac{1}{4}}=7$, to find x and y.

Ans. $x=625$. $y=16$.

16. Given $\sqrt{x}+\sqrt{y} : \sqrt{x}-\sqrt{y} :: 4 : 1$ and $x-y=16$, to find the values of x and y.

Ans. $x=25$. $y=9$.

17. Given $x^2+x^{\frac{4}{3}}y^{\frac{2}{3}}=208$ And $y^2+x^{\frac{2}{3}}y^{\frac{4}{3}}=1053$ } to find the value of x and y.

Assume $x^{\frac{2}{3}}=P$, and $y^{\frac{2}{3}}=Q$.

Square and cube these assumed equations, and we have

$$x^{\frac{4}{3}}=P^2. \quad y^{\frac{4}{3}}=Q^2.$$
$$x^2=P^3. \quad y^2=Q^3.$$

With this substitution, the given equations become

$$P^3+P^2Q=208=13\cdot16. \qquad (1)$$
$$\text{And } Q^3+Q^2P=1053=13\cdot81. \qquad (2)$$

Separate the left hand members into factors,

$$\text{Thus } \quad P^2\ (P+Q)=13\cdot16. \qquad (3)$$
$$\text{And } \quad Q^2\ (Q+P)=13\cdot81. \qquad (4)$$

Divide equation (4) by (3) and we have

$\frac{Q^2}{P^2}=\frac{81}{16}$ Extracting the square root, $\frac{Q}{P}=\frac{9}{4}$.

Or $Q=\frac{9P}{4}$. Take this value of Q in Equation (1) and we have

$$P^3+\frac{9P^3}{4}=13\cdot16.$$
$$\text{Or } \quad 4P^3+9P^3=13\cdot16\cdot4.$$
$$\text{That is } \quad 13P^3=13\cdot16\cdot4.$$
$$\text{Or } \quad P^3=64. \quad P=4.$$

That is $x^{\frac{2}{3}}=4$. $x^{\frac{1}{3}}=\pm2$. $x=\pm8$.

As $Q=\frac{9P}{4}$. $Q=9$, and in short $y=\pm27$.

The above can be solved very well without substitution, but we think the operation would be less refined, and more tedious.

18. Given $\frac{\sqrt{(4x+1)}+\sqrt{4x}}{\sqrt{(4x+1)}-\sqrt{4x}}=9$, to find the value of x.

Ans. $x=\frac{4}{9}$.

Multiply both numerator and denominator of this fraction by $\sqrt{(4x+1)}+\sqrt{4x}$.

And $\left(\frac{\sqrt{(4x+1)}+\sqrt{4x}}{1}\right)^2=9$.

Or $\sqrt{(4x+1)}+\sqrt{4x}=3$.

19. Given $\frac{\sqrt{a+x}+\sqrt{a-x)}}{\sqrt{a+x}-\sqrt{a-x}}=b$. To find the value of x.

Ans. $x=\frac{2ab}{b^2+1}$.

N. B. The preceding problems in this section, are from Bland's Algebraical Problems.

20. Given $\frac{\sqrt{x}+\sqrt{(x-a)}}{\sqrt{x}-\sqrt{(x-a)}}=\frac{n^2 a}{x-a}$ to find the value of x.

(Davies' Bourdon.) Ans. $x=\frac{a(1+n)^2}{1+2n}$.

21. Given $\frac{a+x+\sqrt{(2ax+x^2)}}{a+x}=b$, to find the value of x.

(Davies' Bourdon.)

It is difficult to reduce this equation without resort to substitution.

Assume $a+x=y$. Then $2ax+x^2=y^2-a^2$.

And $\sqrt{2ax+x^2}=\sqrt{y^2-a^2}$.

The given equation is now $\frac{y+\sqrt{y^2-a^2}}{y}=b$

Or $\sqrt{y^2-a^2}=by-y$. Hence $y^2-a^2=b^2y^2-2by^2+y^2$.

$y^2=\frac{a^2}{2b-b^2}$ Or $y=\frac{\pm a}{\sqrt{2b-b^2}}$.

That is $a+x=\pm\frac{\pm a}{\sqrt{2b-b^4}}$ Or $x=\pm\frac{a}{\sqrt{2b-b^2}}-a$.

$x=\pm a\left[\frac{1+\sqrt{2b-b}}{\sqrt{2b-b^2}}\right]$

Without substitution, we must put the equation under this form

$$1+\frac{\sqrt{2ax+x^2}}{a+x}=b.$$

Transpose one and square, and

$$\frac{2ax+x^2}{a^2+2ax+x^2}=b^2-2b+1.$$

Divide the numerator by the denominator, and we have

$$1-\frac{a^2}{a^2+2ax+x^2}=b^2-2b+1.$$

Drop one from both sides, and change signs, and extract the square root, we then have

$$\pm\frac{a}{a+x}=\sqrt{2b-b^2}.$$

Or $a+x=\pm\frac{a}{\sqrt{2b-b^2}}$

The same expression as in the preceding solution, and in fact, this latter solution was indicated by the former.

21. There are two numbers, whose sum is to the greater, as 40 to the less, and whose sum is to the less as 90 to the greater. What are the numbers. Ans. 36 and 24.

Let x=greater, y=less, $x+y$=sum.

$$x+y : x = 40 : y.$$
$$x+y : y = 90 : x.$$

Multiply the two proportions, term by term, *and observe* that the consequents of the new proportion are equal, therefore $(x+y)^2=36\cdot100$. Or $x+y=60$.

22. There are two numbers whose product is 144, and the quotient of the greater by the less is 16. What are the numbers? (Col. Sec. 30.) Ans. 48, and 3.

23. Find two numbers, such that the 2d power of the greater multiplied by the less, gives 448, and the 2d power of the less mnltiplied by the greater, gives 392. (Colburn.)

Let x=greater y= the less.

Then $x^2y=448$. $xy^2=392$

Seperate the known numbers into factors, and multiply the equations together, and we have

$x^3y^3=(8{\cdot}8{\cdot}7)(8{\cdot}7{\cdot}7)=8^3\cdot 7^3$.

Or $xy=8{\cdot}7$. Divide the primitive equations by this last equation, and $x=8$, and $y=7$.

24. A man purchased a field, whose length was to its breadth as 8 to 5. The number of dollars paid per acre was equal to the number of rods in the length of the field; and the number of dollars given for the whole, was equal to 13 times the number of rods round the field. Required, the length and breadth of the field. (Col. Sec. 30.)

Let $8x$=the length, and $5x$=the breadth,

Then $\frac{40x^2}{160}=\frac{x^2}{4}$=No. of Acres.

$\frac{x^2}{4}\times 8x=2x^3$=the whole number of dollars.

Again, $8x+5x=13x$=half round the field.

$13x{\cdot}2{\cdot}13$=thirteen times round, which is equal to the dollars paid.

Therefore, $2x^3=13x{\cdot}2{\cdot}13$.

Divide by $2x$, and $x^2=13{\cdot}13$.

Square root gives $x=13$, and $8x=104$, the length.

N. B. Here, let it be observed that our multiplication, is only indicated not *actualy* done. It is not necessary, yea it would be positively injurious if it were done, and to impress this truth, is, one object of this work.

25. There is a stack of hay, whose length is to its breadth as 5 to 4, and whose highth is to its breadth as 7 to 8. It is worth as many cents per cubic foot as it is feet in breadth; and the whole is worth at that rate 224 times as many cents as there are square feet in the bottom. Required the dimensions of the stack. (Col. sec 30.)

Let $5x$= the length, $4x$= breadth, and $\frac{7x}{2}$= highth.

Then $\left(5x{\cdot}4x{\cdot}\frac{7x}{2}\right)4x$= cost in cents.

Again, $5x \cdot 4x=$ square feet on the bottom.
Hence $224 \cdot 5x \cdot 4x=$ cost in cents.

Therefore $5x \cdot 4x \cdot \frac{7x}{2} \cdot 4x = 224 \cdot 5x \cdot 4x$.

Strike off equal factors, $7x \cdot 2x = 224$,

Or, $x^2=16$. Hence $x=4$.

26. There is a number, to which if you add 7 and extract the second root of the sum, and to which, if you add 16, and extract the second root of sum, the sum of the two roots will be 9. What is the number? (Har. page 199.)

The following is a copied solution, which we believe to be the method that most persons would pursue.

Let $x=$ the number.

Then $(x+7)^{\frac{1}{2}}+(x+16)^{\frac{1}{2}}=9$.

—Take the second power of both members of this equation, and we have

$$x+7+2(x+7)^{\frac{1}{2}}(x+16)^{\frac{1}{2}}+x+16=81.$$

$$2(x+7)^{\frac{1}{2}}(x+16)=58-2x.$$

$$(x+7)^{\frac{1}{2}}(x+16)^{\frac{1}{2}}=29-x.$$

Taking the second power of both members again, and we have

$$(x+7)(x+16)=841-58x+x^2$$
$$x^2+23x+112=841-58x+x^2$$
$$81x+112=841.$$
$$81x+729. \quad x=9.$$

Now let $x^2-7=$ the number. Then to it add 7 and extract the square root, we have x.

By question $x+\sqrt{x^2+9}=9$.

$$\sqrt{x^2+9}=9-x.$$

Square $x^2+9=9 \cdot 9-18x+x^2$.

Drop x^2, and divide by 9, and we have

$$1=9-2x, \text{ or } x=4.$$

But $x^2-7=16-7=9$, the answer.

27. A certain sum of money was put out at interest, and in 8 months amounted to \$297,60, and in 15 months it amounted to \$306 at simple interest. What was the sum and what was the rate per cent?

Ans. \$288; rate 5 per cent.

Let $x=$ the sum, r the rate.

$$\text{Then } x+\frac{2rx}{300}=297{,}60. \qquad (1)$$

$$\text{And } x+\frac{5rx}{400}=306{,}00. \qquad (2)$$

Subtract, and $\frac{5rx}{400}-\frac{2rx}{300}=8{,}40.$

Multiply by 100, and $\frac{5rx}{4}-\frac{2rx}{3}=840.$

Clear of fractions, $15rx-8rx=840\cdot 12.$

Or, $7rx=840\cdot 12.$

$rx=120\cdot 12.$

Hence $5rx=600\cdot 12$, and $\frac{5rx}{400}=18.$

Put this value of $\frac{5rx}{400}$ in equation (2), and $x+18=306$, or $x=288$.

Had these equations been cleared of fractions in the first instance, they would have became lengthy and rather complex.

28. Some hours after a courier had been sent from A to B, which are 147 miles distant, a second was sent who wished to overtake him just as he entered B, in order to which he found he must perform the journey in 28 hours less than the first did. Now the time in which the first travels 17 miles, added to the time in which the other travels 56 miles, is $16\frac{2}{3}$ hours. How many miles does each go per hour?

Ans. The first goes 3, the 2d 7 miles per hour.

Let $x=$ rate of the first, and $y=$ rate of the 2d.

Then $\frac{147}{x}=$ the time on the road,

Per question $\frac{147}{x}-\frac{147}{y}=28$
And $\frac{17}{x}+\frac{56}{y}=\frac{41}{3}$
} It would be very inexpedient to clear these of fractions as many would do.

Divide the first equation by 7, and $\frac{21}{x}+\frac{21}{y}=4$.

Divide this last by 21, $\frac{1}{x}-\frac{1}{y}=\frac{4}{21}$.

Multiply this by 17, and $\frac{17}{x}-\frac{17}{y}=\frac{68}{21}$.

Subtract this from the 2d equation, and we have

$$\frac{73}{y}=\frac{219}{21}. \quad \frac{1}{y}=\frac{3}{21}. \quad y=7.$$

29. Given $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b$, to find the value of x.

Ans. $x=\pm\left\{\left(a^2-\frac{b^3-2a}{3b}\right)^3\right\}^{\frac{1}{2}}$.

30. Given $x+y=a$ and $x^3+y^3=c$ } to find the values of x and y.

This problem is commonly worked by Quadratics, but it is more easy by simple equations. Divide one by the other, &c.

Ans. $\begin{cases} x=\frac{a}{2}+\frac{1}{2}\left(\frac{4c-a^3}{3a}\right)^{\frac{1}{2}} \\ y=\frac{a}{2}-\frac{1}{2}\left(\frac{4c-a^3}{3a}\right)^{\frac{1}{2}} \end{cases}$

31. A person put out a certain sum of money at interest at a certain rate. Another person put out $10,000 *more*, at a rate one per cent higher, and received an income of 800 dollars more.

A third person put out 15,000 dollars more than the first, at a rate 2 per cent higher, and received an income greater by 1,500 dollars.

Required the several sums, and their respective rates of interest.

The solution of this question will involve very high numbers, unless the operator is careful.

To avoid which, Let $a=5000$,

then $2a=10000$,

$3a=15000$,

$\frac{3a}{10}=1500$,

$\frac{16a}{100}=800$.

Put x=A's capital, and $r-1$=A's rate.

$x+2a$=B's " " r=B's "

$x+3a$=C's " " $r+1$=C's "

$$\frac{rx-x}{100}+\frac{16a}{100}=\frac{rx+2ar}{100}. \quad (1)$$

$$\frac{rx-x}{100}+\frac{3a}{10}=\frac{rx+3ar+x+3a}{100}. \quad (2)$$

Reducing (1), gives $x=(16-2r)\,a$. (3)

Reducing (2), gives $x=\frac{(27-3r)\,a}{2}$. (4)

From equations (3) and (4) the value of r can be readily found.

32. A widow possessed 13,000 dollars, which she divided into two parts, and placed them at interest, in such a manner, that the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part 360 dollars interest, and if she had placed the second out at the same rate as the first, she would have drawn for it 490 dollars interest. What were the two rates of interest?

(Bourdon, page 185.) Ans. 7 and 6 per cent.

Let x and y represent the two parcels,

and r and t= the rates,

Then $x+y=13\cdot1000$, (1)

and $rx=ty$ (2)

By the conditions $tx=360\cdot100$, (3)

and $ry=490\cdot100$, (4)

Divide (4) by (3), and $\left(\frac{y}{x}\right)\left(\frac{r}{t}\right)=\frac{49}{36}$. (5)

By equation (2) $\frac{r}{t}=\frac{y}{x}$, substitute this in equation (5) and

$\frac{y^2}{x^2}=\frac{49}{36}$, or $\frac{y}{x}=\frac{7}{6}$. Hence, $y=\frac{7x}{6}$.

Therefore $x+\frac{7x}{6}=13\cdot1000$, or $13x=13\cdot1000\cdot6$.

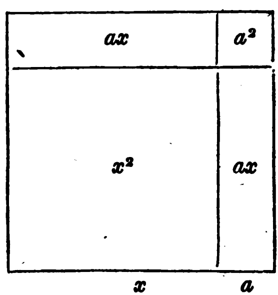
$x=6000$, Answer.

Section VIII.

On Affected Quadratic Equations.

Let the side of a square consist of two parts x and a.

Draw a line to represent the length of x and a, as in the figure below, and on it draw a square, which square consists of four parts, two squares and two equal rectangles



Now if we have an algebraic expression in the form of x^2+2ax, we know that it may be considered *as a part of*, but not a *complete square;* it requires an additional square which *does not* contain x.

Rule 1st. To find this square, take half the coefficient of x in the lowest power, that is one-half of $2a$, or a, *square it* and add it to the above, and we shall have $x^2+2ax+a^2$.

If x^2 has a coefficient, the above rule will not answer, the coefficient *must first* be taken away by division or multiplication.

EXAMPLE.

$$ax^2+bx=c.$$

Divide by a and $\quad x^2+\frac{bx}{a}=\frac{c}{a}.$

Now we have the square of x, and we may consider $\frac{bx}{a}$ as two rectangles put on two of its sides.

To fill up the square, we must add $\frac{b}{2a}$ squared, that is $\frac{b^2}{4a^2}$ and $x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}$, a square.

A square multiplied by a square, produces a square. If therefore we multiply the last equation by $4a^2$, we shall clear it of fractions and still retain its *square form.*

We then have $4a^2x^2+4abx+b^2=4ac+b^2$.

Compare this with the given equation

$$ax^2+bx=c,$$

and we preceive that we can make a square of any general equation of the 2d degree, without removing the coefficient of x^2 and without *making fractions* by the following rule.

RULE 2D. *Multiply by* 4 *times the coefficient of* x^2, *and add the square of the coefficient of* x *to both sides.*

This rule is often very convenient to avoid fractions; *but there are some intricate cases in practice which one may meet with, where neither of the preceding rules appear convenient or practicable.* To master these with skill and dexterity, we must return to a more general and comprehensive knowledge of *binomial squares.*

$x^2+2ax+a^2$ is a complete binomial square.

Let us look at it: We preceive

1st. That it consists of *three terms.*

2d. Two of these terms, the *first* and *last* are *squares.*

3d. The middle term is twice the product of the square roots of the first and last.

Now let us suppose that a^2 is lost, and we have only x^2+2ax. We know these terms cannot make a square, as there are but *two terms*. We know, also, that the last term *being the last*, must be *a square.*

Let it be represented by t^2.

Now by our supposition, $x^2+2ax+t^2$ is a complete binomial square.

But if this be so, *twice the product of the square roots of the first and last is the middle term.* That is $2xt=2ax$. Divide by $2x$, $t=a$. And $t^2=a^2$.

Thus the lost square can be found.

Again, $4a^2+4ab$ are the first and second terms of a binomial square. The third term is lost. Find it. It *must be* a square. Suppose it t^2. Then $4a^2+4ab+t^2$ is a perfect square. *Being so*, $4at=4ab$, or $t=b$. And $t^2=b^2$.

That is t^2 was *equal* to the lost square, b^2 is the identical lost square, and the perfect square is $4a^2+4ab+b^2$. Its root, $2a+b$.

For another example, $36y^2+36y+\cdot\cdot$ are the first and second terms of a square, find the third. Let t^2 be equal to it, and $36y^2+36y+t^2$ is a complete *square.* Therefore $2\cdot6yt=36y$. $t=3$. Hence $t^2=9$.

Hence $36^2+36y+9$ is the square.

Again, $\cdot\cdot+\frac{6}{x}+9$ are the 2d and 3d terms of a binomial square, the first is erased, restore it. Let it equal t^2, then $t^2+\frac{6}{x}+9$ is a square. Being so, $2\cdot3t=\frac{6}{x}$, $t=\frac{1}{x}$, $t^2=\frac{1}{x^2}$

The perfect square then is $\frac{1}{x^2}+\frac{6}{x}+9$. Its root is $\frac{1}{x}+3$.

$9d^2-6dh+\cdot\cdot$ are the 1st and 2d terms of a square, what is the third? *Ans.* h^2.

$9d^2 \pm \cdot\cdot + h^2$ are the first and last terms of a square, what is the middle term? Ans. $\pm 6dh$.

We must prefix the double sign, as the squares are postive in all cases.

We wish to know what to add to or subtract from the following algebraic expression to render it a complete binomial square.

$$\frac{49x^2}{4}+\frac{48}{x^2}-49 .$$

The $\frac{49x^2}{4}$ is a square, we will therefore call that the *first term*, the 49 being *minus* can not be the third term, we will therefore make it the 2d term, and take t^2 for the third term.

Then $\frac{49x^2}{4}-49+t^2$ is a square, and $2\cdot\frac{7x\cdot t}{2}=-49$

$$\text{Or } t=-\frac{7}{x}, \text{ and } t^2=\frac{49}{x^2}.$$

But we have already $\frac{48}{x}$ therefore *add* $\frac{1}{x^2}$ to the given expression, and we shall have a complete square.

In Bland's Problems we find the following equation.

Given $\frac{49x^2}{4}+\frac{48}{x^2}-49=9+\frac{6}{x}$, to find the values of x.

How would one apply the common rules for completing squares to this?

By the preceding explanations, however, we immediately find that $\frac{1}{x^2}$ added to both sides makes squares of both.

$$\text{That is } \frac{49x^2}{4}-49+\frac{49}{x^2}=9+\frac{6}{x}+\frac{1}{x^2}.$$

By extracting the root $\frac{7x}{2}-\frac{7}{x}=3+\frac{1}{x}$.

Or it may be $\frac{7}{x}-\frac{7x}{2}=3+\frac{1}{x}$.

Because $\left(\frac{7x}{2}-\frac{7}{x}\right)$ and $\left(\frac{7}{x}-\frac{7x}{2}\right)$ when squared will give the original square.

From each of these equations there will arise two values of x, and x in the original, must have four values.

$$x=2, \text{ or} -\frac{8}{7}, \text{ or} \frac{-3+\sqrt{93}}{7}, \text{ or} \frac{-3-\sqrt{93}}{7}$$

We shall give more of these difficult cases anon. Every affected quadratic equation, must exist under one of the four following forms:

$$\text{Given}\begin{cases}(1)\ x^2+2ax=b.\\(2)\ x^2-2ax=b.\\(3)\ x^2-2ax=-b.\\(4)\ x^2+2ax=-b.\end{cases} \quad \text{Solution,}\begin{cases}x=-a\pm\sqrt{b+a^2}.\\x=\ \ a\pm\sqrt{b+a^2}.\\x=\ \ a\pm\sqrt{a^2-b}.\\x=-a\pm\sqrt{a^2-b}.\end{cases}$$

In the equations (1) and (2), if we take the plus sign before $\sqrt{b+a^2}$, x will be positive, if the minus sign x will be negative, as $\sqrt{b+a^2}$, is always more than a.

In equations (3) and (4), if b is greater than a^2, the formulæ will require the square root of a negative quantity, which is impossible.

The value of x in such cases is said to be *imaginary*.

After we reduce an equation to one of the preceding forms, the solution is then only a substitution, or giving the particular values to (a) and (b) in the general solution.

We may, *if we please*, avoid reducing an equation to one of the above forms, by completing the square at once. Suppose we have

$$ax^2+bx=c.$$

We may call ax^2 a square, though it be not a rational square.

By this view of the case, we can add t^2 on to both sides, and pronounce the square complete.

That is, $ax^2+bx+t^2=c+t^2$.

As ax^2+bx+t^2, is a complete square, it follows that

$$2\sqrt{a}\cdot xt=bx.$$

$$\text{Or } t=\frac{b}{2\sqrt{a}}.$$

$$\text{Or } t^2=\frac{b^2}{4a}.$$

Substitute this value of t^2 in the above equation, and we have

$$ax^2+bx+\frac{b^2}{4a}=c+\frac{b^2}{4a}.$$

Extract the root, and

$$\sqrt{a}\cdot x+\frac{b}{2\sqrt{a}}=\pm\left(c+\frac{b^2}{4a}\right)^{\frac{1}{2}}$$

$$\text{Or } x=-\frac{b}{2a}\pm\frac{1}{\sqrt{a}}\left(c+\frac{b^2}{4a}\right)^{\frac{1}{2}}$$

But this formula is not convenient, hence it is little known and never used.

N. B. *We may resolve quadratic equations, without any ceremony of completing a square, by making proper substitutions.*

EXAMPLE.

Given $x^2+4x=21$, to find x.

Let $x=y-2$

Then $x^2=y^2-4y+4$

And $4x=\quad 4y-8$

Therefore $x^2+4x=y^2-4=21.$

$y^2=25.$

$y=\pm5.$ But $x=y-2=3$ or -7.

Given $x^2-6x=-9$, to find x.

Assume $x=y+3$

Then $x^2=y^2+6y+9.$

And $-6x=\quad -6y-18.$

Add $x^2-6x=y^2\quad -9=-9.$

$y^2=0\quad y=0$, and $x=3$.

Given $x^2+2ax=b$
Assume $x=y-a$
Then $x^2=y^2-2ay+a^2$
And $2ax=+2ay-2a^2$

$$x^2+2ax=y^2-a^2=b.$$
$$y=\pm\sqrt{b+a^2}.$$

Therefore $x=-a\pm\sqrt{b+a^2}$, the same result as is shown in the general equations.

As a general rule for substituting in *these cases*, assume the value of the unknown term equal to another unknown term, annexed to half the coefficient of the inferior power, with a contrary sign. *

Equations of the third and fourth degree *can sometimes* be reduced to the quadratic form, but this *will depend on the relation of the coefficients.* To determine in any particular case whether this can be done, we must transpose all the members on one side, and if the highest power of the unknown term *is not even*, we must multiply every term of the equation by the unknown letter *to make it even*, and then extract the square root—and if we find a remainder to be any multiple or *aliquot of the root*, the reduction is effected, otherwise, it is impossible and cannot be done.

For example, reduce the following equation to the quadratic form, if it be possible.

$$x^4-8ax^3+8a^2x^2+32a^3x=d.\quad (x^2-4ax.$$
$$x^4$$

$$2x^2-4ax)\quad -8ax^3+8a^2x^2$$
$$-8ax^3+16a^2x^2$$

$$-8a^2x^2+32a^3x=-8a^2(x^2-4ax.)$$

Hence we perceive, that the equation can be expressed thus, $(x^2-4ax)^2-8a^2(x^2-4ax)=d.$

* We are indebted to Harney's Algebra, for this view of quadratics, a singular work, of much merit, published by Messrs. Morton & Griswold, Louisville, Kentucky.

Now Let $(x^2-4ax)=y$. Then $y^2-8a^2y=d$, obviously a quadratic equation, from which y can be found, afterwards x, from the equation $x^2-4ax=y$.

Reduce $x^3+2ax^2+5a^2x+4a^3=0$, to a quadratic.

As the highest power is *not even*, we must multiply by x. And $x^4+2ax^3+5a^2x^2+4a^3x=0$.

By extracting two terms of the square root and observing the *remainder*, the part that *will not come into the root*, we find that

$(x^2+ax)^2+4a^2(x^2+ax)=0$, a quadratic form.

Divide by $(x^2+ax.)$ And $x^2+ax+4a^2=0$.

Or $x^2+ax=-4a^2$.

Given $x^4+2x^3-7x^2-8x+12=0$, to find the value of x.

$$x^4 \qquad\qquad (x^2+x.)$$

$$2x^2+x \qquad 2x^3-7x^2$$

$$2x^3+x^2$$

$$-8x^2-8x+12=-8(x^2+x)+12.$$

Hence our equation becomes

$$(x^2+x)^2-8(x^2+x)+12=0.$$

Putting $y=x^2+x$, we have $y^2-8y=-12$, a quadratic.

$$y=4\pm2=6, \text{ or } 2.$$

Then $x^2+x=6$, and $x^2+x=2$.

$$x=1, \text{ or } 2, \text{ or } -2, \text{ or } -3.$$

Given $x^3-8x^2+19x-12=0$, to find the values of x.

Ans. $x=1$ or 3, or 4.

Section IX.

Miscellaneous Equations and Problems for Practice.

(MOSTLY FROM BLAND'S PROBLEMS.)

1. Given $14+4x-\frac{x+7}{x-7}=3x+\frac{9+4x}{3}$ to find the values of x.

Observe that $\frac{9+4x}{3}=3+x+\frac{x}{3}$.

Now, if we drop equals from both sides, and transpose the *minus* fraction, we have

$11=\frac{x+7}{x-7}+\frac{x}{3}$.

Reducing gives $x=28$, or 9.

2. Given $\frac{x+12}{x}+\frac{x}{x+12}=\frac{78}{15}$ to find the value of x.

Clear of fractions *in form*, observing that the fraction $\frac{78}{15}=\frac{26}{5}$. We then have $(x+12)^2+x^2=\frac{26(x^2+12x)}{5}$

Dividing by 2, gives $x^2+12x+72=\frac{13}{5}(x^2+12x.)$

Multiplying by 5, $5\,(x^2+12x)+72{\cdot}5=13(x^2+12x.)$

$$72{\cdot}5=8(x^2+12x.$$

Dividing by 8, and changing the sides,

$$x^2+12x=45. \quad \text{Hence } x=3, \text{ or } -15.$$

When quadratic equations are reduced to $x^2+ax=b$, and we find that (a) and (b) are *both* whole numbers, x is generally some *integral factor* of (b), and we may often *decide it* without going through a formal solution. In the above example x must be some factor of 45, and as 3 and 15 are the only *integral* factors, there is *strong presumptive evidence* that $x=3$, or 15, or -3, or -15.

Take the equation $x^2+12x=45$, and suppose $x=3$. Then divide the equation by this last and we have $x+12=15$. Drop 12 and $x=3$, a proof that the supposition is true.

3. Given $x+5=\sqrt{x+5}+6$, to find the values of x.

Assume $\sqrt{x+5}=y$

Then $x+5=y^2$, and $y^2-y=6$.

$y=3$, or -2. $x=4$, or -1.

Again, we may transpose 6, and square, and we have

$x^2-2x+1=x+5$, &c.

4. Given $\frac{8}{x^3}+2=\frac{17}{x^{\frac{3}{2}}}$ to find the values of x.

Ans. $x=4$, or $-\frac{1}{2}\sqrt[3]{2}$.

Assume $x^{\frac{3}{2}}=y$, then $x^3=y^2$, and the given equation becomes $\frac{8}{y^2}+2=\frac{17}{y}$. Or $8+2y^2=17y$, &c.

5. Given $x^2+11+\sqrt{x^2+11}=42$, to find the values of x.
Ans. $x=\pm 5$. Or $\pm\sqrt{38}$.

6. Given $2x^2+3x-5\sqrt{2x^2+3x+9}+3=0$, to find the values of x. Before a person can readily solve such equations as this, he must acquire a habit of quick *comparison*. *Survey the equation before you operate.*

Add 6 to both sides, and assume $\sqrt{2x^2+3x+9}=y$.
Then $y^2-5y=6$. This equation gives $y=6$, or -1.
Hence $2x^2+3x+9=36$. Or 1.

$$x=3, \text{ or } \frac{-9}{2}, \text{ or } \frac{-3\pm\sqrt{-55}}{4}.$$

7. Given $9x+4+\sqrt{36x^3+16x^2}=15x^2$. to find the values of x.

Observe that $\sqrt{36x^3+16x^2}=\sqrt{4x^2(9x+4)}=2x\sqrt{9x+4}$.
Let $\sqrt{9x+4}=y$. Then $y^2+2xy=15x^2$.
Add x^2 and $y^2+2xy+x^2=16x^2$. Or $y+x=\pm 4x$.
$y=3x$. Or $-5x$.
That is $\sqrt{9x+4}=3x$. Or $9x^2-9x=4$.
Or $25x^2-9x=4$..

$$x=\frac{4}{3}. \text{ Or} -\frac{1}{3}. \text{ Or } \frac{9\pm\sqrt{481}}{50}.$$

8. Given $x^4-2x^3+x=132$, to find the values of x.

$$\left\{ \begin{array}{c} (x^2-x)^2-(x^2-x)=132 \\ y^2-y=132. \end{array} \right\}$$

Ans. $x=4$. Or -3.

Or $\frac{1\pm\sqrt{-43}}{2}$.

9. Given $4x^4+\frac{x}{2}=4x^3+33$, to find the values of x.

Ans. 2, or $-\frac{3}{2}$, or $\frac{1\pm\sqrt{-43}}{4}$.

10. Given $\frac{x+\sqrt{x}}{x-\sqrt{x}}=\frac{x^2-x}{4}$, to find the values of x.

Ans. $x=4$. Or 1.

Observe that $x^2-x=(x+\sqrt{x})(x-\sqrt{x})$, on the same principle that $a^2-b^2=(a+b)(a-b.)$ Section 6.

11. Given $27x^2-\frac{841}{3x^2}+\frac{17}{3}=\frac{232}{3x}-\frac{1}{3x^2}+5$, to find the values of x.

Remember the *peculiar* manner of completing squares, explained in section 8.

Ans. $x=2$, or $-\frac{14}{9}$, or $\frac{-2\pm\sqrt{-266}}{9}$.

The following is a very interesting equation, and we shall give two solutions, the *common* one, and one by *substitution.*

12. Given $\left(x-\frac{1}{x}\right)^{\frac{1}{2}}+\left(1-\frac{1}{x}\right)^{\frac{1}{2}}=x$, to find the values of x.

Squaring, we have

$$x-\frac{1}{x}+1-\frac{1}{x}+2\left(x-\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{1}{x}\right)^{\frac{1}{2}}=x^2.$$

Transposing,

$$2\left(x-\frac{1}{x}\right)^{\frac{1}{2}}\left(1-\frac{1}{x}\right)^{\frac{1}{2}}=x^2-x-1+\frac{2}{x}.$$

Squaring again and reducing—we obtain this result,

$$x^4-2x^3-x^2+2x+1=0.$$

Taking the square root, we have $x^2-x-1=0$.

Hence, $x=\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$. is Harney's solution.

By substitution,

$$\left.\begin{array}{l}\text{Let}\left(x-\frac{1}{x}\right)^{\frac{1}{2}}=P.\\ \left(1-\frac{1}{x}\right)^{\frac{1}{2}}=Q.\end{array}\right\}\text{Then, } P+Q=x. \qquad (1)$$

Multiply this last equation by $P-Q$, and we have

$$P^2-Q^2=(P-Q)\,x \qquad (2)$$

$$\left.\begin{array}{l}\text{But } P^2=x-\frac{1}{x}\\ Q^2=1-\frac{1}{x}\end{array}\right\}\text{therefore } P^2-Q^2=x-1.$$

Now equation (2) becomes $x-1=(P-Q)x$.

Dividing by x, gives $1-\frac{1}{x}=P-Q. \quad (3)$

Add equation (1) to (3) and we have $1-\frac{1}{x}+x=2P$.

But $x-\frac{1}{x}=P^2$. Hence $1+P^2=2P$.

Transposing the 2P, and $1-2P+P^2=0$.

Square root, $P=1 \quad P^2=1$.

Therefore $x-\frac{1}{x}=1$.

And $x^2-x=1$.

The same as before. Let the reader observe, that in this last operation everything is done that is spoken of, whereas, in the first operation, the most tedious part was omitted, and the result only given. We call attention to this to *recommend the practice of substitution.*

13. Given $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$ to find the values of x.

This is usually done by squaring twice, reducing, then taking the square root, which gives $x^4-a^2x^2-a^4=0$.

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Then $x=\pm a\left(\frac{1\pm\sqrt{5}}{2}\right)^{\frac{1}{2}}$ Assume $P=\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}$

$Q=\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}$ Then $P+Q=\frac{x^2}{a}$. (1)

And $P^2-Q^2=x^2-a^2$. (2)

Divide equation (2) by (1) and we have

$$P-Q=a-\frac{a^3}{x^2}. \quad (3)$$

Add equations (1) and (3), and $2P=\frac{x^2}{a}+a-\frac{a^3}{x^2}$

Multiply by a, and $2aP=x^2-\frac{a^4}{x^2}+a^2$.

But $\left(x^2-\frac{a^4}{x^2}\right)=P^2$. Therefore, $2aP=P^2+a^2$.

Transpose the first member, and $0=P^2-2aP+a^2$.

Take the square root, and transpose, $P=a$, or $P^2=a^2$.

$$\text{But } P^2=x^2-\frac{a^4}{x^2}=a^2$$

Hence, $x^4-a^2x^2-a^4=0$ as before.

14. Given $\frac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}}=(x-2)^2$, to find the values of x.

Ans. $x=5$ or 3.

15. Given $x^4+\frac{17x^3}{2}-34x=16$, to find the values of x.

Ans. $\begin{cases} x=2 \text{ or } -2 \\ x=-8 \text{ or } -\frac{1}{2} \end{cases}$

16. Given $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$, to find the values of x

Ans. $x=4$ or -1.

17. Given $\frac{x^4+2x^3+8}{x^2+x-6}=x^2+x+8$, to find the values of x

Ans. $x=4$ or $-4\frac{1}{3}$.

18. Given $x^{\frac{4}{3}}+7x^{\frac{2}{3}}=44$, to find the values of x.

Ans. $x=\pm8$ or $\pm\sqrt{-11}$.

19. Given $x^4\left(1+\frac{1}{3x}\right)^2-(3x^2+x)=70$, to find the values of x. (Young, page 112.) Ans. 3 or $-3\frac{1}{3}$.

20. Given $\frac{4x}{7}-\frac{30}{7x^8}+\frac{12+\frac{1}{2}x}{3x}=\frac{7}{2x^2}+\frac{x}{2}+\frac{7}{6}$, to find the values of x. (Young, page 111.) Ans. $x=5$ or -12.

21. Given $\frac{18}{x^2}+\frac{81-x^2}{9x}=\frac{x^2-65}{72}$, to find the values of x.

(Young, page 111.)

Multiply by 2, and we have

$$\frac{36}{x^2}+\frac{2(81-x^2)}{9x}=\frac{x^2}{36}-\frac{65}{36}$$

Transpose all the first member, and

$$\frac{36}{x^2}-\frac{x^2}{36}+\frac{65}{36}+\frac{2(81-x^2)}{9x}=0.$$

By adding the first three fractions, we have

$$\frac{(36^2-x^4+65x^2)}{36x^2}+\frac{2(81-x^2)}{9x}=0.$$

Here are two *plus* terms equal to nothing, therefore, each one must equal nothing. For *each being plus*, one cannot be diminished by another, and must of necessity, be nothing in the first place.

Hence, $\frac{2(81-x^2)}{9x}=0$. Or $81-x^2=0$, or $x=9$.

Also by the other fraction

$$x^4-65x^2-36\cdot36=0.$$

Divide this by $x^2-81=0$, (because $x=9$, as above,) and we have $x^2-16=0$, or $x=4$. Hence $x=\pm9$ or ±4.

Section X.

On particular Modes of resolving Quadratic Equations.

The original design of this work was not to theorize at all, but proceed on the supposition, that the student would be in possession of other books of an elementary kind, from which he could learn every principle of operation; but on reflection, we find this supposition may not be true in all cases. We therefore occasionally return to theory.

The following method of resolving a quadratic equation was suggested to the author, by T. J. Matthews, Esq., of Cincinnati, Ohio.

Suppose we have $ax^2+bx=c$, we can multiply by $4a$, &c. as already explained; instead of that, however, assume $x=\frac{u}{a}$. Then $ax^2=\frac{u^2}{a}$, and $bx=\frac{bu}{a}$, and the general equation becomes $u^2+bu=ac$, an equation, which compared with the original equation, we perceive the coefficient of the highest power *is taken away*, without changing the coefficient of the other power, and without making fractions.

In all equations where *b is even*, this transformation will be *very* advantageous.

EXAMPLES.

1. Given $3x^6+42x^3=3321$, to find the values of x.

(R. page 253.)

$$\text{Assume } x^3=\frac{u}{3}$$

Then $u^2+42u=9963$, and $u^2+42u+441=10404$.
Hence $u+21=\pm 102$. That is $u=81$ or -123

But $x^3=\frac{u}{3}=\frac{81}{3}=27$, or $x^3=\frac{-123}{3}=-41$.

Therefore, $x=3$ or $\sqrt[3]{-41}$.

It will be perceived that the preceding required much less labor than the usual way of multiplying by four times the coefficient, &c.

2. Given $21x^2-292x=-500$, to find x.

Assume $x=\frac{u}{21}$. Then $u^2-292u=-10500$.

Therefore $u-146=\pm 104$, or $u=250$, or $+42$.

But $x=\frac{u}{21}$, that is $\frac{250}{21}$ or $\frac{42}{21}=2$, the answer.

This equation will appear again in problem 10 of the next section.

3. Given $8x^4-8x^2=4$, to find the values of x.

Assume $x^2=\frac{u}{8}$. Then $u^2-8u=-32$,

Or $u^2-8u+16=48$. Therefore $u-4=\pm 4\sqrt{3}$.

But $x^2=\frac{u}{8}=\frac{4(1\pm\sqrt{3})}{8}=\frac{1\pm\sqrt{3}}{2}$, or $x=\left(\frac{1\pm\sqrt{3}}{2}\right)^{\frac{1}{2}}$

In the general equation, $ax^2+bx=c$, when b is not even there is not so much advantage in this transformation. In such cases, however, we can multiply the whole equation by 2, and render the 2d coefficient even.

EXAMPLE.

1. Given $6x^2-47x=-35$, to find the values of x.

(R. 249.)

Here as b or 47 is not an even number, multiply by 2, and $12x^2-94x=-70$. Now assume $x=\frac{u}{12}$

Then $u^2-94u=-840$. And $u^2-94u+2209=1369$, or $u=84$ or 10. Therefore $x=\frac{84}{12}=7$ or $\frac{10}{12}=\frac{5}{6}$.

2. Given $4x^2-3x=85$, to find the values of x.

(R. 249.)

Multiply by 2, and $8x^2-6x=170$.

Now assume $x=\frac{u}{8}$. Then $u^2-6x=1360$. $u-3=\pm 37$.

$u=40$ or -34, But $x=\frac{u}{8}=\frac{40}{8}=5$, or $-\frac{34}{8}=-\frac{17}{5}$

There is a pleasant little artifice which may be resorted to in working quadratics, when the roots are whole numbers, and which may sometimes save considerable labor. We can best explain it by particular Examples.

1. Given $x^2+7x=18$, to find the values of x.

Take $2a=7$. That is, assume $2a$ as the coefficient of the first power, and compare it with the absolute term on the other side.

Now consider what number we must multiply $2a$ or 7 by to *approach* 18, *saving room for the square of the multiplier.* In this case, if we multiply $2a$ or 7 by 2, and add the square of 2, that is $7 \cdot 2+4$, we have 18.

The equation then is $x^2+2ax=4a+4$, (because $2a=7$, $4a=14$, and $4a+4=18$.)

Complete the square, and $x^2+2ax+a^2=a^2+4a+4$.

Extract the roots, and $x+a=\pm(a+2)$. That is, $x=2$, or $-2a-2=-9$. Ans.

2. $x^2+8x=20$. Take $2a=8$, then $4a+4=20$.

We then have $x^2+2ax=4a+4$.

Add a^2, and $x^2+2ax+a^2=a^2+4a+4$. Or $x=2$ or -10.

3. Given $x^2+7x=44$, to find the values of x.

Assume $2a=7$. Then to *approach* 44, our multiplier would be 6, but we must *save room*, for the *square of the multiplier.* Therefore we cannot take more than 4 for a multiplier. Try 4, and we find

$8a+16=28+16=44$ Hence $x^2+2ax=8a+16$,

Completing the square, $x^2+2ax+a^2=a^2+8a+16$, or $x+a=\pm(a+4)$, hence $x=4$ or -11.

It may be observed that one of the values of x is always equal to the multiplier.

Take this, one of the preceding examples,

$$u^2-94u=-840. \quad 2a=-94.$$

Now as the square of any number is always positive, we must have a negative product *over* 840, to be brought back by the square of the multiplier, we must therefore take a multiplier as high as 10. Try 10.

$$20a+100=-940+100=-840.$$

Therefore $u^2+2au+a^2=a^2+20a+100$

$$u+a=a+10, \text{ or } u=10, \text{ or } 84.$$

4. Given $y^2-5y=6$, to find the values of y.

$2a=5$. Multiply by 1, and add 1, $2a+1=6$.

$y^2-2ay=2a+1$, or $y^2-2ay+a^2=a^2+2a+1$.

$y-a=a+1$ or $y-a=-a-1$. That is $y=6$ or -1.

5. Given $x^2-3x=4$, to find the values of x.

$2a=3$. $2a+1=4$. Then $x^2-2ax+a^2=a^2+2a+1$.

$x-a=a+1$, or $-a-1$. $x=4$, or -1.

6. Given $x^2-3x=40$, to find the values of x.

Assume $2a=3$. (Multiply by 6, then add 36, and we have too much.) Take 5, and $10a+25=15+25=40$.

Then $x^2-2ax+a^2=a^2+10a+25$,

$x-a=a+5$, or $-a-5$. Hence $x=8$ or -5.

The right multiplier in whole numbers always is to be found, if the value of the unknown term is a whole number, for that multiplier is no other than the root itself, hence in the above examples, or *in any examples*, as soon as the multiplier is found, *one of the values of the unknown is found*, and the problem *is solved*, and unless you wish to find *the other value*, all subsequent labor may be dispensed with.

Now let us examine the theory of this operation. Take the 3d example. $x^2+7x=44$. Assume $2a=7$.

Now, we must find by *inspection*, a multiplier (m) of such a magnitude that $2am+m^2=44$.

Let $2a$ take its original value 7 and $m^2+7m=44$; an equation the same as the one we commenced with, except (m) has taken the place of x. Therefore, finding m is the same as finding x. The whole thing then, is simply *guessing* out x, but doing it in an easy, sly, and stealthy manner.

Notwithstanding this, the operation is of real utility, as will be seen before we arrive at the close of this volume.

SECTION XI.

Mr. Young, in his Algebra, page 115, has given an approximate mode of solving quadratic equations, which is very useful when the coefficients are large or fractional, or more particularly when the root itself is a fraction. The practical operation is not unlike the extraction of square root in arithmetic. Mr. Young appears to want system in the arrangement of his divisors, and for that reason, we think his formula is not much known. We have arranged the divisors agreeable to our taste.

"Let $x^2+ax=n$, be the general form of a quadratic. Let r denote the first figure of the root, and y the remaining part of the root:" that is, if the root be 730, $r=700$, and $y=30$; if the root be 5,27, $r=5$, and $y=.27$, &c.

Now if we *disregard* the y, and take r, the most material part of the root, we shall have $r^2+ar=n$, nearly, or $ar+r^2=n$, nearly, or $r=\frac{n}{a+r}$ an equation sufficiently exact to find r, by trial, as we would find a quotient figure in division. *r is now known.*

To find y. Recollect that $x=r+y$.

Therefore $x^2=r^2+2ry+y^2$

And $\quad ax=ar+ay$

Then $\quad x^2+ax=k+a'y+y^2=n$.

r^2+ar, being known, we call k, and $a+2r$, we call a' for the sake of brevity. Transpose k, and represent $n-k$ by n' and we have $a'y+y^2=n'$.

Now suppose s to represent the most material part of y, and z the remaining part, the same as r represents the root of x. Then $a's+s^2=n'$ *nearly.*

Or $s=\frac{n'}{a'+s}=\frac{n'}{a+2r+s}$ because $a+2r=a'$.

Now, as before $y=s+z$.

Then $y^2=s^2+2sz+z^2$
And $a'y=a's+a'z$

By addition, $y^2+a'y=\ k'+a''z+z^2=n'$.
$a''z+z^2=n'-k'=n''$.

Again, let $z=t+u$. t, being the superior figure of the root of z. Then $a''t+t^2=n''$, *nearly.*

Or $t=\frac{n''}{a''+t}$, but $a'+2s=a''$. And $a+2r=a'$.

Therefore $a+2r+2s+t=a''+t$.

Hence $t=\frac{n''}{a+2r+2s+t}$.

Now, by hypothesis $x=r+y$. $y=s+z$. $z=t+u$, &c. Hence, $x=r+s+t$, &c. r being the figure of the highest denomination, s the next, t the next, &c.

And as above, $r=\frac{n}{a+r}$.

$$s=\frac{n'}{a+2r+s}.$$

$$t=\frac{n''}{a+2r+2s+t}.$$

&c.= &c. &c.

Our divisors may be arranged thus:

1st Divisor $a+r$
add $r+s$

2d Divisor $a+2r+s$
add $s+t$

3d Divisor $a+2r+2s+t$
add $t+u$

4th Divisor $a+2r+2s+2t+u$
add $u+v$

5th Divisor $a+2r+2s+2t+2u+v$ &c. &c.

It will be observed, that r, s, t, u, v, &c., are successive figures in the root, and are found by trial division in succession, and placed in the divisor as soon as found. Let it be observed, that $a+2r$, is a *partial*, or *trial* divisor to find s. Also, $a+2r+2s$, is a trial divisor to find t, &c.

To illustrate this rule, find one value of x in the equation,

$$x^2+37x=5283.$$

After a few trials, we find that x must be more than 50, and less than 60, Therefore $r=50$.

$a+r=37+50$	$=$ 87	1st divisor.
$r=$	50	
Trial divisor $a+2r=$	137	goes in 933, 6 times.
$s=$	6	
2d divisor $a+2r+s$	$=$143	
s	$=$ 6	
Trial divisor to find t	149	
149 in 75, .5, hence $t=$	.5	
3d divisor, $a+2r+2s+t=$	149.5	
	.5	
	150.001	

```
          n      r s t &c.
   87 ) 5283 ( 56,5016
        435
   143 ) 933=n'
         858
   149.5 ) 75,00=n''
            7475
   150,001 ) 250000
             150001
   150,0026 ) 99999.
```

We may now continue the division as in decimals, and we may carry the value of x to eight or ten places of decimals, without any sensible error.

Given $x^2-700x=59829$, to find x.

Here we perceive that x must be more than 700, for, if it were exactly 700, the left hand side of the equation would be $x^2-x^2=0$. But on trial, we find x cannot be 800.

Therefore $r=700$.

$$a+r=-700+700=0$$
$$r=\quad 700$$
$$s=\quad 70$$

$$a+2r+s=\quad 770$$
$$s=\quad 70$$

Trial divisor 840

$$t=\quad 7$$

$$a+2r+2s+t=\quad 847$$

$$\begin{array}{l} n \quad\;\; r\,s\,t \\ 69829\ (\ 777 \\ 000 \\ \hline 5982=n' \\ 5390 \\ \hline 5929=n'' \\ 5929 \\ \hline \end{array}$$

Hence $x=777$. Given $x^2+x=60$, to find x.
Ans. $x=7.26208734$.

Given $x^2+1728\frac{1}{3}x=123\,578$, to find x.
Ans. $x=68.76528+$.

Section XII.

Problems producing Quadratic Equations.

There are many problems put under the rules of quadratics, which may be brought under simple or pure equations, by a little care or artifice at the commencement. Yet by the common way of resolving them, they come out in a quadratic form. Others are *essentially* quadratic.

The first six are of the first class.

1. To find two numbers whose difference shall be 12, and the sum of their squares, 1424. Ans. 20 and 32.

Let $x+6=$ the greater, and $x-6=$ the less. } Difference $=12$.

2. The sum of two numbers is 6, and the sum of their cubes is 72. What are the numbers? Ans. 2 and 4.

Let $3+x=$ the greater, and $3-x=$ the less. } This avoids a quadratic.

3. Divide the number 56 into two such parts, that their product shall be 640. Ans. 40 and 16.

Let $28+x=$ the greater,
$28-x=$ the less.

4. A and B distributed 1200 dollars each, among a certain number of persons. A relieved 40 persons more than B, and B gave to each individual 5 dollars more than A. How many were relieved by A and B?

Let $x+20=$ the number relieved by A.
and $x-20=$ the number relieved by B.

$$\text{Then } \frac{1200}{x+20}+5=\frac{1200}{x-20}.$$

$$\text{Dividing by 5, gives } \frac{240}{x+20}+1=\frac{240}{x-20}.$$

Clear of fractions, and

$$240x-240\cdot20+x^2-20\cdot20=240x+240\cdot20.$$

Drop $240x$, and transpose $-240\cdot20$,
then $x^2=500\cdot20$, $x=100$. Ans. A, \$120. B, \$80.

5. Several gentlemen made a purchase in company, for 175 dollars. Two of them having withdrawn, the bill was paid by the others, each furnishing ten dollars more. What was the number in company at first? Ans. 7.

Let $x\quad=$ the number, and we have a quadratic.
Let $x+1=$ the number, and we avoid a quadratic.

6. A man traveled 72 miles in a certain number of hours. If he had traveled 2 miles more per hour, he would have been 3 hours less on the journey. How many miles did he travel per hour? Ans. 6.

Let $x-1=$ the hours.

7. A gentleman bought a number of pieces of cloth for 675 dollars, which he sold again at 48 dollars by the piece, and gained by the bargain as much as one piece cost him. What was the number of pieces? Ans. 15.

This is a troublesome problem by reason of the large numbers that appear in the solution. To avoid this, the skill of the operator should be directed.

The following solution is copied from Bridge's Algebra, which is a fair sample of common operations.

Let $x=$ the number of pieces.

Then $\frac{675}{x}=$ the number of shillings each piece cost.

By the question, $48x-675=\frac{675}{x}$,

By transposition and division $x^2-\frac{225x}{16}=\frac{225}{16}$,

Complete the square,

$$x^2-\frac{225x}{16}+\frac{50625}{1024}=\frac{225}{16}+\frac{50625}{1024}=\frac{65025}{1024}.$$

Therefore $x-\frac{225}{32}=\frac{255}{32}$, and $x=\frac{480}{32}=15$.

We take $16x^2-225=225$. Observe that 225 appears in two places in the equation, and it is the *square* of 15. Of these circumstances advantage can be taken.

Let $a=16$ and $b=15$

Then $ax^2-b^2x=b^2$. Multiply by $4a$, and add b^4 to both sides, agreeable to observation, sec. 8, and

$$4a^2x^2-4ab^2x+b^4=4ab^2+b^4,$$

Extract the root $2ax-b^2=b\sqrt{4a+b^2}$.

But $\sqrt{4a+b^2}=\sqrt{289}=17$, that is $32x-15\cdot15=17\cdot15$, or $32x=32\cdot15$, and $x=15$, the answer.

8. A set out from C towards D, and traveled 7 miles a day. After he had gone 32 miles, B set out from D towards C, and went every day $\frac{1}{19}$ of the whole journey; and after

he had traveled as many days as he went miles in a day, he met A. Required the distance from C to D.

Ryan's Solution.

Suppose the distance was x miles,

Then $\frac{x}{19}$ = the number of miles B traveled per day, and also the number of days before he met A.

$$\text{Therefore. } \frac{x^2}{361}+32+\frac{7x}{19}=x.$$

By transposition and completing the square,

$$\frac{x^2}{361}-\frac{12x}{19}+36=36-32=4.$$

$$\text{Extracting the root, } \frac{x}{19}-6=\pm 2. \quad \frac{x}{19}=8, \text{ or } 4,$$

$$\text{Or, } x=152, \text{ or } 76,$$

both of which values answer the conditions of the problem.

Query.—*How did he complete the square?* we find no directions in his book that would enable us to do it.

Now let $19x$ = the distance,

Then x = the miles per day, also the days,

x^2 = B's distance, and $7x+32$=A's distance.

Hence, $x^2+7x+32=19x$. $x^2-12x=-32$.

$x-6=\pm 2$, $x=8$, or 4. $19x=152$, as before.

9. A Poulterer going to market to buy turkeys, met with four flocks. In the 2d, were 6 more than 3 times the square root of double the number in the first. The 3d contained 3 times as many as in the 1st and 2d; and the 4th contained 6 more than the square of one-third the number in 3d; and the whole number was 1938. How many were in each flock? Ans. 18, 24, 126, 1770.

Let $2x^2$ = the number in the 1st,

Then $6x+6$= the number in the 2d,

$3(2x^2+6x+6)$ = the number in the 3d,

$(2x^2+6x+6)^2+6$= the number in the 4th,

Assume $2x^2+6x+6=y$, then $y^2+4y+6=1938$.

From this equation y can be found. Afterwards x from the assumed equation.

10. A Vinter sold 7 dozen of sherry and 12 dozen of claret for £50, and finds that he has sold 3 dozen *more* of sherry for £10, than he has of claret for £6. Required the price of each. (Young, page 122.)

Let x = the price of a doz. of sherry in pounds.

Then $\frac{10}{x}$ = the number of doz. of sherry for £10.

and $\frac{10}{x}-3$ = the number of doz. of claret for £6.

Therefore, $\frac{6}{\frac{10}{x}-3}$ = the price of a dozen of claret,

or $\frac{6x}{10-3x}$ = the price of claret per doz.

By the question, $7x+\frac{72x}{10-3x}=50$,

By reduction, $21x^2-292x=-500$, a bad equation to work out. Suppose $a=21$.

$$\text{Then } 14a-2=292, \text{ and } 24a-4=500.$$

Now the equation becomes $ax^2+(2-14a)x=4-24a$.

Multiply by $4a$, &c., and

$$4a^2x^2+A+(2-14a)^2=4-40a+100a^2.$$

Extract the square root, $2ax+2-14a=\pm(2-10a.)$

Therefore $2ax=4a$, or $24a-4$. Or $x=2$, or $\frac{250}{21}$.

This artifice is of *little use* in many equations, but in others it is highly advantageous. In this equation, it will be observed that it comes out $4-40a+100a^2$, which is a complete square. When the right hand side is not an *obvious* square, we must return to the value of (a) before we extract the root.

In the above equation, observe $4a^2x^2+A$, &c., A is made to represent the middle term, which it may do, as that term always disappears in taking the root.

SECTION XIII.

Quadratic Equations containing more than one unknown quantity.

1. Given $x^{\frac{3}{2}}+y^{\frac{2}{3}}=3x$, and $x^{\frac{1}{2}}+y^{\frac{1}{3}}=x$ to find the values of x and y.

Many do not substitute in equations like these, but we think their work would be more brief and clear if they did.

Let $x^{\frac{1}{2}}=P$. $\quad y^{\frac{1}{3}}=Q$.

$x=P^2$. $\quad y^{\frac{2}{3}}=Q^2$.

$x^{\frac{3}{2}}=P^3$.

Then $P^3+Q^2=3P^2$, and $P+Q=P^2$.

Or $Q=P^2-P=(P-1)\,P$. $\quad Q^2=(P-1)^2\,P^2$.

Substitute the value of Q^2 in the 1st equation,

And $P^3+(P-1)^2\,P^2=3P^2$. $\quad P+(P-1)^2=3$.

From which we find $P=2$, or -1.

Then $x=4$, or 1. $\quad y=8$.

2. Given $y^2-64=8x^{\frac{1}{2}}y$, and $y-4=3y^{\frac{1}{2}}x^{\frac{1}{2}}$, to find the values of x and y. Ans. $x=\frac{9}{4}$. $\quad y=16$.

From the 1st equation $y^2-8x^{\frac{1}{2}}y=64$.

From the 2d, $\quad y-2x^{\frac{1}{2}}y^{\frac{1}{2}}=4$.

Consider $-8x^{\frac{1}{2}}$, in one equation, and $-2x^{\frac{1}{2}}$, in the other, as known coefficients to y and $y^{\frac{1}{2}}$, in their respective equations, and complete the squares, &c., and we shall have

$$y=4x^{\frac{1}{2}}\pm4\sqrt{x+4}. \qquad (1)$$

$$y^{\frac{1}{2}}=x^{\frac{1}{2}}\pm\sqrt{x+4}. \qquad (2)$$

Divide equation (1) by (2) and $y^{\frac{1}{2}}=4$. Hence $y=16$.

There is another solution well worthy of notice. It is the following:

From the 1st equation, $x^{\frac{1}{2}}=\dfrac{y^2-64}{8y}$.

2d equation, $x^{\frac{1}{2}}=\frac{y-4}{2\sqrt{y}}$.

Therefore, $y^2-64=4y^{\frac{3}{2}}-16y^{\frac{1}{2}}$.

Or, $y^2-4y^{\frac{3}{2}}+16y^{\frac{1}{2}}-64=0$.

Or, $(y^{\frac{1}{2}}-4)\,(y^{\frac{3}{2}}+16)=0$.

But when the product of several factors equal nothing, one or more of them, must equal nothing.

Hence, $(y^{\frac{1}{2}}-4)\,y^{\frac{3}{2}}=0$. $\quad y^{\frac{1}{2}}=4$.
Therefore, $y=16$,

3. Given $2x^2+3xy+y^2=20$, and $5x^2+4y^2=41$.

Equations like these *where the exponents in every term amount to the same sum*, are called homogeneous; and their solutions can be best effected by assuming $x=vy$.

4. Given $x^2+xy=56$, and $xy+2y^2=60$, to find the values of x and y.

Ans. $\begin{cases} x=\pm4\sqrt{2}, \text{ or } \pm14. \\ y=\pm3\sqrt{2}, \text{ or } \pm10. \end{cases}$

It would be difficult to solve the above without putting $x=vy$.

5. Given $xy+\frac{y^3}{x}=40$, and $\frac{x^3}{y}-xy=96$, to find the values of x and y.

Ans. $x=\pm8$. $y=\mp4$.

6. Given $x(\sqrt{y}+1)+2\sqrt{xy}=55-y(\sqrt{x}+1)$,

and $x\sqrt{y}+y\sqrt{x}=30$, to find the values of x and y.

Ans. $\begin{cases} x=4, \text{ or } 9, \\ y=9, \text{ or } 4. \end{cases}$

7. Given $x^2+2xy+y^2+2x=120-2y$, and $xy-y^2=8$, to find the values of x and y. Ans. $x=6$, or 9. $y=4$, or 1.

Assuming $x+y=s$, the first equation becomes,

$s^2+2s=120$, or $s=10$. $\quad x+y=10$, &c.

8. Given $\frac{x^2}{y}+\frac{y^2}{x}=18$, and $x+y=12$, to find the values of x and y.

Ans. $\begin{cases} x=8, \text{ or } 4, \\ y=4. \text{ or } 8. \end{cases}$

The preceding is an interesting problem, not difficult, but admits of several ways of solution.

Ryan, assumes $x=6+v$, and $y=6-v$, &c. We might assume $x=vy$, but we prefer to clear the first of fractions, and in form, cube the second.

Thus, $x^3+y^3=18xy,$ (1)

$x^3+y^3+3xy(x+y)=12\cdot12\cdot12.$ (2

In equation (2) put the value of x^3+y^3, as found in equation (1), and $18xy+3xy(12)=12\cdot12\cdot12.$

$$54xy=12\cdot12\cdot12.$$

Divide by 6, and $9xy=2\cdot12\cdot12$, or $xy=2\cdot4\cdot4=32$.

9. Given $\frac{x^4}{y^2}+\frac{y^4}{x^2}=\frac{1225}{9}-2xy$, and $x+y=10$, to find the values of x and y.

Transpose $-2xy$ in the first equation, and extract the square root, and we have $\frac{x^2}{y}+\frac{y^2}{x}=\frac{35}{3}$.

Reduce as in the preceding, and we find

$$x=6, \text{ or } 4, \text{ or } 5\pm\sqrt{-\tfrac{13}{11}}.$$
$$y=4, \text{ or } 6, \text{ or } 5\mp\sqrt{-\tfrac{13}{11}}.$$

10. Given $x^2+y^2=13$, and $x^2+xy=10$. Required the values of x and y. Ans. $x=2$. $y=3$.

11. Given $4-\sqrt{x}=3-y$, and $4-x=y-\sqrt{y}$, to find the values of x and y.

From the first, $1+y=\sqrt{x}$. Square, $1+2y+y^2=x$.

Put this value of x in the second equation, and

$$3-2y-y^2=y-\sqrt{y}.$$

Transpose all to one side, having y^2 stand plus,

and $y^2-y^{\frac{1}{2}}+3y-3=0$, or $y^{\frac{1}{2}}(y^{\frac{3}{2}}-1)+3(y-1)=0$.

Here, again, we apply the observation under problem second, and we perceive at once, that $y=1$, which gives $x=4$. We believe this the only direct method of solving this problem.

12. Given $x^{\frac{3}{2}}+x-4x^{\frac{1}{2}}=y^2+y+2$, and $xy=y^2+3y$, to find the values of x and y.

Divide the second by y and subtract the quotient from the first, and $x^{\frac{3}{2}}-4x^{\frac{1}{2}}=y^2-1$. (1)

Divide equation (1) into factors,

$$(x-4)x^{\frac{1}{2}}=(y+1)(y-1) \qquad (2)$$

But $x=y+3$. Subtract 4 from both sides, and

$$x-4=y-1 \qquad (3)$$

Divide equation (2) by (3), and $\sqrt{x}=y+1$, or

$$x=y^2+2y+1=y+3.$$

Reducing, gives $y=1$, or -2, and $x=4$, or 1.

13. Given $y^4-432=12xy^2$, and $y^2=12+2xy$, to find the values of x and y.

This is an important problem, and we give Ryan's solution taken from his *Key*, as exact as we could copy. Afterwards we shall give two other solutions, to show the contrast between going by *Rule* and going perfectly *free* under the guide of true principles. Ryan solved *this*, at least, by *rule*, he obtained two values of one letter, and put them equal to each other, which is a legitimate operation.

"Take the first equation, transpose, and complete the square, and $y^4-12xy^2+36x^2=36x^2+432$.

Extracting the square root, and transpose $6x$, and

$$y^2=6x\sqrt{36x^2+432}=6x+6\sqrt{x^2+12} \qquad (a)$$

From the second equation by transposing, and completing the square $y^2-2xy+x^2=x^2+12$.

Extracting the root, $y-x=\sqrt{x^2+12}$, and

$$y=x+\sqrt{x^2+12}. \qquad (b)$$

Therefore, $y^2=x^2+2x\sqrt{x^2+12}+x^2+12$,

Whence, $y^2=2x^2+2x\sqrt{x^2+12}+12$.

Put this equal to the value of y^2 drawn from the first equation. and we have,

$$2x^2+2x\sqrt{x^2+12}+12=6x+6\sqrt{x^2+12}$$

Or, $x^2+x\sqrt{x^2+12}+6=3x+3\sqrt{x^2+12}$.

By transposition, $x^2+(x-3)\sqrt{x^2+12}=3x-6$,

And $\sqrt{x^2+12}=\frac{3x-6-x^2}{x-3}=-x-\frac{6}{x-3}$

Squaring both sides of this equation, and

$$x^2+12=x^2+\frac{12x}{x-3}+\frac{36}{(x-3)^2}:$$

Therefore, $-36x^2+216x-324=36x-108$.

By transposing, $36x^2-180x=-216$.

$x^2-5x=-6$, Hence, $x=3$, or 2.

Mr. Ryan was too good an Algebraist to have solved this problem in this manner, when his mind was on the subject. He evidently did this mechanically and by rule, and afterwards either *not noticed*, or had no opportunity to improve the printed copy. But it now well serves our purpose. We use it to show the great difference between working by form, and by judgment. But some can never *rise above* forms. To them the spirit of Algebra should never be committed.

But to return to the problem.

In the preceding work, observe the equation marked (*a*), $y^2=6x+6\sqrt{x^2+12}$. Observe, also, equation (*b*), and multiply it by 6, and we have $6y=6x+6\sqrt{x^2+12}$. *But things that equal the same thing, equal one another*, therefore $y^2=6y$. Or $y=6$.

But the 2d original equation gives $x=\frac{y^2-12}{2y}=2$.

Third Method.

From the first equation $2x=\frac{y^4-432}{6y^2}$.

From the second equation $2x=\frac{y^2-12}{y}$.

Therefore $\frac{y^4-432}{6y^2}=\frac{y^2-12}{y}$.

Clear of fractions, and $y^4-432=6y^3-72y$.

Transpose all on one side, and separate into factors, and we have $y^3(y-6)+72(y-6)=0$, or $(y^3+72)(y-6)=0$

But when the product of several factors equal nothing, one or several of them must equal nothing. Therefore, $y-6=0$, or $y=6$.

And $y^3=-72$, or $y=\sqrt[3]{-72}=2\sqrt[3]{-9}$.

We find the following equations in Harney's Algeba, page 283.

14. Given $\left\{\begin{array}{r} x^4-y^4=1280, \\ xy(x^2+y^2)=480. \end{array}\right\}$ Required x and y.

In the Key to the work, we find the following, which we suppose to be the author's solution.

$$x^2+y^2=\frac{480}{xy}$$

Square this last, and subtract $2x^2y^2$ from both members, and we have $x^4+y^4=\frac{(480)^2}{x^2y^2}-2x^2y^2$.

Square this, and subtract from it the square of the first, and we have

$$4x^4y^4=-(1280)^2+\frac{(480)^4}{x^4y^4}-4(480)^2+4x^4y^4 \qquad (a)$$

Clear of fractions, and we get an equation containing only the eighth and fourth powers of xy. This reduced as an affected equation, gives $xy=12$. Put this in the place of xy in the second equation, and reducing, we get

$$x^2+y^2=40.$$

Divide the first equation by this, and we have

$$x^2-y^2=32. \quad 2x^2=72. \quad \text{Hence } x=6.$$

This is the work of a professed mathematician of some eminence, or we should not have copied it, and from other parts of his work, we are convinced that he possesses solid acquirements, but why he solved this in the manner he has, is a *greater problem* than any we have ever met with.

The reader is now requested to observe equation (a), and he will perceive $4x^4y^4$ on both sides, which *may cancel* each other, after which we can divide every term by $(480)^2$, and we have at once $\frac{(480)^2}{x^4y^4}=\frac{64}{9}+4=\frac{100}{9}$.

Extract the square root, and $\frac{480}{x^4y^2}=\frac{10}{3}$.

Or, $x^2y_2^2=3\cdot48=9\cdot16$. Hence $xy=12$.

Here we have no eighth and fourth powers, nor even an affected quadratic equation.

But let us return to the original equations.

$$x^4-y^4=1280. \quad xy\,(x^2+y^2)=480.$$

Divide the first by the second and $\frac{x^2-y^2}{xy}=\frac{1280}{480}=\frac{8}{3}$

Assume $x=vy$, and $\frac{v^2y^2-y^2}{vy^2}=\frac{8}{3}$.

Divide the fraction on the left hand side of this equation by y^2, and $\frac{v^2-1}{v}=\frac{8}{3}$. Therefore $v^2-\frac{8}{3}v=1$.

Reducing, gives $v=3$, or $-\frac{1}{3}$.

Taking the first value, $x=3y$. Put this value of x in the second *of the original equations*, and we immediately find $y=2$: Therefore, $x=6$.

15. Given $x^3+y^3=35$, and $x^9+y^9=20195$, to find the values of x and y. Ans $x=3$; $y=2$.

16. Find x, y, and z, from the following equations.

$$z\,(x+y+z)=6. \quad y\,(x+y+z)=12. \quad x\,(x+y+z)=18.$$

Ans. $x=3$. $y=2$. $z=1$.

17. Find x and y from the equations,

$xy+xy^2=12$, and $x+xy^3=18$. Ans. $\begin{cases} x=2 \text{ or } 16. \\ y=2 \text{ or } \frac{1}{2}. \end{cases}$

18. Find x and y from $(x+y)^2-3y=28+3x$, and $2xy+3x=35$.

Ans. $\begin{cases} x=5, \text{ or } \frac{7}{2}. \\ y=2, \text{ or } \frac{7}{2}. \end{cases}$

19. Find x and y from $x^2+3x+y=73-2xy$, and $y^2+3y+x=44$.

Ans. $\begin{cases} x=4, \text{ or } 16, \text{ or } -12\pm\sqrt{58}. \\ y=5, \text{ or } -7, \text{ or } -1\pm\sqrt{58}. \end{cases}$

The following equations require considerable skill in the student to reduce them, and on that account, are interesting.

Taken from Young's Algebra, page 147.

19. Given $\begin{cases} \dfrac{x+y+\sqrt{x^2-y^2}}{x+y-\sqrt{x^2-y^2}}=\dfrac{9}{8y}\left(x+y\right) \\ (x^2+y)^2+x-y=2x\,(x^2+y)+506. \end{cases}$ to find the values of x and y.

The first equation reduces down without much labor or difficulty, to $y=\frac{3x}{5}$.

Transpose $2x\,(x^2+y)$ in the 2d equation and add x^2 to both sides, $(x^2+y)^2-2x\,(x^2+y)+x^2=x^2+y-x+506$.

The left hand side is a complete square, whose root is

$$((x^2+y)-x)^2=(x^2+y-x)+506.$$

That is $P^2=P+506$, from which P becomes known, and as *the relation* between x and y is known, the equations are substantially reduced. Ans. $x=5$. $y=3$.

21. Given $\left\{\dfrac{1}{1+x}\left(\dfrac{1}{1+x}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}=\dfrac{1}{12}\sqrt{2x}$, to find x.

Ans. $x=8$, or -9.

22. Given $\begin{cases} v+w+x+y+z= 56 \\ vw-x-y-z= 207 \\ wx-v-y-z= -9 \\ xy-v-w-z= -19 \\ yz-v-w-x= 38 \end{cases}$ to find the positive values of v, w, x, y and z.

(Young, page 147.)

Add the second equation to the first—add the third to the first, and so on in succession, and we shall have

$$v+vw+w=236, \text{ or } v=\frac{263-w}{w+1} \quad (1)$$

$$w+wx+x=47, \text{ or } w=\frac{47-x}{x+1} \quad (2)$$

$$x+xy+y=37, \text{ or } y=\frac{37-x}{x+1} \quad (3)$$

$$y+yz+z=94, \text{ or } z=\frac{94-y}{y+1} \quad (4)$$

In equation marked (1), substitute the value of w from equation (2), and we shall have

$$v=\frac{263-\frac{47-x}{x+1}}{\frac{47-x}{x+1}+1}=\frac{263x+263-47+x}{47-x+x+1}=\frac{11x+9}{2} \quad (5);$$

In the same manner, in equation (4), put in the value of y from equation (3), and we have $z=\frac{95x+57}{38}$ (6)

Now add equations (5), (2), (3), (6), and to them add x, and we have

$$v+w+x+y+z=\frac{11x+9}{2}+\frac{47-x}{x+1}+\frac{37-x}{x+1}+x+\frac{95x+57}{38}$$
$=56.$

Multiply by 2, and take the value of $\frac{95x+57}{19}=5x+3$, and we shall have $112=11x+9+\frac{168-4x}{x+1}+2x+5x+3$, or $100=18x+\frac{168-4x}{x+1}$, which equation gives $x=1$.

SECTION XIV.

Problems producing Quadratic Equations containing two or more unknown quantities.

N. B. In working equations containing two unknown quantities (particularly in quadratics,) it evinces a want of skill, in all cases, to let x equal the one, and y the other.

This is the general commencement, but should be departed from in *many instances.*

If we put $x =$ half the sum of two unknown numbers, and $y=$ half the difference of the same numbers, then $x+y$ will represent the greater, and $x-y$ will represent the less.

PROOF.

The sum of two numbers is s, and their difference is d.

Put $P=$ the greater, and $Q=$ the less.

Then - - $P+Q=s$, and $P-Q=d$.

Add, and $2P=s+d$, or $P=\frac{s}{2}+\frac{d}{2}$.

Subtract, and $2Q=s-d$, or $Q=\frac{s}{2}-\frac{d}{2}$.

That is, the greater of two numbers is equal to the *half sum*, plus the *half difference*. And the *half sum* minus the *half difference*, equals the less.

EXAMPLES.

1. The difference of two numbers is 4, and their sum multiplied by the difference of their second powers, gives 1600. What are the numbers ?

Solved in Harney's Algebra, thus

Let $x =$ one, and $y=$ the other,

Then $x-y=4$. (1) $(x+y)(x^2-y^2)=1600$. (2)

Divide (2) by (1) gives $(x+y)(x+y)=400$.

$x^2+2xy+y^2=400$. $x+y=20$. (3)

Take equations (3) and (1), and you can easily get the values of x and y.

Now let $x+y=$ the greater number, and $x-y=$ the less.

Then - - - - $2y=4$, or $y=2$.

The difference of their second powers, is $4xy$. By the second condition, then, $8x^2y=1600$.

And as $y=2$, $x^2=100$, or $x=10$.

The numbers are therefore, 12 and 8.

2. Find two numbers, whose difference multiplied by the

difference of their squares, is 32; and whose sum multiplied by the sum of their squares, gives 352.

In Harney's Algebra we find the following solution.

Let x=one and y= the other.

Then ..... $(x-y)(x^2-y^2)=32.$ (1)

And ..... $(x+y)(x^2+y^2)=272.$ (2)

$x^3-xy^2-x^2y+y^3=32.$ (3)

$x^3+xy^2+x^2y+y^3=272$ (4)

Add (3) and (4) and divide by 2, and we have

$x^3+y^3=152.$ (5)

Subtract (3) from (4), and divide by 2, and we have

(6) $xy^2+x^2y=120$, or, $xy(x+y)=120$ (7)

Multiply this last equation by 3, and add the result to eqution (5), and we have

$x^3+3x^2y+3xy^2+y^3=512.$ (8)

Take the third root of this last equation, and we have

$x+y=8.$ (9)

Substitute this value of $x+y$ in equation (7), and we have

$8xy=120.$ (10) $xy=15.$ (11)

Take equations (9) and (11), and find the values of x and y.

We prefer the following solution.

Let $x+y$= the greater number, and $x-y$= the less.

Then $2x$= their sum, and $2y$= their difference.

Also $4xy$= the difference of their squares, and $2x^2+2y^2$ = the sum of their squares.

Then by the conditions of the problem we must have

(1) $2y\cdot 4xy=32$, and $2x(2x^3+2y^2)=272.$ (2)

Reducing the above by dividing (1) by 8, (2) by 4, and multiplying by x as indicated, we have

(3) $xy^2=4$, and $x^3+xy^2=68.$ (4)

Subtract (3) from (4), gives $x^3=64$, or $x=4$.

Substitute x in equation (3), and $y=1$.

The numbers are therefore, 5 and 3.

3. The product of two numbers multiplied by the sum of

their squares, is 1248; and the difference of their squares, 20. What are the numbers? (H. page 210.)

The following is professor Harney's solution.

"Let x= one, and y= the other.

Then (1) $xy(x^2+y^2)=1248.$

(2) $x^2-y^2=20.$

Or, (3) $x^3y+xy^3=1248.$

(4) $x^2-y^2=20.$

Multiply (4) by xy, and we have $x^3y-xy^3=20xy.$ (5,

Square each member of (3) and (5), and we have

(6) $x^6y^2+2x^4y^4+x^2y^6=1557504.$

(7) $x^6y^2-2x^4y^4+x^2y^6=400x^2y^2.$

Subtract (7) from (6), and we have

$4x^4y^4=1557504-400x^2y^2.$ $x^4y^4+100x^2y^2=389376.$

$x^4y^4+100x^2y^2+2500=391876.$ $x^2y^2+50=626.$

$x^2y^2=576.$ $xy=24.$

Substitute this value of xy in equation (1), and we have

$24(x^2+y^2)=1248.$ (8)

Take the equations (2) and (8) and find the values of x and y."

Not being pleased with the large numbers above, we give the following solution, to which we invite special attention.

Let $x+y$= the greater number, and $x-y$= the less.

Their product is x^2-y^2

The sum of their squares is $2x^2+2y^2$.

The difference of their squares is $4xy$.

Now by the conditions of the problem,

$$(x^2-y^2)(2x^2+2y^2)=1248.$$

Or, $x^4-y^4=624.$ (1) And $4xy=20.$ (2)

From equation (2) $x=\frac{5}{y}\cdot$ $x^4=\frac{625}{y^4}\cdot$

Substitute this in equation 1, and

$$\frac{625}{y^4}-y^4=624, \text{ or } y^8+624y^4=625.$$

Assume $2a=624.$ Then $2a+1=625.$ And the last equation becomes $y^8+2ay^4=2a+1$. Add a^2 to both sides, and $y^8+2ay^4+a^2=a^2+2a+1$.

Extract the root, $y^4+a=a+1$. $y^4=1$. $y=1$.

Therefore, by (2), $x=5$, and 6 and 4 are the numbers.

As one object of this work is to show how to avoid tedious operations among *known*, as well as among *unknown* quantities, we present another problem of precisely the same nature as the last, but the known numbers will require more skill to avoid a high numerical equation.

4. The product of two numbers multiplied by the sum of their squares, is 19968; and the difference their squares, is 80. What are the numbers? H. page 211.

The following is Professor Ray's solution.

Let $x=$ half sum, and $y=$ half difference.

Then $x+y=$ greater, and $x-y=$ less. $x^2-y^2=$ product

$x^2+2xy+y^2+x^2-2xy+y^2=2x^2+2y^2=$ sum of squares,

and $x^2+2xy+y^2-(x^2-2xy+y^2)=4xy=$ diff. of squares.

$2(x^2+y^2)(x^2-y^2)=2(x^4-y^4)=19968$. $x^4-y^4=9984$.

$$4xy=80, \text{ therefore, } xy=20,\ x=\frac{20}{y}$$

$$x^1=9984+y^4, \text{ and } x^4=\frac{20^4}{y^4}.$$

Hence, $9984+y^4=\dfrac{20^4}{y^4}$.

$$9984y^4+y^8=20^4=160000. \quad y^8+9984y^4=16000.$$

$$y^8+9984y^4+(4992)^2=160000+24920064=25080064.$$

$$y^4+4992=5008. \quad y^4=16. \quad y=2.$$

$$x=\frac{20}{y}=10. \quad x+y=12. \quad x-y=8.$$

Observe the large numbers encountered in this mode of operation. However, this is a fair specimen of common solutions. We contend, however, that more care, and more study should be given to known quantities, and numerical relations; and we believe that nothing short of strict attention to the abreviating and cancelling system in Arithmetic, will do it.

The following, is *our Solution.*

Let $x+y=$ the greater, and $x-y=$ the less.

(1) $(x^2-y^2)(2x^2+2y^2)=19968.$ $4xy=80.$ (2)

Or, (3) $x^4-y^4=9984.$ (4) $xy=20.$

From equation (4) $y^4=\frac{160000}{x^4}$ which substitute in equation (3), gives $x^4-\frac{160000}{x^4}=9984.$

Observe that $9984=10000-16.$ Put $2a=10000.$ Then $32a=160000.$ And equation (3) becomes

$$x^4-\frac{32a}{x^4}=2a-16, \text{ or } x^8+(16-2a)x^4=32a.$$

Complete the square, and

$$x^8+(16-2a)x^4+(8-a)^2=64+16a+a^2.$$

Extract the square root, and $x^4+8-a=8+a.$

$$x^4=2a=10000. \quad \text{Hence, } x=10.$$

Now by equation (4) $y=2.$ The numbers are 8 and 12.

Another Solution.

Resume the equations $x^4-y^4=9984$, and $xy=20.$

Assume $x=vy.$

$$\text{Then } y^4=\frac{9984}{v^4-1}, \text{ and } y^2=\frac{20}{v}, \text{ or } y^4=\frac{400}{v^2}.$$

Put the two values of y^4 equal to each other, and divide by 4, and we have $\frac{2496}{v^4-1}=\frac{100}{v^2}.$

Divide again by 4, and clear of fractions, and

$$624v^2=25v^4-25.$$

Transpose 25, and add v^2 to both sides, and we have

$$625v^2+25=25v^4+v^2.$$

Separate each side into factors, and

$$25(25v^2+1)=v^2(25v^2+1).$$

Divide by the common factor, and $v^2=25.$ or $v=5.$

Hence $y=2$, and $x=10$, as before.

5. One man can do a certain piece of work, in 7 hours,

(or a hours,) another man can do the same in 5 hours (or b hours,) how long will it take them if they work both together?

One will accomplish $\frac{1}{7}$ in one hour, or $\frac{1}{a}$.

The other will accomplish $\frac{1}{5}$ or $\frac{1}{b}$ in one hour.

Let t represent the time in which the work would be done, if both work together.

Then $\frac{1}{t}=$ what they would both do in one hour.

Therefore $\frac{1}{a}+\frac{1}{b}=\frac{1}{t}$, or $t=\frac{ab}{a+b}$.

6. Bacchus caught Silenus asleep, by the side of a full cask, and seized the opportunity of drinking, which he continued for two-thirds of the time that Silenus would have taken to empty the whole cask.

After that, Silenus awoke, and drank what Bacchus left. Had they drank both together, it would have been emptied two hours sooner, and Bacchus would have drank only half what he left Silenus.

Required, the time in which they would have emptied the cask separately.

Let $x=$the time in hours for Bacchus.

And $y=$the time in hours for Silenus.

Then $\frac{1}{x}=$the quantity B. drank in one hour.

And $\frac{1}{y}=$ the quantity Silenus drank in one hour.

$\frac{2y}{3}=$ the time B. drank, which, multiplied by what he drank in one hour, gives $\frac{2y}{3x}=$ all he drank. Then $\left(1-\frac{2y}{3x}\right)=$ what Silenus drank. Now the quantity drank, divided by the quantity drank per hour, gives the time.

Therefore, $\frac{2y}{3}+\left(1-\frac{2y}{3x}\right)\frac{y}{1}-2=\frac{xy}{x+y}$ (1)

To find the right hand side of this equation, we refer only to the preceding problem, which was inserted only as a Lemma.

By the 2d conditions,

$$\left(\frac{1}{2}-\frac{y}{3x}\right)\frac{x}{1}=\frac{xy}{x+y} \qquad (2)$$

Divide (2) by x, and afterwards assume $x=vy$, and we have $\frac{1}{2}-\frac{y}{3vy}=\frac{y}{vy+y}$ (3)

Or $\frac{1}{2}-\frac{1}{3v}=\frac{1}{v+1}$ from this equation we find $v=2$. Then $x=2y$.

Substitute this value of x in equation (1) and we immediately find $x=3$. Hence, $y=6$.

7. Two men are employed to do a piece of work, which they can finish in 12 days. In how many days could each do the work, if he undertook it alone, provided it would take one 10 days longer than the other.

(Harney's Alg. 192.) Ans. 20 days, and 30 days.

8. Find two numbers, such that their product shall be equal to the difference of their squares, and the sum of their squares shall be equal to the difference of their cubes.

Ans. $\frac{1}{4}(5\pm\sqrt{5})$ and $\frac{(5\pm\sqrt{5}}{2(1\pm\sqrt{5}}$.

Let x=one and y=the other.

Then $xy=x^2-y^2$. $x^2+y^2=x^3-y^3$. Assume $x=vy$.

Then our equations become,

$vy^2=v^2y^2-y^2$. (1) $v^2y^2+y^2=v^3y^3-y^3$ (2)

Divide these by y^2, and we have $v=v^2-1$. (3)

$v^2+1=(v^3-1)y$. (4)

From equation (3) $v=\left(\frac{1}{2}\pm\frac{1}{2}\sqrt{5}\right)$

From equation (4) $y=\frac{v^2+1}{v^3-1}$. (5)

From equation (3) $v+1=v^2$. Hence $v^2+1=v+2$.

$v^2+v=v^3$.

But $v^2=v+1$. Hence $v^2+v=2v+1=v^3$.

Or $2v=v^3-1$.

Therefore, equation (5) becomes $y=\frac{v+2}{2v}$.

Then $v\,y$ or $x=\frac{v+2}{2}$. But v being known, x is known, and y is known.

9. Find two numbers, such that their sum shall be equal to the difference of their squares, and the sum of their squares shall be equal to the difference of their cubes. Or prove there are no such numbers.

Ans. There are no such numbers, or we may say 1 and 0 are the numbers.

10. A person bought a quantity of cloth of two sorts for £7 18 shillings. For every yard of the better sort, he gave as many shillings as he had yards in all; and for every yard of the worse, as many shillings as there were yards of the better sort more than of the worse. And the whole price of the better sort was to the whole price of the worse as 72 to 7. How many yards were there of each?

Ans. 9 of the better, and 7 of the worse.

11. A man had a vessel containing a mixture of wine and water. He poured out a quantity ot the mixture equal to the number of gallons of wine in the cask. Then there remained in the vessel only $2\frac{2}{5}$ gallons of wine. He then filled up the vessel with water, and found that there were $7\frac{2}{5}$ gallons of water in the cask. How many gallons of each were there in the mixture at first? (Harney 205)

Ans. $4\frac{3}{5}$ gallons of water, and $4\frac{1}{5}$ gallons of wine.

Let x=the wine, y=the water.

Then $x+y$=the gallons of liquid in the cask. Now he draws out x gallons of *liquid* from the cask, we wish to know how much wine this x quantity of liquid contains.

We find it by this proportion:

$$\overset{\text{Liquid.}}{x+y} : \overset{\text{Wine.}}{x} : : \overset{\text{Liquid.}}{x} : \overset{\text{Wine.}}{\frac{x^2}{x+y}} = \text{wine taken out.}$$

Then $x-\frac{x^2}{x+y} = 2\frac{2}{5}=a.$

This equation reduced, gives $xy=a\,(x+y)$ (1)

Again, to find the quantity of water taken out in the x quantity of liquid, we have this proportion.

$$x+y : y : : x : \frac{xy}{x+y} = \text{water taken out.}$$

But x quantity of water, was put back.

Then $x+y-\frac{xy}{x+y} = 7\frac{2}{5}=b.$ (2)

Put $(x+y)=s$, and $xy=p$, then in equations (1) and (2), we have $p=as$ (3)

$s-\frac{p}{s}=b.$ (4)

In (4) put the value of p taken from (3) and $s-a=b$.

Or $s=a+b$. Having now, s and p, we can determine x and y.

Section XV.

A particular case of Cubic Equations.

Sometimes we meet problems under the head of quadratics, which are really of a cubic form, but owing to the relation by *chance existing*, between the coefficients and the absolute term, they can be put under, or reduced to a quadratic form.

For example, an equation in the form of $x^3 \pm ax=b$, is one kind of cubic equations, but such equations can be reduced by quadratics, *whenever b can be divided into two factors, m, and n, and have the factors of such a magnitude that $m^2 \pm a = n$.*

INVESTIGATION.

Given $x^3+ax=b$, to find the values of x.

Assume $m.n=b$, and $m^2+a=n$. Then $a=n-m^2$.

For a and b in the given equation, put in their values, in terms of m and n, and we have $x^3+nx-m^2x=m.n$.

Transpose $-m^2x$ and multiply both sides by x, and

$$x^4+nx^2=m^2x^2+m.nx.$$

Add $\frac{n^2}{4}$, to both sides, and it makes complete squares, thus $x^4+nx^2+\frac{n^2}{4}=m^2x^2+mnx+\frac{n^2}{4}$.

Extract the roots and $x^2+\frac{n}{2}=mx+\frac{n}{2}$.

Drop $\frac{n}{2}$ from both sides, and divide by x, gives $x=m$.

If this theoretical equation is well understood, we need not go through the process of solution, *in particular cases*, for, if we can divide the known term b into factors, answering the condition of $m^2+a=n$. we know at once that the value of the unknown letter, *must be* equal to the factor m.

EXAMPLES.

1. What two numbers are those whose sum is 4, and the difference of their cubes 26? Ans. 3 and 1.

Let x=one, and y=the other.

Then $x+y=4$, and $x^3-y^3=26$.

Now, x^3-y^3 cannot be divided by $x+y$, and every attempt at a direct solution by x and y only, must result in an equation of a high degree.

If the difference of the two numbers were given, we could then divide x^3-y^3, by that difference. Now the difference is *some number.* Let it be represented by D.

Then $x-y=D$, and by division, $x^2+xy+y^2=\frac{26}{D}$ (1)

By squaring this assumed equation, $x^2-2xy+y^2=D^2$ (2)

By subtraction, $3xy=\frac{26}{D}-D^2$. (3)

Or $xy=\frac{26}{3D}-\frac{D^2}{3}$ (4)

Add equations (1) and (4) and we have

$x^2+2xy+y^2=\frac{4\cdot 26}{3D}-\frac{D^2}{3}$. (5)

But as $x+y=4$. $x^2+2xy+y^2=16$. (6)

Therefore by equations (5) and (6), we have

$16=\frac{4\cdot 26}{3D}-\frac{D^2}{3}$.

Or $48D=104-D^3$. Or $D^3+48D=104$.

Now this last equation is in the same form as our theoretical equation, and $a=48$, and $b=104$. b, or 104 has these two factors in whole numbers: 2, and 52, that is, m and n. And $4+48=52$. That is, $m^2+a=n$.

Hence $D=2$. That is, $x-y=2$. But $x+y=4$.

Therefore $x=3$, and $y=1$, the answer.

2. What two numbers are those, whose sum is 6, and the difference of whose cubes is 56? Ans. 4 and 2.

Many other problems of the same kind, might be proposed and worked in the same manner.

3. The product of two numbers is 36, and the difference of their third power is 665. What are the numbers?

(Harney, page 208.)

Let x=one, and y=the other.

Then $xy=36$, and $x^3-y^3=665$.

From the first, $y=\frac{36}{x}$, or $y^3=\frac{36^3}{x^3}$.

Substitute this value in the 2d equation, and we have

$$x^3-\frac{(36)^3}{x^3}=665.$$

Or $x^6-665x^3=(36)^3$, a quadratic equation, in the common form.

But the numbers 665, and the *cube* of 36, are numbers so large and incommensurable, that it would be a very unpleasant and tedious operation to work it out—and in short, such *labor*, when once understood, leads to little or no improvement of mind.

But let us try to work out this question without such large numbers, and for this purpose, assume $x-y=D$.

Then $x^2+xy+y^2=\frac{665}{D}$.

Also, $3xy=108$.

By subtracting this last from the preceding, we have

$$x^2-2xy+y^2=\frac{665}{D}-108.$$

As $x-y=D$. $\therefore$ $x^2-2xy+y^2=D^2$.

Consequently, $D^2=\frac{665}{D}-108$.

Or $D^3+108D=665$.

Now 665 can have only two whole number factors, 5 and 133. But $5^2+108=133$.

Therefore, by our theoretical equation,

$D=5$. That is, $x-y=5$. And $xy=36$.

From these equations, 4 and 9 must be the numbers by *simple inspection.*

4. Given the sum of three numbers, in harmonical proportion, equal to 26, and their continual product equal to 576, to find the numbers.

N. B. Three numbers are said to be in harmonical proportion, when the first is to the third, as the difference between the first and second, is to the difference between the second and third.

Thus a, x, b, are in harmonical proportion, when
$a : b :: a-x : x-b$.

And this proportion gives $x=\frac{2ab}{a+b}$.

But x is the harmonic mean of three numbers. *Therefore, to find the harmonic mean between two numbers, take twice the product of the extremes, divided by their sum.*

We are now ready to solve the problem.

Let $x+y=$ *the greater of two extremes*,
and $x-y=$ the difference of the extremes.

Then $\frac{2(x^2-y^2)}{2x}=$ the middle term.

Now by the conditions of the problem, we have

$$2x+\frac{2(x^2-y^2)}{2x}=26. \qquad (1)$$

$$\text{And } \frac{(x^2-y^2)^2}{x}=576. \qquad (2)$$

Reduce equation (1), and extract the square root of equation (2), and we have $2x^2+x^2-y^2=26x$. (3)

$$\frac{x^2-y^2}{\sqrt{x}}=24. \qquad (4)$$

Multiply equation (4) by the square root of x, and subtract it from equation (3), and we have $2x^2=26x-24\sqrt{x}$, or $x^2=13x-12\sqrt{x}$.

Assume $\sqrt{x}=v$, then $x=v^2$, and $x^2=v^4$. The last equa. then becomes, $v^4-13v^2=-12v$, or $v^3-13v=-12$.

Now -12 has two factors, 3 and -4.

And $3^3-13=-4$. That is $m^2+a=n$ referred to the theoretical equations. Hence $v=3$, and $x=9$.

But $\frac{x^2-y^2}{\sqrt{x}}=24$. Or, $\frac{81-y^2}{3}=24$.

$81-y^2=72$. $9=y^2$, or $3=y$.

Therefore, 6, 8, 12, are the the numbers.

5. Find two numbers such that their sum shall be 12, and the difference of their fourth powers, 1776. Ans. 7 and 5.

Put $x+y=$ the greater, and $x-y=$ the less.

6. Given $x^3+6x=88$, to find the values of x.

$88=4\cdot22$. $4^2+6=22$. Ans. $x=4$.

7. Given $x^3+3x=14$, to find x. Ans. $x=2$.

8. Given $x^3+6x=45$, to find x. Ans. $x=3$.

SECTION XVI.

On Arithmetical Progression

The formulas connected with arithmetical progression, are very simple, and drawn out merely from inspection; any great attempt at explanation, serves rather to confuse, than enlighten; and although this does not profess to be an elementary work, we shall gives all the explanation necessary.

Let a represent the first term of any arithmetical series, and d the common difference.

Then a, $(a+d)$, $(a+2d)$, $(a+3d)$, $(a+4d)$, &c., will be the series itself, if ascending. If decending, a, $(a-d)$, $(a-2d($, $(a-3d)$, $(a-4d)$, &c., will represent the series.

Wherever we stop, is the last term.

The first term exists without, and *independent* of the common difference. Therefore, to obtain any term, we add or subtract the common difference *one less times* than the number of terms.

Let L be the last term, and n the number of terms. Then the general formula for the last term will be

$$L=a+(n-1)\,d \qquad (1)$$

Now let S represent the sum of any arithmetical series.

Then $S=a+(a+d)+(a+2d)+(a+3d)$.

Also, $S=(a+3d)+(a+2d)+(a+d)+a$ by simply changing the order of the terms and adding,

$$2S=(2a+3d)+(2a+3d)+(2a+3d)+(2a+3d).$$

That is $2S$ is equal to the *first* and *last* terms of any series *repeated as many times as there are terms*

Now if L is the last term, and n the number of terms, and a the first term agreeable to our notation,

$$S=(a+L)\frac{n}{2}\cdot \quad \text{But } L=a\pm(n-1)\,d.$$

These two equations are sufficient for every thing relating to arithmetical series, and we use them without modification or change.

Some have as many equations as there are letters or terms expressed, to meet every possible case, but our experience in teaching, has convinced us that it is best to take these alone.

Examples.

1. If the sum of an arithmetical series is 1455, the last term 5, and the number of terms 30; what is the common difference? (Day, 218.) Ans. 3.

Some have an equation with d, the term sought, standing on the left; such an equation can be wrought out of the preceding equations, but it is unnecessary. To solve this problem, we have $1455=(5+L)\,15$. Dividing by 15, we have $97=5+L$, or $L=92$.

Now in the next equation, we have

$$92=5+29d. \quad \text{Or } d=3, \text{ the answer.}$$

2. The sum of an arithmetical series is 576, the first term is 7, and the common difference is 2. What is the number of terms? (Day, 218.) Ans. 21.

By equation, $567=(7+L)\frac{n}{2}$ (1)

$$L=7+(n-1)\,2=5+2n. \quad (2)$$

Put this value of L in equation (1), and we have

$$567=(12+2n)\frac{n}{2}$$

Or $n^2+6n=567$. Or $n^2+6n+9=576$.

Therefore, $n+3=24$, and $n=21$. Answer.

3. Find *six* arithmetic means between 1 and 43. (Ryan 302.)

We could find these numbers *if* we knew the common difference. And we can express the problem in entirely different language.

Thus: The first term of an arithmetic series is 1, the last term 43, and the number of terms 8, (6 between 2.)

What is the common difference. From this

$L=a+(n-1)\ d$ we have $43=1+7d$. Or $d=6$.

Therefore, the numbers are 7, 13, 19, 25, 31, 37.

4. Find the *sum* of 36 terms of the series 40, 38, 36, &c.

Ans. 180.

5. What is the sum of 1, $1\frac{1}{2}$, &c., to 32 terms.

Ans. 280.

The following are not so strictly formula problems, but more generally Algebraic.

1. Find four numbers in arithmetical progression, whose sum shall be 56, and the sum of their squares, 864.

The following is Day's solution:

If x=the second of the four numbers,

And y=the common difference,

The series will be $x-y$, x, $x+y$, $x+2y$.

By the conditions $(x+y)+x+(x+y)+(x+2y)=56$.

And $(x-y)^2+x^2+(x+y)^2+(x+2y)^2=864$.

That is, $4x+2y=56$.

And $4x^2+4xy+6y^2=864$.

Reducing these equations, we have $x=12$, and $y=4$. The numbers required, therefore, are 8, 12, 16, and 20.

We are not pleased with the preceding operation, because it can be done shorter.

But, before we commence the solution, we desire to give this *general direction*.

When *three numbers* are in question, in arithmetical progression, let them be represented by $(x-y)$, (x), $(x+y)$.

Here y is the common difference.

When *four numbers* are in question, let them be represented by $(x-3y)$, $(x-y)$, $(x+y)$, and $(x+3y)$. Here $2y$ is the common difference. So in general, take the expressions for the numbers, such that by addition, *the common difference will disappear*. Now for the preceding problem.

We take, $\left.\begin{array}{l} x-3y=1\text{st.} \\ x-y=2\text{d.} \\ x+y=3\text{d.} \\ x+3y=4\text{th} \end{array}\right\}$ Also $\left\{\begin{array}{l} x^2-6xy+9y^2 \\ x^2-2xy+y^2 \\ x^2+2xy+y^2 \\ x^2+6xy+9y^2 \end{array}\right.$

Then $4x=56$ $\qquad$ $4x^2+20y^2=864$

Or $x=14$ $\qquad$ $x^2+5y^2=216$

But $x^2=196$

$5y^2=20$

$y^2=4$

Therefore, $x-3y=8$. $\qquad$ $y=2$

$x-y=12$, &c., the answer.

2. The sum of three numbers in arithmetical progression is 9, and the sum of their cubes is 153. What are the numbers.

$(x-y)+(x)+(x+y)$, or $3x=9$, or $x=3$.

And $x^3-3x^2y+3xy^2-y^3=$cube of the first.

$x^3 =$ " " 2d.

$x^3+3x^2x+3xy^2+y^3=$ " " 3d.

Sum $3x^3 \quad +6xy^2=153$.

Or $x^3+2xy^2=51$.

But $x^3 \quad =27$.

$2xy^2=24$.

$y^2=4$, or $y=2$.

Therefore, 1, 3, and 5, are the numbers.

3. There are four numbers in arithmetical progression, the sum of the squares of the first two is 34, and the sum of the squares of the last two is 130. What are the numbers?

Ans. 3, 5, 7, 9.

Commence with the convenient expressions, and there can be no difficulty.

4. There are four numbers in arithmetical progression,

whose sum is 28, and their continual product is 585. What are the numbers? Ans. 1, 5, 9, 13.

$$\begin{aligned} \text{Let } x-3y&=\text{1st} \\ x-y&=\text{2d} \\ x+y&=\text{3d} \\ x+3y&=\text{4th} \\ \hline 4x&=28 \\ x&=7 \end{aligned}$$

Multiply the 1st into the 4th and we have x^2-9y^2, multiply 2d into the third and we have x^2-y^2, multiply these products together, and we have $x^4-10x^2y^2+9y^4$, their continual product.

Hence, $x^4-10x^2y^2+9y^4=585$.

But x is known, put in its value and reduce the equation, taking the *smallest* value of y, the *other* will not answer.

5. Suppose a man owes $1000, what sum shall he pay daily so as to cancel the debt, principal and interest, at the end of a year, reckoning it at 6 per cent., simple interest?

(Colburn.)

Divide 1000 dollars by 365, and call the quotient a. This would be the sum he must pay daily, provided there was no interest to be paid.

Cast the interest on a for *one day*, at 6 per cent., and call this interest i.

Then the first day he must pay $a+i$.

The second day, $a+2i$.

The third day, $a+3i$.

and so on, in arithmetical progression.

The last day he must pay $a+365i$.

Altogether, he must pay $\left(\frac{2a+366i}{2}\right)365$.

Or he must pay daily, $a+183i$=the answer.

6. A person travels from a certain place at the rate of *one mile* the first day, 2 the second, 3 the third, and so on; and in six days after, another sets out from the same place,

and travels uniformly, 15 miles a day. In how many days will they be together. (Young, page 122.)

Let x=the days after the second starts.

Then $x+6$=the days, also, the number of miles the first travels the last day.

Hence, $(x+7)\frac{x+6}{2}=15x$.

Or, $x^2-17x=-42$. Assume $2a=17$.

Then, $6a-9=42$. Therefore, $x^2-2ax=-6a+9$.

Complete the square, $x^2-2ax+a^2=a^2-6a+9$.

Square root, $x-a=\pm(a-3)$.

Hence, $x=3$, or 14.

They will be together in 3 days, and in 14 days. Reconcile it.

Section XVII.

Geometrical Proportion and Progression.

The terms of a geometrical series increase by a constant multiplier called the ratio. But if the ratio is a fraction, the series will decrease.

a, ar, ar^2, ar^3, ar^4, &c., is a geometrical series. Take any three consecutive terms, as a, ar, ar^2, or ar^3, ar^4, ar^5, &c., and we perceive that the product of the extremes is equal to the square of the mean. Thus $a^2r^2=a^2r^2$, and $a^2r^8=a^2r^3$, &c. When we take four conscutive terms, as a, ar, ar^2, ar^3, or ar, ar^2, ar^3, ar^4, we find that the product of the two extremes is equal to the product of the two means. Thus $a^2r^3=a^2r^3$, and in the other $a^2r^4=a^2r^4$, &c., &c.

If the terms be odd, the product of the extremes is equal to the square of the middle term, *whatever* be the number of terms.

$a : ar = b : br$ is a geometrical *proportion*, not a geometrical series. But here we perceive that the product of the extremes is equal to the product of the means. As we do not disturb the *equality of an equation* by multiplying or dividing both sides by the same number, so we do not destroy a geometrical proportion by multiplying or dividing *the first* or last couplet by any number whatever.

Thus, $an : anr = b : br$. Or, $a : ar = bn : bnr$. The product of the extremes in both cases is still equal to the product of the means.

We may also multiply or divide either one of the first couplet, by any number, provided we multiply or divide the corresponding term of the other couplet by the same number.

Thus, $a : ar = b : br$.

Then, $na : ar = nb : br$.

This does in fact *change* the ratio, *but it is still a perfect geometrical proportion.*

When we have several proportions, containing relative functions, they may be compounded, varied, and changed, in a varity of ways; but it is neither important nor proper, that we should treat them at length in a work like this. In Day's Algebra, and in an appendix to Ryan's Algebra, and in various other works the student will find this subject fully explained.

Now let us simply cast our eyes once more on a geometrical series.

$$a : ar : ar^2 : ar^3 : ar^4 : ar^5 : ar^6.$$

We find the first term is a *factor* in every term. That the ratio with an exponent *one less* than the number of terms from the commencement, is also a factor. Hence the n^{th} term of any series may be expressed thus, ar^{n-1}.

Call L the last term of n terms. Then $L = ar^{n-1}$.

Let S represent the sum of the series.

Then, $S = a + ar + ar^2 + ar^3$, &c.

Multiply this equation by r, and we have

$$rS = ar + ar^2 + ar^3 + ar^4.$$

But, ... $S = a + ar + ar^2 + ar^3$.

By subtraction $(r-1)S=ar^4-a$. Or, $S=\frac{ar^4-a}{r-1}$.

Therefore, the sum of any series is equal to the last term, multiplied by the ratio, the first term subtracted, and the remainder divided by the ratio *less one.*

More generally, as ar^{n-1} expresses the last term

$$S=\frac{ar^n-a}{r-1}.$$

EXAMPLES.

1. What is the sum of the series, 1, 3, 9, &c., to 12 terms. Ans. 265720.

Here $a=1$. $r=3$. $n=12$.

S is sought, which is, $S=\frac{3^{12}-1}{2}=$ answer.

2 What is the sum of ten terms of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, &c.
Ans. $\frac{174075}{59049}$.

$$a=1. \quad r=\tfrac{2}{3}. \quad n=10. \quad S=\frac{(\frac{2}{3})^{10}-1}{\frac{2}{3}-1}.$$

This will result in a minus quantity divided by a minus, which will give plus.

3. Required seven geometric means betwnen 3, and 768.

$L=ar^{n-1}$, is the formula.

In this case $L=768$, $a=3$, and $n=9$; as 7 numbers is required to be placed between 2, therefore there must be 9 terms, when the series is complete.

In the present example, $768=3r^8$. Or, $256=r^8$.

Take the square root, and $16=r^4$. Or, $r=2$.

Therefore, 6, 12, 24, 48, 96, 192, 384, are the numbers.

4. What is the sum of the series, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c., continued to infinity.

In questions of this kind, conceive the series to be reversed, and call the *first term* the last, and in place of the ratio being $\frac{1}{2}$, in this example, it must be 2. As the terms become smaller and smaller, when we arrive at infinity, the last term will be *absolutely nothing*, not *extremely small* as some suppose.

Now $S=\frac{ar^n-a}{r-1}$ is the formula, or $S=\frac{Lr-a}{r-1}$, L being the last term.

In the present example, $L=1$, $a=0$, $r=2$.

Hence, $S=\frac{2}{1}=2$, the answer.

5. What is the sum of 1, $\frac{1}{3}$, $\frac{1}{9}$, &c., to infinity.

$S=\frac{3}{2}=1\frac{1}{2}$, the answer.

6. What is the sum of 35, 7, $\frac{7}{5}$, &c., to infinity.

$$S=\frac{Lr-0}{r-1}=\frac{35\cdot5}{4}, \text{ answer.}$$

The word infinity is very troublesome to many, and they can hardly conceive that the positive sum of an *infinite* series can be found.

They say it is the sum very nearly, as near as we can express it, &c. They imagine some stop must be made, or something left off. But it is not so, *the whole sum is taken to the utmost verge of infinity*, the formula is true only on *condition* that the series does reach infinity, if it stops short of that, the formula would neither conform to logic or truth.

We can investigate this, in the following manner:

Suppose S=A+B+C+D+E, &c., *to infinity.*

But it being a geometrical series, there is a constant ratio between the terms.

That is, A=rB. B=rC. C=rD. D=rE.

And thus the equations would run on to infinity, and *in our reasoning, we do* carry them to infinity.

Now add the equations together, and we have

A+B+C+D, &c., *to infinity*, =r(B+C+D, &c., to infinity.)

But S=A+B+C, &c. And S—A=(B+C+D+E, &c.,) substitute these values in the equation above, and S=r(—A), an equation which includes the series to absolute infinity.

Or, $s=rs-rA$. Or, $(r-1)\,s=rA$.

$s=\frac{rA}{r-1}$.

7. Find the sum of the series $\frac{1}{5}$, $\frac{1}{25}$, $\frac{1}{125}$, &c., to infinity.

Here $A=\frac{1}{5}$. $r=5$. Therefore $s=\frac{1}{4}$, answer.

Proof. $\frac{1}{4}=\frac{1}{5-1}$. Now actually divide 1, by $5-1$.

Thus $5-1)\ 1\ (\frac{1}{5}+\frac{1}{25}$, &c., without end.

$$\begin{array}{l} 1-\frac{1}{5} \\ \hline \quad\frac{1}{5} \\ \quad\frac{1}{5}-\frac{1}{25}. \end{array}$$

8. Find the value of 1, $\frac{3}{4}$, $\frac{9}{16}$, &c., to infinity.

Here $A=1$. $r=\frac{4}{3}$.

Hence, $s=\frac{\frac{4}{3}}{\frac{4}{3}-1}=\frac{4}{4-3}=4$, answer.

9. Find the exact value of the decimal .3333, &c., *carried to infinity.*

The value of the first figure is $\frac{3}{10}=A$.

The value of the second, is $\frac{3}{100}=B$, &c. &c.

But $s=\frac{rA}{r-1}$. $r=10$.

Hence, $s=\frac{3}{9}=\frac{1}{3}$. Proof, 3) 1.0000

.3333 without end.

10. Find the value of 323232 &c., to infinity.

Here, $A=\frac{32}{100}$. $B=\frac{32}{10000}$, therefore, $r=100$.

And $s=\frac{32}{99}$=the answer.

11. Find the value of .777 &c., without end.

$A=\frac{7}{10}$. $B=\frac{7}{100}$. $r=10$. Therefore, $s=\frac{7}{9}$.

12. Find the value .71333 &c., to infinity. Observe the geometrical series does not commence until we pass $\frac{71}{100}$. Then $A=\frac{3}{1000}$. $B=\frac{3}{10000}$, hence, $r=10$.

$$s=\frac{rA}{r-1}=\frac{\frac{3}{100}}{9}=\frac{1}{300}.$$

Consequently, .71333, &c. $=\frac{71}{100}+\frac{1}{300}=\frac{213}{300}+\frac{1}{300}=\frac{107}{150}$.

From the foregoing, it must be observed, that our mathematical infinities must not be confounded with the Theologian's infinity of time and space. The former may be a very insignificant sum, the latter no mind can grasp.

Miscellaneous problems in geometrical proportion.

1. The product of three numbers in geometrical progression, is 64, and the sum of their cubes is 584. What are the numbers?

Let them be represented by x, xy, xy^2, y being the ratio.

Then per question, $x^3y^3=64$, (1)

And $x^3+x^3y^3+x^3y^6=584$ (2)

From equation (1) $y^3=\frac{64}{x^3}$, and $y^6=\frac{4096}{x^6}$.

Subtract (1) from (2) and in the remainder, put in the value of y^6, and we have

$$x^3+\frac{4096}{x^3}=520.$$

Or $x^6-520x^3=-4096$.

Now if x comes out a whole number, as the unknown term generally does in all such problems as this, we can *sometimes* arrive at the result *more elegantly*, than by the common drudge operation.

Compare the numbers 520, and 4096, one is about eight times the other. Assume $2a=520$.

Then $16a-64=4096$.

Now the equation becomes $x^6-2ax^3=64-16a$.

Completing the square $x^6-2ax^3+a^2=64-16a+a^2$.

Extracting the root $x^3-a=8-a$.

$x^3=8$. Or, $x=2$.

But $x^3y^3=64$, or $xy=4$. Hence, $y=2$.

And the numbers are 2, 4, and 8.

2. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the means, as 7 to 3. Required, the numbers. (Day and others.)

Let x=the first number, and y=the ratio.

Then x, xy, xy^2, xy^3, will be the numbers.

By the conditions, $xy^3-xy=24$.

And $xy^3+x : xy^2+xy=7 : 3$.

Divide by x, $y^3+1 : y^2+y=7 : 3$.

Divide the first couplet by $y+1$, and we have

$y^2-y+1 : y :: 7 : 3$. Or, $3y^2-3y+3=7y$.

Or, $3y^2-10y=-3$.

Now, if the ratio is a whole number; that is, if y will come out *without a fraction*, its value is likely to be some *very simple* factor of 3. But, 1 and 3 are all the simple factors 3 has. We know, from the nature of the problem, that y cannot be one. Therefore, try 3. Dividing one side of the equation by y, the other by 3, and $3y-10=-1$. Or, $3y=9$. Or, $y=3$, a proof that the supposition was right.

In this way, we can very often decide the unknown letter of an equation just as surely, and with much less trouble, than by formally working it out.

y being known, the numbers are soon found to be 1, 3, 9, 27.

In Day's Algebra, however, the student arrives at this problem before he has been instructed in the art of compound division—that is, before he is supposed to be able to divide y^3+1, by $y+1$. Such persons must take another method to solve this problem, but one far more lengthy and difficult. But, as the more circuitous method (exception to the general rule) exhibits much mathematical beauty, we present it.

We wish to express four numbers in geometrical proportion by *two letters only*, and have them stand *symmetrically.*

To accomplish this, let x and y represent the two *middle* numbers, and for the *time being*, for the *moment*, let P and Q be the extremes. Then P, x, y, Q, are four numbers in geometrical progression.

Now, consider only the *first three*, and we have *the product of the extremes*, equal to *the square of the mean.*

$Py=x^2$. Or, $P=\frac{x^2}{y}$.

In the same manner, we find $Q=\frac{y^2}{x}$.

Now, we have, $\frac{x^2}{y} : x :: y : \frac{y^2}{x}$, four numbers, expressed by two letters, *standing symmetrically.*

Five geometrical numbers may be represented by two letters, x and y, *standing symmetrically*,

Thus $\frac{x^3}{y}$, x^2, xy, y^2, $\frac{y^3}{x}$.

Six numbers, thus, $\frac{x^3}{y^2} : \frac{x^2}{y} : x=y : \frac{y^2}{x^2} : \frac{y^3}{x^2}$.

These numbers are often necessary to work difficult problems.

Now, let us return to our problem:

The numbers are $\frac{x^2}{y}$, x, y and $\frac{y^2}{x}$, and observe, that if the series is ascending, y must be greater than x.

By the conditions $\frac{y^2}{x}-x=24$

And $\frac{x^2}{y}+\frac{y^2}{x} : x+y=7 : 3$, multiply the first couplet by xy, and we have $x^3+y^3 : xy(x+y)=7 : 3$.

Multiply the consequents by 3, and add to the antecedents, and $x^3+3xy(x+y)+y^3 : 3xy(x+y)=16 : 9$.

Or, $(x+y)^3 : xy(x+y)=16 : 3$.

Divide the first couplet by $(x+y)$

And $(x+y)^2 : xy=16 : 3$.

Or, $x^2+2xy+y^2 : xy=16 : 3$.

Multiply the consequents by 4, and

$x^2+2xy+y^2 : 4xy=16 : 12$.

Subtract the consequents from the antecedents, and take the remainder and antecedents, and

$x^2-2xy+y^2 : x^2+2xy+y^2=4 : 16$.

Extract the square root of every term, recollecting that y must be greater than x, and $y-x : y+x=2 : 4$.

Adding and substituting terms, and $2y : 2x=6 : 2$.

Or, $y : x=3 : 1$. Therefore, $y=3x$.

But, $\frac{y^2}{x}-x=24$. Or, $y^2-x^2=24x$.

Or, $8x^2=24x$. $x=3$.

Now, the two middle numbers are 3 and 9. Therefore, 1, 3, 9, 27, are the numbers.

3. The sum of four numbers in geometrical progression, is 15, and the sum of their squares is 85. What are the numbers?

Let, $\frac{x^2}{y}+x+y+\frac{y^2}{x}=15=a$.
And $\frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=85=b$

Assume, $x+y=s$.
And $xy=p$.
Then, $x^2+y^2=s^2-2p$.
And, $x^3+y^3=s^3-3sp$.

Transposing $x+y$, in the first equation, and x^2+y^2 in the second, and we have

$\frac{x^2}{y}+\frac{y^2}{x}=a-s$. (1) $\frac{x^4}{y^2}+\frac{y^4}{x^2}=b-s^2+2p$. (2)

Square equation (1) and then transpose $2xy$ from the first member and $\frac{x^4}{y^2}+\frac{y^4}{x^2}=(a-s)^2-2p$. (3)

The left hand members of equations (2) and (3) are equal, therefore, $(a-s)^2-2p=b-s^2+2p$.

Or, $a^2-2as+2s^2-4p=b$. (4)

Clear equation (1) of fractions, and

$x^3+y^3=(a-s)p$. But, $x^3+y^3=s^3-3sp$.

Therefore $ap+2sp=s^3$.

Or $p=\frac{s^3}{a+2s}$. (5)

Put this value of p in equation (4) and

$a^2-2as+2s^2-\frac{4s^3}{a+2s}=b$, clearing of fractions we have

$a^3+2a^2s-2a^2s-4as^2+2as^2-4s^3=ab+2bs$.

Or, $a^3-2as^2=ab+2bs$.

Or, $as^2+bs=\left(\frac{a^2-b}{2}\right)a$. Restore the values of a and b and we shall have $15s^2+85s=70\cdot15$.

An equation which gives $s=6$. Hence, $x+y=6$.

But, $xy=p=\frac{s^3}{a+2s}=\frac{6\cdot6\cdot6}{15+2\cdot6}=\frac{2\cdot6\cdot6}{5+2\cdot2}$.

Or, $xy=8$. Therefore, $x=2$. $y=4$, and the numbers are, 1, 2, 4, 8.

4. Find five numbers in geometrical progression, such that their sum may be 31, and the sum of their squares 341.

(Young, page 151.)

Young's Solution.

Let x, xy, xy^2, xy^3, and xy^4, represent the numbers, then by the question

$$x(1+y+y^2+y^3+y^4)=31.$$

$$x^2(1+y^2+y^4+y^6+y^8)=341,$$

Dividing the 2d by the 1st, we have

$$x(1-y+y^2-y^3+y^4)=11.$$

Or, $11=x(1-y+y^2-y^3+y^4)$, multiply this by the first and we have

$$11(1+y+y^2+y^3+y^4)=31(1-y+y^2-y^3+y^4.$$

Dividing this equation by y^2, and

$$11\left(\frac{1}{y^2}+\frac{1}{y}+1+y+y^2\right)=31\left(\frac{1}{y^2}-\frac{1}{y}+1-y+y^2\right)$$

Assume $y+\frac{1}{y}=s$. Then $y^2+\frac{1}{y^2}=s^2-2$,

By substitution, $11(s^2-2+s+1)=31(s^2-2-s+1.)$
By reduction, this equation becomes

$s^2-\frac{21s}{10}=1.$ Completing the square

$s^2-\frac{21}{10}s+\frac{441}{400}=\frac{841}{400}$

Extracting the root.

$s-\frac{21}{20}=\frac{29}{20}.$ Or $s=\frac{5}{2}.$

But $y+\frac{1}{y}=s=\frac{5}{2}$, this equation gives $y=2$. And from the first equation, after substituting the value of y, we have, $x=1$, therefore, the numbers are, 1, 2, 4, 8, 16.

Another Solution.

Numbers that are in geomeritcal progression, are also in geometrical progression after being squared. Now, agreeable to a preliminary explanation,

Let $\frac{x^3}{y}+x^2+xy+y^2+\frac{y^3}{x}=341=a.$ (1)

Then $\frac{x^{\frac{3}{2}}}{\sqrt{y}}+x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y+\frac{y}{\sqrt{x}}=31=b.$ (2)

Assume $x+y=s$, and $xy=p$, then the equations become

$\frac{x^3}{y}+\frac{y^3}{x}=a-s^2+p.$ (3)

And $\frac{x^{\frac{3}{2}}}{\sqrt{y}}+\frac{y^{\frac{3}{2}}}{\sqrt{x}}=b-s-\sqrt{p}.$ (4)

Clear equations (3) and (4) of fractions, and

$x^4+y^4=ap-ps^2+p^2.$ (5)

$x^2+y^2=(b-s)\sqrt{p}-p.$ (6)

Because $x+y=s$, and $xy=p$, $x^2+y^2=s^2-2p.$

And $x^4+y^4=s^4-4s^2p+2p^2.$

Therefere, $(b-s)\sqrt{p}-p=s^2-2p.$ (7)

: $ap-ps^2+p^2=s^4-4s^2p+2p^2.$ (8)

By reduction, these become

$$(b-s)\sqrt{p}=s^2-p. \qquad (9)$$

$$ap=s^4-3s^2p+p^2. \qquad (10)$$

Square equation (9), and we have

$$(b-s)^2p=s^4-2s^2p+p^2$$

Subtract (10), $ap=s^4-3s^2p+p^2$

And $(b-s)^2p-ap=s^2p.$

Dividing by p, $(b-s)^2-a=s^2$. Or, $b^2-2bs+s^2-a=s^2$.

$$\frac{b^2-a}{2b}=s. \quad \text{Or, } s=10.$$

Putting the values of s and b in equation (9), and we find p, or $xy=16$. Therefore the numbers are 1, 2, 4, 8, 16.

5. There are four numbers in geometrical progessions, such, that the product of the first and second, added to the product of the third and fourth, is 582; and the product of the first and third, added to the product of the second and fourth, is 468. What are the numbers? (Y. page 153.)

Take $\frac{x^2}{y}$, x, y, and $\frac{y^2}{x}$ for the numbers.

Then by the conditions $\frac{x^3}{y}+\frac{y^3}{x}=582=a.$

And $x^2+y^2=468=b.$

Clear the first of fractions, and square the last, and mm

$$x^4+y^4=axy. \qquad (1)$$

Also, $x^4+2x^2y^2+y^4=b^2. \qquad (2)$

Subtract (1) from (2), $2x^2y^2=b^2-axy.$

Or, $x^2y^2+291xy=234\cdot468$. An equation which gives $xy=216$.

But $x^2+y^2=468$. Therefere, $x=12$, or 18, and $y=18$, or 12. And the numbers are 27, 18, 12, and 8. Or, 8, 12, 18, 27.

Try this problem with x, xy, xy^2, xy^3, to represent the numbers.

6. Find six numbers in geometrical progression, such,

that their sum shall be 315, and the sum of the two extremes 165. (Ryan, 309.)

Let $\frac{x^3}{y^2}+\frac{x^2}{y}+x+y+\frac{y^2}{x}+\frac{y^3}{x^2}=315.$

$$\frac{x^3}{y^2}+\frac{y^3}{x^2}=165=a.$$

Therefore, $\frac{x^2}{y}+x+y+\frac{y^2}{x}=150=b.$

Assume $x+y=s.\quad xy=p.$

$$x^3+y^3=s^3-3sp.\quad x^5+y^5=s^5-5s^3p+5sp^2.\quad (1)$$

$$x^5+y^5=ax^2y^2=ap^2.\quad (2)$$

$$\frac{x^2}{y}+\frac{y^2}{x}=b-s.\quad (1)\quad x^3+y^3=bp-ps=s^3-3ps.$$

$$\text{Or, } (b+2s)p=s^3.\quad p=\frac{s^3}{b+2s}$$

But from equation (1) and (2), we have

$$ap^2=s^5-5s^3p+5sp^2\quad (3)$$

Or, $(a-5s)p^2=s^5-5s^3p$. Substitue the value of p, and

$$\frac{(a-5s)s^6}{(b+2s)^2}=s^5-\frac{5s^6}{b+2s}.$$

Divide by s^5, and multiply by $(b+2s)$ at the same time, and $\frac{(a-5s)s}{b+2s}=b+2s-5s=b-3s.$

Or, $as-5s^2=b^2-3bs+2bs-6s^2$, or, $s^2+(a+b)s=b^2$.

This equation gives $s=60$.

But $p=\frac{s^3}{b+2s}=800=xy$, and $x+y=60$.

From these equations, we find that $x=20$, $y=40$. And the numbers are 5, 10, 20, 40, 80, and 160.

7. The arithmetical mean of two numbers exceeds the geometrical mean by 13, and the geometrical mean exceeds the harmonical mean by 12. What are the numbers?

Let x and y be the numbers.

$\frac{x+y}{2}=$ the arithmetical mean.

$\sqrt{xy}=$ the geometrical mean.

and $\frac{2xy}{x+y}=$ the harmonical mean.

By the conditions proposed.

$$\frac{x+y}{2}=\sqrt{xy}+13. \quad \sqrt{xy}=\frac{2xy}{x+y}+12.$$

Assume $x+y=s$. And $xy=p$.

Then $\frac{s}{2}=\sqrt{p}+13$. (1) And $\sqrt{p}=\frac{2p}{s}+12$. (2)

From equation (1), $s=2\sqrt{p}+26$, which place in equation (2), and $\sqrt{p}=\frac{2p}{2\sqrt{p}+26}+12$.

Clear of fractions, $2p+26\sqrt{p}=2p+24\sqrt{p}+12\cdot26$.

Drop common values, and $2\sqrt{p}=12\cdot26$. Or $\sqrt{p}=12\cdot13$

But $s=2\sqrt{p}+26=12\cdot26+26=13\cdot26=13^2\cdot2$.

That is, $x+y=2\cdot13^2$. $xy=13^2\cdot12^2$.

Preserve 13, and 12 as distinct factors to the last, x will come out equal to $18\cdot13$, which is 234, and $y=104$.

Another Solution.

Let $x-y$, and $x+y$ represent the numbers.

Then $x=$ the arithmetical mean, $\sqrt{x^2-y^2}=$ the geometrical mean, and $\frac{x^2-y^2}{x}=$ the harmonical mean.

(1) $x-13=\sqrt{x^2-y^2}$, and $\frac{x^2-y^2}{x}+12=\sqrt{x^2-y^2}$ (2)

Subtract (1) from (2), transpose and multiply by x, and we shall have $y^2=25x$.

Put this value of y^2 in equation (1) and square, and we shall have $x^2-26x+13\cdot13=x^2-25x$.

Hence $x=13\cdot13=169$. Therfore, $y=65$.

8. The sum of five numbers in geometrical progression is 62, their fourth difference is 2. What are the numbers?

(From the Mathematical Diary, No. 1, page 192.)

The proposer demanded that the solution should not involve a higher equation than a quadratic. There are five different solutions in the Diary. The following by Mr. James Foster, then late of Belfast, Ireland, we regard as the best.

Let $x=$ the first term, $r=$ the ratio, $s=$ the sum of the terms, and $d=$ the fourth difference.

Then, - - $x+rx+r^2x+r^3x+r^4x=s.$

$$x-4rx+6r^2x-4r^3x+r^4x=d.$$

By subtracting, $5rx-5r^2x+5r^3x=s-d.$

Or, $$rx-r^2x+r^3x=\frac{s-d}{5}.$$

Subtracting this last from the first gives

$$x+2r^2x+r^4x=\frac{4s+d}{5}.$$

Therefore, $x=\frac{4s+d}{5(r^4+2r^2+1)}$.

And from the 2d equation $x=\frac{d}{r^4-4r^3+6r^2-4r+1}$

$\frac{4s+d}{5(r^2+1)^2}=\frac{d}{(r-1)^4}$. Whence $\left(\frac{4s+d}{5d}\right)^{\frac{1}{2}}=\frac{r^2+1}{(r-1)^2}$.

By putting $\left(\frac{4s+d}{5d}\right)^{\frac{1}{2}}=a=\frac{r^2+1}{r^2-2r+1}$.

Clearing of fractons, $ar^2-2ar+a=r^2+1.$

$$(a-1)r^2-2ar=1-a.$$

By division, $r^2-\frac{2ar}{a-1}=-1.$

Completing the square, and extracting the root, &c.

$$r=\frac{a}{a-1}\pm\left(\frac{2a-1}{(a-1)^2}\right)^{\frac{1}{2}}$$

This question, says the Diary, was solved exactly in the same manner, by Messers. John Delafield, Jun., C. O. Pascalis, Farrell Ward, Mary Bond, William J. Lewis, and George Alsop.

Our Solution.

Observe our expression for five geometrical numbers heretofore explained, and we have,

$$\frac{x^3}{y}+x^2+xy+y^2+\frac{y^3}{x}=62.$$

And - - - $$\frac{x^3}{y}-4x^2+6xy-4y^2+\frac{y^3}{x}=2$$

Subtract and $5x^2-5xy+5y^2=60.$

Or, - - - - - $x^2-xy+y^2=12.$ (1)

Subtract this last from the first, and we have

$$\frac{x^3}{y}+2xy+\frac{y^3}{x}=50. \quad (2)$$

Clear equation (2) of fractions, and

$$x^4+2x^2y^2+y^4=50xy. \quad (3)$$

Transpose xy in equation (1), and then square it, and

$$x^4+2x^2y^2+y^4=144+24xy+x^2y^2 \quad (4)$$

Whence, $x^2y^2-26xy=-144.$

This equation readily gives $xy=18$ or 8, 8 is the value that answers the conditions.

From equation (1), putting in the value of xy, we have $x^2+y^2=20.$ Hence, $x^2+2xy+y^2=36.$ $x+y=6.$

$$x-y=2. \quad \text{Or,}\ y-x=2.$$

As y must be greater than x, we therefore have $x=2$. $y=4.$ $x^2=4.$ $y^2=16.$

The numbers are 2, 4, 8, 16, and 32.

SECTION XVIII.

Cubic Equations.

Cubic equations are of two forms:

Thus, $x^3+ax^2+bx=c.$ And, $x^3+ax=b.$

The first form has no *direct* solution, but all such equations can be reduced to or be made to take the *other form*, and to that, we can find a direct solution.

To make the first equation take the form of the second,
Assume, $x=y-\frac{a}{3}$, that is, the unknown letter put equal to another unknown, and $\frac{1}{3}$ of the coefficient of the second power with a *contrary sign.*

If the equation had been $x^3-ax^2+bx=c$, then we should have assumed $x=y+\frac{a}{3}$.

But to return to our equation, $x=y-\frac{a}{3}$.

Therefore,
$$x^3=y^3-ay^2+\frac{a^2}{3}y-\frac{a^3}{27}.$$
$$ax^2=\quad ay^2-\frac{2a^2}{2}y+\frac{a^3}{9}.$$
$$bx=\quad by-\frac{ab}{3}.$$
$$x^3+ax^2+bx=y^3+a'y+b'=c.$$

A cubic equation wanting the 2d power of the unknown quantity We have shown in section 15, how to solve cubic equations of this form, *where there is* a particular *relation existing* between the known term and the coefficient of the single power of the unknown; but we are now to show a solution *whatever be the relation* between these quantities.

Let us take the general equation, $x^3+ax=b$.

x must have some value, some number must express the value, and as any number can be divided in two parts, we can suppose that $x=y+v$. We can also suppose the value of x to be so divided that the product of the two parts may be *any number* less than $\frac{x^2}{4}$. We assume then $3yv=-a$.

But if the general equation had been $x^3-ax=b$, we should have assumed, $3yv=a$.

Now, because $x=y+v$,
$$x^3=y^3+3y^2v+3yv^2+v^3.$$

$$\text{Or,} \quad \begin{array}{l} x^3=y^3+v^3+3yv(y+v) \\ +ax= \qquad -3yv(y+v) \\ \hline x^3+ax=y^3+v^3=b. \end{array}$$

Square, and $y^6+2y^3v^3+v^6=b^2$.

But, $4y^3v^3 = -\frac{4a^3}{27}$.

Subtract, $y^6-2y^3v^3+v^6=b^2+\frac{4a^3}{27}$.

Extract square root, $y^3-v^3=\left(b^2+\frac{4a^3}{27}\right)^{\frac{1}{2}}$ (1)

But, $y^3+v^3=b.$ (2)

Add, and $2y^3=b+\left(b^2+\frac{4a^3}{27}\right)^{\frac{1}{2}}$

$$y=\left\{\frac{b}{2}+\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$$

Subtracting equation (1) from (2), dividing &c.,

$$v=\left\{\frac{b}{2}-\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$$

Hence $x=\left\{\frac{b}{2}+\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}+\left\{\frac{b}{2}-\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$

But as we have $3yv=-a$. $v=-\frac{a}{3y}$.

Then $x=\left\{\frac{b}{2}+\left(\frac{b^2}{4}+\frac{a^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}-\frac{a}{3\left\{\frac{b}{2}+\left(\frac{b^2}{4}+\frac{v^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}}$

This is Carden's rule. It was first given to the world by an Italian of that name, without explanation, or the process by which it was arrived at, and it troubled the mathematicians a long time, before they were able to prove it.

Prof. J. R. Young, an English mathematician of decided eminence, has appearently attempted to superceed *Carden's*

Rule. He has investigated cubic equations, with ability, and has given a very ingenious theory to enable us to make more brief and expeditious solutions. No doubt Mr. Young and those who once became familiar with his theory, and acquire considerable practical experience in his mode of operation, are able to do so.

But his theory becomes complex when carried far out in detail, and its practical operation, when extended to many places of figures, is quite difficult to learn, and for these reasons, (although it may be the best when learned,) we think it will not generally take the place of Carden's rule, which is easily understood.

Those who wish to study Mr. Young's method, will find it in Young's Algebra, published by Cary & Lea, Philadelphia, 1832; we would extract a portion of it, were it within the limits or spirit of this work.

Examples.

1. Given $x^3+24x=250$, to find x. Ans. $x=5.05$.

All we have to do is to make $a=24$ in the formula, and $b=250$; reduce down and the equation is solved, and so with any other equation in *this form*.

2. Given $x^3-6x^2+18x=22$, to find the values of x.

Before we apply the formula, we must work off $-6x^2$. To do which, assume $x=y+2$, as before explained, and we shall have another equation in the proper form.

y will equal $.3274+$. Hence $x=2.3274+$.

3. Given $x^3+30x=117$, to find x.

Ans. 3, or $-\frac{3}{2}\pm\frac{7}{2}\sqrt{-3}$.

4. Given $x^3+9x=270$, to find x.

Ans. 6, or $-3\pm6\sqrt{-1}$.

These two last and several others are given in Ryan's Algebra, to be solved by Carden's rule, but they can be much more readily solved by directions given in section 15.

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Professor Harney of Louisville, Ky., gives the following very ingenious solution of this kind of cubic equations.

He represents the general equation under its four possible changes. Thus:

(1) $x^3-3px=2s.$ $x^3-3px=-2s.$ (2)

(3) $x^3+3px=2s.$ $x^3+3px=-2s.$ (4)

He now assumes $x=\frac{p}{u}+u$. That is, he takes $x=$ to $\frac{1}{3}$ of the coefficient of x divided by u plus u.

Cube the assumed equation, and we have

$$x^3=\frac{p^3}{u^3}+3p\left(\frac{p}{u}+u\right)+u^3.$$

$$-3px=\quad-3p\left(\frac{p}{u}+u\right)$$

Add - - $x^3-3px=\frac{p^3}{u^3}+u^3=2s.$

Or, $u^6-2su^3=-p^3$, a quadratic.

Hence, $u^3=s\pm\sqrt{s^2-p^3}$. Or, $u=(s\pm\sqrt{s^2-p^3})^{\frac{1}{3}}$

Therefore $x=\dfrac{p}{(s\pm\sqrt{s^2-p^3})^{\frac{1}{3}}}+(s\pm\sqrt{s^2-p^3})^{\frac{1}{3}}$

An equation *essentially the same* as the last form of Carden's rule.

Of course we cannot discuss equations of the fourth, fifth, and higher degrees, and we would only observe, that however ingenious any exact theoretical operation may be, solutions by approximation appear to us, to be the most practical and expeditious, in all very difficult cases.

The Indeterminate and Diophantine Analysis, are not found in many algebras in common use, and as this book may fall into the hands of many who are perfectly able to comprehend these subjects, but never yet had the elementary principles presented to them, we shall, for this reason, treat in an elementary manner.

It is true that these branches of algebra are of little practical utility; but as they require pure logic, and close discriminating thought, we recommend them as a disipline of mind, as highly valuable as any other sections of algebra.

SECTION XIX.

On Indeterminate Equations.

For the complete solution of a problem, we must have as many independent equations as unknown quantities to be determined.

When this is not the case, the problem is indeterminate. For example, $x+y=20$. x may be one or two or three, or $\frac{1}{2}$, or any other number, whole or fractional, under 20, and y will take the remaining part of 20, and the equation is indeterminate in the strictest sense of the term.

If, however, we restrict the values of x and y, to whole or *integer* numbers in the equation $x+y=20$, x cannot have more than 20 different values, when, without this restriction x might take an infinite number of values, and still preserve the equation $x+y=20$.

In some cases, the number of answers to an equation may be infinite, and the particular values restricted to integers. The following is a general case of the kind

$ax-by=c$. This gives $x=\frac{c+by}{a}$, where y may be any value whatever, that will give $\frac{c+by}{a}$, a whole number, but numberless such values of y can be found, as $c+by$ is a magnitude that can rise higher and higher, without limit, according to the assumed value of y.

But take y what we will, and the equation still exists, and therefore, the *number of answers for x is unlimited or infinite.* In such cases, however, *the least values* of x and y are required, and that is a definite problem.

In equations like the following, $ax+by=c$, the number of answers in integer numbers, *may* be *very limited*, may be only one, or may be impossible. The equation gives

$$x=\frac{c-by}{a}.$$

Now, if c is very large, and b and a small, y *may take* many different values before $c-by$ is so small that we cannot divide it by a, and obtain an interger quotient.

When c is not large in reference to b and a, we may obtain only one value of y and x, and if by making $y=1$, we find $\frac{c-b}{a}$, a proper fraction, the problem is impossible.

EXAMPLE.

$3x+5y=13$. $x=\frac{13-5y}{3}$. If we take $y=1$. $x=\frac{8}{3}$, not an integer, therefore, y must not be taken equal to one. Take $y=2$, then $x=1$, both interger values, and the only interger values that will answer the conditions of the equation. $3x+5y=6$, is an equation in which it is impossible to give integer values to both x and y, because $3+5$ is more than 6, or $a+b$ greater than c or $\frac{c-b}{a}$ is a proper fraction.

The equation $ax+by=c$, is always possible in *integers*, when c is greater than $(ab-a-b)$, and a and b prime to each other. The equation *is sometimes* possible, when c is not greater than $(ab-a-b.)$

The equation $7x+13y=71$, is impossible in integers, for both x and y. But the equation $7x+13y=27$, *is possible*, $x=2$ and $y=1$. Here a and b are the same in both equations, but c less. In the first equation, $c=71$; c is not large enough to make the equation possible, for c, 71, is not greater than (7·13—20), but if c was *any number* greater than 71, the equation would be possible in integers, and it is possible with *some numbers* less than 71.

If a and b are not prime to each other in the equation $ax+by=c$, c must be divided by the same number that divides a and b, or the equation is *impossible* in integers for x and y.

EXAMPLES.

1. Given $6x+9y=32$. Or, $2x+3y=\frac{32}{3}$.

But if x is a whole number, $2x$ will be a whole number, and by the same considerations, $3y$ must be a whole number, and two or more *whole numbers added together can never make a fraction,* therefore the equation $2x+3y=\frac{32}{3}$, or $6x+9y=32$, is impossible in integers.

In cases where solutions are possible, our rules of operation rest entirely on these considerations:

1st. *A whole number added to a whole number, the sum is a whole number.*

2d. *A whole number taken from a whole number, the remainder is a whole number.*

3d. *Multiply a whole number by a whole number, and the product is a whole number.*

For instance, if x is a whole number, $2x$, $3x$, $4x$, or any integral number of x is a whole number.

2. Given $3x+5y=35$, to find the x and y in whole numbers. $x=\frac{35-5y}{3}$. But as x is a whole number, its equal, or $\frac{35-5y}{3}$, must equal *some whole number.*

But $\frac{35-5y}{3}=11-y+\frac{2-2y}{3}$.

Now $11-y$ being a whole number take it away, and the remainder $\frac{2-2y}{3}$, must also be a whole number.

But y being a whole number, $\frac{3y}{3}$ is a whole number,

Therefore, $\frac{3y}{3}+\frac{2-2y}{3}=\frac{y+2}{3}$ is a whole number, which number call p. And $\frac{y+2}{3}=p$. Or, $y=3p-2$.

For the least value of y make $p=1$, and y will equal 1, and $x=\frac{35-5y}{3}=10$. Make $p=2$, then $y=4$, and $x=5$. Make $p=3$, then $y=7$, and $x=0$.

Hence, $y=1$ or 4, and $x=10$ or 5, are the only results this equation admits of.

In operating on the fractional expression $\frac{35-5y}{3}$, it was *our object* to *work down* the coefficient of y to 1. To accomplish this object, we cast out whole numbers, add and subtract whole numbers *in the shape of fractions*, &c., only taking care to keep expressions that are equal to integers, until the coefficient of y becomes one.

3. Given $35x-24y=68$, to find the *least values* of x and y in integers.

The number of answers in this case is unlimited.

$$x=\frac{68+24y}{35}=1+\frac{33+24y}{35}.$$

Hence, $\frac{33+24y}{35}=$ some whole number; but $\frac{35y}{35}$ is also a whole number. $\frac{35y}{35}-\frac{33+24y}{35}=\frac{11y-33}{35}=$ an integer.

$$\text{Or, } \frac{33y-99}{35}=\frac{33y-29}{35}-2.$$

$\frac{35y}{35}-\frac{33y-29}{35}=\frac{2y+29}{35}=$ a whole number, multiply by 18, and $\frac{36y+18\cdot29}{35}=y+14+\frac{y+32}{35}=$ a whole number.

Therefore $\frac{y+32}{35}=p$. Or, $y=35p-32$.

Take $p=1$ for the least value of y, and $y=3$. Therefore $x=4$.

4. A man wishes to lay out \$500 for cows and sheep: the cows at 17 dollars per head, and the sheep at 2 dollars. How many of each did he purchase?

Let $x=$ the number of cows, and y the number of sheep. Then $17x+2y=500$. We know this equation is restricted to whole numbers; because the man could not have part of a cow, or part of a sheep.

To find the least number of cows, that he must buy, transpose $17x$, &c., $y=250-8x-\frac{x}{2}$.

Now as y must be a whole number, and $250-8x$ must also equal some whole number, $\frac{x}{2}$ must be a whole number. That is, the number of cows must be *an even number*; because the number must be divided by 2 without a fraction.

Hence, $\frac{x}{2}=p$. Or, $x=2p$. Make $p=1$, and $x=2$, the least number of cows. Then $y=233$, the corresponding number of sheep.

Now if the man wished to purchase as few sheep, and as many cows as possible, we should transpose the other term, thus: - - - - - $x=\frac{500-2y}{17}=29+\frac{7-2y}{17}$.

Therefore, $\frac{7-2y}{17}=$ a whole number. Multiply by 8, and $\frac{56-16y}{17}=$ a whole number, to which add $\frac{17y}{17}$ and we have $\frac{56+y}{17}=3+\frac{5+y}{17}$. Drop off the whole number 3, then $\frac{5+y}{17}=p$. Or, $y=17p-5$. Making $p=1$, gives $y=12$, the smallest number of sheep. This gives $x=28$, the corresponding number of cows.

The number of cows or x, may be any one of the even numbers from 2 to 28.

5. A man wished to spend 100 dollars in cows, sheep, and geese; cows at 10 dollars a piece, sheep at 2 dollars, and geese at 25cts., and the aggregate number of animals to be 100. How many must he purchase of each?

Let $x=$ the number of cows, y the sheep, and z the geese.

Then - - - $10x+2y+\frac{z}{4}=100.$ (1)

And - - - - $x+y+z=100.$ (2)

Clear equation (1) of fractions, and

$$\begin{array}{r} 40x+8y+z=400. \\ x+y+z=100. \\ \hline 39x+7y \quad =300. \end{array}$$

$$x=\frac{300-7y}{39}=7+\frac{27-7y}{39}=\text{a whole number.}$$

Or, $5\left(\frac{27-7y}{39}\right)=\frac{135-35y}{39}=$ a whole number, add $\frac{39y}{39}$

and $\frac{4y+135}{39}$, or $\frac{40y+1350}{39}=y+34+\frac{y+24}{39}=$ a whole number.

Therefore, $\frac{y+24}{39}=p$. Or, $y=39p-24=15$.

This value of y, gives $x=5$. Hence, $z=80$.

If we take $p=2$, we shall have $y=54$; then x will come a minus quantity an inadmissible circumstance in any problem like this. Therefore, 5 cows, 15 sheep, and 80 geese, is the only solution.

6. A person spent 28 shillings in ducks and geese; for the geese he paid 4s. 4d. a piece, and for the ducks, 2s. 6d. a piece. What number had he of each?

Let $x=$ the number of geese, and y the number of ducks.

Then $52x+30y=28\cdot12$. Or, $26x+15y=168$.

We will now show another operation to reduce this kind of equations.

Take the lowest coefficient, (in this example it is 15,) and observe whether it will divide the other numbers or not. If it will divide, *reserve* the whole numbers, and omit the fractions. In this case, 15 *will not* divide 26, and will divide 168. It will go 11 times, *disregard* the remainder.

Now assume $p=x+y-11$. Then, since x and y are whole numbers, p must be a whole number. Multiply by 15, and transpose, and we have

$$15p-15x-15y=-165.$$
$$26x+15y=\ 168.$$

By addition, $15p+11x\quad=3.$ Assume $q=x+p$.
Then - - - - $11q-11p-11x=0.$
Sum - - - - $11q+4p\quad=3.$ Assume $r=p+2q$
Or, - - - - - - $4r-4p-8q=0.$
Sum - - - - - $4r\quad+3q=3.$ Assume $s=r+q-1$
Or, - - - - - - $3s-3r-3q=-3.$
Sum, - - - - - $3s+r\quad=0.$ Or, $r=-3s.$

Now having worked down to unity for a coefficient, the problem is essentially reduced. We can make $s=0$, hence $r=0$. Then in the last assumed equation $q=1$, and in the equation, $r=p+2q$, gives $p=-2$; and in the preceding assumed equation, $q=x+p$, that is $x=3$, and the first assumed equation gives $y=6$.

Hence 3 geese and 6 ducks is the answer, and no other numbers will do.

7. Divide the number 100 into two such parts, that one of them may be divisible by 7, the other by 11.

Let $7x=$ one part, and $11y=$ the other.

Then $7x+11y=100$, and x and y must be whole numbers. Assume $p=x+y-14$. (1)

Then - - - - - - $7p-7x-7y=-98.$
But - - - - - - - $7x+11y=\ 100.$
By addition, - - $7p\quad+4y=2.$

Assume $q=p+y$. (2)

Then - - - - - - $4q-4p-4y=0.$
Add, and - - - - $4q\quad+3p=2.$

$$4q+3p=2.$$

Assume $r=q+p$. (3)

Then - - - - - - $3r-3q-3p=0.$

By addition - - - $3r+q=2.$ Or, $q=2-3r.$

Take $r=0$, then $q=2$, and $p=-2$. And from equation (2), $2=-2+y$, or $y=4$, and $11y=44$, one of the numbers, and of course 56 is the other,

8. Find a number which being divided by 6, shall leave the remainder 2, and the same number divided by 13 shall leave the remainder 3.

Consider that in division, the divisor and quotient multiplied together, and the remainder added, gives the number divided.

Let N represent the number divided, x and y the quotients.

Then $6x+2=N$, and $13y+3=N$.

Consequently, $6x-13y=1$, an equation in which x and y must be whole numbers, because they represent the whole numbers of the division.

Assume $p=x-2y$. We take $2y$ because 6 is contained in 13, twice.

Then - - - - - - $6p-6x+12y=0.$

And - - - - - - $6x-13y=1.$

Add, and - - - - $6p-y=1.$ Or, $y=6p-1.$

For the smallest value of y we must take $p=1$. Then $y=5$, and $13y+3=68$, the answer.

9. What number is that which being divided by 11, leaves a remainder of 3, divided by 19, leaves a remainder of 5, and divided by 29, shall leave a remainder of 10.

Let N be the required number, and x, y, and z the several quotients, and of course they must be whole numbers.

Then $11x+3=N$, and $19y+5=N$, and $29z+10=N$.

Hence, $x=\frac{29z+7}{11}$, and $x=\frac{19y+2}{11}$. $19y=29z+5$.

Or, - - - - - - - - $19y-29z=5$.

Assume $p=y-z$. (1)

Then - - - - $19p-19y+19z=0$.

Add, and - - $19p \quad -10z=5$.

Assume $q=p-z$. (2)

Then - - - - $10q-10p+10z=0$,

By addition - $10q+9p \quad =5$.

Assume $r=q+p$. (3)

Then - - - - $9r-9q-9p=0$.

Add, and - - - $9r+q \quad =5$. Or, $q=5-9r$.

By returning to equations (3) and (2), we find $p=10r-5$, and $z=19r-10$. Not only must z be a whole number, but to make x a whole number, $\frac{29z+7}{11}$ must be a whole number. Substituting the value of z in this last expression, and we have $\frac{551r-283}{11}$ a whole number, or $50r-25+\frac{r-8}{11}$ a whole number. Therefore, $\frac{r-8}{11}$ must be a whole number, which call t; then $r=11t+8$. Let $t=0$, then $r=8$, $z=142$, and $29z+10=N=4148$, the number.

10. Required the least number that can be divided by each of the nine digits without remainders.

Let $x=$ the number.

Then $\frac{x}{2}$, $\frac{x}{3}$, $\frac{x}{4}$, $\frac{x}{5}$, $\frac{x}{6}$, $\frac{x}{7}$, $\frac{x}{8}$, $\frac{x}{9}$, must all be whole numbers.

Now if we make $\frac{x}{8}$ a whole number $\frac{x}{4}$ and $\frac{x}{2}$, the double, and quadruple will be whole numbers of course. Also if $\frac{x}{9}$ is a whole number, $\frac{x}{3}$ *will be* a whole number.

Therefore we have only to find such a value of x as will make $\frac{x}{9}$, $\frac{x}{8}$, $\frac{x}{7}$, $\frac{x}{6}$, $\frac{x}{5}$ whole numbers. $\frac{x}{6}$, may also be cast out, on consideration that $6=2\cdot3$, and $2\cdot3$ are factors, one of 9, the other of 8, in preceding expressions.

Hence we have only to make $\frac{x}{9}$, $\frac{x}{8}$, $\frac{x}{7}$, $\frac{x}{5}$, whole numbers. Put $x=9p$.

Then $\frac{9p}{8}=p+\frac{p}{8}$. Hence, $p=8q$. Then $x=9p=72q$. $\frac{72q}{7}=$ a whole number. $\frac{2q}{7}=$ a whole number, or $\frac{8q}{7}=q+\frac{q}{7}$. Make $\frac{q}{7}=r$, or $q=7r$. In the same way we find $r=5s$. Take $s=1$, then $r=5$. $q=35$. $x=72 \cdot 35=2520$.

N. B. By the aid of the Indeterminate analysis, combined with some well known properties of numbers, we may sometimes solve miscellaneous problems in a very summary manner. For example, we refer to the following.

11. The product of five numbers in arithmetical progression is 10395, and their sum is 35. What are the numbers?

As the number of terms is odd, let x represent the middle term, and y the common difference. Then $(x-2y)$, $(x-y)$, x, $(x+y)$, $(x+2y)$, will be the numbers, and their sum $5x=35$, or $x=7$.

Now as 10395 is the product of all the numbers, 7 must be one of its factors. Therefore divide by 7, and we have 1485 for the product of the four remaining terms, but as this number ends in a 5, it can be divided by 5, and it is more than probable that 5 is one of the numbers, and the one preceding 7, therefore, 2 is the common difference, and 3, 5, 7, 9, and 11, the numbers.

12. Given the sum of the squares of three numbers $=195$, the sum of their cubes $=1799$, and their continued product $=385$, to find the numbers.

By the common rules, without considering any circumstances, this problem would produce equations of high order, and difficult of solution. But let us call to mind the fact, that *the numbers cannot be fractional*, if they were, their squares, cubes, and product *could not all come out whole numbers*. Also, some of the numbers must be under 10;

if even two of them were over 10, the cubes and product must be larger than the numbers mentioned. From considerations like these, we decide that the answer must be whole numbers, and some of them under 10. Now as the product of the three is 385, and the sum of their squares is 195, *both numbers* ending in 5, *it is so probable* that one of the numbers is 5, that we shall so consider it, and let x and y be the other two numbers, then $x^2+y^2+25=195$, and $5xy=385$. Equations from which we readily find one number to be 7, the other 11.

Now by trial we find $5^3+7^3+11^3=1799$, and therefore these *are in fact* the numbers.

SECTION XX.

To determine the number of solutions an equation may admit.

It has already been observed that an equation in the form of $ax-by=c$ admits of an infinite number of solutions, as $x=\frac{c+by}{a}$ a quantity all positive, and the only restriction is to assume y of such a value that the numerator may be divided by a.

But equations in the form of $ax+by=c$. Then $x=\frac{c-by}{a}$ and the numbers of solutions depends on the relative values of c, b, and a.

If c be very large in relation to b and a, as we have before observed, the equation may have many solutions, otherwise not.

We come now in a general manner to determine the number of solutions an equation of this form may have.

Let $ax+by=c$. Assume $ax'-by'=1$. Which equation is always possible, and from which x' and y' can be known in integers.

Multiply the assumed equation by c, and $acx'-bcy'=c$. Put the two values of c equal to each other, and

$$ax+by=acx'-bcy'$$

$$x=cx'-b(y+cy'). \quad \text{Or, } x=cx'-bm. \qquad (1)$$

$$y=a\left(\frac{cx'-x}{b}\right)-cy'. \quad \text{Or, } y=am-cy'. \qquad (2)$$

In these theoretical equations, (1) and (2), m has different values, it being an arbitrary number taken at pleasure, so that cx' may be greater than bm, and am greater than cy' to render x and y positive.

But if no such value of m can be found, it is proof that values of x and y do not exist in positive integers, and on the contray as many suitable values of m as can be found, so many solutions will the equation admit of, and no more.

Now as - - - $mb<cx'$, and $am>cy'$

Or, - - - - - $m<\frac{cx'}{b}$, and $m>\frac{cy'}{a}$

That is m at the same time is found to be greater and less than known quantities, therefore its *limit* or *range* is found.

For instance, if m must be greater than 30, and less than 40, we conclude that it may be any number between 30 and 40, and the *number of different values it can take* is 9.

We perceive that the difference between the integral parts of $\frac{cx'}{b}$ and $\frac{cy'}{a}$ will express the range of m, and the number of different solutions which the equation admits of, (except in certain cases;) as m is more than one of these fractions and less than the other, the difference between the expressions $\frac{cx'}{b}$ and $\frac{cy'}{a}$ is sometimes *one more* than the number of different values of m, such is the case when $\frac{cx'}{b}$ is an integer, in such cases, subtract one from the difference of these quantities for the range of m, but this case very seldom occurs.

N. B. In making use of the expression $\frac{cx'}{b}$ and $\frac{cy'}{a}$ care must be taken, not to take their difference as *factional expressions*, their *absolute difference is not wanted*, it is the difference between the *integral parts of* $\frac{cx'}{b}$ and $\frac{cy'}{a}$.

EXAMPLE.

Required the number of integral solutions to the equation $7x+9y=100$.

Find the least value to x', y' in the equation $7x'-9y'=1$.

$7x'-9y'=1$. Assume $p=x'-y'$

Then - - $7p-7x'+7y'=0$.

Add, and $7p \quad -2y'=1$. Assume $q=3p-y'$

Then - - $2q-6p+2y'=0$.

Add, and $2q+p=1$. Or, $p=1-2q$.

Take $q=0$, then $p=1$. $y'=3$. $x'=4$.

Then $\frac{cx'}{b}=\frac{100\cdot4}{9}$ and $\frac{cy'}{a}=\frac{100\cdot3}{7}$.

That is $\frac{cx'}{b}=44\frac{4}{9}$ and $\frac{cy'}{a}=42\frac{6}{7}$.

Disregarding the fractions, the difference of the *integral parts* is 2, that is *there are two integral solutions to the equation.*

If we had taken the difference between $\frac{400}{9}-\frac{300}{7}$, *in a fractional form*, thus: $\frac{2800}{63}-\frac{2700}{63}=\frac{100}{63}=1\frac{37}{63}$.

Here the *integral difference* is one, which without this caution might be taken for the number of solutions. The *integral difference* in this case is not the *difference of the integrals.*

Observe in this example $44\frac{4}{9}$ and $42\frac{6}{7}$, the *fractionl part* of $\frac{cx'}{b}$ is $\frac{4}{9}$, and the fractional part of $\frac{cy'}{a}$ is $\frac{6}{7}$, the former is less than the latter, in such cases the integral parts must be taken seperately.

But when the fractional part of $\frac{cx'}{b}$ is not less than the fractional part of $\frac{cy'}{a}$, but equal or greater than it, we may find the number of solutions by taking the difference of the expressions $\frac{cx'}{b}-\frac{cy'}{a}$. Reduce to a common denominator, and take the difference of the numerators, and we will have $\frac{c(ax'-by')}{ab}$; but $ax'-by'=1$. Therefore, we have $\frac{c}{ab}$ for the number of solutions, at once.

EXAMPLE.

What number of integral solutions will the equation $9x+13y=2000$ admit of? Ans. 17.

$$9\times 13=117)\,2000\,(17.$$

But as we cannot know whether the fractional part of $\frac{cy'}{b}$ is not less than the fractional part of $\frac{cy'}{a}$ we cannot be sure that dividing c by ab will give the true number of solutions. It either will be the true number *or one less.*

The equation $5x+9y=40$ admits of no solution in whole numbers, $c=40$ will not be divided by $ab=45$.

Now take the equation $5x'-9y'=1$. And we find $x'=2$, and $y'=1$.

Therefore, $\frac{cx'}{b}=\frac{80}{9}=8\frac{8}{9}$, and $\frac{cy'}{a}=\frac{40}{5}=8$. Now as there is no difference between these integral parts, it indicates as it should, that there is no solution.

But let us take $5x+9y=37$, the same equation, except a smaller value of c. If c would not divide by ab before, much less will it now. Yet in this last equation we have a solution. $x'=2$, $y'=1$, as before, and $c=37$, $\frac{cx'}{b}=\frac{74}{9}=8\frac{2}{9}$ and $\frac{cy'}{a}=\frac{37}{5}=7\frac{2}{5}$. Here the difference of the integrals is 1, and indicates one solution. $x=2$, $y=3$.

How many solutions will the equation $2x+5y=40$ admit of?

The auxiliary equation $2x'-5y'=1$, gives $x'=3$, $y'=1$. $\frac{cx'}{b}=24$. $\frac{cy'}{a}=20$. Or, 4 solutions.

But observe that $\frac{cx'}{b}$ in this case, is a complete integral, 24; agreeable then, to previous considerations we must deduct one, and the number of solutions are but 3, as follows: $x=5.\ 10.\ 15.$ $y=6.\ 4.\ 2$, and no other solution can be found.

What number of solutions in whole numbers can be fouud for the equation $3x+5y+7z=100$.

As x and y each cannot be less than one, z cannot be greater than $\frac{100-3-5}{7}=13\frac{1}{7}$. That is, z cannot be greater than 13, in whole numbers. Now suppose $z=1$, and the equation becomes $3x+5y=93$.

The number of solutions for this equation, found as previously directed is 6. That is $\begin{cases} x=26.\ 21.\ 16.\ 11.\ 6.\ 1. \\ y=\ 3.\ 6.\ 9.\ 12.\ 15.\ 18. \end{cases}$

Now x and y can make these 6 changes, and z be constantly equal to 1, *and satisfy the primative equation.*

Take $z=2$, and the equation becomes $3x+5y=86$.

This equation has also 6 solutions, z being though all the changes of x and y equal to 2.

Now take $z=3$, then the original equation is $3x+5y=79$. This equation has five solutions.

Now take $z=4$, then $3x+5y=72$. This equation has four solutions.

Take $z=5$, then $3x+5y=65$. This equation has four solutions.

Take $z=6$, then $3x+5y=58$. This equation has four solutions.

In this manner, by taking z equal to all the integers up to 12 in succession, we find 41 solutions.

The preceding equation is in Bannycastle's Algebra, page 232, he finds 7 solutions, and remarks that they are all the integer values of x, y and z that can be obtained from the given equation.

Ryan, also, in his algebra, page 351, gives the same number of solutions as Bonnycastle, and makes the same remark.

Ryan previously remarked, after showing the greatest possible value of z to be 13, that "13 was also the limit to the number of answers, though they may be considerably less." Yet in Mr. Ryan's Key to Bonnycastle, he very correctly makes 59 solutions to the equation

$$5x+7y+11z=234.$$

But agreeable to his remark on the foregoing, the number of answers should be less than 19.

Given $17x+19y+21z=400$, to find all the integer values of x, y and z, that the equation admits of.

Suppose $z=1$, then the equation is $17x+19y=379$, an equation having only one solution.

Take $z=2$, then $17x+19=358$, an equation having one solution.

We might proceed in this way through the limit of z, and we should find in all ten solutions, but the result is not worth the labor, all that is necessary, is to have a *clear and distinct* understanding how a thing of this sort may be done.

There is a theory, and a very beautiful one too, in Young's Algebra, edited by Samuel Ward, pages 288 to 291, by which this operation may be somewhat abridged, but the result of the theory cannot be given without the theory itself, and the space it would occupy is wholly incompatible with the design of this work.

SECTION XXI.

The Diophantine Analysis.

The Diophantine Analysis teaches how to find square and cube numbers under given conditions, or having given relations to each other.

EXAMPLES.

CASE 1ST. Find such a value of x as will make the expression $ax+b$ a square.

Put $ax+b=n^2$, n^2 being *any square*, it is, therefore, *an indefinite problem.*

From the equation $x=\frac{n^2-b}{a}$; take n equal to any number whatever, and a and b being known, x becomes known.

Eight times a certain number added to 9, makes a square. What is the number?

Let $x=$ the number. Then $8x+9=n^2$, *that is any square.* $a=8$, $b=9$, and $x=\frac{n^2-b}{a}=\frac{n^2-9}{8}$. Assume $n=7$, then $x=5$, the required number. But there are many other numbers that will answer the condition according as we assume n more or less.

Find x, such that the following expressions shall be square numbers.

$$9x+9. \quad 7x+2. \quad 3x-5. \quad x+\tfrac{1}{2}.$$

All these correspond to the general expression $ax+b$.

CASE 2D. Any algebraic expression in the general form of ax^2+bx, may be made a square by supposing its square root equal px; x must be in some part of the root, because the expression contains x^2, that is some function of x. Now if px is the root, $ax^2+bx=p^2x^2$. Divide by x, &c., and we have $x=\frac{b}{p^2-a}$. p may be any assumed value whose square is greater than a.

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e the expression

ll make $6x^2+13x+6$ a

$=169$, $4ac=144$, b^2-4ac

$12x+13=\pm5$. $x=-\frac{2}{3}$,

$2x+3=0$.

Examples.

Six times the square of a certain number, added to five imes the number is a square. What is the number ?

$$6x^2+5x=p^2x^2. \quad \text{Or, } x=\frac{5}{p^2-6}.$$

Here it is obvious that p^2 must be greater than 6, otherwise it is unlimited. Take $p=3$, then $x=\frac{5}{3}=1\frac{2}{3}$.

Find the value of x to make $\frac{x^2}{2}+3x$ a square.

Ans. $x=6$.

Case 3d. Any algebraic expression in the general form of $x^2\pm bx+c$, can be made a square, by putting $x\pm p$ equal its square root.

We can if we please take $x-p$ for the root in all such cases. Then if p is less than x, the square is diminished, if greater, the whole root will be essentially minus, but the square will be plus, and may rise to any amount. Therefore $x-p$ is far more general than $x+p$.

Case 4th. Any expression in the form of $ax^2\pm bx+c^2$, can be made a square, by taking its root equal to $c\pm px$.

It will be observed that x must be in the root of the previous expression, because it has x^2, and c must be in the root of this last expression, because it contains c^2.

In the first we have $x^2\pm bx+c=x^2\pm 2px+p^2$. Or, $x=\frac{p^2-c}{3p\pm b}$.

In the second, $x^2\pm bx+c^2=c^2\pm 2cpx+p^2x^2$. Or, $x\pm b=\pm 3cp+p^2x$. Or, $x=\frac{\pm 2cp\mp b}{1\pm p^2}$ In both cases assume p of any convenient value to render x positive, and as small as possible.

Find a number such, that if it be increased by 2 and 5 seperately, the product of the sums shall be a square.

Let $x=$ the number, then $(x+2)(x+5)=x^2+7x+10$, must be a square. $b=7$, $c=10$. General solution, $x=\frac{p^2-10}{2p\pm 7}$

Now p^2 must be more than 10; hence, take $p=4$, and $x=\frac{6}{15}=\frac{2}{5}$, the *least* number that will answer the conditions.

CASE 5TH. An expression in the form of ax^2+bx+c, where *neither the first nor the last* terms of the expression are squares, neither branch of the root can be directly found, and the expression cannot be made a square, unless we can seperate it into two *rational factors*, or unless we can first subtract from it some simple binomial square, and can then divide the remainder into two rational factors.

By reminding one of the nature of quadratic equations, all may perceive that the expression ax^2+bx+c *must be* the product of two factors, but whether *rational factors* or *not* is the subject of inquiry.

To find the factors which make the product ax^2+bx+c, put this expression equal to 0, and work out the values of x thus, $ax^2+bx+c=0$. Or, $ax^2+bx=-c$. Complete the square, and $4a^2x^2+4abx+b^2=b^2-4ac$.

$$\text{Or, } 2ax+b=\pm\sqrt{\{b^2-4ac.\}}$$

$$\text{Or, } x=\pm\frac{1}{2a}\sqrt{\{b^2-4ac\}}-\frac{b}{2a}.$$

We now perceive that the values of x must be rational, provided $\sqrt{\{b^2-4ac\}}$ is a complete square. If it be so, let $\frac{1}{2a}\sqrt{\{b^2-4ac\}}-\frac{b}{2a}=m$, and $-\frac{1}{2a}\sqrt{\{b^2-4ac\}}-\frac{b}{2a}=n$.

Then the two values of x are $x=m$ and $x=n$, and $(x-m)(x-n)$, are the factors which will give the expression ax^2+bx+c.

EXAMPLES.

1. Find such a value of x as will make $6x^2+13x+6$ a square.

Here $a=6$, $b=13$, $c=6$. $b^2=169$, $4ac=144$, $b^2-4ac=25$ and $\sqrt{\{b^2-4ac\}}=5$. Now $12x+13=\pm 5$. $x=-\frac{2}{3}$, Or, $x=-\frac{3}{2}$. Or, $3x+2=0$, and $2x+3=0$.

That is, $(3x+2)(2x+3)$ will produce the expression $6x^2+13x+6$.

Now to make the expression a square, put

$$(3x+2)(2x+3)=p^2(3x+2)^2$$

Then $2x+3=p^2(3x+2.)$ And $x=\frac{2p^2-3}{2-3p^2}$.

Take $p=1$, and $x=1$. We might have seen at first, that in this *particular expression*, the value of x being 1, would make it a square, as $6+13+6=25$, a square.

That is in all cases when the sum of the coefficients make a square number, the value of x may be one.

2. Find such a value of x as shall render the expression $13x^2+15x+7$ a square.

Here as neither the first nor last terms are squares, nor b^2-4ac a square, the expression cannot be made a square, unless we can seperate the remainder into factors after taking away some simple square. But in this case, $4ac$ is greater than b^2, we must then, in subtracting our square, diminish a and c *and increase* b.

To accomplish this end we will subtract the square of $x-1$, *not* $x+1$.

That is - - - - - -	$13x^2+15x+7.$
Subtract - - - - - -	$x^2-\ 2x+1.$
Remainder, - - - -	$12x^2+17x+6.$

In this last expression, $a=12$, $b=17$, $c=6$. Hence, $b^2-4ac=289-288=1$, a square; we are now sure rational factors can be found to produce the expression

$$12x^2+17x+6.$$

By assuming $12^2+17x+6=0$, and finding the values of x by the quadratic, (*merely to get the factors*,) we find $x=-\frac{2}{3}$, and $x=-\frac{3}{4}$. Or, $3x+2=0$, and $4x+3=0$.

And $(3x+2)(4x+3)=12x^2+17x+6$.*

* The values of x used to obtain these factors have no connection with the values of x to *render* the original expression a square.

Now the original expression is the same as $(x-1)^2+(3x+2)(4x+3.)$ Put its root equal to $(x-1)+p(3x+2,)$ and square this root, and
$(x-1)^2+(3x+2)(4x+3)=(x-1)^2+2p(x-1)(3x+2)+p^2(3x+2)^2$. By reduction, $4x+3=2p(x-1)+p^2(3x+2$.
Or, $(-4+2p+3p^2)x=2p+3-2p^2$.

$$\text{Therefore, } x=\frac{2p+3-2p^2}{3p^2+2p-4}.$$

Take $p=1$, then $x=3$, and $13x^2+15x+7=169$ a square.

Ryan makes x in this example equal $\frac{1}{3}$ and the expression equal $\frac{121}{9}$ a square, but an integer number is always more satisfactory.

3. Find such a value of x as will render $14x^2+5x-39$, a square.

After a few trials this expression is found to be the same as $(2x-1)^2+(5x-8)(2x+5.)$ Assuming its root to be $2x-1+p(5x-8.)$ Then by squaring the root, making it equal to its power, and reducing, we find $x=\frac{8p^2+2p+5}{5p^2+4p-2}$.

Assuming $p=1$, $x=\frac{15}{7}$, and the expression equals 36, a square. Other values can be found, by assuming different values to p.

4. Find such a value of x as shall make $2x^2+21x+28$ a square.

After a little inspection we find this expression equal to $(x+4)^2+(x+1)(x+12.)$ Now if we make

$$(x+4)^2+(x+1)(x+12)=\{(x+4)-p(x+1.)\}^2$$

After reduction, we shall find $x=\frac{12+8p-p^2}{p^2-2p-1}$.

Assume $p=4$, then $x=4$, and the original expression is 144 a square.

If $(x+4)^2+(x+1)(x+12)=\{(x+4)-p(x+12)\}^2$, we shall find $x=\frac{12p^2-8p-1}{1+2p-p^2}$. If we take $p=1$, $x=\frac{3}{2}$. If we take $p=\frac{3}{2}$, $x=8$, and we might find many other numbers that would answer the conditions of the expression.

CASE 6TH. When we have an expression in the form of $a^2x^4+bx^3+cx^2+dx+e$, we can assign a value to x that will make the whole expression a square, *if we can extract three terms of its root.*

Assume such terms as the whole root, square the root so assumed, making it equal to the given expression, and by reducing, we shall have *a value of x* which will make the original expression a square.

EXAMPLE.

Find such a value of x as shall make $4x^4+4x^3+4x^2+2x-6$ a square.

We commence by extracting the square root as far as *three terms*, and find them to be $2x^2+x+\frac{3}{4}$. Now assume

$$4x^4+4x^3+4x^2+2x-6=(2x^2+x+\tfrac{3}{4}.)^2$$

Expanding and reducing, we have $2x-6=\frac{3x}{2}+\frac{9}{16}$.

And $x=13\frac{1}{8}$.

Essentially the same method must be pursued in other problems of the like kind.

CASE 7TH. Find such a value of x as shall render $2x^2+2$ a square.

Expressions of this kind, when neither a nor c are squares, nor b^2-4ac a square, and which cannot be resolved into factors, presents an impossible case, unless we can first find by inspection, some simple value of x that will answer the condition.

In the present example, it is obvious that if $x=1$, the expression is a square. But we wish to find other values than 1 that will render this expression a square, and having found that one will answer, we are now enabled to find other values, thus:

Let $x=1\pm y$. Then $x^2=1\pm 2y+y^2$.

And $2x^2+2=4\pm4y+2y^2$. Here the original expression is transformed into another expression, *having a square* for its first term.

Now we must find such a value of y as shall make $4+4y+2y^2$ a square.

Assume $4+4y+2y^2=(2-my)^2=4-4my+m^2y^2$. Or, $4+2y=-4m+m^2y$. Hence $y=\frac{4(m+1)}{m^2-2}$, m may be any number greater than one. Put $m=2$. Then $y=6$, and $x=1+y=7$, and the original expression $2x^2+2=98+2=100$, a square.

N. B. It often occurs incidentally in the solution of problems, that we must make a square of two other squares. This can be done thus: Let it be required to make x^2+y^2 a square. Assume $x=p^2-q^2$, and $y=2pq$.

Then, - - - - - - $x^2=p^4-2p^2q^2+q^4$.

And, - - - - - - $y^2=\quad 4p^2q^2$.

Add, and - - $x^2+y^2=p^4+2p^2q^2+q^4$, which is evidently a square, whatever be the values of p and q. We can, therefore, assume p and q at pleasure, provided p be greater than q.

Section XXII.

Double and Triple Equalities.

In the preceding section it was only necessary to find such a value of the unknown term as to render a single expression a square. But there are problems where it becomes requisite to find such a value of the unknown term as to render several different expressions squares at the same time. And this is called *double* and *triple equalities*.

Case 1st. As a general equation for double equality, let it be required to find such a value of x that $ax+b$ may be a square, and the *same value* of x give $cx+d$ a square.

Let $ax+b=t^2$. Then $x=\frac{t^2-b}{a}$.

And $cx+d=p^2$. Then $x=\frac{p^2-d}{c}$.

Therefore $\frac{t^2-b}{a}=\frac{p^2-d}{c}$. Or, $ct^2-cb=ap^2-ad$.

Tranpose cb and multiply by c and we have

$$c^2t^2=acp^2+c^2d-acd.$$

As the left hand side of this equation is a square, *whatever may be the values of c and t*, it is now only requisite to find such a value of p^2 as shall render the other side a square, which can be done by some one of the artifices in the preceding section.

To illustrate this we give the following definite problem.

The double of a certain number increased by 4, makes a square, and five times the same number plus one, also makes a square. What is the number

Let x represent the number.

$$\left.\begin{array}{ll}\text{Then} & 2x+4=t^2 \\ \text{And} & 5x+1=p^2\end{array}\right\} \text{From which} \left\{\begin{array}{l} x=\frac{t^2-4}{2} \\ x=\frac{p^2-1}{5}\end{array}\right.$$

Hence $5t^2-20=2p^2-2$. Or, $5t^2=2p^2+18$, multiply by 5, and $25t^2=10p^2+90$.

The left hand side of this last equation being a square, *whatever* be the value of t, it is now only necessary to find such a value of p^2 as to make $10p^2+90$ a square, an expression which corresponds to case 7 of the last section. We therefore *cannot proceed unless* we find by trial, by observation, by intuition as it were, some simple value of p that will make $10p^2+90$ a square, and we do perceive that if $p=1$, the expression will become 100, a square.

Now if $p=1$, it will give a definite and positive value to x, and the problem is solved. If not we must find other values of p.

But we have found that $x=\frac{p^3-1}{5}$, and if $p=1$, $x=0$, and the original expressions $2x+4$, and $5x+1$, become 4 and 1, squares it is true, which answer the *technicalities*, but not the spirit of the question.

To find another value of p. Put $p=1+q$. Then $10p^2+90=100+20p+2q^2$. To make this a square assume $100+20q+2q^2=(10-nq)^2=100-20nq+n^2q^2$.

By reduction, $q=\frac{20(n+1)}{n^2-10}$. Now n must be so taken, that n^2 is more than 10: take $n=5$ and $q=8$, $p=9$, then $x=16$ and the original expressions $2x+4=36$, a square, and $5x+1=81$, a square.

CASE 2D. A double equality in the form of $ax^2+bx=\square$ and cx^2+dx, also equals a square, may be resolved by making $x=\frac{1}{y}$, then the expressions will become $\frac{1}{y^2}(a+by)$ and $\frac{1}{y^2}(c+dy,)$ which must be made squares.

But if we multiply a square by a square, or divide a square by a square, the product or quotient will be square.

Now as each of the preceding expressions are to be squares, and as they obviously have a square factor $\frac{1}{y^2}$ it is only necessary to make $a+by$, and $c+dy$ squares, as in the first case.

We may also take another course and assume $ax^2+bx=p^2x^2$, which gives $x=\frac{b}{p^2-a}$, which value put in the other expression, and we have $c\left(\frac{b}{p^2-a}\right)^2+d\left(\frac{b}{p^2-a}\right)=\square$.

Multiplying this by the square $(p^2-a)^2$, and the expression becomes $cb^2-dbd+abp^2=$ some square, from which the value of p can be found and afterwards x.

Example.

A certain number added to its square, the sum is a square, and the number subtracted from its square, the remainder is a square. What is the number?

Let $x=$ the number.

Then $x^2+x=\square$. And $x^2-x=$ some other square.

$$\text{Assume } x=\frac{1}{y}. \quad \text{Then } \frac{1}{y^2}\left(1+y\right)=\square.$$

$$\text{And } \frac{1}{y^2}\left(1-y\right)=\square.$$

The problem will be solved if we can find such a value of y, as will at the same time make $1+y$ and $1-y$ squares.

Therefore put $1+y=p^2$, and $1-y=q^2$.

From the first, $y=p^2-1$. And $y=1-q^2$.

Therefore, $q^2=2-p^2$. As q^2 is a square, we have only to find such a value of p^2, as shall render $2-p^2$ a square. But this cannot be done *unless* we can find some simple value of p by inspection, and we do observe it must be one. But p being equal to one, gives $y=0$, which will not answer the conditions. Therefore, let $p=1+t$.

Then $2-p^2=1-2t-t^2=(1-ut)^2=1-2ut+u^2t^2$.

Or, $t=\dfrac{2(u-1)}{u^2+1}$. Take $u=2$. $t=\frac{2}{5}$. $p=1+t=\frac{7}{5}$.

$y=p^2-1=\frac{24}{25}$. $x=\dfrac{1}{y}$. $x=\frac{25}{24}$, a number that will answer the given conditions.

Case 3d. To resolve a triple equality.

Equations in the form of $ax+by=t^2$, $ax+dy=u^2$, $ex+fy=s^2$, can be resolved thus:

By expunging y, we find $x=\dfrac{dt^2-bu^2}{ad-bc}$.

Then by expunging x, $y=\dfrac{au^2-ct^2}{ad-bc}$.

Substituting these values of x and y in the third equation and we shall have $\frac{(af-bc)u^2+(de-cf)t^2}{ad-bc}=s^2$.

Assume $u=\pm tz$. Then $u^2=t^2z^2$. Put this value of u^2 in the above, and divide by t^2, and we shall have

$$\frac{(af-bc)z^2+de-cf}{ad-bc}=\frac{s^2}{t^2}.$$

The right hand side of this equation is a square, and therefore all that is now requisite, is to find such a value of z as shall make the other side a square, which when possible, can be done by case 7, section 20.

After z is found t may be assumed of any convenient value whatever. Now u is known, and with t and u known quantities, we know x and y.

The preceding are some of the most comprehensive and general methods yet known; but there are cases in practice where no general rules will be so effectual, as the operator's own judgment and penetration.

Much, *very much* will depend on skill and foresight displayed at the commencement of a problem, by assuming convenient expressions to satisfiy one or two conditions at once, and the remaining conditions can be satisfied by some one of the preceding rules.

EXAMPLES.

1. It is required to find three numbers in arithmetical progression, such, that the sum of every two of them may be a square.

Let x, $x+y$ and $x+2y$ represent the numbers.

Then by the general formula,

$$2x+y=t^2,\ 2x+2y=u^2,\ 2x+3y=s^2.$$

By exterminating x, we have $\frac{t^2-y}{2}=\frac{u^2-2y}{2}$.

Continuing thus after the general equations, we find a long and troublesome process, and in conclusion, we find the numbers to be 482, 3362, and 6242.

The above is according to the common as well as the general method.

The following is Mr. Young's Solution.

Let $x-y$, x, and $x+y$ represent the numbers. Then $2x-y$, $2x$, and $2x+y$ must be squares.

Assume $2x=m^2+n^2$, and $y=2mn$.

Then $2x-x=m^2-2mn+n^2$, and $2x+y=m^2+2mn+n^3$ are evidently squares. It therefore only remains to make $2x$ or m^2+n^2 a square, and this can be done as explained at the close of the last section by assuming $m=r^2-s^2$, and $n=2rs$. Then $2x=m^2+n^2=(r^2+s^2)^2$, an expression in which r and s can be assumed in numbers. But they must be *so assumed* that x shall be greater than y to make $x-y$ the first number, positive and for this reason, we must give the literal expressions for the numbers before taking definite values for r and s. The expressions for the numbers are

$$x-y=\tfrac{1}{2}(r^2+s^2)^2-4rs(r^2-s^2.) \quad x=\tfrac{1}{2}(r^2+s^2.)^2$$

$$x+y=\tfrac{1}{2}(r^2+s^2)^2+4rs(r^2=s^2.)$$

Take $r=9$, $s=1$, and 482, 3362, 6242, are the numbers.

Another Solution.

Let $\frac{x^2}{2}-y, \frac{x^2}{2}$ and $\frac{x^2}{2}+y$ be the numbers.

Then x^2-y, x^2+y, and x^2 must be squares.

But the last being a square, we have only to make x^2-y and x^2+y, squares.

Assume $y=2x-1$. Then $x^2-y=x^2-2x+1$, a square, and we now have only to make x^2+2x-1, a square.

Therefore make $x^2+2x-1=(x+n)^2=x^2+2nx+n^2$.

$$x=\frac{n^2+1}{2(1-n)}.$$

It is manifest that n must be less than one, make it .8.

Then $x=\frac{1.64}{.4}=\frac{41}{10}$. Or, $\frac{x^2}{2}=\frac{1681}{200}$. $y=\frac{72}{10}=\frac{1440}{200}$.

Then $\frac{482}{400}$, $\frac{3362}{400}$, $\frac{6242}{400}$ are the numbers.

Here we have the numbers expressed in fractions, but the denominators are common, and is a square number, we may therefore multiply all three by 400, and we shall have 482, 3362, and 6242 for the numbers, as in the other solutions.

If we take $n=\frac{5}{6}$ in this last result, we shall have 2162, 7442, and 9442 for the numbers.

2. Find two numbers such, that if to each, as also to their sum, a given square, a^2 be added, the three sums shall all be squares.

Let x^2-a^2 and y^2-a^2 represent the numbers: then the first conditions are satisfied.

It now remains to make $x^2+y^2-2a^2+a^2$ a square, or, $x^2+y^2-a^2=\square$. Assume $y^2-a^2=2ax+a^2$. This assumption will make the expression a square, whatever be the values of either x or a. But the assumed equation gives $y^2=2ax+2a^2$, and as y^2 is a square, we must find *such values* of x and a, as shall make $2ax+2a^2$, a square. Put $x=na$. Then $2na^2+2a^2=\square$, or, $a^2(2n+2)=\square$. Hence it is sufficient that we put $2n+2=$ some square. Therefore, assume $2n+2=16$. Hence $n=7$ and $x=7a$. Now take a equal to any number whatever. If $a=1$, $x=7$, $y=4$, and 48 and 15 are the numbers, add 1 to each, and we have 49 and 16, squares; sum, $63+1=64$, a square.

3. Find three square numbers whose sum shall be a square.

Let $x^2+y^2+z^2=\square$. Assume $y^2=2xz$. Then $x^2+2xz+z^2$ is a square. But $2xz=\square$. Let $x=uz$, then $2uz^2=\square$, or $2u=\square=16$, $u=8$, $x=8z$, $z=1$, $x=8$, $y=4$.

Therefore $64+16+1=81=9^2$.

4. Find three square numbers in arithmetical progression.

Let x^2-y, x^2, and x^2+y represent the numbers. Assume $x^2=y^2+\frac{1}{4}$, then the first and last will be squares, and it only remains to make the middle term, or $y^2+\frac{1}{4}$, a square. Therefore, put $y^2+\frac{1}{4}=(y-p)^2$, which gives $y=\frac{p^2-\frac{1}{4}}{2p}$. Take $p=1$, then $y=\frac{3}{8}$, and $y^2+\frac{1}{4}=\frac{25}{64}=x^2$. Therefore, $\frac{1}{64}$, $\frac{25}{64}$, $\frac{49}{64}$, are the numbers; but we can multiply them all by the same square number 64, and their *arithmetical* relation will not be changed, and they will still be squares; hence 1, 25, and 49 may be the numbers, or 4, 100, and 196,

5. Find two whole numbers, such that the sum and difference of their squares, when diminished by unity, shall be a square.

Let $x+1=$ one number, and $y=$ the other. Then by the conditions we must make squares of x^2+y^2+2x, and x^2-y^2+2x. Assume $2x=a^2$, and $y^2=2ax$, then the expressions become $x^2+2ax+a^2$, and $x^2-2ax+a^2$, obvious squares, whatever be the values of x and a. But the equations $2x=a^2$ and $y^2=2ax$ must be satisfied. Take $a=4$, then $x=8$, $y=8$, and $x+1=9$. Therefore 9 and 8 are the numbers.

6. Find three whole numbers, such, that if to the square of each, the product of the other two be added, the three sums shall be squares.

Let x, xy, xv, be the numbers. Then by the conditions, x^2+x^2yv, $x^2y^2+x^2v$, $x^2v^2+x^2y$, must be squares. As each term contains a square factor x^2, it will be sufficient to make $1+yv=\square$, $y^2+v=\square$, and $v^2+y=\square$.

Assume $y=4v+4$, and this will make the first and last expressions squares. Substitute this value of y in the second expression, and we shall have $16v^2+33v+16$, which must be made a square. Hence put $16v^2+33v+16=(4-pv)^2$, which reduced gives $v=\frac{33+8p}{p^2-16}$. Take $p=5$, then $v=\frac{73}{9}$ Now take $x=9$, and we have 9, 73, and 328 for the numbers.

7. Find two whole numbers whose sum shall be an inte

gral cube, and the sum of their squares increased by thrice their sum shall be an integral square.

Let $x+y=n^3$, that is some cube. Then $x^2+y^2+3n^3=\square$. Put $2xy=3n^3$, then $x^2+2xy+y^2$ is a square, whatever may be the values of x and y. But x and y must conform to the equations $x+y=n^3$, and $2xy=3n^3$. Work out the value of x from these equations, on the supposition that n is known, and we shall find $2x=n^3+\sqrt{\{n^6-6n^3.\}}$

Now x will be rational, provided we can find such a value of n as shall render n^6-6n^3 a square, but if we add 9 to this, we perceive it must be a square, and we have two squares, which differ by 9. Therefore one must be 16, the other 25, as these are the only two *integral squares* which differ by 9. Hence $n^6-6n^3+9=25$. Or, $n^3-3=5$. $n^3=8$, $n=2$, and $x=6$, $y=2$.

8. Find three numbers, such that their sums, and also the sum of every two of them, may all be squares.

(Young's Algebra, p. 335.)

Let $x^2-4x=$ the first, $4x=$ second, and $2x+1=$ third. By this notation, all the conditions will be satisfied, except the sum of the last two. That is $6x+1$ must be a square, but to have *three different whole numbers*, no square will answer under 121, the square of 11. Hence put $6x+1=121$. Or, $x=20$. And the numbers will be 320, 80, and 41.

9. Find two numbers, such that their difference may be equal to the difference of their squares, and the sum of their squares shall be a square number.

Let x and y be the numbers. Then $x-y=x^2-y^2$. Divide by $x-y$, and $1=x+y$. Hence $x=1-y$, and $x^2+y^2=1-2y+2y^2$. Which last expression $1-2y+2y^2$ must be made a square. For this purpose, put

$$1-2y+2y^2=(1-ny)^2. \quad \text{Hence } y=\frac{2(n-1)}{n^2-2}.$$

Take n any value to render y less than one in order to give x a positive value. Therefore take $n=3$, and $y=\frac{4}{7}$. Consequently $x=\frac{3}{7}$, answer.

10. Find three numbers in geometrical progression, such that if the mean be added to each of the extremes, the sums in both cases shall be squares. Ans. 5, 20, and 80.

11. Find three numbers, such, that their product increased by unity shall be a square, also the product of any two increased by unity, shall be a square. Ans. 1, 3, and 8.

Assume 1 for the first number, and x and y for the other

12. Find two numbers, such that if the square of each be added to their product, the sums shall be both squares.

Ans. 9 and 16.

13. Find three integral square numbers in harmonical proportion. Ans. 25, 49, and 1235.

14. Find two numbers in the proportion of 8 to 15, and such that the sum of their squares shall be a square number.

Ans. 136 and 255. Bonnycastle's answer, 476 and 1080.

15. Find two numbers such that if each of them be added to their product, the sums shall be both square.

Ans. $\frac{1}{3}$ and $\frac{5}{3}$.

The above require no explanation from us.

There are many severe and tedious problems in the Diophantine Analysis, proposed by Bonnycastle, Young, and others, which require more time and practice than algebraists in general ought to give for the advantage derived, as time and thought may be better employed in Analytical Geometry, the Calculus, or Astronomy.

We would not, however, be understood as speaking disparagingly of this kind of anlysis; as far as its general principles go, it is very improving to the mind, and no person should be considered an algebraist without it, and the fact that it is found in so few of our modern books, is a bad omen of the times. We only condemn a certain class of problems, apparently given, not to exercise or improve the mind, but to defy the powers of the operator.

The Diophantine Analysis is sometime useful in solving general problems where squares and cubes occur, and may be the means of saving a vast amount of labor, as some of the following examples will show.

1. Given $\left\{\begin{matrix} x^2+y=7 \\ y^2-x=7 \end{matrix}\right\}$ to find the values of x and y.

As the right hand side of both equations equal 7,

$$x^2+y=y^2-x. \quad \text{Hence, } x^2+x+y=y^2.$$

As the right hand side of this last equation is a square, the other is a square also, but not in a *sqare from.* We perceive, however, that it will be in a square form, if we assume $y=x+1$.

Put this value of y in the first equation, and we have $x^2+x=6$, hence $x=6$ and $y=2$, the answer.

2. Given $\left\{\begin{matrix} 2x^2-3xy+y^2=4 \\ x^2+3y^2-2xy=9 \end{matrix}\right\}$ to find x and y.

As 4 and 9 are squares, the other side must be squares also. But to make them squares in form as well as in fact, we *must assume* $2x^2-3xy=0$, and $3y^2-2xy=0$, and there will remain $y^2=4$ and $x^2=9$, $y=2$, $x=3$, which values are not inconsistent with the assumed equations.

Mr. Stephen L. Massey of Cincinnati, proposed the following equations to find the values of the several letters.

3. $x+v+x^2v^2=41.$ (1)

$y+u+y^2u^2=21.$ (2)

$v+u+v^2u^2=13.$ (3)

$x+y+x^2y^2=70.$ (4)

Notwithstanding we observe the same regularity in all the equations, yet if we attempt a solution by the common rules of algebra, we shall find that a very high and tedious equation cannot be avoided, we therefore abandon all such

attempts. Look at the equations again, and observe that *all the combinations* of the letters make *integer values* which it is evident could not be the case were any of them fractional.

Now take equation (3) as that has the *smallest numerical value*, put $v+u=s$, and transpose it, we have $u^2v^2=13-s$.

The left hand member of this equation being a square, we must find such a value of s as shall make $13-s$ an *integer* square, s being an integer. Hence take $13-s=9$, then s, or $v+u=4$, and $uv=3$. From these two equations, $v=1$ or 3, and $u=3$ or 1.

Put the value of u in equation (2), and we find at once that u cannot be 3, but it may be 1, which gives $y=4$; y being 4, equation (4) gives $x=2$, and the values are all found. $v=3$, $u=1$, $x=2$, and $y=4$.

The following, by the same proposer, may be expeditiously worked by a similar process.

4\. Given $\left\{\begin{array}{l} 2\sqrt{xy}+v^2=12 \\ x^2y^2-v^2=1296 \\ yv^2+x=3 \end{array}\right\}$ to find x, y, and v.

5\. Given $\left\{\begin{array}{ll} x^2+xy=208. & (1) \\ y^2+yz=69. & (2) \\ z^2+xz=580. & (3) \end{array}\right\}$ to find x and y.

(Bonnycastle, page 272.)

Without aid from both the Indeterminate and Diophantine Analysis it would be very difficult to obtain the values of x, y, and z, in the above equations.

In the first place let us consider it not probable that x and y have *fractional values. If they have*, their combinations in the first equation would not present an integer value, and *at the same time* the two other equations come out integers. Therefore x and y may equal *whole numbers.* In the same way that we decide x may be a whole number, we decide that z may be a whole number; we have now advanced so far as to pronounce x, y, and z, probable *whole numbers.*

Again we observe a symmetry in all the equations, x in the first is involved the same as y in the second, and the same as z in the third. But the second equation has the *least numerical value*, therefore y has the least value, and we perceive that x must be greater, and z the greatest.

Now put $x=ny$. Place this value of x in equation (1), and $n^2y^2+y^2=108$. Or, $y^2=\frac{108}{n^2+n}$. Here it is manifest that n^2+n must be of such a value for a divisor to 108, as will give *a square* for a quotient. But 108 has *only two* divisors, which will give integral *square qoutients*, 27 and 12. Hence $n^2+n=27$, or $n^2+n=12$.

We *must* take this last eqnation as *this only* gives a rational value to n, namely, $n=3$. That is $x=ny$, or $x=3y$, and $y^2=9$, $y=3$, $x=9$, and from equation (4), $z=20$.

6. Given $\left\{\begin{matrix} 4x^2-2xy=12 \\ 2y^2+3xy=\ 8 \end{matrix}\right\}$ to find the values of x and y.

(Young, page 138.)

Put $p=xy$, and transpose $-2xy$ in the first equation, double the second, and transpose $6xy$, and we shall have $4x^2=12+2p$, and $4y^2=16-6p$.

The left hand members of these equations being squares, we have to find such a value of p as shall make at the same time $12+2p$, and $16-6p$, squares. Take $16-6p=4$. Then $p=2$, and $12+2p=16$, a square. Or, $4x^2=16$, $x=\pm2$, $y=\pm1$, answer.

7. Given $\left\{\begin{matrix} 6x^2+2y^2=5xy+12 \\ 3x^2+2xy=3y^2-\ 3 \end{matrix}\right\}$ to find the rational values of x and y.

(Young, page 139.)

To solve this by quadratics is quite laborous; observe how simple by the Diophantine Analysis.

Add the two equations together, and $9x^2=y^2+3xy+9$. The left hand side of this equation is a square, therefore the other side is a square, and to make that side a square *in form* as well as *in fact*, we perceive it is only necessary to make

$x=2$. Or call $y^2+3xy+9$ a binomial square, and decide the value of x agreeable to section 8, This gives $x=2$, the answer.

8. Given $\left\{\begin{array}{l}3x^2+xy=78\\4y^2+3xy=160\end{array}\right\}$ to find the rational values of x and y. (Y. 139.)

Ans. $x=\pm4$, $y=\pm5$.

There are, and may be equations of the preceding forms which have no rational values, such are not susceptible of of this mode of treatment.

Section XXIII.

Miscellaneous Examples.

1. When wheat was 8 shillings a bushel and rye 5, a man wished to fill his sack for the money he had in his purse. Now if he bought 15 bushels of wheat and laid out the rest of his money in rye, he would want 3 bushels to fill the sack: but if he bought 15 bushels of rye, and then filled his sack with wheat, he would have 15 shillings left. How much of each must he purchase to fill his sack, and lay out all his money?

(Colburn, page 50.) Ans. 10 bushels of each.

Solution by Mr. T. J. Matthews.

Let $x=$ the wheat, and $y=$ the rye. Then it is evident that when he buys 15 bushels of wheat he has too much, as he has not money enough left to fill his sack with rye. Now $15-x$ is the excess of the wheat purchased above what he ought to have had, and this excess of quantity, multiplied by the excess of a bushel of wheat above one of rye, will give the dificiency of his money, or equal to 3 bushels of rye at 5 shillings. Consequently, $3(15-x)=15$, or $x=10$.

By similar reasoning, it will appear that when 15 bushels of rye are purchased, he buys too much, and the excess is $15-y$, which multiplied as before, will equal the excess of his money, viz: 15 shillings. Therefore $3(15-y)=15$, and $y=10$.

2. A person bought two cubical stacks of hay, for £41; each of which cost as many shillings per solid yard as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the price of each? (Colburn.)

Solution by T. J. Matthews.

Assume $5x$, and $4x$ the sides of the Cubes.

Then $25x^2-16x^2=9$, by the first condition.

Therefore, $x=1$. $125\cdot4=500$. $64\cdot5=320$.

3 Or, £25. £16. Answer.

3. Given $x+y+z=25$. $xy=6$ $xz=60$, to find x, y and z.

Solution. From the two latter, $z=10y$. Then the first becomes $x+11y=25$. Or $x^2+11xy=25x$, but from the 2d, $11xy=66$. Hence $x^2-25x=-66$. (A)

Assume $2a=-25$, then $6a+9=-66$, and equation (A) becomes $(x^2+2ax+a^2=a^2+6a+9$. Or, $x=3$,)

4. Given $xy=125x+300y$. And $y^2-x^2=90000$, to find x and y. (Young p. 146.) Ans. $x=400$. $y=500$.

Put $y=px$, then $px^2=125x+300px$. (1)

And $p^2y^2-x^2=(300)^2$ (2)

From equation (1) $p=\frac{125}{x-300}$. (3)

From equation (2) $p^2-1=\frac{(300)^2}{x^2}$

The right hand side of this last equation being a squares the other side is also a square, and one accustomed to the analysis will perceive that p must equal $\frac{5}{4}$, to make the expression $p^2-1=\square$. Others can go through the form and they will find that $p=\frac{5}{4}$, which value put in equation (3) gives $x=400$.

It is not imperative that we should resort to the Diaphantine to solve this problem but it is *very convenient.*

5. Find two numbers, such that the fifth power of one may be to the cube of the other, as 972 to 125.

Ans. 6 and 10.

Let x and nx be the numbers.
Then $x^5 : n^3x^3 :: 972 : 125$. Or, $x^2 : n^3 :: 972 : 125$.
Multiply the first and third term by x, and $x^3 : n^3 :: 972x : 125$.

Therefore, $972x = \frac{125x^3}{n^3}$.

The right hand side of this equation is a cube, therefore, $972x=$ a cube: Or, $27 \cdot 36x =$a cube. Hence, $36x$ must be a cube, which it evidently is, when $x=6$, as 36 is the square of 6.

6. Given $\left.\begin{array}{l} x+y+xy(x+y)+x^2y^2=85 \\ xy+(x+y)^2+xy(x+y)=97 \end{array}\right\}$ to find x and y.

Young, 145. Ans. $x=6$ $y=1$.

Put $(x+y)=s$ $xy=p$, then the equations become

$$\left.\begin{array}{l} s+sp+p^2=85 \\ \text{And } p+sp+s^2=97 \end{array}\right\}$$

By addition $(s+p)+s^2+2sp+p^2=182$. (1)
Assume $Q=s+p$, then equation (1) becomes
$Q^2+Q=182$. Hence $Q=13$. Or, $s+p=13$. The remaining steps are obvious.

7. Given $\sqrt{x+12} = \frac{12}{\sqrt{x+5}}$, to find x.

Square $x+12=\frac{144}{x+5}$. Put $x+5=y$.

Then $y+7=\frac{144}{y}$. Or, $y^2+7y=144$.

Put $2a=7$. Then $18a+81=144$.
Hence $y^2+2ay+a^2=a^2+18a+81$.
$y+a=a+9$. Or, $-a-9$.

8. Given, $\begin{cases}(x+y)(xy+1)=18xy. \\ (x^2+y^2)(x^2y^2+1)=208x^2y^2\end{cases}$ to find x and y.

Ans. $x=2\pm\sqrt{3}$. Or $7\pm4\sqrt{3}$.
$y=7\pm4\sqrt{3}$. Or $2\pm\sqrt{3}$.

Solution. Take $x+y=s$. $xy+1=t$ and $xy=p$.

Then $st=18p$. (1) And $(s^2-2p)(t^2-2p)=208p^2$. (2)

Multiply as indicated and take the value of s^2t^2 from equation (1) and we have $324p^2-2p\ (s^2+t^2)+4p^2=208p^2$.

By reduction, $p=\frac{s^2+t^2}{60}$. From equation (1) $p=\frac{st}{18}$.

Therefore, $\frac{s^2+t^2}{60}=\frac{st}{18}$. Assume $s=nt$.

Then this last equation becomes, by a little reduction,

$\frac{n^2+1}{10}=\frac{n}{3}$. Hence $n=3$. Or, $\frac{1}{3}$.

This establishes a relation between $x+y$ and $xy+1$.

Another Solution by Charles E. Matthews.

Multiply the original equations as indicated, and

$x^2y+xy^2+x+y=18xy$.

$x^4y^2+x^2y^4+x^2+y^2=208x^2y^2$.

Divide the first by xy, the 2d by x^2y^2, then

$x+y+\frac{1}{y}+\frac{1}{x}=18$. And $x^2+y^2+\frac{1}{y^2}+\frac{1}{x^2}=208$.

Assume $x+\frac{1}{x}=m$. And $y+\frac{1}{y}=n$.

Then $m+n=18$. And $m^2+n^2=212$.

From which m and n are easily found, afterwards x and y.

9. A square public green is surrounded by a street of uniform breadth. The side of the square is 3 rods less than 9 times the breadth of the street: and the number of square rods in the street exceeds the number of rods in the perimeter of the square by 228. What is the area of the square?

(Day 307.) Ans. 576 rods.

10. A man wishes to purchase a certain number of acres

14*

of land for the money he has at his command. Cleared land is worth 10 dollars per acre; uncleared land is worth $8.—He finds that if he buys 120 acres of cleared land, and lays out the rest of his money for that which is not cleared, he will not get the quantity of land he wants by 25 acres, but, if he buys 220 acres of uncleared land, and then buys a sufficient number of acres of cleared land to make up the number of acres he wants, he will have 4 dollars left. How many acres of each must he buy to have the quantity he wishes, and lay out all his money? (Harney page 203.

Ans. 20 acres cleared, 218 uncleared.

N. B. Call to mind problem first of this section.

In working geometrical problems algebraically much labor may be saved by paying attention to the *relation* of the given numbers.

We give the following as illustrative of these remarks.

1. If the perimiter of a right angled triangle be 720, and the perpendicular falling from the right angle on the hypothenuse be 144; what are the lengths of the sides?

(Day Alg. p. 305.) Ans. 300, 240 and 180.

If we use the identical numbers given 144 and 720, as nine-tenths of our teachers do, they will give large and tedious equations, but if we compare 144 and 720, we shall perceive that one is exactly 5 times the other, and considering the nature of *similar* triangles, we can work on one of only 144th of the linear dimensions of the first, or a triangle whose perimiter is 5, and perpendicular from the right angle 1.

Solution. Let x and y be the two sides, then $5-x-y$ will be the hypothenuse.

And $x^2+y^2=(5-x-y)^2$, and $xy=5-x-y$.

Each member of this last equation expresses the double area of the triangle. Put $x+y=s$. $xy=p$.

Then $s^2-2p=\overline{(5-s)}^2=25-10s+s^2$, and $p=5-s$.
Or, $10s\ -2p=25$
And $2s\ +2p=10$

By addit'n $12s\ =35$. Or, $s=\frac{35}{12}$.

But $s=x+y$, the sum of the two sides which, which, taken from 5, or $\frac{60}{12}$, gives $\frac{25}{12}$, for the hypothenuse of the small triangle, hence $\frac{25}{12}$ by $144=300$, the hypothenuse of the large triangle.

2. The sum of the two sides of a plane triangle is 1155, the perpendicular drawn from the angle included by these sides to the base, is 300; the difference of the segments of the base is 495, what are the length of the three sides?

(Day 305.) Ans. 945, 375, 780.

Write the given members in order, thus 300, 495, 1155. Divide them by 15, and their relation is 20, 33, 77.

The two latter numbers have a common factor 11, which call a. Put $b=20$.

Then the three given lines will be b, $3a$, and $7a$. Let $x=$ the less side. $7a-x=$the greater side, $y=$the shorter segment, and $y+3a=$the longer segment of the base.

Then $y^2+b^2=x^2$. (1
And $y^2+6ay+9a^2+b^2=49a^2-14ax+x^2$. (2)
Subtract (1) from (2), drop $9a^2$ from both sides, and divide by $2a$, and $3y=ab-7x$. (3)

We write $2b$ in place of 40, after dropping $9a^2$.

From the square of (3) subtract 9 times, equation (1,) and we have $-9b^2=a^2b^3-14\ abx+2bx^2$.

Divide by b, afterwards by 2, recollecting that $b=20$, and we have $-90=10a^2-7ax+x^2$.

Add $\frac{9a^2}{4}$, to both sides to complete the square.

Then $\frac{9a}{4}-90=\frac{9}{4}(a^2-40)=\frac{9}{4}\cdot 81=\frac{49a^2}{4}-7ax+x^2$.

Extract square root $\frac{3}{2}\cdot 9=\frac{7a}{2}-x$.

Or, $x=25$. Then $25.15=375$.

3. Divide the number 74 into two such parts that the difference of the square roots of the parts may be 2.

Ans. 25 and 49.

Let $x-1$, and $x+1$ be the square roots, of the two parts. This problem can also be solved by the Diophontine analysis.

4. Given $x^2+y^2=45$ and $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$, to be solved by the Diophontine analysis. Ans. $x=6$. $y=3$.

5. Given $x^2+y^2=45$ and $(x+y)x=54$ to find x and y by the Diophontine analysis. Ans. $x=6$. $y=3$.

The two preceding should also be worked by common algebra.

6. A and B traveled on the same road, and at the same rate, from Huntingdon to London. At the 50th mile stone from London, A overtook a drove of Geese, which were proceeding at the rate of 3 miles in 2 hours, he afterwards met a stage wagon, which was moving at the rate of 9 miles in 4 hours. B overtook the same drove of Geese at the 45th mile stone, and met the same stage wagon exactly forty minutes before he came to the 31st mile stone. Where was B when A reached London?

Solution. Let $x=$ miles traveled by each per hour, and $y=$ distance B was behind A, then $50-2x=$ the distance from London where A met the wagon.

Also, $\frac{y}{x+\frac{9}{4}}=\frac{4y}{4x+9}=$ time elapsed between the meeting of the wagon with A and B, therefore $\frac{9y}{4x+9}=$ distance traveled by the wagon during this time, consequently

$50-2x+\frac{9y}{4x+9}=$distance of the wagon from London, when met by B. But this distance is also $=31+\frac{2x}{3}$ therefore

$50-2x+\frac{9y}{4x+9}=31+\frac{2x}{3}$. Again $y+5:5::x:\frac{3}{2}$,

whence $3y+15=10x$ and $y=\frac{10x-15}{3}$, substituting and reducing, we get the quadratic

$x^2-\frac{123x}{16}=\frac{378}{32}$. Whence $x=9$, consequently $y=25$.

7. Given $x^2+xy=77$, and $xy-y^2=12$, to find x and y, by the Diophontine analysis. Ans. $z=7$. $y=4$.

8. Three equal circles touch each other externally, and enclose between the points of contact a acres of ground, what is the radii of the circle?

Ans. $\frac{\sqrt{a}}{0,16125}$

9. A person has £27 6*s.* in guineas and brown pieces, out of which he pays a debt of £14 17*s.*, and finds that he has exactly as many guineas left as he has paid crowns away; and as many crowns as he has paid away guineas; how many of each had he at first? (Young page 56.)

Ans. 9 crowns paid away; 12 guineas paid away.

Suppose $x=$ the guineas paid away.
And $y=$ the crowns paid away.
Then $21x+5y=297=$ amount paid out } per question.
And $5x+21y=249=$ amount on hand } per question.
Add these equations and divide by 26, &c.

10. The sum of three numbers in harmonical proportion is 191, and the product of the first and third is 4032. What are the numbers? Ans. 72, 63, 56.

11. Is it possible to pay £50 by means of guineas and three shilling pieces only. Ans. Impossible.

12. A merchant drew every year upon the stock he had in trade, the sum of a dollars for the expense of his family. His profits each year, were the nth part of what remain-

ed after this reduction, but at the end of the 3d year he finds his stock exhausted; how much had he at the beginning?

(Totten, 164.) Ans. $\frac{a(3n^2+3n+1)}{(n+1)^2}$

10. Given $\left\{\begin{array}{r}\sqrt{(5\sqrt{x}+5\sqrt{y})}+\sqrt{x}+\sqrt{y}=10 \\ \text{And } x^{\frac{5}{2}}+y^{\frac{5}{2}}=275\end{array}\right\}$ to find x and y.

Put $\sqrt{(5\sqrt{x}+5\sqrt{y})}=n$. Then the first equation becomes $\frac{n^2}{5}+n=10$. Which equation gives $n=5$.

Whence $\sqrt{x}+\sqrt{y}=5$.

From this last and the 2d equation, we find $x=9$, and $y=4$.

11. Three merchants A, B, C, on comparing their gains, find, that among them all they have gained 1444 dollars, and that B's gain added to the square root of A's, made 920 dollars, but if added to the square root of C's, it made 912. What were their several gains?

Ans. A 400, B 900, C 144.

12. A and B purchased a farm containing 900 acres, at 2 dollars per acre, which they paid equally between them, but on dividing the same, A took that part of the farm on which were the best improvements, and agreed to pay 45 cents per acre more than B. How many acres had each, and at what price?

Ans. { A has 400 acres. B 500.
A's price 2,25. B's 1,80.

This problem appears in a variety of forms, sometimes it is a wall to be built, or a ditch to be dug, one end being more expensive per rod than the other, &c., but notwithstanding these different technicalities, it is essentially the same problem, and its *unmerited* notoriety arises from the fact that most persons consider numbers in their abstract, and not in their relative value. For instance, the number 10 is relatively less to add to 100 than to take from 100, and in the above example, as each was at first to pay 200 cents and afterwards, by agreement, one was to pay 45 cents per acre more than the other, many would think it sufficient to add 22,5 cents, and subtract the same from 200, making one pay $225\frac{1}{2}$ cents, the other $177\frac{1}{2}$ cents per acre, but with these prices neither the exact number of acres nor the exact number of dollars will come in, hence, arises the wonder and attention that this problem has excited, but understanding the true nature of numbers at once dispels the mystery.

13. What are the dimensions of a right angled triangle, whose sides are in arithmetical progression; the least side squared and divided by 6, that quotient multiplied by the common difference, will be equal to the area?

Ans. 24, 32 and 40.

14. Given $\begin{cases} x(\sqrt{y}+1)+2\sqrt{xy}=55-y(\sqrt{x}+1) \\ \text{And } x\sqrt{y}+y\sqrt{x})=30. \end{cases}$ to find x & y.

15. Given the sum of the cubes of two numbers $=35$, and the sum of their ninth powers $=20195$, to find the numbers. Ans. 3 and 2.

Let $x^{\frac{1}{3}}$ and $y^{\frac{1}{3}}$ be the numbers, and divide one equation by the other.

16. Given $\begin{cases} x^2+xy=3 \\ (y^2+xy)^{\frac{2}{3}} \quad (y^2+xy)^{\frac{3}{2}}=1. \end{cases}$ to find the values of x and y.

Ans. $x=\frac{3}{2}$. $y=\frac{1}{2}$.

17. Find two numbers, such that the sum of their squares being subtracted from 3 times their product, 11 will remain; and the difference of their squares being subtracted from twice their product, the remainder will be 14.

(Totten's Alg. page 266.) Ans. 5 and 3.