A BRIEF COURSE IN THE CALCULUS

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CONTENTS.

CHAPTER I.

GRAPHS.

CHAPTER II.

GRAPHS CONTINUED. ASYMPTOTES. ALGEBRAIC AND TRANSCENDENTAL FUNCTIONS. CONIC SECTIONS.

CHAPTER III.

LIMITS.

CHAPTER IV.

DERIVATIVES. CONTINUITY.

CHAPTER V.

RATES. TANGENT AND NORMAL.

CHAPTER VI.

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

CHAPTER VII.

HIGHER DERIVATIVES. DERIVED CURVES. APPLICATIONS TO MECHANICS.

CHAPTER VIII.

INFINITE SERIES. TAYLOR'S THEOREM. EXPANSION OF FUNCTIONS.

CHAPTER IX.

MAXIMA AND MINIMA. CONCAVITY.

CHAPTER X.

DIFFERENTIALS. DERIVATIVES OF AREA AND VOLUME. DIFFERENTIALS OF FUNDAMENTAL FORMS.

CHAPTER XI.

INTEGRATION.

CHAPTER XII.

INTEGRATION AS A LIMIT OF A SUM. LENGTHS OF ARCS. AREAS OF CURVES. SOLIDS OF REVOLUTION. CENTER OF MASS. MOMENT OF INERTIA.

CHAPTER XIII.

SINGULAR FORMS. PARTIAL DIFFERENTIATION.

PREFACE.

THIS brief course in the Calculus is intended not only for the class-room, but for the student without a teacher, who hopes to acquire some knowledge of the working principles of the Calculus in a short time. For students who desire to study more than the merest elements of Physics and Electrical Theory or other scientific branches, a brief course in infinitesimal analysis is a necessity. For the general student, it properly rounds out his course in mathematics and exhibits the most interesting as well as the most powerful instrument for research ever devised.

The book presupposes some knowledge of Geometry, a working knowledge of Algebra through logarithms, and a thorough knowledge of the elements of Trigonometry. Two introductory chapters on Graphs will supply the student with all he actually needs of Analytic Geometry to read the book, but it is desirable, if possible, that a brief course in Analytic Geometry should be studied before taking up the study of the Calculus.

The aim has been to write a book for the beginner that should be simple, clear and logical. To this end the method of limits was adopted as giving the only adequate presentation of the subject in all its generality. The method of rates and the method of infinitesimals (when logically presented) both involve the notion of a limit.

There is nothing recondite about it. We cannot grasp with any certainty the idea of a train having a certain velocity at a given instant, without an implied reference to a limiting value. Besides, the student is supposed to be familiar with the many important applications of the theory of limits in Geometry, so that it is by no means a novel subject to him.

The topics selected in this book are mainly those involving fundamental principles. It is best for the student to hammer away at the essentials until they are thoroughly mastered before taking up more advanced subjects. Hence fundamental conceptions have been treated more fully than usual, and a number of illustrative, worked examples have been added.

In preparing this work, the author desires first of all to acknowledge his indebtedness to the French, from Duhamel to Jordan. The English works of Lamb and Gibson are also particularly inspiring and have been consulted, particularly the latter.

WM. CAIN.

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A BRIEF COURSE IN THE CALCULUS.

CHAPTER I.

GRAPHS.

1. Cartesian Coördinates of Points in a Plane. The problem of constructing a *graph* or *locus* from its equation is one of fundamental importance in the Calculus. As introductory to it, Des Cartes' system of coördinates will first be explained. In Fig. 1, $X'OX$ and $Y'OY$ are two straight lines of indefinite length, at right angles and intersecting at O. $X'OX$ is called *the axis of* x, $Y'OY$ *the axis of* y, the two lines together being called *the coördinate axes* and their intersection O, the *origin*. Assume a certain unit of length OM and beginning at O lay it off a number of times to the right and also to the left of O along $X'OX$, also apply the same unit from O up and down along $Y'OY$. Also subdivide each unit division into a convenient number of equal parts (five in the figure) to more accurately lay off fractional parts of a unit.

The position of a point P_1 in the plane XOY is defined as follows: draw a line from P_1 parallel to OY (and therefore perpendicular to OX) to intersection N with the axis of x; the distance NP_1 is called the *ordinate* y, regarded as positive if P_1 is above $X'OX$ and negative if P_1 is below $X'OX$.

The distance ON, from the origin to the foot of the ordinate is called the *abscissa* x and will be considered positive when P_1 is to the right of $Y'OY$, negative when to the left. The directed lengths $x = ON$, $y = NP_1$ are called the *coördinates* of the point P_1, also the axes $X'OX$, $Y'OY$ are called respectively, the *axis of abscissas* and the *axis of ordinates*.

The point P_1 is designated as the point (x, y), meaning the point whose abscissa is x and ordinate y, the abscissa always being written first.

Where P_1 is a fixed point, its coördinates are usually marked with subscripts as in the figure. Thus P_1 is the point (x_1, y_1), P_2 the point $(x_2 = OM, y_2 = MP_2)$ or (x_2, y_2).

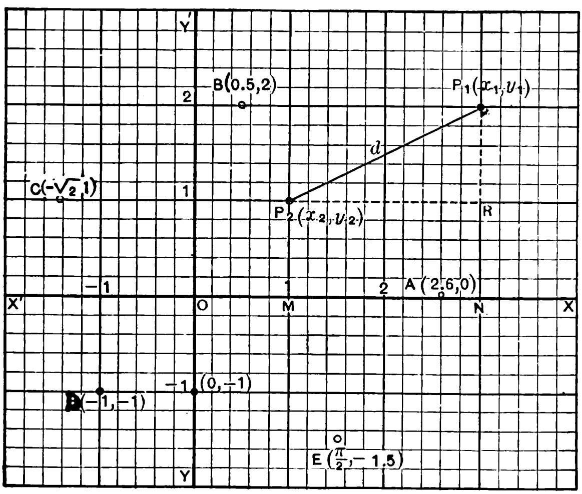

Fig. 1.

When numerical values are assigned with the proper sign to x and y, the point (x, y) is completely determined. Thus to plot the point (3, 2), lay off $x = ON = 3$ units of length to fix N, then draw NP_1 perpendicular to OX a distance $y = NP_1 = 2$, which uniquely determines the point P_1. Similarly, to locate the point B (0.5, 2), lay off 0.5 unit from O along OX and then 2 units upwards, or lay off first 2 units from O along OY and then 0.5 unit to right. To fix $C(-\sqrt{2}, 1)$, lay off from O to left along OX', $\sqrt{2} = 1.41$ nearly (estimating the hundredth of a unit by the eye) and from the end of the abscissa, lay off 1 unit upwards. Similarly, $D(-1, -1)$ is 1 unit to left of $Y'OY$ and 1 unit below OX; $E\left(\frac{\pi}{2}, -1.5\right)$ is 1.57 units of length to right of $Y'OY$ and 1.5 units below OX.

When a point is on the axis of x, $y = 0$: thus A (2.6, 0) is $x = 2.6$ units to right of axis of y, but since $y = 0$, it lies on the axis of x. Similarly the point $(x = 0, y = -1)$ lies on the axis of y at 1 units distance below the origin. The point (0, 0) is the origin O.

2. As a point moves steadily on without hops along the axis of x (say), its abscissa is said to vary *continuously* between the extreme limits attained by the point. This abscissa of the point in any fixed position is a *number* indicating the ratio (with the proper sign) of the distance of the point from O to the unit of length. This ratio may be integral, fractional or incommensurable. In the latter case, the abscissa cannot equal any fraction whose numerator and denominator are finite integers. Thus $\sqrt{2}$ and π are incommensurable numbers. It is thus postulated that to every point on $X'OX$ there corresponds an abscissa expressed as a plus or minus number, commensurable or incommensurable, and conversely, that to every number (abscissa) there corresponds a point.

Similar conclusions hold for the ordinates of a point moving along $Y'OY$. In fact, if a point P_1 traces a *continuous* line (one without breaks) in the plane, its coördinates x and y vary continuously.

3. Distance between Two Points. To find the distance d between $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. Draw P_2R parallel to OX to intersection R with NP_1 (Fig. 1).

$$P_2R = ON - OM = x_1 - x_2, \quad RP_1 = NP_1 - NR = y_1 - y_2,$$

$$\therefore d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad . \quad . \quad . \quad (1)$$

If P_1 and P_2 in the figure are interchanged, $(x_1 - x_2)$ is replaced by $(x_2 - x_1)$ and $(y_1 - y_2)$ by $(y_2 - y_1)$; but the squares being the same, (1) is unaltered.

It is a remarkable fact that this formula is true no matter where P_1 and P_2 are placed in the plane. There are 16 possible cases, for P_1 can be placed in either of the four quadrants and corresponding to each position of P_1, P_2 can be placed in either of quadrants I, II, III, IV as the spaces included in the angles XOY, $X'OY$, $X'OY'$ and XOY' are called.

It will be instructive for the reader to prove the formula for each of these cases. An illustration will be given which shows the simplest way of dealing with negative abscissas or ordinates not only here, but wherever they may be met with.

In Fig. 2, P_1 (x_1, y_1) is placed in the first quadrant, P_2 (x_2, y_2) in the third.

$$\therefore x_1 = ON, \; y_1 = NP_1; \; x_2 = -MO, \; y_2 = -P_2M.$$

Notice particularly that x_2 and y_2 are negative numbers, $\therefore$ $(-x_2) = MO$ is a positive number representing the actual

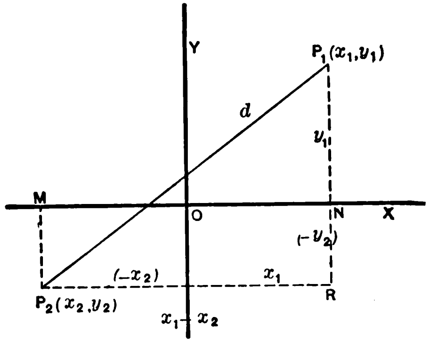

Fig. 2.

distance MO and $(-y_2) = P_2M$ is likewise a positive number, representing the actual distance P_2M.

$$\therefore\ P_2R = ON + MO = x_1 - x_2$$

$$RP_1 = NP_1 + RN = NP_1 + P_2M = y_1 - y_2$$

$$\therefore\ P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

which is formula (1).

If the positive numbers $(-x_2)$, $(-y_2)$ are marked directly on the corresponding lengths in the figure, the formula can be read off at once. Similarly the other cases are very quickly treated.

REMARK. If the point P_2 is at the origin O, $x_2 = 0$, $y_2 = 0$ and formula (1) reduces to

$$d = OP_1 = \sqrt{x_1^2 + y_1^2} \quad . \quad . \quad . \quad (2)$$

which is true wherever P_1 is placed.

1. Find by formula (1), P_1P_2 when $P_1 = (3, 2)$, $P_2 = (1, 1)$.

$$d = \sqrt{(3 - 1)^2 + (2 - 1)^2} = \sqrt{4 + 1} = \sqrt{5}.$$

2. Let $P_1 = (3, 2)$, $P_2 = (-1, -1) = D$ in Fig. 1.

Here, be careful to put $x_2 = -1$, $\therefore -x_2 = +1$; $y_2 = -1$

$\therefore -y_2 = +1$, $\therefore d = \sqrt{(3 + 1)^2 + (2 + 1)^2} = \sqrt{16 + 9} = 5$,

i.e., $P_1D = 5$ units of length.

The signs of the coördinates must be carefully attended to in all substitutions.

3. Draw the figure for each case and find the distance P_1P_2 first by marking the projections of P_1P_2 parallel to the axes in numbers and then computing the hypotenuse $d = P_1P_2$ and, secondly, by substituting the numerical values of (x_1, y_1), (x_2, y_2) directly in formula (1).

(a) $(x_1, y_1) = (3, 2)$ and (x_2, y_2) in turn $(6, 4)$, $(-1, 1)$, $(-2, -2)$, $(1, -2)$.

(b) $(x_1, y_1) = (-3, 2)$; $(x_2, y_2) = (1, 1)$, $(-1, 1)$, $(-1, -3)$, $(1, -2)$.

(c) $(x_1, y_1) = (-3, -2)$; $(x_2, y_2) = (1, 1)$, $(-1, 2)$, $(-1, -3)$, $(2, -1)$.

(d) $(x_1, y_1) = (3, -2)$; $(x_2, y_2) = (4, 6)$, $(-4, 2)$, $(-4, -2)$, $(1, -1)$.

The 16 cases mentioned above are illustrated by examples (*a*), (*b*), (*c*), (*d*). It is easily seen that if any other numbers are used than those given above, the formula would be verified. Hence it is true for any positions of P_1 and P_2.

4. Equation of a Circle. If in Fig. 2, the point $P_1 = (x_1, y_1)$ remains fixed and the point $P_2 = (x_2, y_2)$ is assumed to vary its position, with the proviso that $d = P_1P_2 = r$, shall always remain constant, then the successive positions of P_2 will all lie on the circumference of a circle with center at (x_1, y_1) and radius r.

Since for *any* position of P_2, Eq. (1) is true (Art. 3), we have on squaring (1) and replacing d by r,

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = r^2$$

where (x_2, y_2) represents any point on the circumference.

As it is customary to represent the coördinates of a variable point by (x, y), replace x_2 by x, y_2 by y in the equation above, giving for the equation of the circle,

$$(\mathbf{x} - \mathbf{x}_1)^2 + (\mathbf{y} - \mathbf{y}_1)^2 = \mathbf{r}^2 \quad . \; . \; . \; (3),$$

where (x_1, y_1) represent the coördinates of the center, (x, y) the coördinates of any point on the circumference and r the radius (Fig. 3).

By Art. 3, (x_1, y_1) can have any position, so that (1) and ∴ (3) are true wherever the center is placed in the beginning. The signs of the coördinates though must be strictly attended to.—*See examples.*

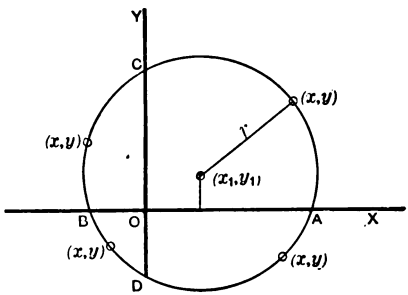

Fig. 3.

If the center is at the origin, $x_1 = 0$, $y_1 = 0$;

$$\therefore \; x^2 + y^2 = r^2$$

is the equation of a circle with center at the origin and radius r. Prove this independently.

EXAMPLE 1. The eq. of a circle with $r = 5$ and the coördinates of the center $(-3, 2) = (x_1, y_1)$ is found by substituting $x_1 = -3, y_1 = 2, r = 5$ in (3) to be, $(x+3)^2 + (v-2)^2 = 25$.

EXAMPLE 2. Suppose the radius remains $r = 5$ but that (x_1, y_1) is (i), $(2, 3)$; (ii), $(-1, -1)$; (iii), $(1, -2)$; find the equations of the three circles and construct.

EXAMPLE 3. The equation $x^2 + y^2 + 2x - 3y = 4$, can be transformed by completing the squares of the x and y terms, thus:

$$(x^2 + 2x + 1) + (y^2 - 3y + \tfrac{9}{4}) = 4 + 1 + \tfrac{9}{4} = \tfrac{29}{4},$$

or

$$(x+1)^2 + (y - \tfrac{3}{2})^2 = \left(\frac{\sqrt{29}}{2}\right)^2,$$

which is the equation of a circle whose center is ($x_1 = -1$, $y_1 = \frac{3}{2}$) and radius $\frac{\sqrt{29}}{2}$, as we find by comparing with (3).

5. Similarly any equation in the form,

$$x^2 + y^2 + Dx + Ey + F = 0,$$

with the coefficients of x^2 and y^2 unity and with no term in xy is the equation of a circle. The coördinates of the center and radius can be found as just explained. Do this for the following circles and construct.

(1) $x^2 + y^2 - 4x - 8y - 5 = 0.$

Ans. $x_1 = 2, y_1 = 4, r = 5.$

(2) $x^2 + y^2 + 6x + 10y = 2.$

Ans. $x_1 = -3, y_1 = -5, r = 6.$

(3) $x^2 + y^2 - 8x + 2y + 20 = 0.$

Ans. $x_1 = 4, y_1 = -1, r = \sqrt{-3}.$

Thus the radius is imaginary and the circle has no real existence.

If r should come out 0, the circle reduces to a point, the center.

(4) $3(x^2 + y^2) - 12x + 6y = 4.$

Divide by 3 and proceed as before.

$$x_1 = 2, y_1 = -1, r = 3.$$

6. Intercepts on the Axes. At the points A and B in Fig. 3 where the curve cuts the axis of x, $y = 0$; therefore on making $y = 0$ in (3) and solving for x, the two values $x = OA$, $x = OB$, corresponding to $y = 0$, are found. They are called the *intercepts* of the curve on the axis of x.

Similarly $x = 0$ in (3) gives the two values of y (OC and OD) which are called the *intercepts* on the y-axis.

Thus in Ex. 1 of Art. 5, $x^2 + y^2 - 4x - 8y - 5 = 0$, on making $y = 0$, we find $x = +5$ and -1, $\therefore OA = 5$, $OB = -1$, or the curve cuts the axis of x at (5, 0) and (−1, 0). When $x = 0$, $y = 4 \pm \sqrt{21}$, the two intercepts on y. If the equation of the circle were $x^2 + y^2 - 4x - 8y + 5 = 0$, the x intercepts, $2 \pm \sqrt{-1}$ are imaginary; which shows that the curve does not cut the axis of x. The y intercepts are real and $= 4 \pm \sqrt{11}$.

If in any case, both the x and y intercepts, as found above, are imaginary, the circle does not cut either axis.

If in Fig. 3, the radius is diminished so that the curve just touches an axis, the intercepts on that axis are equal.

Example. Find the x and y intercepts in circles (2) and (4) of Art. 5.

7. Equation of a Locus. Any equation involving x, y and constants is the equation of a *locus*. Thus (3) is the equation of a locus — the circle. In deriving that equation, use was made of the geometrical property defining a circle. Conversely, the equation of a locus may be given and it may be required to draw the curve from its equation and derive the geometrical property defining it.

Thus if the equation of a locus is,

$$4y = x^2,$$

the equation shows that the square of the abscissa of any point on it is 4 times the ordinate of *that* point.

From the equation we find that

$$\begin{aligned} &\text{if } x = 0, y = 0, \\ &\text{if } x = \pm 1, y = \tfrac{1}{4}, \\ &\text{if } x = \pm 2, y = 1, \\ &\text{if } x = \pm 3, y = \tfrac{9}{4} = 2\tfrac{1}{4}, \\ &\text{if } x = \pm 4, y = 4 \ldots \\ &\text{if } x \doteq \infty, \quad y \doteq \infty. \end{aligned}$$

The notation, "if $x \doteq \infty$, $y \doteq \infty$," is to be read, "if x increases indefinitely or without limit, y increases indefinitely or without limit."

Therefore the points (0, 0), (1, $\frac{1}{4}$), ($-$ 1, $\frac{1}{4}$), (2, 1) ($-$ 2, 1), (3, $2\frac{1}{4}$), ($-$ 3, $2\frac{1}{4}$), (4, 4), ($-$ 4, 4), lie on the curve, for these *simultaneous* values satisfy the equation $4y = x^2$.

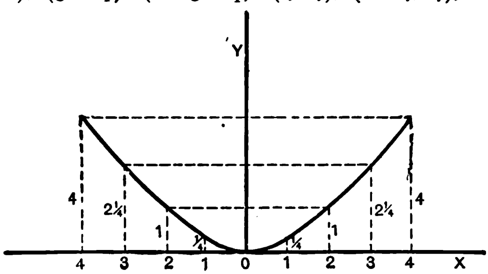

Fig. 4.

On plotting the above points and drawing a curve through them, we have the *graph* corresponding to the equation (see Fig. 4). It is known as the common parabola. For greater accuracy a number of intermediate points should be found.

8. Test to decide whether a Point (x_1, y_1) is on the Curve or not. If the given point is on the curve, its coördinates will satisfy the equation of the curve and conversely. If the coördinates of the point do not satisfy the equation of the curve, the point does not lie on the curve. Thus (3, 2) is not on the parabola $4y = x^2$, since 4×2 is not equal to $3^2 = 9$. The point $(\frac{1}{2}, \frac{1}{16})$ does satisfy $4y = x^2$ and thus lies on the curve.

The x and y, or the *current* coördinates in the equation of any locus, are thus general *variables* and can take an indefinitely great number of values; but of this number, no two values of x and y are to be paired arbitrarily, but only those coördinates that *simultaneously* satisfy the equation of the locus. Thus the equation of a locus is always to be understood with an *if.* If x is given a certain numerical value, the corresponding value (or values) of y with which it is paired, must be found by substituting the given value of x in the equation of the locus and solving for y. Similarly if y is assumed and x computed.

9. Limits of a Curve. The curve only extends over that region where both variables x and y are real. Thus if $x^2 = 4y$ or $x = \pm 2\sqrt{y}$, it is seen that if y is negative, x is imaginary; hence the curve does not extend below the axis of x. When y is positive, x is always real and as y increases uniformly and indefinitely, x increases in the positive direction or decreases (algebraically) in the negative direction for x, continuously and indefinitely. The curve is thus continuous and extends indefinitely upwards.

As another illustration, take the circle

$$x^2 + y^2 = r^2$$

with origin at the center and radius r. Since $y = \pm\sqrt{r^2 - x^2}$, the curve lies in the region corresponding to $x^2 < r^2$ or $-r \leqq x \leqq r$, *i.e.*, x is equal or greater than $-r$ or equal or less than $+r$. This is again expressed by saying that x may equal $-r$ or $+r$ or lies between $-r$ and $+r$ in value. Similarly solving for $x = \pm\sqrt{r^2 - y^2}$, we see that for real values of x, $-r \leqq y \leqq r$. Thus the curve does not extend up or down, to right or to left a greater distance than r which is otherwise evident from a figure. From the equation, when $x = 0$, $y = +r$ or $-r$ and when $y = 0$, $x = +r$ or $-r$, giving the intercepts.

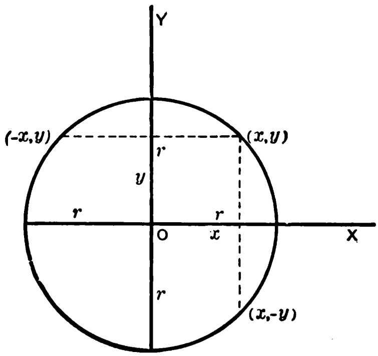

Fig. 5.

10. Symmetry with Respect to an Axis. The point (x, y) is symmetrical as to the axis of x with the point $(x, -y)$; therefore if (x, y), $(x, -y)$ both satisfy a locus, the latter is symmetrical as to x.

Thus in the locus, $x^2 + y^2 = r^2$ (Fig. 5), if a point (x, y) satisfies the equation, the point $(x, -y)$ will satisfy it. Similarly for any point (x, y) on the locus. Hence it is symmetrical as to the axis of x.

In the parabola $4y = x^2$ (Fig. 4), if (x, y) is a point on the locus, $(x, -y)$ is *not* on it; therefore this parabola is not symmetrical as to x, since the term in y is not of an even power, so that it changes sign with y.

The simple test then is this: *if the equation of the locus contains only even powers of y, it is symmetrical as to the axis of x.*

Similarly if (x, y), $(-x, y)$ both satisfy the equation of a locus, it is symmetrical as to the axis of y; hence *if the equation of the locus contains only even powers of x, it is symmetrical as to the axis of y.*

Let the student apply this test to the two curves above.

Also prove, by aid of a figure, that if (x, y), $(-x, -y)$, both satisfy the equation of a curve, then the curve is symmetrical as to the origin. [Hint: show that a straight line from the origin to (x, y) is equal in length to, and a continuation of, the line from $(-x, -y)$, to the origin.] In this case the line from (x, y) to $(-x, -y)$ is called a *diameter* of the curve. It is bisected by the origin.

Prove by this test that the circle $x^2 + y^2 = r^2$ is symmetrical with respect to the center.

The student is now prepared to draw certain simple curves very quickly from their equations.

1. Draw the graph of $y^2 = -x$.

(Art. 9) $y = \pm\sqrt{-x}$ $\therefore$ x must be negative in order that y may be real and the curve extends indefinitely towards the negative side of the x axis, but it does not extend to the right of the origin. Also as x decreases from 0 indefinitely (or is negative and increases numerically) y increases numerically indefinitely. By Art. 10 the curve is only symmetrical as to the x axis. Compute now a few pairs of simultaneous values of x and y and draw the curve, which passes through the origin, since $x = 0$ gives $y = 0$.

2. Discuss and draw the graphs of, the common parabolas, $y^2 = 4x$, $x^2 = -y$; the semi cubical parabola, $y^2 = 4x^3$; the cubical parabola, $y = x^3$.

3. Draw the graphs of $x^2 = 4y^3$, $y^2 = -x^2$, $y^3 = -x$, $y^4 = x^3$, $y^4 = -x^3$.

11. The Equation of a Straight Line. Consider, in Fig. 6, the straight line P_1OP through the origin, making the angle MOP with the axis of x. The tangent of this angle m is called the "slope" or "gradient" of the line. Take *any* point $P\ (x, y)$ in the first quadrant $\therefore x = OM$, $y = MP$, and $MP/OM = y/x = \tan MOP = m \therefore y = mx$. For a point $P_1\ (x_1, y_1)$ on the line in the third quadrant, $x_1 = -M_1O, y_1 = -P_1M_1 \therefore y_1/x_1 = P_1M_1/M_1O = MP/OM = m \therefore y_1 = mx_1$. Hence, if (x, y) *denote any point on the line whatsoever*, whether in the first or third quadrants, the relation

$$y = mx$$

will always be true. It is therefore the equation of the line P_1OP.

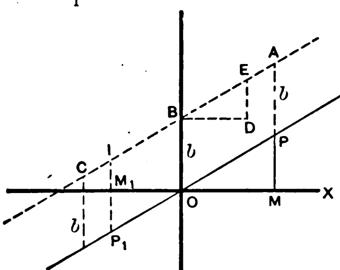

Fig. 6.

If we measure off from the origin a distance $OB = b$ and through B draw CBA parallel to P_1OP, its equation will be,

$$y = mx + b.$$

Thus if $x = OM, y = MA = MP + PA = mx + b$. Similarly, if $x = -M_1O, y = M_1I = -P_1M_1 + P_1I = y_1 + b = mx + b$, from an equation above.

Thus the x and y of *any* point on the line CBA, extended indefinitely in both directions, always satisfies the equation $y = mx + b$ and no point *not* on the line satisfies this equation; hence it is the equation of the straight line whose slope is m, and y intercept b.

EXAMPLE. Construct the line $y = \frac{2}{3}x + 4$. Here comparing with $y = mx + b$, we find, $b = 4$, $m = \frac{2}{3}$ $\therefore$ lay off $OB = 4$ units of length, draw $BD \parallel OX$, 3 units to the right to point D; then draw $DE \parallel OY$, a distance 2 to E. A line connecting B and E is the required line, since $m = \tan MOP = \tan DBE$, has been constructed equal to $\frac{2}{3}$ and $b = OB = 4$.

If the scale adopted is small, since $m = \frac{2}{3} = \frac{4}{6} = \frac{6}{9} =$ etc., lay off $BD = 6$ (say), and $DE = 4$ or $BD = 9$ with $DE = 6$. In fact, for accuracy the point E should be as far from B as the limits of the drawing will admit.

Having drawn this line, as just shown, compute from the equation several simultaneous values of x and y, as $(-9, -2)$, $(-6, 0)$, $(-3, 2)$, $(0, 4)$, $(3, 6)$, plot them and see if they fall on the line as they should.

Next let us consider the line POP_1 in Fig. 7, making the obtuse angle MOP_1 with the axis of x. Let $\tan MOP_1 = m$ be called the slope of this line $\therefore m = -\tan(180 - MOP_1) = -\tan P_1OM_1 = -M_1P_1/M_1O = M_1P_1/(-M_1O) = y_1/x_1$ (if $x_1 = -M_1O$, $y_1 = M_1P_1$) $\therefore y_1 = mx_1$.

Again consider the point P, where $x = OM$, $y = -PM$ $\therefore$ as above, $m = -M_1P_1/M_1O = -PM/OM$ (from similar triangles) $\therefore m = y/x$, or $y = mx$.

Thus the (x, y) of *any* point on POP_1, however extended, always satisfies

$$y = mx,$$

which is thus the equation of the line.

Next, lay off $OB = b$ and draw ABC parallel to POP_1 at a distance b (measured parallel to y) above it $\therefore$ the equation of ABC is

$$y = mx + b,$$

since for any x as OM_1, $y = M_1A = M_1P_1 + P_1A = mx + b$;

or if $x = OM$, $y = -CM = -PM + PC = mx + b$, from the equation for POP_1.

The equation $y = mx + b$, has the same form for the line *ABC* in Fig. 7 as in Fig. 6, but it must be observed that in Fig. 6, $m = \tan MOP$, whereas in Fig. 7, $m = \tan MOP_1 = -\tan P_1OM_1$.

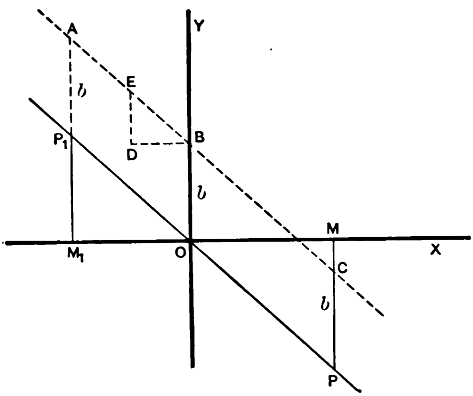

Fig. 7.

In both figures, if the line *ABC* had been drawn parallel to P_1OP at a distance on the axis of $y = b$ *below* the origin, each ordinate of P_1OP would be diminished by b $\therefore$ the equation of *ABC* in that case would be

$$y = mx - b$$

in either figure. Further, if b is supposed positive when measured upwards from O, negative when measured downwards, the one equation, $y = mx + b$, can be said to represent the equation of any straight line with slope m and y intercept b. Note the great generality of this result, for this equation has been proved true wherever (x, y) is taken on the line.

REMARK. In Fig. 7, since the angle MOP is minus and P_1OM_1 is plus, $\tan MOP = -\tan P_1OM_1 = \tan MOP_1$; so that $\tan MOP$ can be said to equal the slope in both figures 6 and 7. If $MOP = \theta$, in either figure, $m = \tan \theta$.

1. Construct the line $y = -\frac{2}{3}x + 4$.

Comparing with $y = mx + b$, we find, $b = 4$, $m = -\frac{2}{3} = -\tan P_1OM_1$ (Fig. 7), ∴ lay off $OB = b = 4$, then $BD = 3$, $DE = 2$, BD being drawn to the left parallel to XO and DE parallel to OY. This gives $\tan EBD = \tan P_1OM_1 = \frac{2}{3}$ ∴ BE is the required line.

2. Construct the lines, $y = \frac{3}{4}x - 5$, $y = -\frac{6}{7}x - 3$, $y = \frac{1}{2}x + 8$, $y = -\frac{1}{3}x$, $y = x$, $y = -x$.

3. Any equation of the first degree in x and y is the equation of a straight line, since it can be put in the form, $y = mx + b$.

Construct, $2x + 3y + 9 = 0$ (or $y = -\frac{2}{3}x - 3$), $3x - 2y + 6 = 0$, $12x + 4y = 9$.

4. If preferred, the intercept method (Art. 6) may be used. Thus, in $10x + 6y = 60$, $x = 0$ gives $y = 10$, ∴ $(0, 10)$ is one point on the line. Again $y = 0$ gives $x = 6$ and $(6, 0)$ is on the line ∴ lay off the x and y intercepts, 6 and 10, and draw the line. Construct in this way, $x + 2y = 20$, $2x - 3y = 12$, $4x + 6y = 24$.

5. Where the intercepts are small, it may be best to compute the (x, y) of extreme points on the line and draw the line through them. To construct, $x + 2y = 1$, $(x = -21, y = 11)$, $(x = 21, y = -10)$, ∴ plot these points and draw the line through them.

Construct, $x + 3y = 5$, $2x - 5y = -1$.

6. Construct $x = a$ (a line parallel to OY at a distance a from it). $y = b$ is a line parallel to OX at a distance b from it, $x = 0$ is the axis of y; $y = 0$ the axis of x.

7. By the intercept method (Art. 6), show that

$$\frac{x}{a} + \frac{y}{b} = 1$$

is the equation of a straight line (see Ex. 3) with intercepts a and b, crossing the first quadrant. Also prove,

$$-\frac{x}{a}+\frac{y}{b}=1$$

crosses the second quadrant with intercepts $(-a)$ and b;

$$-\frac{x}{a}-\frac{y}{b}=1$$

crosses the third quadrant with intercepts $(-a)$ and $(-b)$;

$$\frac{x}{a}-\frac{y}{b}=1$$

crosses the fourth quadrant with intercepts a and $(-b)$.

It follows that $x/a+y/b=1$ can represent any of these cases if a and b admit of plus and minus values.

8. Reduce Exercises 1, 2, 3, 4, to this intercept form and construct.

9. If the point (x_1, y_1) lies on the line

$$y = mx + b$$

$$\therefore\ y_1 = m_1 x_1 + b$$

subtracting, $$\mathbf{y - y_1 = m\,(x - x_1)} \quad . \; . \; . \quad (\alpha)$$

the equation of a line through a point (x_1, y_1) and having a given slope.

10. If the straight line passes through another point (x_2, y_2), its coördinates satisfy (α) $\therefore\ y_2 - y_1 = m\,(x_2 - x_1)$.

$$\therefore\ m = \frac{y_2 - y_1}{x_2 - x_1} \quad . \; . \; . \quad (\beta)$$

a general expression for the slope of any line passing through (x_1, y_1), (x_2, y_2). On substituting in (α),

$$\mathbf{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)} \quad . \; . \; . \quad (\gamma)$$

the equation of a straight line through two points (x_1, y_1) and (x_2, y_2).

The different forms of the equation of a straight line given in Exercises 7, 9 and 10 are true however the line is drawn or wherever (x, y) is taken on it, for the equations have all been derived from the form $y = mx + b$ which was proved generally true in Art. 11. Therefore, one can have perfect confidence in the truth of the results when applied to any example on simply substituting numerical values with their proper signs.

11. Write down by aid of Ex. 10 (γ) the equations of lines passing through the pairs of points $(2, 3)$ and $(-5, 6)$; $(3, -1)$, $(-4, -8)$; $(0, 2)$ and $(-4, 10)$; $(-1, -1)$ and $(-8, -9)$. In the last example (by way of illustration) put $(x_1 = -1, y_1 = -1)$, $(x_2 = -8, y_2 = -9)$ in (γ) and get,

$$y + 1 = \frac{-9+1}{-8+1}(x+1) = \frac{8}{7}(x+1).$$

The slope of this line is 8/7.

12. By aid of the results of Ex. 7, write the equations of lines with x and y intercepts $(6, 5)$, $(-4, 4)$, $(-7, -9)$, $(8, -6)$ and test by Art. 6.

13. By the method of Art. 8, test whether the points, $(0, 4)$, $(2, 3)$, $(6, 1)$, $(8, 0)$, $(2, 2)$ and $(-1, 6)$ lie on the line $x + 2y - 8 = 0$. Find its slope and intercepts.

14. **Angle between two Straight Lines.**

Let the equations of the two lines which make the angle ϕ with each other, be,

$$y = mx + b, \quad y = m'x + b',$$

when,

$$m = \tan\alpha, \qquad m' = \tan\beta \text{ (Fig. 8)};$$

$$\text{if } \alpha > \beta, \qquad \phi = \alpha - \beta$$

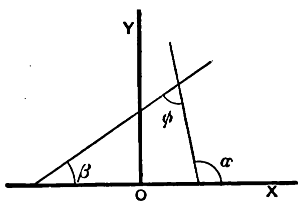

Fig. 8.

$$\therefore \tan\phi = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

$$\therefore \tan \phi = \frac{m - m'}{1 + mm'}.$$

If $\alpha < \beta$, the result, when numerical values are inserted, will come out minus. The arithmetical value will correspond to the acute angle ϕ.

The lines are parallel when $\tan \phi = 0$ or when $m = m'$.

The lines are perpendicular when $\tan \phi = \infty$ or when $1 + mm' = 0$. The last condition can be written, $m' = -\frac{1}{m}$.

What are the equations of lines passing through (2, 1) and (i) parallel, (ii) perpendicular, to the line, $y_1 = 2x + 3$?

The slope of the first line is (by what has just been proved) $m = 2$; of the second line $m = -\frac{1}{2}$. Substitute these values, together with $x_1 = 2, y_1 = 1$ in (a) Ex. 9. $\therefore$ (i) $y - 1 = 2(x - 2)$; (ii) $y - 1 = -\frac{1}{2}(x - 2)$, are the equations of the required lines.

15. **Intersection of Two Lines.** The coördinates of the point or points of intersection *alone* satisfy *both* equations.

Therefore solve the equations of the two loci, regarded as simultaneous, for x and y. Thus call the points of intersection of the parabola $4y = x^2$ (Fig. 2) and the straight line $y = x$, (x_1, y_1) $\therefore$ since $(x_1\ y_1)$ is on both loci, $4y_1 = x_1^2$, $y_1 = x_1$. Solving these two equations, we find the points of intersection to be (0, 0) and (4, 4).

16. Find the coördinates of the intersection of the lines, $2x + y = 10$, $y = 2x + 2$. Construct.

17. Construct the polygon whose vertices are at (1, 1), (0, 2), (1, 3), (2, 2). Prove by Ex. 10 (β) that the opposite sides are parallel; also prove by Ex. 12 that adjacent sides are perpendicular; lastly, show that the figure is a square.

18. Find the equations of the lines through (2, 4) which are respectively parallel and perpendicular to the following lines and construct. (i) $x - 2y = 10$; (ii) $3x + 2y = 12$; (iii) $5x - 3y = 15$.

19. A quadrilateral has the vertices (2, 2), (5, 4), (8, 0), (3, − 3), show that the equations of the sides, in order, are, $3y = 2x + 2$, $3y + 4x = 32$, $5y = 3x - 24$, $y + 5x = 12$.

20. By use of the formula of Ex. 13, find the tangent of the acute angle between the sides of the quadrilateral of Ex. 19, taken in order.

Ans. $\tan\phi = 18, \frac{29}{3}, \frac{14}{5}, \frac{17}{7}.$

CHAPTER II.

GRAPHS. ASYMPTOTES. ALGEBRAIC AND TRANSCENDENTAL FUNCTIONS. CONIC SECTIONS.

12. Algebraic Functions are those involving the variables in a finite number of terms which are affected only by the operations of addition, subtraction, multiplication, division, raising to powers and extraction of roots. *Transcendental functions* include all other cases. Thus x^2, $\sqrt{x}$, $x^{-\frac{1}{2}} + ax$, are algebraic functions of x, whilst, $\sin x$, $\tan^{-1} x$, a^x, $\log x$, are transcendental functions.

When two variables (as x and y) are so related that to a definite value of one of them there corresponds a definite value (or values) of the other, the second is said to be a function of the first. When the relation is expressed by an equation between x and y, the graph can generally be constructed. If arbitrary values are assigned in turn to x and the corresponding values of y computed from the equation, x is termed the *argument* or *independent variable* and y the dependent variable. If, however, it is more convenient to assume values for y and compute the corresponding values of x, then y would be called the independent and x the dependent variable.

13. Graph of 1/x. From the equation, $y = 1/x$, the following pairs of values are found (.3, 3.33), (.4, 2.5,) (.6, 1.67), (.8, 1.25), (1, 1), (1.5, .67), (2, .5), (3, .33), from which the branch of the curve in the first quadrant

(Fig. 9) is plotted. Equal numerical values with minus signs for (x, y) are found for the branch in the third quadrant. The two branches constitute the graph of $y = 1/x$. But this graph possesses peculiarities not hitherto met with on supposing it extended beyond the limits of the figure. Thus let x successively equal 10, 100, 1000, . . . then $y = \frac{1}{x}$ equals successively 0.1, 0.01, 0.001, . . . ; so that as a point moves along the branch in the first quadrant indefinitely to the right, the distance y from the point to the x = axis can become less than any quantity (not zero) assigned in advance, however small that quantity is taken. Thus if y is to become less than 0.001, we have only to make x greater than 1000. For $y < .0001$, x must be $> 10,000$ and so on.

Similarly for the lower branch, it is found that as x increases in the negative direction indefinitely, y approaches zero as near as we please, without ever becoming zero, however great in the negative direction x becomes.

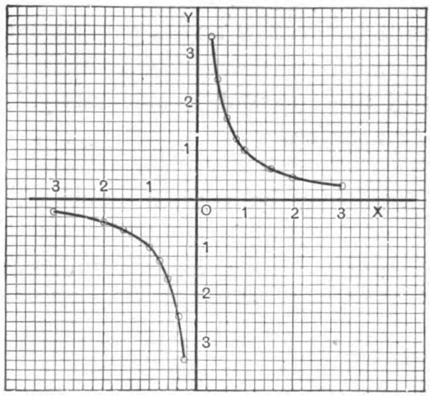

Fig. 9.

In this case, the axis of x is said to be an *asymptote* to both branches of the curve.

On taking y for the independent variable, we see from

$x = \frac{1}{y}$, that as y increases indefinitely, x decreases indefinitely; but x never becomes zero, however great the (finite) value of y is. The axis of y is thus an asymptote to the upper branch. Similarly it is shown to be for the lower branch.

14. Asymptotes. When a curve extends indefinitely from the region of the origin, it may approach a fixed line more and more the farther it recedes from the origin. *If the distance from a point on the curve to a fixed line can be made as near zero as we please, without actually becoming zero, as the point, moving along the curve, recedes indefinitely from the origin, the fixed line is called an asymptote to the curve.*

The following notation is very convenient. When a variable u approaches a finite constant a, so that the difference between u and a can become and remain less than any positive quantity other than zero, however small, u is said to "approach a indefinitely." This is expressed by the notation,

$$u \doteq a,$$

which may be read, u approaches a indefinitely. The notation $u \doteq \infty$ is not to be read in the same way. It is intended to mean that u increases indefinitely.

With this notation applied to the curve of Art. 13, $y = 1/x$ we have, $y \doteq 0$ as $x \doteq \infty$; hence the x axis is an asymptote by the definition. Again, $x \doteq 0$ as $y \doteq \infty$ $\therefore$ the y axis is an asymptote.

The curve $xy = 1$ is a rectangular hyperbola.

When the temperature of a given mass of gas remains constant the product of the pressure p and volume v is equal to a constant k $\therefore$ $pv = k$. If y replaces p and x replaces v, the graph of $xy = k$ is again a rectangular hyperbola referred to its asymptotes as axes.

15. In this book, the symbol ∞ stands for absolute infinity in the sense that space and time are said to be infinite or never ending. However far we may travel in space along a straight line, it is difficult for us to conceive a point in the straight line where there is no space beyond. A never ending line then gives us an idea of what is called infinity as applied to distance. No finite number can express this distance, no matter what finite unit of length is used and the symbol ∞ is used to represent an infinite number as it is called. It is, of course, not subject to arithmetical operations. Adding any finite length to a never ending line gives a never ending line; so the attempt at addition is futile, for it would lead to such results as $a + \infty = \infty$, whether lengths or numbers are used.

In the graph above of $y = \frac{1}{x}$, let x take successive values 10, 100, 1000, . . ., 100 . . . 0. In the last number, suppose a billion noughts to follow the 1, then $1/x$ is extremely small, but it can still be "decreased indefinitely" by increasing the number of noughts. If the number of noughts were increased, so that the string of them of the above size and distance apart, reached to a fixed star, $x = 100 \ldots 0$ would be very large, but $y = 1/x$ would be finite and not zero, though it approaches zero more and more as more noughts are added.

From the consideration then of the equality,

$$\frac{1}{100 \ldots 0} = .000 \ldots 1,$$

(where the denominator has one nought more than the right member) we are led to infer that as the denominator of a fraction, with a constant numerator, increases indefinitely, the value of the fraction approaches zero indefinitely and the imperfect symbol, $\frac{1}{\infty} = 0$, is intended to express this truth. It has no sense except as just defined. There can be no objection to the following abbreviation:

$$\text{if } \frac{1}{x} = y, \text{ then as } x \doteq \infty,\ y \doteq 0.$$

Similarly, from a consideration of the series,

$$\frac{1}{0.1} = 10,\ \frac{1}{0.01} = 100,\ \ \frac{1}{0.001} = 1000 \ldots,$$

we decide that,

$$\text{if } \frac{1}{x} = y, \text{ then as } x \doteq 0,\ y \doteq \infty,$$

which is still more briefly given, $\frac{1}{0} = \infty$, which has no sense by itself (as we cannot divide anything by zero) and can only be interpreted as above.

EXAMPLE 1. The cissoid has the shape given in Fig. 10. From its equation,

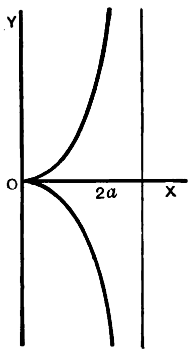

Fig. 10.

$$y^2 = \frac{x^3}{2a - x}$$

(a being constant), we see that it is symmetrical as to the x axis (Art. 10); also y is imaginary either when x is negative or $x > 2a$, but real for $0 \leqq x < 2a$; lastly, as $x \doteq 2a$, $y \doteq \infty$ $\therefore$ the line $x = 2a$ is an asymptote to the two branches of the curve.

EXAMPLE 2. Construct the hyperbola,

$$y - 1 = \frac{1}{4 - x}.$$

As $x \doteq 4$, $y \doteq +\infty$ or $-\infty$ according as $x < 4$ or $x > 4$ $\therefore$ $x = 4$, or a line parallel to the axis of y and 4 units to the right of it, is an asymptote. Again, as $x \doteq \pm\infty$, $y - 1 \doteq 0$ or $y \doteq 1$; hence $y = 1$ or a line parallel to the axis of x and a distance 1 above it, is a second asymptote. Sketch the curve.

EXAMPLE 3. In the locus,

$$y - c = \frac{a}{(x - b)^2},$$

where a, b, c, are constants, prove that $y = c$ and $x = b$, are asymptotes. Roughly indicate the branches on a figure, noting that a curve is always convex to its asymptotes.

EXAMPLE 4. Construct and determine the asymptotes of,

$$y = \frac{1}{x^{\frac{3}{2}}};\ y = \frac{1}{x^{\frac{2}{3}}};\ y = \frac{1}{x^{\frac{5}{3}}}.$$

The above examples represent loci whose asymptotes are easily determined by inspection. For other cases, particularly where the asymptotes are inclined to the axes, advanced treatises on the Calculus must be consulted.

16. Graph of $y = a^x$ or $x = \log_a y$, where a is positive. The graph of $y = 2^x$ or $x = log_2 y$, is shown in part by the full line in Fig. 11. Computing simultaneous values:

$$\begin{array}{ll}
\text{As } x \doteq -\infty, & y \doteq 2^{-\infty} = \frac{1}{2^{\infty}} = 0; \\
\text{if } x = -2, & y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}, \\
\text{if } x = -1, & y = 2^{-1} = \frac{1}{2}, \\
\text{if } x = 0, & y = 2^0 = 1, \\
\text{if } x = 1, & y = 2^1 = 2, \\
\text{if } x = 2, & y = 2^2 = 4; \\
\text{as } x \doteq \infty, & y \doteq 2^{\infty} = \infty.
\end{array}$$

The curve thus approaches the axis of x as an asymptote in the second quadrant, but recedes indefinitely from both axes in the first quadrant. No matter what value is given to a, the curve $y = a^x$ or $x = log_a\ y$ cuts the axis of y at one unit of length above the origin.

Now x is supposed to increase continuously through intermediate values, and its value may be fractional or incommensurable, but the positive value of y is alone to be taken. In fact, $\log_a y$ does not exist when y is negative. The first form $y = a^x$ is an exponential function, the second $x = \log_a y$, a logarithmic function. When $a < 1$, e. g., if $a = \frac{1}{2}$, the graph can be found by rotating the curve about the axis of y, 180°, since $(\frac{1}{2})^x = 2^{-x}$ and x of (2^x) is thus changed to $(-x)$.

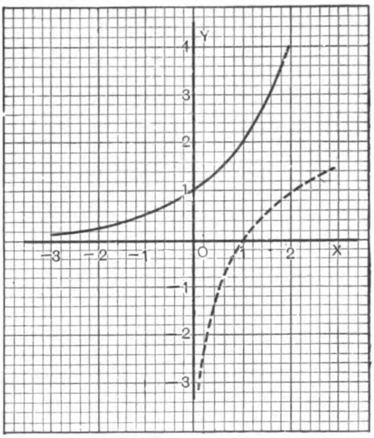

Fig. 11.

The graph of $x = 2^y$ or $y = \log_2 x$ is shown by the dotted curve in Fig. 11. It is asymptotic to the negative side of the y axis. Show this and draw the curve to scale.

The base a of the system of logarithms can be any positive number other than 1; but in practice only two bases are used; that of Briggs or common logarithms, where $a = 10$, and Napier's base, $e = 2.7182818284\ldots$

To change from one system to another, the formulas, deduced in algebra, are:

$$\log_{10} N = .434294 \log_e N,$$
$$\log_e N = 2.302585 \log_{10} N.$$

Hereafter, where no base is indicated, the Napierian base is understood. Thus, $\log x$ means $\log_e x$ or $\log x$ to the base e.

17. Graphs of the Trigonometric Functions. In the Calculus an anglé is not measured in degrees, but in "circular measure," or the length of the arc on a circle of unit radius which subtends the angle. Thus in Fig. 12, the angle AOP is measured by the length of the arc AP in inches if the radius is equal to 1 inch. The length of the circumference being 2π, an angle of 90° is called in circular measure the angle $\pi/2$, an angle of 180°, π or 3.1416, and so on. If the angle AOP in degrees is called $a°$ and arc $AP = x$, on a unit circle, the relation,

$$\frac{x}{\pi} = \frac{a°}{180},$$

will enable us to go from degrees to circular measure, and the reverse. We have $MP = \sin x$, $OM = \cos x$, in Fig. 12.

To draw the graph of $y = \sin x$, we simply lay off along the axis of x in Fig. 13, ON = length of arc AP in Fig. 12 $= x$, and construct the ordinate $y = MP = \sin x$, by taking the distance MP of Fig. 12 in dividers and laying it off *upwards* at N (Fig. 13) when $\sin x$ is positive, which happens when P is in the first or second quadrants, and *downwards* when $\sin x$ is negative, which occurs when P is in the third or fourth quadrants. If the length of a quadrant $\pi/2 = 1.57$ (nearly) is laid off to scale repeatedly as a unit on $X'X$ (Fig. 13) to right and left of O and each part is divided, by repeated bisections, into say 8 parts, and then each quadrant in Fig.

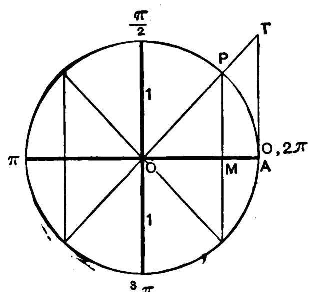

Fig. 12.

12 is divided similarly into 8 equal parts, the length $x = ON$ (Fig. 13) $= AP$ (Fig. 12) is found without computation, and the corresponding $y = \sin x = MP$, can be laid off.

If OP (Fig. 12) revolves counter clockwise x is positive, if clockwise, negative, and x is then laid off to the left of O in Fig. 13. The graph of $\sin x$ is shown by the full line in Fig. 13 that is included between the two lines

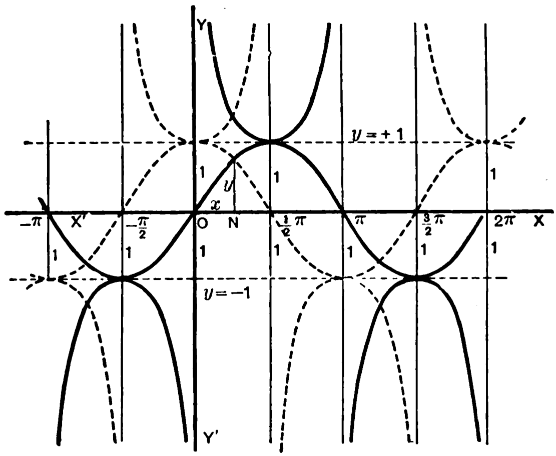

Fig. 13.

parallel to $X'X$ and at a unit's distance above and below X^1X. The curve to the right of $x = 2\pi$ would show the curve from $x = 0$ to 2π, repeated indefinitely. Similarly to the left of O.

To draw $y = \cos x$, proceed as before, except that after laying off any $x = AP$ (Fig. 12) on OX (Fig. 13) to fix the point N, y is then laid off equal to OM (Fig. 12); y being *positive* when M (Fig. 12) is to the right of O, *negative* when to the left of O. The graph is represented by the dotted curve in Fig. 13, lying between the lines $y = \pm 1$.

Above the line $y = +1$ in Fig. 13, and below the line $y = -1$, the graphs of *sec* x (dotted curves) and of *cosec* x (full curves) are drawn. For *sec* x, after dividing up the axis of x as before, we lay off at N (Fig. 13) $y = sec\ x = OT$ of Fig. 12. Similarly, the line values for *cosec* x may be used in constructing its graph.

The graph of $y = tan\ x$ is shown by the full lines in Fig. 14, that of $y = cot\ x$ by the dotted curves. It should be sufficient to indicate the construction for $y = tan\ x$. In Fig. 12, $OA = 1$, $x = arc\ AP$ and $tan\ x = AT$. Hence in Fig. 14, lay off $ON = x$ (as above explained) and $y = AT$ of Fig. 12, to fix a point on the graph of *tan* x.

The straight lines drawn perpendicular to the axis of x at the points on this axis where

$$x = -\frac{\pi}{2}, +\frac{\pi}{2}, \frac{3\pi}{2}, \ldots,$$

are asymptotes to $y = sec\ x$ in Fig. 13 and to $y = tan\ x$ in Fig. 14. The perpendiculars to OX at $x = -\pi, 0, \pi, 2\pi, \ldots$, are asymptotes to $y = cosec\ x$ in Fig. 13, and to $y = cot\ x$ in Fig. 14. Thus considering the graph of $sec\ x = 1/cos\ x$; as x (increasing) $\doteq \pi/2$, $sec\ x \doteq +\infty$; but supposing $x > \pi/2$, as x (decreasing) $\doteq \pi/2$, $sec\ x \doteq -\infty$. At $x = \pi/2$, $cos\ x = 0$ and $sec\ x = 1/cos\ x$ takes the form $1/0$, which has no meaning (Art. 15) $\therefore$ there is no such thing as $sec\ \pi/2$, though for brevity it is convenient to write, $sec\ \pi/2 = \pm\infty$ with the above implied interpretation. Similarly, $cosec\ x = 1/sin\ x$, $tan\ x = sin\ x/cos\ x$, $cot\ x = 1/tan\ x$, all assume the form $1/0$ at the critical values of x

given above; consequently they have no values for such values of x and the interpretation is as just explained for *sec x*.

Where the so-called infinite values for *sec x, cosec x, tan*

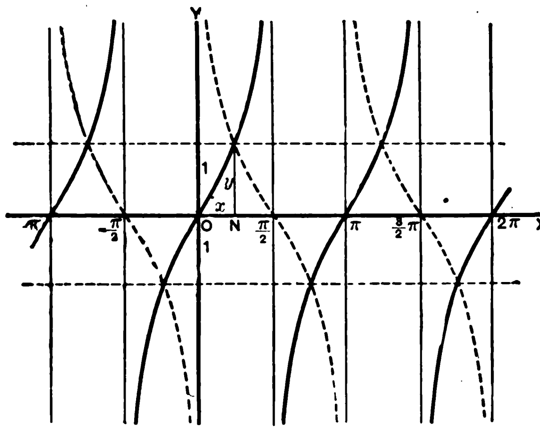

Fig. 14.

x, cot x occur, the corresponding curves are ***discontinuous***; y changing sign from a large positive to a large negative value or the reverse, as x increases through the critical value, as is well illustrated by the graphs. All these functions are *periodic*. Thus if 2π be added to x, *sin x, cos x, sec x* and *coseç x* are unaltered, and the same is true if x is replaced by $(2\,n\,\pi + x.)$ where n is any positive or negative integer. The tangent and cotangent, Fig. 14, are unaltered if x is replaced by $(\pi + x.)$ or generally by $(n.\,\pi + x)$. The *period* here is π, in the other cases $2\,\pi$.

The graphs illustrate many formulas of Trigonometry. Thus, if the curve of *cosines* (Fig. 13) is moved to the right a distance $\frac{1}{2}\,\pi$, it will fit on the curve of *sines*. This illustrates the formula $sin\left(\frac{\pi}{2} + x.\right) = cos\ x.$ The areas between the curve of sines and the axis of x is also equal to the corresponding areas between the curve of cosines and the axis of x, which fact finds an application in the Integral Calculus.

18. Graph of $y = \text{arc sin } x = \sin^{-1}x$. This function is inverse to $x = sin\ y$. Its graph is shown in Fig. 15.

If we rotate the y-axis and the curve of sines of Fig. 13 about $X'X$ through 180°, and then interchange x and y, giving $x = sin\ y$, we shall have the figure 15. Similarly for any inverse function.

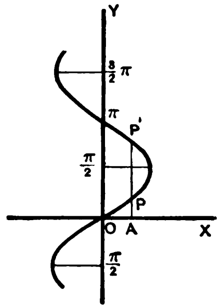

Fig. 15.

The important point to note about any inverse function is, that for a given value of x, y can have an unlimited number of values, unless we restrict the curve to certain limits. Thus in Fig. 15, if $x = OA$, $y = AP, AP', \ldots$; but if the curve is restricted to the part extending from $y = -\frac{\pi}{2}$ to $+\frac{\pi}{2}$, then for any value given to x, plus or minus, y will have but one value. In the inverse functions, $y = sin^{-1}x$, $y = cosec^{-1}x$, $y = tan^{-1}x$, $y = cot^{-1}x$, y will hereafter be supposed to lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, in which case y will be single-valued for values of x that pertain to the function. Similarly $y = cos^{-1}x$, $y = sec^{-1}x$ will be limited to values of the arc y, lying between 0 and π.

19. Polar Coördinates. Polar Equation of a Conic Section. In Fig. 16, let F be a fixed point and FX a fixed line through it; then the position of a point P in a plane is fixed if the angle $MFP = \theta$ and the distance $FP = r$ are given. θ is called the *vectorial angle* and r the *radius vector*, the two being called the *polar coördinates* of P. They are written (r, θ) to designate the point P. Angles will be designated, as usual, *positive* when FP is supposed to have revolved from FX in the counter clockwise direction, *negative* when the rotation from FX is clockwise. $r = FP$ is *positive* in the figure, when laid off from F through the terminal side of the arc θ, *negative* when laid off on PF produced.

Polar Equation of a Conic Section. *A conic section is traced by a point which moves so that its distance from a fixed point bears a constant ratio to its distance from a fixed straight line.*

Call this constant ratio e. In Fig. 16, let F be the fixed point (called the *focus*) and DD' perpendicular to FX at R, the fixed line or *directrix*. Draw FL perpendicular to FX to meet the locus at L, and from L and P (another point on the locus) draw perpendiculars to DD', meeting it at N and D respectively. Also let the locus meet the axis RX at O (*the vertex*). Then from the definition,

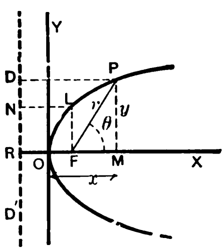

Fig. 16.

$$\frac{OF}{RO} = e; \quad \frac{FL}{LN} = \frac{FL}{RF} = e.$$

Calling $RO = p$ and $FL = l$, these ratios give, $OF = ep$ and $l = e.RF$. For the point P, by the definition, $FP/DP = e \therefore r = e.DP = e(RF + FM) = l + er \cos\theta$, whence

$$r = \frac{l}{1 - e \cos\theta},$$

the polar equation of a conic section. As θ varies from 0 to 360° the point P will describe the curve.

20. Rectangular Equation of a Conic Section. Take the axis of y perpendicular to OX through O, which will now be called the origin. $OM = x$, $MP = y$ for point P. Since,

$$r = \sqrt{FM^2 + MP^2} = \sqrt{(x - ep)^2 + y^2};$$

from the relation above, $r = e.DP$, we have,

$$\sqrt{(x - ep)^2 + y^2} = e.RM = e\,(p + x)\,;$$

or squaring and reducing,

$$y^2 = 2\,e\,(1 + e)\,px - (1 - e^2)\,x^2 \quad . \; . \; . \; (1)$$

Parabola. If $e = 1$, the conic is called a parabola (Fig. 16). Eq. (1) reduces, in this case, to,

$$\mathbf{y^2 = 4\,px} \quad . \; . \; . \; (2)$$

Ellipse. If $e < 1$, the curve is an ellipse. To simplify Eq. 1, for this case, put

$$a = \frac{ep}{1 - e} \text{ and } b^2 = a^2\,(1 - e^2).$$

$$\therefore\; ep\,(1 + e) = a\,(1 - e)\,(1 + e) = a\,(1 - e^2) = \frac{b^2}{a}\,.$$

$$\therefore\; \mathbf{y^2 = \frac{2\,b^2}{a}\,x - \frac{b^2}{a^2}\,x^2} \quad . \; . \; . \; (3)$$

Hyperbola. If $e > 1$, the conic is called a hyperbola. In this case put $a = ep/\,(e - 1)$ and $b^2 = a^2\,(e^2 - 1)$ $\therefore$ the equation of the hyperbola is

$$\mathbf{y^2 = \frac{2\,b^2}{a}\,x + \frac{b^2}{a^2}\,x^2} \quad . \; . \; . \; (4)$$

The three equations (2), (3), (4), can all be represented by

$$\mathbf{y^2 = 2\,Ax + Bx^2} \quad . . . (5)$$

where $B = 0$ for the parabola. For the parabola $2\,A = 4\,p$; for the ellipse and hyperbola, $2\,A = 2\,b^2/a$. It is the expression for the length of the *latus rectum*, $2\,l = 2.FL$ or the double ordinate through the focus. The quantity $B = -\,b^2/a^2$ for the ellipse and $+\,b^2/a^2$ for the hyperbola.

To find where the ellipse cuts the axis of x, make $y = 0$ in (3) and get $x = 0$ or $2\,a$. Therefore the ellipse cuts the axis of x at the vertex marked A in Fig. 17 (the present origin) and at A', where $AA' = 2\,a$. Take a new origin at C the mid-point of AA', CX and CY being the new axes. The old coördinates of a point P are ($x = AM$, $y = MP$); the new, ($x' = CM$, $y = MP$). Then $x = AC + x' = a + x'$, wherever P is placed. On substituting $(a + x')$ for x in eq. (3),

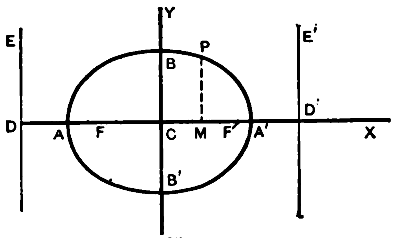

Fig. 17.

$$\frac{x'^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Or dropping accents, with the understanding that x will then equal CM,

$$\frac{\mathbf{x}^2}{\mathbf{a}^2} + \frac{\mathbf{y}^2}{\mathbf{b}^2} = 1 \quad . . . (6);$$

the equation of the ellipse referred to the center and axes.

On making $y = 0$, $x = \pm\,a = CA'$ and CA. On placing $x = 0$, $y = \pm b = CB$ and CB'. AA' is called the major axis, $B'B$ the minor axis. AA' is equal in length to $2a$, BB' to $2b$. The ellipse, which is drawn in Fig. 17, is a closed curve (Art. 9) and from eq. (6) it is symmetrical about both axes. By a similar transformation to the above, the equation of the hyperbola referred to its center C as origin, CX and CY, axes, is found to be,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad . \; . \; . \quad (7)$$

It is thus symmetrical with respect to both axes. Solving for y,

$$y = \pm \frac{b}{a} \sqrt{x^2 - a^2}.$$

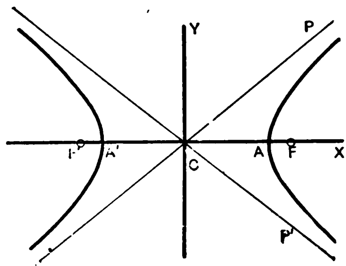

Fig. 18.

Hence x^2 cannot be less than a^2, or there is no curve between the limits $x = -a$ and $x = +a$; but the curve extends indefinitely beyond these limits (Fig. 18). The asymptotes to the hyperbola CP and CP' (Fig. 18) are shown in text-books on conic sections to have the equations,

$$y = bx/a; \;\; y = -bx/a.$$

The hyperbola is "rectangular" or "equilateral" when $a = b$.

From the symmetry of the ellipse and hyperbola about CY, it is seen that they may be regarded as having a second focus, F', symmetrical to F with respect to C, and a second directrix, symmetrical with respect to C, to the first.

21. Convenient practical methods of drawing the parabola and ellipse will now be deduced. The methods pertain more particularly to Projective Geometry, but it will be seen that the analytical proofs are sufficiently simple.

In Fig. 19, let there be given the vertex A, axis AX and a point $I(a, b)$ on a *parabola*, to find other points on the curve. Draw through A, $AY \perp AX$ and through I, $ID \parallel AX$ to intersection D. Then divide $DI = a$ into any number of equal parts and $AD = b$ into the same number of equal parts. Only four divisions of AD and DI are made in the figure. In practice 8 or 16 divisions are desirable. It is to be proved that a line parallel to the axis AX through any point of division, as that marked 3, on AD, will intersect a line from A to the correspondingly marked point 3 on DI, at P, a point on the parabola.

Call the ratio $DB/DI = m \therefore AC/AD = m$. Hence $DB = ma$ and $AC = mb$. The slope of AB is $\dfrac{b}{ma} \therefore$ by Art. 11, its equation is

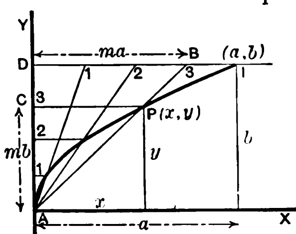

Fig. 19.

$$y = \frac{b}{ma}x.$$

The equation of the line through C parallel to AX is $y = mb$. This line intersects AB at P where the x and y of both lines are the same. Therefore, regarding the two equations as simultaneous, and eliminating m, we find the relation between the x and y of P to be

$$y^2 = \frac{b^2}{a}x.$$

The same relation holds for the other intersections, as the result is true no matter what the numerical value of m is taken; hence all such points as P are points on a parabola. In the figure $m < 1$. For points beyond I, $m > 1$ and the additional construction is obvious. The part of the curve below AX can either be laid off the same way, or it may be made symmetrical with respect to AX to the curve already drawn.

To construct the *ellipse*, Fig. 20, by this method of intersections, having given the two semi-axes, a and b, construct first a rectangle $OADB$ on $a = OA$ and $b = OB$ as sides, and extend BO downwards a distance $OB' = b$. Then divide DA and OA into the same number of equal parts (four in the figure); the intersections of $B1'$ and $B'1$, of $B2'$ and $B'2$, etc., are points on an ellipse. As before, if $O2 = ma$, then $D2' = mb$, where $m = O2/OA = D2'/DA$. By Art. 11, Ex. 9 (γ), the equation of the straight line through B (o, b) and $2'$ $(a, b - mb)$, is

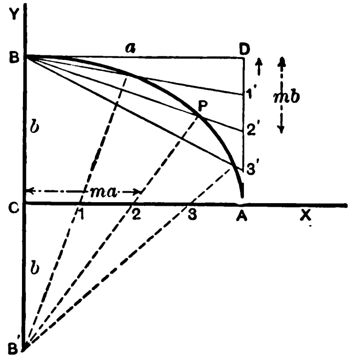

Fig. 20.

$$y - b = \frac{-mb}{a} x,$$

and the equation of $B'2$ with intercepts (ma) and $(-b)$ by Art. 11, Ex. 7, is

$$\frac{x}{ma} - \frac{y}{b} = 1.$$

The intersection of these two lines P, will be the (x, y) of the two equations treated as simultaneous. On eliminating m between them, so that the resulting (x, y) pertains to any one of the intersections noted, we find,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Hence intersections such as $P(x, y)$ lie on an ellipse, with a and b for semi-axes. Symmetric points in the other quadrants can now be marked and the ellipse traced through all the points thus found.

CHAPTER III.

LIMITS, DERIVATIVES, SLOPES, CONTINUITY.

22. **Notation of Functions.** In the Calculus the last letters of the alphabet, from r to z inclusive, are usually supposed to represent variables, the remaining letters, a to q, being generally treated as constant numbers, which do not change in the same investigation. Thus in Art. 20, the current coördinates (x, y) of a point on a conic are variables, connected together, however, by an equation with an *if* implied. Thus *if* x is taken as the *independent* variable and given an arbitrary value, then the corresponding y is to be found by substituting this value of x in the equation and solving for y. We have seen, too, that values assigned to x that make y imaginary, or cause y to assume the form (*constant*)/0, are inadmissible.

y is called the *dependent* variable where x is assumed. But if the values of y are assumed and the corresponding values of x computed from the equation, then y is the independent and x the dependent variable.

In Art. 20, the quantities a, b, e, p, are constants for the same curve and can be looked upon as fixed numbers for any one curve. By assigning different numerical values to these constants, we shall start with different curves, but in the further discussion, the particular values assigned are supposed unchanged.

In Art. 12, functions were classed as either "algebraic" or "transcendental." Certain symbols, $f(x)$, $F(x)$, $\phi(x)$, etc., are used to denote functions of x. Thus $f(x)$ read, "function of x," indicates any expression that contains x. If $f(x)$ represents a particular function as $(2x^2 + ax + b)$ at the beginning of an investigation, it must retain this same form throughout the investigation. It can be supposed to have any form at the start. Similarly where $f(x)$, $F(x)$, $\phi(x)$, etc., occur in the same investigation, each function is supposed to retain the same form throughout the investigation. When used together, they may be read: small f function of x, large F function of x, *phi* function of x, etc.

When two variables as x and y occur in an expression, the notation $f(x, y)$, read function of x and y, is employed.

$f(x, y) = 0$ is called an *implicit* function, and either variable is said to be an implicit function of the other. When the equation is solved for either variable, this one is called an *explicit* function of the other.

Thus take the equation of any conic, referred to its vertex as origin and axis of the curve as the x axis ((5), Art. 20) $y^2 - 2Ax - Bx^2 = 0$. This may be written for brevity, $f(x, y) = 0$ and either x or y is an implicit function of the other; but when solved for y, $y = \pm\sqrt{2Ax + Bx^2}$, it is in the form $y = F(x)$ and y is an explicit function of x.

If $f(x)$ denote a certain function of x, then $f(0)$, $f(1)$, $f(2)$, $f(h)$, $f(x + h)$, denote respectively the values of $f(x)$ when x is replaced in the given function, by 0, 1, 2, h, $(x + h)$ respectively.

Thus if $f(x) = x^2 + \sin x$, $f(0) = 0 + \sin 0 = 0$; $f(1) = 1 + \sin 1 = 1 + \sin 57^\circ.3$ nearly; $f(2) = 4 + \sin 2 = 4 + \sin$

$114°.6$ nearly; $f(h) = h^2 + \sin h$; $f(x+h) = (x+h)^2 + \sin (x+h)$. Similarly, if $f(x, y) = x - y^2$, $f(1, 2) = 1 - 2^2 = -3$; $f(2, 1) = 2 - 1 = 1$; substituting 1 for x and 2 for y in the given function to find $f(1, 2)$, etc.

Be careful to use the comma between x and y in $f(x, y)$ and not write it $f(x. y)$, which means the result of substituting the product (xy) for x in $f(x)$.

1. If $f(x) = \log x$, $f(xy) = f(x) + f(y)$ and $f\left(\frac{x}{y}\right) = f(x) - f(y)$.

2. If $f(x) = x^2 - x + 1$, find $f(-1)$, $f(0)$, $f(2)$.

3. If $f(x)$, in turn, $=$ $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$; write the values of $f\left(\frac{\pi}{6}\right)$, $f\left(\frac{\pi}{2}\right)$, $f(\pi)$.

4. If $f(x) = x^3 \log x$, write down, $f(1)$, $f(x^2)$, $f(\frac{1}{2})$, $f(ax + h)$, $f(\sin x)$. Is $f(-y)$ real?

5. If $f(x, y) = x^2 - y^2$, prove $f(\cos\theta, \sin\theta) = \cos 2\theta$ and $f(\sec\theta, \tan\theta) = 1$.

23. Limits. The student has met with the notion of and the necessity for a limit in geometry. Thus, although a polygon inscribed in a circle can never coincide with the circle, yet the area of the circle was found to be the constant quantity which the area of the polygon could be made to continually approach and differ from by any positive quantity other than zero, however small, provided the number of sides of the polygon was sufficiently increased. This constant is known as the limit of the area of the polygon and is πa^2, where a is the radius of the circle. The method of limits, in this case, enabled us, from a known formula for a variable magnitude, to deduce an expression for a constant magnitude that the variable magnitude could never attain. All the other cases where the method of limits was employed in Plane and Solid Geometry were of this kind, and the student can not do better than to review them.

In Algebra, too, the method of limits was employed, and before taking up the subject formally, some examples from Algebra will be given.

EXAMPLE 1. A proper fraction can be represented by $1/(1+a)$, where a is a positive number. By the binomial formula, $(1+a)^n > 1 + na$, n being a positive integer.

$$\therefore \frac{1}{(1+a)^n} < \frac{1}{1+na}.$$

Using the notation of Art. 14 (which see) it is obvious that as $n \doteq \infty$, $1/(1+na) \doteq 0$, Art. 15;

$$\therefore \left(\frac{1}{1+a}\right)^n \doteq 0 \text{ as } n \doteq \infty.$$

The unattainable limit, which the proper fraction raised to the n^{th} power, approaches indefinitely, is here zero. Using the abbreviation *lim* for limit, the notation employed to express the above result is,

$$\lim \left(\frac{1}{1+a}\right)^n_{n \doteq \infty} = 0.$$

EXAMPLE 2. If α is a finite quantity, which, according to the law of its variation, can be made to differ permanently from zero by as small a quantity as we wish, then $c\alpha$ is such a quantity, c being a constant. For if k is any positive number other than zero, however small, then by hypothesis α can be made less than k/c $\therefore$ $c\alpha$ can be made less than k $\therefore$ as $k \doteq 0$, $c\alpha \doteq 0$. In the language of limits, if $\lim \alpha = 0$, then $\lim c\alpha = 0$.

α is called an *infinitesimal, which is thus a finite quantity whose limit is zero.* Similarly, $c\alpha$ is an infinitesimal. An infinitesimal is thus not zero nor some very small constant, but a variable that can become permanently smaller than any number other than zero, however small that number is taken. Thus in Ex. 1, $\left(\frac{1}{1+a}\right)^n$ is an infinitesimal.

EXAMPLE 3. From Algebra, the sum of n terms of the geometrical series,

$$a + ar + ar^2 + \ldots + ar^{n-1},$$

is,

$$\frac{a - ar^n}{1 - r} = \frac{a}{1-r} - \left(\frac{a}{1-r}\right)r^n.$$

When $r < 1$ numerically, i.e., when r is a proper fraction, $r^n \doteq 0$ as $n \doteq \infty$ by Ex. 1.

Hence by Ex. 2, $\left(\frac{a}{1-r}\right) r^n \doteq 0$ as $n \doteq \infty$

$$\therefore \lim_{n \doteq \infty} [a + ar + ar^2 + \ldots + ar^{n-1}] = \frac{a}{1-r}.$$

This operation is often called summing up the infinite series. Strictly, it is finding the limit of the sum of n terms as n indefinitely increases.

EXAMPLE 4. As an application of this formula,

$$\lim_{n \doteq \infty} \left[1 + \frac{1}{2} + \frac{1}{4} + \ldots + \left(\frac{1}{2}\right)^{n-1}\right] = \frac{1}{1 - \frac{1}{2}} = 2.$$

$$\lim_{n = \infty} \left[\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots + \frac{3}{10}\left(\frac{1}{10}\right)^{n-1}\right] = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{1}{3},$$

or 0.333 . . . 3 approaches indefinitely 1/3 as the number of decimal figures is increased: a result otherwise evident from division.

EXAMPLE 5. To prove that, $\lim_{x \doteq 0} \frac{\sin x}{\tan x} = 1.$

$$\text{Since, } \frac{\sin x}{\tan x} = \cos x \text{ and } \cos x \doteq 1 \text{ as } x \doteq 0,$$

$$\therefore \frac{\sin x}{\tan x} \doteq 1 \text{ as } x \doteq 0, \text{ or, } \lim_{x \doteq 0} \frac{\sin x}{\tan x} = 1.$$

Note the steps here:

(1) It is seen that as x approaches 0, $\sin x/\tan x$ approaches 1, since its equal, $\cos x$, approaches 1.

(2) x can be taken so near 0 that for this and values still nearer zero, the difference between $\sin x/\tan x = \cos x$, and 1 may become and remain less than any given positive number, other than zero, however small.

When these two conditions are satisfied, 1 is said to be the limit of $\sin x/\tan x$. The formal definition is but a generalization from this example.

24. Definition of a Limit. *If, as the argument x approaches a fixed finite value a, the function $f(x)$ approaches a fixed finite value A; further, if x can be taken so near a, that for this and all values of x still nearer a, the numerical difference between $f(x)$ and A may become and remain less than any given positive number, other than zero, however small; then A is said to be the limit of $f(x)$.*

In the special case where $f(x)$ approaches A when $x \doteq \infty$, then it must be shown that x can be made so large (but finite) that for this and all greater values of x, the difference between $f(x)$ and A may become and remain less than any given positive number, other than zero, however small.

The brief notation employed for the two cases is, as given in the examples above,

$$\lim_{x=a} f(x) = A; \quad \lim_{x \doteq \infty} f(x) = A.$$

On referring to the definitions of the symbols $u \doteq a$, $u \doteq \infty$ in Art. 14, it is seen that the first can equally be written *lim* $u = a$; but the second can not be written $\lim u = \infty$, since no finite quantity u can be made to differ from ∞ by a finite quantity, much less one which approaches zero indefinitely.

The definitions are silent as to whether the function reaches its limit or otherwise. As a matter of fact, the theory of limits was invented especially to meet the cases where the variable function did not reach its limit; but for the sake of generality, it is well to include all cases. Thus $\sin x/\tan x$ as $x \doteq 0$, does not reach its limit 1, but takes the form 0/0 when $x = 0$. The same function actually reaches its *limit*, $\frac{1}{2}\sqrt{2}$ as $x \doteq \frac{\pi}{4}$. Again, a function may increase or decrease in approaching its limit, or it may alternately increase or decrease in nearing it. Thus in Ex. 5, above, the function increases towards its limit, whereas in Ex. 1 the function decreases on approaching its limit. The function $\frac{1}{(-2)^n}$, where n is a positive whole number, is alternately plus and minus as $n \doteq \infty$, according as n is even or odd, so that it oscillates about its *limit zero*, being alternately greater and less than its limit. Here the conditions of the definition are satisfied, for a value of n can be found, such that for this and all greater values of n, the numerical difference between $1/(-2)^n$ and 0 can become and remain less than any positive number other than zero, however small. This follows from Ex. 1, above, since it was shown there that

$$\lim_{n \doteq \infty} \left(\frac{1}{2}\right)^n = 0.$$

EXAMPLE 1. Refer to Art. 15 and show that for $a > 1$,

$$\lim_{x \doteq -\infty} a^x = 0;$$

also to Art. 13 and prove that

$$\lim_{x \doteq \infty} \left(\frac{1}{x}\right) = 0,$$

and illustrate both cases with figures.

EXAMPLE 2. The equation, $\frac{x^2 - a^2}{x - a} = x + a$, is true for all values of x other than a. The left member takes the form 0/0 when $x = a$. It thus has no meaning and no definite value when $x = a$. It has a definite limit, however.* For as x approaches a, the quotient approaches $2a$ and x can approach a so closely that the quotient, or its equal $(x + a)$, shall differ from $2a$ by any positive quantity other than zero, however small. Therefore, by the definition, the limit is $2a$.

EXAMPLE 3. $\frac{x^2 + 7x + 12}{x^2 + 2x - 8} = \frac{(x + 3)(x + 4)}{(x - 2)(x + 4)} = \frac{x + 3}{x - 2}$.

Here, we can divide numerator and denominator by $(x + 4)$ except when $x = -4$, for division by zero is senseless and against the laws of algebra. As above, though, the limit of the quotient as $x \doteq -4$ is 1/6.

Transformations are often serviceable in finding the limit of a fraction that takes the form 0/0 for a certain value of x.

EXAMPLE 4. $\frac{\sqrt{x - 1}}{\sqrt{x} - 1} = \frac{\sqrt{x - 1}\,(\sqrt{x} + 1)}{x - 1} = \frac{\sqrt{x} + 1}{\sqrt{x - 1}}$.

The last transformation is true except when $x = 1$. As $x \doteq 1$, the quotient increases indefinitely. Its reciprocal approaches zero as a limit.

EXAMPLE 5. What is the limit of

$$\frac{4x^2 + 2x + 1}{2x^2 + 3x - 4} \text{ as } x \doteq \infty\text{?}$$

Divide numerator and denominator by x^2;

$$\therefore \lim_{x \doteq \infty} \frac{4 + \frac{2}{x} + \frac{1}{x^2}}{2 + \frac{3}{x} - \frac{4}{x^2}} = \frac{4}{2} = 2.$$

* See Art. 103 for the geometrical significance of such forms.

EXAMPLE 6. As an important case, consider the circular arc *ABC*, Fig. 21, of radius unity, whose length in circular measure is $2x$. Divide it into two equal parts $x = AB = BC$ and draw tangents at A and C, intersecting at D on radius OB produced. Let

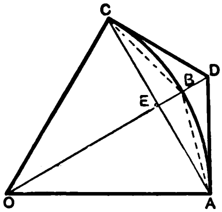

Fig. 21.

$$c = \text{chord } AB = \text{chord } BC.$$

Then, since the radius is taken as unity,

$$EA = EC = \tfrac{1}{2} AC = \sin x$$

and $$AD = DC = \tan x.$$

By geometry,

$$AC < 2c < 2x < AD + DC.$$

Whence, dividing by 2,

$$\sin x < c < x < \tan x.$$

The ratio of the extremes, $\dfrac{\sin x}{\tan x} = \cos x$, approaches 1 as a limit as $x \doteq 0$. Therefore, since c and x are always intermediate in value between $\sin x$ and $\tan x$, and these approach equality indefinitely as $x \doteq 0$, for a stronger reason, the ratio of c and x to each other and to $\sin x$ or $\tan x$, will each have for a limit unity as $x \doteq 0$.

Thus, $$\lim_{x \doteq 0} \frac{\sin x}{x} = 1, \quad \lim_{x \doteq 0} \frac{c}{x} = 1, \text{ etc.}$$

25. Orders of Infinitesimals. In Art. 23, Ex. 2, an infinitesimal was defined to be a variable finite quantity whose limit is zero. *β is defined to be an infinitesimal of order n with respect to a, n being positive but not necessarily integral, when,*

$$\lim_{a \doteq 0} \frac{\beta}{a^n} = k,$$

where k is a finite number, not zero. Thus, when $n = 1$, β is of the same order as α; when $n = 2$, β is of the second order with respect to α, and so on. In Ex. 6 of Art. 24, *sin x, c, x, tan x*, are all of the same order by this test, k in this case being 1.

Let $$\beta = 1 - \cos\alpha = 2\sin^2 \tfrac{1}{2}\alpha.$$

$$\therefore \lim_{\alpha \doteq 0}\left(\frac{1-\cos\alpha}{\alpha^2}\right) = \tfrac{1}{2}\lim_{\alpha \doteq 0}\left(\frac{\sin\frac{1}{2}\alpha}{\frac{1}{2}\alpha}\right)^2 = \tfrac{1}{2},$$

since $$\lim_{x \doteq 0}\left(\frac{\sin x}{x}\right) = 1, \text{ Art. 24, Ex. 6.}$$

$\therefore$ $(1 - \cos\alpha)$ is of the 2d order with regard to α.

Of what order is $(\tan\alpha - \sin\alpha) = \dfrac{\sin\alpha\,(1-\cos\alpha)}{\cos\alpha}$ with respect to α?

We have,

$$\lim_{\alpha \doteq 0}\frac{\tan\alpha - \sin\alpha}{\alpha^3} = \lim_{\alpha \doteq 0}\frac{1}{\cos\alpha}\cdot\frac{\sin\alpha}{\alpha}\cdot\frac{1-\cos\alpha}{\alpha^2}$$
$$= 1 \cdot 1 \cdot \tfrac{1}{2} = \tfrac{1}{2}, \text{ by previous results.}$$

$\therefore$ $(\tan\alpha - \sin\alpha)$ is of the third order with respect to α, since the limit of its ratio to α^3 is finite. In taking limits here, we have tacitly assumed Theorem III. of the next article.

REMARK. In modern books, the reciprocal of an infinitesimal is called an "infinite," which is certainly a most unfortunate misnomer. By reference to Art. 15 and the graph of $\frac{1}{x} = y$, in Art. 13, it is seen that when x is infinitesimal, i.e., a *finite* quantity that approaches zero as near as we wish without ever becoming zero, that y is always *finite*, however large it may become. It is never absolute infinity. By the definition above, it is called an "infinite."

26. Theorems on Limits. To avoid repetition in what follows, the variables, u, v, w, α, β, γ, are all assumed to be functions of x, and to approach their limits indefinitely as x approaches a, according to the definition above. Let us assume,

$$lim\ u = a, \quad lim\ v = b, \quad lim\ w = c,$$
$$lim\ \alpha = 0, \quad lim\ \beta = 0, \quad lim\ \gamma = 0.$$

THEOREM I. *If two variables be always equal, and each approaches a limit, their limits are equal.* That is, if $u = v$ for all values of x, as $x \doteq a$, then $lim\ u = lim\ v$.

This is self-evident, for the two variables, u and v, present but one value, and it is manifestly impossible for the same value to tend indefinitely towards two separate limits. This principle was tacitly assumed in some of the examples above.

The theorems numbered (1), (2), (3), below, will next be proved. They are used to prove the more general theorems that follow, and that include them as special cases. To prove $lim\ (\alpha + \beta + \gamma) = 0$, we have to show that $(\alpha + \beta + \gamma)$ can become less than ω, where ω is a positive number as near zero as we please. By the definition (Art. 25), α, β and γ can each be made less than $\frac{\omega}{3}$; $\therefore (\alpha + \beta + \gamma)$ can be made less than ω; hence by the definition, $\lim (\alpha + \beta + \gamma) = 0$. Similarly for any finite number of variables, whether the variables are positive or negative.

(1) Thus is proved, *the algebraic sum of a finite number of infinitesimals has a limit zero.*

(2) From Ex. 2, Art. 23, we have, *the product of an infinitesimal by a constant k has zero for a limit.*

(3) It follows, since β can be made less than any positive number k and $\therefore a\beta < ka$, that *the product of two infinitesimals has zero for a limit. Similarly for the product of three or more infinitesimals.* We can now prove certain general theorems.

THEOREM II. *The limit of the algebraic sum of any finite number of functions is equal to the sum of the limits of the functions.*

Let the limits of u, v, w, be a, b, c; hence if α, β, γ, are certain infinitesimals,

$$u = a + \alpha,\ v = b + \beta,\ w = c + \gamma,$$

for on taking limits, since $lim\ \alpha = 0$, etc., we reach the hypothesis $lim\ u = a$, etc.

Adding,

$$u + v + w = a + b + c + (\alpha + \beta + \gamma).$$

But $lim\ (\alpha + \beta + \gamma) = 0$ by (1).

$$\therefore \mathbf{lim\ (u + v + w) = a + b + c = lim\ u + lim\ v + lim\ w.}$$

Any of the functions u, v, w, can be positive or negative. A similar proof holds for any finite number of functions.

THEOREM III. *The limit of the product of any number of functions is equal to the product of the limits of the functions.*

$$\begin{aligned} uv &= (a + \alpha)\ (b + \beta) \\ &= ab + a\beta + b\alpha + \alpha\beta. \end{aligned}$$

By (2), $lim\ a\beta = 0$, $lim\ b\alpha = 0$; by (3), $lim\ \alpha\beta = 0$

$$\therefore \text{by (1)},\ lim(a\beta + b\alpha + \alpha\beta) = 0$$

$$\therefore \mathbf{lim\ uv = ab = (lim\ u) \times (lim\ v).}$$

Next, treating (uv) as a single variable, by the same law,

$$\begin{aligned} lim\ uv\ .\ w &= \lim\ (uv) \times \lim\ w \\ &= (\lim\ u) \times (\lim\ v) \times (\lim\ w). \end{aligned}$$

Similarly for the product of any number of functions.

THEOREM IV. *The limit of the quotient of two functions is equal to the quotient of the limits of the functions, provided the limit of the divisor is not zero.*

The proof is simple. Since,

$$u = \frac{u}{v} \cdot v;$$

$$\therefore \ lim\ u = lim \frac{u}{v} \cdot lim\ v,$$

by Theorem III. Hence dividing by *lim v*,

$$\lim \frac{u}{v} = \frac{\lim u}{\lim v}.$$

If *lim v* were zero, and *lim u* not zero, then $u/v \doteq \infty$ and has no limit, which accounts for the restriction above. In fact when $lim\ v = 0$, division by it is inadmissible by the laws of algebra.

EXAMPLE 1.

Prove,

$$\lim_{x \doteq 1} \frac{x - 1}{\sqrt{x^2 - 1} + \sqrt{x - 1}} = 0.$$

When $x = 1$, the fraction takes the form $0/0$, but on dividing the numerator and denominator by $\sqrt{x - 1}$, the limit as $x \doteq 1$ can be found. See Examples 2, 3, 4, of Art. 24. Also Ex. 5 of that article for the case of a fraction that assumes the form ∞/∞ for $x = \infty$.

EXAMPLE 2. The expression, $\sqrt{1 + x} - \sqrt{x}$, takes the form $\infty - \infty$ when $x = \infty$. But for x finite,

$$\sqrt{1 + x} - \sqrt{x} = \frac{1}{\sqrt{1 + x} + \sqrt{x}},$$

which tends to the limit 0 as $x \doteq \infty$.

EXAMPLE 3. *Cosec* $3x$ *sin* $6x$ takes the form $\infty . 0$ when $x = 0$. But it can be transformed into,

$$\frac{\sin 6x}{\sin 3x} = \frac{6}{3} \frac{\left(\dfrac{\sin 6x}{6x}\right)}{\left(\dfrac{\sin 3x}{3x}\right)},$$

whose limit as $x \doteq 0$ is 2. Here we make use of Theorem IV. and the fact proved in Ex. 6, Art. 24, that

$$\lim_{x \doteq 0} \frac{\sin x}{x} = 1.$$

EXAMPLE 4.

$$\lim_{x \doteq 0} \frac{x^{\frac{1}{4}} + x^{\frac{2}{3}} + x^{\frac{3}{4}}}{x^{\frac{1}{2}} + 3\,x^{\frac{1}{3}} + x^{\frac{1}{4}}} = \lim_{x \doteq 0} \frac{1 + x^{\frac{5}{12}} + x^{\frac{1}{2}}}{x^{\frac{1}{4}} + 3\,x^{\frac{1}{12}} + 1} = \frac{1}{1} = 1.$$

Here we divide by the lowest power of x in numerator or denominator.

EXAMPLE 5. Prove, $\lim\limits_{x \doteq 0} \sin 3\,x \cot 2\,x = \frac{3}{2}$.

EXAMPLE 6. In Examples 3, 4 of Art. 24, find the limits as $x \doteq \infty$.

27. $$\lim_{x \doteq a} \frac{\mathbf{x^n - a^n}}{\mathbf{x - a}} = \mathbf{na^{n-1}},$$

for all rational values of n.

This important result will be proved true for (i) n a positive integer; (ii) n a positive proper fraction and (iii) n negative, but either integral or fractional.

(i) If *n is a positive integer,*

$$\frac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \,.\,.\, + a^{n-2}x + a^{n-1}.$$

$\therefore$ since the limit of each of the n terms, as $x \doteq a$, is a^{n-1}; by Th. II., Art. 26,

$$\lim_{x \doteq a} \frac{x^n - a^n}{x - a} = na^{n-1}.$$

(ii) Let $n = p/q$, *a positive proper fraction, p and q being positive integers* $\therefore p = nq$.

If we put, $$x = y^q, \; a = b^q,$$

$$\therefore \frac{x^n - a^n}{x - a} = \frac{y^{nq} - b^{nq}}{y^q - b^q} = \frac{y^p - b^p}{y^q - b^q} = \frac{\dfrac{y^p - b^p}{y - b}}{\dfrac{y^q - b^q}{y - b}}.$$

Since $y = b$ when $x = a$ from the preceding equations, we have to find the limit of the last fraction as $y \doteq b$. But by case (i), the limit of the numerator is pb^{p-1} and that of the denominator is qb^{q-1};

$$\therefore \lim_{x=a} \frac{x^n - a^n}{x - a} = \frac{p}{q} b^{p-q} = \frac{p}{q} a^{\frac{p-q}{q}} = na^{n-1}$$

by Th. IV., Art. 26.

(iii) When n is negative, either integral or fractional, say $n = -m$ where m is positive,

$$\frac{x^n - a^n}{x - a} = \frac{x^{-m} - a^{-m}}{x - a} = -\frac{1}{x^m a^m} \cdot \frac{x^m - a^m}{x - a}.$$

The limit as $x \doteq a$ of the last fraction is, by cases (i) or (ii), ma^{m-1} $\therefore$ by Th. III., Art. 26,

$$\lim_{x \doteq a} \frac{x^n - a^n}{x - a} = -\frac{1}{a^{2m}} ma^{m-1} = -ma^{-m-1} = na^{n-1}.$$

Thus the theorem is proved whether n is positive or negative, whole or fractional.

Cor. When $(x_1 + h)$ is put for x and x_1 for a in this result, whence $x - a = x_1 + h - x_1 = h$;

$$\lim_{h \doteq 0} \frac{(x_1 + h)^n - x_1^n}{h} = nx_1^{n-1},$$

an important formula which leads to the differentiation of x^n.

CHAPTER IV.

DERIVATIVES, SLOPES, CONTINUITY.

28. Increments. If a variable changes from one value to another, the amount by which the *latter* value exceeds the former, is called an increment.

Thus if x changes from the value x_1 to the value x_2, $(x_2 - x_1)$ is called the increment of x and is denoted by the symbol Δx.

Therefore, $$\Delta x = x_2 - x_1$$

is *positive* when x_2 exceeds x_1, *negative* when x_2 is less than x_1. Δx is read *delta* x and must be regarded as an indivisible symbol, Δ by itself having no meaning. When Δx is positive, x has increased algebraically in changing from x_1 to x_2; when Δx is negative, x has decreased, so that x has actually suffered a decrement; but in both cases, Δx is called an increment and its sign will at once indicate whether it is actually an increase or a decrease. The increment that any variable takes is similarly denoted by writing Δ before it. Consider the linear function,

$$y = ax + b;$$

then, if we start with a pair of fixed values of x and y and afterwards suppose x to change to $(x + \Delta x)$, then y must take a new value, which we shall call $(y + \Delta y)$. These new simultaneous values must satisfy the equation,

$$\therefore\ y + \Delta y = a\ (x + \Delta x) + b.$$

Hence on subtracting, $\Delta y = a \cdot \Delta x$.

Similarly if, $y = x^2$

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\,\Delta x + (\Delta x)^2$$

whence the increment Δy of y, corresponding to the simultaneous increment Δx of x, is

$$\Delta y = 2x \cdot \Delta x + (\Delta x)^2.$$

If Δx is assumed to be positive, as it will be hereafter, unless specially noted, the sign of Δy will indicate whether it is an actual increase or a decrease. In the last example, it is plus if x is positive, but it will be minus if x is negative and $2x \cdot \Delta x$ is numerically greater than $(\Delta x)^2$.

In Fig. 22, let the equation of the curve *PQCD* be, $y = f(x)$, (Art. 22).

If $x = OM$, $y = MP = f(x)$, and if $MN = \Delta x$

$$y + \Delta y = NQ = f(x + \Delta x)$$

$$\therefore \Delta y = RQ = f(x + \Delta x) - f(x).$$

In this case Δy is positive.

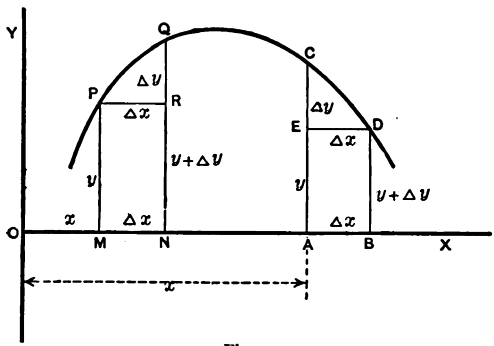

Fig. 22.

But *if* $x = OA$ and $\Delta x = AB$

$$y = AC = f(x)$$

$$y + \Delta y = BD = f(x + \Delta x)$$

$$\therefore \Delta y = BD - AC = -EC = f(x + \Delta x) - f(x)$$

and Δy is found to be negative $\therefore -\Delta y = EC$, *i.e.*, $(-\Delta y)$ is the positive number representing the length of EC.

The notation above is very important and should be thoroughly mastered. For brevity in the examples below, put $h = \Delta x$ in the right member.

EXAMPLE 1. Let $y = f(x) = ax^2 + bx + c$.

Call $x + \Delta x$ and $y + \Delta y$, simultaneous values of x and y and put $h = \Delta x$, where convenient.

$$\begin{aligned} \therefore (y + \Delta y) &= f(x + \Delta x) = a(x+h)^2 + b(x+h) + c \\ &= (ax^2 + bx + c) + 2axh + ah^2 + bh \\ \therefore \Delta y &= f(x + \Delta x) - f(x) = (2ax + b)h + ah^2. \end{aligned}$$

EXAMPLE 2. Let $y = ax^3$, $y = ax^4$; find Δy.

EXAMPLE 3.

$$\text{If } y = \frac{1}{x},\ \Delta y = \frac{1}{x+h} - \frac{1}{x} = -\frac{h}{x(x+h)}.$$

EXAMPLE 4.

$$\text{If } y = \sqrt{x},\ \Delta y = \sqrt{x+h} - \sqrt{x} = \frac{h}{\sqrt{x+h} + \sqrt{x}}.$$

These algebraic results are true whether a graph is implied or not.

29. Derivatives. Let us begin with an example. In Ex. 1 of the last article, divide both sides of the final result by h and replace h by Δx in the left member, from considerations of symmetry. We find,

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = 2ax + b + ah.$$

When $\Delta x = h = 0$, this ratio takes the form 0/0 and has no definite value. There is then no ratio and the result of such a substitution is meaningless. But, as in the many examples of Arts. 25, 26, 27, of functions that assumed the form 0/0 for certain values of the variable, this ratio has *a limit* as $\Delta x = h \doteq 0$; and this limit, that the ratio approaches indefinitely but never reaches, is evidently $2ax + b$. This limit is called *the derivative of y with respect to x* and is written,

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{f(x + \Delta x) - fx}{\Delta x} = 2ax + b.$$

We can find the limit of the right member by putting $h = 0$, since in this example, it can reach its limit; but the right member must be considered in connection with the left member which cannot reach its limit (see in this connection the examples of Arts. 25, 26 and 27). Hence we say, as $h \doteq 0$, $(2ax + b + ah) \doteq (2ax + b)$, but never reaches it, since we cannot make $h = 0$ in the left member and hence cannot consider the possibility in the right member.*

The expression,

$$\lim_{\Delta x \doteq 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

is (in symbols) *the definition of the derivative* of $f(x)$, taken with respect to x as the independent variable, $f(x)$ being any function whatsoever. In words the process indicated, which is called *differentiation*, is as follows:

(1) Give to x in $f(x)$ an arbitrary increment, Δx;

(2) Subtract the first value $f(x)$ of the function from the new value $f(x + \Delta x)$;

* By Theorem IV., Art. 26, we are not authorized in writing, as $\Delta x \doteq 0$,

$$\lim \frac{\Delta y}{\Delta x} = \frac{\lim \Delta y}{\lim \Delta x} = \frac{0}{0},$$

since the case where the limit of the divisor is zero was especially excepted.

(3) Divide by the increment of the independent variable (Δx);

(4) Finally, take the limit of this ratio as the increment of the independent variable tends indefinitely towards zero.

The result is known not only as "the derivative," but as "the differential coefficient" or "the derived function;" and the notations employed, besides that just given, to express the derivative of $y = f(x)$ with respect to x, are,

$$\frac{dy}{dx} = D_x y = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = D_x f(x) = \lim_{\Delta x \doteq 0} \frac{\Delta f(x)}{\Delta x} = y' = f'(x).$$

The first, $\dfrac{dy}{dx}$ (the notation of Leibnitz) is most used, since it is an exact reminder of the operation,

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x},$$

that has been performed. To a less extent is this true of $D_x y$. However, this is often the best form to use, particularly if the *sub* x is dropped, the independent variable being always understood to be x when the function is a function of x.

The notation is similar when different letters are used. Thus if $u = f(t)$, its derivative with respect to t, would be expressed by,

$$\frac{du}{dt}, \quad D_t u, \quad \frac{df(t)}{dt}, \quad D_t f(x), \quad u', f'(t)$$

and similarly for the other forms; x being here replaced by t and y by u.

Apply the process indicated above to find the derivative of $y = \cos x$, using as before h for Δx, for brevity.

$$y = \cos x$$
$$y + \Delta y = \cos(x + h)$$
$$\therefore \frac{\Delta y}{\Delta x} = \frac{\cos(x+h) - \cos x}{h} = -\frac{\sin \frac{1}{2} h}{\frac{1}{2} h} \sin\left(x + \tfrac{1}{2} h\right),$$

on transforming $\cos(x + h) - \cos x$ by use of the formula,

$$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$$

and dividing numerator and denominator by 2. Then, since

$$\lim_{a \doteq 0} \frac{\sin a}{a} = 1,$$

we have on taking the limit as $h \doteq 0$.

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = -\sin x,$$

for,
$$\lim_{h \doteq 0} \sin\left(x + \tfrac{1}{2} h\right) = \sin x.$$

The further consideration of transcendental functions will be deferred to a subsequent chapter.

EXAMPLE 1. From the results of Examples 3 and 4, Art. 28, prove,

if, $$y = \frac{1}{x},\ \frac{dy}{dx} = \quad \frac{1}{x^2};$$

if, $$y = \sqrt{x},\ \frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$

EXAMPLE 2. Find the derivatives, with respect to x,

of $$x,\quad mx + b,\quad ax^2,\quad ax^2 + b,\quad ax^3,\quad ax^3 - b,$$
$$\frac{2}{x},\quad \frac{a}{x} - b,\quad \frac{a}{x} + bx,\quad \sqrt{x} - x^2.$$

EXAMPLE 3. Find $\frac{du}{dt}$ when $u = \frac{1}{4} t^2,\ t^3 - 2t,\ \sqrt{at}.$

EXAMPLE 4. *The derivative of a constant is zero.*

30. Derivative of a Power. It was shown in Art. 27, corollary, that,

$$\lim_{h \doteq 0} \frac{(x+h)^n - x^n}{h} = nx^{n-1}.$$

But this, by definition, is the derivative of x^n.

Therefore if $u = x^n$,

$$\frac{\mathbf{du}}{\mathbf{dx}} = \mathbf{D}_x\mathbf{u} = \frac{\mathbf{d}\,(\mathbf{x}^n)}{\mathbf{dx}} = \mathbf{D}_x\,(\mathbf{x}^n) = \mathbf{nx}^{n-1}.$$

This result was shown in Art. 27 to be true whether n is positive or negative, integral or fractional.

In words, *the derivative of a variable base affected with a constant exponent, is the product of the exponent and the base with its exponent diminished by one.*

Apply this rule to finding the derivatives of Exercises 1, 2, 3 of Art. 29, noting from Ex. 4 that the derivative of a constant is zero.

Where the radical sign is used, change to fractional exponents.

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d(x^{-1})}{dx} = -1\,x^{-2} = -\frac{1}{x^2};$$

$$D\sqrt{x} = Dx^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}};$$

$$Dx = x^0 = 1;\ D\,(mx+b) = m;\ Dax^2 = 2\,ax;$$

$$D\,(ax^2+b) = 2\,ax;\quad D\,(ax^3) = 3\,ax^2;\quad D\,(ax^3 - b) = 3\,ax^2;$$

$$D\frac{2}{x} = D2\,x^{-1} = -\frac{2}{x^2};\quad D\left(\frac{a}{x} - b\right) = -\frac{a}{x^2},$$

$$D\left(\frac{a}{x} + bx\right) = -\frac{a}{x^2} + b;\quad D\,(\sqrt{x} - x^2) = \frac{1}{2\sqrt{x}} - 2\,x.$$

$$D_t\frac{1}{4}t^2 = \frac{1}{2}t;\quad D_t\,(t^3 - 2\,t) = 3\,t^2 - 2;$$

$$D_t\sqrt{at} = D_t\,a^{\frac{1}{2}}t^{\frac{1}{2}} = \frac{1}{2}a^{\frac{1}{2}}t^{-\frac{1}{2}} = \frac{1}{2}\sqrt{\frac{a}{t}}.$$

If, $f(x) = \sqrt{x^3}, \quad f'(x) = \frac{3}{2} x^{\frac{3}{2}-1} = \frac{3}{2} x^{\frac{1}{2}};$

$$f(x) = \frac{2}{\sqrt{x^3}} = 2\,x^{-\frac{3}{2}}, f'(x) = -3\,x^{-\frac{5}{2}};$$

$$f(x) = \frac{a}{x^{\frac{2}{3}}} = ax^{-\frac{2}{3}}, f'(x) = -\frac{2}{3}\,ax^{-\frac{5}{3}};$$

$$y = \frac{3}{x^2} = 3\,x^{-2}, \quad y' = -6\,x^{-3} = -\frac{6}{x^3};$$

$$u = \frac{a^2}{t^5} = a^2t^{-5}, u' = -5\,a^2t^{-6} = \frac{-5\,a^2}{t^6};$$

$$u = \frac{1}{y^{-\frac{4}{3}}} = y^{\frac{4}{3}}, \; u' = \frac{du}{dy} = \frac{4}{3}\sqrt[3]{y};$$

$$x = y^n \;\therefore\; \frac{dx}{dy} = D_y x = \frac{d(y^n)}{dy} = ny^{n-1};$$

$$x = y^{\frac{5}{3}} \;\therefore\; x' = \frac{dx}{dy} = \frac{5}{3}\,y^{\frac{2}{3}}.$$

REMARK. The expression for a derivative, $\frac{dy}{dx}$ is to be regarded simply as a symbol (not to be divided) and not a fraction $dy \div dx$.

Also $$\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x};$$

but the right member can not be written

$$\frac{\lim \Delta y}{\lim \Delta x} = \frac{0}{0};$$

for although by Th. IV., Art. 26, the limit of a quotient is equal to the quotient of the limits, in general, the case where the limit of the denominator is zero, was expressly excluded; besides the result is meaningless. The false notion though evidently gave rise to Bishop Berkeley's celebrated witticism, that "dy and dx were the ghosts of the departed quantities Δy and Δx." Verily there is no other logical view to those who believe that Δy and Δx become changed, as they become "evanescent," in some mysterious way, into dy and dx.

Later on $\frac{dy}{dx}$ will be regarded as a fraction, but for the present, treat it as a whole. In fact $\frac{d}{dx}$ stands for an "operator" like D_x, so that,

$$\frac{d}{dx}(y) = D_x(y) = \frac{dy}{dx},$$

the symbol indicating differentiation with respect to x of the function.

31. Derivative as the Slope of a Curve. In Fig. 23, let CPQ represent the locus $y = f(x)$. Draw perpendiculars PM, QN, from two points on the curve P, Q, to the axis of x; also draw through P a line parallel to the x-axis to meet NQ in R. The tangent to the curve at P, AP makes the angle $XAP = \theta$ with the x-axis.

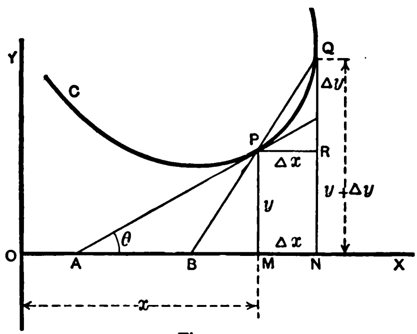

Fig. 23.

If $OM = x$, then $MP = y = f(x)$; also *if* $MN = \Delta x$, then $NQ = y + \Delta y = f(x + \Delta x)$. Hence $PR = \Delta x$ and $RQ = \Delta y = f(x + \Delta x) - f(x)$,

$$\therefore \tan RPQ = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(\Delta x)}{\Delta x}$$

As $\Delta x \doteq 0$, $\Delta y \doteq 0$; N approaches M and Q approaches P indefinitely; the secant BPQ approaches the tangent at P, AP indefinitely; for by *definition*, the tangent AP is the limiting position of the secant BPQ as Q approaches P indefinitely. Moreover, *tan* RPQ = *tan* XBP, tends to a definite limit, *tan* XAP = *tan* θ, when its equal $\Delta y / \Delta x$ tends to a definite limit.

Hence taking limits of the above equation as $\Delta x \doteq 0$, by Th. I., Art. 26,

$$\tan \theta = \frac{dy}{dx} = f'(x).$$

As before observed, Δx and Δy do not become zero, for then the ratio $\Delta y / \Delta x$ is meaningless; hence it would be inexact to say that Q actually reaches P or that the secant PQ takes the position of the tangent at P.

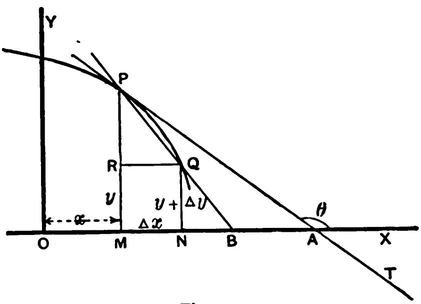

Fig. 24.

In Fig. 24, *if* $OM = x$ and $MN = \Delta x$, then $MP = y$ and $NQ = y + \Delta y$ (Art. 28); hence $\Delta y = NQ - MP = -RP$ or Δy is a negative number. However, remembering that $\tan \alpha = -\tan (180 - \alpha)$,

$$\tan ABP = -\tan QBM = -\tan PQR = \frac{-RP}{RQ} = \frac{\Delta y}{\Delta x}.$$

As Q approaches P indefinitely,

$$\Delta x \doteq 0, \tan ABP \doteq \tan XAP \text{ and } \frac{\Delta y}{\Delta x} \doteq \frac{dy}{dx}.$$

Hence taking limits as $\Delta x \doteq 0$, by Th. I. (Art. 26),

$$\tan \theta = \frac{dy}{dx},$$

as before.

Hence, whether y is increasing or decreasing, the slope *tan* θ, at the point (x, y) is always the derivative of the ordinate with respect to the abscissa at the point in question.

Some authors prefer to regard the tangent line in Fig. 24, as making a *negative* angle XAT with the x-axis, which leads to the same result, since, if XAT is negative, by trigonometry, $\tan XAT = \tan XAP = \tan \theta$.

In the figures, Δx was taken positive; if Δx were taken negative, *i.e.*, if Q is to the left of P, we derive the same formula $\tan \theta = \frac{dy}{dx}$, for the slope. In general the expressions derived for *tan* θ are the same whether Q is taken to the right or left of P, but in very rare cases they may be different, as is illustrated in Fig. 25. In such cases the slope is said to be "discontinuous," as it jumps suddenly from one value to the other.

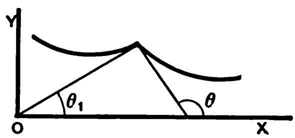

Fig. 25.

EXAMPLE 1. As an illustration of the use of the formula, the slope of the parabola $4y = x^2$ (Fig. 4) at the point (x, y) is,

$$\tan \theta = \frac{dy}{dx} = \frac{1}{2} x.$$

At $x = -2, -1, 0, +1, +2$, the slopes are, $-1, -\frac{1}{2}, 0$ $+\frac{1}{2}, +1$.

Hence, the tangent line to the left of the origin makes an obtuse angle θ with the axis of x; at the origin, through which the curve passes, the tangent line coincides with the x-axis; to the right of the origin, the slope is always positive. At $(-2, 1)$, $\tan\theta = -1 \therefore \theta = 135^\circ$; at $(+2, 1)$, $\tan\theta = +1 \therefore \theta = 45^\circ$.

EXAMPLE 2. The slope of the straight line $y = mx + b$ is, $\frac{dy}{dx} = m$, or independent of x, a result known beforehand. For all other loci, the slope is a function of x.

EXAMPLE 3. The x derivative of $y = \cos x$ was found in Art. 29, to be $(-\sin x) \therefore \tan\theta = -\sin x$. What are the slopes at $x = 0, \pi/6, \pi/2, \pi, 3\pi/2$?

The graph of $y = \cos x$ is given in Fig. 13, Art. 17. The results may be roughly checked by it. The numerical values of $\tan\theta$ can also be read off from the curve of sines.

EXAMPLE 4. The locus $y = x^{\frac{m}{n}}$, has the slope

$$\tan\theta = \frac{m}{n} x^{\frac{m}{n}-1}.$$

Consequently when $m > n$, the slope is always zero at the origin, *i.e.*, the x-axis is a tangent to the curve there.

When $m < n, \left(\frac{m}{n} - 1\right)$ is negative $\therefore$ as $x \doteq 0$, $\tan\theta \doteq \infty$, or the y-axis is tangent to the curve at the origin.

This equation $y = x^{\frac{m}{n}}$ represents a whole family of curves. Any numerical values may be given to m and n. Thus if $m = 2$, $n = 1$, $y = x^2$, the common parabola; if $m = 3$, $n = 1$, $y = x^3$, the cubical parabola; if $m = 3$, $n = 2$, we have $y = x^{\frac{3}{2}}$, the semi-cubical parabola.

EXAMPLE 5. Discuss the slopes of the parabola $y^2 = 4px$ (Fig. 16, Art. 20) and of the hyperbola $y = 1/x$ (Fig. 9, Art. 13) as $x \doteq 0$ and as $x \doteq \infty$ particularly. In $y = 1/x$ the slope $= -1/x^2 \doteq 0$ as $x \doteq \infty$ and since $y \doteq 0$ as $x \doteq \infty$, it is seen that a tangent to the curve at any point (x, y), as $x \doteq \infty$, approaches the x-axis as a limiting position. Similarly is the y-axis the limiting position (never attained) of a tangent at (x, y) as $y \doteq \infty$. As we have seen, these axes are asymptotes as defined in Art. 14.

From the results above, another definition can be given to an asymptote.

An asymptote is the limiting position of a tangent to an infinite branch of a curve as the point of tangency recedes indefinitely from the origin.

There is, of course, no limiting position for the tangent line, unless it crosses the axes, or at least one of them, at a finite distance from the origin; hence an asymptote must cross either axis or both, *at the origin* or *at a finite distance from the origin.* An asymptote may be parallel to an axis, as in Figs. 10, 13 and 14. Has the parabola $y^2 = 4px$ an asymptote?

32. Increasing and Decreasing Functions. We have seen from the graph of $y = f(x)$, Figs. 23 and 24, supposing Δx always positive, that $f'(x)$ is positive when Δy is positive and negative when Δy is negative. The converse is likewise true. For since,

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = f'(x),$$

we must have, from the definition of limit, before limits are taken,

$$\frac{\Delta y}{\Delta x} = f'(x) + a,$$

where a is a (positive or negative) variable whose limit is zero; for on taking limits, we are conducted to the preceding expression. Taking Δx positive and noting that when Δx is small enough, a can be made as small as we wish and therefore numerically less than $f'(x)$, unless $f'(x)$

is zero, we observe that the sign of $f'(x) + a$ and, therefore, of Δy, for sufficiently small values of Δx, will be the same as that of $f'(x)$. This proves that Δy is $+$ when $f'(x)$ is $+$, but Δy is $-$ when $f'(x)$ is $-$; hence $y = f(x)$ *increases when x increases if $f'(x)$ is positive, but decreases when x increases if $f'(x)$ is negative,* algebraical increase being understood.

Thus in Ex. 1, Art. 31, $4y = x^2$, $y = f(x) = \frac{1}{4}x^2$ $\therefore$ $f'(x) = \frac{1}{2}x$. Hence y or $f(x)$ decreases when x is negative, as x increases algebraically, since $f'(x)$ is then negative, but $y = f(x)$ increases when x is positive as x increases. $f(x)$ is thus *a decreasing function* for x negative, but *an increasing function* when x is positive. To find the point where the decrease ends and the increase begins, put $f'(x) = 0$. In this case $f'(x) = 0$ at the origin; hence $y = f(x) = 0$ is the *minimum* value of this function. Repeated applications of these principles will be made in the chapter on maximum and minimum values of a function.

It must be evident to the reader that the derivative can be of great aid in constructing the graphs of functions.

REMARK. It is often a practical necessity or a convenience at any rate, to plot the ordinates to a different scale to that used for the abscissas. Suppose the unit of distance for the ordinates to be 10 times that for the abscissas.

The *slope* of the curve $y = f(x)$, as given by the formula $f'(x)$, must now be multiplied by 10 to agree with the slope as found from the graph as thus plotted (see Art. 39). Similarly for other cases.

33. Continuity. An abscissa x is said to be *continuous* between $x = a$ and $x = b$ when in changing from a to b, it takes once and once only, every value from a to *b.* Generally speaking, a portion of a curve is continuous when there are no breaks in it or sudden jumps, up or down, and no part of it where $y \doteq \infty$ as $x \doteq a$ (a definite value). Before giving a precise definition, let us consider some cases of discontinuity.

The function $y = \dfrac{\sin x}{x}$ has a definite value for every finite value of x but $x = 0$. The limit of $\sin x/x$ as $x \doteq 0$ is 1, but the function is not defined for this value. There is consequently a break at $x = 0$ and the function is discontinuous there.

Part of the locus of,

$$y = \frac{2 - 3^{\frac{1}{x}}}{1 - 3^{\frac{1}{x}}}$$

is shown in Fig. 26. The simultaneous values of (x,y), $(-1, 2\frac{1}{2})$, $(-\frac{1}{2}, 2\frac{1}{8})$, $(\frac{1}{2}, \frac{7}{8})$, $(1, \frac{1}{2})$, $(2, -.37)$ are quickly computed. Then observe that when x is negative and $x \doteq 0$, $3^{\frac{1}{x}} \doteq 0$ $\therefore$ $y \doteq 2$; but when x is positive, on dividing the numerator and denominator above by $3^{\frac{1}{x}}$, then as $x \doteq 0$, $3^{\frac{1}{x}} \doteq \infty$ and

$$y \doteq \frac{-1}{-1} = 1.$$

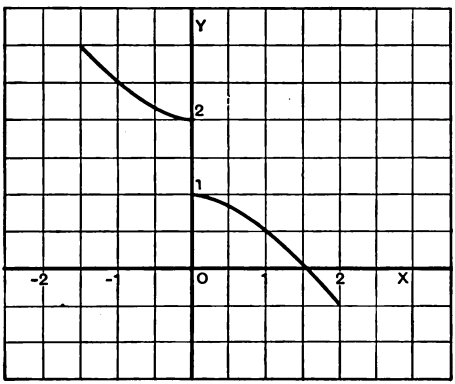

Fig. 26.

There are consequently no definite values at $x = 0$ and further, as x changes from a very small negative number to a very small positive number, y changes from a value near 2, to a value nearly equal to 1. The important point to note is, that as this increment of x tends indefinitely to zero, the change in y does not tend to zero, but to a finite number 1.

Let the student construct the graph of $y = 3^{\frac{1}{x}}$ near the origin and similarly discuss it. Observe in this case (Fig. 26a) that for x negative, $y \doteq 0$ as $x \doteq 0$; but for x positive $y \doteq \infty$ as $x \doteq 0$. There is consequently discontinuity. If Δx is taken across the origin, $\Delta y \doteq \infty$ as $\Delta x \doteq 0$; whereas for a continuous curve, we should have, $\Delta y \doteq 0$ as $\Delta x \doteq 0$.

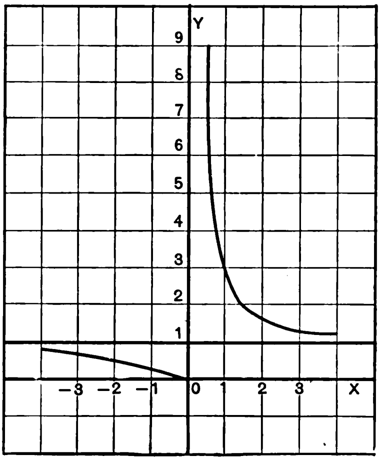

Fig. 26a.

The graph of $1/x$ (Art. 13, Fig. 9) similarly shows discontinuity at $x = 0$, the jump here being from a very large negative value to a very large positive value. Similar conclusions hold for the graphs of *tan* x and *sec* x at $x = \pi/2$, $3\pi/2$, *etc.*, and for the graphs of *cot* x and *cosec* x at $x = 0$, π, *etc.* (Figs. 13 and 14, Art. 17). We conclude from these examples that a function $f(x)$ is ***discontinuous*** at $x = x_1$, (1) *when* $f(x_1)$ *is not a definite finite value;* (2) *when* $f(x)$ *tends to different limiting values according as* x *approaches* x_1 *from the right or left respectively;* (3) *when* $f(x) \doteq \infty$ *as* $x \doteq x_1$. This is included in (1). If any one of the three tests obtains, the function is discontinuous. A function, $f(x)$ is said to be *continuous* for $x = x_1$: (1) *When $f(x_1)$ is a definite finite value*; (2) *when, considering two points on the curve (x_1, y_1), $(x_1 + \Delta x, y_1 + \Delta y)$, it is possible to take Δx so small, that Δy (numerically) can be made less than any assigned quantity other than zero, however small; or more briefly, $\Delta y \doteq 0$* as $\Delta x \doteq 0$. The latter test is sometimes given in the form,

$$\lim_{\Delta x \doteq 0} f(x_1 + \Delta x) = \lim_{\Delta x \doteq 0} (y_1 + \Delta y) = y_1 = f(x_1).$$

It is understood that Δx can be positive or negative. A portion of a curve is continuous when the two requirements above are fulfilled for every point on the portion. At the left end, Δx must be taken positive, at the right end negative, in applying test (2).

34. Continuity of the Elementary Functions. The function $y = x^n$, where n is a positive integer, is continuous. Thus putting $h = \Delta x$, $y + \Delta y = (x + h)^n = x^n + nx^{n-1}h + \ldots + h^n$ by the binomial formula $\therefore$ $\Delta y = nx^{n-1}h + \ldots + h^n$; whence $\Delta y \doteq 0$ as $h \doteq 0$, $\therefore$ since tests (1) and (2) are fulfilled, the function is continuous. It is evidently the same for ax^n and for the rational integral function,

$$ax^n + bx^{n-1} + \ldots + px + q$$

where $a, b, \ldots p, q$, are constants (Art. 26, Th. II.). Also, by Theorems II. and IV., the rational fractional function,

$$\frac{ax^n + a_1x^{n-1} + \ldots + a_n}{bx^m + b_1x^{m-1} + \ldots + b_n},$$

is continuous. For calling this function, $\frac{f(x)}{\phi x}$, it is seen from the above that,

$$\lim_{\Delta x \doteq 0} \frac{f(x_1 + \Delta x)}{\phi(x_1 + \Delta x)} = \frac{\lim f(x_1 + \Delta x)}{\lim \phi(x_1 + \Delta x)} = \frac{f(x_1)}{\phi(x_1)}.$$

The trigonometrical functions are continuous for all values of the angle except for the points of discontinuity before noted. This may be seen from a consideration of the continuous changes in the line values on a unit circle as the angle increases or decreases. The application of the above method to $y = \cos x$ will suffice to illustrate the analytical treatment. In Art. 29, we found,

$$\Delta y = -2 \sin \tfrac{1}{2} h \cdot \sin (x + \tfrac{1}{2} h).$$

Here $\sin (x + \frac{1}{2} h)$ can never exceed unity and $\sin \frac{1}{2}h \doteq 0$ as $h \doteq 0$, $\therefore$ "it is possible to take h so small that the numerical value of Δy can be made less than any assigned quantity, however small"; hence $y = \cos x$ is continuous for all values of x.

It is beyond the scope of this work to prove the continuity of a^x or $\log x$. The interested reader is referred to Lamb's Calculus, pp. 35–48. From the graphs, Art. 16, Fig. 11, it might be inferred that a^x is continuous for all finite values of x and that $\log x$ is continuous for all finite positive values of x, but discontinuous for $x = 0$.

If a function has a "unique" derivative $\lim\limits_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}$, or the same value for the derivative whether Δx is a plus or minus quantity, it follows that $\Delta y \doteq 0$ as $\Delta x \doteq 0$; hence *a function that has a unique derivative is continuous.* The same value of y must be used in computing Δy for Δx positive or negative.

It may be observed, that in constructing the graphs of preceding figures, by plotting a few points on the curve and sketching, with a free hand, a shapely curve through them, it was tacitly assumed that the functions were continuous for the range considered. The assumption can easily be verified, for all the functions considered will be found to have "unique" derivatives, except at the few isolated points noted. If on any curve, $y \doteq \infty$ as $x \doteq a$, then $x = a$ is an asymptote parallel to the y-axis; hence since an asymptote is the limiting position of a tangent (Art. 31), the slope $\frac{dy}{dx} \doteq \infty$ as $x \doteq a$ There is consequently no derivative *at* $x = a$, the point of discontinuity.

EXAMPLE 1. Roughly sketch the graphs of,

$$y = x^{-\frac{7}{5}},\ y = x^{-\frac{4}{3}},\ y = x^{-\frac{5}{4}}.$$

EXAMPLE 2. Find the derivatives of,

$$\frac{1}{x+a},\ \frac{x+a}{x-a},\ \frac{1}{x^2-a^2},\ \frac{2}{x-1}+x,$$

and the equations of the asymptotes.

EXAMPLE 3. Write the equations of the asymptotes to,

$$y = \frac{2x}{1-x^2},\ y = \frac{2x}{1+x^2},$$

by the method of Art. 15 (Lamb's Calculus, p. 31).

EXAMPLE 4. If $f(x) = x^3 - 6x^2 + 9x + 3$,

$$f'(x) = 3(x-3)(x-1)$$

$\therefore$ the tangents to the graphs at $x = 1$, $x = 3$ are parallel to the x-axis, since $f'(x) = 0$ at $x = 1$ or $x = 3$. If $x < 1$ $f'(x)$ is $+$; if $1 < x < 3$, $f'(x)$ is $-$; $\therefore f(x)$ is increasing when $x < 1$ and begins to decrease when x exceeds 1 slightly (Art. 32). Hence $f(1) = 7$ is a *maximum* value of $f(x)$. Prove similarly that $f(3) = 3$ is a *minimum* value of $f(x)$.

EXAMPLE 5. Prove that the minimum value of $y = 4 + (x - 1)^2$ occurs at $x = 1$, by inspection and also by using derivatives.

EXAMPLE 6. If $f(x) = 2x^3 - 9x - 24x + 6$, show that $f(x)$ increases when $x < -1$ and decreases when x lies between -1 and 4. For $x > 4$, $f(x)$ increases. $\therefore f(-1)$ is a maximum, $f(4)$ a minimum value of $f(x)$.

EXAMPLE 7. At what points of the parabolas, $y^2 = 4px$, $y = x^2/4$ are the slopes $+1$, -1?

EXAMPLE 8. The straight line $y = x$ intersects the parabola $y^2 = 4x$ at (0, 0) and (4, 4). Prove that the tangent to the parabola at (0, 0) is perpendicular to the axis of x and hence it makes an angle of 45° with the straight line there. What is the tangent of the angle between the straight line and tangent line to the parabola at the point (4, 4)? See Art. 11, Ex. 12.

Ans. 1/3.

EXAMPLE 9. Prove that $y = 4x$ is a tangent line to the parabola $y = 4 + x^2$ at (2, 8).

35. Differentiation of a Function of a Function.

Let $$y = F(u) \quad . \ . \ . \quad (1)$$

where $$u = f(x) \quad . \ . \ . \quad (2)$$

$F(u)$ and $f(x)$ denoting given functions of u and x respectively, that remain the same throughout the investigation.

If (2) were substituted in (1), y would be expressed as some function of (x), say,

$$y = \phi(x) \quad . \ . \ . \quad (3)$$

x, y and u having the same values in (1), (2) or (3). If we change x to $(x + \Delta x)$ in (2) and (3), u will change to $(u + \Delta u)$ in (2) and (1) and y changes in (1) and (3) to $y + \Delta y$.

We have identically,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

Note now from the assumptions above, that as $\Delta x \doteq 0$, $\Delta u \doteq 0$ and $\Delta y \doteq 0$. Therefore taking limits as $\Delta x \doteq 0$, by Th. I. and III., Art. 26 and Art. 29,

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\mathbf{dy}}{\mathbf{du}} \cdot \frac{\mathbf{du}}{\mathbf{dx}} \quad . \; . \; . \quad (4)$$

In this formula, $\frac{du}{dx}$ is the derivative of u with respect to x as the independent variable, found by differentiating (2) $\frac{dy}{du}$ is the derivative of y with respect to u from (1), just as if u were independent. Consequently, the left member, $\frac{dy}{dx}$ or by (3) the derivative of y with respect to x, can be found without actually forming (3).

Example. Let $y = u^2, \quad u = \cos x,$

$$\therefore \quad \frac{dy}{du} = 2u, \quad \frac{du}{dx} = -\sin x,$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -2 \sin x \cos x = -\sin 2x.$$

36. Derivative of $[F(x)]^n$.

Let $\quad u = F(x), \; y = u^n = [F(x)]^n.$

$$\therefore \frac{du}{dx} = F'(x), \quad \frac{dy}{du} = nu^{n-1},$$

$$\therefore \frac{d[F(x)]^n}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = n\,[F(x)]^{n-1}\,F'(x).$$

Thus, *the derivative with respect to* x *of* $[F(x)]^n$ *is equal to* n *times, the quantity in the parenthesis to the* $(n-1)^{th}$ *power, multiplied by the derivative of the quantity in the parenthesis.*

Apply this rule to the example above:

$$\frac{d}{dx}(\cos x)^2 = 2(\cos x)^{2-1}(-\sin x) = -\sin 2x.$$

Similarly,

$$\frac{d}{dx}(a + bx^n)^m = m(a + bx^n)^{m-1}\, nbx^{n-1};$$

$$\frac{d}{dx}(1 - x^3)^2 = 2(1 - x^3)(-3x^2) = -6(x^2 - x^5).$$

A commonly recurring case of the above formula is that corresponding to $n = 1/2$, which deserves a special rule.

$$\frac{d\sqrt{F(x)}}{dx} = \frac{d[F(x)]^{\frac{1}{2}}}{dx} = \frac{F'(x)}{2\sqrt{F(x)}};$$

or *the derivative of the square root of a function is equal to the derivative of the function divided by twice the square root of the function.*

EXAMPLE. $D_x \sqrt{a - bx^2} = \dfrac{-2bx}{2\sqrt{a - bx^2}}$.

By the first rule,

$$D_x \frac{1}{\sqrt{a - bx^2}} = D_x(a - bx^2)^{-\frac{1}{2}} = -\tfrac{1}{2}(a - bx^2)^{-\frac{3}{2}}(-2bx)$$

$$= \frac{bx}{(a - bx^2)^{\frac{3}{2}}}.$$

37. Let $$\mathbf{y = u + C},$$
u being a function of x and C a constant. Giving x an increment Δx, u becomes $u + \Delta u$ and y, $y + \Delta y$,

$$\therefore\ y + \Delta y = u + \Delta u + C \quad \therefore\ \Delta y = \Delta u$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \quad \therefore\ \frac{dy}{dx} = \frac{du}{dx}.$$

Thus, *an additive constant* (C) *disappears in differentiation.* Hence shifting a curve bodily, parallel to the axis of y, does not change the slope.

Also, two functions which differ only by a constant have the same derivative. This theorem, and the next two, have already been seen, in Art. 29, to follow from the general rule for differentiating. It is well to formally state them.

38. If u, v and s are functions of x, and

$$y = u + v - s;$$

when x changes to $x + \Delta x$,

$$y + \Delta y = (u + \Delta u) + (v + \Delta v) - (s + \Delta s).$$

Whence, on subtracting the first equation, dividing by Δx and taking limits, as $\Delta x \doteq 0$, we obtain, Art. 26, Theorems I. and III.,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{ds}{dx}.$$

Hence, *the derivative of the algebraic sum of two or more functions is found by differentiating each term separately and adding the results with their proper signs.*

39. If C is a constant and u a function of x, then on changing x to $x + \Delta x$,

$$y = Cu,$$

becomes, $y + \Delta y = C(u + \Delta u)$.

$$\therefore \Delta y = C \cdot \Delta u.$$

$$\frac{\Delta y}{\Delta x} = C \frac{\Delta u}{\Delta x}.$$

whence, taking limits as $\Delta x \doteq 0$,

$$\frac{dy}{dx} = C \frac{du}{dx}.$$

Therefore, *a constant factor will appear as a factor in the derivative.*

If $y = f(x)$ represents a locus, then $y = Cf(x)$ represents a new locus, each of whose ordinates is C times the corresponding ordinate of the old locus. The theorem shows us that the slope of the new locus, for the same value of x, is exactly C times that of the old locus (Art. 32, remark). Also, when C is positive, if $C f(x)$ is increasing (or decreasing) for a certain value of x, $f(x)$ is increasing (or decreasing) for the same value of x (Art. 32).

EXAMPLE 1. By Arts. 37, 38 and 30,

$$D_x(a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n) =$$
$$na_0x^{n-1} + (n-1)a_1x^{n-2} + \ldots + a_{n-1}.$$

EXAMPLE 2. By Arts. 30, 37, 38 and 39,

$$D(a + bx^2 - cx^{\frac{1}{3}} + cx^{-\frac{1}{2}})\sqrt{x} = \tfrac{1}{2}ax^{-\frac{1}{2}} + \tfrac{5}{2}bx^{\frac{3}{2}} - \tfrac{5}{6}cx^{-\frac{1}{6}}.$$

EXAMPLE 3. By Arts. 36, 37, 38 and 39,

if, $$u = \sqrt[3]{a + bx^2 + cx^3} = (a + bx^2 + cx^3)^{\frac{1}{3}},$$
$$\therefore D_xu = \tfrac{1}{3}(a + bx^2 + cx^3)^{-\frac{2}{3}}(2bx + 3cx^2).$$

EXAMPLE 4. If $u = \sqrt{x^3 + a\sqrt{x}} = (x^3 + ax^{\frac{1}{2}})^{\frac{1}{2}}$,

$$\therefore D_xu = \tfrac{1}{2}(x^3 + ax^{\frac{1}{2}})^{-\frac{1}{2}}(3x^2 + \tfrac{1}{2}ax^{-\frac{1}{2}})$$
$$= \frac{6\sqrt{x^5} + a}{4\sqrt{x^4 + ax\sqrt{x}}}.$$

EXAMPLE 5. $u = \dfrac{a}{x^2} + \dfrac{1}{x^3} + \dfrac{3}{\sqrt[3]{x^2}}$, find D_xu.

EXAMPLE 6. $u = \dfrac{(a + x)^2}{x^2}$, find D_xu.

EXAMPLE 7. In Art. 20, the equation of any conic, referred to its vertex and axis as origin and x-axis respectively, is,

$$y = \pm\sqrt{2\,Ax + Bx^2}.$$

The slope at (x, y) is,

$$D_x y = \pm\frac{A + Bx}{\sqrt{2\,Ax + Bx^2}} = \frac{A + Bx}{y}.$$

As $x \doteq 0$, $D_x y \doteq \infty$; hence the tangent is perpendicular to the x-axis at the origin.

It is generally more desirable to differentiate the equation of the conic in the form,

$$y^2 = 2\,Ax + Bx^2;$$

for since y is a function of x, therefore the rule for powers (Art. 36) applies, giving,

$$2\,y\frac{dy}{dx} = 2\,A + 2\,Bx;$$

whence we derive the same value of the slope as before on dividing by $2\,y$.

EXAMPLE 8. The equations of the ellipse and hyperbola referred to the center and axes are respectively (Art. 20),

$$a^2y^2 + b^2x^2 = a^2b^2;\quad a^2y^2 - b^2x^2 = -\,a^2b^2$$

$$\therefore\; 2\,a^2y\frac{dy}{dx} + 2\,b^2x = 0;\quad 2\,a^2y\frac{dy}{dx} - 2\,b^2x = 0$$

$$\therefore\; \frac{dy}{dx} = -\frac{b^2x}{a^2y};\quad \frac{dy}{dx} = \frac{b^2x}{a^2y}.$$

The slope of the ellipse is *positive*, when the point of tangency (x, y) is in the 2d or 4th quadrants, negative when (x, y) is in the 1st or 3d quadrants. Discuss similarly the slope of the hyperbola.

EXAMPLE 9. The slope of the hyperbola can be put in the form,

$$\pm\frac{b}{a}\frac{x}{\sqrt{x^2-a^2}}=\pm\frac{b}{a}\frac{1}{\sqrt{1-\frac{a^2}{x^2}}}.$$

As $x \doteq \infty$, the slope $\doteq \pm\frac{b}{a}$ ∴ the slopes of the asymptotes are $\frac{b}{a}$ and $-\frac{b}{a}$ (Art. 31, Ex. 5).

EXAMPLE 10. If $y=F(u)=F(x+a)$ ∴ $u=x+a, \frac{du}{dx}=1$.

∴ by (4) Art. 35, $\frac{dy}{dx}=\frac{dy}{du}$ ∴ $\frac{dF(x+a)}{dx}=F'(u)=F'(x+a)$.

40. Derivative of a Product.

If $$\mathbf{y = uv},$$

where u and v are both functions of x,

$$\begin{aligned}\Delta y &= (u+\Delta u)(v+\Delta v)-uv\\ &= v\Delta u+u\Delta v+\Delta u\Delta v\end{aligned}$$

$$\therefore \frac{\Delta y}{\Delta x}=v\frac{\Delta u}{\Delta x}+(u+\Delta u)\frac{\Delta v}{\Delta x}$$

∴ as $\Delta x \doteq 0$, by Theorems II. and III., Art. 26,

$$\frac{\mathbf{dy}}{\mathbf{dx}}=\mathbf{v}\frac{\mathbf{du}}{\mathbf{dx}}+\mathbf{u}\frac{\mathbf{dv}}{\mathbf{dx}} \quad . . . (1)$$

Learn this as a rule, which state.

Again, if $y = uv.w$, on treating first uv as a single function, we have by the rule,

$$\begin{aligned}\frac{dy}{dx} &= w\frac{d(uv)}{dx}+uv\frac{dw}{dx}\\ &= w\left(v\frac{du}{dx}+u\frac{dv}{dx}\right)+uv\frac{dw}{dx}\end{aligned}$$

$$\therefore \mathbf{D\,(uvw) = vwDu + uwDv + uvDw} \quad . . . (2)$$

Similarly n factors can be treated.

(2) can be read as a rule thus: *the derivative of the product of any number of factors is found by multiplying the derivative of each factor, in turn, by the product of the remaining factors and adding the results.*

EXAMPLE. $u = (x^2 + b)(ax^3 + bx)$

$$\therefore Du = (ax^3 + bx)\, 2x + (x^2 + b)(3ax^2 + b)$$
$$= 5ax^4 + (a + 1)\, 3x^2b + b^2$$

EXAMPLE. $u = x^n = x \cdot x \cdot x \ldots$ (n factors), n being a positive integer. By the rule, since $Dx = 1$,

$$\therefore D(x^n) = x^{n-1} + x^{n-1} + x^{n-1} + \ldots \text{ (n terms)} = nx^{n-1}.$$

41. The Derivative of the Quotient of Two Functions. Let u and v as before be two functions of $x \therefore y = u/v$, will be a function of x. Differentiate, $vy = u$, by the rule for a product (Art. 40):

$$v\frac{dy}{dx} + y\frac{dv}{dx} = \frac{du}{dx}.$$

$$D_x\left(\frac{u}{v}\right) = \frac{dy}{dx} = \frac{1}{v}\frac{du}{dx} - \frac{y}{v}\frac{dv}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2},$$

on substituting $y = \dfrac{u}{v}.$

This can be written,

$$\mathbf{D}\left(\frac{\mathbf{u}}{\mathbf{v}}\right) = \frac{\mathbf{vDu - uDv}}{\mathbf{v}^2},$$

differentiations with respect to x being understood.

Thus, *the derivative of a quotient is, the denominator multiplied by the derivative of the numerator, minus the numerator multiplied by the derivative of the denominator, the difference being divided by the square of the denominator.*

REMARK. If either numerator or denominator is constant, it is shorter to apply the rule (Art. 39) relative to a constant factor.

$$\text{Thus,}\quad D\left(\frac{C}{v}\right) = D\left(C \cdot \frac{1}{v}\right) = C \cdot \frac{-Dv}{v^2} = \frac{-CDv}{v^2};$$

$$\text{also,}\quad D\left(\frac{u^2}{C}\right) = D\left(\frac{1}{C} \cdot u^2\right) = \frac{1}{C} 2u \cdot Du = \frac{2uDu}{C}.$$

It is thus plain that the constant factor C need not be removed from its original place.

Examples under the rule above.

$$D_x \frac{ax^2 + b}{x^2} = \frac{x^2\, 2ax - (ax^2 + b)\, 2x}{x^4} = -\frac{2b}{x^3}.$$

$$D_x \frac{\sqrt{2px}}{x^2} = \frac{x^2 \dfrac{p}{\sqrt{2px}} - \sqrt{2px} \cdot 2x}{x^4} = -\frac{3}{2}\frac{\sqrt{2px}}{x^3}.$$

$$\text{Let } u = \frac{x}{\sqrt{a^2 - x^2}} = \frac{x}{(a^2 - x^2)^{\frac{1}{2}}};$$

$$\therefore D_x u = \frac{(a^2 - x^2)^{\frac{1}{2}} - x \cdot \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)}{a^2 - x^2} = \frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

The rules of Arts. 36, 40 and 41 should be thoroughly memorized before attacking the examples below.

Miscellaneous Examples.

1. $D \cdot x(1 - x) = 1 - 2x.$
2. $D \cdot x^m (1 - x)^n = x^{m-1}(1 - x)^{n-1}\{m - (m + n)x\}.$
3. $D \cdot (a + x)\sqrt{a - x} = \dfrac{a - 3x}{2\sqrt{a - x}}.$
4. $D \cdot (x - 1)(x - 2)(x - 3) = 3x^2 - 12x + 11.$
5. $D \cdot \sqrt{\dfrac{1 + x}{1 - x}} = \dfrac{1}{(1 - x)\sqrt{1 - x^2}}.$

6. $D \cdot \dfrac{x^n - 1}{x^n + 1} = \dfrac{2nx^{n-1}}{(x^n + 1)^2}.$

7. $D \cdot \dfrac{x}{\sqrt{a^2 - x^2}} = \dfrac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$

8. $D \cdot \dfrac{x^m}{(1 - x)^n} = \dfrac{x^{m-1}}{(1 - x)^{n+1}} \{m - (m - n)x\}.$

9. $D \cdot \dfrac{\sqrt{x^2 + 1}}{\sqrt{x^2 - 1}} = \dfrac{-2x}{(x^2 - 1)\sqrt{x^4 - 1}}.$

10. If y is a function of x, so is any expression containing y ∴ by Art. 36,

$$\begin{aligned} D_x[y]^n &= ny^{n-1} \cdot D_x y, \\ D_x(x^m y^n) &= y^n(mx^{m-1}) + x^m(ny^{n-1} \cdot D_x y) \\ &= x^{m-1}y^{n-1}(my + nx . D_x y), \end{aligned}$$

since $(x^m y^n)$ is a product of two functions of x ∴ the rule of Art. 40 applies.

11. $D \cdot (Ax^2 + 2Bxy + Cy^2 - D)$
$= 2Ax + 2By + 2Bx \cdot D_x y + 2Cy \cdot Dy.$

CHAPTER V.

RATES. TANGENT AND NORMAL.

42. Uniform Change. One fundamental problem of the Calculus, that of tangents, has been touched on ; the other, the problem of *rates*, will next engage our attention. Both are closely connected and either one can receive a solution by aid of the other. Before defining rate of change of a function with respect to its argument, clear ideas must be had of what is meant by uniform and non-uniform change or variation and average-rate.

If u is a function of x, (x, u) and (x_1, u_1) being simultaneous values, then for *uniform change*, we must have,

$$\frac{u_1 - u}{x_1 - x} = \text{constant} = a,$$

where (x, u), (x_1, u_1) are *any* two pairs of simultaneous values.

Uniform change is thus defined as such that the increment of the function is in a constant ratio to the corresponding increment of the argument.

The above equation, $u - u_1 = a(x - x_1)$, shows that u is a linear function of the argument x. In fact, if we replace u by y, it is the equation of a straight line through (x_1, y_1).

Conversely, if u is a linear function of x, then u changes uniformly with respect to x. Thus, if $u = ax + b$, and $u_1 = ax_1 + b$,

$$\therefore \frac{u - u_1}{x - x_1} = a,$$

or the change is uniform. The constant ratio a is called the *rate*. Thus again, to refer to a locus, if y replaces u, it is seen by the definition, for the straight line $y = ax + b$, that y varies with respect to x, at the rate $a =$ the slope. In the increment notation, this can be written $\frac{\Delta y}{\Delta x} = a$.

Illustrate this for various points on a straight line.

43. Non-Uniform Change. If $u = f(x)$ and

$$\frac{u_1 - u}{x_1 - x} = \frac{\Delta u}{\Delta x},$$

is not constant for the whole locus, the change is said to be non-uniform, or the function does not vary uniformly with respect to the argument. All other functions than linear ones, vary non-uniformly. The subject will be best understood from a concrete example.

Thus, let $u = x^2$ represent the area of a square ON, Fig. 27, where $x = OB$. That is x^2 is a number, representing the number of square inches in ON if x is the number of linear inches in OB. Similarly for any other unit than the inch.

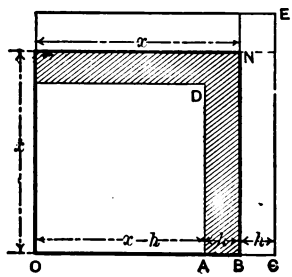

Fig. 27.

Now suppose x to increase to $(x + \Delta x)$ or $(x + h)$ where $h = \Delta x$; then the number of square units in the area OE will be,

$$u + \Delta u = (x + h)^2 = x^2 + 2xh + h^2$$

$$\therefore \frac{\text{Increase in area in sq. units}}{\text{Increase in side in linear units}} = \frac{\Delta u}{\Delta x} = 2x + h.$$

Δu is the number of square units in the area just outside of the square ON.

If $OA = x - h \therefore u - \Delta_1 u = (x - h)^2$ * ;
on subtracting this from $u = x^2$ and dividing by $\Delta x = h$, we obtain,

$$\frac{\Delta_1 u}{\Delta x} = 2x - h.$$

$\Delta_1 u$ represents the shaded area.

Since $\frac{\Delta u}{\Delta x}$ and $\frac{\Delta_1 u}{\Delta x}$ are not equal we have, by the definition, a function u which changes non-uniformly with respect to its argument x.

44. Average-Rate. Definition of Rate. If u is *any* function of x, the ratio of the change in the function to the change in x, when x receives an increment Δx, is defined to be *the average-rate* with which the function varies with respect to its argument for the range considered. Thus starting with an initial value of x, $\frac{\Delta u}{\Delta x}$ is the average-rate of change of u as x varies from the particular value x to $x + \Delta x$.

Thus in the example of Art. 43,

$$\frac{\Delta u}{\Delta x} = 2x + h,$$

is the average-rate of increase of u as x varies from OB to OC, while

$$\frac{\Delta_1 u}{\Delta x} = 2x - h,$$

is the average-rate of increase of u as x varies from OA to OB.

* By the convention of Art. 28, this equation should be written, $u + \Delta_1 u = (x - h)^2$; so that $\Delta_1 u$ would be negative. It was thought best to depart from the usual convention here so that $\Delta_1 u$ should from the start be regarded as positive. A similar remark applies to the example of Art. 46.

It is observed that the *average*-rates, in this example, depend both on x and h and that they approach indefinitely equality as h tends to zero as a limit. But $h = \Delta x$ can never reach zero, for then $\Delta u/\Delta x$ and $\Delta_1 u/\Delta x$ would take the form 0/0 and would be meaningless.

The limit that either ratio approaches indefinitely as $h \doteq 0$, but never attains is,

$$\lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x} = 2x,$$

and this limit is *defined* to be the *rate* at which u varies with respect to x at this particular value of $x = OB$.

By a consideration of the increasing series,

$$\frac{\Delta_1 u}{\Delta x}, \lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x}, \frac{\Delta u}{\Delta x};$$

or their values, $(2x - h)$, $2x$, $(2x + h)$,
the definition given above seems inevitable, since

$$\lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x}$$

is greater than the average-rate $\Delta_1 u/\Delta x$, when x varies from OA to OB and less than $\Delta u/\Delta x$, when x varies from OB to OC, and this is true however near zero h may be.

Definition. *The rate at which any function u of x is said to increase with respect to x when $x = x_1$, is then defined to be,*

$$\lim_{\Delta x \doteq 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx},$$

or the derivative of u with respect to x, provided x_1 is substituted for x after the differentiation is performed.

In the example above, the rate is

$$\frac{d}{dx}(x^2) = 2x,$$

as found before. When $x = 1, 2, 3$, the rate of change of u with respect to x is $2, 4, 6$; it being understood that the change in u is in square units, the change in x being in linear units.

The rate above is called briefly the x-rate of u, and it may be remarked that the element of time does not necessarily enter into it. We shall discuss time-rates later.

45. Rate of Increase of the Ordinate of a Curve with Respect to the Abscissa. In Fig. 27 (a), let $AB = BC = h$, $OB = x$, $BF = y$, $CH = y + \Delta y$, $GH = \Delta y$. Draw DE and FG parallel to OX to intersections E and G with BF and CH respectively.

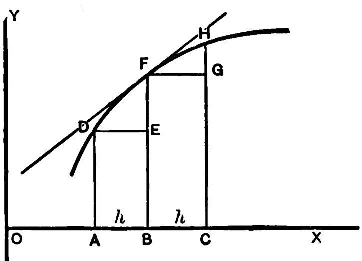

Fig. 27a.

Consider the ratios,

$$\frac{EF}{h}, \frac{dy}{dx}, \frac{GH}{h};$$

the first ratio being (Art. 44), the average-rate of increase of y as x changes from OA to OB, while the last ratio is the average-rate of increase of y as x changes from OB to OC. Also, it is plain that, as $h \doteq 0$,

$$\lim \frac{EF}{h} = \lim \frac{GH}{h} = \frac{dy}{dx} \text{ (Art. 31)},$$

unless the curve is of the kind shown in Fig. 25, Art. 31.*

* Such exceptional curves, with different "right and left hand derivatives" are rarely met with. The slope is said to be discontinuous at such points.

If the investigations of the preceding article had not been made, we should be naturally led, from this result, to the same definition of rate. The rate at which y tends to increase with respect to x is thus, $\frac{dy}{dx}$, or the slope of the curve at the point (x_1, y),

46. Time-Rate or Velocity. If $s = f(t)$, where s represents distance from some origin, and t, the time elapsed from a certain assumed epoch, then from the definition (Art. 44), $\frac{ds}{dt}$ represents the rate of change of s with respect to t. This is a fundamental theorem in Mechanics, and it seems well to give an example illustrating in detail the subject and the meaning of time-rate. When a moving body passes over equal distances in equal times, whatever the time interval is taken, it is said to have a uniform velocity, speed or rate, or to have uniform motion. Hence, the formula for uniform motion,

$$\text{time-rate} = \text{velocity} = \frac{\text{distance}}{\text{time}}.$$

Thus if a train of cars moves uniformly 60 miles in every 3 hours, its velocity or rate is $60/3 = 20$ miles per hour. If a point moves 12 feet every 3 seconds, its velocity is $12/3 = 4$ feet per second.

When the body or point does not pass over equal distances in equal times, its motion is non-uniform, and its *average-rate* in passing over a length s in time t is defined to be s/t, the units of distance and time being stated.

This much being premised, let us consider the case of a body falling in vacuo freely, under the attraction of the earth. If s represents the number of feet passed over, during t seconds of free fall, the formula connecting s and t as given by experiment, is approximately,

$$s = 16\,t^2 \quad . \; . \; . \quad (1)$$

Thus after $t = 1, 2, 3,$ seconds of fall, the spaces described in feet are 16, 64, 144 respectively.

Starting with certain simultaneous values of s and t satisfying (1), let t receive an increment Δt and s the increment Δs, whence,

$$s + \Delta s = 16\,(t + \Delta t)^2.$$

On subtracting (1) and dividing by Δt, we get the "average-rate" of describing the space Δs,

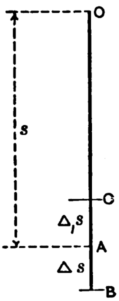

Fig. 28.

$$\frac{\Delta s}{\Delta t} = 16\,(2\,t + \Delta t) \quad . \; . \; . \quad (2)$$

In Fig. 28, $s = OA$, $\Delta s = AB$.

Again, after $(t - \Delta t)$ seconds of fall, the new value of s will be, say, $OC = (s - \Delta_1 s)$, if we take $\Delta_1 s$ as a positive number $= CA$.

$$\therefore \text{ from (1)}, \; (s - \Delta_1 s) = 16\,(t - \Delta t)^2;$$

whence subtracting this from (1) and dividing by Δt, we get the average rate of describing the space CA,

$$\frac{\Delta_1 s}{\Delta t} = 16\,(2\,t - \Delta t) \quad . \; . \; . \quad (3)$$

As $\Delta t \cdot 0$, $CA \doteq 0$ and $AB \doteq 0$ and the limit of either (2) or (3) is $16\,(2\,t)$,

$$\therefore \frac{\Delta_1 s}{\Delta t} < \lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} < \frac{\Delta s}{\Delta t}.$$

However small Δt becomes, provided it is not zero, this inequality holds; the average-rate of describing *CA* is always less than

$$\lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t},$$

and the average-rate of describing *AB* is always greater than this limit, hence it is natural to *define*,

$$\lim_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt},$$

as the *rate* the body has just as it reaches *A*. This is in exact agreement with the definition of rate of Art. 44. This rate $= 32\,t$, is entirely independent of Δt, but it is a function of t. It means that at the end of 1, 2, 3, . . . seconds, the velocity of the body, just at the instant, is 32, 64, 96, . . . feet per second. Thus $32\,t$ is the space that would be passed over in one second if the motion of the body should become uniform at the end of t seconds.

The student of Mechanics will see the meaning of this to be, that if the accelerating force of gravity were removed at the end of the t seconds, the body would move over exactly $32\,t$ feet in each succeeding second. It has an *actual* velocity of $32\,t$ feet per second at the end of t seconds. The body has then at each instant an actual velocity and not an ideal one manufactured by a definition, and this velocity is increasing with t. At the end of t seconds, it must therefore be greater than $16\,(2\,t - \Delta t)$ and less than $16\,(2\,t + \Delta t)$, the average velocities just before and just after the body reaches *A* (Fig. 28). Hence it can only have the value $32\,t$, since Δt can be diminished *ad infinitum* without ever becoming zero.

In the familiar Atwood's machine, the accelerating force acting on a body can be removed at any instant, and the actual distance passed over in each succeeding second measured, giving the actual velocity at the instant considered.

47. Whenever a magnitude (u say), as a distance for example, can be expressed in terms of t, the time reckoned from a certain epoch, its derivative with respect to t gives its change per unit of time at the instant considered.

It will be well for the beginner to conceive Δu to be actually formed; then divided by Δt to give *the average-rate*, whose limit as $\Delta t \doteq 0$ is, by definition, *the rate* at time t.

EXAMPLE 1. Two men, leaving the same point together, walk at right angles, the one at 3 feet, the other at 4 feet, per second. At what rate is the distance between them changing?

At the end of t seconds, the first man is distant $3t$ feet from the starting point, the second $4t$ feet; hence the distance between them is,

$$u = \sqrt{(3t)^2 + (4t)^2} = 5t.$$

Hence the distance between them is changing *uniformly* at the rate of $\frac{du}{dt} = 5$ feet per second.

EXAMPLE 2. Two ships are moving at right angles, the first going east at 3 miles an hour, the second south at 4 miles an hour. At 12 M. the second ship is at the crossing of the two paths, the first ship 2 miles to the east. At what rate is the distance u between them changing t hours after 12 M. At this time, the first ship is $(2 + 3t)$ miles from the crossing point, the second $(4t)$ miles.

$$\therefore \quad u^2 = (2 + 3t)^2 + (4t)^2$$

$$\therefore \quad 2u\frac{du}{dt} = 50t + 12$$

$$\therefore \quad \frac{du}{dt} = \frac{25t + 6}{\sqrt{(2 + 3t)^2 + 16t^2}}.$$

Prove that the first formula (and hence the last) is correct when t is minus, indicating hours before 12 M., distances east and south being taken as $+$, those west and north, $-$. When $t = 2, \frac{du}{dt} = 4.95$, showing that at 2 P.M. the distance between the ships was increasing at the rate of 4.95 miles per hour. When $\frac{du}{dt} = 0$, there is no change in the distance. This happens at $t = -6/25$ or about $\frac{1}{4}$ hour before noon. At 11 A.M., $t = -1$, and the distance between the ships is decreasing (since $\frac{du}{dt}$ is minus) at the rate of $19/\sqrt{17}$ miles per hour. Draw figures.

In many problems, t is not directly introduced, but the argument x is assumed to vary with the time, *i.e.*, to be a function of the time, and consequently, $u = f(x)$ is also a function of the time.

Hence,
$$\frac{dx}{dt} = \text{time-rate of change of } x,$$
$$\frac{du}{dt} = \text{time-rate of change of } u.$$

Examples.

1. A square plate of metal, Fig. 27, has, at a given time, a length of side x inches, and area, $u = x^2$ (sq. inches). It is expanding by heat. When $x = 10$, x is increasing at the rate of 0.001 inches a second; how fast is the area increasing?

∵ Since, by hypothesis,

$$\frac{dx}{dt} = 0.001 \text{ at } x = 10,$$

$$\therefore \text{(Art. 36)}, \quad \frac{du}{dt} = \frac{d}{dt}(x^2) = 2x\frac{dx}{dt} = 20 \times .001 = .02;$$

which means that the area is increasing at the rate, 0.02 sq. in. per second when $x = 10$ inches.

2. If $u = x^3$ represents the volume of a cube whose variable edge is x, find the rate of change of volume at $x = 2$, if each edge is increasing at the rate 0.001 unit a second.

$$\frac{du}{dt} = 3\,x^2\,\frac{dx}{dt} = 3\,(2)^2 \times .001 = .012$$

∴ the volume is changing, just at the instant when $x = 2$ units, at the rate 0.012 cubic units per second.

3. A boat is moving in a direction AB (Fig. 29) at right angles to the shore line BD, uniformly at 10 miles an hour. At what rate is it approaching the point D, BD being 3 miles, when the boat is at A, 4 miles from B?

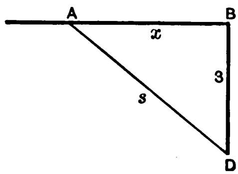

Fig. 29.

Letting $AB = x$, $BD = 3$, $AD = s$; since

$$s = \sqrt{x^2 + 9} \quad \therefore \quad \frac{ds}{dt} = \frac{x}{\sqrt{x^2 + 9}} \cdot \frac{dx}{dt}.$$

Hence when $x = 4$, the boat is approaching D at the rate, $\frac{4}{5} \times 10 = 8$ miles per hour.

4. If a body is forced to run in a groove along AB (Fig. 29) and is acted on by a force applied through a rope AD, attached to a fixed point D, the rope being shortened at the rate of v feet per second, with what velocity is the body moving in the direction AB when $x = 4$ feet?

Substitute $x = 4$, $\frac{ds}{dt} = (-v)$, since s is decreasing, in the formula of Ex. 3, and solve for $\frac{dx}{dt}$.

$$\therefore \frac{dx}{dt} = \frac{\sqrt{x^2 + 9}}{x} \cdot \frac{ds}{dt} = \frac{5}{4}(-v) = -\frac{5}{4}v.$$

If $v = 8$ (feet per second say), $\frac{dx}{dt} = -10$, or x is decreasing (Art. 32) at the rate of 10 ft. per sec.

If $BD = 6$, $x = 8$, $v = 16$, then $\frac{dx}{dt} = -20$. Interpret.

5. The base of a right-angled triangle is constant, but its altitude increases at the rate of 0.1 ft. per second. What is the rate of increase of the area? Let b = base.

Ans. .05 b sq. in. a second (uniform variation).

6. A locomotive headlight at E, Fig. 30, is 10 feet above the straight level track ABC at A. It casts a shadow of a man 6 ft. high at B, of length $s = BC$. If the locomotive moves to the left at 10 feet per second, at what rate is the shadow increasing?

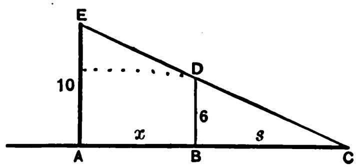

Fig. 30.

If $x = AB$, $s + x : s = 10 : 6$

$$\therefore \frac{ds}{dt} = \frac{3}{2}\frac{dx}{dt} = \frac{3}{2} 10 = 15 \text{ (ft. per sec.)}.$$

7. In Ex. 6, at what rate is AC increasing?

Put $u = AC \therefore u : u - x = 10 : 6$

$$\therefore \frac{du}{dt} = \frac{10}{4}\frac{dx}{dt} = \frac{5}{2}(10) = 25 \text{ (ft. per sec.)}.$$

(This should equal the locomotive's rate (10) + the shadow's rate (15).)

8. In the same example, at what rate is $\tan BCD = y$, decreasing?

$$y = \frac{4}{x} \therefore \frac{dy}{dt} = \frac{-4}{x^2}\frac{dx}{dt} = \frac{-40}{x^2}.$$

In Exs. 6 and 7, the answers are constants, showing uniform motion whatever the value of x. In Ex. 8, y varies non-uniformly, and hence (as in Exs. 1–4) particular values must here be assigned to x. At $x = 10$, $\frac{dy}{dt} = -0.4$, etc.

9. In Ex. 6, Fig. 30, if $I =$ illumination of a sphere at C, is represented by,

$$I = \frac{100}{(CE)^2} = \frac{100}{10^2 + u^2},$$

$$\therefore \frac{dI}{du} = - \frac{200\,u}{(100 + u^2)^2}.$$

This gives the rate of change of I with respect to u, at a value $u = AC$, to be assigned.

10. A spherical balloon is being filled with gas at the rate of c cubic metres a second. When the diameter is 1 metre, at what rate is it increasing?

If $u =$ diameter and $x =$ volume,

$$x = \frac{\pi}{6} u^3 \quad \therefore \frac{du}{dt} = \frac{1}{3} \left(\frac{6}{\pi x^2} \right)^{\frac{1}{3}} \frac{dx}{dt}.$$

$$\frac{dx}{dt} = c; \text{ also when } u = 1,\ x = \pi/6 \quad \therefore \frac{du}{dt} = \frac{2c}{\pi}.$$

The material of this fictitious balloon is supposed to be perfectly elastic and to offer no increased resistance to expansion.

11. Consider a straight metal rod of length unity at the temperature of melting ice. On being heated θ degrees Centigrade above this temperature, its new length x (it is found by experiment) can be represented by the expression,

$$x = 1 + a\theta + b\theta^2,$$

where a and b are constants. The rate of change of length per degree is,

$$\frac{dx}{d\theta} = a + 2b\theta,$$

which is called "the coefficient of expansion for the temperature θ."

If we take increments as usual (Art. 28) we find,

$$\frac{\Delta x}{\Delta \theta} = a + 2b\theta + b \cdot \Delta\theta.$$

This ratio gives the average increase of length per degree, as θ varies from θ to $\theta + \Delta\,\theta$. The distinction between $\frac{dx}{d\theta}$ and $\frac{\Delta\,x}{\Delta\,\theta}$ has been made before, but it is well to recur to it repeatedly to fix the idea in the mind permanently. When $\Delta\,\theta$ is small, compared with θ, either expression will give the same result with sufficient accuracy for practical purposes.

12. One ship is moving N. 60° E. at 8, another south, at 10, miles per hour. The first ship is x miles from the intersection of the paths when the second is y miles from it. If u is their distance apart in miles, find the rate of change of u when $x = 20$, $y = 20$.

$$u^2 = x^2 + y^2 + xy,$$

$$\therefore \frac{du}{dt} = \frac{1}{2\,u}\left[(2\,x + y)\frac{dx}{dt} + (2\,y + x)\frac{dy}{dt}\right].$$

The ships are [redacted] 34.5 miles [redacted] apart (value of u).
. separating at rate [redacted] per hour, or about 15.4 p.h.

48. Coördinates of a Locus Expressed in Terms of *t*.

When x and y are expressed in terms of t, the locus can be drawn either by computing simultaneous values of x and y for assumed values of t and plotting them, or else t can be eliminated (as in the examples below) and the curve drawn from the resulting equation of the locus.

(i) $x = a + bt,\ y = c + dt \quad \therefore\ y - c = \frac{d}{b}(x - a).$

(ii) $x = 2\,t,\ y = 16\,t^2 \quad \therefore\ y = 4\,x^2.$

(iii) $x = t,\ y = 2\,t^3 \quad \therefore\ y = 2\,x^3.$

(iv) $x = a\cos\frac{2\,\pi\,t}{T},\ y = b\sin\frac{2\,\pi\,t}{T} \quad \therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

(v) $x = a\cos\frac{2\,\pi\,t}{T},\ y = b\cos\frac{4\,\pi\,t}{T} \quad \therefore\ \frac{y}{b} + 1 = \frac{2\,x^2}{a^2}.$

(vi) $x = a\sec t,\ y = b\tan t \quad \therefore\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

EXAMPLE. (i) represents a straight line through the point (a, c) and with a slope d/b.

EXAMPLE. (ii) represents a parabola, (iii) a cubic parabola, (iv) an ellipse, (v) a parabola and (vi) a hyperbola. In Exs. (iv) and (v) the constant T is called the periodic time.

If we suppose the axis of x horizontal, the axis of y, vertical; then $\dfrac{dx}{dt}$ is the time-rate of change of x or the horizontal component of the velocity of a point moving along the curve, $\dfrac{dy}{dt}$, the vertical component.

Thus in Ex. (i), a point moving according to the law, $x = a + bt$, $y = c + dt$, is moving horizontally at rate b feet per second and vertically at rate d feet per second, if distances, as x and y, are measured in feet and t in seconds.

In Ex. (ii), the point is moving horizontally 2 feet and vertically $(32\,t)$ feet per second. Thus at the end of 1, 2, 3, . . . seconds, its vertical rate is 32, 64, 96, . . . feet per second, but the horizontal rate is always 2 feet per second. What are the horizontal and vertical rates in Ex. (iii) at the end of 1, 2, 3, seconds?

To get the resultant velocity along the curve the following preliminary theorem will first be proved.

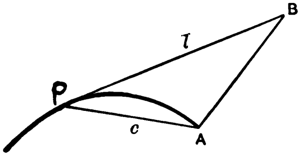

Fig. 31.

49. The Limit of the Ratio of any Infinitesimal Arc to its Chord is Unity.

Let the arc PA, Fig. 31, be supposed either to lie wholly in a plane or otherwise. A tangent is drawn at P of variable length $l = PB$ and the points A and B are connected by a straight line.

Suppose the *constant* angle PAB to be made so great that $l >$ arc PA and as A approaches P, PB shall always remain greater than arc PA. Call the length of the chord PA, c. We have, by the law of sines,

$$\frac{l}{c} = \frac{\sin PAB}{\sin ABP};$$

which ratio approaches unity for a limit as arc $PA \doteq 0$ and $\therefore$ angle $APB \doteq 0$; for, since $PAB = 180 - (ABP + APB)$, the angles PAB and ABP tend to become supplementary and hence their sines approach equality indefinitely.

Since, $$\frac{\text{arc } PA}{\text{chord } PA} < \frac{l}{c}$$

and the limit of l/c is unity $\therefore$ the ratio of arc PA to chord PA has a limit unity as arc $PA \doteq 0$, since the latter ratio is always nearer unity than the ratio l/c.

50. Resultant Velocity along a Curve. In Fig. 23, Art. 31, call the distance CP from any arbitrary point C of the curve to P, s, and its increment, arc PQ (corresponding to $\Delta x = PR$, $\Delta y = RQ$), Δs. The coördinates (x, y) of P and the length s will be regarded as functions of t, the time from a fixed epoch. If we conceive a point moving along the curve CPQ; at time t it is at P, for which $x = OM$, $y = MP$, $s = CP$; when the time has changed to $t + \Delta t$, the point is at Q and $x + \Delta x = ON$, $y + \Delta y = NQ$, $s + \Delta s = CPQ$.

Let the length of chord PQ be denoted by c. Then from the right triangle PRQ, we have,

$$(\Delta x)^2 + (\Delta y)^2 = c^2 = \left(\frac{c}{\Delta s}\right)^2 \cdot (\Delta s)^2 \quad . \quad . \quad . \quad (1)$$

$$\therefore \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 = \left(\frac{c}{\Delta s}\right)^2 \cdot \left(\frac{\Delta s}{\Delta t}\right)^2 .$$

Hence, as $\Delta t \doteq 0$, and $\therefore \Delta x, \Delta y, \Delta s$, as well as c, tend to the limit zero,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 \ldots (2),$$

since by the preceding article, $lim\ (c/\Delta s) = 1$.

In this formula, $\left(\frac{dx}{dt}\right)$ is the velocity of the point parallel to the x-axis, $\left(\frac{dy}{dt}\right)$ the velocity parallel to the y-axis, and $\left(\frac{ds}{dt}\right)$ the velocity along the curve just as the point reaches P. It tends to go in the direction of the tangent there, as can be easily seen. Thus let P' be a point on the curve just to the left of P. As the point moves from P' to P along the curve, the direction of the line $P'P$ approaches indefinitely that of the tangent at P as its limit. The limiting position of $P'P$ or the tangent AP is defined as the direction of the curve at P or the direction of the moving point just as it reaches P.

In examples (i), (ii), (iii) of Art. 48, the component velocities can be found at once and substituted in (2) to find the resultant velocity $\left(\frac{ds}{dt}\right)$.

Thus in Ex. (i), $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{b^2 + d^2}$;

in Ex. (ii), $\frac{ds}{dt} = \sqrt{4 + (32\,t)^2}$;

in Ex. (iii), $\frac{ds}{dt} = \sqrt{1 + 36\,t^4}$.

The direction of motion at time t or at the point (x, y) corresponding, can be found by differentiating the equation of the curve expressed in terms of x and y.

Thus in (i), $\frac{dy}{dx} = \frac{d}{b}$, the slope of the line;

in (ii), $\frac{dy}{dx} = 8x = 16t$;

in (iii), $\frac{dy}{dx} = 6x^2 = 6t^2$.

Similarly, examples (iv), (v), (vi), can be treated as soon as methods of differentiating the circular functions are known.

If it is assumed that a point moves *uniformly* along the curve with a velocity $v = \frac{ds}{dt}$, the component velocities can be found.

(vii). Thus if the curve is a parabola, $y^2 = 4px$,

we have,
$$\frac{dy}{dt} = \frac{2p}{y}\frac{dx}{dt}, \quad v = \frac{ds}{dt}.$$

Substituting in general formula (2) above, we find,

$$\frac{dx}{dt} = \frac{yv}{\sqrt{y^2 + 4p^2}}, \quad \frac{dy}{dt} = \frac{2pv}{\sqrt{y^2 + 4p^2}},$$

for the velocities horizontally and vertically.

(viii). In the circle, $x^2 + y^2 = a^2$, we readily find,

$$\frac{dx}{dt} = \frac{vy}{a}, \quad \frac{dy}{dt} = -\frac{vx}{a},$$

the signs corresponding to a point moving (with velocity v) clockwise along the curve.

Thus $\frac{dx}{dt}$, having the same sign as y, is + in quadrants II and I and − in quadrants IV and III. In the first case the motion is to the right, in the second case to the left. Similarly $\frac{dy}{dt}$ is + when x is minus, or in quadrants II and III, where the motion is upwards, and − in quadrants I and IV (x being plus) where the motion is downwards.

(ix). Referring to Art. 19 and Fig. 16, for the case of the parabola ($e = 1$), we have (Fig. 16)

$$FP = DP \therefore r = p + x \therefore \frac{dr}{dt} = \frac{dx}{dt}.$$

If a comet is supposed to describe the parabola with the sun at the focus F, we see that it approaches or recedes from the sun at the same rate as it moves parallel to the axis of its orbit. If the velocity along the orbit is taken as constant and $= v$, a formula above (Ex. vii) gives,

$$\frac{dr}{dt} = \frac{dx}{dt} = \frac{yv}{\sqrt{y^2 + 4p^2}}.$$

At the vertex $y = 0 \therefore \frac{dr}{dt} = 0$, as is evident otherwise.

Remark. To find the slope of a curve where x and y are given as functions of t without forming the equation of the curve in (x, y), we start with the equality,

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}},$$

and take limits as $\Delta t \doteq 0$, $\Delta x \doteq 0$, $\Delta y \doteq 0$.

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Thus in Ex. (ii), Art. 48,

$$\frac{dx}{dt} = 2, \frac{dy}{dt} = 32\,t;$$

hence,

$$\frac{dy}{dx} = \frac{32\,t}{2} = 16\,t = 8\,x,$$

as can be found directly from the equation $y = 4\,x^2$.

51. Derivative of the Arc. Recurring to formula (1), Art. 50, and dividing both sides of the equation first by $(\Delta x)^2$, then by $(\Delta y)^2$, we have,

$$1 + \left(\frac{\Delta y}{\Delta x}\right)^2 = \left(\frac{c}{\Delta s}\right)^2 \cdot \left(\frac{\Delta s}{\Delta x}\right)^2,$$

$$\left(\frac{\Delta x}{\Delta y}\right)^2 + 1 = \left(\frac{c}{\Delta s}\right)^2 \cdot \left(\frac{\Delta s}{\Delta y}\right)^2.$$

Whence taking limits as $\Delta x \doteq 0$ or $\Delta y \doteq 0$, and interchanging members,

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2,$$

$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2.$$

In the first result, x is the independent variable, in the second, y.

From Fig. 23, Art. 31, we have also,

$$\cos\theta = \lim_{c \doteq 0} \frac{\Delta x}{c} = \lim_{c \doteq 0} \left(\frac{\Delta x}{\Delta s} \cdot \frac{\Delta s}{c}\right) = \frac{dx}{ds};$$

$$\sin\theta = \lim_{c \doteq 0} \frac{\Delta y}{c} = \lim_{c \doteq 0} \left(\frac{\Delta y}{\Delta s} \cdot \frac{\Delta s}{c}\right) = \frac{dy}{ds};$$

since we know from Art. 49 that

$$\lim_{c \doteq 0} \left(\frac{\Delta s}{c}\right) = \lim_{c \doteq 0} \frac{\text{arc } PQ}{\text{chord } PQ} = 1$$

It will be seen that the formulas for *cos* θ and *sin* θ, are true also for Fig. 24, Art. 31, provided θ = the negative angle XAT and not the positive angle XAP; for cos XAT is positive and $= \frac{\lim_{c \doteq 0} \Delta x}{c}$, which is positive; whereas sin XAT is negative, agreeing in sign with its equal, $\lim_{c \doteq 0} \frac{\Delta y}{c}$, Δy being negative.

If in the formulas above for *cos* θ and *sin* θ, x and y are functions of t, divide the numerator and denominator of the fractions $\frac{\Delta x}{\Delta s}$, $\frac{\Delta y}{\Delta s}$, by Δt before going to the limit.

$$\therefore \cos\theta = \left(\frac{dx}{dt}\right) \Big/ \left(\frac{ds}{dt}\right), \quad \sin\theta = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{ds}{dt}\right).$$

EXAMPLE 1. It is easily found from the general formula,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

that for the semi-cubical parabola, $ay^2 = x^3$,

$$\frac{ds}{dx} = \sqrt{1 + \frac{9x}{4a}}.$$

The expression which differentiated will give this result is,

$$s = \frac{8a}{27}\left(1 + \frac{9x}{4a}\right)^{\frac{3}{2}} + C,$$

where C is a constant.

It was remarked in Art. 50, that s, the length of the arc, was measured from any arbitrary point on the curve to the point $P(x, y)$. In this instance, since (0, 0) is on the curve, take the origin for the point from which s is measured $\therefore$ when P is moved to the origin, $s = 0$ when $x = 0$ $\therefore$ from the last equation,

$$0 = \frac{8a}{27} + C \therefore C = -\frac{8a}{27}.$$

whence the length of the arc from the origin to any point $P(x, y)$ on the curve is,*

$$s = \frac{8a}{27}\left(1 + \frac{9x}{4a}\right)^{\frac{3}{2}} - \frac{8a}{27}.$$

* The semi-cubical parabola was the first curve rectified. It was effected without the use of the Calculus (which was not then invented) by Neil in 1657.

This process of finding s from its derivative is called Integration. The lengths of curves are found by integration, after the method illustrated above.

EXAMPLE 2. Prove, for the parabola, $y^2 = 4px$,

$$\frac{ds}{dx} = \sqrt{1 + \frac{p}{x}}; \quad \frac{ds}{dy} = \sqrt{1 + \frac{y^2}{4p^2}}.$$

EXAMPLE 3. The equation of the ellipse,

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

Art. 20 (6), can be put in the form,

$$y^2 = (1 - e^2)(a^2 - x^2),$$

since $$b^2 = a^2(1 - e^2).$$

It is easily shown that $$\left(\frac{ds}{dx}\right)^2 = \frac{a^2 - e^2x^2}{a^2 - x^2}.$$

EXAMPLE 4. Similarly from Art. 20 (6) and the relation $b^2 = a^2(e^2 - 1)$, the equation of the hyperbola is,

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2) = (e^2 - 1)(x^2 - a^2);$$

whence, $$\left(\frac{ds}{dx}\right)^2 = \frac{e^2x^2 - a^2}{x^2 - a^2}.$$

EXAMPLE 5. The equation of the four-cusped hypocycloid, Fig. 32, is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

$$\therefore \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \quad \therefore \frac{ds}{dx} = a^{\frac{1}{3}}x^{-\frac{1}{3}}.$$

whence, integrating,

$$s = \tfrac{3}{2}a^{\frac{1}{3}}x^{\frac{2}{3}} + C.$$

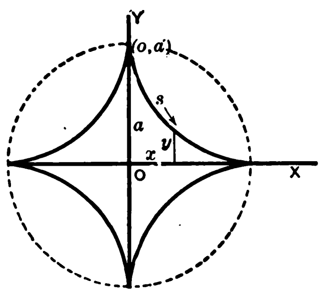

Fig. 32.

The point (o, a) is on the curve; counting s from this point, $s = 0$, when $x = 0 \therefore C = 0$.

The value of s (when $C = 0$) gives the length of the curve from (o, a) to a point in the first quadrant whose abscissa is x.

52. Equations of Tangent and Normal. The Calculus methods afford great facility in dealing with problems concerning tangents and normals. Let $y = f(x)$ represent the equation of any locus, Figs. 33 and 34 (full or dotted curve). The equation of any line passing through P (x_1, y_1) is of the form (Art. 11, Ex. 9),

$$y - y_1 = m(x - x_1),$$

where m is the slope. In order that this should be the equation of the tangent line to the curve at P, we must have

$$m = \tan\theta = \frac{dy_1}{dx_1}.$$

This notation means, that the derivative of $y = f(x)$ must be found and the particular values x_1, y_1 substituted for x, y in it to obtain $\frac{dy_1}{dx_1}$.

The equation of the tangent line at P (x_1, y_1) (Figs. 33 and 34) is therefore

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1).$$

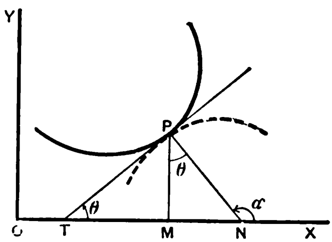

Fig. 33.

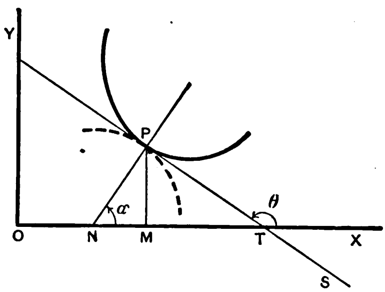

Fig. 34.

Thus, the equation of the tangent line to the ellipse, $a^2y^2 + b^2x^2 = a^2b^2$ at (x_1, y_1) is,

$$y - y_1 = \frac{-b^2x_1}{a^2y_1}(x - x_1).$$

Since (x_1, y_1) is on the curve, $a^2y_1^2 + b^2x_1^2 = a^2b^2$; hence clearing of fractions and substituting for $a^2y_1^2 + b^2x_1^2$ its value a^2b^2, the equation of the tangent line becomes,

$$a^2yy_1 + b^2xx_1 = a^2b^2.$$

Since the normal PN is perpendicular to the tangent PT, we have for its slope, in Fig. 33,

$$\tan\alpha = \tan(90 + \theta) = -\cot\theta = -\frac{1}{\tan\theta};$$

and in Fig. 34,

$$\tan\alpha = \tan(\theta - 90) = -\cot\theta = -\frac{1}{\tan\theta};$$

the same expression, which is minus the reciprocal of $\frac{dy_1}{dx_1}$. Hence the equation of the normal at $(x_1\ y_1)$ is,

$$y - y_1 = \frac{-1}{\left(\frac{dy_1}{dx_1}\right)}(x - x_1).$$

Thus, the equation of the normal to the ellipse above, at (x_1, y_1) is,

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1).$$

53. Subtangent, Subnormal, Tangent, and **Normal.** In Figs. 33 and 34, draw the ordinate PM; then TM is called the *subtangent* and MN the *subnormal*. The expressions below for them are easily found, provided TM and MN are taken as positive in Fig. 33 and negative in Fig. 34. The finite segments PT and PN are called the *tangent* and *normal*. As they are simply lengths, the positive sign is to be given to the radicals.

In the next four formulas, the point P is simply designated (x, y), and for *tan* $\theta = \frac{dy}{dx}$, the briefer notation, y' is adopted.

$$\text{Subtangent} = TM = y \cot \theta = \frac{y}{y'};$$
$$\text{Subnormal} = MN = y \tan \theta = yy';$$
$$\text{Tan} = TP = \sqrt{MP^2 + TM^2} = \frac{y\sqrt{1+(y')^2}}{y'};$$
$$\text{Nor} = NP = \sqrt{MP^2 + MN^2} = y\sqrt{1+(y')^2}.$$

Examples.

Show that the equations of the tangent and the normal (Art. 52) at (x_1, y_1) to,

1. The circle, $x^2 + y^2 = a^2$, are $yy_1 + xx_1 = a^2$ and $x_1 y = y_1 x$.
2. The hyperbola, $a^2y^2 - b^2x^2 = -a^2b^2$ (Art. 20), are,

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1, \; y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1).$$

3. The hyperbola $xy = c$ (Art. 13) are,

$$x_1y + y_1x = 2c, \; y_1y - x_1x = y_1^2 - x_1^2.$$

The x-intercept of the tangent is found to be $2x_1$: prove from this that "the segment of any tangent to a hyperbola between the asymptotes, is bisected by the point of contact."

4. The subtangents and subnormals are found by aid of the formulas of Art. 53.

Curve.	Subtangent.	Subnormal.
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$	$\frac{x^2 - a^2}{x}.$	$-\frac{b^2x}{a^2}.$
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$	$\frac{x^2 - a^2}{x}.$	$\frac{b^2x}{a^2}.$
$x^2 + y^2 = a^2.$	$-\frac{y^2}{x}.$	$-x.$
$y^2 = 4px.$	$2x.$	$2p.$
$xy = c.$	$-x.$	$-\frac{y^3}{c}$
$ay^2 = x^3.$	$\frac{2}{3}x.$	$\frac{3x^2}{2a}.$
$y = a^2x^3.$	$\frac{1}{3}x.$	$3a^4x^5.$

In all the expressions above for subtangent and subnormal, (x, y) is the point of tangency. Roughly sketch the curves and write the proper values on the corresponding segments. Also note, for the last four curves, that on laying off TM from M (assumed) equal to the values above ($2.OM$, $-OM$, $\frac{2}{3}OM$, $\frac{1}{3}OM$, respectively) the tangent TP can be drawn.

5. The equation to the tangent line of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (Fig. 32, Art. 52), is,

$$\frac{x}{x_1^{\frac{1}{3}}} + \frac{y}{y_1^{\frac{1}{3}}} = a^{\frac{2}{3}};$$

the x and y intercepts are thus, $a^{\frac{2}{3}}x_1^{\frac{1}{3}}$, $a^{\frac{2}{3}}y_1^{\frac{1}{3}}$; whence the distance along the tangent from the x-axis to the y-axis, is everywhere constant, and equal to a.

6. The slope at (x_1, y_1) of the common parabola,

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}} \quad . \; . \; . \quad (1), \text{ is } \frac{dy_1}{dx_1} = -\frac{y_1^{\frac{1}{2}}}{x_1^{\frac{1}{2}}}.$$

The points (a, o), (o, a), lie on the curve, Fig. 35. At (a, o) the slope is zero, showing that the x-axis is tangent to the curve there; also as $x_1 \doteq 0$, $\frac{dy_1}{dx_1} \doteq -\infty$, hence the y-axis is tangent to the curve at (o, a). The equation of the tangent line at (x_1, y_1) is found to be,

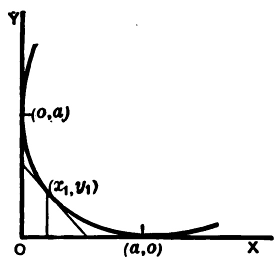

Fig. 35.

$$\frac{x}{x_1^{\frac{1}{2}}} + \frac{y}{y_1^{\frac{1}{2}}} = a^{\frac{1}{2}}.$$

Prove that the sum of the two intercepts of the tangent on the two axes is equal to a.

7. The *adiabatic* curves, $x^m y^n = c^{m+n}$, have the subtangent $= -\frac{nx}{m}$. The ratio of the abscissa of the point of tangency

(x) to this is $\frac{m}{n}$ (ignoring sign); hence, show that the part of the tangent intercepted between the axes is divided at its point of contact into segments, which are to each other in the constant ratio $m:n$. When $m = n = 1$, the curve is a hyperbola (Art. 13). The locus includes also all the parabolas.

Thus, if $m = -1$, $n = 2$, $y^2 = cx$, the common parabola;
if $m = -3$, $n = 1$, $c^2y = x^3$, the cubical parabola;
if $m = -3$, $n = 2$, $cy^2 = x^3$, the semi-cubical parabola.

8. The equations of the tangent and normal to the parabola, $y^2 = 4px$, are,

$$yy_1 = 2p(x + x_1);\quad (y - y_1)\,2p + (x - x_1)\,y_1 = 0.$$

Find the equations of the tangent and normal to the parabola $y^2 = 8x$ at the points whose common abscissa is 6.

9. The equation of any conic whose axes are parallel to the x and y axes, can be written in the form,

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0.$$

The derivative (using the method of Art. 41, Ex. 10) is,

$$2\left(Ax + By\frac{dy}{dx} + G + F\frac{dy}{dx}\right) = 0.$$

$$\therefore \frac{dy_1}{dx_1} = -\frac{Ax_1 + G}{By_1 + F}.$$

On substituting this in the equation of the tangent line, Art. 52, it may be written,

$$Ax_1x + By_1y + Gx + Fy = Ax_1^2 + By_1^2 + Gx_1 + Fy_1.$$

But (x_1, y_1) being on the curve satisfies its equation;

$$\therefore Ax_1^2 + By_1^2 + 2Gx_1 + 2Fy_1 + C = 0.$$
$$\therefore Ax_1^2 + By_1^2 + Gx_1 + Fy_1 = -Gx_1 - Fy_1 - C.$$

On substituting this value above, and transposing, the equation of the tangent line can be written,

$$Ax_1 x + By_1 y + G(x + x_1) + F(y + y_1) + C = 0.$$

To show the application of this formula, take the case of the circle, Ex. 3, Art. 4, and compare its equation directly with that of the conic,

$$Ax^2 + By^2 + 2Gx + 2Fy + C = 0;$$
$$x^2 + y^2 + 2x - 3y = 4.$$
$$\therefore A = B = 1, \ G = 1, \ F = -3/2, \ C = -4.$$

Hence the equation of the tangent to this circle at (x_1, y_1) is found to be, on substituting these values in the general formula above,

$$x_1 x + y_1 y + x + x_1 - \tfrac{3}{2}(y + y_1) - 4 = 0.$$

At a point on the circle where $y_1 = 2$, we find from the equation of the curve, $x_1 = -1 \pm \sqrt{7}$; hence on substituting these values in the preceding equation, we find the equations of the tangent lines at $(-1 \pm \sqrt{7}, 2)$. Similarly we proceed with any conic for any desired point of tangency (see Tanner and Allen's Analytic Geometry for a full discussion of the conics given by the above general formula).

CHAPTER VI.

DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

54. Derivatives of the Trigonometric Functions. Let the circle, Fig. 36, have a unit radius; then any angle AOP in circular measure is expressed by the length of the arc $AP = x$, measured in the same unit as the radius. If in the formula of Art. 17, $x/\pi = a^\circ/180$, we put $x = 1$ (= radius), we find $a^\circ = 57.^\circ 2957795 +$. Therefore, in Fig. 36, the angle 1 in circular measure = $AO1$ = length of arc $AP1$, corresponds to about $57.^\circ 3$; *i.e.*, $AO1$ in degrees = $57.^\circ 3$ nearly. This angle is called a *radian* and is the unit in circular measure. In this system, which

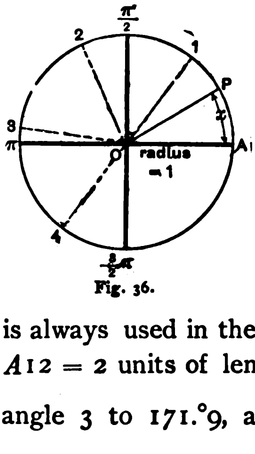

Fig. 36.

is always used in the Calculus, angle 2 = $AO2$ (where arc $A12$ = 2 units of length) corresponding to about $114.^\circ 6$, angle 3 to $171.^\circ 9$, angle $\frac{\pi}{2}$ to 90°, angle π to 180°, etc., where π = length of semi-circumference (in the same unit as the radius) = 3.141592. . . .

Example. How many degrees correspond to the angles

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{10}, \frac{\pi}{6}, \frac{\pi}{4}?$$

In what follows, u equals any function of x.

(1). $\mathbf{D_x \sin u = \cos u \cdot D_x u.}$

If $y = \sin u$, on giving x an increment Δx, $y + \Delta y = \sin (u + \Delta u)$,

$$\therefore \frac{\Delta y}{\Delta x} = \frac{\sin (u + \Delta u) - \sin u}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

Reduce the numerator by use of the formula, $\sin A - \sin B = 2 \sin \frac{1}{2} (A - B) \cos \frac{1}{2} (A + B)$, divide numerator and denominator by 2 and take the limits as $\Delta x \doteq 0$, bearing in mind that $\lim_{a \doteq 0} (\sin a/a) = 1$ and that $\lim_{\Delta x \doteq 0} \cos (u + \frac{1}{2} \Delta u) = \cos u$, since Δu and Δy tend to zero indefinitely with Δx.

$$\therefore \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \doteq 0} \frac{\sin \frac{1}{2} \Delta u}{\frac{1}{2} \Delta u} \cdot \cos (u + \tfrac{1}{2} \Delta u) \cdot \frac{\Delta u}{\Delta x}$$

$$\therefore D_x y = \cos u \cdot D_x u.$$

(2). $\mathbf{D_x \cos u = - \sin u \cdot D_x u.}$ This can be proved by use of the last formula thus:

Let, $$y = \cos u = \sin \left(\frac{\pi}{2} - u\right)$$

$$\therefore D_x y = D_x \sin \left(\frac{\pi}{2} - u\right) = \cos \left(\frac{\pi}{2} - u\right) \cdot D_x \left(\frac{\pi}{2} - u\right)$$

$$\therefore D_x \cos u = - \sin u \cdot D_x u.$$

(3). $\mathbf{D_x \tan u = \sec^2 u \cdot D_x u.}$ Here, use is made of (1) and (2).

Let, $$y = \tan u = \frac{\sin u}{\cos u},$$

$$\therefore D_x y = \frac{\cos u \,.\, D_x \sin u - \sin u \,.\, D_x \cos u}{\cos^2 u}$$

$$= \frac{\cos^2 u \cdot D_x u + \sin^2 u \cdot D_x u}{\cos^2 u} = \frac{1}{\cos^2 u} D_x u.$$

$$\therefore D_x \tan u = \sec^2 u \cdot D_x u = \frac{1}{\cos^2 u} D_x u.$$

(4). $\mathbf{D_x \cot u = - cosec^2 u \cdot D_x u.}$

Thus, $\cot u = \tan\left(\frac{\pi}{2} - u\right)$ $\therefore$ by (3),

$$D_x \cot u = \sec^2\left(\frac{\pi}{2} - u\right) \cdot D_x\left(\frac{\pi}{2} - u\right)$$

$$= - \operatorname{cosec}^2 u \cdot D_x u = - \frac{1}{\sin^2 u} D_x u.$$

(5). $\mathbf{D_x \sec u = \tan u \sec u \cdot D_x u.}$

For, $$\sec u = \frac{1}{\cos u}$$

$$\therefore D_x \sec u = \frac{\sin u \cdot D_x u}{\cos^2 u} = \sec u \tan u \cdot D_x u.$$

(6). $\mathbf{D_x \operatorname{cosec} u = - \operatorname{cosec} u \cot u \cdot D_x u.}$

Thus, $$\operatorname{cosec} u = \frac{1}{\sin u}$$

$$\therefore D_x \operatorname{cosec} u = \frac{- \cos u \cdot D_x u}{\sin^2 u} = - \operatorname{cosec} u \cot u \cdot D_x u.$$

(7). $\mathbf{D_x \operatorname{vers} u = \sin u \cdot D_x u.}$

For, $D_x (1 - \cos u) = \sin u.\ D_x u.$

(8). $\mathbf{D_x \operatorname{covers} u} = D_x (1 - \sin u) = \mathbf{- \cos u \cdot D_x u.}$

These eight formulas should be memorized. As an aid to this end, write out the results for formulas (1) to (8), when $u = x, ax, ax^2, ax + b, x/a$. The method to follow when the power of a trigonometric function has to be differentiated is seen from the following example.

By Art. 36, $D_x [F(x)]^n = n [F(x)]^{n-1} \cdot D_x F(x)$.

$\therefore \mathrm{D}_x \sin^n (ax + b) = D_x [\sin (ax + b)]^n = na [\sin (ax + b)]^{n-1} \cos (ax + b)$.

Examples.

1. $D \sin \frac{cx}{a} = \frac{c}{a} \cos \frac{cx}{a}$.

2. $D \sin (\cos x) = \cos (\cos x)\ D \cos x = - \sin x \cos (\cos x)$.

3. $D\,2\tan^4 x = D\,2(\tan x)^4 = 8\tan^3 x \sec^2 x.$

4. $D(2\sin x + \cos^2 x \sin x) = 2\cos x + 2\cos x(-\sin x)$ $\sin x + \cos^2 x \cdot \cos x = 2(1 - \sin^2 x)\cos x + \cos^3 x = 3\cos^3 x.$

5. $D \operatorname{cosec}(mx) = -m \cot mx \operatorname{cosec} mx.$

6. $D\cot\sqrt{1-x^2} = \operatorname{cosec}^2\sqrt{1-x^2} \cdot \dfrac{x}{\sqrt{1-x^2}}.$

7. $D\sin\sqrt{x} = \dfrac{\cos\sqrt{x}}{2\sqrt{x}}.$

8. $D(\cos ax)^{\frac{1}{2}} = -\dfrac{a}{2}\dfrac{\sin ax}{\sqrt{\cos ax}}.$

9. $D\sin x\cos x = \cos 2x.$

10. $D\sin^2 x = \sin 2x.$

11. $D\dfrac{\sin x}{1+\cos x} = \dfrac{1}{1+\cos x} = \dfrac{1}{2}\sec^2\dfrac{x}{2}.$

12. $D(\tan x - \cot x) = \dfrac{1}{(\sin x\cos x)^2} = \dfrac{4}{\sin^2 2x}.$

13. $D\sqrt{\dfrac{1-\cos x}{2}} = \dfrac{1}{2}\sqrt{\dfrac{1+\cos x}{2}}.$

14. $D\dfrac{1-\cos x}{\sin x} = \dfrac{1}{1+\cos x} = \dfrac{1}{2}\sec^2\left(\dfrac{x}{2}\right).$

15. The derivative of, $\sin 3x = 3\sin x - 4\sin^3 x$, is $\cos 3x = 4\cos^3 x - 3\cos x.$

16. $D(\frac{1}{3}\tan^3 x - \tan x) = \sec^2 x(\tan^2 x - 1).$

17. $D_\theta(\tan\theta + \sec\theta) = \sec\theta(\tan\theta + \sec\theta).$

18. $D_\theta(\theta\sin\theta + \cos\theta) = \theta\cos\theta.$

19. $D_\theta(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta) = \sin^2\theta.$

20. $D_\theta\dfrac{1-\cos\theta}{1+\cos\theta} = \dfrac{2\sin\theta}{(1+\cos\theta)^2}.$

21. Prove that the subtangent of the sinusoid, $y = \sin x$ is $\tan x$; of the curve of secants $y = \sec x$, $\cot x$. See Fig. 13, Art. 17.

55. Derivatives of the Inverse Trigonometrical Functions. The symbol $y = \sin^{-1} x$, is to be read, $y =$ *arc whose sine is x.* This notation is common in England and America; in Europe, the longer but more suggestive notation, $y = \text{arc}\ \sin x$, is used. A glance at the graph of $y = \sin^{-1} x$ (Art. 18, Fig. 15) shows that y has more than one value corresponding to any given value OA of x. If a limited portion of the curve, lying between $y = -\pi/2$ and $+\pi/2$, is alone considered, then y is single-valued for all admissible values of x.

It is in the line of simplicity to give such values to the arc or angle) that all the inverse functions $\sin^{-1} x$, $\cos^{-1} x$, etc., may be single-valued. In what follows then the range of the arc y will be restricted to values from $-\pi/2$ to $\pi/2$ for $\sin^{-1} x$, $\text{cosec}^{-1} x$, $\tan^{-1} x$ and $\cot^{-1} x$, but from 0 to π for $\cos^{-1} x$ and $\sec^{-1} x$.

(1). *Derivative of* $\sin^{-1} u$, *where* $u = f(x)$.

If $y = \sin^{-1} u$ $\therefore$ $\sin y = u$.

$\therefore$ differentiating with respect to x,

$$\cos y \frac{dy}{dx} = \frac{du}{dx}.$$

Now since y lies between $-\pi/2$ and $\pi/2$, $\cos y = +\sqrt{1 - u^2}$, or the positive sign only must be given to the radical. Hence solving for

$$\frac{dy}{dx} = D_x y,$$

$$D_x \sin^{-1} u = \frac{D_x u}{\sqrt{1 - u^2}}.$$

(2). *Derivative of* $\cos^{-1} u$.

Let $\qquad y = \cos^{-1} u \quad \therefore \ \cos y = u.$

$$\therefore \ -\sin y \cdot \frac{dy}{dx} = \frac{du}{dx}.$$

But $\qquad 0 < y < \pi \quad \therefore \ \sin y = +\sqrt{1 - u^2}.$

$$\therefore \ \mathbf{D}_x \cos^{-1} \mathbf{u} = \frac{-\mathbf{D}_x\mathbf{u}}{\sqrt{1 - \mathbf{u}^2}}$$

(3) *Derivative of* $\tan^{-1} u$.

Let $\quad y = \tan^{-1} u \quad \therefore \ \tan y = u.$

Differentiating, $\qquad \sec^2 y \cdot \frac{dy}{dx} = \frac{du}{dx}.$

But $\qquad \sec^2 y = 1 + \tan^2 y = 1 + u^2$

$$\therefore \ \mathbf{D}_x \tan^{-1} \mathbf{u} = \frac{\mathbf{D}_x\mathbf{u}}{1 + \mathbf{u}^2}.$$

(4). $\qquad \mathbf{D}_x \cot^{-1} \mathbf{u} = D_x\left(\frac{\pi}{2} - \tan^{-1} u\right) = -\frac{\mathbf{D}_x\mathbf{u}}{1 + \mathbf{u}^2}.$

(5) *Derivative of* $\sec^{-1} u$.

Let $\qquad y = \sec^{-1} u \qquad \therefore \ \sec y = u.$

$$\therefore \ \sec y \tan y \cdot \frac{dy}{dx} = \frac{du}{dx}.$$

In this case if y lies between 0 and $\frac{\pi}{2}$, *sec* y and *tan* y are both positive; if y lies between $\pi/2$ and π, both *sec* y and *tan* y are negative. In both cases their product $= u\sqrt{u^2 - 1}$ is positive; hence

$$\mathbf{D}_x \sec^{-1} \mathbf{u} = \frac{\mathbf{D}_x\mathbf{u}}{\mathbf{u}\sqrt{\mathbf{u}^2 - 1}},$$

with the understanding that when $u = \sec y$ is negative, $\sqrt{u^2 - 1} = \tan y$, must be taken so likewise.

If the student will draw the graph of $y = \sec^{-1}x$ (see Art. 18), he will note that the slope is always positive between $x = 0$ and $x = \pi$; $\therefore$ $D_x \sec^{-1} u$ is always positive when $y = \sec^{-1} u$ lies between 0 and π.

Similarly, it can be shown that $D_x \operatorname{cosec}^{-1} x$ is always negative when x lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$$(6) \qquad \mathbf{D_x csc^{-1} u} = D_x\left(\frac{\pi}{2} - \sec^{-1} u\right) = -\frac{\mathbf{D_x u}}{\mathbf{u}\sqrt{\mathbf{u}^2 - 1}}.$$

Let the student prove the formulas (7) and (8) below.

$$(7) \qquad \mathbf{D_x\, vers^{-1} u} = \frac{\mathbf{D_x u}}{\sqrt{2\mathbf{u} - \mathbf{u}^2}}.$$

$$(8) \qquad \mathbf{D_x\, covers^{-1} u} = \frac{-\mathbf{D_x u}}{\sqrt{2\mathbf{u} - \mathbf{u}^2}}.$$

The derivatives (1) to (8) become discontinuous for values of u that cause the denominator to become zero, the numerator remaining finite.

Examples.

The derivatives are all taken with respect to x.

1. $D \sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2 - x^2}}$; $D \cos^{-1}\frac{x}{a} = \frac{-1}{\sqrt{a^2 - x^2}}$;

   $D \tan^{-1}\frac{x}{a} = \frac{a}{a^2 + x^2}$; $D \cot^{-1}\frac{x}{a} = \frac{-a}{a^2 + x^2}$;

   $D \sec^{-1}\frac{x}{a} = \frac{a}{x\sqrt{x^2 - a^2}}$; $D \csc^{-1}\frac{x}{a} = \frac{-a}{x\sqrt{x^2 - a^2}}$;

   $D \operatorname{vers}^{-1}\frac{x}{a} = \frac{1}{\sqrt{2ax - x^2}}$; $D \operatorname{covers}^{-1}\frac{x}{a} = \frac{-1}{\sqrt{2ax - x^2}}$.

2. $D \cos^{-1}(x^3) = \frac{-3x^2}{\sqrt{1 - x^6}}$.

3. $D \sin^{-1} \dfrac{2x}{1+x^2} = \dfrac{2}{1+x^2}$.

4. $D\, \frac{1}{2} \sin^{-1} \dfrac{x^2}{a^2} = \dfrac{x}{\sqrt{a^4 - x^4}}$.

5. $D\,(\sin^{-1} x - \sqrt{1-x^2}) = \dfrac{\sqrt{1+x}}{\sqrt{1-x}}$.

6. $D\, 2 \tan^{-1} \sqrt{\dfrac{1-x}{1+x}} = \dfrac{-1}{\sqrt{1-x^2}}$.

7. $D \sec^{-1} \dfrac{1}{\sqrt{1-x^2}} = \dfrac{1}{\sqrt{1-x^2}}$.

8. $D \sin^{-1} \sqrt{1-x^2} = \dfrac{-1}{\sqrt{1-x^2}}$.

9. $D \operatorname{vers}^{-1} (6\, x^{\frac{1}{3}}) = \dfrac{1}{\sqrt{3\, x^{\frac{4}{3}} - 9\, x^2}}$.

10. $D \left\{ (a^2 + x^2) \tan^{-1} \dfrac{x}{a} \right\} = 2\, x \tan^{-1} \left(\dfrac{x}{a} \right) + a.$

11. $D \tan^{-1} \left(\dfrac{1 - \cos x}{1 + \cos x} \right)^{\frac{1}{2}} = \dfrac{1}{2}$.

Here, $u = \left(\dfrac{1 - \cos x}{1 + \cos x} \right)^{\frac{1}{2}};$

$$\therefore\ D_x u = \frac{1}{2} \left(\frac{1 - \cos x}{1 + \cos x} \right)^{-\frac{1}{2}} \frac{(1 + \cos x) \sin x + (1 - \cos x) \sin x}{(1 + \cos x)^2}$$

$$= \left(\frac{1 + \cos x}{1 - \cos x} \right)^{\frac{1}{2}} \frac{\sin x}{(1 + \cos x)^2}.$$

$$\therefore\ D \tan^{-1} u = \frac{D_x u}{1 + u^2} = \frac{1 + \cos x}{2} \left(\frac{1 + \cos x}{1 - \cos x} \right)^{\frac{1}{2}} \frac{\sin x}{(1 + \cos x)^2}$$

$$= \frac{\sin x}{2 \sqrt{1 - \cos^2 x}} = \frac{1}{2}.$$

12. $D \sin^{-1} (3\, x - 4\, x^3) = \dfrac{3}{\sqrt{1 - x^2}}$

56. Angular Velocity. Crank and Connecting Rod. Referring to Fig. 36, Art. 54, suppose that a solid body rotates about an axis perpendicular to the plane of the paper at O. Lay off a unit length on any line OA, from O to A, in the plane of the paper and consider the motion of a point describing the arc of a circle AP 12. . . , At end of time t, let $x = AP =$ distance the point has moved. In an additional time Δt, the point moves a distance Δx. Hence $\Delta x/\Delta t$ represents the average-rate of describing Δx, and reasoning as in Art. 44, the limit to which $\Delta x/\Delta t$ approaches as $\Delta t \doteq 0$ or $\frac{dx}{dt}$, is called the velocity of the point at the end of the time t. If the rate is uniform $x/t = \Delta x/\Delta t$ is constant

$$\therefore \lim_{\Delta t \doteq 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

is constant and represents the rate here as well as in the case of variable motion.

The velocity of the point P at any instant, is called the *angular velocity* of the supposed body at the instant. It is the velocity with which a point at a unit's distance from the axis moves for uniform rate or tends to move at any instant for non-uniform rate, and its value, in any case, is $\frac{dx}{dt}$.

Let us now discuss the case of motion presented by the *crank*, *connecting rod* and *slide*.

In Fig. 37, CP of length r, is the crank, P the crank pin, PQ of length l, the connecting rod, and Q the slide (attached to the piston) which moves in the direction of the center C. The point P of course moves in a circle. When P is at A, Q is at

D, $\therefore AD = l$. Let $QD = s$. Call θ the angle ACP expressed in circular measure. We have,

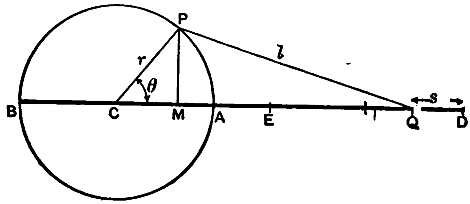

Fig. 37.

$$CM + MQ + QD = CA + AD = l + r.$$

$$\therefore r\cos\theta + \sqrt{l^2 - r^2\sin^2\theta} + s = l + r.$$

$$\therefore s = r(1 - \cos\theta) + l\left(1 - \sqrt{1 - \frac{r^2\sin^2\theta}{l^2}}\right).$$

This formula is exact, but a little too complicated for practical use, so an approximation is resorted to. The term

$$\left(\frac{r\sin\theta}{l}\right)^2 = \left(\frac{PM}{PQ}\right)^2.$$

is generally quite small, so that if we develop the radical expression to two terms only by the binomial formula giving,

$$\sqrt{1 - \frac{r^2\sin^2\theta}{l^2}} = 1 - \frac{1}{2}\frac{r^2\sin^2\theta}{l^2},$$

the error is only a very small per cent.

This reduces the value of s to,

$$s = r(1 - \cos\theta) + \frac{1}{2}\frac{r^2\sin^2\theta}{l} \quad \dots \quad (1)$$

To get the velocity of the slide Q, we differentiate this with respect to t.

$$\frac{ds}{dt} = \left(r\sin\theta + \frac{r^2}{2l}\sin 2\theta\right)\frac{d\theta}{dt} \quad \dots \quad (2)$$

As we have just seen, $\frac{d\theta}{dt}$ is the angular velocity or the velocity of a point on CP at a unit's distance from C. Call it ω and assume it to be *constant*, so that P moves with the constant rate ωr.

The speed of Q by (2) is zero at $\theta = 0$ and $\theta = \pi$ or at the *dead points* A and B.

The time-rate of change of the speed of Q is called its *acceleration*. Its value is,

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) = \left(r \cos\theta + \frac{r^2}{l}\cos 2\theta\right)\omega^2 \quad . \; . \; . \quad (3)$$

What are the accelerations of Q at $\theta = 0, \frac{\pi}{2}, \pi$? The forces acting on the slide to cause motion are proportional to the accelerations and thus vary with ω^2.

If we call the constant rate at which P moves v, we have $v = \omega r$; further as

$$l \doteq \infty\ , \quad \frac{ds}{dt} \doteq (r \sin\theta)\,\omega = \frac{vy}{r}, \text{ if } y = MP.$$

This is a case of *simple harmonic motion* and agrees with the result of Art. 50 (viii).

57. $\lim\limits_{m \doteq \infty}\left(1 + \frac{1}{m}\right)^m = e$.

When m is a positive integer, we have by the binomial theorem,

$$\left(1+\frac{1}{m}\right)^m = 1 + m\cdot\frac{1}{m} + \frac{m(m-1)}{1\cdot 2}\cdot\frac{1}{m^2} + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{m^3} + \cdots,$$

there being $(m + 1)$ terms in all.

Writing this in the form,

$$\left(1+\frac{1}{m}\right)^m = 1 + 1 + \frac{1\left(1-\frac{1}{m}\right)}{1\cdot 2} + \frac{1\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)}{1\cdot 2\cdot 3} + \cdots,$$

and taking limits as

$$m \doteq \infty \text{ and } \therefore \frac{1}{m} \doteq 0, \text{ etc.},$$

$$\lim_{m \doteq \infty} \left(1 + \frac{1}{m}\right)^m = 1 + 1 + \frac{1}{2} + \frac{1}{2, 3} + \frac{1}{2, 3, 4} + \cdots$$
$$\doteq 2.718281829 \cdots$$

This limit is universally denoted by *e*. It can be computed to any desired precision on taking a sufficient number of terms. The computation is easy, as each term of the series can be derived from the preceding on dividing it by an evident factor. Let the student compute the sum of ten terms of the series to six decimal figures.

REMARK. The above demonstration is incomplete, for the result is not only true when *m* is "a positive integer," but also for *m* positive or negative, integral, fractional or incommensurable; besides, then the theorems of limits are held to apply to infinite sums and products. See Byerly's Differential Calculus for a clear and complete proof.

58. Derivative of $\log_a u$, u being a function of x.

If $y = \log_a u$; on giving x an increment Δx

$$\therefore y + \Delta y = \log_a (u + \Delta u)$$

$$\Delta y = \log_a (u + \Delta u) - \log_a u$$

$$= \log_a \left(\frac{u + \Delta u}{u}\right) = \log_a \left(1 + \frac{\Delta u}{u}\right)$$

$$\therefore \frac{\Delta y}{\Delta x} = \log_a \left(1 + \frac{\Delta u}{u}\right) \cdot \frac{1}{\Delta x}.$$

On multiplying the second member by $\frac{1}{u} \cdot \frac{u}{\Delta u} \cdot \frac{\Delta u}{1} = 1$,

$$\frac{\Delta y}{\Delta x} = \frac{1}{u} \cdot \frac{u}{\Delta u} \log_a \left(1 + \frac{\Delta u}{\Delta x}\right) \cdot \frac{\Delta u}{\Delta x}.$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{u} \cdot \log_a \left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}} \cdot \frac{\Delta u}{\Delta x}.$$

As $\Delta x \doteq 0$, $\Delta u \doteq 0$ and $\Delta y \doteq 0$; also $\frac{u}{\Delta u} \doteq \infty$. On putting $m = \frac{u}{\Delta u}$, we note that $m \doteq \infty$ as $\Delta u \doteq 0$.

$$\therefore \log_a \left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}} = \log_a \left(1 + \frac{1}{m}\right)^m,$$

approaches the limit $\log_a e$ as $\Delta x \doteq 0$ by Art. 57. Hence taking limits as $\Delta x \doteq 0$ (Art. 26, Th. III.),

$$\frac{dy}{dx} = \frac{1}{u} \log_a e \cdot \frac{du}{dx}$$

$$\therefore \mathbf{D}_x \log_a \mathbf{u} = \log_a \mathbf{e} \cdot \frac{\mathbf{D}_x \mathbf{u}}{\mathbf{u}} \quad . \; . \; . \; (1)$$

$$\text{If } a = e, \; \mathbf{D}_x \log_e \mathbf{u} = \frac{\mathbf{D}_x \mathbf{u}}{\mathbf{u}} \quad . \; . \; . \; (2)$$

Lastly, if in (2), $u = x$,

$$\mathbf{D}_x \log_e \mathbf{x} = \frac{1}{\mathbf{x}} \quad . \; . \; . \; (3)$$

Hereafter, *$\log_e u$ will be written $\log u$, the Napierian base e being understood.*

Formula (2) is so frequently used, it should be stated as a rule as follows: The derivative of $\log u$ is equal to the derivative of u divided by u.

EXAMPLE. Differentiate,

$$\log \frac{\sqrt{1 + x^2} + x}{\sqrt{1 + x^2} - x} = \log (\sqrt{1 + x^2} + x)^2,$$

on rationalizing the denominator.

$$D \log (\sqrt{1 + x^2} + x)^2 = D\, 2 \log (\sqrt{1 + x^2} + x)$$

$$= 2\frac{D(\sqrt{1 + x^2} + x)}{\sqrt{1 + x^2} + x} = 2\frac{\dfrac{x}{\sqrt{1 + x^2}} + 1}{\sqrt{1 + x^2} + x} = \frac{2}{\sqrt{1 + x^2}}.$$

59. Derivative of a^u, u being a function of x.

$$\text{Let} \quad y = a^u$$

$$\therefore \log y = u \log a.$$

Differentiating as to x, using (2) above,

$$\frac{D_x y}{y} = \log a\, D_x u$$

$$\therefore \mathbf{D_x a^u = \log a \cdot a^u \cdot D_x u} \ \ldots \ (1)$$

When $a = e$, $\log a = \log e = 1$

$$\therefore \mathbf{D_x e^u = e^u \cdot D_x u} \ \ldots \ (2)$$

A remarkable case is when $u = x$,

$$\therefore \mathbf{D_x e^x = e^x} \ \ldots \ (3)$$

(i) The derivative of u^n, which in Art. 30 was not proved for n incommensurable, can be easily proved by use of (2) Art. 58, whether n is positive or negative, integral, fractional or incommensurable.

$$\text{Thus if } y = u^n \therefore \log y = n \log u \therefore \frac{Dy}{y} = \frac{nDu}{u}$$

$$\therefore Du^n = \frac{ny}{u} Du = nu^{n-1} Du.$$

(ii) In a similar manner $y = u^v$ (u and v being functions of x) can be differentiated by first taking logarithms.

Thus, if $y = x^x \therefore \log y = x \log x$;

$$\therefore \frac{Dy}{y} = \log x + x\left(\frac{1}{x}\right) \therefore Dx^x = x^x (\log x + 1).$$

(iii) If preferred, logarithms may be taken before differentiating in the case of functions that may be differentiated as they stand.

Thus, if $y = \dfrac{x^n}{(a+x)^n}$ $\therefore \log y = n \log x - n \log (a+x)$.

$$\therefore \frac{Dy}{y} = \frac{n}{x} - \frac{n}{a+x} \quad \therefore Dy = \frac{nax^{n-1}}{(a+x)^{n+1}}.$$

(iv) If u, v, z, w, are functions of x,

$$D_x \log \frac{uv}{zw} = D_x \{\log u + \log v - \log z - \log w\}$$
$$= \frac{D_x u}{u} + \frac{D_x v}{v} - \frac{D_x z}{z} - \frac{D_x w}{w}.$$

For example,

$$D \log \frac{x \cdot \sqrt{x}}{\sqrt{1+x^2}} = \frac{1}{x} + \frac{1}{2x} - \frac{x}{1+x^2}.$$

(v) $D_x \{e^{ax} \sin (bx+c)\} = ae^{ax} \sin (bx+c) + be^{ax} \cos (bx+c)$.

To write this result in a briefer form, put,

$$R \cos \theta = a, \quad R \sin \theta = b.$$

Regarding R as positive; if a and b are both positive, θ is in the first quadrant; if a is negative and b positive, θ is in the second quadrant, etc. This substitution is always possible, since the equations give,

$$R = \sqrt{a^2+b^2}, \quad \tan \theta = b/a,$$

and $\tan \theta$ can take any $+$ or $-$ value.

The derivative now takes the form,

$$D_x \{e^{ax} \sin (bx+c)\} = Re^{ax} \{\cos \theta \sin (bx+c) + \sin \theta \cos (bx+c)\}$$
$$= Re^{ax} \sin (bx+c+\theta).$$

Similarly, $D_x \{e^{ax} \cos (bx+c)\} = Re^{ax} \cos (bx+c+\theta)$.

(vi) The locus, $2y = a\,(e^{x/a} + e^{-x/a})$ is called a "catenary," it being the curve assumed by a flexible cord of uniform weight per linear unit when freely suspended by its extremities (Fig. 38).

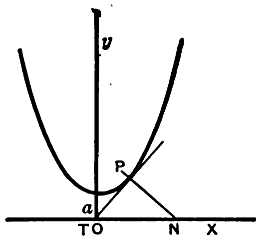

Fig. 38.

$$2\,Dy = (e^{x/a} - e^{-x/a})$$

$$\therefore\ 2\,D_x s = (e^{x/a} + e^{-x/a})$$

by Art. 51, $\therefore\ s = \dfrac{a}{2}\,(e^{x/a} - e^{-x/a}) + C.$

If s is taken equal to zero when $x = 0$, it follows that $C = 0$; hence the length of the arc of the catenary from $x = 0$ to $x = x$ is,

$$s = \frac{a}{2}\,(e^{x/a} - e^{-x/a}).$$

Since $D_x s = y/a\ \therefore$ "the normal," Art. 53, $= y.D_x s = y^2/a$. Also note that when $x = 0$, $y = a$ and $Dy = 0$ or the tangent is parallel to the x-axis.

Examples.

1. $Dx^n \log x = x^{n-1}\,(1 + n \log x)$.
2. $D \log (\sin x) = \cot x$.
3. $D \log (\cos x) = -\tan x$.
4. $Dx^n e^x = x^{n-1}\,(x + n)\,e^x$.
5. $y = ce^{x/a}\ \therefore$ subt. $= a$; subn. $= \dfrac{c^2}{a} e^{2x/a}$ (Art. 53).
6. $y = \log x\ \therefore$ subt. $= x \log x$; subn. $= \dfrac{1}{x} \log x$.
7. $De^{\frac{1}{x}} = -\dfrac{1}{x^2} e^{\frac{1}{x}}$.
8. $Da^{\log x} = \dfrac{\log a}{x} a^{\log x}$.
9. $Dx^n a^x = nx^{n-1} a^x + x^n a^x \log a$.
10. $D \log (\log x) = \dfrac{1}{x \log x}$.

11. $D \log (ax^2 - bx)^{\frac{1}{3}} = \frac{1}{3} \frac{2ax - b}{ax^2 - bx}$.

Write, $\log (ax^2 - bx)^{\frac{1}{3}} = \frac{1}{3} \log (ax^2 - bx)$.

12. $D \log \{\sqrt{x+1} + \sqrt{x-1}\} = \frac{1}{2\sqrt{x^2-1}}$.

13. $D \log (x + \sqrt{x^2 + a^2}) = \frac{1}{\sqrt{x^2+a^2}}$ (Ex. Art. 58).

14. $D \log \{\sqrt{x+1}\,(1-x)\} = D\{\log \sqrt{x+1} + \log (1-x)\}$

$$= \frac{1}{2(x+1)} + \frac{1}{x-1} = \frac{3x+1}{2(x^2-1)}.$$

15. $D \log \sqrt{ax^3 - c} = D\frac{1}{2} \log (ax^3 - c) = \frac{1}{2} \frac{3ax^2}{ax^3 - c}$.

16. $D \log \frac{\sqrt{1-x^2}}{x} = D\{\log \sqrt{1-x^2} - \log x\} = \frac{-1}{x(1-x^2)}$.

17. $y = x^{\frac{1}{x}} \therefore \log y = \frac{1}{x} \cdot \log x$;

$$\therefore \frac{Dy}{y} = -\frac{1}{x^2} \log x + \frac{1}{x^2} = \frac{1 - \log x}{x^2}.$$

$$\therefore Dx^{\frac{1}{x}} = \frac{1 - \log x}{x^2} \cdot x^{\frac{1}{x}}.$$

18. The symbol *sinh* is to be read *hyperbolic sine of*, or *h sine of; cosh*, *hyperbolic cosine of*, and so on. The hyperbolic functions are defined as follows:

$$\sinh x = \frac{1}{2}(e^x - e^{-x});\ \cosh x = \frac{1}{2}(e^x + e^{-x});$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}};\quad \coth x = \frac{\cosh x}{\sinh x};$$

$$\text{sech}\, x = \frac{1}{\cosh x};\quad \text{cosech}\, x = \frac{1}{\sinh x}.$$

Show that,

$D_x \sinh x = \cosh x$; $D_x \cosh x = \sinh x$;

$D_x \tanh x = \text{sech}^2 x$; $D_x \coth x = -\text{cosech}^2 x$;

$D_x\, \text{sech}\, x = -\text{sech}\, x \tanh x$;

$D_x\, \text{cosech}\, x = -\text{cosech}\, x \coth x$.

60. List of Fundamental Derivatives, all taken with respect to x, where $u = f(x)$.

$$D(u)^n = n(u)^{n-1} Du.$$

$$D \log_a u = \log_a e \cdot \frac{Du}{u}.$$

$$D \log_e u = \frac{Du}{u}.$$

$$D a^u = \log_e a \cdot a^u Du.$$

$$D e^u = e^u Du.$$

$$D \sin u = \cos u \cdot Du.$$

$$D \cos u = - \sin u \cdot Du.$$

$$D \tan u = \sec^2 u \cdot Du.$$

$$D \cot u = - \csc^2 u \cdot Du.$$

$$D \sec u = \sec u \tan u \cdot Du.$$

$$D \csc u = - \csc u \cot u \cdot Du.$$

$$D \operatorname{vers} u = \sin u \cdot Du.$$

$$D \operatorname{covers} u = - \cos u \cdot Du.$$

$$D \sin^{-1} u = \frac{Du}{\sqrt{1-u^2}}.$$

$$D \cos^{-1} u = - \frac{Du}{\sqrt{1-u^2}}.$$

$$D \tan^{-1} u = \frac{Du}{1+u^2}.$$

$$D \cot^{-1} u = - \frac{Du}{1+u^2}.$$

$$D \sec^{-1} u = \frac{Du}{u\sqrt{u^2-1}}.$$

$$D \csc^{-1} u = - \frac{Du}{u\sqrt{u^2-1}}.$$

$$D \operatorname{vers}^{-1} u = \frac{Du}{\sqrt{2u-u^2}}.$$

$$D \operatorname{covers}^{-1} u = - \frac{Du}{\sqrt{2u-u^2}}.$$

CHAPTER VII.

HIGHER DERIVATIVES. DERIVED CURVES. APPLICATIONS TO MECHANICS.

61. Higher Derivatives. The first derivative of $f(x)$ or $f'(x)$, is either a function of x or a constant. It can therefore be differentiated with respect to x. The result is called the second derivative of $f(x)$ and is denoted by $f''(x)$. Its derivative with respect to x is denoted by $f'''(x)$ and so on.

Thus if $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$ and $f^{iv}(x) = 0$. The subsequent derivatives are all zero.

Besides the above notation, the following symbols are used:

Successive derivatives of $y = f(x)$ with respect to x:

First,	$f'(x)$,	y',	$D_x y$,	$\dfrac{dy}{dx}$;
Second,	$f''(x)$,	y'',	$D_x(D_x y) = D_x^2 y$,	$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$;
Third,	$f'''(x)$,	y''',	$D_x^3 y$,	$\dfrac{d^3y}{dx^3}$;
. . .	. . .	. . .	. . .	. . .
n^{th},	$f^{(n)}x$,	$y^{(n)}$,	$D_x^n y$,	$\dfrac{d^ny}{dx^n}$.

In $f^{(n)}(x)$ or $y^{(n)}$, n is inclosed in a parenthesis to distinguish it from y^n.

Where D is used without a subscript, it is understood to mean D_x.

The n^{th} derivative is required in certain cases, as of development into series. In the case of only a few functions can the n^{th} derivative be readily found. Some of these are given below.

(i) $f(x) = e^x \therefore f'(x) = e^x, f''(x) = e^x, \ldots; \therefore f^{(n)}(x) = e^x.$
(ii) $y = a^x \therefore y' = \log a. a^x, y'' = (\log a)^2. a^x, \therefore y^{(n)} = (\log a)^n a^x.$
(iii) $y = e^{ax} \therefore y' = ae^{ax}, y'' = a^2e^{ax}, \ldots, y^{(n)} = a^n e^{ax}.$
(iv) In Art. 59, Ex. (v), we found,

$$De^{ax}\sin(bx+c) = Re^{ax}\sin(bx+c+\theta),$$
$$De^{ax}\cos(bx+c) = Re^{ax}\cos(bx+c+\theta),$$

where $R\cos\theta = a$, $R\sin\theta = b \therefore R = \sqrt{a^2+b^2}$, $\tan\theta = b/a$.

In words, the derivative of either function is found by multiplying by R and increasing the angle by θ.

$$\therefore D^2e^{ax}\sin(bx+c) = R^2e^{ax}\sin(bx+c+2\theta),$$
$$D^3e^{ax}\sin(bx+c) = R^3e^{ax}\sin(bx+c+3\theta);$$

whence, $D^ne^{ax}\sin(bx+c) = R^ne^{ax}\sin(bx+c+n\theta).$
Similarly, $D^ne^{ax}\cos(bx+c) = R^ne^{ax}\cos(bx+c+n\theta).$
If $a = 0 \therefore R = b, \tan\theta = \infty \therefore \theta = \pi/2;$

$$\therefore D^n\sin(bx+c) = b^n\sin\left(bx+c+\frac{n\pi}{2}\right).$$
$$D^n\cos(bx+c) = b^n\cos\left(bx+c+\frac{n\pi}{2}\right).$$

The last two results are easily proved independently. Thus, $D\sin(bx+c) = b\cos(bx+c) = b\sin(bx+c+\pi/2)$, from which the law for writing out successive derivatives is obvious.

(v) $f(x) = x^m \therefore f'(x) = mx^{m-1}, f''(x) = m(m-1)x^{m-2},$
$f'''(x) = m(m-1)(m-2)x^{m-3}$, etc.

The law here is, that the exponent of x is m minus the order of the derivative, and the last factor in m is m minus a number one less than the order of the derivative.

$$\therefore f^n(x) = m(m-1)\ldots(m-(n-1))\,x^{m-n}$$
$$= m(m-1)(m-2)\ldots(m-n+1)\,x^{m-n}.$$

(vi) $f(x) = \log(1+x) \therefore f^1(x) = (1+x)^{-1}$,

$f''(x) = (-1)(1+x)^{-2}$,

$f'''(x) = (-1)^2\lfloor 2\,(1+x)^{-3}, f^{iv}(x) = (-1)^3\lfloor 3\,(1+x)^{-4}, \ldots;$

$\therefore f^n(x) = (-1)^{n-1} \cdot \lfloor n-1 \cdot (1+x)^{-n}$.*

(vii) Successive derivatives of implicit functions are formed as illustrated in this example.

If $y^2 - 4px = 0 \therefore 2y\,Dy - 4p = 0 \therefore Dy = 2p/y$.

$$D^2y = \frac{-2pDy}{y^2};\ \text{substituting } Dy = \frac{2p}{y} \therefore D^2y = -\frac{4p^2}{y^3}.$$

$$D^3y = \frac{4p^2D(y)^3}{y^6} = \frac{12p^2y^2\,Dy}{y^6} = \frac{24p^3}{y^5}.$$

Examples.

1. $y = ax^4 + bx^3 + c \therefore y^{iv} = \lfloor 4\,a$.

2. $f(x) = x^2 \log x \therefore f'''(x) = 2x^{-1}$.

3. $f(x) = x \log x \therefore f^n(x) = (-1)^{n-2} \cdot \lfloor n-2 \cdot x^{1-n}$.

4. $f(x) = \tan x \therefore f'''(x) = 2 + 8\tan^2 x + 6\tan^4 x$.

5. $y = \dfrac{x^2}{1-x} = -[x + 1 + (x-1)^{-1}] \therefore y''' = 6(x-1)^{-4}$.

6. If $y = ax^2 + bx \therefore \dfrac{d^2y}{dx^2} - \dfrac{2}{x}\dfrac{dy}{dx} + \dfrac{2y}{x^2} = 0$.

7. If $y = ae^{cx} + be^{-cx} \therefore \dfrac{d^2y}{dx^2} - c^2y = 0$.

8. If $x^2 + y^2 = a^2 \therefore \dfrac{d^2y}{dx^2} = -\dfrac{r^2}{y^3}$. See (vii).

* $\lfloor m = 1 \cdot 2 \cdot 3 \cdot 4 \ldots (m-1)\,m$.

9. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \therefore D_x^2 y = -\frac{b^4}{a^2 y^3}$.

10. If $y = xe^x \therefore y^{(n)} = e^x (x + n)$.

11. If $y = x^2 e^x \therefore y^{\text{iv}} = e^x (x^2 + 8x + 12)$.

12. By the formula for powers, $D[F(x)]^n = n[F(x)]^{n-1} DFx$.

$$\therefore D[Dy]^2 = 2\,Dy \cdot D^2 y;\ D(Dy)^n = n\,(Dy)^{n-1} D^2 y;$$

$$D(y')^4 = 4\,(y')^3 y'';\ \frac{d}{dx}\left(\frac{dy}{dx}\right)^n = n\left(\frac{dy}{dx}\right)^{n-1}\frac{d^2y}{dx^2}.$$

13. In Art. 35, it was shown, that if $y = F(u)$, $u = f(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad . \; . \; . \; (1)$$

If $\qquad x = e^u \therefore u = \log x \therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{1}{x}$.

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{du}\right) - \frac{1}{x^2}\frac{dy}{du}.$$

But by (1), $\qquad \frac{d}{dx}\left(\frac{dy}{du}\right) = \frac{d\left(\frac{dy}{du}\right)}{du} \cdot \frac{du}{dx} = \frac{d^2y}{du^2} \cdot \frac{du}{dx}$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) = \frac{1}{e^{2u}}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right).$$

The independent variable is thus changed from x to u.

62. Acceleration. Acceleration is time-rate of velocity. Referring to Art. 46 (which re-read) for the familiar illustration of falling bodies, we have given the equation,

$$s = 16\,t^2,$$

where s is the distance fallen through in feet in t seconds of free fall.

Calling v the velocity acquired, precisely at the end of t seconds, expressed in feet per second,

$$v = \frac{ds}{dt} = 32\,t.$$

In $(t + \Delta t)$ seconds, call the new velocity $v + \Delta v$,

$$\therefore\ v + \Delta v = 32\,(t + \Delta t)$$

$$\therefore\ \frac{\Delta v}{\Delta t} = 32.$$

The change of velocity Δv, divided by Δt, the time in acquiring the additional velocity Δv, is the average-rate of change of velocity per second. In this instance, it is constant and equal to 32, which means that the velocity increases 32 feet per second in each second of time. The

$$\underset{\Delta t \doteq 0}{\text{limit}} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = 32,$$

is the acceleration or time-rate of velocity. In cases like this, where the acceleration is *constant*, the acceleration means the actual change of velocity during a unit of time.

The following values computed from the formulas, $s = 16\,t^2$, $v = 32\,t$, $\quad$ = acceleration = 32, may make the subject plainer.

t	s	v	a
1	16	32	32
2	64	64	32
3	144	96	32
4	256	128	

Thus at the end of 4 seconds, the body has fallen through 256 feet, the velocity at the instant is 128 feet per second, and the velocity changes 32 feet a second during each second of fall.

If for some other case of motion,

$$s = ct^3,$$

$$\therefore\ v = \frac{ds}{dt} = 3\,ct^2,$$

$$\therefore\ v + \Delta v = 3\,c\,(t + \Delta\,t)^2,$$

$$\therefore\ \frac{\Delta\,v}{\Delta\,t} = 3c\,(2\,t + \Delta\,t).$$

This gives the average-rate or change per second of velocity in the time $\Delta\,t$ succeeding the time t. This approaches a limit,

$$\frac{dv}{dt} = 6\,ct,$$

which is the time-rate of v and is defined as the acceleration.

Acceleration then is what the average-rate of change (or change per second) of velocity approaches indefinitely as the time of the change, $\Delta\,t$, diminishes indefinitely. It will be denoted by a.

From the above, it will be observed that whether the acceleration is constant or variable, $\frac{dv}{dt}$ will always give its value. This equals $\frac{d^2s}{dt^2}$. Thus,

if, $\quad s = 16\,t^2;\ v = \frac{ds}{dt} = 32\,t;\ a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 32;$

if, $\quad s = ct^3;\ v = \frac{ds}{dt} = 3\,ct^2;\ a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 6\,ct.$

It will be necessary for the beginner to recur again and again to the fundamental ideas and processes involved in this article and Art. 46, to thoroughly grasp the meaning of velocity and acceleration in connection with their symbolic equivalents.

EXAMPLE 1. If $s = 16\,t^{\frac{3}{2}}$ find v and a at the end of 4 seconds.

EXAMPLE 2. If $s = 16\,t^{\frac{1}{2}}$, find v and a at the end of 9 seconds. a here is negative, indicating that the velocity is decreasing as t increases.

EXAMPLE 3. In Art. 50 (viii), the velocity components parallel to OX and OY of a point moving in a circle $x^2 + y^2 = a^2$, clockwise, with a constant velocity v along the curve, were found to be,

$$\frac{dx}{dt} = \frac{v}{a}y; \quad \frac{dy}{dt} = -\frac{v}{a}x,$$

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{v}{a}\cdot\frac{dy}{dt} = -\frac{v^2}{a^2}x;$$

$$\frac{d^2y}{dt^2} = -\frac{v}{a}\cdot\frac{dx}{dt} = -\frac{v^2}{a^2}y.$$

$\dfrac{d^2x}{dt^2}$ is the acceleration parallel to OX;

$\dfrac{d^2y}{dt^2}$ is the acceleration parallel to OY.

Since both are negative in quadrant I, $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are each decreasing as x increases (Art. 32).

EXAMPLE 4. From (vii), Art. 50, for a point moving with constant velocity v around a parabola clockwise, prove the axial accelerations to be,

$$\frac{d^2x}{dt^2} = \frac{8\,p^3v^2}{(y^2 + 4p^2)^2}; \quad \frac{d^2y}{dt^2} = -\frac{4\,p^2yv^2}{(y^2 + 4p^2)^2}.$$

Angular Acceleration. In Art. 56, angular velocity was defined. If θ = arc described by a point in a rotating body, at a unit's distance from its axis of rotation, reckoned from any fixed radius of the unit circle, then $\dfrac{d\theta}{dt}$ was found to represent the angular velocity.

By analogy to what has preceded

$$\frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}$$

represents the *angular acceleration* or time-rate of change of angular velocity. If this is uniform, it represents the change of velocity of the moving point in each unit of time; if variable, it represents what the average-rate of change of the velocity of the point, at time t, approaches indefinitely as the time of the change (Δt) diminishes indefinitely. The reasoning above given for rectilinear motion applies throughout with the exception that the moving point now describes a circle having unit radius.

63. Derived Curves. To represent graphically distance passed over, velocity and acceleration in the case of falling bodies, replace s by y for convenience and let the axis of abscissas be the t-axis. Also, as the ordinates of the actual loci are too large to plot conveniently, divide by 64 and graph the "plotted loci" below.

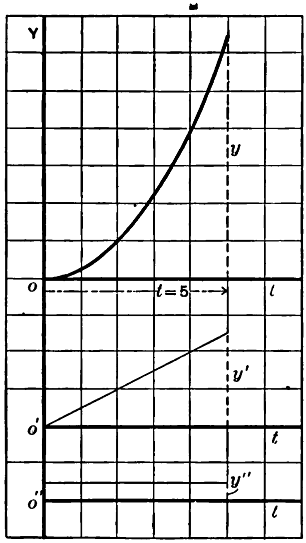

Fig. 39.

Actual Loci.	Plotted Loci.
$y = 16t^2$.	$y = \frac{1}{4}t^2$.
$y' = 32t$.	$y' = \frac{1}{2}t$.
$y'' = 32$.	$y'' = \frac{1}{2}$.

In these formulas, $y' = \frac{dy}{dt} = v$, $y'' = \frac{dv}{dt} = a$. In Fig. 39, with origin at O and axes Ot and OY, $y = \frac{1}{4} t^2$ is plotted to scale as usual; with O' as origin, $y' = \frac{1}{2} t$ is drawn; finally, with O'' as origin, $y'' = \frac{1}{2}$ is constructed. The points O' and O'' lie on YO produced, and the new axes of abscissas are parallel to the old. By a comparison of the formulas above, it is seen that the ordinates y, y', y'' of the "plotted loci," when measured to scale, must each be multiplied by 64 to give the true result. Thus at the end of t seconds, the body has fallen through 64 y feet, the velocity is 64 y' feet per second and the acceleration $64 \times \frac{1}{2} = 32$ feet per second, the ordinates y, y', y'' being measured to scale on the drawing and their numerical values used.

The upper curve, with O as origin, is called the *primitive curve;* the (straight) line through O', the *first derived curve* and the lower (straight) line the *second derived curve.* Evidently the slope of the primitive curve at any point whose ordinate is y, is $y' = \frac{dy}{dt}$ (measured to scale), and the slope of the first derived curve at y' is $y'' = \frac{dy'}{dt} = \frac{1}{2}$ or constant.

For the "actual loci," such values must be multiplied by 64. The advantage of a graph is that for any value of t, integral or fractional, the values of y, y' and y'' can be read off at a glance; besides, it presents a complete picture of the variations of the functions as the time varies.

Ordinarily the first and second derived curves are not straight lines. A more general case is shown by Fig. 40, where the primitive curve with O as origin has the equation $y = \sin x$; therefore the first derived is $y' = \cos x$ and the second derived, $y'' = -\sin x$, plotted with O' and O'' as origins respectively. Here the slope of each curve for a given x is given by the ordinate of the curve below, measured to scale (the same scale was used here for lines parallel or perpendicular to OX). Two theorems may be stated: (1) where the tangent lines are parallel to OX in any curve, its derived (the curve just below it) crosses its axis of abscissas and conversely; (2) corresponding to points of inflection in any curve (as π in the primitive) are tangent lines parallel to OX in its derived curve. The first theorem is evident; the reason for the second will be made plain later. See Gibson's Calculus, pp. 183, 190, for the construction of both derived and integral curves when the original curve is not given by an equation.

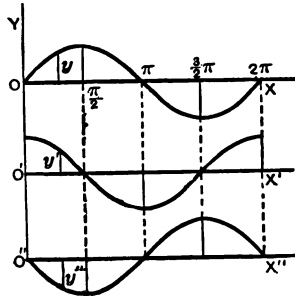

Fig. 40.

As an illustration of how these curves may be of use in practice, suppose a point P, Fig. 41, to move around the circle counter-clockwise with a *constant* velocity v. If the radius of the circle is a, the velocity of a point moving in a unit circle with center at O is v/a. This is the "angular velocity." If $\theta =$ angle XOP in circular measure, then $\frac{d\theta}{dt} = \frac{v}{a}$.

Projecting the point $P(x, y)$ on OY at N, let us find the velocity and acceleration of N, which moves up or down along OY as P moves around the circle.

We have,

$$y = a \sin \theta,$$

$$y' = \frac{dy}{dt} = a \cos \theta \cdot \frac{d\theta}{dt} = v \cos \theta,$$

$$y'' = \frac{d^2y}{dt^2} = -v \sin \theta \cdot \frac{d\theta}{dt} = -\frac{v^2}{a} \sin \theta.$$

The velocity y' of N decreases from v for $\theta = 0$, to $y' = 0$ at $\theta = \frac{\pi}{2}$; it then increases numerically as N moves downwards, until at $\theta = \pi$, $y' = -v$ (P is now at B and N at O). It becomes zero at $\theta = \frac{3}{2}\pi$, and then as P describes the last quadrant, N moves from D to O, with an increasing velocity which again attains the value v just as P reaches A. Let the student trace the changes in the acceleration y'' as θ varies from 0 to 2π.

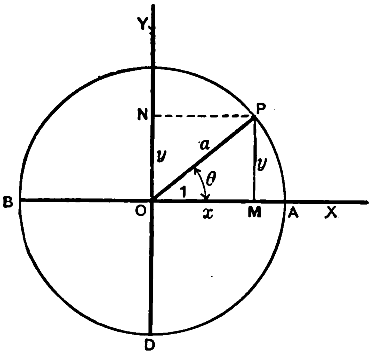

Fig. 41.

These *variations* are exactly represented in the curves of Fig. 40 on changing x to θ, but to get the actual values just found, from the curves of Fig. 40, y must be magnified a times; y^1, v times, and y'', $\left(\frac{v^2}{a}\right)$ times.

Thus, as θ increases from 0 to 2π, the ordinates y, y', y'', of Fig. 40, magnified a, v and $\frac{v^2}{a}$ times respectively, represent exactly, for any assumed θ, the distance of N from O in Fig. 41, the velocity of N along OY and its acceleration respectively. The acceleration is seen to be 0 at $\theta = 0$, $(-v^2/a)$ at $\theta = \pi/2$, 0 at π, (v^2/a) at $3\pi/2$ and 0 at 2π. For all intermediate values, the second derived curve shows whether y'' is increasing or decreasing as θ increases.

The motion of N, Fig. 41, is a case of *simple harmonic motion;* the motion of M has already been discussed in Art. 56, for $l \doteq \infty$. Both are given in Ex. 3, Art. 62, for clockwise motion,

EXAMPLE 1. Draw the curve $s = a \cos\theta$ with its first and second derived curves, regarding θ as the abscissa and s as the ordinate. This will show the motion of M, Fig. 41.

EXAMPLE 2. Assuming $y = x + \frac{1}{2}x^2$ as the equation of the primitive curve, construct it for x positive; also its first and second derived curves.

EXAMPLE 3. Construct, for positive values of x only, $y = x^3$; also its first and second derived curves.

64. Applications to Mechanics.

In Mechanics, the force F acting on a body of mass m and producing in it an acceleration f feet per second in each second, is given by the relation,

$$F = mf = m\frac{dv}{dt},$$

PROBLEM I. Suppose a body of mass m to be subjected to a continuously acting force F which is directly proportional to the distance s of the body from O (Fig. 42) and is always directed towards O whether s is positive or negative. Denote the force acting on unit mass at one foot from O, μ. The force acting on the body is then numerically $\mu m s$ and is directed to the left when

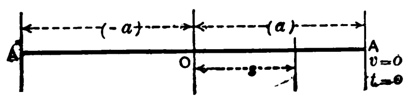

Fig. 42.

s is positive and to the right when s is negative. In the latter case, its value is positive and equals $(-\mu m s)$. In either case, the force acting to the right is $-\mu m s$, which equals mf, since the force in the direction in which s increases is always mf.

Whatever the mass, the two equations,

$$v = \frac{ds}{dt} \text{ and } \frac{dv}{dt} = f = -\mu s,$$

are true. Multiplying them together,

$$v\frac{dv}{dt} = -\mu s \cdot \frac{ds}{dt} \quad . . . \ (1)$$

$$\therefore v^2 = -\mu s^2 + C,$$

since on differentiating the latter equation and dividing by 2, we obtain the preceding equation.

To determine the constant C, let us make the assumption that $v = 0$ when $s = a$.

$$\therefore 0 = -\mu a^2 + C \quad \therefore C = \mu a^2,$$

$$\therefore v = \mu^{\frac{1}{2}}(a^2 - s^2)^{\frac{1}{2}} \quad . . . \ (2)$$

or,

$$\frac{-1}{\sqrt{a^2 - s^2}}\left(\frac{ds}{dt}\right) = \mu^{\frac{1}{2}}.$$

Hence,

$$\cos^{-1}\frac{s}{a} = \mu^{\frac{1}{2}}t + C',$$

for on differentiating this, as to t, we derive the preceding equation.

If we count the time from the instant when the body is at A $\therefore t = 0$ when $s = a$; hence $C' = 0$,

$$\therefore t = \mu^{-\frac{1}{2}}\cos^{-1}\frac{s}{a} \quad . . . \ (3)$$

or,

$$s = a\cos(\mu^{\frac{1}{2}}t),$$

which is a case of simple harmonic motion Since $\cos(\mu^{\frac{1}{2}}t)$ varies from $+1$ to -1 and back again, as t increases from 0 to $2\pi/\mu^{\frac{1}{2}}$, the body starting at A $(s = a)$ passes to A' $(s = -a)$ and then returns to A and repeats indefinitely.

From (2), $v = 0$ at $s = +a$ or $-a$; at O $(s = 0), v = a\mu^{\frac{1}{2}}$, the maximum velocity.

From (3), when $s = 0, t = \frac{1}{2}\pi\mu^{-\frac{1}{2}}, \frac{3}{2}\pi\mu^{-\frac{1}{2}}$, etc.; also, the time in going from A $(s = 0)$ to A' $(s = -a)$ is $\mu^{-\frac{1}{2}}\pi$.

If a body could fall through a vertical shaft, passing through the center of the earth to the other side, with no resistances (as of the air) to free motion, the motion would correspond to the case just given; for by Mechanics the acceleration f in such a case, varies directly as the distance from the center of the earth. If we call the acceleration due to gravity at the surface of the earth g and r the radius of the earth in feet, $g = \mu r$. Therefore as above, the velocity in feet per second with which the body reaches the center of the earth is, putting $a = r$,

$$v = r\sqrt{\mu} = \sqrt{gr},$$

and the time required to fall through the earth is,

$$2t = \frac{\pi}{\sqrt{\mu}} = \pi\sqrt{\frac{r}{g}}.$$

Placing $r = 20919360$ and $g = 32.17$ (corresponding to the latitude of New York) we find $2t = 42$ min., 13.4 seconds. Of course the resistance of the air would change this result very greatly.

PROBLEM II. Find the velocity with which a body would strike the earth in falling from a point above the surface of the earth, assuming that the acceleration of the body varies inversely as the square of the distance from the earth's center.

In Fig. 42, let O represent the center of the earth, its radius being r, s (now $> r$) is the distance from O to the body at any point of its fall from A. Thus s varies from $s = a = OA$ to $s = r$ during the fall.

The acceleration f at the distance s from O, by the law stated, is given in terms of s by the equation,

$$\frac{f}{g} = \frac{r^2}{s^2} \therefore f = \frac{gr^2}{s^2} = -\frac{d^2s}{dt^2} \quad . \; . \; . \quad (1)$$

the minus sign being used since $\left(\frac{ds}{dt}\right)$ diminishes as s increases, and $\therefore$ its derivative must be negative. Multiplying (1) by $\left(\frac{ds}{dt}\right)$ and re-arranging,

$$\left(\frac{ds}{dt}\right)^1 \frac{d}{dt}\left(\frac{ds}{dt}\right) = -gr^2s^{-2}\frac{ds}{dt}.$$

This equation can be obtained from the following, by differentiating the latter with respect to t, Art. 36.

$$\frac{1}{2}\left(\frac{ds}{dt}\right)^2 = \frac{gr^2}{s} + C.$$

Since $v = 0$ when $s = a \therefore C = -\frac{gr^2}{a}$. Hence the velocity v, when the body strikes the earth at $s = r$, is given by,

$$\frac{1}{2}v^2 = gr^2\left(\frac{1}{r} - \frac{1}{a}\right) \quad . \; . \; . \quad (2)$$

As $a \doteq \infty$, $1/a \doteq 0$ and (2) tends to the value,

$$v = \sqrt{2gr},$$

or 6.95 + miles a second. Hence the velocity with which a falling body can reach the earth, can never exceed 7 miles a second, however great the fall.

For a small fall h, putting $a = r + h$ in (2),

$$\frac{1}{2}v^2 = gr\frac{h}{r+h} = gh\frac{1}{1+\frac{h}{r}}.$$

This shows exactly the small error made in the common formula,

$$v^2 = 2gh,$$

corresponding to a fall of, say, a few hundred feet.

CHAPTER VIII.

INFINITE SERIES. TAYLOR'S THEOREM. EXPANSION OF FUNCTIONS.

65. Infinite Series. Let us consider the sum,

$$u_1 + u_2 + u_3 + \cdot \cdot \cdot + u_{n-1} + u_n + \cdot \cdot \cdot,$$

where u_1, u_2, u_3, etc., are connected by some law, and call the sum of the first n terms s_n;

$$\therefore\ s_n = u_1 + u_2 + u_3 + \cdot \cdot \cdot + u_n.$$

The successive terms may be positive or negative. If the number of terms is *finite*, the value of the series is simply the sum of the terms; but if the number of terms is *infinite*, the *value*, denoted by s, is the limit of the sum of the first n terms, as n increases indefinitely.

$$\therefore\ s = \lim_{n \doteq \infty} s_n.$$

When the number of terms is infinite, the series is called an *infinite series.** Where s_n tends to a definite finite limit s, the infinite series is said to be *convergent*, and to converge to the value s.

* Infinite is here used in the sense of never ending, just as space and time are never ending. Thus an infinite series is a never ending series. A never ending series cannot properly be called an infinite series if by an infinite number we mean the reciprocal of an infinitesimal (Remark, Art. 45). Such an "infinite series" always ends! To avoid this double meaning to "infinity," the word, in this book, is restricted throughout to mean but one thing, absolute infinity, corresponding to never ending, without limits or bounds.

If, however, s_n increases numerically beyond all bounds as $n \doteq \infty$, the series is *divergent*, and s_n has no limit. Finally s_n may *oscillate* as $n \doteq \infty$, between certain finite values, and s_n therefore has no *definite* limit, and the series is *non-convergent.*

An example of an oscillatory series is,

$$s_n = 1 - 1 + 1 - 1 + \ldots + (-1)^{n-1}\, 1,$$

which takes the value 0 or 1 according as n is even or odd.

For the ordinary geometric series,

$$a + ax + ax^2 + \ldots + ax^{n-1} + ax^n + \ldots,$$

it is shown in Algebra that,

$$s_n = a(1 + x + x^2 + \ldots + x^{n-1}) = \frac{a}{1-x} - \frac{ax^n}{1-x}.$$

If x is intermediate in value to -1 and $+1$, or in brief, if $-1 < x < +1$,

$$\therefore \lim s_n = s = \frac{a}{1-x},$$

since by Art. 23, Exs. 1, 2, 3,

$$\lim_{n \doteq \infty} \frac{a}{1-x} x^n = 0.$$

In fact, by actual division,

$$\frac{a}{1-x} = a\left(1 + x + x^2 + \ldots + x^{n-1} + \frac{x^n}{1-x}\right)$$

which is true for all values of x but $x = 1$, since division by 0 is expressly excluded by the laws of Algebra. The term $\dfrac{ax^n}{1-x}$ in the last equation, is called *the remainder after n terms*, and indicates the error made in summing only

n terms. Thus if $x = 0.1$, $a = 1$, $n = 5$, the sum of the first 5 terms in the right member differs from $\frac{1}{1 - 0.1}$ by $\frac{(0.1)^5}{.9} = .000011 +$.

When $-1 < x < 1$, the remainder tends to zero as its limit, and the *infinite* series $a(1 + x + x^2 + \ldots)$ represents or defines the function $a/(1 - x)$. But if x is numerically greater than 1, the remainder increases numerically with n, and the sum of n terms of the series differs more and more from $a/(1 - x)$; the *infinite* series $a(1 + x + x^2 + \ldots)$ is *divergent*, since it increases without limit, and it does not represent the function $a/(1 - x)$ at all.

When $x = -1$, we have the oscillatory series given above, which does not converge to a definite value.

We conclude that the *infinite* series (with no remainder term),

$$a(1 + x + x^2 + \cdots)$$

can represent the function $a/(1 - x)$ *only* when x lies between -1 and $+1$ in value.

Only convergent series are of use in practice.

The most important formula ever devised for the expansion of functions into series is the one named after the discoverer, "Taylor's formula." As preliminary to its derivation, the following important theorem will be proved.

66. Rolle's Theorem. *Let $F(x)$ be a single valued function of x, then if $F(x)$ and $F'(x)$ are continuous as x varies from a to b, and if $F(x)$ is zero when $x = a$ and when $x = b$, then $F'(x)$ will be zero for at least one value of x between a and b.*

In Fig. 43, let $y = F(x)$ represent the continuous curve *APB* which crosses the x-axis at A $(x = a)$ and B $(x = b)$, then the theorem states, what is evident from the figure, if $F'(x)$ is continuous from A to B, that for at least one point P on the curve, between A and B, the tangent is parallel to the x-axis.

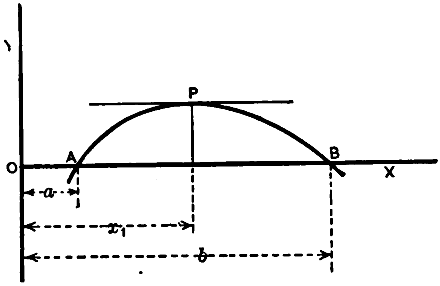

Fig. 43.

The student should draw a wavy curve from A to B to show that where there are several points as P where the tangent is parallel to the x-axis, the number of such points is always odd.

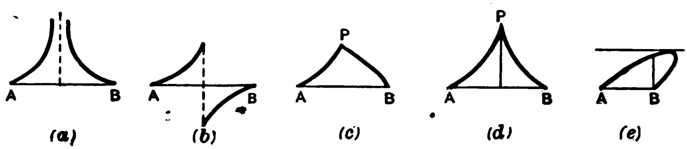

Fig. 44.

Fig. 44 illustrates the necessity for continuity in both $F(x)$ and $F'(x)$ between $x = a$ and $x = b$. Thus in Fig. 44 (a) and (b), y is discontinuous (Art. 33), in (c) and (d) the slope is discontinuous, changing suddenly at P in (c) and becoming infinite at P in (d). At Fig. 44 (e), y is not single-valued at B. In none of these cases is $F'(x) = 0$ between A and B or when x lies between a and b.

67. Taylor's Series. To expand $f(x)$ in a series of the form,

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + \cdots + A_n(x-a)^n + \cdots,$$

where $A_0, A_1, A_2, A_3, \ldots A_n, \ldots$ are constants to be determined, we have to assume the convergency of the expansion and also that the above infinite series can be successively differentiated term by term exactly as if it was a finite series. The last is proved in larger treatises on the Calculus. It is also assumed that $f(x)$ and all of its derivatives are continuous, from $x = x$ to $x = a$ inclusive.

The object in this article is to get the form of the expansion. It will lead to a rigorous proof to be given subsequently.

On differentiating successively the above identity with respect to x, we obtain,

$$f'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \cdots,$$
$$f''(x) = 2A_2 + 2 \cdot 3A_3(x-a) + \cdots,$$
$$f'''(x) = 2 \cdot 3A_3 + \text{terms in } (x-a),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f^{(n)}(x) = \underline{|n} A_n + \text{terms in } (x-a),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

On making $x = a$ in each of the identities, since the derivatives are by assumption continuous in the vicinity of $x = a$, none are infinite or indefinite, and we have,

$$A_0 = f(a);\ A_1 = f'(a);\ A_2 = \frac{f''(a)}{1 \cdot 2};$$
$$A_3 = \frac{f'''(a)}{1 \cdot 2 \cdot 3};\ \ldots,\ A_n = \frac{f^n(a)}{\underline{|n}};\ \ldots.$$

Hence substituting in the first equation,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{1 \cdot 2}f''(a) + \frac{(x-a)^3}{1 \cdot 2 \cdot 3}f'''(a)$$
$$+ \cdots + \frac{(x-a)^{n-1}}{\underline{|n-1}}f^{(n-1)}(a) + \frac{(x-a)^n}{\underline{|n}}f^{(n)}(a) + \cdots \quad (1)$$

This is one form of *Taylor's series,* which is named after Dr. Brook Taylor who discovered it. Another form is obtained from (1) by putting $(x' + h)$ for x and x' for a $(\therefore x - a = h)$ and then dropping accents.

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) = \mathbf{f}(\mathbf{x}) + \mathbf{h}\mathbf{f}'(\mathbf{x}) + \frac{\mathbf{h}^2}{1 \cdot 2}\mathbf{f}''(\mathbf{x}) + \frac{\mathbf{h}^3}{1 \cdot \ldots 3}\mathbf{f}''' \ldots$$
$$+ \cdots + \frac{\mathbf{h}^n}{\lfloor \mathbf{n}}\mathbf{f}^{(n)}(\mathbf{x}) + \cdots$$

In this form, $f(x)$ and all of its derivativ… to be continuous from x to $x + h$.

EXAMPLE 1. Let $f(x+h) = (x+h…$ $= 3x^2$; $f''(x) = 2.3\, x$; $f'''(x) = 2…$ derivatives are each zero; hence in t… nates. On substituting these values …

$$(x + h)^3 = x^3 + 3x^2h \ldots$$

This example is a particular case …

EXAMPLE 2. Let

$$f\ (x + h) = (x + h)^m \therefore \ldots$$
$$f'\ (x) = mx^{m-1};$$
$$f'''(x) = m(m - 1)(m - \ldots$$

Substituting in (2), we derive the …

$$(x + h)^m = x^m + mx^{m-1} \ldots$$
$$+ \frac{m(m \ldots}{\ldots}$$

EXAMPLE 3. Let $f(x + h)$ …

Put $m = \log_a e \quad \therefore f\ldots$

$$f'''(x) = \frac{2m}{x^3},\ f^{iv}(x \ldots$$

∴ substituting in (2), we get the *logarithmic series*,

$$\log_a (x + h) = \log_a x + m\left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \cdots\right).$$

Here x must not be 0, since $f(x), f'(x), f''(x), \ldots$ are then discontinuous.

We derive the *Napierian logarithmic series* from this by putting $a = e, x = 1, \therefore m = 1$;

$$\log(1 + h) = \frac{h}{1} - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \cdots.$$

EXAMPLE 4. Let

$$f\ (x + h) = \sin(x + h) \quad \therefore f\ (x) = \sin x,$$
$$f'\ (x) = \cos x, \qquad f''\ (x) = -\sin x,$$
$$f'''(x) = -\cos x, \ldots,$$
$$\therefore \sin(x + h) = \sin x + h\cos x - \frac{h^2}{2}\sin x - \frac{h^3}{6}\cos x + \cdots$$

If $\quad x = 0, \quad \sin h = h - \frac{h^3}{|\underline{3}} + \frac{h^5}{|\underline{5}} - \cdots$

The angle must of course be expressed in radians. Thus to get sin 20°,

$$h = \frac{\pi}{9} = 0.34906585.$$

On substituting in the last formula and using only three terms we get, $\sin \pi/9 = 0.34202$, as given in a 5 place table.

EXAMPLE 5. Prove,

$$a^{x+h} = a^x\left\{1 + \log a\,\frac{h}{1} + (\log a)^2\,\frac{h^2}{|\underline{2}} + (\log a)^3\,\frac{h^3}{|\underline{3}} + \cdots\right\}$$

EXAMPLE 6. Develop $(x - a + h)^{\frac{2}{3}}$ by Taylor's series.

$$f(x) = (x - a)^{\frac{2}{3}}, \quad f'(x) = \tfrac{2}{3}(x - a)^{-\frac{1}{3}}, \text{ etc.}$$

$$\therefore (x - a + h)^{\frac{2}{3}} = (x - a)^{\frac{2}{3}} + \frac{2h}{3(x-a)^{\frac{1}{3}}} - \frac{4h^2}{18(x-a)^{\frac{4}{3}}} \cdots$$

This development is not true for $x = a$ as was foreseen, since then $f'(x), f''(x), \ldots$ are all discontinuous. The value of the function for $x = a$ is $h^{\frac{2}{3}}$.

67*a*. Maclaurin's Series. In (1) of Art. 67, on putting $a = 0$, we obtain Maclaurin's series,

$$\mathbf{f(x) = f(0) + x f'(0) + \frac{x^2}{1 \cdot 2} f''(0) + \frac{x^3}{1 \cdot 2 \cdot 3} f'''(0) + \cdots} \quad \mathbf{(1\,a)}.$$

In this formula, to find $f^r(0)$, the r^{th} derivative of $f(x)$ must be first found and in the result, x put equal to zero.

EXAMPLE. Let

$$\begin{array}{llll} & f\ (x) = \sin x & \therefore & f\ (o) = 0. \\ \therefore & f'\ (x) = \cos x & \therefore & f'\ (o) = 1. \\ & f''\ (x) = -\sin x & \therefore & f''\ (o) = 0. \\ & f'''(x) = -\cos x & \therefore & f'''(o) = -1. \\ & f^{iv}\ (x) = \sin x & \therefore & f^{iv}\ (o) = 0. \\ & f^{v}\ (x) = \cos x & \therefore & f^{v}\ (o) = 1. \\ & \cdot\ \cdot\ \cdot\ \cdot & & \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

Hence substituting in (1 *a*),

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \cdots$$

Examples.

Prove by Maclaurin's series the following:

1. $\tan x = x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \cdots$

2. $\sec x = 1 + \frac{x^2}{\underline{|2}} + \frac{5\,x^4}{\underline{|4}} + \cdots$

3. $\log(\cos x) = -\frac{x^2}{\underline{|2}} - \frac{2\,x^4}{\underline{|4}} - \cdots$

4. $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$

Why cannot log (sin x) be expanded?

5. $\sin^2 x = x^2 - \frac{2x^4}{\lfloor 3} + \frac{32x^6}{\lfloor 6} + \ldots$

Hint: $2 \sin x \cos x = \sin 2x$.

6. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$

Hint: $f'(x) = (1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots$

$\therefore f''(x) = -2x + 4x^3 - 6x^5 + \ldots$, etc.

The expansion is true when $-1 < x < 1$ (Art. 72, Ex. 5).

7. If $\quad x = \frac{1}{\sqrt{3}},\ \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$$= \frac{1}{\sqrt{3}}\left(1 - \frac{1}{9} + \frac{1}{45} - \frac{1}{189} + \ldots\right)$$

$$\therefore \pi = 3.14 \ldots$$

Other series have been devised that converge much more rapidly than this.

8. $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$

9. $\log(1 + e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{2^3} - \frac{x^4}{2^3 \lfloor 4} + \ldots$

By Taylor's formula (2), Art. 67, find:

10. $\text{Cos}\,(x + h) = \cos x \left(1 - \frac{h^2}{\lfloor 2} + \frac{h^4}{\lfloor 4} - \ldots\right)$

$$- \sin x \left(h - \frac{h^3}{\lfloor 3} + \frac{h^5}{\lfloor 5} - \ldots\right)$$

$$= \cos x \cos h - \sin x \sin h.$$

11. $\log \sin (a + x) = \log \sin x + a \cot x - \frac{a^2}{2} \operatorname{cosec}^2 x$

$$+ \frac{a^3}{3} \cot x \operatorname{cosec}^2 x + \ldots$$

12. In Art. 61 (iv), on putting $a = \cos \alpha$, $b = \sin \alpha$, $c = 0$, we find $R = 1$, $\tan \theta = \tan \alpha$ $\therefore$ $\theta = \alpha$ and $D^n e^{x \cos \alpha} \cos (x \sin \alpha) = e^{x \cos \alpha} \cos (x \sin \alpha + n\alpha)$.

$\therefore$ make $n = 1, 2, 3, \ldots$ and prove, by Maclaurin's formula

$$e^{x \cos \alpha} \cos (x \sin \alpha) = 1 + x \cos \alpha + \frac{x^2}{\underline{|2}} \cos 2\alpha + \frac{x^3}{\underline{|3}} \cos 3\alpha + \ldots$$

The student is referred to Gibson's Calculus, pp. 375–407, for instructive chapters on Infinite Series.

The examples above sufficiently illustrate the use of the formulas and their extreme generality. We have yet to ascertain when the series obtained are convergent and represent the functions. To aid in this a general expression for "the remainder after n terms" (see illustration in Art. 65) will now be found.

68. Taylor's Theorem. Remainder after n Terms. Suppose $f(x)$ and its first n derivatives to be continuous from $x = a$ to $x = b$; also note that $f'(a), f''(a), \ldots$, mean that the derivatives of $f(x)$ with respect to x, $f'(x)$, $f''(x), \ldots$, must be formed and a put for x in the results.

Let us write,

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{1 \cdot 2} f''(a) + \ldots$$
$$+ \frac{(b-a)^{n-1}}{\underline{|n-1}} f^{n-1}(a) + R,$$

where R is such a number that added to the value of the first n terms of the right member, the result shall exactly equal the value of $f(b)$. By proceeding as indicated below, it is found that a general expression for R can be found which accounts for our assuming the particular form of series chosen.

To get the first form of "remainder after n terms," place $R = (b - a)^n Q$.

$$\therefore f(b) = f(a) + (b - a) f'(a) + \frac{(b - a)^2}{1 \cdot 2} f''(a) + \ldots + \frac{(b - a)^{n-1}}{\lfloor n - 1} f^{n-1}(a) + (b - a)^n Q \ldots (1)$$

Q is here a constant, since every term in (1) is a constant. The object is to find an expression for Q.

Let us consider a function of x, $F(x)$, given by the equation,

$$F(x) = f(b) - f(x) - (b - x) f'(x) - \tfrac{1}{2}(b - x)^2 f''(x) - \ldots - \frac{(b - x)^{n-1}}{\lfloor n - 1} f^{(n-1)}(x) - (b - x)^n Q \ldots (2)$$

The reason for taking this form of a function is that $F(a) = 0$ by (1) and $F(b) = 0$ identically, hence by Rolle's theorem (Art. 66) $F'(x)$ must be zero for a value of x, say x_1, between a and b, since $F(x)$ and $F'(x)$ are continuous from $x = a$ to $x = b$, for by assumption, $f(x)$ and its first n derivatives are continuous from $x = a$ to $x = b$ (Art. 34).

If the student find difficulty here, let him note that the right member of (2) is the sum of a finite $(n + 1)$ number of terms containing x in some form or other; therefore $y = F(x)$ can be supposed drawn from $(x = a, y = 0)$ to $(x = b, y = 0)$ and is sufficiently represented by Fig. 43, Art. 66, to show that $F'(x) = 0$ for some value of x, x_1, between a and b.

In differentiating (2), note that the derivative of $f'(x) = f''(x)$, etc., and that the rules for the sum and the product

(Arts. 37–40) must be applied. The derivative of (2) with respect to x is,

$$F'(x) = 0 - f'(x) + f'(x) - (b-x)f''(x) + (b-x)f''(x) + \dots$$
$$- \frac{(b-x)^{n-1}}{\lfloor n-1} f^{(n)}(x) + n(b-x)^{n-1} Q \dots (3)$$

It will be observed that the terms destroy each other in pairs, so that only the last two terms are left. Since $F'(x) = 0$ for $x = x_1$ where x_1 lies between a and b, we derive from (3) the desired value of Q.

$$Q = \frac{1}{\lfloor n} f^{(n)}(x_1) \dots (4)$$

If θ = a positive proper fraction, *i.e.*, if $0 < \theta < 1$, any number x_1 between a and b can be written

$$x_1 = a + \theta(b-a).$$

On substituting the value of Q just found in (1), we have,

$$f(b) = f(a) + (b-a)f'(a) + \tfrac{1}{2}(b-a)^2 f''(a) + \dots$$
$$+ \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{(b-a)^n}{\lfloor n} f^n[a + \theta(b-a)] \dots (5)$$

The demonstration above holds whatever the value of b we start with, provided $f(x)$ and its first n derivatives are continuous from $x = a$ to $x = b$. Hence we can replace b by x and write,

$$f(x) = f(a) + (x-a)f'(a) + \tfrac{1}{2}(x-a)^2 f''(a) + \dots$$
$$+ \frac{(x-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{(x-a)^n}{\lfloor n} f^{(n)}\{a + \theta(x-a)\} \dots (6)$$

provided $f(x)$, $f'(x)$, $f''(x)$, . . ., $f^{(n)}(x)$ are continuous from $x = a$ to $x = x$.

This equation may be said to express a theorem which is known as *Taylor's Theorem.*

The number θ is unknown except that it is some positive proper fraction. It has generally different values for different values of x and n.

If we denote the sum of the first n terms of (6) by $S_n(x)$ and the last term, *the remainder after n terms*, by $R_n(x)$,

$$f(x)=S_n(x)+R_n(x);\ R_n(x)=\frac{(x-a)^n}{\underline{|n}}f^{(n)}[a+\theta(x-a)]\ldots(7)$$

$R_n(x)$ shows the error made in expressing $f(x)$ by the sum of only n terms of the series. If n can be made so large that the value of $R_n(x)$ for this and all greater values of n, will be less than any assigned number, however small, the series will be *convergent.* Otherwise expressed, as $n \doteq \infty$ in (6) above, if *lim* $R_n(x) = 0$, the series (6) approaches the infinite series (1) of Art. 67, which is then said to be convergent. In this case, since $f(x)$ and its first n derivatives are by hypothesis continuous, *every* derivative must be continuous when $n = \infty$. When these conditions are fulfilled, (1) of Art. 67 exactly represents the function $f(x)$.

To decide then whether a series is convergent and represents the function, we consider $R_n(x)$. If its limit as $n \doteq \infty$ is zero, the series is convergent and represents the function. If $R_n(x)$ increases with n, the series is divergent and worthless. The infinite series in this case does not represent the function. An illustration was given in Art. 65.

69. Second Form of Remainder. The form of the remainder just found, is called Lagrange's, after the discoverer *Lagrange.* *Cauchy's* form for the remainder is easily found by writing $(b - a)\,Q$ instead of $(b - a)^n\,Q$ in Eq. (1) of the last article and proceeding as there indicated to determine Q. The last term of Eq. (3) now becomes Q, (4) is replaced by

$$Q = \frac{(b - x_1)^{n-1}}{\lfloor n - 1} f^{(n)}(x_1),$$

and the last term of (5) by,

$$(b - a)\,Q = \frac{(b - a)\,(b - x_1)^{n-1}}{\lfloor n - 1} f^{(n)}(x_1)$$
$$= \frac{(b - a)^n\,(1 - \theta)^{n-1}}{\lfloor n - 1} f^{(n)}\,\{a + \theta\,(b - a)\},$$

since $(b - x_1) = b - a - \theta\,(b - a) = (b - a)\,(1 - \theta)$.

Hence *Cauchy's form of the Remainder* is,

$$R_n(x) = \frac{(x - a)^n\,(1 - \theta)^{n-1}}{\lfloor n - 1} f^{(n)}\,[a + \theta\,(x - a)] \quad . \; . \; . \; (8)$$

This replaces the last term of (6), Art. 68. If in (5), Art. 68, we put $b = x + h$, $a = x$ and consequently $(b - a) = h$, we get,

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{1.2} f''(x) + \; . \; . \; .$$
$$+ \frac{h^{n-1}}{\lfloor n - 1} f^{n-1}(x) + R_n(x) \quad . \; . \; . \; (9)$$

The two forms of the Remainder are,

$$R_n(x) = \frac{h^n}{\lfloor n} f^n(x + \theta h); \quad R_n(x) = \frac{h^n\,(1 - \theta)^{n-1}}{\lfloor n - 1} f^n(x + \theta h).$$

70. Maclaurin's Theorem.* Maclaurin's Theorem is only another form of Taylor's Theorem, found by putting $a = 0$ in (6) of Art. 68. It is as follows:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{1.2}f''(0) + \frac{x^3}{1.2.3}f'''(0) + \ldots$$
$$+ \frac{x^{n-1}}{\lfloor n-1}f^{(n-1)}(0) + \frac{x^n}{\lfloor n}f^{(n)}(\theta x) \quad \ldots \quad (10)$$

Lagrange's and Cauchy's forms for the remainder are respectively,

$$R_n(x) = \frac{x^n}{\lfloor n}f^{(n)}(\theta x);\ R_n(x) = \frac{x^n(1-\theta)^{n-1}}{\lfloor n-1}f^{(n)}(\theta x) \quad \ldots \quad (11)$$

The first form is written in (10) and can be replaced by the second, obtained from (8), when desired. Here we form $f^{(n)}(x)$ and then replace x by (θx) to get $f^n(\theta x)$.

71. To prove $\lim\limits_{n \doteq \infty} \frac{y^n}{\lfloor n} = 0.$

Suppose y finite and equal to or less numerically than m, where m is a positive integer; then we have numerically,

$$\frac{y^n}{1.2.3 \ldots n} = \frac{y^m}{1.2.3 \ldots m} \cdot \frac{y}{m+1} \cdot \frac{y}{m+2} \cdot \ldots$$
$$\frac{y}{n-1} \cdot \frac{y}{n} < \frac{y^m}{\lfloor m}\left(\frac{y}{m+1}\right)^{n-m},$$

since there are $(n - m)$ factors following $y^m/\lfloor m$, each, after the first $(y/(m+1))$, less than $y/(m+1)$.

* Taylor's Series was published in 1715 in Taylor's *Methodus Incrementorum*. Lagrange found the Remainder term named after him and published it in 1772. Cauchy's form was published in 1826. Maclaurin published his series in 1742. Stirling, however, anticipated him, having published the series that now is called Maclaurin's in 1730. It is more properly called Stirling's Series. Neither author found $R_n(x)$.

But $y/(m+1)$, by hypothesis, is a proper fraction; hence by Examples 1, 2, 3, Art. 23,

$$\lim_{n \doteq \infty} \frac{y^m}{\lfloor m} \left(\frac{y}{m+1}\right)^{n-m} = 0 \therefore \lim_{n \doteq \infty} \frac{y^n}{\lfloor n} = 0.$$

A factor of the form $y^n/\lfloor n$, appears in all the expressions for the remainder, and its limit as $n \doteq \infty$ is invariably zero; hence if $f^n(x)$ remains finite as $n \doteq \infty$, Taylor's or Maclaurin's series gives the expansion of and represents the function $f(x)$.

72. Examples. The n^{th} derivatives of several functions given below have been already found in Art. 61, Examples (i) to (vi).

1. To expand a^x by Maclaurin's formula.

$$f(x) = a^x \therefore f'(x) = \log a \cdot a^x, f''(x) = (\log a)^2 \cdot a^x, \ldots$$

$$f^n(x) = (\log a)^n \cdot a^x \therefore f^n(\theta x) = (\log a)^n a^{\theta x}.$$

Also,

$$f(0) = a^0 = 1, f'(0) = \log a, f''(0) = (\log a)^2, \ldots;$$

hence substituting in (10) of Art. 70, we get,

$$a^x = 1 + \frac{x}{1} \log a + \frac{x^2}{1.2} (\log a)^2 + \ldots + \frac{x^n}{\lfloor n} (\log a)^n a^{\theta x}.$$

$$\therefore R_n(x) = \frac{(x \log a)^n}{\lfloor n} a^{\theta x}.$$

The factor $a^{\theta x}$ is finite; the other factor is of the form $y^n/\lfloor n$ whose limit as $n \doteq \infty$ is 0 by Art. 71 $\therefore \lim\limits_{n \doteq \infty} R_n(x) = 0$ and a^x is represented by the infinite series,

$$a^x = 1 + \frac{x}{1} \log a + \frac{x^2}{1.2} (\log a)^2 + \frac{x^3}{1.2.3} (\log a)^3 + \ldots$$

If $a = e$ the Napierian base, $\log e = 1$,

$$\therefore e^x = 1 + \frac{x}{1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots$$

2. $f(x) = \sin x \therefore f'(x) = \cos x, f''(x) = -\sin x,$

$f'''(x) = -\cos x, f^{iv}(x) = \sin x, f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right).$

$\therefore f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{iv}(0) = 0$

$f^{(n)}(0) = \sin\frac{n\pi}{2},\ f^{(n)}(\theta x) = \sin\left(\theta x + \frac{n\pi}{2}\right).$

$R_n(x) = \frac{x^n}{\underline{|n}} f^{(n)}(\theta x) = \frac{x^n}{\underline{|n}} \sin\left(\theta x + \frac{n\pi}{2}\right).$

Therefore, since the last factor cannot exceed 1 numerically and $\lim_{n \doteq \infty} \frac{x^n}{\underline{|n}} = 0$ by Art. 71, $\therefore$ limit $R_n(x) = 0$ and the infinite series, obtained by substituting the above values in Maclaurin's formula, or,

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} \cdot \cdot \cdot$$

is convergent for all finite values of x.

3. Prove in the same way that,

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \cdot \cdot \cdot$$

for every finite value of x.

It may be remarked that the last series may be obtained from the former by differentiation.

4. $f(x) = \log(1 + x), f'(x) = (1 + x)^{-1}, f''(x) = -(1 + x)^{-2}, \ldots f^n(x) = (-1)^{n-1} \cdot \underline{|n-1} \cdot (1 + x)^{-n}.$
$\therefore f(0) = 0,\ f'(0) = 1,\ f''(0) = -1, \ldots f^{(n)}(0) = (-1)^{n-1} \cdot \underline{|n-1}.$

Hence by Maclaurin's formula,

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdot \cdot \cdot \because R_n(x)$$

The *infinite* series evidently diverges if x is numerically greater than 1, since the ratio of the n^{th} term to the one before it approaches $-x$ as $n \doteq \infty$. In Algebra this is proved; also that the series is divergent when $x = -1$ and convergent when $x = 1$.

Lagrange's remainder is,

$$R_n(x) = (-1)^{n-1} \cdot \frac{1}{n} \cdot \left(\frac{x}{1+\theta x}\right)^n.$$

For x *positive* and < 1, the last factor is of the form r^n, where r is a proper fraction $\therefore$ its limit is zero; also the limit of $1/n$ is zero. $\therefore \lim\limits_{n \doteq \infty} R_n(x) = 0$.

Cauchy's remainder is,

$$R_n(x) = (-1)^{n-1} \cdot x^n \frac{1}{1+\theta x} \cdot \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

For x *negative* and numerically < 1, the last parenthesis is a proper fraction, since $1 + \theta x > 1 - \theta$; θx being negative, but numerically $< \theta$. Hence by Ex. 1, Art. 23, the limit as $n \doteq \infty$ of the last factor is zero; also *lim* $x^n = 0$ and $1/(1 + \theta x)$ is finite. $\therefore \lim\limits_{n \doteq \infty} R_n(x) = 0$.

Hence we have shown, that for $-1 < x < 1$, the infinite series,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

is convergent.

As this series converges too slowly for computation, write $-x$ for x, giving

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$$

By subtraction,

$$\log\frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right\}.$$

Let, $$\frac{1+x}{1-x} = \frac{y+1}{y} \qquad \therefore\ x = \frac{1}{2y+1};$$

hence when y is positive, x lies between 0 and + 1. On making the substitution, we get,

$$\log(y+1) = \log y + 2\left\{\frac{1}{2y+1} + \frac{1}{3}\left(\frac{1}{2y+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2y+1}\right)^5 + \ldots\right\},$$

a rapidly converging series for the computation of logarithms.

Thus for $y = 1$, $\log 2 = 2\left\{\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \ldots\right\}$.

$\therefore$ summing 6 terms, $\log 2 = 0.693147 +$.

Similarly $y = 2$, gives $\log 3 = \log 2 +$ etc.; $\log 4 = \log 2^2 = 2 \log 2$; $\log 5 = \log 4 +$ etc., on putting $y = 4$; $\log 6 = \log 3 \times 2 = \log 3 + \log 2$, and so on.

In this way a table of Napierian logarithms can be computed. In Algebra, it is shown that the Napierian logarithm of a number must be multiplied by

$$m = \frac{1}{\log 10} = \frac{1}{2.302585} = 0.434294 +$$

to find the logarithm of the number to base 10.

5. $f(x) = (1+x)^m \;\therefore f'(x) = m(1+x)^{m-1}$, etc. (Art. 61, v.) $f^n(x) = m(m-1)\ldots(m-n+1)(1+x)^{m-n}$

$\therefore f(0) = 1, f'(0) = m, f''(0) = m(m-1), \ldots$

$f^{(n)}(\theta x) = m(m-1)(m-2)\ldots(m-n+1)(1+\theta x)^{m-n}$

$$\therefore (1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \ldots$$
$$+ \frac{m(m-1)\ldots(m-n+2)}{\lfloor n-1}x^{n-1} + R_n(x).$$

When m is a positive integer, $f^{m+1}(x) = 0, f^{m+2}(x) = 0, \ldots$ hence the series ends, and the expansion is a finite series of

$(m + 1)$ terms. When m is not a positive integer, the series is infinite, and we have to prove for what values of x it is convergent by a consideration of the remainder. Cauchy's form,

$$R_n(x) = \frac{m(m-1)\ldots(m-n+1)}{\lfloor n-1}x^n(1-\theta)^{n-1}(1+\theta x)^{m-n},$$

can be written,

$$R_n(x) = mx(1+\theta x)^{m-1} \cdot \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot$$
$$\frac{(m-1)(m-2)\ldots(m-n+1)}{1.2\ldots(n-1)}x^{n-1}.$$

When x, whether positive or negative, is numerically less than 1, the first factor $mx(1+\theta x)^{m-1}$, is finite. The parenthesis term in the second factor is a proper fraction (see Ex. 4, for x negative); hence the limit of this factor as n and $\therefore (n-1) \doteq \infty$, is 0 (Ex. 1, Art. 23). Finally the last factor can be written as the product of $(n-1)$ factors,

$$\frac{(m-1)x}{1} \cdot \frac{(m-2)x}{2} \cdot \frac{(m-3)x}{3} \cdots \frac{(m-n+1)x}{n-1}, \ldots \text{(i)},$$

each one of the type,

$$\frac{m-r}{r}x = \left(\frac{m}{r} - 1\right)x,$$

as we see, on putting r successively $= 1, 2, \ldots (n-1)$.

Now $$\lim_{r \doteq \infty}\left(\frac{m}{r} - 1\right)x = -x,$$

a proper fraction; hence for some value r_1 of r, $\left(\frac{m}{r} - 1\right)x$ becomes and remains, less than unity numerically as r still further increases. Therefore calling the *finite* product of the first r_1 fractions in (i), A and the product of the remaining fractions (each one a proper fraction) B, we see that AB is finite for all values of n. (In fact, by the reasoning of Art. 71, *lim* $AB = 0$ as $n \doteq \infty$, though this is not necessary to our conclusion.) It follows that $\lim_{n \doteq \infty} R_n(x) = 0$ so long as $-1 < x < 1$ and the infinite series is convergent. See Chrystal's Algebra, vol. 2, ch. 26, § 6, for $x = +1$ or -1.

If a is greater than b, $(a+b)^m$ can be written $a^m(1 + b/a)^m$ and expanded by the formula for $(1+x)^m$ on putting the proper fraction, b/a for x. On doing this and then multiplying through by a^m as indicated, we derive the *binomial formula,*

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{\lfloor 2} a^{m-2}b^2 + \ldots$$

which is thus proved true whether m is positive or negative, whole or fractional.

If a is less than b, interchange them in the last formula. The largest number is written first before developing.

6. Using the results in Exs. 1, 2, 3, Art. 72, prove,

$$e^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x,$$

also,
$$e^{-x\sqrt{-1}} = \cos x - \sqrt{-1}\sin x.$$

73. Theorem of Mean Value. In (9), Art. 69, the last term gives the remainder over after n terms. Let $n = 1$ and (9) takes the form,

$$f(x+h) = f(x) + hf'(x + \theta h) \quad . \; . \; . \; (1)$$

If $n = 2$,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x+\theta h) \quad . \; . \; . \; (2).$$

The first formula is called *The Theorem of Mean Value.* It is likewise easily proved by aid of a figure.

Thus in Fig. 45 let $y = f(x)$ and also $f'(x)$, the slope of the curve, be continuous from x to $x + h$. Then at some point of the curve between these limits, the tangent is parallel to the secant. The abscissa of this point can be represented by $x + \theta h$, where $0 < \theta < 1$.

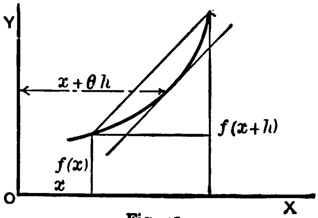

Fig. 45.

$$\therefore \frac{f(x+h) - f(x)}{h} = f'(x + \theta h),$$

which gives formula (1) above.

EXAMPLE. For the parabola, $y = f(x) = x^2$, we shall find that $x + \theta h = x + \frac{1}{2} h$. Thus,

$$\frac{f(x+h) - f(x)}{h} = 2x + h,$$

which is equal to $f'(x) = 2x$ when we put $x + \frac{1}{2} h$ for x $\therefore\ \theta = \frac{1}{2}$ in this case.

CHAPTER IX.

MAXIMA AND MINIMA. CONCAVITY.

74. Maxima and Minima. Definitions. First Method. Let us suppose that $y = f(x)$ is represented by a graph — take Fig. 46 as an illustration; then points on the curve as M, S, V, where, as x increases algebraically, the ordinates cease to decrease and begin to increase, are called *turning points* of the function, and the corresponding ordinates BM, GS, KV, are called *minimum* ordinates, and their scaled values, *minimum* values of the function. Also points on the curve, such as Q and T, where, as x increases algebraically, the ordinates cease to increase and begin to decrease, are again called *turning points*, but the corresponding values of the function, represented by the ordinates EQ and HT, are known as *maximum* values of the function.

Another definition is as follows:

Let h be a positive quantity, other than zero, but as near zero as we choose; then $f(a)$ is a maximum value of $f(x)$ when $f(a)$ is algebraically greater than both $f(a - h)$ and $f(a + h)$; $f(a)$ is a minimum value of $f(x)$ when $f(a)$ is algebraically less than both $f(a - h)$ and $f(a + h)$.

Thus in Fig. 46, *if* $OB = a$ and $AB = BC = h$, then since $f(a) = BM$, is less than both $f(a - h) = AL$ and $f(a + h) = CN$, however small h is, it is a minimum value of $f(x)$. If however we call $OE = a$, $DE = EF = h$, then since $f(a) = EQ$, is greater than both $f(a-h) = DP$ and $f(a+h) = FR$, however small h is taken, it follows that $f(a)$ is a maximum value of $f(x)$.

The two definitions give the same idea, though expressed differently. The Calculus turning values generally give the greatest or least values of the function in a certain

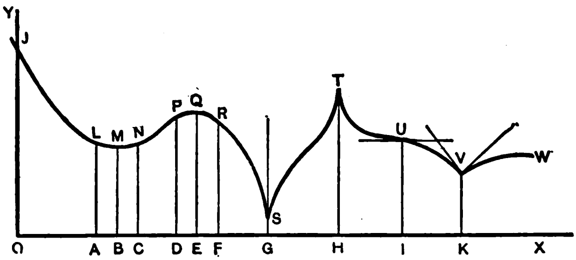

Fig. 46.

range, but not always. Thus in Fig. 46, the *greatest* value of $f(x)$ as x varies from o to OK is OJ, which is not a Calculus maximum; for $f(0) = OJ$ is not greater than $f(0-h)$ as well as $f(0+h)$. In physical applications, this point must be carefully noted. For example, to design a bridge member, we wish to know the greatest stress in it that can be caused by a passing load as it occupies successive positions on the bridge and not simply the Calculus maximum unless the latter agrees with the former.

Observe that the definition states that $f(a)$ is a minimum when it is *algebraically* less than both $f(a-h)$ and $f(a+h)$. Thus for the upper curve of Fig. 40, Art. 63, $y = \sin x$ is a minimum (not a maximum) at $x = 3\pi/2$. This follows too from the first definition.

The abscissas as OB, OE (Fig. 46) corresponding to turning values are called *critical* values of x. They may be found by aid of the principles of Art. 32, which are in brief, that if $f(x)$ is increasing, as x increases, $f'(x)$ is positive; if decreasing, $f'(x)$ is negative. Thus at L (Fig. 46), $f'(x)$ is $-$, at N, $f'(x)$ is $+$ and $f(x)$ is a minimum at M; at P, $f'(x)$ is $+$, at R, $f'(x)$ is $-$ and $f(x)$ is a maximum. If the function $f'(x)$ is continuous, it can only change sign by passing through zero; hence in this case, we put $f'(x) = 0$ to find the critical values of x. This is otherwise evident from the figure, since the tangent is parallel to the x-axis at either M or Q.

When $f'(x)$ is discontinuous as at V, this test fails to give the critical value OK. At S or at T, the tangent is vertical $\therefore$ if $f'(x) \doteq \infty$ as $x \doteq a$, a is a critical value. Hence the following test may be applied to ascertain maximum or minimum values, if any exist.

Having found a critical value a for x, either by solving $f'(x) = 0$ or by noting that as $x \doteq a$, $f'(x) \doteq \infty$; then,

If $f'(a-h)$ is positive and $f'(a+h)$ is negative, $f(a)$ is a maximum of $f(x)$. If $f'(a-h)$ is negative and $f'(a+h)$ is positive, $f(a)$ is a minimum of $f(x)$. If $f'(a-h)$ and $f'(a+h)$ have both the same sign, $f(a)$ is neither a maximum nor a minimum of $f(x)$.

The latter case is shown at U, where the tangent is parallel to the x-axis, but the ordinate continually decreases as x passes through the value OI.

Exercise. Let the student construct a curve having $f'(x) = 0$ at U, but with the ordinate increasing as x increases through OI; also one with a vertical tangent as at S or T, but (*i*) with the ordinate decreasing as x increases through OG and (*ii*) increasing as x increasing, passes through OH. At such a point called *a point of inflection*, there is neither a maximum nor a minimum.

At a "*point saillant*" as at V, the critical value of x is found by inspection of $f'(x)$, since $f'(x)$ must change sign there. Sometimes the turning values can be found from a consideration of $f(x)$ alone. Thus $y = \sin x$ has maximum values 1 at $x = \pi\ 2$, $2\pi + \pi/2$, etc., and minimum values at $3\pi/2$, $2\pi + 3\pi/2$, etc.

Examples.

1. $f(x) = 2ax - x^2 \therefore f'(x) = 2(a - x)$.
$f'(x) = 0$ at $x = a$.
$f'(a - h) = 2h, f'(a + h) = -2h$.

$\therefore$ By the rule, $f(x)$ is a maximum at $x = a$, since $f'(a - h)$ is $+$ and $f'(a + h)$ is $-$.

As only the signs are needed, they alone will generally be given in what follows.

The maximum value of $f(x)$ is $f(a) = a^2$

2. $f(x) = b + (x - a)^5 \therefore f'(x) = 5(x - a)^4 = 0$ for $x = a$.
$f'(a - h)$ and $f'(a + h)$ are both $+$ $\therefore$ there is no turning value.

In any case when $f'(x)$ is an even power of $(x - a)$ it cannot change sign and hence there can be no turning value.

3. $f(x) = b + (x - 3)^{\frac{4}{3}}$. Since $(x - 3)^{\frac{4}{3}}$ is always $+$, the minimum value is evidently b, corresponding to $x = 3$.

Prove this also by the method used in Ex. 1. The shape of the curve about $x = 3$ is shown at M, Fig. 46.

4. $f(x) = b + (x - 3)^{\frac{2}{3}} \therefore f'(x) = \frac{2}{3}\frac{1}{(x - 3)^{\frac{1}{3}}}$.

Thus

$f'(x) \doteq \infty$ as $x \doteq 3$

$f'(3-h)$ is $-$, $f'(3+h)$ is $+$ $\therefore f(x)$ is a minimum at $x = 3$. This is evident too from the form of $f(x)$; for since the last term is always $+$, min. $f(x) = b$.

The curve about S, Fig. 46, represents $y = f(x)$ in the vicinity of $x = 3$.

5. $f(x) = b + (x-3)^{\frac{1}{3}}$ has a point of inflection with a vertical tangent at $x = 3$.

$f'(x)$ being always $+$, y is always increasing.

Sketch the curve.

6. $f(x) = (x-1)^2(x+2)$.

$\therefore f'(x) = 2(x-1)(x+2) + (x-1)^2 = 3(x-1)(x+1)$.

The critical values are 1 and -1.

$f'(1-h) = + - + = -, f'(1+h) = + + + = +$

$f'(-1-h) = + - - = +, f'(-1+h) = + - + = -$

$\therefore x = 1$ gives a min., $x = -1$, a max. value.

$f(1) = 0, f(-1) = 4$.

7. $f(x) = (x-2)(x-3)(x-4)$. Putting $f'(x) = 0$, we find, $x = 3 \pm \frac{1}{\sqrt{3}}$ $\therefore f'(x) = 3\left(x - 3 - \frac{1}{\sqrt{3}}\right)\left(x - 3 + \frac{1}{\sqrt{3}}\right)$.

$\therefore$ when $a = 3 + \frac{1}{\sqrt{3}}$, $f'(a-h)$ is $-$, $f'(a+h)$ is $+$

$\therefore x = 3 + \frac{1}{\sqrt{3}}$ corresponds to a minimum.

When $a = 3 - \frac{1}{\sqrt{3}}$, $f'(a-h)$ is $+$, $f'(a+h)$ is $-$

$\therefore x = 3 - \frac{1}{\sqrt{3}}$ gives a maximum.

8. $f(x) = \pm\sqrt{(x-2)(x-3)(x-4)}$.

The present function is the square root of the function of Ex. 7.

$$f'(x) = \frac{\pm 3\left(x-3-\frac{1}{\sqrt{3}}\right)\left(x-3+\frac{1}{\sqrt{3}}\right)}{2\sqrt{(x-2)(x-3)(x-4)}}.$$

As before, taking the upper sign, $x = 3 - \frac{1}{\sqrt{3}}$ is found to give a max. and using the lower sign a min. value. The value $x = 3 + \frac{1}{\sqrt{3}}$ can no longer be used, as it makes the denominator imaginary.

In fact, the locus $f(x)$ does not exist for $x < 2$ or when $3 < x < 4$ as then $f(x)$ is imaginary. Roughly sketch the curve, noting that it is symmetrical about the x-axis, forms an oval from $x = 2$ to $x = 3$ and is real and continuous when $x = 4$ or > 4.

The branch above the x-axis, corresponding to the + sign of the radical, must be considered separately from the part below the x-axis, in finding max. or min. values.

9. $f(x) = (x-2)^2(x+1)^3 \therefore f'(x) = (x-2)(x+1)^2(5x-4)$.

$x = 2$ gives a min., $x = 4/5$ a max.,

$x = -1$ gives neither a max. nor a min.

10. If, $$f(x) = \frac{x^2 - 7x + 6}{x - 10},$$

prove that $x = 4$ gives a maximum, $x = 16$ a minimum.

75. Maxima and Minima continued. Second Method. By aid of Taylor's formula, the criteria for ascertaining maxima or minima values, are easily established independently of previous considerations.

From (2), Art. 73, we have,

$$k_1 = f(a+h) - f(a) = hf'(a) + \frac{h^2}{2}f''(a+\theta h) \quad . \; . \; . \; (1)$$

$$k_2 = f(a-h) - f(a) = -hf'(a) + \frac{h^2}{2}f''(a-\theta_1 h) \quad . \; . \; . \; (2)$$

In these equations, $f(x)$, $f'(x)$ and $f''(x)$ are all supposed continuous (and therefore finite) when x is between $(a-h)$ and $(a+h)$ in value and $0 < \theta < 1$, $0 < \theta_1 < 1$. The right member of (1) can be written,

$$h\left[f'(a) + \frac{h}{2}f''(a+\theta h)\right];$$

from which it is seen that h can be made so small that $\frac{h}{2}$ $f''(a+\theta h)$ can be made numerically less than $f'(a)$ and consequently the sign of the right member of (1) will be the same as that of $hf'(a)$. Similarly the sign of the right member of (2) can be made the same as that of $-hf'(a)$ when h is sufficiently diminished. Hence since h is a positive number, $k_1 = f(a+h) - f(a)$ will have the same sign

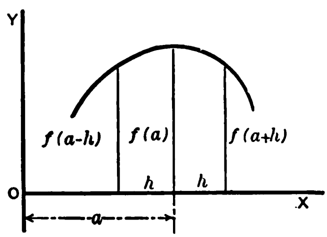

Fig. 47.

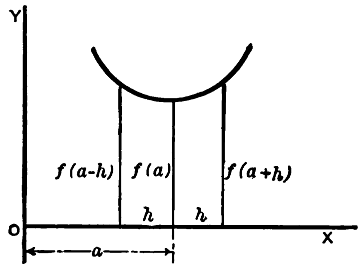

Fig. 48.

as $f'(a)$, but $k_2 = f(a-h) - f(a)$ will have a different sign from $f'(a)$. But by reference to Fig. 47, it is seen that for a maximum, k_1 and k_2 must both be negative, and from Fig. 48, that for a minimum, both k_1 and k_2 must be positive; *i.e.*, in either case, k_1 and k_2 must both have the same sign. Hence we cannot have a maximum or a minimum unless $f'(a) = 0$.

Suppose this necessary condition fulfilled; then from (1) and (2) the signs of both k_1 and k_2 will be the same as that of $f''(a)$ when h is made small enough, since $f''(a + \theta h) \doteq f''(a)$ as $h \doteq 0$.

Hence if $f''(a)$ is negative, $f(a)$ is a maximum; if $f''(a)$ is positive, $f(a)$ is a minimum.

Thus in Ex. 1, Art. 74, $f''(x) = -2 \therefore f''(a) = -2$ and $f(x) = 2ax - x^2$ is a maximum at $x = a$.

In Ex. 6, $f''(x) = 6x$. This is $+$ for $x = 1 \therefore f(1)$ is a min. but $f''(x)$ is $-$ for $x = -1 \therefore f''(-1)$ is a maximum, as found before.

The present method is inapplicable to Ex. 3 of Art. 74, since $f''(x)$ is discontinuous at the critical value.

When $f''(a) = 0$, we revert to Taylor's formula again and carry the expansion a little farther. Thus let $n = 4$ in (9), Art. 69, and put $x = a$; also write the corresponding series when h is changed to $-h$ and omit the terms in $f'(a)$, $f''(a)$ which are now supposed to be zero.

$$k_1 = f(a + h) - f(a) = \frac{h^3}{\underline{|3}} f'''(a) + \frac{h^4}{\underline{|4}} f^{\text{iv}}(a + \theta h) \quad . . . (3)$$

$$k_2 = f(a - h) - f(a) = -\frac{h^3}{\underline{|3}} f'''(a) + \frac{h^4}{\underline{|4}} f^{\text{iv}}(a - \theta h) \quad . . . (4)$$

Reasoning as before, we can make h so small that the signs of the right members will be the same as that of their first terms. Hence, if $f'''(a)$ is not zero, there can be no maximum nor minimum, since then k_1 and k_2 will have unlike signs. If $f'''(a)$ is zero, both k_1 and k_2 will have the same sign as $f^{iv}(a)$; hence as before, $f(a)$ is a maximum when $f^{iv}(a)$ is negative, a minimum when $f^{iv}(a)$ is positive.

When $f^{iv}(a)$ is zero, we extend the series two more terms and reason as before.

It is now plain that the following general conclusions hold in testing $f(x)$ for maximum and minimum values.

Let a be a critical value or root of $f'(x) = 0$; substitute it for x in $f''(x)$, $f'''(x)$, etc. If the first derivative that does not vanish is of an even order, $f(a)$ is a maximum or a minimum of $f(x)$ according as this derivative is negative or positive. If, however, the first derivative that does not vanish is of an odd order, $f(a)$ is neither a maximum nor a minimum of $f(x)$.

Examples.

1. In Ex. 2, Art. 74, $f(x) = b + (x - a)^5 \therefore f^{v}(x) = 120$. The first four derivatives vanish for the critical value $x = a$ obtained from $f'(x) = 0$. Hence as the last derivative that does not vanish is of an odd order, there is no turning value.

2. $f(x) = x^3 - 3x^2 - 9x \quad \therefore f'(x) = 3x^2 - 6x - 9 = 0$, giving the critical values 3 and -1.

$f''(x) = 6x - 6 \quad \therefore f''(3) = 18$, is positive $\therefore$
$f(3)$ is a min.; $f''(-1) = -12$, is negative $\therefore$
$f(-1)$ is a maximum.

3. $f(x) = 3x^4 - 28x^3 + 84x^2 - 96x$,
$f'(x) = 12(x-1)(x-2)(x-4)$,
$f''(x) = 12(3x^2 - 14x + 14)$.
$f'(x) = 0$ gives $x = 1, 2, 4$.
$f''(1) = 36$, $f''(2) = -24$, $f''(4) = 72$.
$\therefore f(2)$ is a maximum, $f(1)$ and $f(4)$ minima.

4. $f(x) = 3x^4 - 8x^3 + 6x^2 \therefore f'(x) = 12x(x-1)^2$.

Prove that at $x = 0, f(x)$ is a max., but at $x = 1, f(x)$ is neither a max. nor a min.

5. $y = \frac{b}{a}\sqrt{2ax - x^2}$. Show that $x = a$ gives a max.

If the minus sign before the radical is used, $x = a$ gives a minimum.

6. Given, $f'(x) = x(x+3)^2(x-1)^3$.

Here the second method is laborious; the first easily shows that $f(0)$ is a max., $f(1)$ is a min., whereas $x = -3$ gives no turning value.

7. $\frac{ax}{a^2 + x^2}$ is a max. for $x = a$, a min. for $x = -a$.

8. $\sec x/\tan x$ is a min. for $x = \frac{1}{2}\pi$.

9. Show that $1/e$ is the max. value of $(\log x)/x$.

10. Find the max. and min. values of,

$$\frac{x^2 - 7x + 6}{x - 10}, \qquad 12x - 9x^2 + 2x^3.$$

11. x^x is a max. at $x = 1/e$; $\frac{\log x}{x}$ is a max. at $x = e$; xe^x is a min. at $x = -1$.

12. $x + a^2x^{-1}$ is a max. at $x = a$, min. at $x = -a$.

13. $\frac{2x}{1 + x^2}$ is a max. at $x = 1$, min. at $x = -1$.

14. $-x + 2 - \frac{2}{1 + x}$ is a max. at $-1 + \sqrt{2}$, a min. at $-1 - \sqrt{2}$.

15. The max. value of $(\sin x + \cos x)$ is $\sqrt{2}$.

16. $f(x) = 12x^5 - 45x^4 + 40x^3 + 6$.
$f(1)$ a max.; $f(2)$ a min.; $f(0)$ neither max. nor min.

Problems in Maxima and Minima.

76. Certain kinds of problems can be solved very simply by algebraic or trigonometrical methods. See Chrystal's Algebra, vol. 2, chap. xxiv. Space can only be given here to the following.

From the identities,

$$(1)\quad xy = \tfrac{1}{4}\left[(x+y)^2 - (x-y)^2\right],$$
$$(2)\quad (x+y)^2 = 4xy + (x-y)^2,$$
$$(3)\quad x^2 + y^2 = \tfrac{1}{2}\left[(x+y)^2 + (x-y)^2\right],$$

we see, when $(x + y)$ is a constant or when the sum of two numbers is given, that by (1), their product $(x \cdot y)$ is greatest when they are equal and also by (3) the sum of their squares is least when they are equal. Thus when $(x + y) = 10$, xy is a maximum when $x = y = 5$; also $x^2 + y^2$ is a minimum for $x = y = 5$.

From (2) we conclude that if xy, or the product of two numbers, is constant, their sum is least when they are equal. Thus if $xy = 25$, $x + y$ is least when $x = y = 5$. Let the student test these results by assuming $x = 1, 2, 3, \ldots$ in turn and computing y from the conditions given.

In Geometry, Physics, Engineering, etc., the determination of maximum or minimum values is often of the highest practical importance. Hundreds of problems covering a wide range could be given, but we have space for only a few.

In the solution of problems, we try first to obtain an expression in terms of *one variable* and constants, for the function whose maximum or minimum values are desired, though sometimes the use of two variables is to be preferred. Again, the nature of the problem will often show, without forming $f''(x)$, whether a maximum or minimum value has been attained.

1. A wall whose width is b and height h, incloses a rectangular piece of ground of given area a, the length of the sides of the rectangle being x and y. What is the ratio of x to y when the cubic contents of the wall is a minimum? $xy = a$; cubic contents $= [2y + 2(x + 2b)]bh$.

Substituting $y = a/x$ in the last result, we see that,

$$f(x) = 2bh\left[\frac{a}{x} + x + 2b\right] \text{ is to be a min.}$$

$$f'(x) = 0 \text{ gives } x = \sqrt{a} \therefore y = \sqrt{a}.$$

Hence $x = y$, or the rectangle is a square. $f''(x) = 2a/x^3$, hence a min.; otherwise the nature of the problem indicates a min, since from the first two eqs., as $y \doteq 0$, $x \doteq \infty$ and as $x \doteq 0$, $y \doteq \infty$, $\therefore f(x) \doteq \infty$, or the function increases as either $x \doteq 0$ or $y \doteq 0$, the condition $xy = a$, being always satisfied; hence $f(\sqrt{a})$ cannot be the greatest value $\therefore$ it is the least. When a function has but one turning value, it is necessarily either the *greatest* or the *least* value of the function.

2. A person in a boat at D, 3 miles from the shore at B, wishes to reach P, 5 miles from B in the shortest time. He can row 4 miles and walk 5 miles per hour. At what point of the shore BP must he land? Call E the point.

$$\therefore \frac{DE}{4} + \frac{PE}{5} = \text{total hours, to be a min.}$$

$$\text{If } x = BE, f(x) = \frac{\sqrt{9 + x^2}}{4} + \frac{5 - x}{5} \text{ to be a min.}$$

Ans. He must land 1 mile from P.

3. To find the shortest straight line AB (Fig. 49) that can be drawn through a given point P (a, b), within the right angle XOY and limited by the axes OX, OY. Let $PAO = BPC = \theta$.

From the figure, we have, by definition of *sec* θ and *cosec* θ,

$$\sec\theta = \frac{BP}{a}, \quad \operatorname{cosec}\theta = \frac{AP}{b},$$

$$\therefore \; BP + PA = a\sec\theta + b\operatorname{cosec}\theta = f(\theta),$$

to be a min.

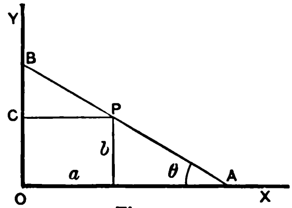

Fig. 49.

Putting $f'(\theta) = 0 \; \therefore \; \tan\theta = \sqrt[3]{b/a}$.

This value evidently gives a min., for as $\theta \doteq 0$, $AB \doteq \infty$; also as $\theta \doteq 90°$, $AB \doteq \infty$; hence there are greater values for AB than the one that corresponds to the above value of tan θ, which is consequently the least, since it is either the greatest or the least and we have just seen that it cannot be the greatest.

4. The volume V of a circular cylindrical vessel with open top is constant; what is the ratio of its altitude y to the radius of its base x when its inner surface is a minimum?

$$V = \pi x^2 y \quad \therefore \; f(x) = 2\pi xy + \pi x^2 = \frac{2V}{x} + \pi x^2.$$

Ans. The altitude = the radius of the base.

5. A Norman window consists of a rectangle (whose width is $2x$ and height y) surmounted by a semicircle (of radius x). Given the perimeter p, required the height and the breadth of the window so that the quantity of light admitted should be a maximum.

The manner of procedure when two variables are used will be sufficiently indicated by the following. Call u = area,

$$\therefore \; u = \tfrac{1}{2}\pi x^2 + 2xy; \; \frac{du}{dx} = \pi x + 2y + 2x\frac{dy}{dx}.$$

Also, $\pi x + 2x + 2y = p \therefore \pi + 2 + 2\dfrac{dy}{dx} = 0.$

Substitute $\dfrac{dy}{dx} = -\dfrac{\pi}{2} - 1$ derived from this equation in the value of $\dfrac{du}{dx}$, which then place equal to zero.

$$\frac{du}{dx} = \pi x + 2y + 2x\left(-\frac{\pi}{2} - 1\right) = 0 \therefore x = y.$$

This relation evidently corresponds to a maximum u since as $x \doteq 0, p \doteq 2y$ and $u \doteq 0$ from the equations above. Complete the reasoning. It is very easy, however, to find the second derivative. Its sign indicates a maximum,

$$\frac{d^2u}{dx^2} = \pi + 2\frac{dy}{dx} + 2\left(-\frac{\pi}{2} - 1\right) = \pi + 4\left(-\frac{\pi}{2} - 1\right) = -\pi - 4.$$

6. Work Ex. 4 after the method of Ex. 5 (not eliminating y).

7. If the volume of a circular cylindrical vessel, closed top and bottom, is given, prove that the entire inner surface is a minimum when the diameter is equal to its height.

8. The cross-section of a canal is given in Fig. 50, the sides of length a, being each inclined at an angle θ with the horizontal. The length of base $= b$, altitude $= h$. If the area of the cross-section A is given, when is the wetted perimeter a minimum?

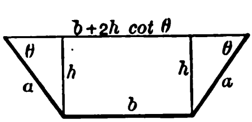

Fig. 50.

We see from the figure that, $h = a \sin \theta$, and

$$A = (b + h \cot \theta)\, h \therefore b = \frac{A}{h} - h \cot \theta$$

$$2a + b = \frac{2h}{\sin \theta} + \frac{A}{h} - h \cot \theta.$$

(1) *Suppose h constant, θ variable* and $f(\theta) = 2a + b$.

$$\therefore f'(\theta) = \frac{-2h \cos \theta}{\sin^2 \theta} + \frac{h}{\sin^2 \theta} = 0,$$

giving for a minimum $\cos\theta = \frac{1}{2}$, $\theta = \frac{\pi}{3}$. This gives the least value of $(2a + b)$ and not the greatest, since as $\theta \doteq 0$, $(2a + b)$ increases. Otherwise $f''\left(\frac{\pi}{3}\right) = 2h / \sin(\pi/3) =$ a positive number.

(2) Suppose θ constant, h variable, $f(h) = 2a + b$

$$f'(h) = -\frac{A}{h^2} - \cot\theta + \frac{2}{\sin\theta},$$

$$f''(h) = +\frac{A}{h^3}.$$

$$f'(h) = 0, \text{ gives } h = \sqrt{\frac{A\sin\theta}{2 - \cos\theta}},$$

corresponding to a minimum.

These conclusions correspond to a maximum flow of water, since the *resistance* to flow, along the wetted perimeter, is least when the perimeter is least.

9. Find the maximum rectangle that can be inscribed in an ellipse $x^2/a^2 + y^2/b^2 = 1$.

The sides are $a\sqrt{2}$ and $b\sqrt{2}$.

10. Prove that the altitude of the greatest cylinder that can be inscribed in a right cone is one-third that of the cone.

11. Find the greatest cone that can be inscribed in a sphere.

Ans. The altitude $= \frac{4}{3}$ the radius of the sphere.

12. Let the cost per hour of driving a steamer against a current of v miles per hour be, $a + bx^3$, where x is the velocity of the steamer, relatively to the water. In going a distance of c miles, find x for a minimum cost.

The ship's rate, relatively to the bank of the river, is $x - v$, the time in going the distance c is $\frac{c}{x - v}$, hence the cost is,

$$f(x) = \frac{c(a + bx^3)}{x - v}.$$

$$f(x) \text{ is a minimum when } 2\,bx^3 - 3\,bvx^2 = a.$$

If $a = 0$, $x = \frac{3}{2}v$.

13. Perry, in his "Calculus for Engineers," gives the following: Voltaic cell of $E.M.F. = e$ and internal resistance r; external resistance R. The current is $C = e/(r + R)$. The power given out is $P = RC^2$. What value of (the variable) R will make P a maximum?

$$P = \frac{e^2R}{(r+R)^2} \text{ a max. or } \frac{(r+R)^2}{e^2R} \text{ a min.}$$

Ans. $R = r$.

14. There are two point sources of heat at A and B respectively, their intensities at unit distance being a and b. From what point P on the straight line AB (of length c) is the amount of heat received the least, the intensity of heat radiation varying inversely as the square of the distance?

Let $AP = x \therefore PB = c - x \therefore$ the total intensity of heat at P is,

$$f(x) = \frac{a}{x^2} + \frac{b}{(c-x)^2}.$$

This is a minimum at

$$x = \frac{c\sqrt[3]{a}}{\sqrt[3]{a} + \sqrt[3]{b}}.$$

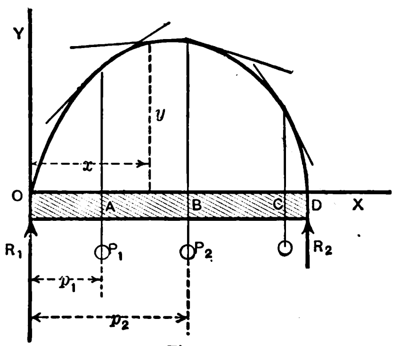

Fig. 51.

15. In Fig. 51, OD represents a horizontal beam, supported at the ends, loaded with a uniform load w lbs. per foot over the whole span, also with a load P_1 lbs. at A, P_2 lbs. at B, etc. Let $OA =$

p_1, $OB = p_2$, $OC = p_3$ and call the left reaction R_1. Then the bending moment at a point x feet to right of O is,

from $x = 0$ to $x = p_1$, $y = R_1 x - \frac{1}{2} wx^2$;
$x = p_1$ to $x = p_2$, $y = R_1 x - \frac{1}{2} wx^2 - P_1 (x - p_1)$;
$x = p_2$ to $x = p_3$, $y = R_1 x - \frac{1}{2} wx^2 - P_1 (x - p_1) - P_2 (x - p_2)$.

If we represent these moments y by a curve (Fig. 51) it will be continuous, but the slope will be discontinuous for $x = OA$, OB, etc.

Thus the slopes of the three curves are,

$$\begin{aligned} x &= 0 \text{ to } x = p_1, & y' &= R_1 - wx, \\ x &= p_1 \text{ to } x = p_2, & y' &= R_1 - wx - P_1, \\ x &= p_2 \text{ to } x = p_3, & y' &= R_1 - wx - P_1 - P_2, \end{aligned}$$

from which it is seen that the slopes of the first two curves for $x = OA$, are not the same; neither are the slopes of the second and third curves the same at $x = OB$ (a common point). Yet although there may be no horizontal tangent anywhere the maximum y is easily seen to be at the point where y' to the left of the point is positive and to the right negative; *i.e., it is where y' (or the shear) changes sign.* If we should put $y' = 0$, say for the second curve, we tacitly assume that x can take values beyond the limits p_1 and p_2 (contrary to the hypothesis), and it can very well happen that we find from $y' = 0$, $x > p_2$ or even $x > OD$, indicating a turning value for *this* curve *extended* beyond the limits imposed. If, however, $p_1 < x < p_2$, the critical value x corresponds to a maximum moment.

77. Concavity. Points of Inflection. When the slope of a portion of a curve IQE, Fig. 52, increases algebraically in going to the right, from I to E, it is said to be *concave upwards*. The tangent line at any point of IQE

turns to the left (counter-clockwise) as the point of contact moves to the right; hence any point of the curve *IQE* lies *above* (measured along an ordinate) the tangent at any other point of it, which property may be taken as the geometric definition for a curve that is *concave upwards*.

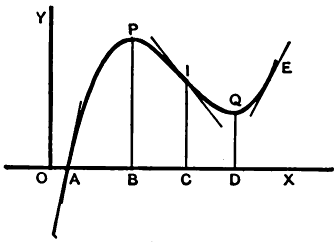

Fig. 52.

When the slope of a portion of a curve *API*, Fig. 52, decreases algebraically in going to the right, from *A* to *I*, it is said to be *concave downwards*. The tangent line at a point of *API*, turns to the right (clockwise) as the point of contact moves to the right; hence any point of the curve *API* lies *below* (measured along an ordinate) the tangent at any other point of it, which geometric property thus defines a curve that is *concave downwards*.

If the equation of the curve is $y = f(x)$, the slope $f'(x)$ increases (algebraically) as x increases (algebraically) when the curve, as *IQE*, is concave upwards; therefore by Art. 32, its derivative $f''(x)$ is positive.

When the curve as *API* is concave downwards, $f'(x)$ decreases (algebraically) as x increases (algebraically); therefore $f''(x)$ is negative.

Where the concavity changes from upwards to downwards (as x increases) or the reverse, as at I, the tangent necessarily crosses the curve, and the point I is called a *point of inflection*. On opposite sides of I, $f''(x)$ has opposite signs; hence at I, $f'(x)$ has a turning value (Art. 74).

Since $f'(x)$ is a function of x, call it $\phi(x)$; *i.e.*, let $f'(x) = \phi(x)$. Then to ascertain if a curve has a point of inflection, we apply the methods of Art. 74 or Art. 75 to test $\phi(x)$ for maximum or minimum values. If $\phi(x) = f'(x)$ has no turning values, the curve $y = f(x)$ has no points of inflection.

EXAMPLE I. $f(x) = x^3 - 7x^2 + 16x - 10$

$$\phi(x) = f'(x) = 3x^2 - 14x + 16 = 3(x - 6/3)(x - 8/3)$$

$$\phi'(x) = f''(x) = 6x - 14; \quad \phi''(x) = f'''(x) = 6.$$

When x is $< 7/3$, $f''(x)$ is $-$ and the curve is concave downwards; when x is $> 7/3$, $f''(x)$ is $+$ and the curve is concave upwards $\therefore x = 7/3$ is a point of inflection. It follows too from Art. 74 that $\phi(x)$ is a min. at $x = 7/3$, since $\phi'(x)$ changes from $-$ to $+$, as x increasing, passes through the value $7/3$. Otherwise by the method of Art. 75, since $\phi''(x) = 6$ is $+$, $\phi(x)$ is a minimum.

NOTE. The general shape of a curve can be determined very quickly by finding the points where the tangent is parallel to the x-axis and the points of inflection. Thus in the last example $f'(x) = 0$ at $x = 2$ and $x = 8/3$, $f(2) = +2$, $f(8/3) = +50/27$. For the point of inflection, $x = 7/3 = \frac{1}{2}(2 + \frac{8}{3})$, or the mean of the abscissas of the points where the tangent is parallel to the x-axis. (This is true for any cubic.) Hence we conclude that the curve has the shape of Fig. 52 and that it can only cross the x-axis once; consequently there is only one real root of $f(x) = 0$. By trial, we can soon find two values of x differing by, say, 0, 1, for one of which $f(x)$ is $+$, for the other $-$; then by Newton's or Horner's method (given in Algebra) an approximate value of the real root of $f(x) = 0$ can be found. See Gibson's Calculus, p. 244, for a full discussion of Newton's method.

If the curve is such that the x-axis touches the curve at Q, there will be three real roots of $f(x) = 0$, two being equal to

$x = OD$. If the x-axis cuts $PIQE$, there will be three real roots. By aid of the Calculus we thus have an independent method of ascertaining the number of real roots of a rational integral function of x.

EXAMPLE 2. $f(x) = x^3 - 5x^2 + 9x - 5$.

$$f'(x) = 3x^2 - 10x + 9 = 0 \text{ at } x = \tfrac{1}{3}(5 \pm \sqrt{-2});$$

hence this curve has no tangent parallel to the x-axis anywhere.

$$f''(x) = 6x - 10.$$

To the left of $x = 5/3$, $f''(x)$ is $-$ $\therefore$ the curve is concave downward; to the right of $x = 5/3$, $f''(x)$ is $+$ $\therefore$ the curve is concave upward. The point of inflection is at $x = 5/3$ for which $f(x) = f(5/3) = 20/27$ $\therefore f(x) = 0$ has only one real root at x a little less than $5/3$.

The slope at the point of inflection $(\frac{5}{3}, \frac{20}{27})$ is $f'(\frac{5}{3}) = \frac{2}{3}$, which is an aid in drawing the curve (see Art. 11 on constructing a line passing through a given point and having a given slope).

Examples.

Find the points (if any) where the tangent is parallel to the x-axis, the points of inflection (if any) and the slope there, the concavity from $-\infty$ to $+\infty$ and the points (or near the points) where the curve cuts the axes, and from these data, roughly sketch the curve.

1. $fx = 3x^3 - 4x + 5$.

$f(-\frac{2}{3}) = \max \cdot f(x)$; $f(\frac{2}{3}) = \min.$; $(0, 5) =$ pt. inflection, etc.

$$f(x) = \frac{x^4}{24} - \frac{x^3}{3} - 3x^2 + 1 = 0.$$

$f'(x) = 0$ at $x = 0$, $x = 3 \pm \sqrt{45} = -3.7+, 9.7+$.

$f''(x) = \frac{1}{2}(x+2)(x-6)$, $+$ for $x < -2$, $-$ for $-2 < x < 6$, $+$ for $x > 6$.

3. $a^2y = x^3$ (cubical parabola).

4. $y - 2 = (x + 1)^3$.

5. $y = x^{2n+1}, \quad y = x^{2n}$.

6. $y = \dfrac{2x}{1 + x^2}, \quad \dfrac{x^2}{1 + x^2}$.

7. $y = \sin x, \quad y = \cos x, \quad y = \tan x$.

8. $y^2 = 2Ax + Bx^2$ (conic section, Art. 20) has no points of inflection.

Thus:

$$2y\frac{dy}{dx} = 2A + 2Bx \quad \therefore f'(x) = \frac{dy}{dx} = \frac{A + Bx}{y};$$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{1}{y^2}\left(yB - (A + Bx)\frac{dy}{dx}\right) = \frac{By^2 - (A + Bx)^2}{y^3};$$

which reduces to, $f''(x) = -A^2/y^3$ $\therefore$ when y is $+$, the conic is concave downwards; when y is $-$, the conic is concave upwards. There are consequently no points of inflection on either portion. The origin is not a point of inflection, since the vertical tangent does not cross the curve there.

9. $(y - c)^2 = x^3$ (semi-cubical parabola).

10. In Fig. 46 it is observed that maxima and minima values occur alternately. Why? There are points of inflection between M and Q, S and T, T and V. Why? Why is there no point of inflection between Q and S?

11. Find the points of inflection of,

$$y = e^{-x^2}; \qquad\qquad x(x^2 - ay) = a^3.$$

Ans. $x = \pm\dfrac{1}{\sqrt{2}}$; $y = 0$.

CHAPTER X.

DIFFERENTIALS. AREAS. VOLUMES.

78. Differentials. Heretofore we have regarded $\frac{dy}{dx}$ as an indivisible symbol, like its equivalents, $D_x y$, $f'(x)$, y'; but after this that restriction will be removed and $\frac{dy}{dx}$ will be looked upon likewise as a fraction with a finite numerator (not zero) and finite denominator (not zero) whose value is always exactly equal to $f'(x)$ or $\lim\limits_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x}$.

$$\therefore\ dy = f'(x)dx \quad . \quad . \quad . \quad (1)$$

dy and *dx* are now called *differentials.*

In Fig. 53, if $y = f(x)$ is the equation of a curve PQ, $OM = x$, $MP = y$, draw a tangent at P to the curve making the angle θ with the x-axis and cutting the ordinate NQ at T; also draw PR parallel to OX and cutting NQ at R. Then *if* $PR = MN$ is put equal to dx, RT will be dy, since their ratio dy/dx is exactly equal to $\tan\theta = f'(x)$. Also, since,

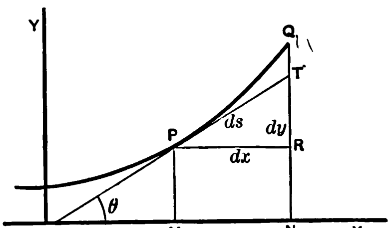

Fig. 53.

$$\frac{dy}{dx} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = f'(x);$$

from the nature of a limit, $\Delta y/\Delta x$ will not equal $f'(x)$ (except for the linear function $y = ax + b$) but will differ from it by an infinitesimal α (which may be positive or negative) whose limit, as $\Delta x \doteq 0$, is consequently zero.

$$\therefore \frac{\Delta y}{\Delta x} = f'(x) + \alpha.$$

On taking limits we reach the preceding identity. We have consequently,

$$\Delta y = [f'(x) + \alpha] \Delta x \quad \cdots \quad (2)$$

Therefore in Fig. 53, *if* we let $MN = \Delta x = dx$, then $RQ = \Delta y = [f'(x) + \alpha] \Delta x$. On comparing this with $dy = f'(x)\, dx = RT$, we note that,

$$TQ = \Delta y - dy = \alpha \Delta x \quad \cdots \quad (3)$$

Again, we found in Art. 51,

$$\cos\theta = \frac{dx}{ds}, \quad \sin\theta = \frac{dy}{ds};$$

consequently, if $PR = dx$, whence $RT = dy$, it follows that $PT = ds$.

It is immaterial what length we assume for dx, so it be finite (and not zero); but having assumed a certain length $MN = dx$, we can then only have $RT = dy$, $PT = ds$.

The triangle PRT may be called the differential triangle. Its sides may sometimes be regarded as fixed in length; at other times, infinitesimals according to the nature of the investigation.

From the figure,

$$ds^2 = dx^2 + dy^2 \quad \cdots \quad (4)$$

Treat now, dx, dy, ds, as infinitesimals and let $dx = \Delta x$. As $\Delta x \doteq 0$, the point Q travels down the curve towards P and from (1) and (2),

$$\lim_{\Delta x \doteq 0} \frac{\Delta y}{dy} = \lim_{\Delta x \doteq 0} \frac{(f'(x) + a)\,\Delta x}{f'(x)\,dx} = \lim_{\Delta x \doteq 0} \frac{f'(x) + a}{f'(x)} = 1 \quad \cdots \quad (5)$$

since by hypothesis, *lim* $a = 0$.

Also calling the length of chord PQ, c and the length of arc PQ, Δs, we have from the figure and from (4),

$$\frac{\Delta s}{ds} = \frac{\Delta s}{c} \cdot \frac{c}{ds} = \frac{\Delta s}{c} \cdot \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\sqrt{dx^2 + dy^2}} = \frac{\Delta s}{c} \cdot \frac{\sqrt{1 + (\Delta y/\Delta x)^2}}{\sqrt{1 + (dy/dx)^2}}.$$

By Art. 24, Ex. 6,

$$\lim_{\Delta x \doteq 0} \frac{\Delta s}{c} = 1;$$

hence, taking limits as $\Delta x \doteq 0$, since *lim* $(\Delta y/\Delta x) = dy/dx$

$$\therefore \lim_{\Delta x \doteq 0} \frac{\Delta s}{ds} = 1 \quad \cdots \quad (6)$$

79. The limit of the ratio of two variables is not changed when either is replaced by any other variable, the limit of whose ratio to it is unity.

To prove this fundamental principle, let a, β, be two variables and a', β', two other variables, not equal to the first, but bearing such relations to them that,

$$lim\,\frac{a}{a'} = 1, \quad lim\,\frac{\beta}{\beta'} = 1;$$

consequently the limits of their inverse ratios, α'/α, β'/β, will be 1. We have the identity,

$$\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'} \cdot \frac{\beta'}{\beta} \cdot \frac{\alpha}{\alpha'}.$$

Since the limits of equal quantities are equal and the limit of a product is the product of the limits (Art. 26), we have on taking limits,

$$\lim \frac{\alpha}{\beta} = \lim \frac{\alpha'}{\beta'} \cdot \lim \frac{\beta'}{\beta} \cdot \lim \frac{\alpha}{\alpha'} = \lim \frac{\alpha'}{\beta'},$$

which was to be proved.

Let the student show that α alone can be replaced by α', also that β alone can be replaced by β'.

This theorem is of great utility in mathematics, for it is often difficult or impossible to find an expression for a quantity α, but comparatively easy to find another α', the limit of whose ratio to the first is unity. In any problem concerning the limit of a ratio, α' can then replace α. Thus we have seen in Art. 78 that,

$$\lim \frac{\Delta s}{c} = 1, \quad \lim \frac{\Delta y}{dy} = 1, \quad \lim \frac{\Delta s}{ds} = 1;$$

hence the chord can replace the arc, dy can replace Δy and ds can replace Δs in any problem concerning the limit of a ratio. Thus is rigidly demonstrated the scientific basis for the substitutions above.

The student will come across in reading many old authors and some recent ones, the use of dy and ds as meaning the actual increments Δy and Δs, following the lead of Leibnitz, one of the originators of the Calculus. With the meaning given above to dy and ds, they can never equal Δy and Δs except for linear functions, since they are never supposed to cease to exist or become zero. But it is easy now to see that similar substitutions can always be made when the restrictions of the theorem are attended to. The student must be prepared to do this in reading books that follow the Leibnitz school.

It is not proposed to enter further into the theory of infinitesimals in this book, particularly as it is a little confusing until the method of limits is firmly grasped. Then a course in differentials and infinitesimal theory is desirable, and it will be found to be, when logically presented, a corollary to and founded on the method of limits. In the Integral Calculus, the use of differentials is of great practical convenience.

80. Derivative of an Area. The area considered is $AMPE$, Fig. 54, between the fixed ordinate AE, the variable ordinate $y = MP$, the curve $y = f(x)$ and the axis of x. Let $x = OM$, $\Delta x = MN$, $\Delta y = RQ$; also let the number of square units in the area $AMPE = A$ and the number in its increment $MNQP = \Delta A$.

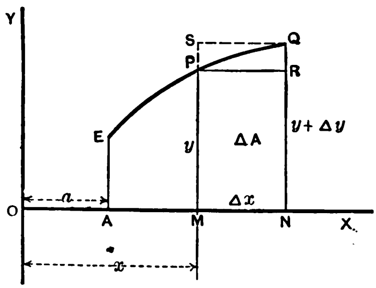

Fig. 54.

The area ΔA, when Δx is small, lies between area rectangle MR and area rectangle MQ or between $y \Delta x$ and $(y + \Delta y) \Delta x$, the limit of whose ratio is,

$$\lim_{\Delta x \doteq 0} \frac{y + \Delta y}{y} = \lim_{\Delta x \doteq 0} \left(1 + \frac{\Delta y}{y}\right) = 1.$$

$$\therefore \text{ a fortiori, } \lim_{\Delta x \doteq 0} \frac{\Delta A}{y \Delta x} = \frac{1}{y} \frac{dA}{dx} = 1.$$

$$\therefore \ \frac{\mathbf{dA}}{\mathbf{dx}} = \mathbf{y} \quad . \ . \ . \ (1)$$

or,
$$\mathbf{dA} = \mathbf{ydx} \ . \ . \ . \ (2)$$

$y = f(x)$ is supposed to be a continuous function between the two extreme ordinates. If it is not single-valued, only one branch of the curve must be taken at a time. On *integrating* or finding the value of A, the sign of y, if always positive, will give a positive area; if always negative, a negative area. If for the range of x considered, y is sometimes positive and sometimes negative, the points where the curve cuts the x-axis must be found and the positive and negative areas separately determined.

It follows from (1) that y is the x rate of change of the area.

In physical investigations, the abscissas and ordinates are frequently made to represent various quantities, as time, velocity, etc. Thus if x is replaced by t and y by v, we have from (1), $dA/dt = v$ (velocity); whence as $ds/dt = v$, Art. 46, we see that the number of square units of area A will equal the number of linear units of distance traversed by a moving point in the time t, since A corresponds to s.

EXAMPLE 1. As an application of (1) to finding an area, let $y = mx$, the equation of a straight line through the origin.

$$\therefore \ \frac{dA}{dx} = mx \ \ \therefore \ A = \frac{mx^2}{2} + C,$$

an arbitrary constant C being always added. The derivative of the right member is mx, as it should be.

If we let $OA = 0$ or count the area from the origin, since $x = OM$ (Fig. 54) is a variable we can suppose it to diminish to zero, $\therefore$ in the last equation put $A = 0$, $x = 0$, $\therefore C = 0$. This determines the constant. Returning to the original equation where $x = OM = b$ (say), the area of the triangle OMP $= \frac{1}{2} mb^2 = \frac{1}{2} b. \ mb = \frac{1}{2} OM \cdot MP = \frac{1}{2}$ base $\times$ altitude.

We can otherwise determine the constant C on the supposition that

$A = 0$ when $x = OA = a \ \therefore 0 = \frac{1}{2} ma^2 + C \ \therefore \ C = -\frac{1}{2} ma^2$.

The area,

$$A = \tfrac{1}{2} mx^2 - \tfrac{1}{2} ma^2 = \tfrac{1}{2}(x - a)(mx + ma),$$

now represents the area of the trapezoid $AMPE$, bounded by the straight line EP, the fixed ordinate AE, the variable ordinate MP and the x-axis. Let $x = OM = b$.

$$\therefore \ A = \tfrac{1}{2}(b - a)(mb + ma) = \tfrac{1}{2}(AE + MP)\,AM,$$

the area as given in Geometry.

The values a and b here are called " end values " of x.

The process above is in reality integration, which will be more fully developed in the following chapters. Let the student observe that,

$$D_x \frac{x^{n+1}}{n+1} = x^n, \ D_x \log x = \frac{1}{x}, \quad D_x(-\cos x) = \sin x, \text{ etc.,}$$

and the finding a function that differentiated will give a stated function, for the simple examples selected, should not prove difficult.

EXAMPLE 2. Parabola. $y^2 = 4px$ (Fig. 19, Art. 21). Find the area between the part of the curve lying above the x-axis, the ordinates at $x = 0$, $x = x$ and the x-axis.

$$\frac{dA}{dx} = y = 2p^{\frac{1}{2}}x^{\frac{1}{2}} \ \therefore \ A = \frac{4}{3}p^{\frac{1}{2}}x^{\frac{3}{2}} + C.$$

$A = 0$ when $x = 0$ $\therefore$ $C = 0$ $\therefore$ $A = \frac{2}{3}x \cdot 2p^{\frac{1}{2}}x^{\frac{1}{2}} = \frac{2}{3}xy$,

or $\frac{2}{3}$ the circumscribing rectangle.

EXAMPLE 3. Exponential Curve. $y = e^x$ (Fig. 11, Art. 16).
$A = e^x + C$. If $A = 0$ when $x = 0$, $C = -1$.

EXAMPLE 4. Hyperbola. $y = \frac{1}{x}$ (Fig. 9, Art. 13).

$$\frac{dA}{dx} = \frac{1}{x} \quad \therefore \ A = \log x + C.$$

If $A = 0$ when $x = 1$, $C = 0$ $\therefore$ $A = \log x$ = Naperian logarithm of end value. From this relation, Naperian logarithms are often called hyperbolic logarithms. Show the area included on a figure.

EXAMPLE 5. Find the areas between the x-axis, the ordinates at $x = 0$, $x = \pi/2$ and the following curves (Fig. 13, Art. 17).

$$y = \sin x, y = \cos x.$$

In each case the answer is 1.

EXAMPLE 6. $y = \sec^2 x$ with end values, $x = 0$, $x = \pi/4$. $A = \tan x + C$, $C = 0$, if $A = 0$, $x = 0$ $\therefore$ required area $= \tan \pi/4 = 1$.

EXAMPLE 7. Parabola. $4y = x^2$ (Fig. 4, Art. 7). (1) Let $A = 0$, for $x = 0$, then the area from the origin to $x = 2$ is 2/3. But (2) if $A = 0$ for $x = 1$, the area from $x = 1$ to $x = 3$ is $2\frac{1}{6}$ square units.

EXAMPLE 8. Cubical Parabola. $y = x^3$, $A = \frac{x^4}{4} + C$.

If $A = 0$ when $x = 0$, $C = 0$. The area from $x = 0$ to $x = 2$ is 4 square units. Show that the area from $x = 1$ to $x = 3$ is 20 square units.

EXAMPLE 9. In Ex. 8, find the area from $x = -2$ to $x = 0$; also from $x = -3$ to $x = -1$. Why is the result negative?

EXAMPLE 10. In Ex. 8, if we find the area from $x = -3$ to $x = +3$, the result is 0. Explain this.

The constant C is determined here by supposing $A = 0$ when $x = -3$ $\therefore C = -81/4$.

EXAMPLE 11. Find the area of the curve,

$$y = x^3 - 7x^2 + 16x - 10 \text{ (Art. 76, Ex. 1, Fig. 52)},$$

for the end values, $x = 1$ (when $y = 0$) and $x = 2$.

EXAMPLE 12. Find A for, $y = x^3 - 5x^2 + 9x - 5$ when the end values are $x = 1$ ($\therefore y = 0$) and $x = 2$. Art. 76, Ex. 2.

81. Derivative of a Volume. Fig. 55 is intended to show a geometrical solid cut by planes perpendicular to OX at A, B and C. The axis of x is not necessarily an axis of symmetry; in fact, it need not meet the solid at all so far as the demonstration below is concerned. The assumption will be made, however, that the area of any cross-section, made by a plane perpendicular to the x-axis, as that at B, is a function of OB; *i.e.*, calling $OB = x$ and area of cross-section at B, X, then X is supposed to be expressed as a function of x.

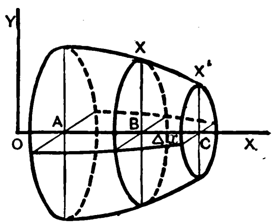

Fig. 55.

Let $BC = \Delta x$, then similarly the area of the cross-section at $C = X'$ is a function of $(x + \Delta x)$. The volume of the solid between a fixed plane at A and the plane at B (both perpendicular to OX) will be called V. Then if $BC = \Delta x$, the volume between the cross-sections at B and C will be ΔV.

Imagine a cylinder constructed on X as a base and another on X' as a base, both of altitude Δx; then when Δx is small enough, it is evident that the value of ΔV is between $X \Delta x$ and $X' \Delta x$, the volumes of the cylinders mentioned.*

But as $\Delta x \doteq 0$, $X' \doteq X$,

$$\therefore \lim_{\Delta x \doteq 0} \frac{X' \Delta x}{X \Delta x} = \lim_{\Delta x \doteq 0} \frac{X'}{X} = 1;$$

a fortiori,
$$\lim_{\Delta x \doteq 0} \frac{\Delta V}{X \Delta x} = \frac{1}{X}\frac{dV}{dx} = 1;$$

$$\therefore \frac{\mathbf{dV}}{\mathbf{dx}} = \mathbf{X} \quad . \; . \; . \quad (1)$$

EXAMPLE 1. To find the volume of a cone, Fig. 56, the area of whose base is A and altitude $OD = a$.

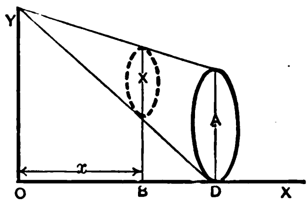

Fig. 56.

Let the vertex be placed anywhere on the y-axis. Then by Geometry,

$$X : A = x^2 : a^2$$

$$\therefore X = Ax^2/a^2.$$

$$\therefore \text{ By (1),} \qquad \frac{dV}{dx} = \frac{Ax^2}{a^2} \quad \therefore V = \frac{Ax^3}{3a^2} + C.$$

If when $x = 0$, $V = 0$, then $C = 0$ and the last equation gives the volume of a cone of altitude x, having X for a base. Now let $x = OD = a$ and we have for the volume of a cone with base A square units and altitude a linear units,

$$V = \tfrac{1}{3} Aa, \text{ or } \tfrac{1}{3} \text{ the base by the altitude.}$$

Precisely the same investigation and result hold for a pyramid.

* If for some abnormal surface this is not true, imagine cylinders (perhaps oblique) of altitude Δx and bases A and A', inscribed and circumscribed about ΔV $\therefore A \Delta x < \Delta V < A' \Delta x$. When X is a continuous function of x, then as $\Delta x \doteq 0$ $A \doteq X$ and $A' \doteq X$ $\therefore$ *lim* $A \Delta x / A' \Delta x = 1$ $\therefore$ *lim* $\frac{\Delta V}{A \Delta x} = \frac{1}{X}\frac{dV}{dx} = 1$; or, $\frac{dV}{dx} = X$.

EXAMPLE 2. If X is given $= \pi \frac{c}{a}(a^2 - x^2)$, prove that between the end values $x = 0$, $x = a$, $V = \frac{2}{3}\pi c a^2$.

82. Solids of Revolution. If Fig. 55 is supposed to represent a solid of revolution about the x-axis, each cross-section will be a circle with center on the x-axis. If $y = f(x)$ is the equation of the generating curve, whose revolution about OX generates the curved surface of the solid, the area X = area of a circle with radius y corresponding to $x = OB$ ∴ $X = \pi y^2$ and (1) of the last article becomes,

$$\frac{dV}{dx} = \pi y^2 \quad . \; . \; . \quad (2)$$

If x and y are interchanged in the figure, we find for a solid of revolution about OY,

$$\frac{dV}{dy} = \pi x^2 \quad . \; . \; . \quad (3)$$

EXAMPLE 1. If an ellipse, $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, revolves about its major axis, it generates a *prolate spheroid.* Here, by (2),

$$\frac{dV}{dx} = \pi \frac{b^2}{a^2}(a^2 - x^2) \quad \therefore \quad V = \pi \frac{b^2}{a^2}\left(a^2 x - \frac{x^3}{3}\right) + C.$$

Suppose $V = 0$ when $x = 0$ ∴ $C = 0$. Now let $x = a$ to get half the volume of the prolate spheroid, $\frac{2}{3}\pi a b^2$; hence the volume of the entire spheroid is, $\frac{2}{3}(2a \cdot \pi b^2)$ or $\frac{2}{3}$ the volume of its circumscribed cylinder of revolution.

The volume of a sphere is found from this, by making $a = b$ = radius of the sphere, to be $\frac{4}{3}\pi a^3$.

EXAMPLE 2. Prove similarly by aid of (3), that the volume of the solid generated by the revolution of an ellipse about its minor axis (an *oblate spheroid*) is,

$$\frac{4}{3}\pi a^2 b = \frac{2}{3} \text{ circumscribing cylinder of revolution.}$$

EXAMPLE 3. Find the part of the volume formed by revolving the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ about the x-axis, included between the planes at $x = 1$, $x = 2$. Proceeding as in Ex. 1, we find,

$$V=\frac{4}{9}\pi\left(9x-\frac{x^3}{3}\right)+C.$$

In place of determining the constant as hitherto, the following method may be used.

For

$$x = 2,\quad V_2 = \tfrac{4}{9}\pi\,(\tfrac{46}{3}) + C.$$
$$x = 1,\quad V_1 = \tfrac{4}{9}\pi\,(\tfrac{26}{3}) + C.$$

V_2 and V_1 are both reckoned from the same fixed plane to the planes at $x = 2$ and $x = 1$, respectively. On subtracting, C is eliminated, and we find for the required volume, $\frac{80}{27}$ (3.1416) cubic units. This method may be used in any of the examples if preferred.

EXAMPLE 4. The solid generated by revolving the parabola $y^2 = 4px$, about its axis is called a paraboloid. Prove that its volume is $\frac{1}{2}$ the circumscribing cylinder.

EXAMPLE 5. Let the cubical parabola, $y^3 = a^2 x$, revolve about the axis of x; the volume generated from $x = 0$ to $x = x$,

is,

$$\tfrac{3}{5}\pi a^{\frac{4}{3}}x^{\frac{2}{3}}\cdot x = \tfrac{3}{5}\pi y^2\cdot x$$

or $\frac{3}{5}$ the circumscribing cylinder.

If $x = 8$, $a = 1$, vol. $= \frac{96}{5}$ (3.14 . . .) cubic units.

EXAMPLE 6. If $y = e^x$, $V=\frac{\pi}{2}e^{2x}+C$.

What volume does this represent when $C=-\frac{\pi}{2}$? $C = 0$?

EXAMPLE 7. Let the straight line $\frac{x}{6}+\frac{y}{4}=1$ revolve about the x-axis. Find the volume generated by the part of the line intercepted by the axes.

$$V = -32\left(1 - \frac{x}{6}\right)^3 + C$$

when $x = 6$, $V_2 = C$; when $x = 0$, $V_1 = -32 + C$ $\therefore$ the volume required $= V_2 - V_1 = 32$ cubic units.

EXAMPLE 8. From Ex. 1 we have for the volume of the prolate spheroid included between some fixed plane (not assigned) and the plane distant x from O,

$$V = \pi\,\frac{b^2}{a^2}\left(a^2x - \frac{x^3}{3}\right) + C.$$

To find the volume of a segment of one base of height h, we determine the volume V_2 for $x = a$ and then the volume V_1 for $x = (a - h)$ and subtract the latter from the former, giving

$$V_2 - V_1 = \pi\,\frac{b^2}{a^2}\left(ah^2 - \frac{1}{3}h^3\right).$$

If we put $a = b$, we find the volume of a *spherical* segment of one base of height h equal to, $\pi h^2 (a - \frac{1}{3}h)$.

The radius of the sphere is a.

EXAMPLE 9. The *methods* of deriving the general formulas in Arts. 80 and 81, are of great value in all kinds of problems. The following shows the application to a physical problem.

Let us assume Newton's law of gravitation, that the attraction between two bodies of masses M and M' respectively, r linear units apart, is MM'/r^2. In Fig. 55, Art. 81, let a material point of mass M at O and a thin cylindrical rod AC of mass m per unit of length, exert a mutual attraction. To determine this attraction F, for a length of the rod $AB = b$.

Let $OA = a$, $OB = x$, $BC = \Delta x$. The mass of the rod $BC = m\,\Delta x$. The attraction ΔF between M and $m\,\Delta x$ by Newton's law, is evidently less than $Mm\,\Delta x/x^2$ and greater than $Mm\,\Delta x/(x + \Delta x)^2$.

$$\therefore\ \frac{\Delta F}{\Delta x} \text{ is between } \frac{Mm}{x^2} \text{ and } \frac{Mm}{(x + \Delta x)^2} \text{ in value.}$$

But the last two ratios tend indefinitely towards equality as $\Delta x \doteq 0$;

$$\therefore \lim_{\Delta x \doteq 0} \frac{\Delta F}{\Delta x} = \frac{dF}{dx} = \frac{Mm}{x^2}$$

$$\therefore F = -\frac{Mm}{x} + C,$$

which gives the attraction between the mass M at O and a rod AB of length $x - a$.

Since $F = 0$ when $x = a \therefore C = Mm/a$. On substituting and putting $x = a + b$, *the attraction for a rod of length* $AB = b$, is,

$$F = Mm\left(\frac{1}{a} - \frac{1}{a+b}\right).$$

83. Differentials of Fundamental Forms. As preliminary to integration, it is advisable for the student to become familiar with the differential notation and to write out a table of fundamental differentials.

From Art. 78, we have, if $u = f(x)$,

$$\frac{du}{dx} = D_x u = f'(x),$$

and treating $\frac{du}{dx}$ as a fraction,

$$du = D_x u \cdot dx = f'(x) \cdot dx \quad . \quad . \quad . \quad (1)$$

$D_x u = f'(x)$, appearing here as the coefficient of dx, is often called a "differential coefficient" as well as a derivative.

To find then the differential of u, du, we multiply the derivative of u with respect to x by dx, the differential of x. Thus from Arts 37–41, if u, v and s are functions of x and differentiation with respect to x is assumed, we have,

$$\text{if, } y = C,\ dy = 0;$$
$$\text{if, } y = u + v - s,\ dy = du + dv - ds;$$
$$\text{if, } y = Cu,\ dy = Cdu;$$
$$d(uv) = vdu + udv;$$
$$d(u/v) = (vdu - udv)/v^2.$$

Also from the fundamental *derivatives* of Art. 60, on multiplying by dx and observing from (1), $du = Du \cdot dx$, that $dy = Dy \cdot dx$, where y can equal in turn, u^n, $\log_a u$, etc., we find the fundamental *differentials* given below.

$$d\,(u)^n = n\,(u)^{n-1}\,du.$$
$$d\,\log_a u = \log_a e \cdot \frac{du}{u}.$$
$$d\,\log_e u = \frac{du}{u}.$$
$$da^u = \log_e a \cdot a^u\,du.$$
$$de^u = e^u\,du.$$
$$d\,\sin u = \cos u \cdot du.$$
$$d\,\cos u = -\sin u \cdot du.$$
$$d\,\tan u = \sec^2 u \cdot du.$$
$$d\,\cot u = -\csc^2 u \cdot du.$$
$$d\,\sec u = \sec u \tan u \cdot du.$$
$$d\,\csc u = -\csc u \cot u \cdot du.$$
$$d\,\text{vers}\, u = \sin u \cdot du.$$
$$d\,\text{covers}\, u = -\cos u \cdot du.$$
$$d\,\sin^{-1} u = \frac{du}{\sqrt{1 - u^2}}.$$
$$d\,\cos^{-1} u = -\frac{du}{\sqrt{1 - u^2}}.$$
$$d\,\tan^{-1} u = \frac{du}{1 + u^2}.$$
$$d\,\cot^{-1} u = -\frac{du}{1 + u^2}.$$
$$d\,\sec^{-1} u = \frac{du}{u\sqrt{u^2 - 1}}.$$

$$d\csc^{-1}u = -\frac{du}{u\sqrt{u^2 - 1}}.$$

$$d\,\text{vers}^{-1}u = \frac{du}{\sqrt{2u - u^2}}.$$

$$d\,\text{covers}^{-1}u = -\frac{du}{\sqrt{2u - u^2}}.$$

If in the formulas for $d\sin^{-1}u$, $d\cos^{-1}u$, we put $u = \frac{bx}{a}$, $du = \frac{bdx}{a}$, we derive after reduction,

$$d\frac{1}{b}\sin^{-1}\frac{bx}{a} = \frac{dx}{\sqrt{a^2 - b^2x^2}}; \quad d\frac{1}{b}\cos^{-1}\frac{bx}{a} = -\frac{dx}{\sqrt{a^2 - b^2x^2}}.$$

Similarly prove,

$$d\frac{1}{ab}\tan^{-1}\frac{bx}{a} = \frac{dx}{a^2 + b^2x^2}; \quad d\frac{1}{ab}\cot^{-1}\frac{bx}{a} = -\frac{dx}{a^2 + b^2x^2};$$

$$d\frac{1}{a}\sec^{-1}\frac{bx}{a} = \frac{dx}{x\sqrt{b^2x^2 - a^2}}; \quad d\frac{1}{a}\csc^{-1}\frac{bx}{a} = -\frac{dx}{x\sqrt{b^2x^2 - a^2}};$$

$$d\frac{1}{b}\text{vers}^{-1}\frac{2b^2}{a^2}x = \frac{dx}{\sqrt{a^2x - b^2x^2}}; \quad d\frac{1}{b}\text{covers}^{-1}\frac{2b^2}{a^2}x = \frac{-dx}{\sqrt{a^2x - b^2x^2}}.$$

Prove,
$$d\frac{1}{2a}\log\frac{x - a}{x + a} = \frac{dx}{x^2 - a^2}.$$

Here we must have $x^2 > a^2$, in order that $(x - a)/(x + a)$ should be positive and the logarithm real. If $x^2 < a^2$, $(a - x)/(a + x)$ is positive and

$$d\frac{1}{2a}\log\frac{a - x}{a + x} = \frac{dx}{x^2 - a^2}.$$

$$d\log\sec u = \tan u\,du; \quad d\log\sin u = \cot u\,du.$$

$$d\log\tan\frac{u}{2} = \csc u\,du; \quad d\log\tan\left(\frac{u}{2} + \frac{\pi}{4}\right) = \sec u\,du.$$

$$d\log\left(u + \sqrt{u^2 \pm a^2}\right) = \frac{du}{\sqrt{u^2 \pm a^2}}.$$

CHAPTER XI.

INTEGRATION.

84. Integration. In the Differential Calculus a function $F(x)$ was given and its differential $f(x)\,dx$ was obtained. The reverse operation is known as *integration* or *anti-differentiation.* The symbol $\int$ is used to indicate integration. Thus,

$$\int f(x)\,dx = F(x) + C \quad . \quad . \quad . \quad (1)$$

an arbitrary constant being added to $F(x)$, since $d(F(x) + C) = f(x)\,dx$, the constant disappearing in the differentiation.

$\int f(x)\,dx$ is to be read "integral of" $f(x)\,dx$; it indicates the operation of finding a function $(F(x) + C)$, that differentiated will give $f(x)\,dx$. The differential $f(x)\,dx$ is called the *integrand.*

The *constant of integration* C may be any finite number positive or negative, or zero. The result of the integration, in the form $F(x) + C$, is called the *general* or *indefinite integral.* In problems, such as those given in Arts. 80–82, the constant C is determined from some convenient hypothesis and the result is a *definite* integral.

If we differentiate (1), noting from the hypothesis, that $d(F(x) + C) = f(x)\,dx$, we have,

$$d\int f(x)\,dx = f(x)\,dx;$$

also from $d\,(F(x) + C) = f(x)\,dx$, on integrating both sides, we have,

$$\int d\,(F(x) + C) = \int f(x)\,dx = F(x) + C.$$

From these two results we see that the signs d and $\int$ annul each other.

The two main problems of the Integral Calculus are:

(1) To find the quantity which differentiated gives $f(x)\,dx$;

(2) To find the limit of the sum of infinitesimals of the type $f(x)\,dx$ which vary in a certain manner to be given subsequently.

The following general theorems are readily deduced from the principle that d and $\int$ annul each other:

(i) since $\quad d\,(au) = adu \therefore au = \int adu;$

or, since $\quad u = \int du \therefore a\int du = \int adu.$

Hence, *a constant factor a, can be transposed from one side of the sign* $\int$ *to the other, without altering the value of the integral.*

(ii) Since, $d\,(u + v - s) = du + dv - ds$; on integrating,

$$u + v - s = \int (du + dv - ds).$$

$\therefore$ writing, $\int du$ for u, $\int dv$ for v, $\int ds$ for s,

$$\int (du + dv - ds) = \int du + \int dv - \int ds.$$

Hence, *the integral of an algebraic sum is equal to the algebraic sum of the integrals of the separate terms,* or they can only differ by an added constant.

(iii) From (i) it follows that,

$$\int du = \int \frac{1}{a}\, adu = \frac{1}{a}\int adu = a\int \frac{du}{a}.$$

The useful rule follows that, *we can multiply by a constant a to one side of the sign* $\int$, *provided we divide by the same constant on the other side.*

85. Standard Forms. By aid of the formulas given in Art. 83, we can write down at once the following table of standard forms; since, by what has been given above, the differential of the right member should equal the quantity to the right of the sign of integration in the left member. This test applies to all formulas or examples.

[1] $\int u^n du = \frac{u^{n+1}}{n+1} + c$ (provided n is not -1).

[2] $\int \frac{du}{u} = \log u + c = \log u + \log c' = \log c'u.$

[3] $\int a^u du = \frac{a^u}{\log a} + c.$

[4] $\int e^u du = e^u + c.$

[5] $\int \sin u\, du = -\cos u + c, = \text{vers}\, u + c'.$

[6] $\int \cos u\, du = \sin u + c, = -\text{covers}\, u + c'.$

[7] $\int \sec^2 u\,du = \tan u + c.$

[8] $\int \csc^2 u\,du = -\cot u + c.$

[9] $\int \sec u \tan u\,du = \sec u + c.$

[10] $\int \csc u \cot u\,du = -\csc u + c.$

[11] $\int \tan u\,du = \log \sec u + c.$

[12] $\int \cot u\,du = \log \sin u + c.$

[13] $\int \sec u\,du = \log \tan\left(\frac{u}{2} + \frac{\pi}{4}\right) + c.$

[14] $\int \csc u\,du = \log \tan \frac{u}{2} + c.$

[15] $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c, = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c'.$

[16] $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\log\frac{u-a}{u+a} + c, \text{ if } u^2 > a^2,$

$= \frac{1}{2a}\log\frac{a-u}{a+u} + c, \text{ if } u^2 < a^2.$

[17] $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + c, = -\cos^{-1}\frac{u}{a} + c'.$

[18] $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log\left(u + \sqrt{u^2 \pm a^2}\right) + c.$

[19] $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + c, - \frac{1}{a}\csc^{-1}\frac{u}{a} + c'.$

[20] $\int \frac{du}{\sqrt{2au - u^2}} = \text{vers}^{-1}\frac{u}{a} + c, = -\text{covers}^{-1}\frac{u}{a} + c'.$

From Art. 83, we likewise obtain the following very useful special forms:

[21] $$\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b} \sin^{-1} \frac{bx}{a} + c, = -\frac{1}{b} \cos^{-1} \frac{bx}{a} + c'.$$

[22] $$\int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab} \tan^{-1} \frac{bx}{a} + c, = -\frac{1}{ab} \cot^{-1} \frac{bx}{a} + c'.$$

[23] $$\int \frac{dx}{x\sqrt{b^2x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{bx}{a} + c, = -\frac{1}{a} \operatorname{cosec}^{-1} \frac{bx}{a} + c'.$$

[24] $$\int \frac{dx}{\sqrt{a^2x - b^2x^2}} = \frac{1}{b} \operatorname{vers}^{-1} \frac{2\,b^2x}{a^2} + c,$$
$$= -\frac{1}{b} \operatorname{covers}^{-1} \frac{2\,b^2x}{a^2} + c'.$$

The two forms of the integral given in several of the formulas, correspond to different values of c and c'. Thus to take [17]; by trigonometry,

$$\sin^{-1} \theta + \cos^{-1} \theta = \frac{\pi}{2} + 2n\pi = c' - c.$$

Similarly for the others.

Since the inverse functions are many-valued, the restrictions on the range of the angle given in Art. 18, must be carefully observed. Also if $x < a$, the integral of $dx/(x - a) = -dx/(a - x)$ is $\log(a - x)$ and if x is negative the integral of dx/x is $\log(-x)$ as we verify by differentiating. This change is necessary since there is no logarithm of a negative member.

This accounts, too, for the restriction in [16].

86. Use of Standard Formulas. In the differential calculus rules have been deduced for differentiating any known function. In the integral calculus, on the contrary, no such general rules or formulas exist for integrating any known function. We have here to endeavor by various devices, as algebraical and trigonometrical transformations or substitutions, to reduce the given function to one of the standard forms. As a consequence, integration is nearly always more difficult than differentiation. In fact many functions cannot be integrated at all.

On integrating a given function by different methods, the answers are sometimes found to have different forms. On reduction, however, they will always be found to differ (if at all) only by a constant.

For brevity the constant is omitted in the examples below, though it is always to be supplied in any application.

The most important formula above is [1] and it should be stated in words, thus: $\int (u)^n\,du$ *is equal to u raised to the one higher power divided by the new exponent.* For example,

$$\int ax^2dx = \frac{ax^3}{3};$$

$$\int \frac{adx}{x^2} = \int ax^{-2}\,dx = ax^{-1}/(-1) = -\frac{a}{x};\quad \int \sqrt{x^3}\,dx = \int x^{\frac{3}{2}}dx$$

$$= x^{\frac{5}{2}} \Big/ \frac{5}{2} = \frac{2}{5}\,x^{\frac{5}{2}};\quad \int \frac{dx}{\sqrt{x^3}} = \int x^{-\frac{3}{2}}\,dx = x^{-\frac{1}{2}} \Big/ \left(-\frac{1}{2}\right)$$

$$= -2x^{-\frac{1}{2}} = -\frac{2}{\sqrt{x}}.$$

To find $\int \frac{x^{n-1}\,dx}{(a+bx^n)^m} = \int (a+bx^n)^{-m}\,x^{n-1}\,dx;$

put $u = a + bx^n \therefore du = nbx^{n-1}\,dx$. Hence to write the given integral in the typical form $(u)^n du$ we must multiply and divide by nb (see(*iii*), Art. 84) and apply the rule.

$$\frac{1}{nb}\int (a+bx^n)^{-m}\,(nbx^{n-1}\,dx) = \frac{(a+bx^n)^{1-m}}{nb\,(1-m)}.$$

This result is not true when $m = 1$; then [2] alone applies.

When the numerator can be made (by introducing a *constant* factor) the exact differential of the denominator, formula [2] always applies.

$$\int \frac{x^{n-1}\,dx}{a + bx^n} = \frac{1}{nb}\int \frac{nbx^{n-1}\,dx}{a + bx^n} = \frac{1}{nb}\log\,(a + bx^n);$$

since, if $u = a + bx^n$, $du = nbx^{n-1}\,dx$; so that the integral

$$= \frac{1}{nb}\int \frac{du}{u} = \frac{1}{nb}\log u = \frac{1}{nb}\log\,(a + bx^n).$$

In the examples below, rules (i), (ii), (iii), of Art. 84 are frequently applied. Where the answers are not given, the results should be tested by differentiation, Art. 84 (1).

Examples.

1. $\displaystyle\int ax^4\,dx = a\,\frac{x^5}{5};\quad \int \frac{a\,dx}{5\,x^3} = -\,\frac{a}{10\,x^2}.$

2. $\displaystyle\int \left(3\,x - 20\,x^{\frac{2}{3}} - \frac{3\,a}{x^4} + \frac{a^2}{x^3} + a^3\right) dx,$

   by (i) and (ii), Art. 84, $\displaystyle = \frac{3}{2}x^2 - 12\,x^{\frac{5}{3}} + \frac{a}{x^3} - \frac{a^2}{2\,x^2} + a^3x.$

3. $\displaystyle\int (a^2 + x^2)^{\frac{1}{2}}\,x\,dx = \frac{1}{2}\int (a^2+x^2)^{\frac{1}{2}}\,(2\,x\,dx) = \frac{1}{3}\,(a^2+x^2)^{\frac{3}{2}}.$

4. $\displaystyle\int \sqrt{a+cx^3}\cdot 2\,bx^2\,dx = \frac{2\,b}{3\,c}\int (a + cx^3)^{\frac{1}{2}}\,3\,cx^2\,dx =$

   $\displaystyle\frac{4\,b}{9\,c}\,(a+cx^3)^{\frac{3}{2}}.$

5. $\displaystyle\int (a + bx^3)^2\,dx.$ Here the quantity outside the parenthesis cannot be made the exact differential of that inside by multiplying by a *constant* factor (we have no right to multiply

by a *variable* factor and then divide by it before the $\int$ sign); hence expand and integrate.

$$\int (a^2 + 2\,abx^3 + b^2x^6)\,dx = a^2x + \frac{1}{2}abx^4 + \frac{b^2x^7}{7}.$$

6. $\int 5\,(6\,ax^3 - 3\,bx^2)^{\frac{1}{3}}\,(6\,ax^2 - 2\,bx)\,dx =$

$$\frac{5}{3}\int (6ax^3 - 3\,bx^2)^{\frac{1}{3}}\,(18\,ax^2 - 6\,bx)\,dx = \frac{5}{4}\sqrt[3]{(6\,ax^3 - 3\,bx^2)^4}\,.$$

7. $\int \frac{ax^2\,dx}{\sqrt[3]{3 + x^3}} = \frac{a}{2}\,(3 + x^3)^{\frac{2}{3}}.$

8. $\int (a + bx^2)^2\,xdx\,;\int 8\,x^4\,\sqrt[5]{2 + 3x^5}.$

9. $\int (a+x)^n\,dx = \frac{(a+x)^{n+1}}{n+1};\quad \int (a-x)^n dx = -\frac{(a-x)^{n+1}}{n+1}.$

When $n = -1$ in Ex. (9), formula [2] is alone applicable.

10. $\int \frac{dx}{(a + x)^2};\quad \int \frac{dx}{(a + x)^3};\quad \int \frac{dx}{(a - x)^2};\quad \int \frac{dx}{(a - x)^3}.$

11. $\int \frac{dx}{a+x} + \int \frac{dx}{a-x} = \int \frac{d\,(a+x)}{a+x} - \int \frac{d\,(a-x)}{a-x} = \log(a+x)$

$$- \log(a - x) = \log\frac{a + x}{a - x}\,(x^2 < a^2) = \log\frac{x + a}{x - a}\,(x^2 > a^2).$$

The last result is found for $x^2 > a^2$, since

$$\int \frac{dx}{a - x} = -\int \frac{d\,(x - a)}{x - a} = -\log\,(x - a),\ \text{etc.}$$

Notice that whether a or x is positive or negative, if $x^2 > a^2$, then $(x + a)/(x - a)$ is positive, since the signs of both numerator and denominator are both + or both −, according as the sign of x is + or −. Similarly $(a + x)/(a - x)$ is always positive (and its logarithm real) when $x^2 < a^2$.

12. Adding the integrals in Ex. 11 (see (ii), Art. 84) and integrating by use of [16], we reach the above results. Thus :

$$2a\int\frac{dx}{a^2-x^2} = -\log\frac{x-a}{x+a} = \log\frac{x+a}{x-a}\,(x^2>a^2),\ \text{etc.}$$

Also on differentiating both members, an equality results which proves most easily the generality of the results in Ex. 11.

13. $\displaystyle\int\frac{dx}{a+bx} = \frac{1}{b}\log(a+bx) = \log(a+bx)^{\frac{1}{b}}.$

14. $\displaystyle\int\left(\frac{b}{x}-\frac{2ax}{bx^2}+\frac{x^2}{ax^3}\right)dx = b\log x - \frac{a}{b}\log bx^2 + \frac{1}{3a}\log x^3,$

or $\displaystyle\int\left(\frac{b}{x}-\frac{2a}{bx}+\frac{1}{ax}\right)dx = \left(b-\frac{2a}{b}+\frac{1}{a}\right)\log x.$

The answers only differ by a constant, since

$$\frac{a}{b}\log b\cdot x^2 = \frac{a}{b}\log b + \frac{2a}{b}\log x \text{ and } \frac{1}{3a}\log x^3 = \frac{3}{3a}\log x.$$

15. $\displaystyle\int\frac{2xdx}{ab-7x^2};\quad \int\frac{(4x+2)\,dx}{x^2+x+1};\quad \int\frac{\sin\ xdx}{\cos x}.$

16. $\displaystyle\int(\log x)^n\frac{dx}{x}.$ Let $u=\log x\ \therefore du=\dfrac{dx}{x}$;

$$\therefore \int(\log x)^n\frac{dx}{x} = \int(u)^n\,du = \frac{u^{n+1}}{n+1} = \frac{(\log x)^{n+1}}{n+1}.$$

17. $\displaystyle\int\frac{1+x^2}{1-x}dx = \int\left(-x-1-\frac{2}{x-1}\right)dx =$

$$-\frac{x^2}{2}-x-\log(x-1)^2.$$

18. $\displaystyle\int\frac{\log(3x+1)}{3x+1}dx = \frac{1}{3}\int udu,$ if $u=\log(3x+1)$,

$$= \frac{1}{6}u^2 = \frac{1}{6}\{\log(3x+1)\}^2.$$

19. From [3] and [4] we readily find,

$$\int na^x\,dx = n\int a^x\,dx = n\frac{a^x}{\log a}.$$

20. $$\int na^{nx}\,dx = \frac{m}{n}\int a^{nx}\,d\,(nx) = \frac{m}{n}\,\frac{a^{nx}}{\log a}.$$

When $a = e$; $\log a = \log e = 1$.

21. $$\int\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)dx = c\int e^{\frac{x}{c}}\,d\left(\frac{x}{c}\right) - c\int e^{-\frac{x}{c}}\,d\left(-\frac{x}{c}\right)$$

$$= c\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$$

22. $$\int\frac{e^{2x}\,dx}{e^x + 1} = \int\left(e^x - \frac{e^x}{e^x + 1}\right)dx = e^x - \log\,(e^x + 1).$$

23. In formulas [5] – [14], if $u = ax + b$, $du = adx$

$$\therefore \int\sin\,(ax+b)\,dx = \frac{1}{a}\int\sin\,(ax+b)\,d\,(ax+b) = -\frac{\cos\,(ax+b)}{a};$$

$$\int\tan\,(ax + b)\,dx = \frac{1}{a}\log\sec\,(ax + b)\text{; etc.}$$

24. $$\int\sec^2\,(ax^2)\,xdx = \frac{1}{2a}\tan\,(ax^2).$$

25. $$\int\frac{\tan\,(ax^2)}{\cos\,(ax^2)}\,xdx = \frac{1}{2a}\int\sec^2\,(ax^2)\tan\,(ax^2)\,d\,(ax^2),$$

$$\text{by [9]} = \frac{1}{2a}\sec^2\,(ax^2).$$

26. $$\int\frac{dx}{\sin^2\,(a + 2x)} = -\frac{1}{2}\cot\,(a + 2x),\text{ by [8].}$$

27. $$\int\frac{dx}{\sin x\cos x} = \int\csc\,(2x)\,d\,(2x) = \log\tan x,\text{ by [14].}$$

28. $\int \sin^2 x dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{x}{2} - \frac{1}{4}\sin 2x.$

$$\int \cos^2 x dx = \frac{1}{2}(x + \frac{1}{2}\sin 2x).$$

29. $\int \sin^3 x dx = \int \left(\frac{3}{4}\sin x - \frac{1}{4}\sin 3x\right) dx = -\frac{3}{4}\cos x$

$$+ \frac{1}{12}\cos 3x,$$

or, $\int \sin^3 x dx = \int (1 - \cos^2 x)\sin x dx = \int \sin x dx +$

$$\int \cos^2 x d\cos x = -\cos x + \frac{1}{3}\cos^3 x.$$

30. $\int \sin^3 x \cos^3 x = \int \sin^3 x(1 - \sin^2 x) d\sin x = \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6}.$

31. $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x.$

32. $\int \tan^3 x dx = \int \tan x(\sec^2 x - 1) dx = \frac{1}{2}\tan^2 x + \log\cos x.$

33. Formulas [21]–[24] apply where a and b are any constant quantities. Thus if $a = \sqrt{3}$, $b = \sqrt{2}$,

$$\int \frac{dx}{\sqrt{3 - 2x^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\left(x\sqrt{\frac{2}{3}}\right); \int \frac{dx}{3 + 2x^2} = \frac{1}{\sqrt{6}}\tan^{-1}\left(x\sqrt{\frac{2}{3}}\right).$$

34. By completing the square of the terms in x, such expressions as the following can be integrated by [15]–[20].

$$\int \frac{dx}{x^2 + 2x + 3} = \int \frac{d(x+1)}{(x+1)^2 + 2} = \int \frac{du}{u^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}.$$

35. $\int \frac{dx}{\sqrt{1 - x^2 + x}} = \int \frac{dx}{\sqrt{\frac{5}{4} - (x - \frac{1}{2})^2}} = \int \frac{du}{\sqrt{\frac{5}{4} - u^2}}$

(where $u = x - \frac{1}{2}$), $= \sin^{-1}\frac{u}{\frac{\sqrt{5}}{2}} = \sin^{-1}\frac{2x - 1}{\sqrt{5}}.$

36. $\int \frac{dx}{x^2 + 2x + 3} = \int \frac{dx}{(x+2)^2 - 1} = \frac{1}{2} \log \frac{x-1}{x+3}.$

37. $\int \frac{dx}{\sqrt{x^2 + 2ax}} = \log (x + a + \sqrt{x^2 + 2ax}).$

38. $\int \frac{(2x+3)\,dx}{x^2 + 2x + 3} = \int \frac{(2x+2)\,dx}{x^2+2x+3} + \int \frac{dx}{(x+1)^2 + 2}$

$$= \log (x^2 + 2x + 3) + \frac{1}{\sqrt{2}} \tan^{-1} \frac{x+1}{\sqrt{2}}.$$

39. $\int \frac{(x+1)\,dx}{x^2 + x + 1} = \frac{1}{2} \log (x^2 + x + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}}.$

40. $\int \frac{2(x+1)\,dx}{\sqrt{x^2+x+1}} = \int \frac{(2x+1)\,dx}{\sqrt{x^2+x+1}} + \int \frac{dx}{\sqrt{(x+\frac{1}{2})^2 + \frac{3}{4}}}$

$$= 2\sqrt{x^2 + x + 1} + \log \left[x + \tfrac{1}{2} + \sqrt{x^2 + x + 1}\right].$$

Integrals of the forms of Exs. 38 and 40 can be integrated in a similar manner.

87. Integration by Substitution. The following trigonometric substitutions are often useful.

$x = a \sin \theta$ in the forms, $a^2 - x^2$, $\sqrt{a^2 - x^2}$;
$x = a \tan \theta$ " " " $x^2 + a^2$, $\sqrt{x^2 + a^2}$;
$x = a \sec \theta$ " " " $x^2 - a^2$, $\sqrt{x^2 - a^2}$.

EXAMPLE 1. Let $x = a \sin \theta \;\therefore\; dx = a \cos \theta \, d\theta$.

$$\int \sqrt{a^2 - x^2}\, dx = a^2 \int \cos^2 \theta \, d\theta = \frac{a^2}{2} (\theta + \sin \theta \cos \theta)$$

(by Ex. 28, Art. 86).

$$\therefore \int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}.$$

EXAMPLE 2. Let $x = a \tan\theta$ $\therefore$ $dx = a \sec^2\theta\, d\theta$, $x^2 + a^2 = a^2\sec^2\theta$.

$$\int \frac{dx}{(x^2+a^2)^2} = \int \frac{a\sec^2\theta\, d\theta}{a^4\sec^4\theta} = \frac{1}{a^3}\int \cos^2\theta\, d\theta = \frac{1}{2a^3}(\theta + \sin\theta\cos\theta)$$
$$= \frac{1}{2a^3}\left[\tan^{-1}\frac{x}{a} + \frac{ax}{a^2+x^2}\right].$$

EXAMPLE 3. Let $x = a\sec\theta$ $\therefore$ $dx = a\sec\theta\tan\theta\, d\theta$,

$$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{1}{a^2}\int \cos\theta\, d\theta = \frac{1}{a^2}\sin\theta = \frac{\sqrt{x^2-a^2}}{a^2 x}.$$

This integral can likewise be determined by the substitution $x = 1/y$. Use the latter substitution in the next example.

EXAMPLE 4.

$$\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \int \frac{-y\,dy}{(a^2y^2-1)^{\frac{3}{2}}} = -\frac{1}{2a^2}\int (a^2y^2-1)^{-\frac{3}{2}}(2a^2y\,dy)$$
$$= +\frac{2}{2a^2}(a^2y^2-1)^{-\frac{1}{2}} = \frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}}.$$

EXAMPLE 5. $\displaystyle\int \frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{3}}}dx$. Put $x = z^6$, where 6 is the least common multiple of the denominators of the fractional exponents. This device will always rationalize expressions of this kind.

$$\int \frac{x^{\frac{1}{2}}-1}{x^{\frac{1}{3}}}dx = \int \frac{z^3-1}{z^2}6z^5\,dz = 6\int (z^6-z^3)\,dz$$
$$= \frac{6}{7}z^7 - \frac{6}{4}z^4 = \frac{6}{7}x^{\frac{7}{6}} - \frac{3}{2}x^{\frac{2}{3}}.$$

Similarly proceed if the differential contains no surd except $(a+bx)$ affected with fractional exponents.

EXAMPLE 6. To integrate $x\,dx/(a+bx)^{\frac{3}{2}}$, put $a + bx = u^2$;

$$\therefore \int \frac{x\,dx}{(a+bx)^{\frac{3}{2}}} = \frac{1}{b^2}\int \frac{(u^2-a)}{u^3}2u\,du = \frac{2}{b^2}\int (1-au^{-2})\,du$$
$$= \frac{2}{b^2}\left(u + \frac{a}{u}\right) = \frac{2}{b^2}\frac{u^2+a}{u} = \frac{2}{b^2}\frac{2a+bx}{\sqrt{a+bx}}.$$

General methods are given in the standard books for rationalizing and integrating differentials which contain no surd except one of the form $\sqrt{a + bx + cx^2}$ and for dealing with *binomial* differentials of the form $x^m (a + bx^n)^p\, dx$ and various trigonometric differentials. The student desiring further information in an elementary treatise, may consult Edward's "Integral Calculus for Beginners." It is desirable to have at hand, too, Peirce's "Short Table of Integrals" (Ginn & Co.).

88. Partial Fractions. This subject belongs properly to Algebra, but two of the cases more frequently met with will be illustrated by examples which will indicate the general method to follow.

EXAMPLE 1. To resolve $x/(x^2 - 1)$ into partial fractions.

Factor the denominator and place,

$$\frac{x}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}.$$

Now perform *mentally* the operation of multiplying both members of this identity by $(x + 1)$ and then put $x + 1 = 0$.

$$\therefore \left[\frac{x}{x-1}\right]_{x=-1} = A \quad \therefore\ A = \frac{-1}{-2} = \frac{1}{2}.$$

Similarly, multiply (mentally) the above identity by $(x - 1)$ and then place $(x - 1) = 0$.

$$\therefore B = \left[\frac{x}{x+1}\right]_{x=1} = \frac{1}{2}.$$

$$\therefore \frac{x}{x^2 - 1} = \frac{1}{2}\left(\frac{1}{x+1} + \frac{1}{x-1}\right).$$

The general rule is, to every non-repeated factor $(x - a)$ of the denominator, corresponds a fraction of the form $A/(x - a)$.

If the numerator of the original fraction is not of lower degree than the denominator, divide by the latter and thus separate the fraction into an entire part and a rational fraction whose numerator is of lower degree than the denominator and proceed with the latter as above.

Thus, $$\frac{x^2-1}{x^2-4} = 1 + \frac{3}{x^2-4} = 1 + \frac{3}{4}\left(\frac{1}{x-2} - \frac{1}{x+2}\right).$$

EXAMPLE 2. If $(x-a)^n$ is a factor of the denominator, write n partial fractions corresponding,

$$\frac{A}{(x-a)^n} + \frac{B}{(x-a)^{n-1}} + \cdots + \frac{N}{x-a}.$$

The non-repeated factors (if any) are treated as above. Thus we write,

$$\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x+1)^2} + \frac{C}{x+1}.$$

On multiplying (mentally) by $(x-1)$ and then putting $(x-1)=0$,

$$A = \left[\frac{1}{(x+1)^2}\right]_{x=1} = \frac{1}{4}.$$

On multiplying (mentally) the same identity by $(x+1)^2$ and then placing $(x+1)=0$, we get,

$$B = \left[\frac{1}{x-1}\right]_{x=-1} = -\frac{1}{2}.$$

Substitute these values of A and B and since the equation is an identical one, it is true for any value of x. Hence substitute any convenient value of x and thus determine C.

Thus if $x=0$, we find $C = -1 + \frac{1}{4} + \frac{1}{2} = -\frac{1}{4}$.

$$\therefore \frac{1}{(x-1)(x+1)^2} = \frac{1}{4}\left(\frac{1}{x-1} - \frac{2}{(x+1)^2} - \frac{1}{x+1}\right).$$

We are able now to integrate the following differentials.

$$\int \frac{x^2 - 1}{x^2 - 4}\, dx = \int dx + \frac{3}{4} \int \left(\frac{dx}{x-2} - \frac{dx}{x+2}\right) = x + \log\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}$$

$$\int \frac{dx}{(x-1)(x+1)^2} = \frac{1}{4}\left(\log(x-1) - \log(x+1) + \frac{2}{x+1}\right)$$

$$= \log\left(\frac{x-1}{x+1}\right)^{\frac{1}{4}} + \frac{1}{2}\,\frac{1}{x+1}.$$

See Gibson's Calculus, p. 291.

89. Integration by Parts. If u and v denote any functions of x, we have,

$$d(uv) = udv + vdu.$$

Integrating and transposing.

$$\int udv = uv - \int vdu \quad . \; . \; . \quad (1)$$

This is the formula for integration by parts.

If in any example, $\int vdu$ is known or is more readily determined than $\int udv$, the latter integral can either be found or the solution will be advanced. u and dv must be chosen with that end in view, as the following examples illustrate.

EXAMPLE 1. $\int \log x dx.$

Put, $u = \log x, \quad dv = dx,$

$$\therefore du = \frac{dx}{x}, \quad v = x.$$

$\therefore$ by (1), $\int \log x dx = x \log x - \int x \frac{dx}{x} = x \log x - x.$

EXAMPLE 2. $\int xe^{ax}\, dx = x\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot dx = \frac{e^{ax}}{a^2}(ax - 1).$

Here we put, $u = x, \quad dv = e^{ax}dx \quad \therefore du = dx, \quad v = \frac{e^{ax}}{a}.$

Examples.

1. $\int x \cos x dx = x \sin x + \cos x.$

2. $\int x \log x dx = \frac{1}{2} x^2 \log x - \frac{1}{4} x^2.$

3. $\int x^n \log x dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right).$

4. $\int x \sin x dx = \sin x - x \cos x.$

5. $\int x^2 \sin x dx = - x^2 \cos x + 2\int x \cos x dx.$

(By Ex. 1) $= - x^2 \cos x + 2x \sin x + 2 \cos x.$

6. $\int x^2 \cos x dx = 2x \cos x + (x^2 - 2) \sin x.$

7. $\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1 - x^2}.$

8. $\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \log (1 + x^2).$

9. $\int \cos^{-1} x dx = x \cos^{-1} x - \sqrt{1 - x^2}.$

10. $\int \cot^{-1} x dx = x \cot^{-1} x + \frac{1}{2} \log (1 + x^2).$

11. $\int \sqrt{a^2 + x^2}\, dx = x\sqrt{a^2 + x^2} - \int \frac{x^2}{\sqrt{a^2 + x^2}} dx;$

or since, $\sqrt{a^2 + x^2} = \frac{a^2 + x^2}{\sqrt{a^2 + x^2}},$

$$\int \sqrt{a^2 + x^2}\, dx = \int \frac{a^2 dx}{\sqrt{a^2 + x^2}} + \int \frac{x^2 dx}{\sqrt{a^2 + x^2}}$$

$$= \frac{a^2}{2} \log (x + \sqrt{a^2 + x^2}) + \int \frac{x^2 dx}{\sqrt{a^2 + x^2}}.$$

Adding and dividing by 2,

$$\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left(x + \sqrt{a^2 + x^2}\right).$$

12. $$\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left(x + \sqrt{x^2 - a^2}\right).$$

13. $$\int x^n a^{mx}\,dx = \frac{x^n a^{mx}}{m \log a} - \frac{n}{m \log a}\int x^{n-1} a^{mx}\,dx.$$

For $n = 1, \quad a = e, \quad \int x e^{mx}\,dx = \frac{x e^{mx}}{m} - \frac{1}{m^2} e^{mx}$

$n = 2, \quad a = e, \quad \int x^2 e^{mx} = \frac{x^2 e^{mx}}{m} - \frac{2}{m}\int x e^{mx}\,dx$

$$= \frac{e^{mx}}{m}\left(x^2 - \frac{2x}{m} + \frac{2}{m^2}\right).$$

Similarly for $n = 3, 4, \ldots$, making use in each case of the result previously determined.

90. Definite Integrals. If the indefinite integral of $f(x)\,dx$ is $F(x) + c$, the *definite* integral between the *end values* a and b of x is found by substituting in $F(x) + c$, first $x = b$, then $x = a$ and subtracting the last result from the former, giving $[F(b) + c] - [F(a) + c] = F(b) - F(a)$. The arbitrary constant c has now disappeared, and the result is a definite value.

The following is the notation employed:

$$\int_a^b f(x)\,dx = \Big[F(x) + c\Big]_a^b = F(b) - F(a).$$

This is called integrating between limits, for a range of x from a to b; b being designated the upper limit (or end value) and a the lower limit (or end value). $f(x)$ is assumed to be continuous between $x = a$ and $x = b$.

If the limits are interchanged, the sign of the result is changed, since,

$$\int_b^a f(x)\,dx = F(a) - F(b).$$

Also if any other variable than x is used the result is the same, *i.e.*,

$$\int_a^b f(u)\,du = F(b) - F(a);$$

or a definite integral is a function simply of the end values and not of the variable of integration.

Since the constant c disappears in integrating between limits, it is not necessary to set it down.

It is sometimes best to write the values of the integral for the upper and lower limits, the one below the other and then subtract. Thus taking the result of Ex. 1, Art. 89,

$$\int_1^2 \log x dx = \Big[x \log x - x\Big]_1^2 \begin{array}{l} = 2\log 2 - 2 \\ = \quad\quad\; 0 - 1 \\ \hline \end{array}$$

$$\int_1^2 \log x dx = 2 \log 2 - 1.$$

EXAMPLE 1. $\displaystyle\int_a^b \frac{dx}{x^n} = \left[-\frac{1}{n-1}\frac{1}{x^{n-1}}\right]_a^b = \frac{1}{n-1}\left(\frac{1}{a^{n-1}} - \frac{1}{b^{n-1}}\right),$

provided n is not $= 1$. If $n > 1$, the definite integral

$$\doteq \frac{1}{n-1} \cdot \frac{1}{a^{n-1}} \text{ as } b \doteq \infty;$$

but if $0 < n < 1$, the integral $\doteq \infty$, since $1/b^{n-1} = b^{1-n} \doteq \infty$ as $b \doteq \infty$. Similarly discuss the change as $a \doteq 0$.

If $n = 1, \quad \int_a^b \frac{dx}{x} = \Big[\log x\Big]_a^b = \log \frac{b}{a}$

and this $\doteq \infty$ as $b \doteq \infty$, or as $a \doteq 0$.

Construct the graphs of

$$y = \frac{1}{x^2}, \quad y = \frac{1}{x} \text{ (Art. 13) and } y = \frac{1}{x^{\frac{1}{2}}},$$

corresponding to $n = 2$, $n = 1$, $n = 1/2$ respectively.

EXAMPLE 2. From Exs. 5 and 9, Art. 89, find,

$$\int_0^{\frac{\pi}{2}} x^2 \sin x dx = \pi - 2;$$

$$\int_0^1 \cos^{-1} x dx = 1.$$

EXAMPLE 3. To prove

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

If $\int f(x)\, dx = F(x) + c$, then the right member is,

$$F(c) - F(a) + F(b) - F(c) = F(b) - F(a)$$

which equals the left member. *Q.E.D.*

Similarly it can be shown that,

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^g f(x)\, dx + \int_g^b f(x)\, dx;$$

and so on for any number.

CHAPTER XII.

INTEGRATION AS A LIMIT OF A SUM. LENGTHS OF ARCS. AREAS OF CURVES. SOLIDS OF REVOLUTION. CENTER OF MASS. MOMENT OF INERTIA.

91. Integration Viewed as a Limit of a Sum. In Chapter XI. we have given some of the simpler methods of finding anti-differentials, integration being viewed as the reverse of differentiation. In this chapter a definite integral will be proved to represent the limit of a sum, and some applications will be made to illustrate the working of the theory.

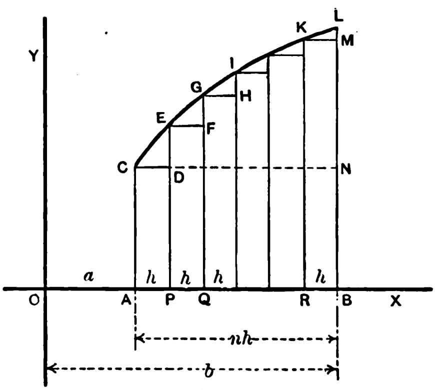

Fig. 57.

Let $F(x)$ denote any function of x which is continuous, as well as its derivative, from $x = a$ to $x = b$, and suppose the graph of $y = F(x)$ to be constructed between these limits (Fig. 57).

Also let,

$$\frac{dF(x)}{dx} = \frac{dy}{dx} = f(x) \quad . \quad . \quad . \quad (1)$$

$$\therefore \text{ Art. } 78, \quad \frac{\Delta y}{\Delta x} = f(x) + \alpha,$$

where $\alpha \doteq 0$ as $\Delta x \doteq 0$ $\therefore$ Putting $h = \Delta x$,

$$\Delta y = f(x)\, h + \alpha h \quad . \; . \; . \quad (2)$$

In the figure, let $OA = a$, $CB = b$, and suppose $AB = b - a$ to be divided into n parts, each equal to $h = \Delta x$ $\therefore$ $b - a = nh$.

Also $F(a) = AC$, $F(b) = BL$; $F(b) - F(a) = NL$.

From (2), if $x = a$, $\Delta y = DE = f(a)\, h + \alpha_1 h$;
if $x = a + h$, $FG = f(a + h)\, h + \alpha_2 h$;
if $x = a + 2h$, $HI = f(a + 2h)\, h + \alpha_3 h$;

. . . .

if $x = a + (n - 1)\, h$, $ML = f(a + (n - 1)\, h) + \alpha_n h$.

The values of α will generally be different, hence each is marked with a subscript.

On adding these equations, we have, since $DE + FG + HI + \ldots + ML = NL = F(b) - F(a)$,

$$F(b) - F(a) = [f(a) + f(a + h) + f(a + 2h) + \ldots$$
$$+ f(a + (n - 1)\, h)]\, h + R$$

where, $R = (\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n)\, h$.

If α_r is numerically the greatest of the infinitesimals $\alpha_1, \alpha_2 \ldots \alpha_n$ (n in number), and $\bar{\alpha}_r$ is its numerical value,

$$\bar{R} < n \bar{\alpha}_r h \;\therefore\; \bar{R} < \bar{\alpha}_r (b - a).$$

Since, by assumption, the limit of any α is zero as $h \doteq 0$ $\therefore$ $\lim\limits_{h \doteq 0} R = 0$.

Hence taking limits, as $h = \Delta x \doteq 0$ and $\therefore$ $n \doteq \infty$,

$$F(b) - F(a) = \lim_{h=0} [f(a) + f(a + h) + f(a + 2h) + \ldots$$
$$+ f(a + (n - 1)\, h)]\, h.$$

But from (1) we have,

$$dF(x) = f(x)\,dx \;\therefore\; \int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

$$\therefore \int_a^b \mathbf{f}(\mathbf{x})\mathbf{dx} = \lim_{h \doteq 0} [\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{a}+\mathbf{h}) + \mathbf{f}(\mathbf{a}+2\mathbf{h}) + \ldots + \mathbf{f}(\mathbf{a}+(\mathbf{n}-1)\mathbf{h})]\mathbf{h} \quad \ldots \quad (3)$$

where it must be remembered that $h = \Delta x$.

The right member is often written in the abbreviated notation,

$$\lim_{\Delta x \doteq 0} \sum_a^b f(x)\,\Delta x,$$

which is to be read "limit of the sum, as $\Delta x \doteq 0$, of quantities of the type $f(x)\,\Delta x$, as x varies from a to b."

$$\therefore \int_a^b f(x)\,dx = \lim_{\Delta x \doteq 0} \sum_a^b f(x)\,\Delta x \quad \ldots \quad (4)$$

It follows, if we take $dx = \Delta x$, that the symbols

$$\int_a^b, \quad \lim \sum_a^b,$$

stand for the same thing. Thus a definite integral is not a sum, but the limit of a sum. The limit is never attained, since as $h \doteq 0$, $n \doteq \infty$ and

$$\lim_{\Delta x \doteq 0} \sum_a^b f(x)\,\Delta x \doteq \text{the form } 0 \cdot \infty,$$

since there are n terms, and each approaches zero indefinitely.*

* The sign $\int$ was originally the old elongated S, the first letter of *Summa*. Leibnitz used it to mean "sum," exactly what "sigma" or Σ, stands for. It is now seen to represent the limit of a sum.

Although a graph was used above to render the meaning plainer, it is not at all necessary to derive the important conclusions (3) and (4). The analytical proof, in brief, is as follows:

Assume (1) from which (2) follows; then since

$$\Delta y = F(x+h) - F(x); \text{ from (2) we have,}$$
$$F(x+h) - F(x) = f(x)h + \alpha h \quad . \quad . \quad . \quad (2')$$

If $f(x)$ is continuous from $x = a$ to $x = b$, $f(x)$ for any value of x within this range is finite $\therefore$ the right member of $(2') \doteq 0$ as $h \doteq 0$, which entails that $F(x)$ should be a continuous function as well as $f(x)$ within the range considered (Art. 33).

Let x now take successively the values a, $a+h$, $a+2h$, . . . , $a+(n-1)h$ in $(2')$ above.

$$F(a+h) - F(a) = f(a)h + \alpha_1 h,$$
$$F(a+2h) - F(a+h) = f(a+h)h + \alpha_2 h,$$
$$F(a+3h) - F(a+2h) = f(a+2h)h + \alpha_3 h.$$

. . . .

$$F(a+nh) - F(a+(n-1)h) = f(a+(n-1)h)h + \alpha_n h.$$

Adding and assuming $a + nh = b$,

$$F(b) - F(a) = \sum_a^b f(x)h + R,$$

where $$R = (\alpha_1 + \alpha_2 + \alpha_3 + \ldots \alpha_n)h.$$

Hence, exactly as above, on taking limits as $h \doteq 0$, we deduce (3) or (4).

92. It may prove instructive to verify the equality (3) of Art. 91 in a special case.

Thus if $$f(x) = e^x, \int_a^b e^x dx = [e^x]_a^b = e^b - e^a.$$

This gives the left member of (3). Also, since $f(a) = e^a$, $f(a+h) = e^{a+h}$, etc., the right member is,

$$\lim_{h=0} [e^a + e^{a+h} + e^{a+2h} + \ldots + e^{a+(n-1)h}]h.$$

The quantity in brackets is the sum of n terms of a geometrical series whose first term is e^a and common ratio e^h; hence the last expression is equal to,

$$\lim_{h \doteq 0} \frac{e^{a+nh} - e^a}{e^h - 1} h = \lim_{h \doteq 0} (e^b - e^a) \frac{h}{e^h - 1},$$

since $b = a + nh$ (Art. 91).

To evaluate this limit, expand e^h (Art. 72):

$$\therefore \lim_{h \doteq 0} \frac{h}{e^h - 1} = \lim_{h \doteq 0} \frac{h}{h + \frac{1}{2}h^2 + \ldots} = \lim_{h \doteq 0} \frac{1}{1 + \frac{1}{2}h + \ldots} = 1;$$

hence the right member of (3) Art. 91, in this case, equals $e^b - e^a$, or the same as the left member.

Edwards, in his "Integral Calculus for Beginners," has given a number of similar illustrations.

In fact, Mr. Homersham Cox in his "Integral Calculus" has determined the fundamental integrals from the "summatory definition."

93. Areas of Curves. In Fig. 57, let us now write $y = f(x)$, for the equation of the curve CL; then the ordinates at $C, E, G, \ldots$, will be $f(a), f(a+h), f(a+2h), \ldots$, and the sum of the areas of the inscribed rectangles $AD + PF + QH + \ldots$ will be,

$$S = [f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)]h.$$

This differs from the true area $A = ACEGLB$, by the sum of the curvilinear triangles, $CDE + EFG + GHI +$

$\ldots + KML$, which is less than $h\,[DE+FG+\ldots+ML]$ or less than $h \cdot NL$ or, if $0 < \theta < 1$, equal to $\theta\, h \cdot NL$.

$$\therefore\ A = S + \theta h \cdot NL.$$

The limit of this expression as $h \doteq 0$ and $\therefore$

$$n = \frac{b-a}{h} \doteq \infty, \text{ is } A = lim\ S.$$

$$\therefore \text{ by (3) Art. 91, } \mathbf{A} = \int_a^b \mathbf{f}\,(\mathbf{x})\,\mathbf{dx} \quad . \quad . \quad . \quad (1)$$

This result, obtained by the summatory method, may likewise be found from the differential equation (2) of Art. 80, $dA = f(x)\,dx$; for if

$$\int f(x)\,dx = F(x) + C,$$

$$\therefore\ A = F(x) + C,$$

A being the area from some fixed ordinate up to the ordinate whose abscissa is x.

If we estimate the area from the fixed ordinate AC (Fig. 57), then $A = 0$, when $x = a$.

$$\therefore\ 0 = F(a) + C.$$

Also calling A = area $ACLB$, when $x = b$, to agree with the previous designation of this article,

$$\therefore\ A = F(b) + C.$$

On subtracting the former equation from the latter, we find,

$$A = F(b) - F(a),$$

which equals $\int_a^b f(x)\,dx$ or the right member of (1).

The two independent methods then lead to the same result. Generally the use of the definite integral is to be preferred.

Examples.

1. By use of (1) of this article, the area of the trapezoid of Ex. 1, Art. 80, between the end values a and b is,

$$A = \int_a^b mxdx = \left[\frac{mx^2}{2}\right]_a^b = \frac{m}{2}\left[b^2 - a^2\right].$$

Sometimes the end value b is replaced by x as in some of the exercises in Art. 80.

2. Work Exs. 2–11 of Art. 80 by use of (1) of this article.

3. The area included between the arc of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

in the first quadrant, the ordinates $x = 0$, $x = c$ and the x-axis is, Art. 87, Ex. 1,

$$\int_0^c \frac{b}{a}\sqrt{a^2 - x^2}\, dx = \frac{b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^c$$

$$= \frac{b}{a}\left[\frac{c\sqrt{a^2 - c^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{c}{a}\right].$$

Thus let $a = 5$, $b = 3$, $c = 4$, all in feet.

$$\therefore \sin^{-1}\frac{c}{a} = \sin^{-1}\frac{4}{5}.$$

The angle in degrees and minutes (to the nearest minute) whose sine is $4/5 = .80000$ is $53° 8'$; the corresponding arc on a unit circle (Art. 17) is, $\frac{\pi \times 53.13}{180} = .927 \ldots$, which represents the angle expressed in *radians* $\therefore \sin^{-1}4/5 = .927 \ldots$; hence the required area equals,

$$\frac{3}{5}\left[\frac{4}{2}(3) + \frac{25}{2}(.927)\right] = 10.55 \text{ square feet.}$$

If $c = a$ in the general formula, the area of the first quadrant is found to be,

$$\frac{ab}{2} \sin^{-1} 1 = \frac{ab}{2} \cdot \frac{\pi}{2};$$

hence the area of the whole ellipse is 4 times this or πab. When $a = b$, the ellipse becomes a circle, whose area is consequently πa^2, where a is the radius of the circle.

4. The area between the two parabolas, $y^2 = 4ax$, $x^2 = 4ay$ which intersect at (0, 0), $(4a, 4a)$ is,

$$A = \int_0^{4a} \left(\sqrt{4a}\, x^{\frac{1}{2}} - \frac{1}{4a} x^2\right) dx = \frac{16}{3} a^2.$$

5. Find the area between $y^2 = 9x$ and $y = x$. They intersect at (0, 0), (9, 9). Let the inch be the unit.

$$\therefore A = \int_0^9 (3x^{\frac{1}{2}} - x)\, dx = 13.5 \text{ sq. inches.}$$

Construct the loci.

6. Find the areas between the following curves, the axis of x and the ordinates specified.

(i) $y = \dfrac{8a^3}{x^2 + 4a^2}$; $x = 0$ to $x = 2a$.

(ii) $y = \log x$; $x = 1$ to $x = 2$.

(iii) $y = \sin^{-1} x$; $x = 0$ to $x = 1$.

(iv) $y = \dfrac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$; $x = 0$ to $x = x$.

(v) $x^2 y = a^2(x - y)$; $x = 0$ to $x = a$.

(vi) $x(x^2 - ay) = a^3$; $x = 1$ to $x = 2$.

Answers. (i) $a^2\pi$; (ii) $2 \log 2 - 1$; (iii) $\dfrac{\pi}{2} - 1$;

(iv) *as* (Art. 59, (vi)); (v) $\dfrac{a^2}{2} \log 2$; (vi) $\dfrac{7}{3a} - a^2 \log 2$.

7. Since $A = lim\ S$, $A =$ limit of the sum of the inscribed rectangles $\therefore$ if abscissas are drawn at C and L (Fig. 57) to meet the y-axis, the area between these abscissas, the curve and the y-axis, is the limit of the sum of the inscribed rectangles corresponding, or since the area of any rectangle is $x\,\Delta y$,

$$\text{Area } A = \lim_{\Delta y \doteq 0} \sum_{AC}^{BL} x\,\Delta y = \int_{AC}^{BL} x dy.$$

94. Volumes of Solids of Revolution. Again referring to Fig. 57, Art. 91, let the equation of the curve be $y = f(x)$; call $OA = a$, $OB = b$ and let $AB = b - a = nh$, where $h = \Delta x$, so that as $h \doteq 0$, $n = (b - a)/h \doteq \infty$. The successive ordinates AC, PE, QG, . . . RK are respectively $f(a)$, $f(a + h)$, $f(a + 2h)$, . . . $f(a + (n - 1)h)$. *As the figure ACLB revolves about the x-axis,* it will describe a solid of revolution whose volume we shall call V.

The successive inscribed rectangles AD, PF, QH, . . . RM, in the revolution, will describe cylinders whose total volume is (since h is the altitude of each),

$$S = \pi\,(AC^2 + PE^2 + QG^2 + \ . \ . \ . \ RK^2)\,h.$$

The curvilinear triangles CDE, EFG, . . . will describe (as $ACLB$ makes a complete revolution) a series of rings. Imagine these rings moved to the right until they are found included in the solid (of altitude RB) formed by revolving $RBLK$ about the x-axis, BL being supposed the greatest ordinate from A to B. The sum of the volumes of these rings will then be less than the volume generated by the rectangle RL, or less than $\pi\ .\ BL^2\ .\ h$, or if $0 < \theta < 1$, equal to $\theta\pi\ .\ BL^2\ .\ h$, when θ is given its proper value which is always < 1.

$$\therefore\ V = S + \theta h \cdot BL^2 \cdot \pi.$$

But as $h = \Delta x \doteq 0$, $lim\ \theta h \cdot BL^2 \cdot \pi = 0$ $\therefore$ $V = lim\ S$.

$$\therefore V = \pi \lim_{\Delta x \doteq 0} \sum_a^b [f(x)]^2 \Delta x = \pi \int_a^b (f(x))^2\, dx.$$

This is made plain by putting $\phi(x) = [f(x)]^2$; then by Art. 91,

$$\lim_{\Delta x \doteq 0} \sum_a^b \phi(x) \Delta x = \int_a^b \phi(x)\, dx.$$

The formula for V can be written,

$$\mathbf{V} = \pi \int_a^b \mathbf{y^2 dx} \quad . \; . \; . \; (1)$$

If the curve $y = f(x)$ between the end ordinates has one or more maximum or minimum values, (1) is applicable from one turning-point to the next, using for a and b, the proper end values, and hence over the whole range from $x = a$ to $x = b$ (Art. 90, Ex. 3). A similar remark applies to (1) of Art. 93.

If the curve revolves about the axis of y, we evidently get the volume generated by taking the limit of the sum of quantities of the type $\pi x^2 dy$.

$$\therefore \mathbf{V} = \pi \int_{a'}^{b'} \mathbf{x^2\, dy} . \; . \; . \; (2)$$

where the end values are now $y = b' = BL$, $y = a' = AC$.

If in Fig. 57, PE is taken as a variable ordinate, (1) can be written,

$$V = \pi \int_a^b PE^2 \cdot d(OP) \quad . \; . \; . \; (3)$$

which is true however the axes may be shifted, OB remaining fixed in position.

The volume V is always the limit of the sum of the volumes of the inscribed cylinders.

Examples.

1. Find the volume generated by revolving the *hypocycloid* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ about the x-axis between the limits 0 and a. Use (1)

$$\pi\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx = \frac{16\,\pi\,a^3}{105}.$$

2. Given, that in the *tractrix*,

$$dx = -\frac{\sqrt{a^2 - y^2}}{y}\,dy,$$

to find the volume generated by revolving the curve about the x-axis. It is simpler here to change the independent variable from x to y and substitute in (1) the above value of dx; then taking the end values of y as 0 and a,

$$V = -\pi\int_0^a \sqrt{a^2 - y^2}\cdot y\,dy = \frac{1}{3}\pi\,a^3.$$

3. Find the volume generated by revolving about the y-axis the arc of the catenary,

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right),\ \text{from } (0,\,a) \text{ to } \left(a, \frac{a}{2}(e + e^{-1})\right).$$

See Fig. 38, Art. 59, (vi).
Substitute in (2),

$$dy = \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)dx,$$

so that the variable is changed from y to x and the limits from $y = a$, $y = \frac{a}{2}(e + e^{-1})$ to the corresponding x limits, $x = 0$, $x = a$.

$$\therefore\ V = \frac{\pi}{2}\int_0^a \left(x^2e^{\frac{x}{a}} - x^2e^{-\frac{x}{a}}\right)dx.$$

$\int x^2e^{x/a}\,dx$, can be found by putting $n = 2$, $a = e$, $m = \frac{1}{a}$ in Ex. 13, Art. 89. For $\int x^2e^{-x/a}\,dx$, put $n=2$, $a=e$, $m= -\frac{1}{a}$.

The answer is,

$$V = \frac{\pi a^3}{2}\left(e + \frac{5}{e} - 4\right).$$

4. As an application of (3), let it be required to find the volume generated by an arc of a parabola $y^2 = 4px$ from (0, 0) to $(a, 4\sqrt{pa})$, revolving about the line $x = a$ or the line AB, Fig. 58. From any point E (x, y) of the curve, drop a perpendicular to the axis of revolution AB, meeting it at P, and take some convenient point A on this axis to correspond to the origin O of (3); then by (3),

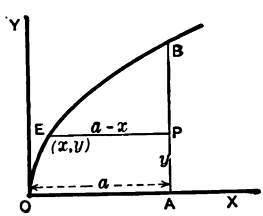

Fig. 58.

$$V = \pi \int_0^{AB} PE^2\, d\,(AP) = \pi \int_0^{\sqrt{4pa}} (a - x)^2\, dy$$

$$= \pi \int_0^{\sqrt{4pa}} \left(a - \frac{y^2}{4p}\right)^2 dy = \frac{16\,\pi a^2\,\sqrt{pa}}{15}.$$

The parabolic spindle generated by the whole arc subtended by the double ordinate at A is twice this.

5. Draw the locus $ay^2 = x^3$ (a semi-cubical parabola) and letter the part above the x-axis as in Fig. 58, making $OA = a$ and $\therefore AB = a$. What is the volume of the solid generated by revolving the arc OEB about AB? Proceeding as in Ex. 4, we find,

$$V = \pi \int_0^a (a - x)^2\, dy = \frac{3\,\pi}{2\sqrt{a}} \int_0^a \left(a^2 x^{\frac{1}{2}} - 2\,a x^{\frac{3}{2}} + x^{\frac{5}{2}}\right) dx$$

$$= \frac{8}{35}\pi a^3.$$

95. Fundamental Principle of Infinitesimals. *If the limit of the sum of n infinitesimals of the same sign, as $n \doteq \infty$, be finite, the limit is not altered if each infinitesimal is replaced by another, provided the limit of the ratio of the two corresponding infinitesimals is unity.*

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$, be n related infinitesimals and suppose

$$\lim_{n \doteq \infty} (\alpha_1 + \alpha_2 + \ldots + \alpha_n) = \text{finite constant} = c;$$

also let $\beta_1, \beta_2, \ldots, \beta_n$ be another set of infinitesimals such that, as $n \doteq \infty$,

$$\lim \frac{\beta_1}{\alpha_1} = 1, \lim \frac{\beta_2}{\alpha_2} = 1, \ldots, \lim \frac{\beta_n}{\alpha_n} = 1 \quad \ldots (1)$$

Whence,

$$\frac{\beta_1}{\alpha_1} = 1 + \epsilon_1, \frac{\beta_2}{\alpha_2} = 1 + \epsilon_2, \ldots, \frac{\beta_n}{\alpha_n} = 1 + \epsilon_n,$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, are infinitesimals, each having the limit zero.

On clearing the last set of equations of fractions and adding, etc., we obtain,

$$(\beta_1 + \beta_2 + \ldots + \beta_n) - (\alpha_1 + \alpha_2 + \ldots + \alpha_n)$$
$$= \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 + \ldots + \alpha_n \epsilon_n \quad \ldots (2)$$

Now if ϵ_r is the numerically greatest of the ϵ's, and $\bar{\epsilon}_r$ is its numerical value, then numerically,

$$\alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 + \ldots + \alpha_n \epsilon_n < (\alpha_1 + \alpha_2 + \ldots + \alpha_n)\bar{\epsilon}_r.$$

But $\lim_{n \doteq \infty} (\alpha_1 + \alpha_2 + \ldots + \alpha_n)\bar{\epsilon}_r = c \lim_{n \doteq \infty} \bar{\epsilon}_r = c \cdot 0 = 0,$

since the limit of any ϵ is zero $\therefore$ the limit of the right member of (2) is zero.

Hence taking the limit of (2) and transposing,

$$\lim_{n \doteq \infty} [\beta_1 + \beta_2 + \ . \ . \ . \ + \beta_n] = \lim_{n \doteq \infty} [\alpha_1 + \alpha_2 + \ . \ . \ . \ + \alpha_n],$$

which proves the proposition.

This very valuable theorem enables us in a limit of a sum or an integral, to replace the true expression for a term (which is often difficult or impossible to obtain) by another (perhaps very easy to obtain) when, and only when, the limit of the ratio of corresponding terms is unity.

(i) Thus in Art. 91, Fig. 57, $NL = DE + FG + \ . \ . \ . + ML$; *i.e.*, NL is equal to the sum of the successive Δy's. But

$$y = F(x); \ \frac{dy}{dx} = f(x); \ \therefore dy = f(x) \ h \text{ and}$$

$$\lim_{n \doteq \infty} \frac{\Delta y}{dy} = \lim_{\Delta x \doteq 0} \frac{\Delta y}{dy} = 1,$$

by Art. 78. Therefore dy can replace the corresponding Δy in the *limit* of the sum (not the sum).

$$\therefore NL = \lim_{h \doteq 0} [f(a) + f(a+h) + \ . \ . \ . \ + f(a + (n-1)h)]\, h,$$

as was proved otherwise in Art. 91.

By reference to Art. 78 and to Fig. 53, it is seen that each increment of y is replaced by the corresponding increment of the tangent. Let the student illustrate this on a figure similar to Fig. 57.

(ii) Referring to Art. 93 and Fig. 57, where now, $y = f(x)$,

$$\frac{\text{Area } PEGQ}{\text{Area } PEFQ} < \frac{GQ \cdot h}{PE \cdot h} \text{ or } \frac{GQ}{PE};$$

$$\text{but as } h \doteq 0, \ \lim GQ/PE = 1 \ \therefore \lim \frac{PEGQ}{PEFQ} = 1.$$

Therefore $PEFQ = yh = f(x)\,dx$, can replace the true area $PEGQ$ in the value of A when the limit is taken. As this applies to any inscribed rectangle, by the theorem,

$$A = \lim_{h \doteq 0} S = \lim_{h \doteq 0} \sum_a^b f(x)\,dx = \int_a^b f(x)\,dx.$$

(iii) In the case of solids of revolution, Art. 94 and Fig. 57, calling $PE = y$, $QG = y + \Delta y$, the volume of the solid generated by $PEGQ$ is between $\pi y^2 h$ and $\pi\,(y + \Delta y)^2 h$, whose ratio is $\left[\dfrac{y}{y + \Delta y}\right]^2$, which has a limit 1 as $h \doteq 0$ and $\therefore \Delta y \doteq 0$; *à fortiori*, the ratio of the volume of the solid generated by $PEGQ$ and that generated by the rectangle PF, has a limit 1. Hence $\pi y^2 h$ can replace the volume generated by $PEGQ$ when the limit of the sum, as $h \doteq 0$, of such volumes, is taken $\therefore$ the solid generated by $ACLB$ revolving about OX has the volume,

$$\lim_{\Delta x \doteq 0} \sum_a^b \pi y^2 \Delta x = \int_a^b \pi y^2 dx,$$

as found in Art. 94.

(iv) In Fig. 57, Art. 91, let AB be supposed divided into n equal or unequal parts $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ in length, each having a limit zero as $n \doteq \infty$. Call the lengths of the corresponding arcs, CE, EG, . . ., KL, $\alpha_1, \alpha_2, \ldots, \alpha_n$ and the lengths of the chords, CE, EG, . . ., KL, $\beta_1, \beta_2, \ldots, \beta_n$. Then by Art. 49, as $n \doteq \infty$, the ratios, $\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots; \alpha_n/\beta_n$, have each 1 for a limit. Hence since as $n \doteq \infty$, $\lim\,(\alpha_1 + \alpha_2 + \ldots + \alpha_n)$ = length of arc CL, is finite; by the theorem, as $n \doteq \infty$, $\lim\,(\alpha_1 + \alpha_2 + \ldots + \alpha_n) = \lim\,(\beta_1 + \beta_2 + \ldots + \beta_n)$, or *the length of any arc is the limit towards which tends the perimeter of any inscribed polygon as the sides diminish in length and their number augments indefinitely.*

96. Lengths of Arcs. In Fig. 57, Art. 91, let $\Delta s_1 = CE$. $\Delta s_2 = EG, \ldots, \Delta s_n = KL$, then $(\Delta s_1 + \Delta s_2 + \ldots, + \Delta s_n) = arc\ CL$. Also as $h \doteq 0$, $\lim (\Delta s_1 + \Delta s_2 + \ldots + \Delta s_n) = arc\ CL$, a finite constant, so that the first requirement of the theorem of Art. 95 is satisfied, Δs corresponding to a in that article.

It is evident that for *any* x as OP and $\Delta x = PQ$, that $\Delta s = EG$, and by Art. 78, if ds is the corresponding differential of the arc, then

$$\lim_{\Delta s \doteq 0} \frac{\Delta s}{ds} = 1;$$

hence by the theorem, any ds can replace the corresponding Δs in the limit of the sum above.

$$s = arc\ CL = \lim_{\Delta x \doteq 0} \sum_a^b ds = \int_a^b ds \quad . . . (1)$$

From Art. 51, we have, in the differential notation,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \quad \text{or,} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

By aid of the first expression, the "integrand" ds can be expressed in terms of x, by aid of the second in terms of y. That one should be used that will lead to the easiest integration.

Of course s can be found, as an indefinite integral, by integrating either value of ds above; but the method by which (1) was derived is instructive and should be easily understood, as it is one of the simplest applications of the theorem of Art. 95. Observe here that in place of adding portions $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$ of the arc CL, we add corresponding portions of successive tangents, that may be designated by $ds_1, ds_2, \ldots, ds_n$ (Art. 78). If we stopped here the result would be in error, however small Δx is taken, but all error vanishes on taking the limit of the sum and we reach an exact result. This case is typical of all the applications of the summatory method.

Examples.

1. Find the length of an arc of a parabola $y^2 = 4px$ from the origin (o, o) to the point (x, y). Take y as the independent variable,

$$\therefore s = \int_0^y ds = \int_0^y \left(1 + \left(\frac{dx}{dy}\right)^2\right)^{\frac{1}{2}} dy = \int_0^y \sqrt{1 + \frac{y^2}{4p^2}}\,dy$$

$$= \frac{1}{2p}\int_0^y \sqrt{4p^2 + y^2}\,dy = \frac{y}{4p}\sqrt{4p^2 + y^2}$$

$$+ p \log \frac{y + \sqrt{4p^2 + y^2}}{2p}.$$

2. Work Examples 1 and 5, Art. 51, and (vi) Art. 59 by use of formula (1).

3. For the circle $x^2 + y^2 = r^2$,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \frac{rdx}{\sqrt{r^2 - x^2}}$$

$\therefore$ by (1), the length of a circumference is equal to 4 times the length of a quadrant, or,

$$s = 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = 4r\left[\sin^{-1}\frac{x}{r}\right]_0^r = 2\pi r.$$

4. Ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. By Ex. 3, Art. 51,

$$ds = \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}\,dx.$$

This differential cannot be integrated directly, but $(a^2 - e^2 x^2)^{\frac{1}{2}}$ can be expanded by the binomial theorem and the separate terms can be integrated, as is shown in larger treatises, giving a slowly converging series, unless the eccentricity e is very small.

5. The method of expansion can be illustrated by the case of a flat arc of a parabola (Fig. 4) $y = cx^2$.

Let $\quad x = l$ when $y = a \therefore cl^2 = a \therefore c = a/l^2$

$$\therefore y = \frac{a}{l^2}x^2; \ \frac{dy}{dx} = \frac{2\,ax}{l^2} \therefore ds = \sqrt{1 + \frac{4\,a^2x^2}{l^4}}\,dx.$$

When $\frac{4\,a^2x^2}{l^4} < 1$, the radical expression can be expanded by the binomial formula.

$$\therefore ds = \left(1 + \frac{2\,a^2x^2}{l^4} - \frac{2\,a^4x^4}{l^8} + \ . \ . \ . \right) dx.$$

Integrate this between $x = 0$ and $x = l$,

$$\therefore s = l\left(1 + \frac{2}{3}\left(\frac{a}{l}\right)^2 - \frac{2}{5}\left(\frac{a}{l}\right)^4 + \ . \ . \ . \right).$$

When a/l is a small proper fraction, the series is rapidly convergent and can be used to compute an approximate value of s.

When the cable of a suspension bridge is loaded uniformly horizontally, the cable takes the form of an arc of a parabola. If the half span is l and the rise from the lowest point to the top of the tower a, the length of cable from tower to tower is $2\,s$, as given by the above formula approximately.

97. Extension of a heavy Vertical Bar suspended from its upper extremity and acted on only by its own weight. In Fig. 59, let AB represent a homogeneous bar of uniform cross-section of length l feet, suspended from B and acted on only by its own weight. Let the weight of the bar be w pounds per linear foot, so that the weight of a length $x = AC$ is wx pounds. Also call the extension of the bar per foot, when resting on a horizontal support, from a pull of 1 pound, e.

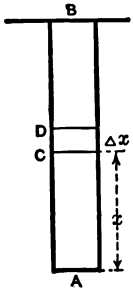

Fig. 59.

If we call $CD = \Delta x$, then the portion CD is extended from the weight wx lbs. of the part below C, an amount $wx \cdot e\Delta x$. However, CD is extended, not only by the weight of AC, but to some extent, by its own weight; but the total extension from both causes cannot exceed $w(x + \Delta x)e\Delta x$, for this assumes that the whole of the weight of AD, $w(x + \Delta x)$, acts upon *every* part of CD, whereas the weight of any portion of CD acts only upon the part of CD above it. The true extension of CD is thus between $wex\Delta x$ and $we(x + \Delta x)\Delta x$, whose ratio, $x/(x + \Delta x)$ has a limit 1 as $\Delta x \doteq 0$. All the more will the limit of the ratio of the true extension of CD to $wex\Delta x$ be unity; hence by the theorem of Art. 95, $wex\Delta x$ can replace the true extension of CD in taking the limit of the sum of such extensions, as x takes the usual successive values from 0 to l and $\Delta x \doteq 0$.

Thus suppose AB divided into n parts, each of length $h = \Delta x$, so that $l = nh$; then as x takes the successive values, $0, h, 2h, 3h, \ldots (n-1)h$, we find the total extension E of the bar, due to its own weight,

$$E = \lim_{h \doteq 0}[weh + we \cdot 2h + we \cdot 3h + \cdots + we(n-1)h]h$$

$$= \lim_{h \doteq 0} we[1+2+3+ \cdots +(n-1)]h = we\lim_{h \doteq 0} \tfrac{1}{2}n(n-1)h^2.$$

But, $n(n-1)h^2 = (nh)^2 - nh \cdot h = l^2 - l \cdot h$, whose limit as $h \doteq 0$ is l^2 $\therefore E = \frac{1}{2}wel^2$.

It was thought instructive to the student to give, in this simple example, the whole process implied by the integral sign. More briefly,

$$E = \lim_{\Delta x \doteq 0} \sum_0^l wex\,\Delta x = \int_0^l wexdx = \frac{1}{2} wel^2.$$

98. Center of Mass. In the case of a solid body, made up of material particles, if the masses (Art. 64) of the separate particles are denoted by m_1, m_2, m_3, . . . and their distances from any plane by x_1, x_2, x_3 . . . respectively, then the distance x_0 of *the center of mass* from the plane, for all the particles constituting the solid body, is defined to be,

$$x_0 = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma\, mx}{\Sigma\, m} \quad \ldots \ (1)$$

A similar formula holds if a second plane of reference is taken at right angles to the first and also for a third plane of reference at right angles to the other two. The three coördinates corresponding, say x_0, y_0 and z_0, thus determine uniquely *the center of mass.*

To apply the Calculus to the determination of the center of mass, the matter will have to be considered as distributed *continuously* (along a line, surface or volume, as the case may be), so that the limit of the sums in (1), as each $m \doteq 0$, will have to be taken.

Thus if the rectangle AB, of length l, width b, Fig. 59, be supposed to have matter distributed uniformly over it (as it would be if AB represented a thin sheet of paper of that form), the mass corresponding to any area, as that at CD of width b and height Δx, will be proportional to that area and can be replaced by it in finding center of mass.

Center of mass is coincident with center of gravity for actual bodies, and where the particle of mass m is supposed of infinitesimal dimensions, it can be shown, from the laws of Mechanics, that its center of gravity is at an infinitesimal distance from any of the points of the particle. In fact, if we combine successively the forces acting on the atoms which compose the particle, since the successive resultants act always *between* the two points of application of the forces corresponding, it is seen that the point of application of the final resultant, which passes through the center of gravity of the particle of mass m, will always be at an infinitesimal distance from all the points of the particle.

The values of x in (1) are strictly estimated to the center of mass of each particle; but when matter is supposed continuous, x can be taken as the distance from the fixed plane to any point on the surface or interior of the body of mass m, since as $m \doteq 0$, the limit of the ratio of the true to the substituted value of x is 1 (from what we have seen above) and it is the same for the ratio of the true to the substituted value of mx.

This follows from the theorem of Art. 95. Therefore in the formula for continuously distributed matter,

$$x_0 = \frac{\lim\limits_{m \doteq 0} \Sigma mx}{\lim\limits_{m \doteq 0} \Sigma m} \quad \cdots \quad (2)$$

x will be supposed measured from the fixed plane up to the *surface* of the particle of mass m. For any elementary mass m in (2), x is called the *arm* and mx the *moment* about the fixed plane.

99. Center of Mass of Plane Surfaces. When the matter is supposed distributed uniformly over an area A; Δa, the area corresponding to m takes the place of m in (2). The denominator is then $= A$ and (2) can be written

$$x_0 A = \lim_{\Delta a \doteq 0} \sum (x . \Delta a) \quad \cdots \quad (3)$$

$x_0 A$ is called the moment of the area. For *plane surfaces*, the "fixed plane" is taken perpendicular to the surface and x is thus measured from their intersection, which line is called the *axis*.

(3) states the theorem for plane surfaces: *The area multiplied by the distance from its center of mass to an axis is equal to the limit of the sum of the moments, about the axis, for each elementary area of the surface.*

EXAMPLE. The distance x_0 from A to the center of mass of the rectangle AB (Fig. 59) whose length $AB = l$, width b, is from (3), since $\Delta a = bdx$,

$$x_0 = \frac{\int_0^l bxdx}{bl} = \frac{1}{2} l.$$

Similarly we find that the center of mass is $\frac{1}{2} b$ distant from either side parallel to AB; hence it is at the center of figure.

To apply the theorem to the area $ACDB$ of Fig. 60, contained between the curve CD, whose equation is $y = f(x)$, the ordinates at $x = a$, $x = b$, and the $x =$ axis, we substitute for the true element of area $\Delta a = IKQP$ the rectangle $IPLK = y \Delta x$ and take for the arm, $x = OI$; then as in (ii) Art. 95, $A =$ area $ACDB = \int_a^b ydx$ and, by Art. 95, we can replace the true moment by $x\,(y\Delta x)$. Therefore by (3),

$$x_0A = \int_a^b xydx \cdots (4)$$

To find y_0, the distance of the center of mass of $ACDB$ from OX, we note from the example above, that the center of mass of the rectangle IL is $y/2$ above OX and of the rectangle constructed on IK and KQ as sides, $\frac{1}{2}\,(y + \Delta y)$ above OX; hence the center of mass of $IPQK$ lies between these two, and its distance above OX can be expressed by $\frac{1}{2}\,(y + \theta\,\Delta y)$, where $0 < \theta < 1$.

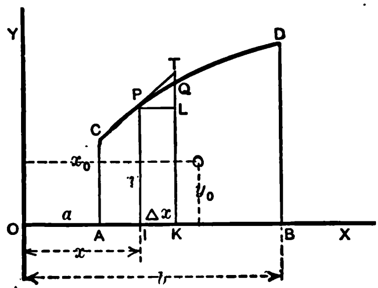

Fig. 60.

By the theorem above, the moment of the area $IPQK$ about OX, equals its area multiplied by the distance of its center of mass from OX, which product lies between, $y\Delta x \cdot \frac{y}{2}$ and $(y + \Delta y)\,\Delta x\,\frac{1}{2}\,(y + \theta\,\Delta y)$, whose ratio, as $\Delta x \doteq 0$, $\Delta y \doteq 0$, has a limit unity. Hence the limit of the ratio of the true moment to

$\frac{y^2\Delta x}{2}$ is 1. $\therefore$ by Art. 95, $\frac{y^2\Delta x}{2}$ can replace the true moment of $IPQK$ about OX in the limit of a sum.

The theorem above gives,

$$y_0A = \frac{1}{2}\lim_{\Delta x \doteq 0}\sum_a^b y^2\,\Delta x = \frac{1}{2}\int_a^b y^2dx \cdots (5)$$

EXAMPLE 1. To find the center of mass of an elliptic quadrant, the equation of the curve being, $ay = b\sqrt{a^2 - x^2}$.

By Ex. 3, Art. 93, $A = \dfrac{\pi ab}{4}$.

From (4), $x_0 \dfrac{\pi ab}{4} = \dfrac{b}{a}\displaystyle\int_0^a \sqrt{a^2 - x^2}\,x dx = \dfrac{1}{3} ba^2 \therefore x_0 = \dfrac{4}{3}\dfrac{a}{\pi}$.

From (5), $y_0 \dfrac{\pi ab}{4} = \dfrac{1}{2}\dfrac{b^2}{a^2}\displaystyle\int_0^a (a^2 - x^2)\,dx = \dfrac{1}{3} ab^2 \therefore y_0 = \dfrac{4}{3}\dfrac{b}{\pi}$.

On making $a = b$, x_0, y_0, for a circular quadrant of radius a are each, $\dfrac{4}{3}\dfrac{a}{\pi}$.

EXAMPLE 2. Determine (x_0, y_0) for a portion of a plane above the x-axis, bounded by the parabola, $y^2 = 4px$, the x-axis and the ordinate at $(x = b, y = c = \sqrt{4pb})$. See Art. 80.

Ans. $x_0 = \dfrac{3}{5}b, \quad y_0 = \dfrac{3}{8}c.$

EXAMPLE 3. Make a similar determination when the curves are,

(i) $ay^2 = x^3$; (ii) $a^2y = x^3$, from $x = 0$ to $x = b$.

Ans. (*i*) $x_0 = \dfrac{5}{7}b, \quad y_0 = \dfrac{1}{8}\dfrac{b^4}{a}$; (*ii*) $x_0 = \dfrac{4}{5}b, \quad y_0 \dfrac{2}{7}\dfrac{b^3}{a^2}$.

EXAMPLE 4. Find (x_0, y_0) for the trapezoid formed by the straight line $y = mx$, the ordinates at $x = a$ and $x = b$ and the x-axis.

Ans. $x_0 = \dfrac{2}{3}\dfrac{b^3 - a^3}{b^2 - a^2}, \quad y_0 = \dfrac{m}{3}\dfrac{b^3 - a^3}{b^2 - a^2} \therefore y_0 = \dfrac{1}{2}mx_0$

When $a = 0$, a triangle is formed for which,

$x_0 = \dfrac{2}{3}b, \; y_0 = \dfrac{1}{3}mb = \dfrac{1}{3}$ extreme ordinate.

100. To Find the Center of Mass of Any Plane Curve *CD*, Fig. 60. If $IK = dx$, then $LT = dy$, $PT = ds$, $PQ = \Delta s$ and (Art. 78) $lim\ ds/\Delta s = 1$ as $dx \doteq 0$ ∴ ds can replace m in (2) above which is now proportional to Δs.

$$\therefore \mathbf{x}_0 = \frac{\int_a^b \mathbf{xds}}{\int_a^b \mathbf{ds}} = \frac{\int_a^b \mathbf{xds}}{\mathbf{s}}; \quad \mathbf{y}_0 = \frac{\int_a^b \mathbf{yds}}{\mathbf{s}};$$

where s = length of arc *CD*.

Example 1. Let the curve be,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}, \text{ Ex. 5, Art. 51.} \quad \therefore ds = a^{\frac{1}{3}} x^{-\frac{1}{3}} dx;$$

∴ between the limits $x = 0$, $x = a$,

$$x_0 = \frac{2}{5}a, \; y_0 = \frac{2}{5}a.$$

Example 2. To find the center of mass of a circular arc *DBC*, Fig. 61, take y as the independent variable and integrate between the limits,

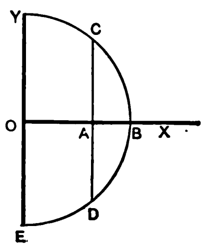

Fig. 61.

$$y = AC = c \text{ and } y = -c = -DA.$$

$$x^2 + y^2 = a^2 \; \therefore dx = -\frac{y}{x}dy;$$

$$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy = \frac{a}{x}dy.$$

$$\therefore x_0 = \frac{\int_{-c}^{c} xds}{s} = \frac{\int_{-c}^{c} ady}{s} = \frac{2c \cdot a}{DBC} = \frac{DC \cdot OB}{DBC}.$$

$y_0 = 0$ from considerations of symmetry

For a semi-circle *EBY*,

$$x_0 = \frac{2a \cdot a}{\pi \cdot a} = \frac{2a}{\pi}.$$

101. Center of Mass for a Solid of Revolution. In Fig. 60, let *ACDB* make a complete revolution about *OX*, thus generating "a solid of revolution." The element of volume was shown in Art. 95 (iii) to be $\pi y^2 dx$ or the volume generated by the rectangle *IL*. Therefore by the theorem of Art. 95, $\pi y^2\, dx$ can replace *m* in (2) Art. 98, which thus becomes,

$$\mathbf{x}_0 = \frac{\pi \int_a^b \mathbf{xy^2dx}}{\textbf{volume}} = \frac{\int_a^b \mathbf{xy^2dx}}{\int_a^b \mathbf{y^2dx}}.$$

$y_0 = 0$ or the center of mass is on the *x*-axis, as follows from the symmetry of the solid about the axis.

EXAMPLE 1. To find the center of mass of a right cone formed by revolving the line $y = ax$ about *OX* from $x = 0$ to $x = b$.

$$x_0 = \frac{\int_0^b a^2x^3dx}{\int_0^b a^2x^2dx} = \frac{3}{4}\, b.$$

EXAMPLE 2. To find the center of mass of a paraboloid of revolution. Let $y^2 = 4px$. Take limits from $x = 0$ to $x = b$.

$$x_0 = \frac{\int_0^b 4\, px^2dx}{\int_0^b 4\, pxdx} = \frac{2}{3}\, b.$$

EXAMPLE 3. Let the generating curve be the conic, $y^2 = 2\,Ax + Bx^2$, Art. 20 (5) and Fig. 16, the origin being at a vertex.

$$x_0 = \frac{\int_0^x (2\,Ax^2 + Bx^3)\,dx}{\int_0^x (2\,Ax + Bx^2)\,dx} = \frac{8\,Ax + 3\,Bx^2}{12\,A + 4\,Bx}.$$

For the parabola,

$$B = 0,\ 2A = 4p \therefore x_0 = \frac{2}{3}x \text{ (as in Ex. 2).}$$

For the ellipse,

$$2A = \frac{2b^2}{a},\quad B = -\frac{b^2}{a^2};$$

and for the hyperbola,

$$2A = \frac{2b^2}{a},\quad B = \frac{b^2}{a^2};$$

therefore using the upper sign for the hyperbola, the lower sign for the ellipse,

$$x_0 = \frac{8ab^2x \pm 3b^2x^2}{12ab^2 \pm 4b^2x},$$

which gives the distance from the vertex to the center of mass. If the extreme ordinate is at the center of the ellipse, $x = a$ and $x_0 = \frac{5}{8}a$. This result pertains to a semi-prolate spheroid.

If for the ellipse, we put $a = b$, we get for a spherical segment of one base of altitude x,

$$x_0 = \left(\frac{8a - 3x}{3a - x}\right)\frac{x}{4}.$$

$a - x_0$ gives the distance from the center of the sphere to the center of mass of the spherical segment.

102. Moment of Inertia. Space forbids entering extensively into this subject, which is of great importance in Mechanics. Moment of Inertia strictly involves mass, but in a special case, the discussion of the bending of beams or wherever the theory of elasticity is employed, the term "moment of inertia" is applied to areas with no reference to mass. In this case, it may be thus defined: if Δa represents a portion of the area of a given plane figure and r is its distance from a fixed line of the plane, called the axis, then I, the moment of inertia of the area of the figure about the axis, is,

$$I = \lim_{\Delta a \doteq 0} \sum r^2 \Delta a \quad . \; . \; . \quad (1)$$

where the summation extends over the whole area.

One or two examples will illustrate the application of the formula to simple cases.

EXAMPLE 1. To find the moment of inertia of the rectangle Fig. 59, Art. 99, about the lower edge, the width being b, the height h. Call $AC = x$, $CD = \Delta x$ and the element of area $\Delta a = b\Delta x$.

$$\therefore I = \lim_{\Delta x \doteq 0} \sum_0^h x^2 \cdot b\Delta x = b\int_0^h x^2 dx = \frac{bh^3}{3}.$$

EXAMPLE 2. If the axis is taken parallel to the base and $h/2$ above it, then the only change is in the limits of the integration, which are now $-h/2$, $h/2$, since x is to be estimated from the new axis.

$$\therefore I = b\int_{-h/2}^{h/2} x^2 dx = \frac{bh^3}{12}.$$

EXAMPLE 3. Find the moment of inertia of a right triangle OBC, Fig. 62, about OBX. Let $OB = b$, $BC = h$ and the equation of OC be, $y = ax$. Consider two points (x, y), $(x + \Delta x, y + \Delta y)$, on OC. The inscribed rectangle has a width Δx and altitude y,

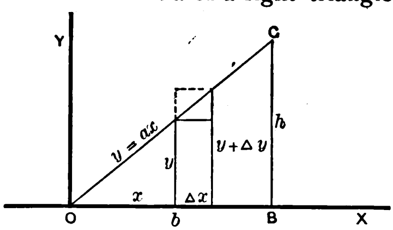

Fig. 62.

and its moment of inertia about OX by Ex. 1, is

$$\frac{y^3}{3}\Delta x = \frac{a^3x^3\Delta x}{3}.$$

The partly circumscribed rectangle has a width Δx and altitude $(y + \Delta y)$ and its moment of inertia about OX is,

$$\frac{(y + \Delta y)^3}{3} \Delta x.$$

The limit of the ratio of these,

$$\frac{(y + \Delta y)^3}{y^3},$$

as $\Delta x \doteq 0$, $\Delta y \doteq 0$, is unity. Hence, since the moment of inertia of the true area (the trapezoid with parallel sides y and $y + \Delta y$) is between that for the two rectangles, the limit of its ratio to $\frac{y^3}{3} \Delta x$ is unity. Hence by the theorem of Art. 95, $\frac{1}{3} a^3 x^3 \Delta x$ can be used for the true moment in taking the limit of the sum of all such moments from $x = 0$ to $x = b$.

$$\therefore \ I = \int_0^b \frac{1}{3} a^3 x^3 dx = \frac{a^3 b^4}{12} = \frac{bh^3}{12}.$$

It will be observed here that the moment of inertia of the inscribed rectangle $\frac{1}{3} y^3 \Delta x$ must be expressed as a function of x by substituting for y its value from the equation of the locus, $y = ax$ in this case. This rule also applies when the locus is a curve whose equation is $y = f(x)$. We have then to take the limit of the sum of all such moments as x changes (Δx at a time) from the inferior to the superior limit assumed.

CHAPTER XIII.

SINGULAR FORMS. PARTIAL DIFFERENTIATION.

103. Singular Forms. In Articles 24, 25 and 26 are given examples of functions that assume the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$ and $\infty . 0$ when a certain number is substituted for the variable. Other functions assume the forms 1^{∞}, ∞^{0}, 0^{0}, for certain values of the variable. The function, in all these cases, is said to assume a *singular* or *indeterminate* form, though the latter term is rather unfortunate, since the Calculus supplies general methods by which the limit of the function can be determined as the variable approaches the value in question.

Before developing the Calculus methods, let us give a geometrical interpretation which throws a full light upon the meaning of the results. McMahon and Snyder in their Differential Calculus, have given this interpretation for the function $\frac{x^2 - a^2}{x - a}$ which takes the form $\frac{0}{0}$ when a is put for x. What follows is simply an extension of the method to any function that assumes the form $\frac{0}{0}$.

Let us consider first the function,

$$y = \frac{x^2 - 8x + 15}{x^2 - 7x + 12} = \frac{(x-3)(x-5)}{(x-3)(x-4)} \quad . \; . \; . \; (1)$$

which takes the form $\frac{0}{0}$ for $x = 3$.

The value of y can be computed for any x but $x = 3$. For any other value than 3, the factor $(x - 3)$ can be struck out, giving,

$$y = \frac{x-5}{x-4} \quad . \; . \; . \quad (2)$$

For $x = 3$, this cancellation cannot be effected by any rule of Algebra, since division by 0 is not admissible.

Clear (1) of fractions and arrange as follows:

$$(x - 3)\,[y\,(x - 4) - (x - 5)] = 0 \quad . \; . \; . \quad (3)$$

This locus is made up of the straight line,

$$x = 3 \quad . \; . \; . \quad (4)$$

parallel to the y-axis and 3 units from it and the hyperbola (2) above, which is more conveniently plotted in the form,

$$y = 1 - \frac{1}{x-4} \quad . \; . \; . \quad (5)$$

The graphs of the hyperbola and straight line are shown in Fig. 63. When $x = 3$, the y of the hyperbola (2) or (5) is exactly 2; but for the straight line (4), y can have any value, in fact an infinite number of values, $y = 2$ being one of the infinite number.

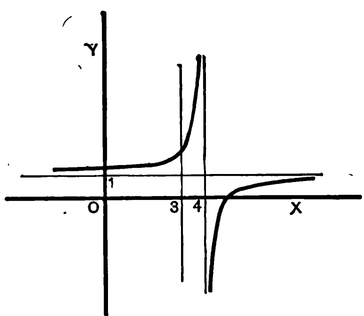

Fig. 63.

This explains perfectly why (1) assumes the form $\frac{0}{0}$ for $x = 3$; for then, where both loci are considered, y can have any value, corresponding to points on the straight line $x = 3$ or (4) above. It is usual to *define* the *value* of y for $x = 3$ as the limit to which (1) or (2) approaches indefinitely as $x \doteq 3$. It is convenient to do this, for this gives a value to y in (1) for $x = 3$, that is *continuous* with the values assumed by it as x passes through 3, corresponding to ordinates of the hyperbola only; but it must not be forgotten that the definition excludes part of the locus (1), viz., part of the straight line $x = 3$ and is to that extent arbitrary. The point of intersection (3, 2) of the two loci (4) and (5) corresponds to the value of y as given by the above definition. To get *lim* y as $x \doteq 3$ from (1), a limit that is actually attained, we simply cancel the factor $(x - 3)$ in (1) and put $x = 3$ in the result.

If the original function had been

$$y = \frac{(x-4)\,(x-5)}{(x-4)\,(x-4)},$$

the hyperbola locus will be found unchanged, but the straight line locus is now $x = 4$. On canceling $(x - 4)$, it is seen that $y \doteq \infty$ as $x \doteq 4$ as is evident from the figure since $x = 4$ is an asymptote. In this case for $x = 4$, $y = \frac{0}{0}$ from the original equation, but on cancellation, it is seen that $y \doteq \infty$ as $x \doteq 4$. It has no definite limit in this exceptional case. The function is discontinuous at $x = 4$.

Generally, *if*,

$$y = \frac{(x-a)\,F(x)}{(x-a)\,f(x)}$$

take the form $\frac{0}{0}$ *for* $x = a$, *its value is defined to be the value (if it has a definite value) of the right member, when* $(x - a)$ *is canceled and we put* $x = a$ *in the result.*

On clearing, etc., we have,

$$(x - a)\,[y f(x) - F(x)] = 0,$$

which gives the two loci,

$$x = a, \quad y = \frac{F(x)}{f(x)},$$

whose intersection corresponds to the value of y at $x = a$ as defined.

If *either* $F(x)$ or $f(x)$ are irrational, *either* $F(a)$ or $f(a)$ may be imaginary, in which case,

$$y = \frac{F(x)}{f(x)}$$

does not intersect the line $x = a$ and the original fraction is imaginary for this value of x.

Many expressions that assume the form $\frac{0}{0}$ can be evaluated by the rule above, on first expanding by Taylor's Theorem.

Thus, $$y = \frac{e^x - 2 + e^{-x}}{1 - \cos x},$$

has a limit 2 as $x \doteq 0$.

By Art. 72

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{\underline{|3}} + \ldots; \quad e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{\underline{|3}} + \ldots;$$

$$1 - \cos x = \frac{x^2}{\underline{|2}} - \frac{x^4}{\underline{|4}} + \ldots$$

$$\therefore y = \frac{x^2 + \frac{x^4}{12} \ldots}{\frac{x^2}{2} - \frac{x^4}{24} \ldots} = \frac{1 + \frac{x^2}{12} \ldots}{\frac{1}{2} - \frac{x^2}{24} \ldots}.$$

As all the terms of the infinite converging series in numerator and denominator contain x except the first,

$$lim\ y\ (\text{as } x \doteq 0) = \frac{1}{(\frac{1}{2})} = 2.$$

This sufficiently illustrates the method to pursue in similar examples.

In this way prove,

$$\lim_{x \doteq 0} \frac{e^x - e^{-x}}{\log(1+x)} = 2; \quad \lim_{x=0} \frac{x - \sin x}{x^3} = \frac{1}{6}; \quad \lim_{x \doteq 0} \left(\frac{\sin mx}{x}\right)^n = m^n.$$

$$\lim_{x \doteq 0} \frac{x - \tan^{-1} x}{x^3} = \frac{1}{3}; \qquad \lim_{x \doteq 0} \frac{a^x - b^x}{x} = \log \frac{a}{b}.$$

NOTE. If $f(x)$ is any continuous function of x, then by Taylor's series, Art. 67, assuming the series convergent when x is near a,

$$\frac{f(x) - f(a)}{x - a} = \frac{(x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \ldots}{x - a} \ldots \quad (1)$$

The division by $(x - a)$ in the right member can be effected except when $x = a$.

Let $y =$ left member and $F(x) = f'(a) + \frac{1}{2}(x-a)f''(a) + \ldots$ Here $F(x) =$ slope of secant of $f(x) = (f(x) - f(a))/(x-a)$, except when $x = a$, when $F(x) = f'(a) =$ slope of tangent line.

$$\therefore\ y = \frac{(x-a)\,F(x)}{(x-a)}.$$

$\therefore\ (x - a)\,[y - F(x)] = 0$ and the graph of this locus consists of the straight line $x = a$ and the curve $y = F(x)$, which intersect at $x = a$.

But, $$\lim_{x \doteq a} y = \lim_{x \doteq a} \frac{f(x) - f(a)}{x - a} = f'(a), \qquad \text{Art. 29,}$$

which is equal to, $$\lim_{x \doteq a} F(x) = f'(a).$$

Thus the derivative of $f(x)$ at $x = a$, is the one value of y (amongst the infinite number on the line $x = a$) which is continuous with the other values of y along the curve $y = F(x)$.

Thus the derivative of $f(x)$ at $x = a$, is referred to the general case above of functions that assume the form o/o if a is put for x. The result is in agreement with the fact that although a line through two points has a definite slope, a line through one point only can have an infinite number of slopes; but if the two points lie always on a curve, the slope of the line through them has a definite limit as the second point approaches indefinitely the first.

104. Form $\frac{0}{0}$. Let $F(x)$, $f(x)$, represent continuous functions of x in the vicinity of $x = a$, so that $F(x + h)$, $f(x + h)$, can be expanded by Taylor's formula. Also suppose $F(a) = 0$, $f(a) = 0$. By the theorem of mean value, Art. 73,

$$\frac{F(a+h)}{f(a+h)} = \frac{F(a) + hF'(a+\theta h)}{f(a) + hf'(a+\theta' h)} = \frac{F'(a+\theta h)}{f'(a+\theta' h)},$$

where the values of θ and θ' are between 0 and 1.

When $h = 0$, the left member assumes the form $\frac{0}{0}$. Put $x = a + h$ and note that as $h \doteq 0$, $x \doteq a$; then taking limits, as $h \doteq 0$,

$$\lim_{x \doteq a} \frac{F(x)}{f(x)} = \frac{F'(a)}{f'(a)}.$$

Hence, if $\frac{F(x)}{f(x)}$ *assumes the form* $\frac{0}{0}$ *when* $x = a$, *to find its limit as* $x \doteq a$, *the derivative of the numerator divided by the derivative of the denominator is formed and* x *put equal to* a *in the result.*

It often happens that $\frac{F'(a)}{f'(a)}$ assumes the form $\frac{0}{0}$, in which case, the same rule applies to this function, so that its limit as $x \doteq a$, is $\frac{F''(a)}{f''(a)}$ and so on for similar cases. The case where $a \doteq \infty$ is treated in standard works and leads to the rule given above.

EXAMPLE. $\frac{\sin x - \sin x \cos x}{x^3}$ takes the form $\frac{0}{0}$ for $x = 0$.

$$\therefore \lim_{x \doteq 0} \frac{\sin x - \sin x \cos x}{x^3} = \lim_{x \doteq 0} \frac{\cos x + \sin^2 x - \cos^2 x}{3x^2}.$$

On noting that this takes the form $\frac{0}{0}$ for $x = 0$, we differentiate again. The result again comes under the rule, so the process is repeated a third time. The previous result then equals,

$$\lim_{x \doteq 0} \frac{-\sin x + 4 \sin x \cos x}{6x} = \lim_{x \doteq 0} \frac{-\cos x + 4(\cos^2 x - \sin^2 x)}{6} = \frac{1}{2}.$$

See Art. 25 for another way of reaching this result.

Examples.

1. $\lim_{x \doteq \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \doteq \pi/2} \frac{-\cos x}{-\sin x} = \frac{0}{1} = 0.$

2. $\lim_{x \doteq 0} \frac{2 \sin x - x}{x \sin x} = \lim_{x \doteq 0} \frac{2 \cos x - 1}{\sin x + x \cos x}.$

This expression has no definite limit but as $x \doteq 0$, its value $\doteq \infty$.

3. $\lim_{x \doteq 0} \frac{x \sin x - x^2}{2 \cos x + x^2 - 2} = -2.$

This example can be more easily worked by expansion.

4. Find limits, as $x \doteq 0$, of,

$$(i)\ \frac{e^x - e^{-x}}{\sin x};\quad (ii)\ \frac{x \log(1+x)}{1-\cos x};\quad (iii)\ \frac{1-\cos x}{x^2};\quad (iv)\ \frac{a-\sqrt{a^2-x^2}}{x^2}.$$

Ans. (i) 2; (ii) 2; (iii) $\frac{1}{2}$; (iv) $\frac{1}{2a}$.

5. After differentiating the first time, reduce and cancel the factor $\sqrt{x-a}$ in the following:

$$\lim_{x \doteq a} \frac{\sqrt{x-a}+\sqrt{x}-\sqrt{a}}{\sqrt{x^2-a^2}} = \lim_{x \doteq a} \frac{1}{2} \frac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}} \cdot \frac{\sqrt{x+a}}{x}$$

$$= \frac{\sqrt{2a}}{2a} = \frac{1}{\sqrt{2a}}.$$

6. $\displaystyle \lim_{x \doteq 1} \frac{x^{\frac{3}{2}} - 1 + (x-1)^{\frac{3}{2}}}{(x^2-1)^{\frac{3}{2}} - x + 1} = -\frac{3}{2}.$

7. $\displaystyle \lim_{x \doteq 1} \frac{b^{1-x} - a^{1-x}}{1-x} = \log \frac{b}{a}.$

For brevity we shall write below $1/0 = \infty$, $1/\infty = 0$, with the interpretation already given in Art. 15.

105. Forms ∞/∞, $0 \cdot \infty$ *and* $\infty - \infty$.

Where $F(x)/f(x)$ takes the form ∞/∞ for $x = a$, write,

$$\frac{F(x)}{f(x)} = \frac{1/f(x)}{1/F(x)},$$

which last expression assumes the form $0/0$ for $x = a$; hence by the rule above, the limit, as $x \doteq a$, of the given function, is the limit, as $x \doteq a$, of the following:

$$\frac{F(x)}{f(x)} = \frac{-\frac{1}{(f(x))^2} f'(x)}{-\frac{1}{(F(x))^2} F'(x)} = \left(\frac{F(x)}{f(x)}\right)^2 \cdot \frac{f'(x)}{F'(x)}.$$

On dividing through by $\lim \dfrac{F(x)}{f(x)}$, and solving for $\lim F(x)/f(x)$, we find,

$$\lim_{x \doteq a} \frac{F(x)}{f(x)} = \lim_{x \doteq a} \frac{F'(x)}{f'(x)} = \frac{F'(a)}{f'(a)},$$

giving the same rule as for functions that assume the form $0/0$ for $x = a$.

The division above by $\lim F(x)/f(x)$ when it $\doteq \infty$ as $x \doteq a$, or when its limit is 0, is not admissible. It is shown, however, in larger treatises (see Todhunter or Byerly in Differential Calculus) that the same rule given above holds for this case as well as for previous cases.

Transformations are allowed at any stage not only in this, but in all cases.

$$\text{Ex.}\quad \lim_{x \doteq 0} \frac{\log \cot x}{\log x} = \left.\frac{-\dfrac{\csc^2 x}{\cot x}}{x^{-1}}\right]_{x=0} = -\left.\left(\frac{x}{\sin x}\right)\frac{1}{\cos x}\right]_{x=0} = -1.$$

The given function here takes the form ∞/∞ when $x = 0$. Functions assuming the forms $0 \cdot \infty$, $\infty - \infty$ for $x = a$ can be transformed so as to take the form $0/0$. Thus if $F(x) \doteq 0$ and $f(x) \doteq \infty$ as $x \doteq a$,

$F(x) \cdot f(x)$ can be changed to $\dfrac{F(x)}{\dfrac{1}{f(x)}}$, which tends to the form $0/0$ as $x \doteq a$.

EXAMPLE 1. $x \log x$ takes the form $0 \times -\infty$ for $x = 0$, but $\dfrac{\log x}{1/x}$ takes the form $\dfrac{\infty}{\infty}$ to which the rule applies.

$$\therefore \quad \lim_{x \doteq 0} x \log x = \lim_{x \doteq 0} \frac{1/x}{-1/x^2} = \lim_{x \doteq 0} (-x) = 0.$$

EXAMPLE 2. Substituting $1/y$ for x is sometimes serviceable. $[(a^{\frac{1}{x}} - 1)\,x]$ takes the form $0 \cdot \infty$ for $x = \infty$; but if $y = \frac{1}{x}$, as $x \doteq \infty$, $y \doteq 0$ and,

$$\lim_{x \doteq \infty} (a^{\frac{1}{x}} - 1)\,x = \lim_{y \doteq 0} \frac{a^y - 1}{y} = \lim_{y \doteq 0} \frac{a^y \log a}{1} = \log \text{a}.$$

EXAMPLE 3. $\frac{1}{\log x} - \frac{x}{x-1} \doteq \infty - \infty$ as $x \doteq 1$; but this is equal to, $\frac{x - 1 - x\log x}{\log x\,(x-1)}$ which for $x = 1$ takes the form $\frac{0}{0}$.

$\therefore$ its limit as $x \doteq 1$, is,

$$\lim_{x \doteq 1} \frac{1 - \log x - 1}{\frac{x-1}{x} + \log x} = \lim_{x \doteq 1} \frac{-\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} = -\frac{1}{2}.$$

EXAMPLE 4. $\lim_{x \doteq \pi/2} \left(x \tan x - \frac{\pi}{2} \sec x\right) = -1.$

EXAMPLE 5. $\lim_{x \doteq 1} (1 - x) \tan \frac{\pi x}{2} = \frac{2}{\pi}.$

106. Forms $0^0, \infty^0, 1^\infty$. Functions that assume these forms for $x = a$, can, by use of logarithms, as illustrated in the examples below, be reduced to other functions that assume some of the forms already discussed.

EXAMPLE 1. $x^{\sin x}$ takes the form 0^0 when $x = 0$.

Now since, $\quad x = e^{\log x}, \quad x^{\sin x} = e^{\sin x \log x}.$

As $x \doteq 0$, $\lim \sin x \log x = \lim \left(\frac{\sin x}{x} \cdot x \log x\right) = 1 \times 0 = 0,$

since $\lim\,(\sin x/x) = 1$, Arts. 24 and 26 and Ex. 1, Art. 105.

$$\therefore \lim_{x \doteq 0} x^{\sin x} = e^0 = 1.$$

38. The volume generated by revolving the arc of this parabola from (0, a) to (a, 0) about the axis of x is,

$$\pi \int_0^a y^2 dx = \frac{7}{5} a^3 \pi.$$

39. The cycloid is defined by the equations,

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta).$$

$$\therefore\ dx = a(1 - \cos\theta)\, d\theta, \quad dy = a \sin\theta\, d\theta.$$

$$\therefore\ ds^2 = dx^2 + dy^2 = 2a^2(1 - \cos\theta) = 4a^2 \sin^2\frac{\theta}{2} \cdot d\theta^2.$$

Hence the length of the arc of the cycloid from $\theta = 0$ to $\theta = \pi$ is,

$$\int_0^\pi ds = 4a \int_0^\pi \sin\frac{\theta}{2} \cdot \frac{d\theta}{2} = -4a \cos\frac{\theta}{2}\Big]_0^\pi = 4a.$$

40. The equation $(x - a)^2 + y^2 = a^2$, represents a circle of radius a, with origin at the left extremity of a horizontal diameter. Find the area of a quadrant of this circle. See Ex. 24.

$$\int_0^a \sqrt{2ax - x^2} \cdot dx = \frac{\pi a^2}{4}.$$

41. What is the area above the x-axis included between the curves $y^2 = 2ax - x^2$ and $y^2 = ax$?

The circle (just considered) and the parabola touch at (0, 0) and cut at (a, a).

$$\therefore\ \text{Area} = \int_0^a (\sqrt{2ax - x^2} - \sqrt{ax})\, dx = \frac{\pi a^2}{4} - \frac{2}{3} a^2.$$

42. A woodman cuts to the center of a vertical tree trunk, 2 feet in diameter. The lower face of the cut is horizontal, the upper face is inclined 45° to the horizontal. What is the volume of the wood removed? Use (1), Art. 81.

Ans. $\frac{2}{3}$ cu. ft.

Trigonometric Formulas.

$$\sin^2\theta + \cos^2\theta = 1; \quad \tan\theta = \frac{\sin\theta}{\cos\theta}; \quad \sec\theta = \frac{1}{\cos\theta}.$$

$$\sec^2\theta = 1 + \tan^2\theta;$$

$$\operatorname{cosec}^2\theta = 1 + \cot^2\theta; \quad \cot\theta = \frac{\cos\theta}{\sin\theta}; \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta.$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).$$

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).$$

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

$$\sin 2\theta = 2\sin\theta\cos\theta; \quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$2\sin^2\tfrac{1}{2}\theta = 1 - \cos\theta; \quad 2\cos^2\tfrac{1}{2}\theta = 1 + \cos\theta.$$

$$\tan\frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}.$$

$$\sin(-\theta) = -\sin\theta; \ \cos(-\theta) = \cos\theta; \ \tan(-\theta) = -\tan\theta.$$

$$\sin(\pi-\theta) = \sin\theta; \qquad \sin(\pi+\theta) = -\sin\theta.$$

$$\cos(\pi-\theta) = -\cos\theta; \qquad \cos(\pi+\theta) = -\cos\theta.$$

$$\tan(\pi-\theta) = -\tan\theta; \qquad \tan(\pi+\theta) = \tan\theta.$$

$$\sin(\pi/2-\theta) = \cos\theta; \qquad \sin(\pi/2+\theta) = \cos\theta.$$

$$\cos(\pi/2-\theta) = \sin\theta; \qquad \cos(\pi/2+\theta) = -\sin\theta.$$

$$\tan(\pi/2-\theta) = \cot\theta; \qquad \tan(\pi/2+\theta) = -\cot\theta.$$

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14
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