NOTES

UPON

LEAST SQUARES

AND

GEODESY

PREPARED FOR USE IN

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PART I.

LEAST SQUARES.

CHAPTER I.

METHOD OF LEAST SQUARES.

1. INTRODUCTION. The method has for its object the finding of the best or most probable values, for a set of unknown quantities depending upon physical measurement, which can be obtained from a given set of observations; and to find the degree of confidence which can be placed in the results, as determined from the agreement of the observations among themselves. This agreement may be very misleading as to the actual accuracy of the results unless the circumstances under which the observations were taken are known.

The observations are subject to several classes of errors as follows:

1st. Constant errors, or those which under the same circumstances, and in the measures of the same quantity, have the same value: or those in which the value can be made to depend upon the circumstances by some definite law. They are usually subdivided into; theoretical, such as refraction and curvature in leveling, etc., whose effects, when their causes are once thoroughly understood, can be computed in advance, and hence they cease to exist as errors; instrumental, such as the line of collimation of a level not being horizontal when the bubble is in the center, etc., which are discovered by an examination of the instruments; or of the observations made with them and may be removed, when their causes are understood, either by a proper method of using the instruments or by subsequent computation; personal, such as always setting a target a little too high, etc., and which depend upon the peculiarities of the observer. These latter are often the subject of special investigation under the name of "personal equation"; while not strictly constant they are nearly so with trained observers.

These errors are sought out and eliminated or corrections applied as far as possible.

2nd. Mistakes or abnormal errors, such as reading a circle a degree out of the way, the slipping of a clamp, the sighting at a wrong object, etc.

3rd. Accidental errors, or the necessary inaccuracies which cannot be computed in advance from the circumstances of the observations and eliminated.

The limit of the first class is fixed by the limit of knowledge of instruments and of physical phenomena.

The limit of the second class can only be approximately fixed, as there are no means of distinguishing between inaccuracies and small mistakes.

In what follows the third class should be understood, unless otherwise stated.

The following may be assumed as axioms:

1. Small errors occur more frequently, or are more probable than large ones.
2. Positive and negative errors of the same magnitude are equally probable, and in a large number of observations are equally frequent.
3. Very large errors do not occur.

2. MEAN-SQUARE ERROR. The square root of the average square of the errors, is called the mean-square error, denoted by m.s.e. or ϵ. It is used in comparing different sets of observations.

Thus if $\Delta_1, \Delta_2, \cdots \Delta_n$ be the true errors committed in a series of n equally good observations,

$$\epsilon^2 = \frac{\Delta_1^2 + \Delta_2^2 \cdots + \Delta_n^2}{n} = \frac{[\Delta^2]}{n}^{\times} \qquad (1)$$

Ex. 1. In Gradmessung in Ostpreussen the excess over 180° plus the spherical excess, is given in seconds for the measured angles in 22 triangles as follows:

$^{\times}$The square brackets are used to denote summation.

No.	Δ	Δ²	No.	Δ	Δ²	No.	Δ	Δ²	No.	Δ	Δ²
				For.	6.249		For.	10,855			24.411
1	+ .36	+ .130	7	+1.76	3.098	13	-1.36	1.850	19	+1.67	2.789
2	+ .93	+ .865	8	+0.92	+.846	14	+1.86	+ 3.460	20	-0.72	0.518
3	- .51	+ .260	9	+ .56	+.314	15	-0.42	+ .176	21	1.35	1.822
4	-1.46	+2.132	10	.00	.000	16	+ 1.68	+ 2.822	22	-0.98	0.960
5	- .95	+0.902	11	- .59	+.348	17	+ 1.62	+ 2.624			
6	-1.40	+1.960	12	.00	+ .000	18	+1.62	+ 2.624			
		6.249			10.855			24.411			30.500

$\therefore$ the m.s.e., $\varepsilon = \sqrt{30.500/22} = 1''.18$

3. LAW OF PROPOGATION OF ERROR.

$$\text{Let } x = a_1 M_1 \pm a_2 M_2 \pm \ldots\ a_n M_n \qquad (2)$$

where $a_1, a_2 \ldots a_n$, are constants unaffected by error, and $M_1, M_2, \ldots M_n$, are observed independent quantities with the m.s.e's., $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$

If $\Delta_1', \Delta_1'', \Delta_1''', \ldots \Delta_2', \Delta_2'', \Delta_2''', \ldots \Delta_n', \Delta_n'', \Delta_n'''$, are the errors for different observed values of $M_1, M_2 \ldots M_n$, the errors in the corresponding values of x will be,

$$\left.\begin{aligned} \pm\Delta_x' &= \pm a_1\Delta_1' \pm a_2\Delta_2' \pm \cdots\ a_n\Delta_n' \\ \pm\Delta_x'' &= \pm a_1\Delta_1'' \pm a_2\Delta_2'' \pm \cdots\ a_n\Delta_n'' \\ \pm\Delta_x''' &= \pm a_1\Delta_1''' \pm a_2\Delta_2''' \pm \cdots\ a_n\Delta_n''' \end{aligned}\right\} \qquad (a)$$

Squaring each line and adding,

$$[\Delta_x^2] = a_1^2[\Delta_1^2] + a_2^2[\Delta_2^2] \cdots + a_n^2[\Delta_n^2] \pm 2a_1a_2[\Delta_1\Delta_2] \cdots \pm 2a_1a_n[\Delta_1\Delta_n] \cdots \pm 2a_2a_n[\Delta_2\Delta_n] \qquad (b)$$

Positive and negative errors of the same magnitude being equally liable to occur, by axiom 2, §1, the products

$$\pm 2\, a_1a_2[\Delta_1\Delta_2] \cdots \pm 2a_1\, a_n[\Delta_1\Delta_n] \cdots \pm 2a_2\, a_n[\Delta_2\Delta_n]$$

will tend to foot up zero (approaching it nearer the greater the number of observed values) and may be neglected.

$\therefore$ dividing (b) by n, and remembering the definition of m.s.e., §2,

$$\varepsilon_x^2 = a_1^2\, \varepsilon_1^2 + a_2^2\, \varepsilon_2^2 + \ldots\ldots a_n^2\, \varepsilon_n^2 \qquad (3)$$

In the general case,

$$x = f(M_1, M_2, \ldots M_n) \qquad (4)$$

where f ( ) denotes any function. —

If the different observed values be substituted for the true values of the observed quantities, we shall have

$$x \pm \Delta_x' = f(M_1 \pm \Delta_1', M_2 \pm \Delta_2', \cdots M_n \pm \Delta_n')$$
$$x \pm \Delta_x'' = f(M_1 \pm \Delta_1'', M_2 \pm \Delta_2'', \cdots M_n \pm \Delta_n'')$$

Expanding the second members by Taylor's theorem, and supposing the observations accurate enough so that the squares products and higher powers of the Δ's may be neglected,*

$$\Delta_x' = \pm\frac{df}{dM_1}\Delta_1' \pm \frac{df}{dM_2}\Delta_2' \cdots \pm\frac{df}{dM_n}\Delta_n'$$
$$\Delta_x'' = \pm\frac{df}{dM_1}\Delta_1'' \pm \frac{df}{dM_2}\Delta_2'' \cdots \pm\frac{df}{dM_n}\Delta_n''$$

---

*If $u = f(x, y)$, and x be increased by h,

$$f(x+h, y) = u + \frac{du}{dx}\cdot\frac{h}{1} + \frac{d^2u}{dx^2}\cdot\frac{h^2}{1.2} + \cdots$$

where $\frac{du}{dx} = f'(y)$, which we may place $= z$.

If y be increased by k,

$$z' = f'(y+k) = z + \frac{dz}{dy}\cdot\frac{k}{1} + \frac{d^2z}{dy^2}\ \frac{k^2}{1.2} + \cdots\cdots = \frac{du}{dx} + \frac{d\frac{du}{dx}}{dy}k + \cdots$$

Hence,

$$f(x+h, y+k) = f(x, y+k) + \frac{du}{dx}h + \frac{d\frac{du}{dx}}{dy}hk + \frac{d^2u}{dx^2}\cdot\frac{h^2}{2} + \cdots$$

But,

$$f(x, y+k) = f(x, y) + \frac{du}{dy}\cdot\frac{k}{1} + \frac{d^2u}{dy^2}\cdot\frac{k^2}{1.2} + \cdots$$

Substituting,

$$f(x+h, y+k) = f(x, y) + \frac{du}{dy}k + \frac{du}{dx}h \cdots$$

Similarly we may extend to 3 or more variables, as assumed above.

These correspond to (a), ∴ from (4),

$$\epsilon_x^2 = \left(\frac{df}{dM_1}\epsilon_1\right)^2 + \left(\frac{df}{dM_2}\epsilon_2\right)^2 + \cdots \left(\frac{df}{dM_n}\epsilon_n\right)^2 \qquad (5)$$

Ex.1. Find the m.s.e. in the length of a city block 500 ft. long measured with a 100 ft. tape having a m.s.e. in its length of .01 ft.
Ans. 0.05 ft.

Ex.2 Find the m.s.e. in the length of a city block 500 feet long measured with 5 100-ft. tapes each with a m.s.e. of .01 ft.
Ans. 0.02 ft.

Ex.3. In the triangle ABC, AC or b = 1080 ft. with ϵ_1 = 0.1 ft.; A = 50° with ϵ_2 = 10" (in arc, =10 sin 1"); B = 64° with ϵ_3 = 10". Find the m.s.e. for BC, or a.

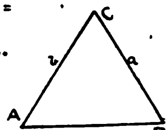


Fig.1.

(4) reduces to $a = b \dfrac{\sin A}{\sin B}$ = 903.44 ft.

$$\frac{df}{dM_1} = \frac{da}{db} = \frac{\sin A}{\sin B} = \frac{a}{b} = .85$$

$$\frac{df}{dM_2} = \frac{da}{dA} = \frac{b \cos A}{\sin B} = a \cot A = 758$$

$$\frac{df}{dM_3} = \frac{da}{dB} = -\frac{\sin A \cos B}{\sin^2 B}b = -a \cot B = -441$$

Substituting in (5)

$$\epsilon_a^2 = a^2\epsilon_1^2/b^2 + a^2\cot^2 A\,\epsilon_2^2 + a^2\cot^2 B\,\epsilon_3^2$$
$$= (.85 \times .1)^2 + (441 \times 10 \times .00000485)^2 + (758 \times 10 \times .00000485)^2$$
$$= .0072 + .0005 + .0013 = .009$$

ϵ_a = .095 ft.

4. LAW OF PROPOGATION OF ERROR, b. In (b) of §3 it may be noted that in summing the products of the Δ's, one of the factors, Δ, may contain constant error, or otherwise differ from the accidental errors of observation included in class 3, §1, and the sum of the products will still approximate zero, so that the value of ϵ_x^2 will be given by (3) or (5).

Ex. 1 If the m.s.e. in placing a 50-ft. tape length is .02 ft., and the m.s.e. in the length is .01 ft.; find that in a 1500-ft. line measured with the tape, due to both causes.

From the first cause by (3),

$\epsilon_x'^2 = (.02)^2 + (.02)^2 + \ldots$ for 1500/50 = 30 terms (each measurement giving an M). Or,

$\epsilon_x' = .02\sqrt{30}$ = .1 ft.

From the second cause, by (3),

ϵ_x'' = 1500/50 .01 = .3 ft.

From both together,

$\epsilon_x = \sqrt{(.1)^2 + (.3)^2}$ = .32 ft.

Reducing the m.s.e. from the first source to one-half its value would only reduce the final m.s.e. to .3 ft., while reducing that from the second source to one-half would reduce the final m.s.e. to .18 ft.

It is thus seen that but little is gained in accuracy by reducing one source of error when there is another much larger one unnoticed; the great gain comes from reducing the large source.

5. THE SIMPLE ARITHMETIC MEAN. When a number of equally good, direct, and independent observations are taken for the value of an unknown quantity, the arithmetic mean is always taken for the best or most probable value, there being no reason for giving more influence to one than to another of the observations.

Thus if the observed values are $M_1, M_2, \ldots M_n$, the most probable value,

$$x_0 = [M]/n \qquad (6)$$

This can be written

$$x_0 = 1/n\, M_1 + 1/n\, M_2 + 1/n\, M_3 \ \ldots\ldots$$

so that if ε = m.s.e. for an observed M and ε_0 = the m.s.e. for x_0, we have from (3),

$$\varepsilon_0^2 = (\varepsilon/n)^2 + (\varepsilon/n)^2 + \text{to } n \text{ terms}, \quad = n(\varepsilon/n)^2$$

Or,

$$\varepsilon_0^2 = \varepsilon^2/n \tag{7}$$

i.e., the m.s.e. of the arithmetic mean decreases as the square root of the number of observations increases.

The difference between the arithmetic mean and the different observed values are called residuals.

If they be denoted by v_1, v_2 ..., and the error of the arithmetic mean by δ, we shall have,

Residuals, $v_1 = x_0 - M_1, v_2 = x_0 - M_2 \ldots$

True errors, $\Delta_1 = (x_0 \pm \delta) - M_1$, $\Delta_2 = (x_0 \pm \delta) - M_2, \ldots$

or $\Delta_1 = v_1 \pm \delta$, $\Delta_2 = v_2 \pm \delta$, ...

squaring and adding,

$$[\Delta^2] = [v^2] \pm 2\delta[v] + n\delta^2$$

(6) can be written,

$$(x_0 - M_1) + (x_0 - M_2) + (x_0 - M_3) \ldots = 0$$

$$\text{or } [v] = 0 \tag{8}$$

Substituting and dividing by n,

$$\varepsilon^2 = [v^2]/n + \delta^2$$

The most probable value of δ, the error of x_0, is usually assumed to be the m.s.e. of the mean itself, or $\varepsilon_0 = \varepsilon/\sqrt{n}$. Substituting,

$$\varepsilon^2 = [v^2]/n + \varepsilon^2/n$$

$$\varepsilon^2 = [v^2]/(n-1) \tag{9}$$

From (7),

$$\varepsilon_0^2 = [v^2]/(n(n-1)) \tag{10}$$

Ex. 1. The following values are given in Pri. Tri. U.S. Lake Survey, p. 895, for the observed difference in longitude between Detroit and Cambridge.

					$-v$	$+v$	v^2
June	21	0^h	47^m	$41^s.154$	.040		.0016
"	22			41.171	.057		.0030
	23			41.138	.024		.0006
"	24			41.110		.004	.0000
"	29			40.995		.119	.0142
mean, x_0		0	47	41.114	.121	.123	.0194

$$\varepsilon_0 = \sqrt{[v^2]/(n(n-1))} = \sqrt{.0194/20} = 0^s.031$$

6. THE WEIGHTED ARITHMETIC MEAN. An observation is said to have the weight w, when its m.s.e. is equal to that of the mean of w observations of weight unity. If then ε is the m.s.e. of an observation of weight unity, and $\varepsilon_1, \varepsilon_2, \ldots$, are the m.s.e's. for weights w_1, w_2, we have from (7),

$$\varepsilon_1^2 = \varepsilon^2/w_1, \quad \varepsilon_2^2 = \varepsilon^2/w_2$$

or,

$$w_1/w_2 = \varepsilon_2^2/\varepsilon_1^2 \tag{11}$$

i.e., the weights are inversely as the squares of the m.s.e's.

If the different values of a quantity, $M_1, M_2, M_3, \ldots$, have the weights $w_1, w_2, w_3, \ldots$, each value being supposed to be the mean of w values of weight unity, the sum of the original values can be found by multiplying each mean by the number and adding; the average can then be found by dividing by the total number. I.e., the arithmetic mean,

$$x_0 = (M_1 w_1 + M_2 w_2 + M_3 w_3 \ldots)/(w_1 + w_2 + w_3 + \ldots) = [Mw]/[w] \tag{12}$$

The m.s.e. of the mean,

$$\varepsilon_0 = \varepsilon/\sqrt{[w]} \tag{13}$$

(12) can be written,

$$(x_0 - M_1)w_1 + (x_0 - M_2)w_2 + (x_0 - M_3)w_3 + \ldots = 0, \text{ i.e., } [vw] = 0 \tag{14}$$

As in §5, $\Delta_1 = v_1 \pm \delta$, $\Delta_2 = v_2 \pm \delta$, $\Delta_3 = v_3 \pm \delta$.

If each equation be squared, then multiplied by its corresponding w, and added,

$$[w\Delta^2] = [w v^2] \pm 2\delta[wv] + \delta^2[w] \tag{a}$$

The observations with weights w give errors Δ; the corresponding errors for weight unity would most probably be $\Delta\sqrt{w}$, from the relation (11) between weights and m.s.e's.

$[w\Delta^2]$ is ∴ the sum of the squares of the errors for weights unity, $= n\varepsilon^2$ by (1).

By §5, $\delta = \varepsilon_0$, $= \varepsilon/\sqrt{[w]}$ by (13)

Substituting in (a),

$$n\varepsilon^2 = [w\ v^2] + \varepsilon^2$$

$$\varepsilon^2 = [w\ v^2]/(n-1) \qquad (15)$$

$$\varepsilon_0^2 = [w\ v^2]/([w](n-1)) \qquad (16)$$

Ex. 1. The following values are given in Pri.Tri. U.S.Lake Survey, p. 895, for the observed difference in longitude between Detroit and Cambridge.

			w	v	wv	wv^2
May 13	$0^h\ 47^m$	$41^s.163$	0.5	$-.117$	$-.059$	.00684
23		40.986	0.5	$+.080$	$+.040$	.00320
24		41.038	1.0	$+.008$	$+.008$	.00006
26		41.030	1.0	$+.016$	$+.016$	.00026
June 4		41.084	1.0	$-.038$	$-.038$	.00144
11		41.012	1.0	$+.034$	$+.034$	.00116
Mean = 0	47	41.048		Sums	$+.001$	.01298

$$\varepsilon_0 = \sqrt{.01298/25} = 0^s.023.$$

7. CONTROLS.

Simple Arithmetic Mean.

Since, $v_1 = x_0 - M_1$, $v_2 = x_0 - M_2$, $v_3 = x_0 - M_3$ . . . , and $n\ x_0 = [M]$,

$$[v^2] = n\ x_0^2 - 2\ x_0[M] + [M^2]$$

Or, $[v^2] = [M^2] - [M]^2/n$

Also, from (8), $[v] = 0$ (17)

Weighted Arithmetic Mean.

Since $v_1 = x_0 - M_1$, $v_2 = x_0 - M_2$, $v_3 = x_0 - M_3$, and by (12), $[w]\,x_0 = [w\ M]$,

$$[w\ v^2] = x_0^2[w] - 2\ x_0[w\ M] + [w\ M^2]$$

$[w\ v^2] = [w\ M^2] - [w\ M]^2/w$

Also, from (14), $[v\ w] = 0$ (18)

It may be noted that the left hand places as far as they agree may be left off from the values of M, or any constand subtracted, whenever it will simplify the numerical computation for (17) or (18).

In Ex. 1, §5, we have for the different values of M, subtracting 41 from each; .154, .171, .138, .110, -.005.

Squaring and adding,

$[M^2]$ = ~~Adding and squaring,~~	.0841
$[M]^2/n$ =	.0645
$[M^2] - [M]^2/n$ =	.0196 nearly checking $[v^2]$.

The mean x_0 when maltiplied by 5 is +.002 greater than $[M]$ so that $[v]$ should = +.002 instead of 0.

Ex. 2. In Ex. 1, §6, subtracting 40 from each M,

$[w\ M^2]$ =	5.48041
$[w\ M]^2/[w]$ =	5.46744
	0.01297, checking $[w\ v^2]$

8. CLOSENESS OF COMPUTATION. If the most probable value x as computed by a rigorous method, have the errors $\Delta_1, \Delta_2, \Delta_3$ ...., the value $x \pm c$, computed by an approximate method, will have the errors, $\Delta_1 \pm c, \Delta_2 \pm c, \Delta_3 \pm c$, ..

Hence

$$\varepsilon_{x\pm c}^2 = ((\Delta_1 \pm c)^2 + (\Delta_2 \pm c)^2 + \ldots)/n$$
$$= [\Delta^2]/n + c^2, \quad = \varepsilon_x^2 + c^2$$
$$\varepsilon_{x\pm c} = \varepsilon_x(1 + c^2/2\varepsilon_x^2), \quad \text{(approximately)}$$

If we allow the difference between $\varepsilon_{x\pm c}$ and ε_x to be $0.01\varepsilon_x$, i.e., allow the m.s.e. to be increased 1% by inaccuracy in computation, which would appear safe, then

$$.01 = c^2/2\varepsilon_x^2 \text{, or } c = 14\varepsilon_x \qquad (19)$$

or, the error of computation can be 14% of the m.s.e. without sensibly increasing the inaccuracy of the result.

Ex. 1. In a 7-place log table the error in the last place will vary from 0 to .5, all values within these limits occurring with equal frequency. a, The m.s.e. for this method of distribution of error is $a/\sqrt{3}$, where a = the greatest error. This would give, m.s.e. = $.5/\sqrt{3}$ = .29 in the 7th. place. An interpolated value, expressed as $M_1 + (M_2 - M_1)m$, where M_1 and M_2 are the adjacent tabular quantities and m the percentage interval between the corresponding numbers, would have the following m.s.e's. for different values of m, the 7th place only being retained in the interpolation (Annals of Mathematics, II, pp. 54-59: or Geographical Tables, p. lxxxvi).

m	= 1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10
m.s.e.	=.29	.41	.35	[illegible]	.37	.38	.38	.39	.39	.39

The average m.s.e. will thus be well within 0.4. In geodetic work a m.s.e. of .3 second is about the minimum value for Horizontal angles. A triangulation will be most exact, or the test most severe, when the angles of each triangle = 60°. The change in log sin 60° for a change of 1″ is 12.2 in the 7th. place so that the m.s.e due to inaccuracy of measurement = .3 × 12.2 = 3.7; i.e., c/ε_x = .4/3.7 = 11%, instead of the 14% allowed by (19).

Again a m.s.e. of 1 : 1,000,000 is excellent base line work. The log of 1,000,000 is changed 4.3 in the 7th. place by a change of unity in the number so that c/ε_x = .4/ 4.3 = 9%, instead of the 14 allowed by (19).

7-place logs are thus ample for the best geodetic work.

6-place logs are ample for, (19),

ε_x = .4/.14 = 2.9; 2.9/1.22 = 2.4″ in angle,

2.9/ .43 = 7 in 1,000,000 in distance,

or for the best city work.

5- place logs are ample for, 24″ in angle,

7 in 100,000 in distance,

or for the best railroad, or ordinary first-class field work.

4-place logs are ample for, 240″, or say 4′. in angle,

7 in 10,000 in distance,

or for the best chain and compass work, and much of the stadia work.

With suitable tables, like Vega, 7-place; Bremiker 6-place; Gauss 5-place; Encke says the times required for the same computation are as 3, 2, 1, respectively. He also says, 4 places are sufficient for minutes and 1:4000 in sides; 5 places for 5″ and 1 : 40,000; 6 places for 1/2″; and 7 places for 1/20″, limits not as conservative as the above.

9. INDEPENDENT OBSERVATIONS UPON INDEPENDENT QUANTITIES. In the general case of indirect observations let the equations be of the form,

$$\begin{aligned} f'(X, Y, Z \ldots) - M_1 &= 0 \text{ weight } w_1 \\ f''(X, Y, Z \ldots) - M_2 &= 0 \text{ weight } w_2 \end{aligned} \qquad (20)$$

in which the number n of the observed quantities $M_1, M_2, \ldots$, is greater than m, that of the required ones, X, Y, Z, ...

The observations being imperfect, no set of values can be found for the unknowns which will not leave residuals, so that (20) would be more correctly written,

$$\begin{aligned} f'(X, Y, Z, \ldots) - M_1 &= v_1 \\ f''(X, Y, Z, \ldots) - M_2 &= v_2 \end{aligned} \qquad (21)$$

which are sometimes called error or residual equations.

We first find approximate values, by partial solution or otherwise, for X, Y, Z, ..., so that $X = X_0 + x$, $Y = Y_0 + y$, ..., (x, y, ..., being so small that terms containing the squares, products and higher powers may be neglected without sensible error), then expand by Taylor's theorem, as in § 3; (21) thus becomes

$$a_1x + b_1y + c_1z + \ldots + l_1 = v_1$$

$$\left.\begin{aligned} a_2x + b_2y + c_2z + \ldots &+ l_2 = v_2 \\ a_3x + b_3y + c_3z + \ldots &+ l_3 = v_3 \end{aligned}\right\} \quad (22)$$

Where $a = df/dX_o$, $b = df/dY_o$, $c = df/dZ_o$, ... = constants.

$$l = f(X_0, Y_0, \ldots) - M.$$

The most probable values for the corrections, x, y, z, ...,(it will be proved later) will be those which will make $[w\,v^2]$ = minimum.
Hence since x, y, z, ..., are independent,

$$d[w\,v^2]/dx = 0,\ d[w\,v^2]/dy = 0,\ d[w\,v^2]/dz = 0, .$$

$$\left.\begin{aligned} \text{or}, w_1v_1\, dv_1/dx + w_2v_2\, dv_2/dx + \ldots &= 0 \\ w_1v_1\, dv_1/dy + w_2v_2\, dv_2/dy + \ldots &= 0 \end{aligned}\right\} (a)$$

Substituting the values of v from (22),

$$\left.\begin{aligned} [w\,a^2]x + [w\,a\,b]y + [w\,a\,c]z + \ldots + [w\,a\,l] &= 0 \\ [w\,a\,b]x + [w\,b^2]y + [w\,b\,c]z + \ldots + [w\,b\,l] &= 0 \\ [w\,a\,c]x + [w\,b\,c]y + [w\,c^2]z + \ldots + [w\,c\,l] &= 0 \end{aligned}\right\} (23)$$

These are called normal equations, or better <u>final</u> equations.
They can be more briefly written by substituting in (a) the values of the differential coefficients from (22).

$$[wva] = 0,\ [wvb] = 0,\ [wvc] = 0, \ldots \quad (24)$$

If the weights are equal or unity, w will disappear as a factor, giving

$$\begin{aligned} [a^2]x + [ab]y + [ac]z + \ldots + [al] &= 0 \\ [ab]x + [b^2]y + [bc]z + \ldots + [bl] &= 0 \end{aligned} \quad (25)$$

The solution of (23), (24), or (25), will give definite values for x, y, z,.. which applied to the approximate values X_0, Y_0, Z_0, ... will give the most probable ones which can be found from the given equations or observations.
Linear equations can be arranged in the form of (22) without approximate values whenever it will lessen the numerical work, the loss of higher powers occurring in the reduction to linear form, and not in the later work.

10. CONTROL, NORMAL EQUATIONS. If in (22) we place

$$\begin{aligned} a_1 + b_1 + c_1 + \ldots\ l_1 &= s_1 \\ a_2 + b_2 + c_2 + \ldots\ l_2 &= s_2 \end{aligned}$$

and treat s similarly to l, i.e., multiply each by its w a, and add the products; each by its w b and add; etc.; the terms of the first members will be the coefficients of the normal equations and the second members check terms for them, as below:

$$\left.\begin{aligned} [w\,a^2] + [w\,a\,b] + [w\,a\,c] + \ldots + [w\,a\,l] &= [w\,a\,s] \\ [w\,a\,b] + [w\,b^2] + [w\,b\,c] + \ldots + [w\,b\,l] &= [w\,b\,s] \\ [w\,a\,c] + [w\,b\,c] + [w\,c^2] + \ldots + [w\,a\,l] &= [w\,c\,s] \end{aligned}\right\} (28)$$

Ex. 1. Jordan, Vermessungskunde, I, p. 35, gives barometer readings, as the means of 12 years meteorological observations, at 9 stations, as follows:

1.	Bruchsal,	h = 120.2	B = 751.18	6.	Heiden h=	492.4	B=718.13
2.	Cannstatt	225.1	742.37	7.	Isny	708.1	700.48
3.	Stuttgart	270.6	738.50	8.	Freuden	733.5	697.64
4.	Calw	347.6	731.27	9	Schop.	763.9	695.23
5.	Freidrich	406.7	726.99				

Plotting these values with height h above sea level and barometer reading B as coordinates, the curve will be nearly or quite a straight line. On this account Jordan assumes,

$$B = X + hY, \text{ or } X + hY - B = v$$

(the theoretic function is a logarithmic one).
Assume $X_0 = 750^{mm}$, $Y_0 = -.05$, and to equalize coefficients, put $h/100\ (100y) = h'y'$. Then

Table for Forming the Normal Equations.

No	a	b	l	s	b^2	bl	bs
1	1	1.20	-0.80	+ 1.40	1.44	-.96	1.68
2	1	2.25	-0.38	+ 2.87	5.06	-.85	6.46
3	1	2.71	-0.15	+ 3.53	7.34	-.11	9.65
4	1	3.48	+0.92	+ 5.40	12.11	+3.20	18.79
5	1	4.07	+0.47	+ 5.54	16.56	+1.91	22.55
6	1	4.92	+2.45	+ 8.37	24.21	12.05	41.18
7	1	7.08	+2.87	+10.95	50.13	20.32	77.53
8	1	7.34	+3.68	+12.02	53.88	27.01	86.23
9	1	7.69	+3.28	+11.95	59.14	25.07	91.90
	9	40.74	+12.32	+62.03	229.87	37.34	357.97

$$9\ x + 40.74\ y' + 12.32 = 0 \qquad \text{Check} = 62.06$$

$$40.74x + 229.87\ y' + 67.34 = 0 \qquad 357.97$$

Solving, $x = 1.78$; $y' = -.695$; $y = -.00695$; $X = X_0 + x = 761.78$; $Y = Y_0 + y = -.08695$.

Substituting, the required equation becomes,

$$B^{mm} = 761.78^{mm} - .08695\ h^{m}.$$

11. **M.S.E.'S OF THE UNKNOWNS.** If in solving (25) the elimination was fully carried out, each unknown would be finally expressed as a linear function of $l_1, l_2 \ldots$, and the m.s.e's. of the latter being the same as those of $M_1, M_2, \ldots$, and known, those of the former would follow from §3. To effect this elimination use indeterminate multipliers, i.e., multiply the first of (25) by Q' the second by Q'', ..., and add the products. Then to find x, give such values to Q', Q'' .... that in the sum or final, equation the coefficients of the unknowns shall be zero, except those of x which shall be unity. This gives,

$$\left.\begin{aligned} [a^2]\,Q' + [ab]\,Q'' + [ac]\,Q''' + \ldots &= 1 \\ [ab]\,Q' + [b^2]\,Q'' + [bc]\,Q''' + \ldots &= 0 \\ [ac]\,Q' + [bc]\,Q'' + [c^2]\,Q''' + \ldots &= 0 \end{aligned}\right\} \qquad (a)$$

so that the sum equation reduces to

$$x + [al]\,Q' + [bl]\,Q'' + [cl]\,Q''' + \ldots = 0 \qquad (b)$$

The coefficients of the unknowns in (25) and (a) are the same. Hence if the values $x, y, z, \ldots$, are found from (25) in terms of $l_1, l_2, \ldots$, those of $Q', Q'' \ldots$, would result from them by putting $[al] = -1$, $[bl] = [cl] = 0$. This is also evident from (b). We now wish to show that if ε = m.s.e. of an observation of weight unity, ε_x = m.s.e. of the value of x found from the normal equations, then,

$$\varepsilon_x^2 = Q'\varepsilon^2$$

In (b), x being a linear function of $l_1, l_2, \ldots$, we may place,

$$x + \alpha' l_1 + \alpha'' l_2 + \alpha''' l_3 + \ldots = 0 \qquad (c)$$

in which by comparing coefficients,

$$\left.\begin{aligned} \alpha' &= a_1 Q' + b_1 Q'' + c_1 Q''' + \ldots \\ \alpha'' &= a_2 Q' + b_2 Q'' + c_2 Q''' + \ldots \\ \alpha''' &= a_3 Q' + b_3 Q'' + c_3 Q''' + \ldots \end{aligned}\right\} \qquad (d)$$

If each of these equations be multiplied by its a and added, each by its b, etc.: then by (a),

$$[a\alpha] = 1, \quad [b\alpha] = 0, \quad [c\alpha] = 0 \ \ldots\ldots \qquad (e)$$

The number of these equations is m.
Multiply each of d by its α and add, then by (e);

$$[\alpha^2] = Q' \qquad (f)$$

From the value of x in (c), we have by §3,

$$\varepsilon_x^2 = \alpha'^2\varepsilon^2 + \alpha''^2\varepsilon^2 + \alpha'''^2\varepsilon^2 + \ldots$$

or,

$$\varepsilon_x^2 = [\alpha^2]\varepsilon^2 = Q'\varepsilon^2 \qquad (27)$$

Hence to find the m.s.e. of x in terms of that of an observation: write $-1, 0, 0, \ldots$, for the absolute terms of the normal equations and solve for x: the value thus found multiplied by the square of the m.s.e. of an observation will give the square of the m.s.e. required.

In the same way it may be shown that the m.s.e. of y can be found by using 0, - 1, 0, ... , for the absolute terms; etc.

If the observations have different weights, $w_1, w_2, w_3, \ldots$, the multiplication of each by its $\sqrt{w}$ will reduce the m.s.e's., $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots$ to ε', the m.s.e. for weight unity, by (11). The observations now all having the m.s.e. ε', (27) will apply to (23), or to the normal equations with weights, ε being replaced by ε'. (27) could also have been derived directly from (23).

12. M.S.E. OF AN OBSERVATION. The most probable values of the unknowns substituted back in (22) will give the residuals, $v', v'', \ldots$, while the true values, $x + dx, y + dy, \ldots$ if known, would give the true errors,

$$\left.\begin{aligned} a'(x + dx) + b'(y + dy) + \ldots\ldots + l' &= \Delta' \\ a''(x + dx) + b''(y + dy) + \ldots\ldots + l'' &= \Delta'' \end{aligned}\right\} \quad (28)$$

and we should at once have, $\varepsilon^2 = [\Delta^2]/n$

If the first equation be multiplied by a', the second by a'', etc., then by b', b'', etc., we will have by (25)

$$\begin{aligned} [aa]\,dx + [ab]\,dy + [ac]\,dz + \ldots\ldots - [a\Delta] &= 0 \\ [ab]\,dx + [bb]\,dy + [bc]\,dz + \ldots\ldots - [b\Delta] &= 0 \\ [ac]\,dx + [bc]\,dy + [cc]\,dz \quad \ldots\ldots - [c\Delta] &= 0 \end{aligned}$$

These being the same form as (25), the value of dx can be found from that of x, by substituting $-\Delta$ for l in (c), §11, giving,

$$dx - \alpha'\Delta' - \alpha''\Delta'' - \alpha'''\Delta''' - \qquad = 0 \qquad (\alpha)$$

If we multiply (28) by $v', v'', \ldots$, respectively, the sum of the products will be by (24),

$$[vl] = [v\Delta],$$

and similarly from (22), $[v\,l] = [v^2]$, from which,

$$[v\Delta] = [v^2] = [v\,l] \qquad (29)$$

Again, multiply (22) by $\Delta', \Delta'', \ldots$, respectively and add;

$$[a\Delta]x + [b\Delta]\,y + [c\Delta]\,z + \ldots + [l\Delta] = [v\Delta] = [v^2]$$

Multiply (28) similarly,

$$[a\Delta]x + [b\Delta]y + [c\Delta]z.. + [l\Delta] + [a\Delta]dx + [b\Delta]dy + [c\Delta]dz.. = [\Delta^2]$$

From these two equations,

$$[\Delta^2] = [v^2] + [a\Delta]dx + [b\Delta]dy + [c\Delta]dz + \ldots \qquad (30)$$

The value of $[a\Delta]dx$ can be found by multiplying

$$[a\Delta] = a'\Delta' + a''\Delta'' + a'''\Delta''' + \ldots \text{ and } (a),$$
$$dx = \alpha'\Delta' + \alpha''\Delta'' + \alpha'''\Delta''' + \ldots$$

giving, since the sum of the products, $\alpha'a''\Delta'\Delta''$, .. $\alpha''a'\Delta'\Delta''$ will approximate zero,

$$[a\Delta]dx = \alpha'a'\Delta'^2 + \alpha''a''\Delta''^2 + \ldots$$

If we substitute the average value of Δ^2, which is ε^2, for $\Delta'^2, \Delta''^2, \ldots$, this reduces to (e), §11,

$$[a\Delta]\,dx = \varepsilon^2$$

Similarly, the mean value of the other terms, $[b\Delta]dy$, $[c\Delta]dz$, will be ε^2. Substituting in (30),

$$n\,\varepsilon^2 = [v^2] + m\,\varepsilon^2, \text{or } \varepsilon^2 = [v^2]/(n - m) \qquad (31)$$

If the observations have different weights, they can be reduced to the same weight by multiplying by $\sqrt{w}$, as in §11, giving

$$\varepsilon'^2 = [wv^2]/(n - m) \qquad (32)$$

Having ε or ε', the m.s.e's. for the unknowns can be found from §11.

13. SOLUTION OF NORMAL EQUATIONS.- The ordinary methods answer well when there are but few unknowns. Indeterminate multipliers are convenient in special cases, while the method of successive approximation

will often involve the least labor. But the method of substitution, due to Gauss, will generally be found preferable, as below:

NORMAL EQUATIONS.

$$\begin{array}{llllll} [aa]x + [ab]y + [ac]z + \ldots & + [al] & = 0 & \text{Check } [as] \\ [ab]x + [bb]y + [bc]z + \ldots & + [bl] & = 0 & [bs] \\ [ac]x + [bc]y + [cc]z + \ldots & + [cl] & = 0 & [cs] \end{array}$$

From the first equation,

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z \cdot - \frac{[al]}{[aa]} \qquad \frac{[as]}{[aa]}$$

Substituting,

$$\left.\begin{array}{lll} [bb.1]y + [bc.1]z \ldots + [bl.1] = 0 & & [bs.1] \\ [bc.1]y + [cc.1]z \ldots + [cl.1] = 0 & & [cs.1] \end{array}\right\} \quad (a)$$

where, $[bb.1] = [bb] - \frac{[ab]}{[aa]}[ab]$; $[bc.1] = [bc] - \frac{[ab]}{[aa]}[ac]$; $[bl.1] = [bl] - \frac{[ab]}{[aa]}[al]$;

$[cc.1] = [cc] - \frac{[ac]}{[aa]}[ac]$; $[cl.1] = [cl] - \frac{[ac]}{[aa]}[al]$; $[bs.1] = [bs] - \frac{[ab]}{[aa]}[as]$;

$[cs.1] = [cs] - \frac{[ac]}{[aa]}[as]$. . . .

From the first of (a),

$$y = -\frac{[bc.1]}{[bb.1]}z \ldots - \frac{[bl.1]}{[bb.1]} \qquad \text{Check } \frac{[bs.1]}{[bb.1]}$$

Substituting,

$$[cc.2]z + \ldots + [cl.2] = 0 \qquad [cs.2] \qquad (b)$$

where,

$[cc.2] = [cc.1] - \frac{[bc.1]}{[bb.1]}[bc.1]$; $[cl.2] = [cl.1] - \frac{[bc.1]}{[bb.1]}[bl.1]$

$[cs.2] = [cs.1] - \frac{[bs.1]}{[bb.1]}[bs.1]$

Bringing down the first equation of each group, we have the derived normal equations.

$$\left.\begin{array}{llll} [aa]x + [ab]y + [ac]z + \ldots + [al] & = 0 & \text{Check } [as] \\ \qquad [bb.1]y + [bc.1]z + \ldots + [bl.1] & = 0 & [bs.1] \\ \qquad\qquad [cc.2]z + \ldots [cl.2] & = 0 & [cs.2] \end{array}\right\} \quad (33)$$

14. FORM FOR SOLUTION. A problem in astronomy is taken to also illustrate the method of reducing a set of time transits, for clock error, azimuth error and collimation error. The observed time of transit t_1 requires:

Correction for azimuth error, $x_1 = x \sin(\phi - \delta) \sec\delta = xa$ (a) (b)

Correction for inclination telescope axis, $i_1 = i \cos(\phi - \delta)\sec\delta = iI$

Correction for collimation error, $y_1 = y \sec\delta = yb$ (c)

to give the true clock face time t, where ϕ = latitude, δ = declination of the star.

Then $t = t_1 + ax + iI + by$

If t_2 = true time of transit (computed from right ascension).

Clock correction, $\Delta t = t_2 - t$; or $\Delta t = t_2 - (t_1 + ax + Ii + by)$ (d)

If clock correction at time $t_0 = \Delta t_0$, and rate = r,

$$\Delta t = \Delta t_0 + (t - t_0)r = \Delta t'_0 + z + (t - t_0)r$$

where z is a correction to the assumed value $\Delta t'_0$.

Substituting in (d),

$$\Delta t'_0 + (t - t_0)r + t_1 - t_2 + iI + ax + by + z = 0$$

or, $$ax + by + z + l = 0 \qquad (e)$$

where $$l = \Delta t'_0 + (t - t_0)r + t_1 - t_2 + iI$$

Each observed transit gives an equation (e), in which a and b can be computed from (a) and (b); l can be computed from the transit data and clock rate after assuming $\Delta t'_0$; while the most probable values of x, y, z are to be found.

The following data was obtained by the Class in Astronomy, Oct. 2d., 1895.

Eq. 33.)

SOLUTION OF PROBLEM.

$t_1 - t_2$		i I	a	b
7^m	$52^s.33$	$+0^s.32$	-0.07	+1.41
7	51.68	+0.14	+0.68	+1.00
7	51.70	+0.17	+0.52	+1.02
7	48.38	-0.34	+2.51	-2.67
7	53.46	+0.42	-0.73	+2.13
7	51.84	+0.14	+0.75	+1.01
7	51.69	+0.18	+0.53	+1.02
7	51.33	+0.23	+0.68	+1.00
7	51.55	+0.16	+0.81	+1.02
7	53.43	+0.33	+0.09	+1.27

The clock rate is small; assume $r = 0.4t_0' = -7^m 52^s$ at 7 p.m.

$l_1 = -7^m 52^s + 0 + 7^m 52^s.33 + 0^s.32 = +0.65$; $l_2 = -0.18$; etc., as below.

Table for Forming the Normal Equations.

a	b	c	l	s	aa	ab	al	as	bb	bl	bs
-0.07	+1.41	1	+0.65	+2.99	.005	- .099	- .045	- .209	1.988	+ .917	+4.216
+0.68	+1.00	1	-0.18	+2.50	.463	+ .680	- .122	+ 1.700	1.000	- .180	+2.500
+0.52	+1.02	1	-0.13	+2.41	.270	+ .531	- .068	+ 1.253	1.040	- .133	+2.458
+2.51	-2.67	1	-3.96	-3.12	6.300	-6.702	-9.940	- 7.831	7.129	+10.573	+8.330
-0.73	+2.13	1	+1.85	+4.28	.533	-1.555	-1.372	+3.125	4.537	+4.004	+9.116
+0.75	+1.01	1	-0.02	+2.74	.563	+ .758	- .015	+2.055	1.020	- .020	+2.767
+0.53	+1.02	1	-0.13	+2.42	281	+ .541	- .070	+1.283	1.040	- .133	+2.468
+0.68	+1.00	1	-0.44	+2.24	.462	+ .680	- .299	+1.523	1.000	- .440	+2.240
+0.81	+1.02	1	-0.29	+2.54	.656	+ .826	- .235	+2.057	1.040	- .296	+2.591
+0.09	+1.27	1	+0.76	+3.12	.008	+ .114	+ .068	+ .281	1.613	+ .965	+3.962
+5.77	+8.21	10	- 1.86	+22.12	9.541	-4.226	-12.098	-1.013	21.407	+15.257	40.648

Normal Equas.

$$\begin{cases} 9.541x - 4.226y + 5.77z - 12.098 = 0 & -1.013 \text{ Check} \\ -4.226x + 21.407y + 8.21z + 15.257 = 0 & +40.648 \quad " \\ 5.77x + 8.21y + 10.00z - 1.86 = 0 & +22.12 \quad " \end{cases}$$

Solution Normal Equations.

No	x	y	z	l	Q_x	Q_y	Q_z	Check	Remarks
I	9.541	-4.226	5.77	-12.098	1			- .013	
II	-4.226	21.407	8.21	15.257		1		41.648	
III	5.77	8.21	10.	- 1.86			1	23.12	
			Solution						
IV		-.4429	.6048	-1.2680	.1048			- .0014	I/9.541
		-1.872	2.556	-5.359	442			- .006	IV × 4.226
II		21.407	8.21	15.257		1		41.648	
V		19.535	10.766	9.898	.442	1		41.642	-1
		2.555	-3.489	7.316	-.605			.008	IV×(-5.77)
III		8.21	10.	-1.86			1	23.12	
VI		10.765	6.511	5.456	-.605		1	23.128	-1
VII		1	.5511	.5067	.0226	.0512		2.1315	V/19.535
			-5.932	-5.455	-.243	-.551		-22.948	VII×(-10.765)
VI			6.511	5.456	-.605		1	23.128	
VIII			.579	.001	-.848	-.551	1	.180	+1
			1	.0017	-1.4646	-.9516	1.7254		VIII/.579

$z = -.0017 \qquad Q_z = 1.73$

V, with $z = -.0017$ gives $y = -.5058$; with $z' = -.9516$, $Q_y = .58$; with $z = -1.4646$, $Q_y' = .8290$.

I, with $z = -.0017$ and $y = -.5058$ gives $x = 1.045$; with $z = -1.4646$ and $y = .8290$, $Q_x = 1.36$.

Collecting results, $x = 1.04$; $y = -.51$; $z = -.00$; $Q_x = 1.36$; $Q_y = .58$; $Q_z = 1.73$.

Substituting back in the observation equations (as inferred from the table for forming the normal equations).

$-v$	$+v$	v^2	$-v$	$+v$	v^2	$-v$	$+v$	v^2
.14		.020	.25	.03	.032	.35	.30	.101
	.02	.000		.03	.001	.24		.058
.11		.012		.24	.058		.03	.001
	.01	.000	.10		.010		.20	.040
.25	.03	.032	.35	.30	.101	.59	.53	.200

$\varepsilon^2 = .20/7 = .029$; $\varepsilon_x = \sqrt{1.38 \times .029} = 0.20$; $\varepsilon_y = \sqrt{.58 \times .029} = 0.13$;

$\varepsilon_z = \sqrt{1.73 \times .029} = 0.22$.

Clock correction at 7 p.m., Oct. 2, 1895, = $+7^m 52^s \pm 0.22$; Azimuth correction = $+1^s.04 \pm .20$; collimation correction = $-0^s.51 \pm .13$.

15. INDEPENDENT OBSERVATIONS UPON INDEPENDENT QUANTITIES. If there are m' equations, or as they are usually called, rigid conditions, connecting the m unknowns, the case can be reduced to §9 by eliminating m' unknowns, leaving the remaining $m - m'$ independent until connected by observation equations. This method is usually used when m' is small and for indirect observations; when m' is large and the observations are direct the elimination by indeterminate multipliers will involve less labor as below.

Let the m rigorous equations be,

$$f'(V_1, V_2, \ldots V_m) = 0$$
$$f''(V_1, V_2, \ldots V_m) = 0 \qquad (34)$$

where $V_1, V_2 \ldots$, are the most provable values of the unknowns.

For each V substitute the observed value M plus a correction v ($V = M + v$), expand by Taylor's theorem as in §3, and put

$$df'/dM_1 = a_1\ ;\ df''/dM_1 = b_1 \ldots\ ;\ df'/dM_2 = a_2;\ df''/dM_2 = b_2 \ldots;$$
$$f(M_1, M_2, \ldots M_m) = q\ ;$$

giving

$$a_1 v_1 + a_2 v_2 + a_3 v_3 \ldots + q_1 = 0$$
$$b_1 v_1 + b_2 v_2 + b_3 v_3 \ldots + q_2 = 0 \qquad (35)$$
$$c_1 v_1 + c_2 v_2 + c_3 v_3 \ldots + q_3 = 0$$

These equations must be rigorously satisfied by $v_1, v_2, \ldots$

The observation equations are;

$V_1 - M_1 = v_1$; $V_2 - M_2 = v_2$; . . ; or, $(V_1 - M_1)\sqrt{w_1} = v_1\sqrt{w_1}$; $(V_2 - M_2)\sqrt{w_2} = v_2\sqrt{w_2}$.

The most probable corrections, v_1 v_2, ..., will be those which make

$$w_1 v_1^2 + w_2 v_2^2 + w_3 v_3^2 + \ldots \text{ a minimum,}$$

or
$$w_1 v_1 dv_1 + w_2 v_2 dv_2 + w_3 v_3 dv_3 + \ldots = 0 \qquad (36)$$

This minimum is conditioned by (35). Differentiating,

$$a_1 dv_1 + a_2 dv_2 + a_3 dv_3 + \ldots = 0$$
$$b_1 dv_1 + b_2 dv_2 + b_3 dv_3 + \ldots = 0$$
$$c_1 dv_1 + c_2 dv_2 + c_3 dv_3 + \ldots = 0$$

which must be satisfied at the same time with (36)

The number of these equations is m'; the number of terms in (36) is m; and as $m > m'$, we can find the values of m' differentials in terms of the $m - m'$ others and substitute in (36). The remaining differentials being independent their coefficients will separately equal zero. This elimination is effected by indeterminate multipliers; i.e., multiply the first equation by A, the second by B, etc., and (36) by - 1, then add the products and give A, B, C ...., such values that m' coefficients of dv's shall equal zero. The other $m - m'$ dv's being independent their coefficients must = 0.

$$Aa_1 + Bb_1 + Cc_1 \ldots - w_1 v_1 = 0$$
$$Aa_2 + Bb_2 + Cc_2 \ldots - w_2 v_2 = 0 \qquad (37)$$
$$Aa_3 + Bb_3 + Cc_3 \ldots - w_3 v_3 = 0$$

Multiply the first by a_1/w_1, the second by a_2/w_2 ...., then by b_1/w_1, b_2/w_2, ..., etc., and add the products. This will give by comparison with (35) m' normal equations containing m' unknowns.

$$[a\,a/w]\ A + [ab/w]\ B + [a\,c/w]\ C \ldots + q_1 = 0$$
$$[a\,b/w]\ A + [bb/w]\ B + [b\,c/w]\ C \ldots + q_2 = 0 \qquad (38)$$
$$[a\,c/w]\ A + [bc/w]\ B + [c\,c/w]\ C \ldots + q_3 = 0$$

in which $w = 1$ for equal weights.
A, B, C, ... , are called correlatives of the equations of condition. Their values from (36) substituted in (37) give

$$v_1 = (a_1A + b_1B + c_1C \ldots) \, 1/w_1 \qquad (39)$$
$$v_2 = (a_2A + b_2B + c_2C \ldots) \, 1/w_2$$

from which $V_1 = M_1 + v_1$, $V_2 = M_2 + v_2$, $V_3 = M_3 + v_3$.

Since there were m observations and m observed quantities, while m'. quantities have been eliminated, the difference between the number of observations and that of the unknowns is m', so that (31) and (32) become

$$\varepsilon^2 = [v^2]/m', \qquad \varepsilon'^2 = [wv^2]/m' \qquad (40)$$

16. CONTROL. If in (37) we place

$$a_1 + b_1 + c_1 + \ldots\ldots = s_1$$
$$a_2 + b_2 + c_2 + \ldots\ldots = s_2$$

and treat s the same as one of the other terms in deriving (38), the following checks will result. It should be noted that they do not contain the absolute terms as in (28).

$$[a\,a/w] + [a\,b/w] + [a\,c/w] \ldots\ldots = [a\,s/w]$$
$$[a\,b/w] + [b\,b/w] + [b\,c/w] \ldots\ldots = [b\,s/w] \qquad (41)$$
$$[a\,c/w] + [b\,c/w] + [c\,c/w] \ldots\ldots = [c\,s/w]$$

To check $[w\,v^2]$ multiply (39) by $\sqrt{w}$, square and add.

$$[w\,v^2] = [a^2/w]\,A^2 + 2[a\,b/w]\,AB + 2[a\,c/w]\,AC + \ldots$$
$$[b^2/w]\,B^2 + 2[b\,c/w]\,BC + \ldots$$
$$+ [c^2/w]\,C^2$$

By (38) $[wv^2] = -Aq_1 - B\,q_2 - C\,q_3 \ldots\ldots \qquad (42)$

Similarly for independent observations upon independent quantities, § 9, multiply (22) by $\sqrt{w}$, square and add.

$$[wv^2] = [w\,a^2]x^2 + 2[wab]\,xy + 2[wac]\,xz \ldots\ldots + 2[wal]\,x$$
$$[wb^2]y^2 + 2[wbc]\,yz \ldots\ldots + 2[wbl]\,y$$
$$[wc^2]\,z^2 \ldots\ldots + 2[wcl]\,z$$
$$+ [wl^2]$$

By (23),

$$[wv^2] = x[wal] + y[wbl] + z[w\,cl] \ldots\ldots + [wl^2] \qquad (43)$$

Applying (43) to the example of § 14.

$$[v^2] = -12.532 - 7.781 + 20.560 = .197$$

nearly checking $[v^2]$ as found on page 11.

17. EXAMPLE. In the U.S.C. & Geodetic Survey Report, 1880, App. 6, are given the following differences of longitude..

Dates					Observed Differences	Cor.
1851	Cambridge-Bangor	0h	9m	23s.080	± 0.042	v_1
1857	Bangor-Calais		6	00.316	± 0.015	v_2
1866	Calais-Hts. Content		55	37.973	± 0.066	v_3
1866	Hts. Cont. Foilh.	2	51	56.355	± 0.029	v_4
1866	Foilhommer-Green.		41	33.336	± 0.049	v_5
1872	Brest-Greenwich		17	57.593	±0.022	v_6
1872	Brest-Paris		27	18.512	±0.027	v_7
1872	Greenwich-Paris		9	21.000	±0.038	v_8
1872	St. Pierre-Brest	3	26	44.810	±0.027	v_9
1872	Camb. St. Pierre		59	48.608	±0.021	v_{10}

1869-70	Camb.-Duxbury		1	50.191	± 0.022	v_{11}
1870	Duxbury-Brest	4	24	43.276	± 0.047	v_{12}
1867-72	Washington-Cambridge		23	41.041	± 0.018	v_{13}
1872	Washington-St. Pierre	1	23	29.553	± 0.027	v_{14}

Number of conditions (34) or (35) = l - n + 1 = 14 - 11 + 1 = 4, (l = no. of observed differences of longitude, n = no. of stations).

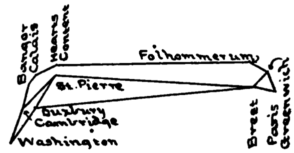


Fig. 2.

$$-v_6 + v_7 - v_8 = ^s.085 = 0$$

$$-v_1 - v_2 - v_3 - v_4 - v_5 + v_6 + v_9 + v_{10} - .045 = 0$$

$$-v_9 - v_{10} + v_{11} + v_{12} + .049 = 0$$

$$v_{10} + v_{13} - v_{14} = +.096 = 0$$

with weights inversely as the squares of the uncertainties.

Table for Normal Equations.

v	$\sqrt{1/w}$	100/w' = 1/w	a	b	c	d	s	bb/w	bs/w
v_1	.043	.18		-1			-1	.18	.18
v_2	.015	.02		-1			-1	.02	.02
v_3	.066	.44		-1			-1	.44	.44
v_4	.029	.08		-1			-1	.08	.08
v_5	.049	.24		-1			-1	.24	.24
v_6	.022	.05	-1	-1			0	.05	
v_7	.027	.07	1				1		
v_8	.038	.14	-1				-1		
v_9	.027	.07		1	-1		0	.07	
v_{10}	.021	.04		1	-1	1	1	.04	.04
v_{11}	.022	.05			1	-	1		
v_{12}	.047	.22			1		1		
v_{13}	.018	.03				1	1		
v_{14}	.027	.07				-1	-1	1.12	1.00

Normal Equations

.26 A	-.05B	-.086			= 0	.21	Check
-.05 A	+1.12B	-.11C	+.04D	-.045	= 0	1.00	"
	-.11B	+.38C	-.04D	+.049	= 0	.23	"
	+.04B	-.04C	+.14D	+.096	= 0	.14	"

from which A = .342; B = .063; C = -.191; D = -.763.

Substituting in (39),

v_1 = .18(-.063) = ^{s}-.011	v_8 = .14(-342)		= ^{s}.048
v_2 = .02(-.063) = -.001	v_9 = .07(.063+.191) -		= .018
v_3 = .44(-.063) = -.028	v_{10} = .04(.063+.191-.763)		= -.020
v_4 = .08(-.063) = -.005.	v_{11} = .05(-.192)		= -.010
v_5 = .24(-.063) = -.015	v_{12} = .22(-.191)		= -.042
v_6 = .05(-.342+.063) = -.014	v_{13} = .03(-.763)		= -.023
v_7 = .07(.342) = +.024	v_{14} = .07(.763)		= .053

Adding each v to the corresponding observed value will give the most probable value for the difference in longitude between two adjacent points, while the same difference will be found between any two distant points by any circuit.

If we square each v, multiply by w and add, $[wv^2]$ = .1148.

Computing by (42), $[wv^2]$ = .1149.

(40), $\epsilon' = \sqrt{.1149/4}$ = 0.170.

The ε for each observation can be found by dividing ε' by $\sqrt{w}$.

18. M.S.E. OF A FUNCTION OF THE REQUIRED QUANTITIES. For the case of indirect observations, the unknowns being independent they can be expressed in terms of the observed values as below.

$$F = f(X,Y,\ldots) = f(X_0+x, Y_0+y,\ldots) = f(X_0,Y_0,\ldots) + \frac{df}{dX_0}x + \frac{df}{dY_0}y + \ldots$$

$$= H + G_1x + G_2y + \ldots$$

From §11,(c),

$$x = -[\alpha l],\quad y = -[\beta l],\ldots$$

Substituting,

$$F = H - (G_1\alpha' + G_2\beta' + \ldots)\, l_1 - (G_1\alpha'' + G_2\beta'' + \ldots)\, l_2$$

$$§3 \qquad \varepsilon_F^2 = (G_1\alpha' + G_2\beta' + \ldots)^2\varepsilon_{l_1}^2 + (G_1\alpha'' + G_2\beta'' + \ldots)^2\varepsilon_{l_2}^2 + \qquad (44)$$

The values of α are given in §11,(d); those of β would be found similarly from the Q's. obtained by putting 0, −1, 0, .. for the absolute terms of the normal equations, as in the problem of §14; etc.

Eq.(44) can be transformed so that more of the numerical work of solving the normal equations can be utilized, but the transformation is long and will be omitted.

For the case of direct observations let,

$$F_1 = f_1(V_1, V_2, \ldots V_m)$$

$$= f_1(M_1, M_2, \ldots M_m) + (df_1/dM_1)v_1 + (df_1/dM_2)v_2 + (df_1/dM_m)v_m, \qquad (45)$$

$$= H_1 + g_1v_1 + g_2v_2 + \ldots g_mv_m \qquad (a)$$

where the v's are connected by (35), i.e.,

$$a_1v_1 + a_2v_2 + \ldots + a_mv_m + q_1 = 0$$
$$b_1v_1 + b_2v_2 + \ldots + b_mv_m + q_2 = 0 \qquad (46)$$

with $[w\,v^2]$ = a minimum.

Multiply the first of (46) by k_1, the second by k_2, etc., then add to (a), giving,

$$F_1 = H_1 + (g_1 + a_1k_1 + b_1k_2 + \ldots)v_1 + (g_2 + a_2k_1 + b_2k_2 +)v_2 + \ldots q_1k_1 + q_2k_2 + (b)$$

If now proper values be given to the correlatives $k_1, k_2, \ldots$, we can treat $v_1, v_2, \ldots$, as if independent as in §15, giving, §3,

$$\varepsilon_{F_1}^2 = (g_1 + a_1k_1 + b_1k_2 + \ldots)^2\varepsilon_{v_1}^2 + (g_2 + a_2k_1 + b_2k_2 + \ldots)^2\varepsilon_{v_2}^2 + \qquad (47)$$

or using weights,

$$1/w_{F_1} = (g_1 + a_1k_1 + b_1k_2 + \ldots)^2/w_1 + (g_2 + a_2k_1 + b_2k_2 + \ldots)^2/w_2 \qquad (48)$$

If the most probable values of the v's. are substituted in the value of F_1, this function will have its most probable value, .. by §29, $\varepsilon_{F_1}^2$ will be a minimum. This condition will determine $k_1, k_2, \ldots$, by differentiating (47) with respect to them as independent variables.

giving
$$d\varepsilon_{F_1}^2/dk_1 = 0 \qquad d\varepsilon_{F_1}^2/dk_2 = 0$$

$$[\varepsilon_v^2aa]k_1 + [\varepsilon_v^2ab]k_2 + \ldots [\varepsilon_v^2ag] = 0$$
$$[\varepsilon_v^2ab]k_1 \; [\varepsilon_v^2bb]k_2 + \ldots [\varepsilon_v^2bg] = 0 \qquad (49)$$

or using weights,

$$[aa/w]k_1 + [ab/w]k_2 + \qquad [ag/w] = 0$$
$$[ab/w]k_1 + [bb/w]k_2 + \ldots \qquad [bg/w] = 0 \qquad (50)$$

These equations have the same coefficients as the normal equations (38), so that the values of k can be easily found by adding a column of absolute terms in the solution as in §14.

Ex. 1. Find the m.s.e. in a triangle side due to the m.s.e's. of the measured angles.

The function equation (45) is,

$$F_1 = a = b \sin A/\sin B = b \sin(M_1 + v_1)/\sin(M_2 + v_2)$$

$\therefore \quad g_1 = df/dM_1 = a \cot M_1 \; ; \; g_2 = df/dM_2 = -a \cot M_2$

The rigorous equation to be satisfied in closing the triangle is,

$$A + B + C - (180 + s) = 0$$

$\therefore \quad a_1 = a_2 = a_3 = 1$, and (49) gives $k_1 = [\varepsilon^2 g]/[\varepsilon^2]$

substituting in (47),

$$\varepsilon_{F_1}^2 = a^2 \sin^2 1'' \left((1 - \varepsilon_1^2/[\varepsilon^2]) \cot M_1 + (\varepsilon_2^2/[\varepsilon^2]) \cot M_2\right)^2 \varepsilon_1^2$$
$$+ a^2 \sin^2 1'' \left((\varepsilon_1^2/[\varepsilon^2]) \cot M_1 + (1 - \varepsilon_2^2/[\varepsilon^2]) \cot M_2\right)^2 \varepsilon_2^2$$
$$+ a^2 \sin^2 1'' \left((\varepsilon_1^2/[\varepsilon^2]) \cot M_1 - (\varepsilon_2^2/[\varepsilon^2]) \cot M_2\right)^2 \varepsilon_3^2$$

$$\varepsilon_{F_1}^2 = a^2 \sin^2 1'' \left((\varepsilon_1^2 - \varepsilon_1^4/[\varepsilon^2]) \cot^2 M_1 + (\varepsilon_2^2 - \varepsilon_2^4/[\varepsilon^2]) \cot^2 M_2 + 2\varepsilon_1^2 \varepsilon_2^2/[\varepsilon^2] \cot M_1 \cot M_2\right)$$

If $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$,

$$\varepsilon_{F_1}^2 = 2/3\, a^2 \sin^2 1'' (\cot^2 M_1 + \cot^2 M_2 + \cot M_1 \cot M_2) \varepsilon^2$$

If the triangle is equilateral,

$$\varepsilon_{F_1}^2 = 2/3\, a^2 \sin^2 1'' \, \varepsilon^2$$

If the base has the m.s.e. ε_{b_1} by §3, $\varepsilon_{F_1}^2$ would be increased by $(a^2/b^2)\varepsilon_b^2$

Ex. 2 Find the m.s.e's. of the adjusted angles of a triangle in terms of those of the measured ones.

CHAPTER II.
THEORY.

19. PRINCIPLES OF PROBABILITY. The mathematical probability of the occurrence of an event is defined as the ratio of the number of ways it may happen to the total number of ways in which it may either happen or fail; each being supposed independent and equally liable to occur. Thus if an urn contain a white balls, b black and c red ones; in a single draw

Probability of drawing a white ball	$= a/(a + b + c)$	(51)
Of failing to draw a white ball	$= (b + c)/(a + b + c)$	
Giving sum of both probabilities	$= (a + b + c)/(a + b + c) = 1$	
Of drawing either a white ball or a black	$= (a + b)/(a + b + c)$	
Of drawing a black, white, or red	$= (a + b + c)/(a + b + c) = 1$	
Of drawing a green ball	$= 0/(a + b + c) = 0$	

We thus see that the probability is an abstract number which varies with the degree of confidence which can be placed in the occurrence of an event, zero denoting impossibility and unity certainty; that the probability of occurrence plus that of failure must always equal unity; and that the probability of the occurrence of an event which can happen in several independent ways is the sum of the separate probabilities.

If a second urn contain a' white balls, b' black and c' red ones, the number of possible combinations or cases in a single draw from each urn $= (a + b + c)(a' + b' + c')$, while the number of favorable cases for two white balls $= aa'$. Hence in two successive draws, one from each urn,

$$\text{Prob. of drawing 2 white balls} = aa'/((a+b+c)(a'+b'+c')) \qquad (52)$$

or by (51), equals the product of the separate probabilities. The same could be proved for any number of events.

We thus see that the probability of a compound event, produced by the occurrence of several simple and independent events, equals the product of the separate probabilities.

20. PROBABILITY CURVE. With the accidental errors of observation, the following axioms derived from experience, were stated in §1:

1. Small errors occur more frequently, or are more probable than large ones.
2. Positive and negative errors of the same magnitude are equally probable, and in a large number of observations are equally frequent.
3. Very large errors do not occur.

From the first axiom,it may be assumed that the probability p,of an error ,is some function of the error

From the first axiom,it may be assumed that the probability p,of an error Δ,is some function of the error,or

$$p = f(\Delta) \qquad (a)$$

Practically there is a limit to the graduation and use of instruments by which Δ can have only definite numerical values differing by the finest reading: $d\Delta$,so that the probability of an error Δ is the probability that the error lies between Δ and $\Delta + d\Delta$,a value which will vary with $d\Delta$, ∴ (a) would be more correctly written

$$p = f(\Delta)\,d\Delta \qquad (53)$$

Mathematically we have to treat Δ as a continuous variable.

Taking p as a continuous function of Δ, (53) represents a curve of the general form Fig.3; for, by the first axiom above,small values of Δ must have the largest probabilities,p ;by the second,the curve must be symmetrical about the axis of P; and by the third,p must be zero for all values of Δ greater than a given limit $\pm$ l,an impossibility except for $l = \infty$,although it can be closely approximated.

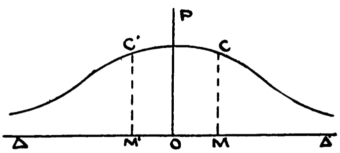


Fig.3.

21. FORM OF f(Δ).- Observations may be direct or indirect,i.e., the observed quantities may be the required ones or they may be functions of them. As the first is but a special case of the second,only the latter need be considered.

Let us take the observation equations of § 9.

$$\left.\begin{array}{l} f'(X,Y,\ldots) - M_1 = v_1 \\ f''(X,Y,\ldots) - M_2 = v_2 \end{array}\right\} \qquad (54)$$

there being n equations and m unknowns with n > m.

The probability of the occurrence of a given series of errors, $\Delta_1, \Delta_2, \ldots$ in $M_1, M_2, \ldots$ will be by (52) and (53)

$$p = f(\Delta_1)\,d\Delta_1\, f(\Delta_2)\,d\Delta_2 \ldots \qquad (55)$$

But the true values of X,Y,... are unknown,and since $\Delta_1, \Delta_2 \ldots$ are found from them by substituting in (54), their true values are also unknown. The most probable values,which if the number of observations is great, may be taken as the true ones,of the errors and hence also of the unknowns,will be those which make p a maximum: or since log p varies with p,and the unknowns are independent,except as connected by the observations themselves, the derivatives of log p with reference to X,Y,...,must equal zero.

This gives, since

$$\log p = \log f(\Delta_1) + \log f(\Delta_2) + \log d\Delta_1 + \log d\Delta_2 + \cdots$$

$$\left.\begin{array}{l} f'(\Delta_1)\,d\Delta_1/dX + f'(\Delta_2)\,d\Delta_2/dX \ldots\ldots = 0 \\ f'(\Delta_1)\,d\Delta_1/dY + f'(\Delta_2)\,d\Delta_2/dY \ldots\ldots = 0 \\ \text{-- -- -- --} \end{array}\right\} \qquad (56)$$

in which

$$f'(\Delta) = df(\Delta)/(f(\Delta)\,d\Delta) \qquad (57)$$

The number of these equations being the same as that of the unknowns, they will serve to determine them when $f'(\Delta)$ is known.

$f(\Delta)$ and $f'(\Delta)$ being general,they must hold whatever the number of unknowns.

When the number is one,the unknown is directly observed,giving for the errors,

$$\Delta_1 = X - M_1 \quad , \Delta_2 = X - M_2 \ldots\ldots$$

from which

$$d\Delta_1/dX = d\Delta_2/dX = \ldots\ldots = 1$$

and (56) reduces to

$$f'(\Delta_1) + f'(\Delta_2) + \ldots\ldots = 0$$

or, $$(f'(\Delta_1)/\Delta_1)\Delta_1 + (f'(\Delta_2)/\Delta_2)\Delta_2 + (f'(\Delta_3)/\Delta_3)\Delta_3 + \ldots = 0 \qquad (58)$$

It is usually assumed that the arithmetic mean is the best,or most probable value that can be found for a single quantity from a set of direct observations all equally good. Making this assumption, and also that the number of observations is great, it may be called the true value, or $X = (M_1 + M_2 + M_3 + \ldots)/n$

transposing,

$$(X - M_1) + (X-M_2) + (X-M_3)\ldots = 0$$

i.e., $\Delta_1+\Delta_2+\Delta_3+\cdots = 0$

Comparing this with (58), and remembering that each must hold whatever the value of n,

$$f'(\Delta)/\Delta = \text{a constant} = k$$

$\therefore$ by (57) $df(\Delta) / f(\Delta) = k\Delta\, d\Delta$

Integrating

$$\log f(\Delta) = k\Delta^2/2 + \log C$$

or $f(\Delta) = Ce^{k\Delta^2/2}$

in which e is the base of the Naperian system of logarithms.

Since as $f(\Delta)$ increases, Δ diminishes, k must be essentially negative. As its value is unknown we may replace it by another unknown constant, i.e., place $k = -1/\varepsilon^2$, giving

$$p = f(\Delta)\, d\Delta = C\, e^{-\Delta^2/(2\varepsilon^2)}\, d\Delta \qquad (59)$$

22. CONSTANT C. In deriving (53), the probability of an error between Δ and $\Delta + d\Delta$, it was assumed that p increased directly with $d\Delta$, which would be true for small intervals. For larger intervals the probability varies with Δ, so that the sum of the separate probabilities would have to be taken, giving,

$$p_a^b = \int_a^b f(\Delta)d\Delta = C\int_a^b e^{-\Delta^2/(2\varepsilon^2)}d\Delta \qquad (a)$$

Since all errors are included between $\pm\infty$, the probability of an error between these limits = 1, and of an error between 0 and ∞ (plus and minus errors being equally probable) = 1/2.

$$\therefore\ 1/2 = C\int_0^\infty e^{-\Delta^2/(2\varepsilon^2)}d\Delta$$

If $\Delta^2/(2\varepsilon^2) = t^2$, $d\Delta = \varepsilon\sqrt{2}dt$, $\therefore$ $t = \infty$ for $\Delta = \infty$, and $1/2 = C\varepsilon\sqrt{2}\int_0^\infty e^{-t^2}dt$

Since the definite integral is independent of the variable, we may also put,

$$1/2 = C\varepsilon\sqrt{2}\int_0^\infty e^{-u^2}du$$

giving

$$1/8 = C^2\varepsilon^2\int_0^\infty\int_0^\infty e^{-t^2-u^2}dt\,du \qquad (b)$$

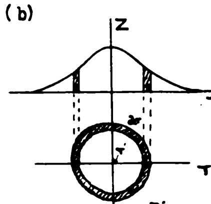


Fig. 4.

To integrate, take a surface of revolution generated by a curve with equation $z = e^{-t^2}$ in the ZT plane, or $z = e^{-u^2}$ in the ZU plane.

Its equation will be $z = e^{-t^2-u^2}$, or $z = e^{-r^2}$

Its differential volume above the plane TU, as found by dividing into elementary prisms, will be,

$$dV = z\,dt\,du = e^{-t^2-u^2}\,dt\,du$$

giving

$$V = 4\int_0^\infty\int_0^\infty e^{-t^2-u^2}\,dt\,du \qquad (c)$$

Its differential volume, as found by dividing the plane TU into elementary rings of area $= 2\pi r\,dr$, and erecting hollow cylinders of heights z, will be

$$dV = 2\pi r\,dr\,z = 2\pi r\,dr\,e^{-r^2}, \text{ giving } V = \pi\int_0^\infty e^{-r^2}2r\,dr$$

or

$$V = -\pi(e^{-r^2})_0^\infty = \pi \qquad (d)$$

By (c) it is seen that the required integral $= V/4$, which by (d) $= \pi/4$.

Substituting this value in (b),

$$1/8 = C^2\varepsilon^2\pi/4, \text{ or } C = 1/\varepsilon\sqrt{2\pi}$$

$\therefore$ (59) becomes

$$p = f(\Delta)d\Delta = d\Delta\, e^{-\Delta^2/(2\varepsilon^2)}/\varepsilon\sqrt{2\pi} \qquad (60)$$

23. VALUE OF PROBABILITY INTEGRAL BY SERIES.—Substituting the value of

C in § 22(a), with the limits changed to −a and + a.

$$P_{-a}^{+a} = 1/(\varepsilon\sqrt{2\pi})\int_{-a}^{+a} e^{-\Delta^2/(2\varepsilon^2)}d\Delta$$

or, with $\Delta^2/(2\varepsilon^2) = t^2$; $d\Delta = \varepsilon\sqrt{2}\,dt$, and the limits changed to $-t = -\Delta/(\varepsilon\sqrt{2})$ and $+t = \Delta/(\varepsilon\sqrt{2})$,

$$P_{-t}^{+t} = (p)_t = (1/\sqrt{\pi})\int_{-t}^{+t} e^{-t^2}dt = (2/\sqrt{\pi})\int_0^t e^{-t^2}dt$$

Expanding e^x by Maclaurin's theorem, $e^x = 1 + x/1! + x^2/2! + x^3/3! + \cdots$

or $\quad e^{-t^2} = 1 - t^2/1! + t^4/2! + t^6/3! + t^8/4! + \cdots$

Substituting, $(p)_t = (2/\sqrt{\pi})(t - t^3/(3\times1!) + t^5/(5\times2!) - t^7/(7\times3!) + t^9/(9\times4!) - \cdots)$ (61)

which converges rapidly for small values of t.

For large values of t, a more rapidly converging series is obtained by integrating by parts, thus:

$$\int e^{-t^2}dt = \int(-1/2t)\,de^{-t^2} = -(1/2t)e^{-t^2} - (1/2)\int(e^{-t^2}/t^2)dt$$

$$= -(1/2t)e^{-t^2} + (1/2^2t^3)e^{-t^2} + (1.3)/2^2\int(e^{-t^2}/t^4)dt$$

$$\int_t^{\infty} e^{-t^2}dt = (e^{-t^2}/2t)\left(1 - 1/(2t^2) + 1.3/(2t^2)^2 - 1.3.5/(2t^2)^3 + \cdots\right)$$

But $\int_0^t e^{-t^2}dt = \int_0^{\infty}e^{-t^2}dt - \int_t^{\infty}e^{-t^2}dt = \sqrt{\pi}/2 - \int_t^{\infty}e^{-t^2}dt$

Substituting,

$$(p)_t = 1 - (e^{-t^2}/t\sqrt{\pi})\left(1 - 1/(2t^2) + 1.3/(2t^2)^2 - 1.3.5/(2t^2)^3 + \cdots\right) \quad (62)$$

From (61) and (62) Table VII has been constructed from which $(p)_t$ can be found for any value of t or $\Delta/\varepsilon\sqrt{2}$.

In a given set of observations errors of different magnitude should occur in proportion to their probabilities as found from Table VII. This gives a method of testing theory by practice, as below: in the 18 independently observed values for the angle Mednicken-Fuchsberg at station Trenk, given in Gradmessung in Ostpreussen, p. 73.

Angle		−v	+v	v²	Angle		−v	+v	v²
83° 30′	38″.25	−1.38		1.90	forward	49″.38	−8.50		40.74
	7.50	−2.63		6.92				+7.84	
	6.00	−1.13		1.28	83° 30′	3.16		+1.71	2.92
	4.77		+0.10	0.01		4.57		+0.30	0.09
	3.75		+1.12	1.25		4.75		+0.12	0.01
	0.25		+4.62	21.34		6.50	−1.63		2.66
	3.70		+1.17	1.37		5.00	−0.13		0.02
	6.14	−1.27		1.61		4.75		+0.12	0.01
	4.04		+0.83	0.69		4.25		+0.62	0.38
	6.96	−2.09		4.37		5.25	−0.38		0.14
sums	49.36	−8.50	+7.84	40.74		87.59	−10.69	10.71	46.97

Mean = 83° 30′ 34″.87: $[v^2] = 46.97$ $\quad\varepsilon$ is found = 1.66

For probability of error < 1″, $t = \Delta/\varepsilon\sqrt{2} = 1/(1.66\sqrt{2}) = .426$. ∴ (p)t from Table VII = 45%. Number of errors < 1″ = n(p)t = .45 × 18 = 8.

Similarly as below

				Theory	Actual
No. errors < 0″.5:	t = .5 × .426;	p = 24%;	np =	4.3	8
< 1:	t = 1 × .426;	p = 45;	np =	8.1	8
< 2:	t = 2 × .426;	p = 77;	np =	13.8	14
< 3:	t = 3 × .426;	p = 93;	np =	16.8	17
< 4:	t = 4 × .426;	p = 99;	np =	17.8	17
> 4:	t =	p = 0.01;	np =	.2	1

With a larger number of observations a closer agreement would be expected.

24. DEGREE OF PRECISION. It should be noted that the value of p in §23 for a given value of Δ depends not on Δ but on $t = \Delta/(\varepsilon\sqrt{2})$; so that in two sets of observations the probability of an error less than δ in the first will be equal that of an error less than δ′ in the second, if $\delta/\varepsilon = \delta'/\varepsilon'$: e.g. if $\varepsilon' = 2\varepsilon$, the probability of an error less than δ in the

first will be the same as that of one less than $\delta'/2$ in the second, or the probability of an error less than, say 1″ in the first will be as great as that of one less than 2″ in the second, or the degree of precision of the second is said to be only one-half as great as that of the first.

The degree of precision is then inversely as ε, and observations can be reduced to the same degree of precision, and their errors directly compared by dividing them by their corresponding ε's.

These quotients must in fact be abstract numbers, since $\Delta^2/(2\varepsilon^2)$ is the exponent of e in (60).

25. CONSTANT ε. In a large number of observations errors of different values will appear in proportion to their probabilities (as found to be nearly the case for a small number of observations in §23, so that in n observations, or errors, there should be by (60)

$$n d\Delta' \; e^{-\Delta'^2/2\varepsilon^2}/(\varepsilon\sqrt{2\pi}) \quad \text{errors of value } \Delta',$$

$$n d\Delta'' \; e^{-\Delta''^2/2\varepsilon^2}/(\varepsilon\sqrt{2\pi}) \quad \text{" " " } \Delta'', \text{ etc.}$$

Squaring each error, adding, and dividing by the total number n, we have for the average square the sum of a series of terms of the form,

$\Delta^2 n \, d\Delta \, e^{-\Delta^2/2\varepsilon^2}/(\varepsilon\sqrt{2\pi})$; and since the limits of Δ are $\pm\infty$, we will have, with $\Delta^2/2\varepsilon^2 = t^2$ and $d\Delta = \varepsilon\sqrt{2}\,dt$,

$$\text{Average square} = [\Delta^2]/n = 2\varepsilon^2/\sqrt{\pi}\int_{-\infty}^{+\infty} e^{-t^2} t^2 dt = 4\varepsilon^2/\sqrt{\pi}\int_0^{\infty} e^{-t^2} t^2 dt$$

Integrating by parts,

$$\int_0^\infty (-t/2)\, d\, e^{-t^2} = -((t/2)e^{-t^2})_0^\infty + (1/2)\int_0^\infty e^{-t^2} dt, = 0 + \sqrt{\pi}/4 \quad \text{by §22.}$$

Substituting, $[\Delta^2]/n = \varepsilon^2$ (63)

which by comparison with (1) shows that the constant ε of §21 is the m.s.e. of §2.

26. AVERAGE ERROR. Similarly to §25, we have the mean value of the errors taken without regard to sign,

$$\eta = [\pm\Delta]/n = 2\varepsilon\sqrt{2}/\sqrt{\pi}\int_0^\infty e^{-t^2}\, t\,dt = 2\sqrt{2}\,\varepsilon\,\pi^{-1/2}(-1/2\, e^{-t^2})_0^\infty$$

$$\left.\begin{array}{ll} \text{or} & \eta = \varepsilon\sqrt{2}\,\pi^{-1/2} = .7979\varepsilon \\ \text{or} & \varepsilon = 1.2533\,\eta = 1.2533[\pm\Delta]/n \end{array}\right\} \quad (64)$$

27. PROBABLE ERROR, r. If a series of errors be arranged in order of magnitude, the central one is called the probable error. There thus being as many errors with less values as with greater, the probability that any error taken at random will be less than r will be the same as that it is greater, and each equals one-half.

Its value is found by placing p = 1/2 in §23, and solving for $t(=\Delta/(\varepsilon\sqrt{2}))$, giving

$$\Delta/(\varepsilon\sqrt{2}), \text{or } r/(\varepsilon\sqrt{2}) = 0.4769$$

from which

$$r = 0.6745\,\varepsilon \quad (65)$$

The p.e. and m.s.e. are both used in expressing the precision of observations.

28. GRAPHIC REPRESENTATION. If in (60), $\varepsilon = 1/\sqrt{2}$ which reduces Δ to t,

$$p/(dt) = f(t) = \pi^{-1/2} e^{-t^2}$$

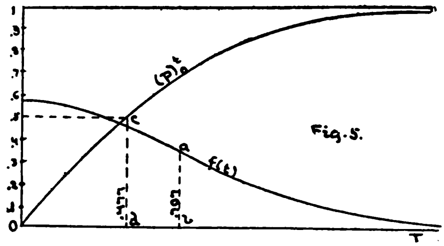
from which the curve f(t) of Fig.5 can be plotted by assuming values for t and solving for f(t), as below. Its general form was shown in Fig.3.

t	f(t)	t	f(t)	t	f (t)
0.0	0.564	0.6	0.394	1.5	0.059
.2	.542	.8	.297	2.0	.010
.4	.481	1.0	.208	3.0	.000

Since p/dt is an ordinate, p, the probability of an error t, will be an area = f(t)dt; while (p)t of §23, the probability of an error between 0 and t, will be the area from 0 to t below the f(t) curve. Laying these values of (p)t off as ordinates for given values of t by Table VII, we have the curve

$(p)^t_0$.

If $\Delta = \epsilon$, $t' = 1/\sqrt{2} = 0.707$, corresponding to the m.s.e. If $\Delta = .6745\epsilon$, $t'' = .6745/\sqrt{2} = 0.477$, corresponding to the p. e. . These ordinates are laid off at ab and cd; the latter will bisect the area between the f(t) curve and the axes, and cut the $(p)^t_0$ curve at the height 0.5, from the definition of p.e.



The former will give the point of inflection of the f(t) curve, for, placing the second differential coefficient equal zero,

$$d^2 f(t)/(dt^2) = 4\,t^2 \pi^{-1/2} e^{-t^2} - 2\pi^{-1/2} e^{-t^2} = 0$$

or

$$t = 1/\sqrt{2} = t', \text{ as above.}$$

29. PRINCIPLE OF LEAST SQUARES. In §21 we saw that with n unknowns dependent upon observation, their most probable values were those which made

$$p = f(\Delta_1)\,d\Delta_1\; f(\Delta_2)\,d\Delta_2\; f(\Delta_3)\,d\Delta_3 \cdots$$

a maximum; or substituting the values of $f(\Delta_1)\ f(\Delta_2)\ldots\ldots$ from (60),

$$p = (d\Delta_1\, d\Delta_2\, d\Delta_3 \ \ldots\ldots)(\epsilon_1^{-1}\epsilon_2^{-1}\epsilon_3^{-1}\ldots)(2\pi)^{-\frac{n}{2}} e^{[-\Delta^2/2\epsilon^2]}$$

a maximum which since $d\Delta_1, d\Delta_2, \ldots \epsilon_1, \epsilon_2 \ldots$ are constants, or are known from the observations, will be a maximum when

$$(1/2)[\Delta^2/\epsilon^2]$$

is a minimum; i.e., each error being divided by its m.s.e., or reduced to a standard degree of precision, § 24, the most probable values of the unknowns will be those which make the sum of the squares of the quotients a minimum. Hence the name <u>Least Squares</u>.

If the degrees of precision are equal, ϵ can be factored out, leaving $[\Delta^2]$ a minimum.

When $[\Delta^2]$ is a minimum $[v^2]$ will also be a minimum.

For, §5,

$$n\epsilon^2, \text{ or } [\Delta^2] = [v^2] + n\delta^2. \text{ But, } §5,\ n\delta^2 = \epsilon^2 = [\Delta^2]/n.$$

Substituting,

$$[v^2] = ((n-1)/n)[\Delta^2] \qquad (66)$$

Hence we may also say that each residual being divided by its m.s.e., or reduced to a standard degree of precision, the most probable values of the unknowns will be those which make the sum of the squares a minimum.

We may also note that, since it was assumed as an axiom that the arithmetic mean of a number of equally good observations is the most probable value, the arithmetic mean must make the sum of the squares of the residuals a minimum. To test this, take some other value of the unknown as $x_0 + \delta$. The residuals will be $v'_1 = v_1 + \delta$, $v'_2 = v_2 + \delta$. Squaring and adding,

$$[v'^2] = [v^2] + 2\delta[v] + n\delta^2$$

which since $[v] = 0$, and $n\delta^2$ is positive, will always be greater than $[v^2]$.

30. RELATION BETWEEN AVERAGE, MEAN SQUARE, AND PROBABLE ERRORS. To find the average error of § 28 in terms of the residuals v, with one unknown, directly observed, we have from (66),

$$[v^2] = ((n-1)/n)[\Delta^2]$$

$\therefore$ it may be concluded that on the average,

$$\Delta^2/v^2 = n/(n-1), \text{ and } \Delta/v = \sqrt{n/(n-1)}$$

or if v and Δ are added without regard to sign,

$$[\pm\Delta] = \sqrt{n/(n-1)}\,[\pm v]$$

$\therefore$ Average error, $\eta = [\pm\Delta]/n = [\pm v]/\sqrt{n(n-1)}$
From (7) and (64) $\eta_0 = [\pm v]/n\sqrt{n-1}$ (67)

Substituting these values of η and η_0 in (64),

$\epsilon = 1.2533\,[\pm v]/\sqrt{n(n-1)}$ $\epsilon_0 = 1.2533\,[\pm v]/n\sqrt{n-1}$ (68)

which are known as Peter's formulas.

From (9) and (10).

$\epsilon = \sqrt{[v^2]/(n-1)}$; $\epsilon_0 = \sqrt{[v^2]/n(n-1)}$ (69)

From (65),
$r = .6745\sqrt{[v^2]/n-1}$; $r_0 = .6745\,[v^2]/\sqrt{n(n-1)}$

which are known as Bessel's formulas.

For the general case of indirect observations, we have (31),

$\epsilon^2 = [v^2]/(n-m)$ while, $\epsilon^2 = [\Delta^2]/n$

$\therefore$ as above

$\Delta/V. = \sqrt{n/(n-m)}$; $[\pm\Delta]/[\pm V] = \sqrt{n/(n-m)}$

$\eta = [\pm\Delta]/n = [\pm v]/\sqrt{n(n-m)}$ (70)

$\epsilon = 1.2533[\pm v]/\sqrt{n(n-m)}$; $r = .8454[\pm v]/\sqrt{n(n-m)}$

which are known as Luroth's formulas.

For the case of direct observations upon independent quantities m' takes the place of $n-m$ as in (49), giving,

$\epsilon = 1.2533[\pm v]/\sqrt{nm'}$; $r = .8454[\pm v]/\sqrt{nm'}$ (71)

(32) and (40) being,

$\epsilon = \sqrt{[v^2]/(n-m)}$ and $\epsilon = \sqrt{[v^2]/m'}$, with $r = .6745\epsilon$

The values derived from the first powers of the residuals are often used because they are more easily computed: they are not, however, as accurate as those derived from the second powers. Weights are readily introduced, if desired.

31. LIMIT OF ACCURACY. In deriving the preceeding formulas it has been been assumed ;(a) That the number of observations is great; (b) That Δ can be regarded as a continuous variable; (c) That all constant errors have been eliminated. With but few observations, (a) and (b) are only partially satisfied; still if (c) is satisfied, the computed m.s.e. will, on the average, be the true one, although in an individual case it may be somewhat in error.

But as constant error is often present, the computed m.s.e. may be very misleading, unless the circumutances under which the observations were taken, or the reputation of the observer, are known. Again, when the number of observations, n, is great, an increase in n does not reduce the m.s.e. as rapidly as theory would indicate ($\epsilon_0 = \epsilon/\sqrt{n}$), and finally there is in every species of observations an ultimate limit of accuracy beyond which no mass of accumulated observations can ever penetrate. As stated by Wright (Adj. of Observations) " Experience, however, shows that in a long series of measurements we are never certain that our result is nearer the truth than the smallest quantity the instrument will measure."

In a word we cannot measure what we cannot see". He then quotes from Pfr. Rogers, who found with the meridian circle the p.e. of a single complete determination of the declination of a star = ± 0".36 and of the right ascension of an equatorial star ± 0^s.026, who says: "If therefore the p.e. can be taken as a measure of the accuracy of the observations, there ought to be no difficulty in obtaining from a moderate number of observations the right ascension within 0^s.02 and the declination within 0".2. Yet it is doubtful, after continuous observations in all parts of the world for more than a century, if there is a single star in the heavens whose absolute coordinates are known within these limits.' The reason is that the observations are not arranged so that constant error is eliminated, but only the accidental errors.

In explanation of the statement that "we cannot measure what we cannot see", it may be said that the axiom 1, §1 (small errors occur more frequently or are more probable than large ones), applies only down to the limit of appreciation or measurement, and that below this limit another law of distribution of error applies in which the m.s.e. of the mean does not increase as $\sqrt{n}$.

32. REJECTION OF DOUBTFUL OBSERVATIONS. This is one of the most difficult points in connection with the adjustment of observations. An observer is at liberty to arrange the observations and choose the conditions under which he will observe as his experience and best judgment may dictate. Having begun the observations, if he finds the conditions unfavorable he is at liberty to stop, reject the work already done, and begin again under more favorable auspices. When it comes to individual results in a set, if there is reason to suspect that an observation is poor before obtaining the result, a note should be made to that effect and a line drawn through the result. If the only reason for suspecting it is because it differs from the others, the young observer should hesitate about rejection unless the discrepancy is so great that a mistake is certain. The attitude of an observer should be that of perfect honesty and fairness, directing his effort each time to obtaining the best possible value of the quantity sought without being biased by the preceeding results, and without regard to them except to know in a general way that no great mistakes are being made.

Having the different results together, and being familiar with the circumstances under which the observations were made, the observer can decide which if any he will leave out in making up the mean.

The computer in revising the work, usually assumes the right to revise the rejection of observations. For this purpose he, if not the observer, will usually require a criterion. Several have been proposed. Peirce's is perhaps in most common use, but the following based upon Table VII has able advocates and is the simplest.

If $\Delta = 3\varepsilon$ in Table VII, $t = 3/\sqrt{2} = 2.12$, giving $p = .997$; i.e., only 3 errors in 1000 should exceed 3 times the m.s.e. On this account, the criterion calls for rejecting errors greater than 3ε in limited series of observations. Many object to any criterion, and leave the matter to the judgment of the observer, or to the computer in cases where more data is obtained by subsequent observations or by an advance in theoretical knowledge.

See on this subject Wright, p 131-8

. C H A P T E R . I I I.

APPLICATION TO TRIANGULATION.

33. TRIANGULATION. This is the most common method of obtaining the true relative positions of distant points when considerable accuracy is desired. High points when possible are chosen for stations or vertices, and signals are erected to make them intervisible.

The horizontal angles between the signals are measured, and usually the vertical also. One or more base lines are measured, which allows of computing all the other sides. The triangles are usually solved as plane by taking one-third the spherical excess of the triangle from each angle.

The latitude and longitude of one or more stations are observed and the azimuth of one or more sides. The latitudes, longitudes and azimuths can then be computed throughout the chain by formulas developed in Part. II.

In adjusting these horizontal angles of a triangulation, there are two classes of errors or discrepancies which arise; one from the adjustment of the observed angles at a station, the other from their adjustment in the triangulation. Strictly both should be considered together, but much labor is saved by adjusting the angles at a station first, and with these corrected values adjusting the angles of the triangulation without reference to the first adjustment: and as the discrepancies in the first adjustment are small compared with those in the second, this method is usually chosen.

34. STATION ADJUSTMENT. The adjustment of the angles at a station can be avoided by measuring the angles independently, and without checks. This can be done by measuring, say the angles between adjacent stations, as in Fig.6, and using them directly in the second adjustment, or by measuring the angle from a reference line around to the right to each station as in Fig. 7. In the latter case each measured angle would correspond to a bearing or direction of the line to its right, although for convenience the differences are sometimes treated as angles.

In the first case, if the angles should close the horizon, the adjustment would reduce to dividing the discrepancy equally among the angles if of equal weight, or inversely as the weights, if the weights are unequal.

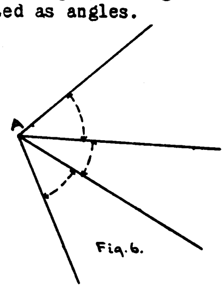


Fig.6.

If instead of closing the horizon, the sum of all is measured, the discrepancy would be divided equally among the angles including the sum, if of equal weight, or inversely as the weights if unequal. The angles may be observed as in Fig.8, swinging from the left hand signal to each of the $n-1$ others, then from the second to the $n-2$ others, etc., for $n-1$ sets, giving a total of $n(n-1)/2$ angles between n stations.

Denoting the observed values by M_1, M_2, ..., and the required ones by $X, Y, Z, \ldots$ or rather by $X_o + x$, $Y_o + y$, $Z_o + z$, the adjustment is readily effected by §9.

Another method of measuring the angles at a station is, with circle fixed, to read upon each station in order to the right, then reverse the telescope and read in the reverse order. Other sets are taken in other positions of the circle. The instrument arranged for this work is called a direction instrument, and the method, the method of directions.

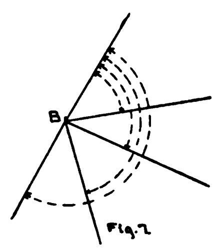


Fig.7.

Denote the required __directions__ of the signals, or the angles which they make with the reference line, by $Y, Z, U, \ldots$. or by $Y_o + y$, $Z_o + z$, $U_o + u$, ... where Y_o, Z_o, U_o ... are approximate values; also the angle between the zero of the circle for each position and the reference line by $X'_o + x'$, $X''_o + x''$, ... Then if the readings of the circle on 1 are M'_1, M'_2, ..., on 2, M''_1, M''_2, etc., the observation equations will be

$$X'_o - M'_1 = v'_1$$

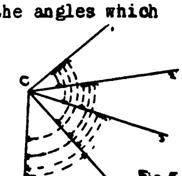


Fig.8

$X'. + Y - M'_2 = v'_2$, for the first position

$X' - M'_1 \quad = v''_1$

$X'' + Y'' - M''_2 = v''_2$, for the second position, etc.

Or denoting the values of the first members when $X_0, Y_0, \ldots$, are substituted by 1, as in §9, (22) becomes

$$x'. + 1'_1 = v'_1$$
$$x'. + y + 1'_2 = v'_2$$
$$x'' + 1''_1 = v''_1$$
$$x'' + y + 1''_2 = v''_2$$

from which corrections can be found as in §9.

35. WEIGHTING. In §34 each M is the mean of quite a number of observations; the m.s.e. of each can be found from the separate observations by (10); the squares of the reciprocals will give the weights for the observation equation.

With these the m.s.e's. of the computed angles can be found with the values of the angles as in the problem of §14. If the angles are observed independently the m.s.e's. can be found by (10) as above.

Having the adjusted angles or directions, the next step is to make up the triangles. In order to determine the number connecting different groups of points we may note the following:

The number of lines required to connect p points with a closed figure is p, and this gives one check upon the observed angles. Every additional line will give an additional check, so that with l lines and p points,

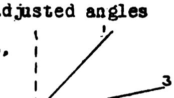
$$\text{No. of angle checks} = l - p + 1 \qquad (72)$$

This will usually give the number of triangles; in exceptional cases triangles cannot be found and polygons will have to be used instead. When there is an excess in the number of triangles the best shaped ones should usually be taken. The triangle errors can then be computed by comparing the sums of the three angles in each with 180° + spherical excess. Squaring these, adding and dividing by the number of triangles will give the average square, or ϵ^2 for a triangle. Dividing by 3 will give the average ϵ^2 for an angle, or by 6 the average ϵ^2 for a direction*, by §3. Comparing this with the average ϵ^2 found for the adjusted angles or directions at the station, and it will usually be found greater. The reason is that the former include only the observing errors, while the latter include both the observing and triangle errors, or those due to eccentricity of signal and instrument, lateral refraction, one sided illumination, etc. Subtracting the former from the latter will give the ϵ^2 due to triangle error which must be regarded as constant. Adding this to the ϵ^2 due to observing error for each angle we have the total for each angle; the weights for the triangle adjustment will be proportional to these reciprocals.

In case more than 3 of the adjusted angles are required to form a triangle, the sum of the squares of the triangle errors should be divided by the total number of angles used, for the average ϵ^2 for an angle; while in forming the sum of the ϵ^2 for the adjusted angles at a station, each should be repeated as many times as the angle is used in different triangles and the total number of ϵ^2 used as a divisor in obtaining the average. Polygons can be included with the triangles in following out this method for angles or directions, if there are not triangles enough to satisfy (72).

The effect of the triangle error is to make the weights more nearly equal; if it is to be neglected, nearly as good results will be obtained by neglecting weights as by taking them from the ϵ^2 of the adjusted angles and with less labor.

*An angle is made up of the difference of two directions, the same as by the difference of two bearings. Thus the angle 1-2-3 = -1/2 + 3/2, where 1/2 and 3/2 denote the directions of the stations 1 and 3.



Care should be taken to have the angles about equally well measured.

36. FIGURE ADJUSTMENT. The geometrical conditions to be satisfied in the triangulation are:

(a). The sum of the angles in each triangle = 180° + spherical excess, or in each polygon, 180° times number of sides, + spherical excess, less 360°.

(b). The length of a side which can be found by computing through different triangles must have the same length by each.

(a) gives rise to <u>angle equations</u>, (b) to <u>side equations</u>, both coming under §15.

Thus in the following pentagon:

Angle equations.

There are 5 triangles besides the station condition that the angles about f must remain equal to 360°. (35) becomes

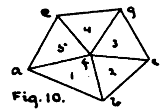


Fig. 10.

$$(a_1) + (b_1) + (f_1) + q_1 = 0$$

. . . .

$$(e_5) + (a_5) + (f_5) + q_5 = 0$$

$$(f_1) + (f_2) + (f_3) + (f_4) + (f_5) = 0$$

where q_1 ..... q_5 are the sums of the observed, or station adjusted angles, in the triangles, less 180° + spherical excess, or the triangle errors; and (a_1), (b_1) ...., are the corrections to the angles, or the v's.

Side equations.

The triangles which give a side equation, or a check upon the length of a side, will usually have one vertex in common, called a pole, while the sides radiating from it will each be common to two triangles.

In making up the check equation, the two radiating sides of each triangle are written as a fraction, beginning with any one and taking the adjacent ones in order in either direction around to the first again, the denominator of the last can each time be taken for the numerator of the next when the last denominator will be the same as the first numerator, giving unity for the continued product. Each fraction can be replaced by the ratio of the sines of the opposite angles in the same triangle, giving the required check on the angles.

Thus 5 triangles have a common vertex at f, giving

$$\frac{af}{bf}\ \frac{bf}{cf}\ \frac{cf}{gf}\ \frac{gf}{ef}\ \frac{ef}{af} = 1$$

or, $$\frac{\sin b_1 \sin c_2 \sin g_3 \sin e_4 \sin a_5}{\sin a_1 \sin b_2 \sin c_3 \sin g_4 \sin e_5} = 1$$

Taking logs.

$$\log\sin b_1 - \log\sin a_1 + \log\sin c_2 - \log\sin b_2 + \log\sin g_3 -$$

$$\log\sin c_3 + \log\sin e_4 - \log\sin g_4 + \log\sin a_5 - \log\sin e_5 = 0$$

The df'/dM_1, of §15, $= d(\log\sin b_1)/db_1 = \text{Mod.}\cos b_1/\sin b_1 = \text{Mod.}\cot b_1$

where Mod. = the modulus of the common system of logarithms.

$d(\log\sin b_1)/db_1$ = ratio of change in log sin to change in arc. $= d_1/\sin 1''$

where d = tabular difference of log sin for 1".

∴ (35) becomes

$$d_1(b_1) - d_2(a_1) + d_3(c_2) - d_4(b_2) + d_5(g_3) - d_6(c_3) + d_7(e_4) - d_8(g_4) + d_9(a_5)$$

$$- d_{10}(e_5) + q_7 = 0$$

where q_7 = the value of the log sin equation, when the observed angles are substituted.

For convenience the decimal point is moved either six or seven places to the right for d and q.

37. ADJUSTMENT OF QUADRILATERAL. Seneca Lake, 1882. Angles observed independently. Weights found as in §35. Spherical excess inappreciable.

Triangle 34				
N^{34}_{w}	70°	27′	53″.0	1/w = 1.5
L^{34}_{e}	60	19	06.5	0.6
O^{34}_{w}	49	13	05.4	0.6
	180	00	4.9	

Triangle 35				
L^{35}_{e}	48°	53′	29.3	1/w = 0.4
M^{35}_{e}	60	41	56.9	= 0.5
O^{35}_{w}	70	24	31.0	= 0.6
	179	59	57.2	

Triangle 36

N_w^{36}	41°	21′	25″.8	1/w = 1.1
M_e^{36}	29	25	52.7	= 1.0
L_e^{34} {	60	19	06.5	= 0.6
L_e^{35}	48	53	29.3	= 0.4
	179	59	54.3	

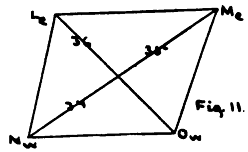


Fig. 11.

giving 3 angle equations with $q' = +4''.9$, $q'' = -2''.8$, $q''' = -5''.7$.

Side equation　Pole at Le　$(L_eN_w/L_eO_w)(L_eO_w/L_eM_e)(L_eM_e/L_eN_w)=1$

	Log sine +	d for 1″		Log sine −	d for 1″
O_w^{34}	9.8792119	18.1	N_w^{34}	9.9742517	7.5
M_e^{35}	9.9405473	11.8	O_w^{35}	9.9741006	7.5
N_w^{36}	9.8200378	23.9	M_e^{36}	9.6914172	37.4
	9.6397970			9.6397695	

695
+ 275 = q_{iv}

Table for Normal Equations (38)

v	1/w	a	b	c	d	s	a²/w	ac/w	ad/w	as/w	b²/w	bc/w	bd/w	bs/w	c²/w	cd/w	cs/w	d²/w	ds/w
(N_w^{34})	1.5	1			-7.5	-6.5	1.5		-11.25	-9.75								84.37	73.13
(L_e^{34})	0.6	1		1		2	0.6	0.6		1.2					0.6		1.2		
(O_w^{34})	0.6	1			18.1	19.1	0.6		10.86	11.46								196.57	207.43
(L_e^{35})	0.4		1	1		2					0.4	0.4		0.8	0.4		.8		
(M_e^{35})	0.5		1		11.8	12.8					0.5		5.9	6.4				69.62	75.52
(O_w^{35})	0.6		1		-7.5	-6.5					0.6		-4.5	-3.9				33.76	29.25
(N_w^{36})	1.1			1	23.9	24.9									1.1	26.29	27.39	628.33	654.52
(M_e^{36})	1.0			1	-37.4	-36.4									1.0	-37.4	-36.4	1398.76	1361.36
							2.7	0.6	-0.39	2.91	1.5	0.4	1.4	3.3	3.1	-11.11	-7.01	2411.4	2401.11

NORMAL EQUATIONS　　Check

2.7 A		+ 0.6 C	−0.39 D	+ 4.9	= 0		− 2.91
	1.5 B	+ 0.4 C	+1.4 D	− 2.8	= 0		− 3.3
0.6 A	0.4 B	+ 3.1 C	−11.11 D	− 5.7	= 0		+ 7.01
0.39 A	+1.4 B	− 11.11 C	+2411.4 D	+275.0	= 0	−	2401.3

These equations are more readily solved with a **smaller coefficient for D** in the fourth. Thus let $D_1 = 10^{-1} D$, giving,

2.7 A		+0.6 C	− 0.04 D_1	+ 4.9	= 0	−8.16
	1.5 B	+0.4 C	+ 0.14 D_1	− 2.8	= 0	+ .76
0.6 A	0.4 B	+3.1 C	− 1.11 D_1	− 5.7	= 0	+2.71
0.04 A	0.14 B	−1.11 C	+24.11 D_1	27.5	= 0	−50.60

from which $A = -2.20$, $B = +1.52$, $C = +1.68$, $D_1 = -1.08$, $D = -.108$

(39) becomes

$(N_w^{34}) = 1.5\,(-2.20 - 7.5\,(-.108)) = -2''.1$

$(L_e^{34}) = 0.6\,(-2.20 + 1.68) = -0.3$

$(O_w^{34}) = 0.6\,(-2.20 + 18.1\,(-.108)) = -2.5$

$-4.9 = -q'$

(L_e^{35})	= 1″.3	(N_w^{36})	= 1.0
(M_e^{35})	= .1	(M_e^{36})	+5.7
(O_w^{35})	= 1.4	$(L_e^{34}) + (L_e^{35})$	1.0
	+2.8 = +q″		+5.7 = − q‴

(O_w^{34}) −2.5 × 18.1 = −45.2		(N_w^{34}) − 2.1 × 7.5	= −15.7
(M_e^{35}) +0.1 × 11.8 = + 1.2		(O_w^{35}) + 1.4 × 7.5	= +10.5
(N_w^{36}) −1.0 × 23.9 = −23.9		(M_e^{36}) + 5.7 × 37.4	= +213.2
−67.9			208.0

-208.0
-275.9　should　$= -q^{iv}$

These corrections applied to the observed angles will give the adjusted ones.

38. NUMBER AND FORMATION OF THE SIDE EQUATIONS. When in any system the first two points are determined by the length of the line joining them, the determination of any additional point requires two sides or two directions so that in any system of p points we have to determine p - 2 points, which requires 2(p - 2) directions, or by adding the first 2p - 3. Hence in a system of l sides and p points,

$$\text{No. of side equations} = l - 2p + 3 \qquad (73)$$

where each side requires to be observed over from one end only.

Stations between which side equations exist form systems about a central point or pole including it in a triangle or polygon.

Frequently the pole falls outside which makes no difference in the solution. In either case there is one characteristic property; i.e., at every station three lines meet, save one, where p - 1 meet, there being p stations. Complications arise from systems within systems. It is less work to take the pole where the least angles have been observed, in cases which permit of choice.

In a completed quadrilateral where the angles are measured independently, it is best to take the pole at the vertex of the three triangles giving the angle equations; in other cases where the adjacent angles are used giving 4 to a triangle the pole is conveniently taken at the intersection of the two diagonals.

The number of angle equations was found in (72), each side requiring to be sighted over in both directions.

In a chain of triangles where two bases have been measured, both being regarded perfect, the absolute term of the side equation becomes the ratio of the bases instead of unity.

39. ADJUSTMENT OF SECONDARY TO PRIMARY WORK. The primary work having been adjusted by itself, the entire discrepancy would be thrown into the secondary. This would be accomplished by placing the correction to the adjusted or perfect angle, or its v, equal zero, so that the term containing it would disappear from (35).

Thus in the following figure, we have given the angles of the primary triangle 1-2-3, and those of the secondary triangles 1-2-4, 2-3-4, 3-4-1, derived as differences of direction.

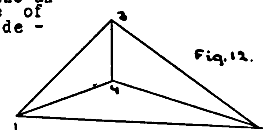


Fig. 12.

Angle equations

$$+(4/2) - (2/4) + (1/4) - (4/1) + q' = 0$$

$$+(4/3) - (3/4) + (2/4) - (4/2) + q'' = 0$$

$$(4/1) - (1/4) + (3/4) - (4/3) + q''' = 0$$

Side equations

$$(1\text{-}4)/(2\text{-}4) \times (2\text{-}4)/(3\text{-}4) \times (3\text{-}4)/(1\text{-}4) = 1$$

$$\text{or} \quad + d_1(4/2) + d_2(4/1) + d_3(4/3) + d_4(4/2) + d_5(4/1) + d_6(4/3) + q'''' = 0$$

From these the corrections can be derived as usual.

If a secondary chain connects at each end with a primary side, and in many other cases, the checks due to the connection are often brought in as a side equation, azimuth equation, latitude equation, and longitude equation; thus making the computed side of the same length as, parallel to, and coinciding with, the primary side.

40. M-S.E. OF ANY SIDE. In §18, Ex. 1, it was found that

$$\epsilon_{a_1}^2 = a_1^2 \sin^2 1''(\cot^2 A_1 + \cot^2 B_1 + \cot A_1 \cot B_1)\epsilon^2 2/3 + \epsilon_b^2 a_1^2/b^2$$

Similarly for the next side.

$$\epsilon_{a_2}^2 = a_2^2 \sin^2 1'' (\cot^2 A_2 + \cot^2 B_2 + \cot A_2 \cot B_2)\epsilon^2 2/3 + \epsilon_{a_1}^2 a_2^2/a_1^2$$

Substituting .

$$\epsilon_a^2 = a^2 \sin^2 1''([\cot^2 A] + [\cot^2 B] + [\cot A \cot B])\epsilon^2 2/3 + \epsilon_b^2 a^2/b^2 \quad (74)$$

This will give the m.s.e. of any side in a chain of triangles, b being the measured base, and A and B the angles used in computing the side. If more than one series of triangles can be used the shortest or the one giving the smallest m.s.e. should be taken.

41. APPROXIMATE ADJUSTMENT FOR AZIMUTH. An azimuth equation may come from connecting to two sides of a triangulation which has previously been adjusted, as already indicated, or it may come from the observed azimuths of two triangle sides. Strictly the azimuth equation should be included with the others in the figure adjustment; but much labor is saved, and often sufficient accuracy attained, by considering it separately after the first adjustment has been made.

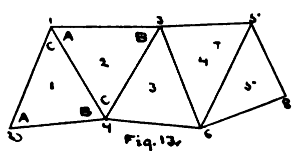


Fig. 13.

In Fig. 13 the angles A and B are used in computing the side 5-8 from the base 1-2. The azimuth of 5-8 can be computed from that of 1-2 by using only the angles C. Counting azimuth clockwise as usual, the azimuth of 1-4 would be found from that of 1-2 by subtracting C_1. The azimuth of 4-1 would differ from that of 1-4 by 180° less the convergence of the meridians, and can be computed from formulas in Part II. The azimuth of 4-3 can be found from that of 4-1 by adding C_2, etc. If q_z = computed azimuth of 5-8 less the observed or direct value,

Azimuth equation

$$a, \quad -(C_1) + (C_2) - (C_3) \; .... \; + q_z = 0$$

Angle equations

$$b, \; (A_1) + (B_1) + (C_1) = 0; \quad c, \; (A_2) + (B_2) + (C_2) = 0, \cdots$$

for n triangles.

Forming the normal equations as usual.

$$nA - B + C - D \ldots + q_z = 0$$
$$-A + 3B = 0$$
$$+A + 3C = 0$$

Finding the value of B, C, etc..., in terms of A and substituting in the first equation,

$$nA - A/3 - A/3 - \ldots + q_z = 0$$
$$nA - nA/3 + q_z = 0$$

$$A = - 3\,q_z/2n, \quad B = -q_z/2n, \quad C = +q_z/2n, \quad D = -q_z/2n, \text{ etc.}$$

Corrections

$$(A_1) = -q_z/2n, (B_1) = -q_z/2n, \quad (C_1) = +q_z/n$$
$$(A_2) = +q_z/2n, \quad (B_2) = +q_z/2n, \quad (C_2) = -q_z/n$$

i.e., divide the excess of the computed over the observed azimuth by the number of the triangles, and apply one-half of this quantity to each of the angles used in computing distance through the chain, and the total quantity, with the sign changed, to the third angle, the latter being so applied each time as to reduce the discrepancy.

42. APPROXIMATE ADJUSTMENT BETWEEN BASES.- Strictly this equation should be added with the others in the figure adjustment, but frequently it is omitted, until the other adjustment has been made in order to see how close the bases will check, or the check base may not have been measured until the figure adjustment has been completed. In such cases the base adjustment can be made separately as below.

Base equation

$$a \qquad [d_A (A) - d_B (B)] + q_b = 0$$

where, as in Fig. 13, the A angles are opposite the required sides and the B angles opposite the known ones in passing from the first to the second base, d_A and d_B are the differences of the log sines of the angles for 1", and q_b is the discrepancy in the logs of the bases when the observed values are substituted.

Angle equations.

$$b \quad (A_1) + (B_1) + (C_1) = 0, \quad c \quad (A_2) + (B_2) + (C_2) = 0, \ldots$$

for n triangles.

Normal equations.

$$([d_A^2] + [d_B^2])A + (d_{A_1} - d_{B_1})B + (d_{A_2} - d_{B_2})C + q_b = 0$$
$$(d_{A_1} - d_{B_1})A + 3B = 0$$
$$(d_{A_2} - d_{B_2})A + 3C = 0$$

From the 2nd equation, $B = -(d_{A_1} - d_{B_1})A/3$

From the 3rd equation, $C = -(d_{A_2} - d_{B_2})A/3$

Substituting in the first,

$$A = - 3\,q_b / 2[d_A^2 + d_A d_B + d_B^2]$$

Substituting in (69),

$$(A_1) = (2d_{A_1} + d_{B_1})A/3 \quad (B_1) = -(d_{A_1} + 2d_{B_1})A/3, \quad (C_1) = -(d_{A_1} - d_{B_1})A/3$$

The corrections to the C angles will tend to foot up zero, the differences for 1" for the A and B angles averaging about equal in a triangulation. The disturbance in the azimuth adjustment will thus be small. By calling the C corrections zero ($d_A = d_B$) the angle equations become,

$$b \quad (A_1) + (B_1) = 0, \quad c \ (A_2) + (B_2) = 0, \cdots$$

Normal equations,

$$([d_A^2] + [d_B^2])\, A + (d_{A_1} - d_{B_1})B + (d_{A_2} - d_{B_2})C + \dots + q_b = 0$$

$$(d_{A_1} - d_{B_1})\, A + 2B = 0$$

$$(d_{A_2} - d_{B_2})\, A + 2C = 0$$

From which, $B = -(d_{A_1} - d_{B_1})A/2, \quad C = +(d_{A_2} + d_{B_2})A/2$

$$A = -2q_b / [d_A^2 + 2d_A d_B + d_B^2]$$

$$(A_1) = (d_{A_1} + d_{B_1})A/2 \ ; \ (B_1) = -(d_{A_1} + d_{B_1})A/2; \dots$$

These corrections when applied will not disturb the azimuth adjustment so that the length and direction of any line will be the same computed from either end of the chain.

43. ADJUSTMENT FOR LATITUDE AND LONGITUDE. The observed latitude and longitude would not check throughout the chain due to local deflection of the plumb line.

In joining new work to old adjusted work at two points, as in filling in secondary triangulation, the junction side computed through the new work must be parallel to the old (azimuth equation), must have the same length (base line equation), and must coincide in position at one end, which is best effected by a latitude and longitude equation. This last can be introduced in the figure adjustment, but the discrepancy in good work will be so small that the equation can be omitted in the first adjustment, and the error in latitude and in longitude distributed as in a land survey without serious loss of accuracy. Each station can then be reduced to center by the method given in Part II, making the figure consistent throughout.

44. TRIGONOMETRIC LEVELING. There are three methods of determining the difference in level trigonometrically; from non-simultaneous readings at the two stations; from simultaneous readings; and from readings at one of the stations only. Approximate formulas for the 3 cases are

$$h_1 = k \tan 1/2(\delta_2 - \delta_1) + (m_2 - m_1)\, k^2/2R_z$$

$$h_2 = k \tan 1/2(\delta_2 - \delta_1)$$

$$h_3 = k \cot \delta_1 + (1 - 2m_1)k^2/2R_z$$

where k = horizontal distance; δ_1, δ_2 = observed zenith distances; m_1, m_2 = coefficients of refraction; R_z = radius of curvature of the arc joining the two stations.

The m.s.e. for each result can be found as in §3, remembering that k is well known, and that δ is nearly 90°,

$$\varepsilon_{h_1}^2 = k^2 \sin^2 1''\, \varepsilon_\delta^2/2 + k^4 \varepsilon_m^2/2R_z^2 \qquad (75)$$

$$\varepsilon_{h_2}^2 = k^2 \sin^2 1''\, \varepsilon_\delta^2/2 + k^4 \varepsilon_m^2/2R_z^2 \qquad (76)$$

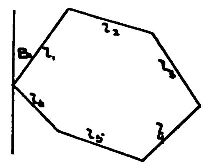
$$\varepsilon_{h_3}^2 = k^2 \sin^2 1''\, \varepsilon_\delta^2 + k^4 \varepsilon_m^2/R_z^2 \qquad (77)$$

In adjusting a net, the algebraic sum of the h's in going around a triangle should = 0, giving for the number of the equations, the same as for the number of angle equations, l - p + 1.

There will usually be enough reciprocal observations so that the value of m can be computed for the lines observed at each station, assigning weights to each reciprocal set by Bessel's empirical formula,

$n_1 n_2 \sqrt{K}/(n_1 + n_2)$, where n_1, n_2, are the numbers of observations for δ_1, δ_2.

The weights to be given to the differences in height would be the reciprocals of the ε^2's found above. Wright, p.392, assumes $\varepsilon_\delta = 2''$, $\varepsilon_m = 0.02$, as being fair averages.



·Fig.14.

45. ADJUSTMENT OF A COMPASS SURVEY. For each side the length and bearing are directly measured, while the latitude and departure are computed.
The latitude equation is,

$$L = l \cos B \qquad (a)$$

$$dL/dl = \cos B \quad dL/dB = - l \sin B \quad (b)$$

§3. $\varepsilon_L^2 = \varepsilon_l^2 \cos^2 B + \varepsilon_B^2 l^2 \sin^2 B$ (c)

For the departure,

$$D = l \sin B \qquad (d)$$

$$dD/dl = \sin B \quad dD/dB = l \cos B \quad (e)$$

$$\varepsilon_D^2 = \varepsilon_l^2 \sin^2 B + \varepsilon_B^2 l^2 \cos^2 B \qquad (f)$$

If we assume as was practically done by Dr. Bowditch that

$$\varepsilon_l^2 = \varepsilon_B^2 l^2 = l \times \text{constant} = l\,C \qquad (78)$$

which reduces (b) and (d) to

$$\varepsilon_L^2 = \varepsilon_D^2 = l\,C \qquad (79)$$

i.e., the squares of the m.s.e's. in latitude and in departure are each proportional to the lengths of the sides.
In §29 it is shown that for equal weights the most probable corrections will be those which make the sum of the squares a minimum; and for unequal weights the sum of the squares of the quotients found by dividing each correction by its m.s.e. Hence denoting the corrections in latitude for the different sides by $v_{L_1}, v_{L_2}, v_{L_3}, \ldots$,

$$v_{L_1}^2/l_1 + v_{L_2}^2/l_2 + v_{L_3}^2/l_3 + \quad = \text{minimum.}$$

Differentiating,

$$dv_{L_1} v_{L_1}/l_1 + dv_{L_2} v_{L_2}/l_2 + dv_{L_3} v_{L_3}/l_3 + \quad = 0 \qquad (80)$$

The sum of the corrections must equal the total error in latitude with its sign changed, $-q_1$, a constant, $v_{L_1} + v_{L_2} + v_{L_3} + \quad = q_1$

Differentiating, $dv_{L_1} + dv_{L_2} + dv_{L_3} + \quad = 0$ (81)

Comparing (80) and (81), and remembering that each must hold whatever the number of sides, or v's.,

$$v_{L_1}/l_1 = v_{L_2}/l_2 = v_{L_3}/l_3 = \qquad (82)$$

i.e., the corrections in latitude are proportional to the lengths of the sides, according to the Dr. Bowditch rule.
The same can be found for the corrections in departure, giving,

$$\left.\begin{array}{l} v_{D_1} + v_{D_2} + v_{D_3} + \quad = q_2 \\ v_{D_1}/l_1 = v_{D_2}/l_2 = v_{D_3}/l_3 \end{array}\right\} \qquad (83)$$

If the corrections are required for computing area, they can be applied directly to the values in the latitude and departure columns; but if they are required for a geometrically consistent map or record the corresponding corrections must be found for the distances and bearings. This can be done by dividing the corrected departure by corrected latitude for the tangent of the corrected bearing, then dividing departure by sine and latitude by cosine for the corrected distance giving weight to the value having the larger numerator, and using the other as a check. This requires the use of as many decimal places as the original computation.
From the differential equations (b) and (e), the total correction to the side,

$$\left.\begin{array}{l} dl = dL/\cos B + dD/\sin B \\ \quad = dL\, l/L + dD\, l/D \end{array}\right\} \qquad (84)$$

The total correction to the bearing,

$$dB = - dL/D + dD/L, \qquad \text{or in minutes,}$$

$$dB' = - dL/D \sin 1' + dD/L \sin 1' \qquad (85)$$

Equations (84) and (85) are readily computed with a slide rule, or even by inspection from the coordinate sheet.
In equation (78) an uncertainty in chaining which would amount to 1 ft. in 500 would give, $\varepsilon_{500} = 1 = \sqrt{500\,C}$; or $C = 0.0447$

.. $\varepsilon_l = 0.0447\sqrt{l}$ $\quad \varepsilon_B = \text{(in minutes)} = 0.0447/\sqrt{l} \sin 1'$

Substituting for different distances,

Distance.	Uncertainty in chaining.	Uncertainty in bearing.
10 feet	0.14 feet	0° 48′
50	0.32	0 21
100	0.45	0 15
500	1.00	0 07
1 000	1.41	0 05
2 000	2.00	0 03

An examination of these results shows that the assumption is fairly reasonable, although it gives too great weight to the bearings of long lines, and perhaps too small to those of very short ones.

48. ADJUSTMENT OF A TRANSIT SURVEY. In an ordinary transit survey no bearings are observed, but the horizontal angles between the lines are measured. In computing coordinates a meridian is observed or assumed and the bearings found from the angles. To express these bearings in terms of the measured angles in the adjustment equations, as should be done for accuracy, involves too much labor. To use them as observed quantities will give different bearings and different coordinates, depending upon the direction taken around the figure for each in case the angles do not "close".

In ordinary work the m.s.e. of an angle need not exceed 1 minute, if care is taken in setting over the points and in plumbing the flag poles, using tacks on the stakes for all lines of less than 300 ft., swinging without delay from the back sight to the front sight, and lining in a "range" point to swing from for all lines of less that 50 ft. With these precautions, the m.s.e. need not increase with the shortness of the line, as with the compass with which it is a waste of time to guard against errors of eccentricity in setting up or flagging.

On very rough ground, or in going through brush, where the flag pole is partly hidden, it may be difficult to keep the m.s.e. below 2 minutes; while, for careful work, the m.s.e. can be readily kept within 1/2 minute.

For good work the length of sight should be limited to about 1200 ft.

It is believed that the time required to swing back by the lower motion and forward by the upper for a second measure of the angle is well repaid by the freedom from mistakes and increased accuracy secured.

Ordinarily, it will be more difficult to measure distances to 1:500 than angles to minutes, while an accuracy of 1:1 000 is seldom reached except on level groud or for city work.

The accuracy of angle work is thus considerably greater than that of chaining, 1 minute in angle giving 0.15 ft. in 500 as compared with 1 ft. in chaining; or, 0.5 minute, 0.15 ft. in 1 000, as compared with the 1 ft. due to the more accurate chaining.

On this account it will be admissible to adjust the angles to close the figure (i.e., so that the sum of the interior angles shall equal twice as many right angles, less four, as the figure has sides) by distributing the error equally among the angles to the nearest 1/2 or 1/4 minute if they all been equally well measured, or concentrate the corrections somewhat upon the poorer angles if not equally well measured. The bearings or azimuths are then computed and assumed to be correct in the final adjustment.

This leaves only the two conditions:

Sum of latitudes equal zero. Sum of departures equal zero.

That is,

$$\left.\begin{aligned} l_1 \cos B_1 + l_2 \cos B_2 + l_3 \cos B_3 + &= 0 \\ l_1 \sin B_1 + l_2 \sin B_2 + l_3 \sin B_3 + &= 0 \end{aligned}\right\} \quad (86)$$

where the total corrections are to be applied upon the basis of inaccuracy in chaining.

Denote the observed distances by $M_1, M_2, M_3, \ldots$, and the required corrections by $v_1, v_2, v_3, \ldots$. The corrected distances will be

$$l_1 = M_1 + v_1, \quad l_2 = M_2 + v_2, \quad l_3 = M_3 + v_3 \ldots$$

Substituting in (86),

$$(M_1 + v_1)\cos B_1 + (M_2 + v_2)\cos B_2 + (M_3 + v_3)\cos B_3 = 0$$

$$(M_1 + v_1)\sin B_1 + (M_2 + v_2)\sin B_2 + (M_3 + v_3)\sin B_3 = 0, \text{ or}$$

$$\left.\begin{aligned} v_1 \cos B_1 + v_2 \cos B_2 + v_3 \cos B_3 \quad + q_1 &= 0 \\ v_1 \sin B_1 + v_2 \sin B_2 + v_3 \sin B_3 \quad + q_2 &= 0 \end{aligned}\right\} \quad (87)$$

where $q_1 = M_1 \cos B_1 + M_2 \cos B_2 + M_3 \cos B_3 +$ = error in latitude,

$q_2 = M_1 \sin B_1 + M_2 \sin B_2 + M_3 \sin B_3 +$ = error in departure.

For convenience change (87) to

$$\left.\begin{aligned} v_1 L_1/l_1 + v_2 L_2/l_2 + v_3 L_3/l_3 + \quad + q_1 &= 0 \\ v_1 D_1/l_1 + v_2 D_2/l_2 + v_3 D_3/l_3 + \quad + q_2 &= 0 \end{aligned}\right\} \quad (88)$$

where $L_1, L_2, \ldots, D_1, D_2, \ldots$, denote latitudes and departures.

If ϵ^2 for chaining increase as l, or the weights inversely as l, (38) becomes

$$[L^2/l]\,A + [L\,D/l]\,B + q_1 = 0$$

$$[L\,D/l]\,A + [D^2/l]\,B + q_2 = 0$$

Solving,

$$\left.\begin{aligned} A &= (q_2 [L\,D/l] - q_1 [D^2/l])/([D^2/l][L^2/l] - [L\,D/l]^2) \\ B &= (q_1 [L\,D/l] - q_2 [L^2/l])/([D^2/l][L^2/l] - [L\,D/l]^2) \end{aligned}\right\} \quad (89)$$

(39) become,

$$\left.\begin{aligned} v_1 &= L_1 A + D_1 B \\ v_2 &= L_2 A + D_2 B \end{aligned}\right\} \quad (90)$$

Adding, $[v] = A\,[L] + B\,[D] = 0$, nearly.

Also, $v_{L_1} = v_1 L_1/l_1 = A\,L_1^2/l_1 + B\,L_1 D_1/l_1$

$v_{L_2} = v_2 L_2/l_2 = A\,L_2^2/l_2 + B\,L_2 D_2/l_2$

$$v_{D_1} = v_1 D_1/l_1 = A\ D_1 L_1/l_1 + B\ D_1^2/l_1$$
$$v_{D_2} = v_2 D_2/l_2 = A\ L_2 D_2/l_2 + B\ D_2^2/l_2$$

with $[v_L] = -q_1$ and $[v_D] = -q_2$.

If the inaccuracy in chaining increases directly with the distance (ϵ varying as l), or the weights inversely as l^2, (88) become,

$$\left.\begin{array}{l}[L^2]A + [L\ D]B + q_1 = 0\\ [L\ D]A + [D^2]B + q_2 = 0\end{array}\right\} \qquad (91)$$

$$\left.\begin{array}{l}\text{with } A = (q_2[L\ D] - q_1[D^2])/([D^2][L^2] - [L\ D]^2)\\ B = (q_1[L\ D] - q_2[L^2])/([D^2][L^2] - [L\ D]^2)\end{array}\right\} \qquad (92)$$

$$v_1 = L_1 l_1 A + D_1 l_1 B$$
$$v_2 = L_2 l_2 A + D_2 l_2 B$$

In order to equalize numbers so as to retain the same number of decimal places throughout, 100 l is used in place of l in (89), making the values of A and B 100 times too great and requiring the values of v to be divided by 100.

If it is assumed that the error in chaining increases directly with the distance, (78) may be changed to

$$\epsilon_l = \epsilon_B l = l \times \text{constant} = l\ C \qquad (93)$$

which changes (79) to $\epsilon_L^2 = \epsilon_D^2 = l^2 C$ (94)

(80) to $dv_{L_1} v_{L_1}/l_1^2 + dv_{L_2} v_{L_2}/l_2^2 + dv_{L_3} v_{L_3}/l_3^2 + \ \ = 0$ (95)

(82) to $v_{L_1}/l_1^2 = v_{L_2}/l_2^2 = v_{L_3}/l_3^2$ } (96)

(83) to $v_{D_1}/l_1^2 = v_{D_2}/l_2^2 = v_{D_3}/l_3^2$ }

i.e., the corrections in latitude are proportional to the squares of the sides, as also for the corrections in departure.

An examination of (93) shows that an error of 1:500 in distance will give $1/500 = \epsilon_B l$,

or $\epsilon_B' = 1/500 \times .00029 = 7'$, or for 1/1 000, $\epsilon_B' = 3.5'$.

These ratios are more reasonable for transit work than those tabulated from (78), but it would require an accuracy of 1/10 000 in chaining, or the best grade of level ground city work to reduce the correspondeng angle error to a value easily attained in ordinary transit work, unless the figure has a very large number of sides.

In this method the error of closure of the angles would first have to be distributed before computing the coordinates.

Example 1. The following field measurements were made with transit and tape:

Sta. 1, 44°38.5′ R, 287.24 ft.; sta. 2, 8°04′ R, 451.75 ft.; sta. 3, 129°17.5′ R, 921.60 ft.; sta. 4, 89°26′ R, 212 ft.; sta. 5, 2°35.5′ L, 317.3 ft.; sta. 6, 91°9.5′ 443.6 ft.

The deflections foot up 360° requiring no adjustment for angle closure. The line 6-1 is nearly north and south and it is taken for the meridian.

In computing the coordinates columns are added for $L^2/100\ l$, $D^2/100\ l$, $L\ D/100\ l$, made up with slide rule from the distances and coordinates as given below:

Station	Bearing.	Distance	Latitude, L. +	Latitude, L. −	Departure, D. +	Departure, D. −	$\frac{L^2}{100\ l}$	$\frac{D^2}{100\ l}$	$\frac{L\ D}{100\ l}$
1	N 44°38.5′ E	287.24	204.37		201.83		1.45	1.42	1.44
2	N 52 42.5 E	451.75	273.70		359.40		1.65	2.85	2.18
3	S 2 00 W	921.60		921.04		32.16	9.20	0.01	0.32
4	N 88 34 W	212	5.30			211.93	0.00	2.11	-0.05
5	S 88 50.5 W	317.3		6.41		317.24	0.01	3.17	0.06
6	North	443.6	443.60				4.44	0.00	0.00
	Totals		926.97	927.45	561.23	561.33	16.75	9.56	3.95
				926.97		561.23			
			$q_1 =$	- 0.48	$q_2 =$	- 0.10			

$$A = (-0.10\times3.95 + 0.48\times9.56)/(160 - 15.55) = +0.029$$
$$B = (-0.48\times3.95 + 0.10\times16.75)/(160 - 15.55) = -0.001$$

$v_1 = +0.05$	$v_{L_1} = +0.04$	$v_{D_1} = +0.04$
$v_2 = +0.08$	$v_{L_2} = +0.05$	$v_{D_2} = +0.06$
$v_3 = -0.26$	$v_{L_3} = +0.26$	$v_{D_3} = +0.01$
$v_4 = 0.00$	$v_{L_4} = 0.00$	$v_{D_4} = 0.00$
$v_5 = 0.00$	$v_{L_5} = .00$	$v_{D_5} = 0.00$
$v_6 = +0.13$	$v_{L_6} = +0.13$	$v_{D_6} = 0.00$
$[v] = 0.00$	$[v_L] = +0.48 = -q_1$	$[v_D] = +0.11 = -q_2$

If any line is regarded as perfect, as in connecting with a survey already adjusted, the corresponding correction is made zero and the corresponding $L^2/100\ l$, $D^2/100\ l$, and $L\ D/100\ l$ omitted in the summation for A and B.

PART II. GEODESY

CHAPTER I.

INTRODUCTION

1. GEODETIC SURVEY. Geodesy is the science and art of making the measurements and reductions required in relatively locating, with accuracy on the earth's surface, points which may be widely separated. It hence supposes a knowledge of the figure of the earth, of the various phenomena which effect physical measurements and of the construction and use of instruments, in addition to the accuracy of sight and touch so characteristic of the good observer.

A triangulation-net, or chain of triangles, is usually employed as giving the best results, both in quantity and accuracy, for the expenditure. Elevated points are chosen for the triangle vertices, at distances apart varying with the character of the survey from a few miles up to a hundred, one or more level lines shorter than the others are selected for base-lines, in such positions that they can be readily connected with the main net; signals are established which define the vertices accurately, yet are conspicuous enough to be seen by the aid of a telescope from the adjacent stations; the horizontal angles of the triangles, and usually the vertical also, or the inclinations of the sides, are then accurately measured with a theodolite, and the base-lines with a base apparatus. All the triangle sides and the differences in elevation of the vertices can then be computed.

Usually the elevations above sea-level of one or more vertices are measured; while astronomical observations are taken to determine the latitudes, and the distances in longitude from some observatory or reference station of one or more vertices, and the azimuths of one or more sides. The actual positions on the earth's surface, both horizontally and vertically, can then be computed.

The objects of a geodetic survey are usually twofold:

(a). The location or recovery of boundary and division lines or monuments, and the furnishing of a net with which to connect a topographic or hydrographic survey so that the inaccuracies of the latter cannot accumulate over large areas.

(b). The accurate determination of the figure of the earth. The distance between the parallels or meridians through any two stations or vertices results from the triangulation, and their difference in latitude and longitude, from astronomical observations. Dividing the difference in latitude in linear units by the angle in degree measure, or in π=measure, will give the length of a degree, or the radius of curvature, of the meridian. From these values in different latitudes the semi-axes a and b, the meridian quadrant Q, or the semi-major axis a and the eccentricity e or ellipticity ε ($\varepsilon = (a-b) \div a$) can be computed, assuming the section an ellipse, or the actual form can be approximated. Similarly, the parallels can be computed, assuming circles, -giving an ellipsoid of revolution, -or their actual shape can be approximated.

2. HISTORIC OUTLINE. (a). In glancing at the development of the science of geodesy we may note as of special interest:

The first authenticated hypothesis of the spherical form of the earth by Pythagoras, who is supposed to have been born about 582 B.C.

The first determination of the circumference by Eratosthenes, 230 B.C. He originated the method of deducing the size of the earth from a measured meridional arc, for he found that while the sun's rays were vertical at noon during the summer solstice at Syene in southern Egypt they made an angle $2\pi \div 50$ with the vertical at Alexandria in northern Egypt, and reasoned from this that the earth's circumference must be 50 times the distance between the points. The distance according to the statements of travelers was 5000 stadia, giving 250,000 stadia for the circumference by assuming both points to be on the same meridian (Syene is nearly 3° east of Alexandria) Jordan (Vermessungkunde, Stuttgart, 1890, Vol.3, p.2) estimates this value to be about 16% in excess by taking 1 stadium = 185^m, the exact value of a stadium being unknown.

Omitting the arc of 2 degrees of the meridian which was directly measured with wooden rods under the direction of an Arabian Caliph in 827; and the measurement made by Fernel in 1525 by counting the number of revolutions made by a carriage wheel in going from Paris to Amiens and reducing the broken line to the meridian, - giving by an unusual compensation of errors, a computed circumference only 0.1% in excess; we come to,

The arc measured by Snellius of Holland, 1615, it being the first in which the principle of triangulation was employed. He used 33 triangles; measured his base-line with a chain; his angles with a sector having sights attached; and found a meridional arc of about 1° 11′. His computed circumference was 3.4% too small.

The introduction of cross-hairs in the telescope and its adaptation to angle instruments, by Picard in 1669. He extended a triangulation over an arc of about 1° 23′, from a base-line nearly seven miles long; and derived the most accurate degree length thus far given. His angles were carefully measured with a sector of 10 feet radius, to which a telescope was attached.

The facts reported by Richer, on his return from an astronomical expedition in 1672, viz, that his clock, which beat seconds at Paris before starting, lost about two minutes per day while at the island of Cayenne, S. America, and could only be corrected by shortening the pendulum 1 1/4 Paris lines.

The announcement by Newton, Principia, 1687, of the theory of universal gravitation, and of the corollary of the oblate spheroidal form of the earth. The first was confirmed by Picard's more accurate degree-length; for with the diameter of the earth thus given, the force of gravity at the surface and the force required to hold the moon in its orbit, were to each other inversely as the squares of the distances from the earth's center. The second was confirmed by the behavior of Richer's pendulum; for since by Church's Mechanics, §78,

$$t = \pi\sqrt{\frac{l}{g}\left(1+\frac{h}{8l}\right)}$$

(where t is the time of oscillation in seconds in vacuo; l the length; g the acceleration of gravity; and h the versed sine of the semiarc of oscillation, supposed small) an increase in t for a given value of l in the lower latitude, indicated a decrease in g or an increase in distance from the earth's center in approaching the equator.

The extension of Picard's triangulation each way from the vicinity of Paris to include a meridional arc of 8° 31′, between 1683 and 1716 by J. and D. Cassini; from which the length of a degree of the meridian was found to be less at the northern end than at the southern. The earth would thus be a prolate spheroid, and not an oblate, as advocated by Newton, Huygens, and others. Huygens had published in 1691 the results of experiments whereby he found that a flexible hoop when rotated about one of its diameters would become flattened at the poles if unrestrained. The controversy which arose finally induced the French Academy,- as the French at this time took the lead in Geodesy,-to send out two expeditions, one to Peru under the equator in 1735, the other to Lapland under the Arctic circle in 1736, to definitely settle the question. The degree length in Lapland, when made known in 1737, was found to be greater than at Paris; Cassini's arc when revised in 1744, gave a greater length for a degree of the meridian at the northern end than at the southern; so that when the result from Peru was received about a year later all agreed in confirming the oblate hypothesis. The details of the measures of these arcs are extremely interesting. The first is described by Maupertius in La Figure de la Terre, Paris, 1738, and by Outhier in Journal d'un Voyage au Nord en 1736 -7, while its remeasure by Svanberg, 1801-3, is described in Exposition des Operations faites en Lapponie.. par J. Svanberg, Stockholm, 1805. The second, by Cassini de Thury, in La meridienne de l'Observatorie de Paris, verifiée.. Paris, 1744. And the third in La figure de la terre, par M. Bouguer, Paris, 1749, and Mesure, des trois premiers Degrés du Méridien par M. de la Condamine, Paris, 1751. Clarke, Geodesy - Oxford, 1880, pp. 3-13, gives an excellent resume of the work in Lapland and Peru.

The triangulation to connect the observatories of Paris and Greenwich proposed 1783; and that to determine the earth's meridian quadrant, 1791, from the measure of an arc of about 9° 40′. extending south from the extreme northern end of France, one ten-millionth part of this quadrant was to be used as a standard unit of length to be called a meter The French introduced the repeating circle (see §24) on the first and the Borda base apparatus (see §52) on the second. With the one, the angle to be measured between two signals is added on the circle as many times as desired, or as there are repetitions, -as may be done with an ordinary railroad transit, -when, subtracting the initial reading from the final, with 360° added for each full circumference passed, and dividing by the number of repetitions, the value of the angle is found with the errors of gradnation and of reading divided by the number of repetitions, or by as great a number as desired. With the other, the change in length of the measuring rod due to a change in temperature is inferred from the actual change with reference to a companion rod having a different rate of expansion, forming a metallic, or Borda, thermometer. While the theoretic advantages have never been fully realized in either case, the importance of the principles developed may be inferred from the fact that both have held an important place in geodetic work from that time to the present. (For descriptions of the French portions of the work see Exposé des Operations faites en France en 1787 pour la jonction des Observatories de Paris et Grenwich, by Mm. Cassini, Mechain, and Legendre; and the three volumes entitled Base du systeme métrique decimale, by Delambre,) Paris, 1806-10.) On the part of the triangulation which fell to the English, a Ramsden theodolite was introduced, of such excellent quality that the repeating circle, and the corresponding method of repeating angles, has never crossed the Channel.. This instrument has remained in use, on primary triangulation in England and in India to the present time; and Col. Clarke, in 1880 (Geodesy, p. 14) says, that with the exception of some very trifling repairs, it is as good as when first used. The circle, 36 inches in diameter, was graduated with a dividing engine by dots into spaces of 15′; it is read by three micrometer microscopes to single seconds. The telescope has a focal length of 36 inches, and is supported by an axis two feet long. For a description of the work see, Account of the Observations and Calculations of the Principal Triangulation,, by Capt. A.R. Clarke, R.E., London, 1858.

3. HISTORIC OUTLINE. (b). The increased accuracy introduced by the French and English on the survey to connect Paris and Greenwich, and on the survey to determine the length of the meter, mark the close of the eighteenth century as the beginning of the era of modern geodesy. General interest in the subject became awakened and geodetic surveys began to extend over Europe; while the degree of accuracy attained, in some respects at least, compares not unfavorably with that of the present time. E.G., large triangles were easily closed within 3″ with the 36-inch Ramsden theodolite; a maximum limit which has long been prescribed by the U. S. Coast Survey for primary triangles, although the average error is very much less.

In England, the Ordnance Survey developed from the triangulation connecting Paris and Greenwich; it has extended over the entire kingdom with a triangulation and detailed topography, under Gen. Roy, Capt. Mudge, Col. Colby, and Gen. James, respectively as directors. See account of the Trigonometrical Survey of England and Wales, 1799,, also Account of the Observations and Calculations of the Principal Triangulation,, by Capt. A.R. Clarke, London, 1858.

In India, work was commenced in 1802 under Col. Lambton, - a short arc was measured in 1790 by Burrow (Montliche Correspondenz XII, 488) - ; it has been continued under Col. Everest, Sir Waugh, Lieut. Gen. Walker, and Col. Thuilhier. The objects have been mainly topographic, but in order to properly check the work over such large areas, chains of primary triangles, with an occasional tie-chain. at right angles have been carried along meridian lines at such distances apart that the intervening country can readily be covered by secondary triangles. A meridional arc of about 23° 49′. has resulted, and an arc of the parallel of some 30°; the first is of value in degree determination; but the difference in longitude has not been determined with sufficient accuracy to warrant the use of the second.

See, An Account of the Measurement of an Arc of the Meridian between the Parallels of 18° 03′ and 24° 07′,** ,by Col. Everest, London, 1830; also An Account of the Measurement of Two Sections of the Meridional Arc of India.**, by Lieut. Col. Everest, 1847; and Account of the Great Trigonometric Survey of India, by Lieut Gen. Walker to Vol. X, and under the order of Col. Thuillier from Vols. X to XIV in 1890, inclusive.

On the Continent, geodetic work was begun in Prussia in 1802, by von Zach. In Switzerland and Italy work was begun in 1811, the object being to join the French Triangulation and secure an arc of the parallel from the Atlantic Ocean to the Adriatic sea; when completed in 1832 it was not found very satisfactory and has never received much credit.

In Russia, the first work of value was begun in 1817 under Tenner and Struve; in 1855 a meridional arc of about 25° 20′., extending from the Danube to the North Sea, had been completed. The report of the work in the two volumes, Arc du Meridien, de 25° 20′. entre le Danube et la mer glaciale mesure depuis 1816, jusqu'en 1855; Ouvrage compose sur les different materiaux et redige, par F.C.W. Struve, St. Petersburg, 1860, is considered the greatest contribution yet made to the subject of the figure of the earth, and should be studied by all who are interested in geodesy.

In Hanover, Gauss measured a meridional arc for a degree measure, 1821-23, and extended the triangulation over the country, 1824-44. His work is classic; to it is due the first application of the method of least squares in the adjustment of a triangulation net; the theory of conical coordinates; the general theory of geodetic lines on curved surfaces; and the invention and use of the heliotrope.

In 1831, Bessel and Bayer, began a triangulation to connect the chains of France, Hanover, Denmark, Prussia and Bavaria, with that of Russia, and to serve for degree-measurements. This work is also classic; the publication of the report, Gradmessung in Ostpreussen and ihre Verbindung.., by F.W. Bessel, Berlin, 1838, is thought by Col. Clarke to mark an era in the science of Geodesy, on account of the precision of the book, and of the work of which it treats; many of the methods which are there for the first time described being still in use.

The Russian and Austrian chains were connected between 1847 and 1851; and the Swiss and Lombardian chains at about the same time. The English and Belgian were joined in 1861.

About 1862 the Permanente Commission der International Erdmessung, -The International Geodetic Association, -was organized largely through the efforts of Gen. Baeyer, Bessel's colaborer. (Pfr. Helmert of Berlin, is director and A. Hirsch, of Nuremburg, permanent secretary.) For an account of the recent work in Europe, reference may be had to the yearly reports of this Association, which includes some twenty-four countries.

But little work was done in Italy until the formation of the Italian Commission, 1865. Work was begun in Spain in 1858, and excellent results have been obtained under Col. Ibañez. A remeasure of the French arc of Delambre and Mechain was begun in 1870 under the direction of M. Perrier, and this was followed by an extension of the French and Spanish chain across the Mediterranean to Algiers in 1879, giving a meridional arc of 27° extending from the Shetland Islands to the desert of Sahara.

The chains of Russia and England have just been connected through Central Prussia with small discrepancies between the ten base-lines joined. Accurate topographic surveys and lines of geodetic levels have also been extended over the greater part of Europe.

The development of least squares has added much to the precision of geodetic work. The theory was first stated by Legendre in 1805; it was added to by Adrian in 1808; but its full developement was due to Gauss in 1809, and its first application to the adjustment of a triangulation was made by him in adjusting the Hanover arc as already noted.

The method as now extended and perfected is applied in the reduction of every important geodetic survey.

4. GEODETIC WORK IN THE UNITED STATES. The English Astronomers, Mason and Dixon, in running out the celebrated line bearing their name, found the position of the division line between Maryland and Delaware which coincides approximately with the meridian to be on low and level ground, and hence well adapted to direct measurement for a degree determination. Ac-

cordingly, with the aid of the Royal Society of London, they made a direct measurement with wooden rods, starting at the south-west corner of Delaware and extending into Pennsylvania, of about 1° 29'., and determined the azimuths of the different portions of the line and the latitudes of its extremeties. The work, described in London Philosophical Transactions, 1768, by Mason and Maskeline, is not accepted with much confidence.

The U.S. Coast Survey was authorized by Congress, in 1807; but, owing to lack of funds, work was not commenced until 1817, and but little was done except in detached surveys along the coast, until 1832. The triangulation, which was commenced in the vicinity of New York Harbor, has been gradually extended along the entire Atlantic coast, along the Gulf coast and along the greater part of the Pacific coast, not including Alaska. In 1871, the project was authorized of connecting the Atlantic and Pacific systems and of furnishing trigonometric surveys to such states as should make the necessary provision for carrying on the topographic and geologic portions of the work.

The transcontinental chain, which extends approximately along the thirty-ninth parallel, was soon begun and is now completed, (1898) giving an arc of about 22° in latitude, and of about 49° in longitude.

The opportunity afforded for state surveys has been improved by quite a number of states, while the country will eventually be covered with a triangulation net which will compare favorably with any in Europe.

Since the extension to include interior work, the survey has been known as the Coast and Geodetic Survey. It is under the Treasury Department.

The superintendents, and times of their appointments, have been, F.R. Hassler, 1807; A.D. Bache, 1843, Benjamin Pierce, 1867; C.P. Patterson, 1874; J.E. Hilgard, 1881; F.M. Thorn, 1886, T.C. Mendenhall, 1889; W.W. Duffield, 1894; H. S. Pritchett, 1897; O.H. Tittmann, 1900. The yearly reports contain much valuable material, especially in the appendices.

The survey of the Northern and Northwestern Lakes was commenced in 1841, under the War Department; better instruments and methods were introduced in 1851, and the character of the work was gradually improved to 1870, when the survey passed under the charge of Gen. C.V. Comstock of the Corps of Engineers. From that date to the close in 1881 a continuous chain of triangulation, depending upon 8 carefully measured bases, was extended from St. Ignace Island, on the north shore of Lake Superior, to Parkersburg in Southern Illinois, a distance in latitude of 10°, and from Duluth, Minn., via. Chicago, to the east end of Lake Ontario, a distance along its axis of 1,300 miles, or in longitude of 16°. Some very excellent base-line work has been done and the triangulation has been carefully executed. See, Primary Triangulation U.S. Lake Survey, 1882, by Gen. C.B. Comstock; or see the yearly reports of the Chief of Engineers.

Many of the states are now engaged in geodetic surveys. Massachussetts took the lead, under Borden, in 1831.

CHAPTER II.

TRIANGULATION, RECONNOISSANCE, SIGNALS.

5. PRIMARY, SECONDARY, TERTIARY, TRIANGULATION. When a triangulation is to be extended over a large tract of country, or between two or more distant points, a system of primary triangles is employed; which is characterized by the maximum development of which the topography will admit. This in level or slightly undulating country, will allow of triangle sides of only 15 to 25 miles, on account of the height of signal, and of observing stand, required to overcome the earth's curvature; while in mountainous country, sides of from 40 to 60 miles are common, and those from 100 to 150 miles are not unknown. Distances are determined with an accuracy of about 1 : 100,000, the range being from about 1 : 60,000 to 1 : 200,000.

If points are required nearer together than the primary stations, secondary ones are established. The triangles connecting them with the primary ones, or with each other are called secondary triangles. Their sides usually vary from 5 to 25 or more miles; while an accuracy of from 1 : 20000 to 1 : 50000 is usually attained.

If an accurate topographic or hydrographic survey is to follow, points not more than from 1 to 3 miles apart will be required; the triangles connecting them with the secondary ones are called tertiary triangles. Their sides do not usually exceed about 5 miles; while an accuracy of from 1:5,000 to 1:20,000, or an average of 1 : 10,000 is usually attained.

For surveys of less extent, the primary triangulation, and the secondary also is sometimes omitted. Greater care and accuracy will then be required in the tertiary triangulation, as it must check its own work. In primary work, the base-lines are usually from 4 to 12 miles long and they are placed from 200 to 600 miles apart measured along the triangulation. In secondary work, which does not start from primary work or check upon it at sufficiently small intervals, they are about 2 to 3 miles long and are placed at distances apart of from 50 to 150 miles. In tertiary work, which is not sufficiently checked by secondary, they are from 1/2 to 2 miles long, and are placed at intervals of from 10 to 40 miles. These distances vary with the character of the work and of the country, as well as with the indivuality of the person conducting the survey.

6. TRIANGULATION SYSTEMS. In connecting two distant points, or in following a line as a coast or boundary, a principal chain of triangulation should be laid out, along which distances and azimuths or directions can be carried with the greatest accuracy and directness. At the end of the chain, and at as many intermediate points as may be thought necessary, a check is had by measuring a base and observing an astronomical azimuth, and comparing the measured length and direction with those computed through the chain.

In covering a large area with a network of triangulation, the method often employed is to extend around the area, a main chain, which is checked by closing upon itself, and which serves as a framework with which to connect longitudinal chains. These in turn serve for transverse chains, which complete the gridiron of primary triangulation and allow the intervening areas to be reached by secondary and tertiary triangles. The discrepancy due to imperfect measurement are adjusted for each series, in order, and each is then considered perfect in fitting the next lower to it. The adjustment is thus comparatively simple while if the whole area were covered with a series of continuous triangles all measured with the same accuracy, the labor would increase so rapidly with the number of triangles as soon to become prohibitory except by subdividing into more or less arbitrary sections.

The above methods should be flexible enough to allow of taking advantage of routes most favorable for the triangulation, even though they are some distance from the boundary, or do not give cross chains at right angles, or at uniform distances apart.

The composition of the chain also deserves attention. In order to make a comparison of strings of practically the same length, Mr. C.A. Schott (C. and G. Survey Report, 1876, App. 20) takes a string of 10* equilateral triangles with sides of unity; 3 regular hexagons with sides of unity, each divided into 6 equilateral triangles by joining a central point with the vertices; and 7 quadrilaterals, with diagonals of unity, Fig. 1, and finds that: The actual lengths of the strings will be 5, 5.2 and 4.93 respectively.

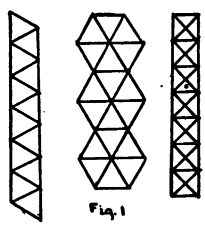


Fig. 1

The numbers of stations will be 12, 17, and 16.

The numbers of the sides to be sighted will be 21, 34, and 36.

The total lengths of the sides will be 21, 34, and 29.6

The areas covered will be 5, 9, and 4.04.

The numbers of checks upon the observed angles, due to geometric conditions, will be 10, 21, and 28. While these regular figures and separate systems usually are not feasible, the above comparison, the above comparison indicates that the single string of triangles is the most favorable for rapidity and economy; the differ -

*The 9 which Mr. Schott used is here changed to 10, to give the same actual advance of triangulation rather than that of extreme points. A quadrilateral in geodesy is a four-sided polygon having all the vertices joined.

ence being more apparent in level or prairie country, as only about two-thirds as many expensive elevated signals will be required; while if the level ground be wooded, the additional saving in clearing only about two-thirds the length of lines will usually compel its adoption even for the best grade of work. The string of hexagons, or other polygons having their vertices joined to an interior point, commands attention when greater width and accuracy are desirable; while the string of quadrilaterals affords greater accuracy with less stations and less labor, and is the system usually adopted by the C. and Geodetic Survey except for densely wooded level country.

7. ELEVATION OF SIGNAL. Usually the question of intervisibility of stations is best settled by actual observation; but when the station points are not intervisible, and signals can only be rendered so by elevation, the required heights may be difficult to determine by observation, unless there is a tree or other elevated object near, from the top of which the desired view may be had. In such cases, if the heights of the stations are known, and that of the intervening ground, which obstructs the view, can readily be determined, as would be the case for level ground or for a line passing over the water, the required heights can be readily computed. In the vertical section through the two stations C and C′, Fig. 2; let AA′ be a straight line tangent at D; BDB′, the line of sight, between the two intervisible points B and B′, concave downwards on account of refraction. Denote the distances AD, A′D, in miles by k, k′; the required heights BC, B′C′, in feet by h, h′; the radius in miles by R(log R = 3.597317); and the coefficient of refraction, with mean value 0.07, by m, or the refraction angle ADB by m × AOD. Then in the right triangle AOD,

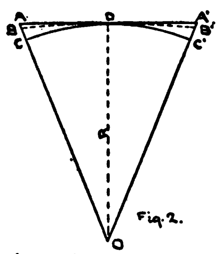


Fig. 2.

$$(AC + R)^2 = k^2 + R^2, \text{ or } AC, \text{ in miles} = \frac{k^2}{2R}, \text{ nearly}$$

$\because$ ADB = mAOD = 2mADC, and the angles are small,

$\therefore$ AB = 2m AC, and BC, in miles, or $\frac{h}{5280}$ =

$$AC - AB = \frac{k^2}{2R}(1 - 2m); \quad h = \frac{k^2}{2R} \times 0.86 \times 5280$$

$$k^2 = 1.743\ h \qquad (1)$$

where k is in miles and h is in feet.

I.e., <u>the square of the distance in miles is about 1 3/4 times the required elevation in feet</u>; - a convenient rule easily remembered.

For k in kilometers and h in meters, (1) reduces to

$$k^2 = 14.807\ h \qquad (2)$$

The line of sight should not pass nearer the surface than 10 feet at the tangent point, on account of the lack of transparency and danger of lateral refraction, due to the disturbed lower air.

Ex. 1. Two stations of the U.S. Lake Survey, Buchanan on the north side of Lake Superior, and Brulé River on the south, are 10 and 19 feet above lake level, respectively, and 16 miles apart. A signal 35 feet high was used at Brulé.

How high should the instrument and observing stand be elevated at Buchanan, in order to see the upper 20 feet of the signal at Brulé?

19 + 35 - 20 = 34; 34 - 10 = 24, the available height at Brulé.

Placing h = 24 in (1), $k = \sqrt{1.743 \times 24} = 6.5$ miles, the distance from Brulé to the tangent point. 16 - 6.5 = 9.5 miles, the distance from the tangent point to Buchanan.

$$h' = \frac{(9.5)^2}{1.743} = 52 \text{ feet.}$$

52 + 10 = 62, the required height above lake level, or 52 feet above the ground.

8. HINTS IN SELECTING STATIONS. Choose the highest elevations, even if

at greater first cost on account of inaccessibility. They will then the better command new ground, if at any time it becomes necessary to extend the work beyond its original limits; while high lines of sight meet less atmospheric disturbance.

Use as long lines as the topography of the country, and the visibility of the signals, will admit of in order to increase the accuracy.

Avoid low lines and lines passing over cities, furnaces, etc.

Form triangles which shall be as nearly equilateral as may be; the usual limits for an angle are from 30° to 120°, but Capt. Boutelle now recommends for C. and G. Survey practice (Report 1885, App. 10) an extension of from 10° to 15° each way in quadrilaterals or other well checked systems of primary triangulation when necessary.

The nearer an angle to 90° the less does a change in its value affect its sine, while the nearer to 0° or 180°, the greater in an increasing ratio does a change in its value affect its sine. Hence a triangle side will be least affected by angle errors, when the angles on which it depends are near 90°

The nearest approach to this, when two sides of a triangle are required in terms of the third, will be 60° for each angle, as given above. If, however, one side is not common to any other triangle- as when advancing by a single string of triangles- an error in its length will not be transmitted into the chain, so that a small opposite angle will not be as objectionable as when both sides are required with equal accuracy.

When a point is to be located by cuts from two or more known stations the lines should intersect as nearly at right angles as may be.

In finally locating stations, make certain that those intended to be intervisible really are so, even at the expense of time and patience in waiting for clearing weather ; otherwise the observing party will suffer vexatious and expensive delays.

Select stations so that permanent station-marks can be placed and protected, or so that accurate references can be had to permanent objects.

Advance by quadrilaterals, when the greatest accuracy is desired.

Locate secondary and tertiary stations so as to command a sweep of the area to be surveyed, in order to readily locate, by intersections, points for the topographic and hydrographic parties.

9. BASE LINES. A base line site should be selected with reference to securing suitable ground for measurement and a convenient expansion, by well shaped triangles or quadrilaterals, to reach a side of the main triangulation.

The line should be free from obstructions, and quite smooth for a width of at least 12 feet; longitudinal slopes up to 3° to 5° are admitted without serious inconvenience, even when making the most accurate measurements; the ends need not be intervisible from the ground, if they can be made intervisible by signals and observing stands of moderate elevation. The measurements can be made along two straight segments, not differing widely in direction, if better ground will thus be secured. Narrow ravines can be crossed by bridges or trestlework with complete success; while a wide one, or a bog or similar obstruction to direct measurement, can be passed by triangulation without very serious decrease of accuracy. Subsidary bases which are to be measured with a long steel tape can be located on rougher ground if necessary.

The selection of the system of triangles by which the side of a main triangle can be computed from the base, with the greatest accuracy for the expenditure, requires considerable skill. Auxiliary stations will be required in the expansion; working down from a side and locating the auxiliaries and base line to correspond in a level country, or up from a base-line to the main side, modified to adapt it to expansion if necessary, in case of rough country.

In case several sites are available, the cost of preparation and of measurement, and the cost of the connecting triangulation, should be estimated for each; this when compared with the relative accuracy of the triangle side which each can furnish, will allow of selecting the one most desirable.

Ex. 1. The Buffalo base of the U.S.L. Survey, measured near Buffalo .

N.Y., in 1875, is shown in Fig.3, together with the connection to, and a portion of, the main triangulation.

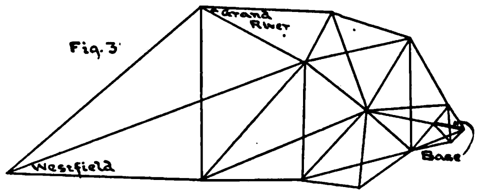


Fig.3

The gradual enlargement from the base to a side of the main triangulation, and the different triangles which may be used in finding the length of any side, as Grand River-Westfield, from the base may be noted.

The Edisto base of the C.Survey, shown in Fig.5, §11, on the other hand consists of the side of a main primary triangle; the other sides being short because the country is level and heavily timbered.

10. RECONNOISSANCE, PRIMARY TRIANGULATION. A general reconnoissance should precede the selection of stations, in order to become sufficiently familiar with the topography to be able to recognize the most prominent features and elevations, as seen from different points of view, and in order to determine the general scheme of triangulation, and the general routes best suited to the ground, for aid in conducting the detailed reconnoissance.

Unless the surface is level and unbroken, points will be found which from their position or elevation, will offer such advantages that they probably must be used for stations. Starting from these, others must lie within prescribed areas, in order to fulfill the required geometric conditions, and make use of the longest feasible sides.

From each of these probable station points, sights should be taken to the others if visible, and also to such points in the prescribed areas as will possibly serve for stations.

Other available points can be occupied, and the process repeated, if necessary. Should a point be occupied which has not been cut from at least two other stations, sights must be taken upon at least three known points, when its position can be determined by §12.

Magnetic bearings often aid in orientation on arriving at a new station; and in identifying objects already located, by giving approximate directions; while they sometimes aid in plotting when insufficient angles have been taken.

A hasty outline profile sketch of the ground in the vicinity of each object sighted will aid very materially in identification from surrounding stations, while if the estimated distance in miles, is written near the point, and the circle reading is written above on a vertical through it, see Fig.4, very clear and concise notes will result. The obstructed arcs at a station should be noted; as also the cause, and whether they can be removed by cutting, or by signal elevation. Should the location be likely to prove difficult; vertical angles should be taken to aid in deciding upon the intervisibility of signals by giving differences of elevation.

Fig.4.

A plat of this preliminary triangulation should be kept up by angles, starting from a known or assumed side; or by computed triangle sides, if greater accuracy is desired. Then working from probable station points, or from stations already located, the possible point in a given area is picked out which will best fulfill the conditions imposed, as to length of line, intervisibility, etc. In the same manner as many new ones are chosen from the plot as desired.

Without experience, it is quite difficult on reaching an elevated point, to orient one's self and be able to identify signals and topographic features at distances of 40 to 50 miles, even under the most favorable conditions. When, as is often the case, the features are not prominent, and the air is thick with haze and smoke for days at a time, the skill and patience of the experienced are fully taxed. With wooded elevations the observations must be usually taken from the top of a tree, or if none can be found of sufficient height, from the top of a ladder formed by splicing several together and supporting them by guys.

High elevations with the summits free from timber afford the best station sites. Wooded summits require sight-lines to be cut through. These for pole signals should be about 100 feet wide and they should be extended back of the station far enough so that the signal will not be seen against near woods.

As the summits broaden, or the timber becomes valuable, elevated signals and observing stands should be considered before clearing the lines, although they generally should not be adopted unless a considerable saving will result.

Parallel wooded ridges may present much difficulty, if so near together that the triangle sides must reach over an intermediate ridge instead of spanning an intermediate valley. The direction across the ridge to an invisible station can be found from the plat, or from §14; when the required signal elevation can be found from the vertical angle, or from carefully taken aneroid barometer readings; but if two or more ridges intervene, actual tests, from ladder tops, or an examination of the entire line will be necessary.

In level country, an elevation of 70 feet for signal and observing stand will allow of 20-mile sides. If wooded, these had best be used in a chain of nearly equilateral triangles having all the lines cut through; but if clear, as on prairie, quadrilaterals with diagonals of 21 miles and sides of about 15, will add only one more station in 30 miles of progress, which will be more than compensated for by the increased precision attained.

If the level ground be cultivated and contain patches of valuable timber, the difficulties will be so much increased, even if the ground be rolling, that the greatest care and skill will be required to avoid insuperable obstacles. Sometimes chains of secondary triangles along the water courses have proved effective.

Full notes and sketches should be taken of the points most important for the subsequent work. Among these are the means of access; the timber which can be found at the site for the signal; the roads which have to be opened by the angle party in occupying the station; the places nearby where board can be had; etc.

The efficiency and economy of the survey will depend very materially upon the skill, good judgment and experience of the person who conducts the reconnoissance.

11. SECONDARY AND TERTIARY TRIANGULATION. Starting with the long primary sides as bases, points of the first order are taken, which will shorten the triangle sides and command the area to be surveyed. From these shorter sides, points of the second order are taken so that they will command every prominent object visible. From the short sides thus obtained, tertiary points are located by cuts from at least 2, preferably 3, stations.

These points should include as many prominent objects, usually from 1 to 3 miles apart, as may be needed by the topographer in tying up his work, or by the hydrographer in taking angles to locate soundings, etc.; such as church spires, cupolas, chimneys, flags in prominent trees, large white crosses or triangles painted upon rocky cliffs, etc.

Well-shaped triangles are not so important as the securing of a sufficient number of convenient points for the topographer, since the errors introduced do not accumulate over large areas, being checked by the primary system. If the latter is omitted, better shaped secondary triangles should of course be employed.

Ex. 1. Fig. 5 shows a portion of the primary and secondary triangulation near the Edisto base of the C and G. Survey, South Carolina, on a scale of 1 : 400,000, taken from the Report for 1895, App. 10. The country is flat and wooded, no elevations of 20 feet being available. The use for secondary sides of the lines cleared for primary ones may be noted.

In the same App. may be found a sketch of the secondary triangulation of Boston Bay, an open country with suitable elevations.

12. N-POINT PROBLEM. To determine the position of a point, when only angles at the point have been observed between known stations. Lay off the angles on tracing cloth in order around a point; place the cloth on the plat and move it until each line shall pass through the station to which it belongs; when the vertex can be picked through. Two angles will locate a point, giving the 3-point problem, except when the point lies on or near a circle passing through the three stations on which the sights are taken; 3 or more angles are better, forming a check. A 3-armed protractor is often used in place of the tracing cloth; also a sheet of paper, by cutting out a narrow strip along each line near the portion to be used.

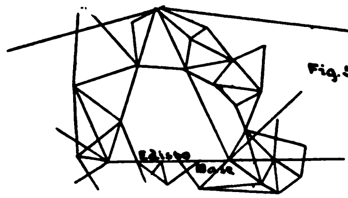


Fig. 5.

When a more accurate solution is desired than can be had from a careful plat on a large scale, a numerical one is used. In Fig. 6, let S be the required point at which the angles P, P', P'', ---..., have been observed upon the known stations, F, G, H, ... B is also known, it being the angle between known stations.

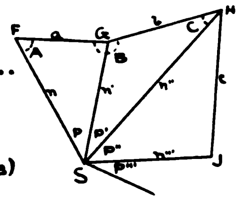


Fig. 6.

In the triangles, SFG; SGH, by Formula 18)*

$$n' = \frac{a \sin A}{\sin P} = \frac{b \sin C}{\sin P'} \qquad (a)$$

$\therefore$ a sin A sin P' - b sin C sin P = 0 (b)

In the polygon SFGH, P+P' +A +B + C = 360°

C = Q - A, where Q = 360° - (P+ P' + B) (3)

Substituting in (b), with the expansion of sin(Q-A) from Formula 3),

$$a \sin A \sin P' - b \sin P(\sin Q \cos A - \cos Q \sin A) = 0$$

$$a \sin P' - b \sin P \sin Q \cot A + b \sin P \cos Q = 0$$

$$\cot A = \cot Q + \frac{a \sin P'}{b \sin P \sin Q}$$

$$\cot A = \cot Q\left(1 + \frac{a \sin P'}{b \sin P \cos Q}\right) \qquad (4)$$

Having A, all the angles of the triangles become known, when

$$n = \frac{a \sin (P+A)}{\sin P} \;;\; n' = \frac{b \sin(P' + C)}{\sin P'} \;; \text{etc.} \qquad (5)$$

Ex. 1. At Sheldrake Point, Cayuga Lake, N.Y., the following angles were observed upon 3 known stations W_e (Willets) J_e (King's Ferry), and J_w (Kidders) Required the position of Sheldrake.

Observed.			Given.	
~~Kid.~~ Wil.-Shel.- King's = P	= 118°	15'.2	Wil-King's = a	= 7150.2
King's-Shel.- Kid. = P'	= 58	31.2	King's-Kid. = b	=3050.7
			Kid.-King's-Wil. = B	= 101° 8'.

From which by (3) Q = 82° 5.3'.

(4) a = 7150.2 3.85432
P' = 58° 31'.2, sin 9.93086
3.78518

*See Table I.

b = 3050.7	3.48440	forward	3.73818
P = 118° 15.2, sin	9.94490		
Q = 82 05.3, cos	9.13877	....	2.56807
+ 16.486	2.56807	....	1.21711
1			
17.486		....	1.24269
Q = 82° 05.3, cot		....	9.14292
A = 22 22.1, cot		....	0.38561
C = 59 43.2 = Q - A by (6)			

a = 7150.2 .	3.85432	b = 3050.7	3.48440
P+A = 140° 37.3, sin ...	9.80239	C = 59° 43.2, sin	9.93630
	3.65671		3.42070
P = 118° 15.2, sin	9.94491	P′ = 53° 31.2, sin......	9.93086
n = 5149.9	3.71180	n′ = 3039.2	3.48984

Computing n′ by the first equas. of (a) the same value is found as above.

13. TWO-POINT PROBLEM. If two unknown stations, C and D, Fig. 7, see each other, and also two known stations, A and B, their positions can be determined by measuring the angles ACB, BCD, CDA, ADB, as follows:

Draw the line C′D′ of convenient length on tracing cloth and at C′ and D′ lay off the measured angles; the intersection of the two lines which pass through A will determine its position on the cloth, and similarly for B; join A and B′; place the cloth on the plat so that A′ will coincide with station A and B′ will fall on the line AB of the map, produced if necessary; prick through the points B′, C′, and D′. Then through B draw ∥'s to B′C′ and B′D′; their intersections with AC′ and AD′ will determine C and D on the map.

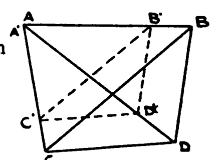


Fig. 7.

If more accuracy is desired; assume CD as unity and compute AC and AD in the triangle ACD, and BC and BD in the triangle BCD. Having two sides and the included angle in ACB, AB can be found (formula 20): the ratio of the true value to the computed one will be the ratio which the other sides bear to their computed values.

Ex. 1. The following angles were observed at Giles and Elm of the C.U. Skaneateles Lake Survey in 1892 upon the known stations Haight and Olmstead.

Haight - Giles - Elm	= 50°	02′.	17″
Olm. - Giles - Elm	= 35	05	03
Giles - Elm - Olm.	= 88	01	27
Giles - Elm - Haight	= 50	04	29

Haight-Olmstead = 12944 feet.

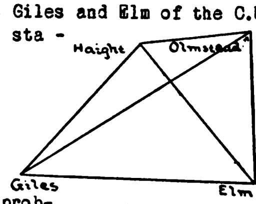


Fig. 8

For fuller treatment of the N-and two-point problems, see Zeit.- fur Vermes, 1888, P. 140.

14. DIRECTION OF INVISIBLE STATIONS. If enough angles have been taken so that the stations can be platted by methods already given, the direction of the line joining any two can be taken directly from the plat with a protractor. Or, starting from some known side, the sides of the preliminary triangles can be computed from the observed angles; when by assuming a meridian, the distance in latitude and in longitude of each point from an initial one can be computed as in an ordinary land survey. The tangent of the azimuth of the line joining any two points can then be found by dividing the difference in longitude by that in latitude. The line can then be cleared from either end if obstructed by timber, or the height of signal for intervisibility can be determined if the obstruction is an intervening ridge.

For an example in difficult country in northern Alabama, see U.S.C. & G.S. Report, 1885, App. 10.

If two stations C and D each see two points A and B, Fig. 7 §13, the direction to trim from one to the other can then be found as follows: At A measure BAD and DAC, and at B, DBC and CBA. Compute AD in the triangle ABD and AC in the triangle ABC, calling AB unity; then in ACD two sides and the included angle are known from which the angles at C and D can be found by formula 20]. Or , the directions can be found by platting.

15. OUTFIT. When accurate angles are required a light transit with a good telescope is most convenient. The needle will give bearings, while by adding a level to the telescope tube and a gradienter screw or good vertical circle, elevation angles can be measured with sufficient accuracy for determining intervisibility. An aneroid barometer is also convenient for determining differences of elevation. For distances over 25 miles, a reconnoitering glass with stand will be found desirable on account of the larger telescope. If care is taken in setting up to place the tripod head level, the small horizontal circle will give angles quite accurately.

In a wooded country where angles have to be measured from tree tops, a sextant will be necessary; also a telescope or field glass for identifying the stations, and a set of spurs or creepers for climbing. An azimuth or pocket compass is convenient; also the best available map of the region.

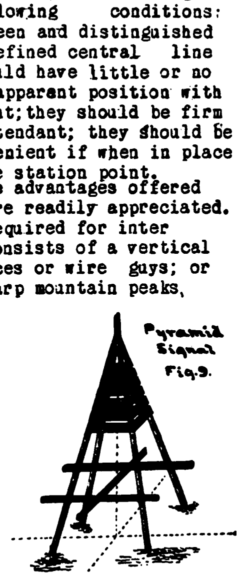
To these should be added some 100 feet of about 3/8 inch manilla rope, a ball of twine, an axe, and material for different colored flags to be spread out upon trees or other objects for temporary signals. An assistant, who is quick and handy at all kinds of work and who is used to climbing, and a horse and covered wagon, will complete the outfit. Much of the traveling will necessarily be on foot or possibly on horseback, if the country is hilly or wooded.

If away from all supplies, a cook and the usual camp outfit will be necessary; while for primary triangulation, in rough country with good railroad facilities, like much of New England, it may be more convenient to travel the long distances between stations by rail, hiring a horse when use can be made of one.

16. SIGNALS. After the exact station points have been located, the signals which are to be erected over them, to give definite points for sighting in measuring the angles should fulfill the following conditions:

They should be conspicuous, so as to be readily seen and distinguished from surrounding objects; they should have a well defined central line or point upon which to fix the cross-hairs; they should have little or no phase, i.e., this line or point should not change in apparent position with the direction of the illumination by direct sunlight; they should be firm in position unless of the class which require an attendant; they should be cheap, or light and portable; while often it is convenient if when in place they will allow an instrument to be set up over the station point. With these general requirements in mind, the relative advantages offered by the different signals to be described will be more readily appreciated.

17. POLE SIGNALS. When height is not required for inter visibility, one of the most common forms of signal consists of a vertical pole set in or on the ground, and supported by braces or wire guys; or of a pyramid or tripod surmounted by a pole. On sharp mountain peaks, where only small, stunted timber can be found, the rectangular pyramid, Fig. 9, is convenient. A signal with height of apex of from 12 to 18 feet and legs from 3 to 5 inches at the top, can be erected and a center pole 8 to 12 feet long inserted by 3 men, without tackle. By inclosing the top with boards, cloth or slats made from small poles, visibility can be given; while the apex and pole remain for accurate bisection. The pole can be increased to any desired diameter by nailing on slats or poles after erection; while the signal can be anchored to the rock, by wiring the legs to anchorbolts, or by wire guys extending from the top



Pyramid Signal Fig. 9.

of the pole.

On flatter peaks, more height must be given for visibility, rendering the tripod signal, Fig. 11, more convenient. By bolting all four pieces together on the ground, with a 1 to 1 1/4 inch bolt, as shown in Fig. 10 or better with the head raised 6 feet on a bent or staging, 5 men can raise a 25 to 35 foot signal of round timber, each piece being 5 or 6 inches in diameter at the top, with no special outfit except about 30 feet of rope. Pits are dug, or stones piled up to prevent the feet a and b from slipping; the head c is then lifted and pushed to position by the third leg when the pole is made vertical by pulling down the large end with a rope; it is secured by spiking braces to the tripod legs.

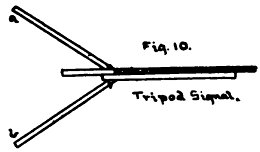


Fig. 10. Tripod Signal.

If the angles at the station are to be measured with the signal in place, the legs should be so placed as not to obstruct the lines of sight to the other stations. They should extend a couple of feet into the ground; or if on rock, be securely tied to anchor bolts by wire rope, or notched and horizontal cross pieces attached and loaded with stone. Wire guys from the top of the pole may also be desirable

A tin cone or barrel of larger diameter than the pole is often placed at the top, especially when the tripod head will not be seen against the sky.

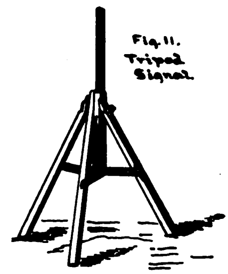


Fig. 11. Tripod Signal.

The pole should not be more than 6 to 8 inches at the tripod head, even for a large signal, on account of the weight in erection; it can afterwards be increased, or the pole straightened, by nailing on light slats. Or, when lumber is available, a square box of 8-inch plank in place of the pole will give diameter without increased weight; one or more slats along the center of each side will make it more nearly cylindrical.

A very convenient and portable signal for tertiary work can be made by supporting a pole on a tripod having a light cast iron head and about 10-ft. legs.

By holding the pole in position by wire guys, a signal 15 to 20 feet high can be made very stable while there is room enough underneath to set up an instrument. Any portion of the pole can be enlarged to any desired diameter by light slats.

18. DIAMETER AND HEIGHT. The diameter of pole for short lines may be large enough to subtend an angle as seen by the observer of 4 or 5 seconds; but as the distance and the power of the telescope increase the angle should diminish, according to Coast Survey practice, down to one second for about 15 miles, and not fall below this value for greater distances (see also §19).

Diameter to subtend one second at,

1 mile	= 0.307 inch,	40 miles	= 12.3 inches
10 "	= 3.100 "	60 "	= 18.4 "
20 "	= 6.100	80 "	= 24.6 "

Increased diameter beyond that necessary for visibility gives increased range to the cross-hairs in bisection, and introduces the uncertain element of phase with cylindrical signals which do not show against the sky.

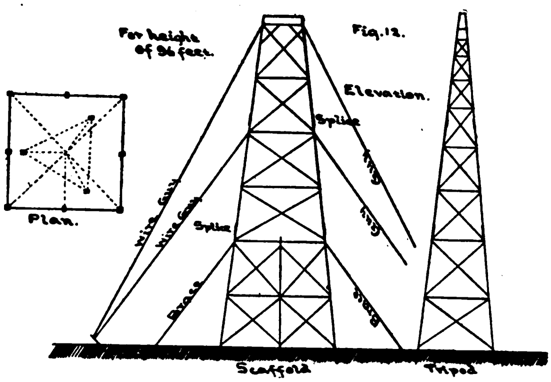
The height of signal in feet should be about one-half the distance in miles, plus 10. Less height may answer for long lines, or for signals on sharp peaks with a sky back ground, but height adds to visibility without diminishing accuracy, and with only the increased cost of construction.

A signal to be seen against the sky should be painted black or wound with black cloth, one to be seen against the ground should be painted white or wound with white cloth; unless two colors are needed on the same signal for ready identification from surrounding objects, when the pole, or pole and tripod, can be painted in alternate rings of black and white, or red and white, each ring being several feet wide.

19. SIGNALS WITHOUT PHASE.- Various signals have been devised to avoid phase or the effect produced by the unequal illumination by direct sunlight of the portion of the signal facing the observer, whereby the apparent and real centers do not coincide. One devised by Bessel for the Prussian triangulation in 1831, and used on the U.S. Lake Survey, consists of a board in place, or in front of, the pole with its face ⊥ to the line of sight. On the latter survey a width was given of about 4 seconds as seen by the observer, yet good angles were obtained. The station must be visited and the board changed each time the observing party move to a new station.

Another designed in 1881, and used on the Mississippi River Survey for distances of from 5 to 12 miles, gave excellent results. It consists of a horizontal board 6 inches in diameter, to the circumference of which are attached 4 stiff vertical wires, 90° apart, each 5 feet long. These wires are held in position by a wire ring at the top and another one-third the distance from the top; each joint being well soldered. Two opposite wires are connected for the upper and lower thirds by a white cloth, and the other two for the central third by a black cloth; 4 guy wires are attached at the central ring, and the board rests on a tripod or other support.

20. ELEVATED SIGNALS AND OBSERVING STANDS. When the signal and instrument at the station require elevating, and no existing structure can be made use of, a suitable one must be erected. The standard tripod and scaffold adopted for C and G. Survey work, for heights of floor from 32 to 96 feet, increasing by multiples of 16, are shown in Fig. 12. The scaffold is removed from the tripod in elevation for clearness; their relative positions can be seen from the plan.



For full details see Capt. Boutelle's excellent paper in Report, 1882, App. 10. See also, App. 9, page 158, and Pri. Tri. U.S. L. Survey, page 318.

The tripod, which supports the instrument when observing and the pole or other signal when observed upon, starts with a firm cap; the posts are 6 by 8 inches; they are scarf-spliced with a 3-foot lap, held by 6 5/8-inch bolts and 4 5-inch boat spike, at points 33 ft. apart starting from the top with 36-ft. sticks; batter 1 in 8; and braced by joists from 2 by 3 to 3 by 3 ins. spiked with 6-inch boat spikes.

The observing scaffold, which is placed outside of but not in contact with the tripod, starts with a floor 12 ft. square about 4 feet below the tripod head; the posts are 6 by 6 ins.; in sections of the same length and spliced in the same manner as for the tripod, using half-inch bolts; batter 1 in 6 measured diagonally; braces from 3 by 3 to 4 by 4 ins, in 16 ft. tiers. The posts above the floor are connected by a railing; while the flight of stairs connects the landing on the top of one set of horizontal braces with that on the top of the next. The short central posts starting on the ground in Fig. 12 are only used for tall scaffolds.

The posts for both tripod and scaffold rest on wooden shoes 12 by 15 inches. They are all placed on the same level, about 3 feet below the station point; and at the proper distance apart and from the center, by plumbing down from a templet placed on the ground.

To erect a structure of 3 sections : a derrick boom about 30 feet long and 6 inches in diameter is set up and held by guy ropes, advantage being taken of a tree if convenient in erecting it, the lower lengths of the tripod posts are then lifted upright, one by one, and held by guys with the lower ends in position, a workman ascends each post by means of cleats fastened to it, and the tops are sprung to relative positions and nailed to a templet, the templet is then shifted until a plumb hung from its center will fall over the station-point; when the bracing is spiked on and a floor laid on the upper horizontal joists. The pulley block is shifted to the top of a post and the lower end of the boom drawn up to the floor, it being kept upright by paying out the guys attached to the top; the next lengths of posts are drawn up and the splices bolted; the tops put in place and the bracing attached as before. The derrick is lowered and the lower two sections of the scaffold erected and braced as above; a floor is laid over the horizontal braces of tripod and scaffold; the derrick is drawn up and the upper section of each put in place and braced. About 12 days will be necessary, with workmen familiar with the work. In exposed situations the guys shown in Fig. 12 should be attached: 3/8 in. wire rope, each with turn buckle, is used.

Round timber can be used if more convenient The method of erection on the U.S. Lake Survey, for heights to 140 feet, was to put together one side of the observing tower on the ground; attach radiating ropes at different points, all leading to the rope through the block; erect a derrick boom and haul the side to position with teams; the side was then held by guys and the block shifted to it and one side of the inner tripod hauled up and held in the same way; when the third leg of the tripod was hauled up and the braces attached to the side already in position; then the opposite side of the tower was raised and the braces attached. Sills some 3 feet underground were used for the tower but not for the tripod. The station mark was placed after the signal was up. The work was let by the vertical foot; the contractor with 15 men and 2 teams would frame, erect and complete a signal in two days.

The tripod is often protected from the wind while observing by stretching cotton cloth over the windward side of the scaffold. With this precaution, the tripod is very steady in windy weather, and as good results have been obtained, even with large instruments, as from the ground. In sunny weather the tripod will twist in azimuth, following the sun during the day and returning at night, and some observers use the cotton screens to protect from the sun rather than from the wind; but the observations can be so arranged as to eliminate the effect of twist from the result A portable tripod and scaffold, having a floor about 12 feet high, is shown in Fig. 13. The tripod legs are 6 by 8 inches 18 feet long; held by an inch bolt 16 inches long, and by three horizontal braces. The scaffold posts are 5 by 5 inches, 16 1/2 feet long; the horizontal braces are 7 feet long, and the diagonal ones 10 feet. The posts are interchangeable and the braces are held by wood screws. The posts all extend about 2 feet into the ground, and the floor is placed from 2 to 3 feet below the top. Only a few hours are required for erection, after everything is in readiness.

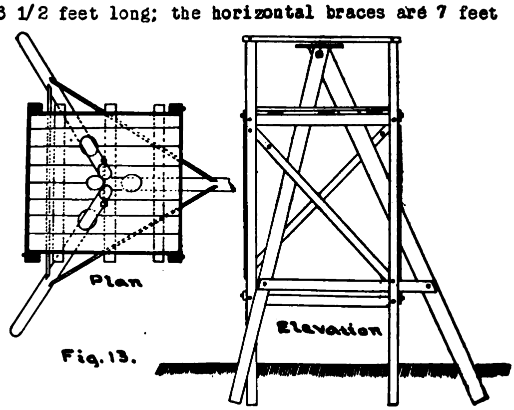


Fig. 13.

In India, hollow masonry towers 50 feet or more in height were extensively used for the support of the instrument in crossing the plains; while in the early French surveys, church spires, large towers, etc., were often used with inaccurate results due to phase.

21. HELIOTROPES. One of the most common forms in use is called the gas pipe heliotrope, Fig. 14. A piece of 2-inch iron pipe serves as a telescope tube, while it carries 2 rings or diaphrams, each with about an inch opening, and a 2 1/4 inch plate glass mirror having motion about a horizontal and a vertical axis. The whole instrument is supported by a wood- screw which can be screwed into a tripod head or other block. It is set up directly over the station mark, or on line and a few feet in front of it, and the cross hairs of the telescope brought on the distant observing party; the mirror is then turned so that the reflected sunlight will pass through the first or near diaphram and give a concentric ring of light around the second which is a little smaller; and this is continued by gently tapping the mirror at intervals of from 1/2 to 2 minutes.

Gas Pipe Heliotrope

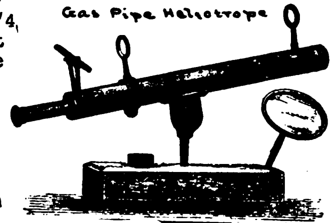


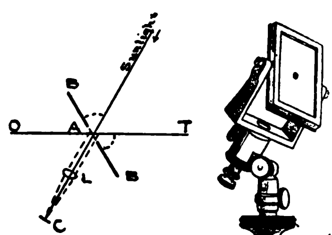
Fig. 14

The adjustment of the instrument should be tested by bringing the cross hairs on an object within a few hundred feet, throwing the light as above and noting if it falls as far above the object as the rings are above the cross hairs.

The Steinheil heliotrope differs from that already described in having only one mirror and no rings, making it very simple and convenient for reconnoissance work.

The axis of the frame is hollow and it contains a small lens, L, Fig. 15, and a white reflecting surface C, usually chalk, at the focus of the lens.

By turning this axis towards the sun, a hole through the silvering of the mirror allows a beam of sunlight to reach the lens and be concentrated upon the white surface. It is reflected from the surface back to the lens and emerges in parallel rays which reach the back of the mirror in a direction just opposite to that of the incident rays. Enough of these rays will be reflected from the back to give an image of the bright spot C, and in a direction AO directly opposite to the reflection of sunlight from the face of the mirror. Hence if the eye be placed at O so as to see the observing party through the opening A in the direction AT, and the mirror be turned until the bright spot C is seen (the axis pointing towards the sun) the sunlight will be reflected in the direction OAT to the observing party.



Steinheil Heliotrope
Fig. 15.

The distance from the reflecting surface to the lens is adjustable for focus.

When the alignment has been once secured, if there is no natural landmark in range, a pole should be set up at a distance of 100 to 200 feet so that its sharp top will be on or a little below the line; the light can then be shown, and often used by the observing party on days when haze and smoke will prevent the heliotroper from seeing even the outline of the hill or mountain at the observing station.

A second mirror is usually supplied which can be screwed up and light reflected from it to the first, if at any time the first falls in shadow or its angle of incidence becomes so great that the reflected beam will not fill the diaphram.

Extreme accuracy in pointing is not essential, the range being about the diameter of the sun, or 32 minutes.

About a 2-inch mirror is used for lines from about 20 to 60 miles, and usually in connection with pole or other signals. For shorter lines, a pasteboard or other screen with a smaller opening should be attached to the second ring. For longer lines larger mirrors are used. Thus on the U.S. Lake Survey for the longest lines a common mirror 9 by 12 inches was set up and light thrown through a circular hole in a wooden screen some 20 ft. distant in the direction of the observing station, this having a diameter of from 8 to 10 ins. on sides of 90 to 100 miles. On the longest line ever observed, Mts. Shasta-Lola in northern Cal., 192 miles, a helio 12 ins. square was used.

Wilson, Topographic Surveying, gives

$$x = .046\ d \qquad (6)$$

for the length of the side of the mirror in inches, where the distance d is in miles, and $d > 10$.

Too much light gives by irradation a diameter too large for accurate bisection and increases the unsteadiness; an opening suited to the distance or one which will subtend from one-fourth to one-fifth of a second, will give in quiet air a small bright disk easy to bisect.

An intelligent and very faithful person should be picked out for the heliotroper; otherwise delay and vexation will result. If he is to occupy the station a long time he can usually be picked up in the locality with economy, if for only a short time it may be more economical to have one who is familiar enough with the work and with instruments to go to new stations and establish himself without assistance, when directed by the observing party.

22. NIGHT SIGNALS. Lamps with 10-in. reflectors for short lines and the Drummond light for long ones were used on the English Ordnance Survey in the last century; while night signals have been extensively used in the recent prolongations of the Nouvelle méridienne de France by M. Perrier, and Argand lamps and heliotropes are exclusively used in India.

The electric light, in the focus of a reflector 20 inches in diameter and 24 inches focal length, proved very successful recently on a line of 168 miles across the Mediterranean where on account of fog and mist a 12-inch heliotrope had failed to once show during a three months' trial.

Some recent experiments made with the magnesium light* indicate that it is sufficiently powerful for long lines; while, unlike the Drummond or electric light, it is exceedingly portable(the instrument used weighing only 5 lbs.) and can be operated by an ordinary heliotroper.

The apparatus consists of an 8-inch reflector, a small lamp, a clock work, and a reel of magnesium tape which is fed by the clock to the lamp and burned in the focus of the reflector. For accurate bisection a pasteboard screen was used to reduce the diameter on all but hazy nights on a line of 60 miles. The tape was burned intermittently by time table to save expense; it costing about 2 1/3 cents per minute for a steady continuous light.

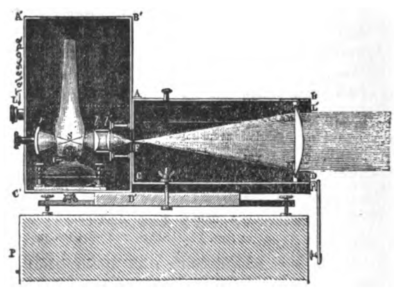


Fig. 16. — Collimateur optique.

Two of M. Perrier's lamps were also used, See Fig. 16. Each consists of a box containing a flat wick petroleum lamp in the focus of an 8 inch lens of 24 inches focal length. The emergent rays subtend an angle of about 1°. The intensity of light as compared with the magnesium was about as 2 to 5. It made a very pretty mark to point upon on clear nights, but at a distance of 43 miles it would often be scarcely visible in the telescope, and would not allow of illuminating the cross hairs, when the magnesium light was clearly visible. A student lamp was also tried; and with an 8-inch reflector it was visible in the telescope at 31 miles when the outline of the mountain was invisible at sunset.

The accuracy in these experiments proved to be equal or greater than

*U.S.C.& G.S. Report, 1880, App 8.

for day signals; while the time for good observing in favorable weather extended from about one hour after sunset to from 10 o'clock to midnight.
Collimators and reflectors, with kerosene lamps, were both successfully used on the N.Y. State survey for distances up to about 50 miles. The field, however, was left dark and the cross hairs illuminated from behind, giving light lines in the dark field.
The whiteness and intensity of the acetyline light, and the simplicity of the portable lamp, should place it in the first rank for night signals.

23. STATION REFERENCE. In referencing a station, the object should be to render the recovery of the locality and of the exact station point, as easy and certain as possible, at any time and by any one unfamiliar with the country but familiar with the kind of work. The station point is usually marked by an underground and by a surface mark. The underground mark should be placed below frost and plow, or some 3 or 4 feet below the surface. It may consist of any material which is durable, foreign to the locality, and capable of receiving and retaining an exact center mark.

Jugs and bottles, cut stone blocks, and hollow cones of stoneware are among the most common. The stone block, holding a copper bolt, and surrounded by masonry is much used at the ends of base-lines, where a very accurate mark is essential on account of working up from so short a side.

The surface mark should not be in contact with the underground mark, while it should project enough above the surface to be readily found. A stone post, with the top dressed some 4 to 6 inches square, and the center marked by a cross or hole is much used: often the number of the station, or the initials of the survey, are cut near the top. On the Coast Survey, 3 other marks are used, two in the meridian and one ⊥ to it at a distance of 6 feet when practicable; each has an arrow pointing toward the center.
Should the station be on firm rock, a hole is drilled some 12 to 15 inches deep and filled with lead or sulphur; or a copper bolt is inserted with a wedge at the bottom which tightens as the bolt is driven down.
Along coasts and rivers where stations are forced out within reach of the action of the water, and on soft yielding and shifting soil, much difficulty may be met in securing proper station marks without undue expense. Screw piles protected by masonry or riprap, etc., are among the expedients resorted to when reference cannot be had to near, permanent objects or to reference marks set for the purpose. A stake driven down in soft, wet soil; a hole made with a bar and filled with quicklime in impervious soil, or with charcoal; mounds; references to trees; etc.; are among the marks often used for the less important stations.

A topographic sketch of the station and its surroundings should be given; on which are shown the features likely to aid in identification, and especially those objects which can be used for reference points. This should be accompanied by the distances to these points, taken with steel tape if near enough, or by including them in a sweep of angles which includes one or more distant objects and a magnetic bearing. If to these are added the kind of a signal; with the heights above the station mark of the points most convenient for sighting in measuring vertical angles, as tripod head, top of pole, etc. ; the name of the land owner or person who has been requested to look after the station, or of those who would know most of its position ; the name of the nearest railroad station and the best method of approach; the description will be reasonably complete.

The various tertiary points sighted upon should be described, to aid the topographer in identifying stations with ease and certainty, and to aid in securing the stations for use in future topographic and hydrographic work.
A station should be named from the popular name of the hill or locality, or from some well known peculiarity of the ground; or from the owner of the land; or in such a way as to best call attention to the special locality. Numbers are sometimes used in the computations and records, as being more concise.

CHAPTER III.

INSTRUMENTS AND OBSERVING

24. DEVELOPMENT OF ANGLE INSTRUMENTS. When Snellius of Holland introduced the principle of triangulation in 1615, angles were measured with quadrants, rectangles or semi-circles graduated on their peripheries, and having alidades with sights attached. Defects in graduation were early detected, and efforts made to remedy them by using large radii; 6 to 7 feet was the smallest radius for a sector while 180 feet were not uncommon with the Arabian astronomers.

A means of measuring parts of a division was devised by Nunéz in 1542; the present form of the vernier was first used by Vernierus in 1631; the entire circle was first used by Roemer in 1672; and the first micrometer and cross-hairs in the telescope were used by Picard, although construct- by Auzout in 1666.

The great advance in instrument construction dates from 1783 when the survey to connect the observatories of Paris and Greenwich was begun. The French brought out the repeating circle constructed upon the principle pointed out by Tobias Mayer in 1752. Fig. 17 shows the general construction. The horizontal circle just above the leveling screws is an auxiliary not essential in the measurement of angles. The long vertical axis is forked at the upper end to carry the short horizontal axis which supports the repeating circle on one side and a counter weight on the other. The circle and weight are connected by an axis ⊥ to the circle and to the horizontal axis, and it is rigidly attached to the latter.

By rotation around the horizontal axis the circle can be set at any inclination from horizontal to vertical; this in connection with the vertical axis will allow of bringing the circle into any plane. The circle carries two telescopes, one above, the other below, both eccentric, each capable of rotation about the axis, with independent clamps and tangent screws; the lines of collimation are // to the circle and the position of the upper telescope can be read by means of verniers.

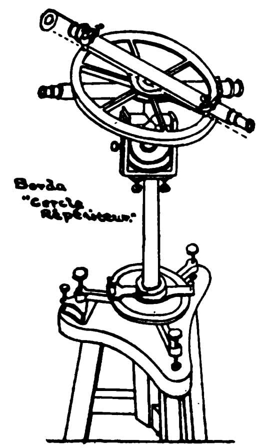


Fig. 17.

To measure an angle the following steps are necessary; bring the plane of the circle into the plane of the objects; clamp the upper telescope at zero: rotate the circle until the upper telescope bisects the right object and clamp the circle (the old French circles were graduated counter clockwise); bring the lower telescepe to the left object and clamp; unclamp circle and rotate until lower telescope bisects right object and clamp; loosen upper telescope, and bring onto left object. The reading will now be twice the angle, for in rotating the circle so that the lower telescope changes from the left to the right object the zero rotates through the same angle to the right of the right object, and the upper telescope must be brought over once the angle to reach the right object and once again from the right to the left, giving a reading of twice the angle. The above steps are continued until a sufficient number of repetitions have been taken when the last reading (increased by the proper number of 360°'s) is divided by the number of repetitions for the value of the angle.

In measuring vertical angles a level on the side of the lower telescope comes up in position, not shown in Fig. 17, to serve for the reference horizon when the circle is vertical.

At the same time the English brought forward the celebrated Ramsden theodolite, partially described in § 2, which in its essential principles is the same as the modern theodolite and does not need separate description

The different parts of an instrument will be taken up in detail, beginning with the telescope.

25. NORMAL VISION. The eye is an optical instrument, consisting essentially of a series of transparent refracting media bounded by curved surfaces, forming a lens, and a delicate network of nerve fibers, spreading out from the optic nerve, forming the retina. A pencil of light entering the eye is refracted by the lens and brought to a focus upon the retina, and the impression is carried to the brain along the optic nerve.

The normal eye at rest is supposed to be adjusted for parallel rays, the curvature of the lens and its distance from the retina will increase with the nearness of the object up to the limit of distinct vision, which is some 8 to 10 inches; the pupil or aperture for the admission of light is also adjustable. The distance from the center of the lens to the retina is about 0.6 inch.

With this ratio of distances of retina and object from lens (0.6 to 8) the image will be only 0.6/8 = .075 times as large as the projected object.

The angular magnitude for 1μ in the projected object at the distance of 8 inches, where μ = the millionth part of a meter, =0.000,0394 inches

$$= \frac{0.000,0394}{8 \sin 1''} = 1''$$

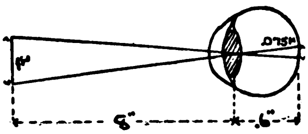


Fig. 18.

The minimum angle between two bright points or lines upon a dark ground, or the reverse, which the eye can distinguish without running them together is found to be about 60″. This would give the distance between the images,

$$= 60 \times .075 = 4.5\mu$$

The surface of the retina is made up of minute papilla or nerve elements called <u>rods</u> and <u>cones</u> from 2μ to 6μ in diameter, with an average of 4.5μ; showing no power to distinguish impressions on parts of a papillus.

A single dark line upon a bright ground can be distinguished, it is said, when the visual angle is only 1/50th as large as the above (image 0.09μ).

According to Pfr. Forster's investigateons as given in Jordan's Handbuch der Vermess., Vol. II, p. 147, the minimum distance b, between a hair and scratch, which can be distinguished in bisecting a division mark upon a bright scale, as with the cross hairs of a micrometer microscope, Fig. 19, is 2.5μ measured upon the retina. With this width of line the probable error of the bisection, with a power of 25, was found to be 0.25μ measured upon the retina.

Fig. 19.

This width referred to the object and unaided vision, would correspond to b = 2.5/.075 = 34″, or a visual angle of 34″; while the probable error of bisection would be one-tenth as great. A power of 34 would thus give a probable error of 0.1μ in bisecting a division.

If b be increased 16 fold, or so as to cover 8 papilla or nerve elements a power of 85 is necessary for a probable error of 0.1μ in bisection; and if widened to cover 15, a power of 150 is necessary.

26. THE ASTRONOMICAL TELESCOPE. This in its simplest form consists of two biconvex lenses fixed in a tube; the eyepiece and the object glass. Its advantages over the unaided eye in accurately sighting an instrument upon a point, are; (a) increased light; (b) magnifying power; and (c) the use of cross hairs.

The following are from Geometric Optics:

A lens is a portion of a refracting medium bounded by two surfaces of revolution having a common axis; this axis is called the <u>axis</u> of the lens. The surfaces of revolution are usually spherical or plane; if they do not intersect, the lens is supposed to be bounded by a cylinder in addition having the same axis. The <u>thickness</u> is the distance between the bounding surfaces measured on the axis. The <u>optical center</u> is a point of the axis, usually within the lens, through which if any ray of light pass, the direction after passing through the lens will be parallel to its direction before, a slight offset taking place for oblique rays on account of the refraction towards the normal on entering the lens.

For spherical surfaces this point is found by drawing any two parallel radii, joining the points where each cuts its own surface, and noting the intersection of this line with the axis. The ratio of the distances of the centers of curvature from the optical center equals the ratio of the radii. When one surface is plane, the optical center is found at the other surface.

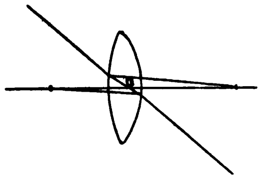


Fig. 20.

The principal focal length of the lens, f, is found from,

$$\frac{1}{f} = (n-1)\left(\frac{1}{r} + \frac{1}{r'}\right) \qquad (7)$$

where r and r′ are the radii, and n the index of refraction.

The fundamental equation connecting conjugate focii is,

$$\frac{1}{f} = \frac{1}{f''} + \frac{1}{f'} \qquad (8)$$

where f'' is the distance of the object and f' that of the image.

27. MAGNIFYING POWER. In Fig. 21, let O be the object glass and O′ the eyepiece. The rays of light from the arrow head, A will be brought to a focus at A′ where the ray through the optical center O meets the focal plane, and those from C at C′, these rays preserving their direction beyond the lens but suffering a slight offset as indicated in §26. Join A′ and C′ with the optical center of the eyepiece. All the rays of light coming from A and C which pass through the telescope will emerge in pencils parallel with, or slightly diverging from, these two directions A′O′, C′O′, if adjusted for distinct vision for a normal eye. Without the telescope, the angular magnitude of the object with the eye at O would be β.

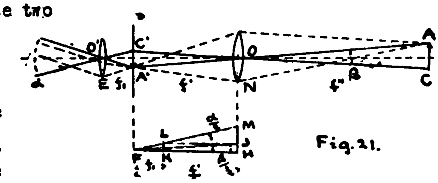


Fig. 21.

With the telescope, the angular magnitude is α. Draw FH = f′, the focal length of the objective; erect the ⊥ HJ = A′C′/2; take FK = f_1, the focal length of the eyepiece; erect the ⊥ KL = HJ; join J and L with F, giving HFJ = β/2, and HFM = α/2.

Extending FL to M to refer both images to the same distance, the apparent magnitudes will be as HM to HJ.

$$\text{But } HM : HJ = FH : FK = f' : f_1, \text{ or}$$

$$G = f'/f_1 \qquad (9)$$

i.e., <u>the magnifying power equals the focal length of the object glass over that of the eyepiece.</u>

$$\text{Also,} \quad HM : HJ :: \tan\alpha/2 : \tan\beta/2.$$

$$\therefore \quad G = \tan\alpha/2 \,/\, \tan\beta/2, = \alpha/\beta \text{ nearly} \qquad (10)$$

i.e., <u>the magnifying power equals the angular magnitude as seen through the telescope over the angular magnitude as seen with the naked eye, nearly.</u>

Since by (8), f′ increases with the nearness of the object, G will be greater for a near than for a distant object; f′ for parallel rays is taken as the standard.

For normal eyes the eye piece would be focussed for a virtual image at the distance of most distinct vision, or about 8 inches; myopic eyes, unless corrected by glasses, would require the eyepiece to be pushed in, and hypermetropic eyes, pulled out, thus changing f_1 and G.

In Fig. 22, it may be noted that the extreme rays from a point A striking the object glass at the distance apart D will intersect at a′ in the focal plane and emerge in parallel lines ($\alpha = 0, \beta = 0$) at the distance apart d′.

From similar triangles, neglecting the thickness of the lenses,

$$D/d' = f'/f_1$$

from (9), $G = D/d'$ (11)

i.e., the magnifying power equals the diameter of the clear aperture of the object glass over that of the emergent cylinder of rays from a point.

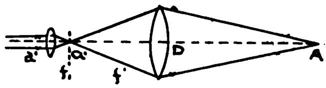


Fig.22

For the magnifying glass, or simple microscope,

Take FH = 8 inches, the distance for normal ~~voice~~ vision, FK = f_1; HJ = AC/2. Then

$$G = 8/f_1 \quad (12)$$

If an objective is added, making a compound microscope, it will magnify the image AC in the ratio f''/f''' (see Fig.21)

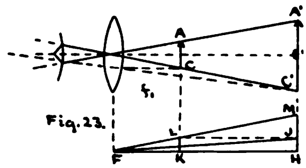


Fig.23.

$$G = \frac{8}{f_1}\,\frac{f''}{f'''} \quad (13)$$

Ex.1. Find the power of a magnifying glass having a focal length of 1″.

28. MEASUREMENT OF MAGNIFYING POWER. (a) Set up the telescope where two prominent well defined objects can be seen symmetrically with reference to the center of the field, on looking through the object end, and focus for parallel rays. Set up a transit back of the telescope, and measure the angle A subtended by the objects as seen through the telescope. Remove the telescope; set the center of the transit in the position occupied by the eye-piece and measure the angle A′ between the same objects as seen directly.

Then by (10)

$$G = \frac{\tan 1/2\,A'}{\tan 1/2\,A} = \frac{A'}{A} \text{ (nearly)} \quad (14)$$

(b) Focus the telescope for parallel rays; point it towards the sun, or a bright sky, and measure the diameter d′ of the emergent cylinder at the eye-piece as thrown upon a paper screen; measure the clear diameter D of the objective by pushing a pencil in from the edge until it will just cast a shadow on the screen, and noting the reduction from the apparent diameter. Square pieces of paper of different sizes, moistened and placed around the circumference, will show the clear diameter more accurately than the pencil point.

By (11) $D/d' = 1$ (approx).

(c). Sight to a speaking rod, a clapboarded house, or other object which will answer for a scale of equal parts. While looking through the telescope at a scale unit with one eye count the number of units which it covers as seen by the other or free eye; this number will be the power G′ for the given distance.

To find G, the power for parallel rays; measure the distance f'' from the center of the objective to the front of the cross-hair diaphragm, when focussed for the above scale reading, and the distance f′ when focussed on a distant object.

$$\therefore \text{ from (9)} \quad G = f'G'/f'' \quad (15)$$

The method (a) is the most accurate, (b) will give fair results except for high powers for which it is difficult to measure d′ with sufficient accuracy; (c) is the most convenient for low powers.

Ex.1. The angle subtended by two objects when seen looking into the object end of the telescope focussed for parallel rays was A = 2′ 11″. The angle subtended, as seen directly was A′ = 1° 18′ 06″. Required G.

By (14) approx. $G = \dfrac{1°\ 18'\ 06''}{2'\ 11''} = \dfrac{4686}{131} = 36$

By (14) $G = \dfrac{\tan 1°\ 18'\ 06''}{\tan 2'\ 11''} = 36$

29. INTENSITY AND BRIGHTNESS. Let D = diameter of the object glass; d = that of the eyepiece; d_1 = that of the pupil of the eye, assumed = 0.2 in. by Chauvenet for astronomical work, and .224 mm. or .09 in. by Jordan for geodetic work, the actual size varying with the individual and with the brightness over a greater range than indicated by the above values, m = percentage of light striking the object glass, from a given point in the optical axis which passes through the lenses, = 85% for the best telescopes, and often falling to 60%.*

With the unaided eye, the cone of rays which can enter it from a given object has a diameter d_1. With the telescope, the diameter of the cone which may be condensed to enter it is D. The quantities of light, for same distance from object, will vary as the squares of these diameters, or allowing for the loss due to absorption and reflection of the lenses, the increased percentage of light due to the use of the telescope,

$$I = m D^2/d_1^2$$

For all this light to enter the eye, $d \leqq d_1$, or, substituting the value of d from (11), $d_1 \leqq D/G$.

If $d_1 < D/G$, as may be the case with telescopes designed for special purposes, the effective diameter of the object glass will be reduced as far as light is concerned to $d_1 G$. This value substituted in the value of I, gives,

$$\left.\begin{aligned} I &= m D^2/d_1^2, \text{ when } d_1 \geqq D/G \\ &= m G^2, \text{ when } d_1 < D/G \end{aligned}\right\} \quad (16)$$

Owing to the magnifying power, this light appears to come from an area G^2 times as large as without the telescope.

∴ The brightness, or light per unit area as compared with the naked eye,

$$\left.\begin{aligned} B = I/G^2 &= m \frac{D^2}{d_1^2 G^2}, \text{ when } d_1 \geqq D/G \\ &= m, \text{ when } d_1 < D/G \end{aligned}\right\} \quad (17)$$

Tabulating (17) d for different values of D and G, we have the following,

Table for Brightness

G	m	d_1 inches	Aperture in Inches, D = 1	1 1/2	2	2 1/2	3	3 1/2	4
10	.85	.09	.85	.85	.85	.85	.85	.85	.85
		.20	.21	.48	.85	.85	.85	.85	.85
20	.85	.09	.26	.59	.85	.85	.85	.85	.85
		.20	.05	.12	.21	.33	.48	.65	.85
30	.85	.09	.12	.26	.47	.73	.85	.85	.85
		.20	.02	.05	.09	.15	.21	.29	.38
40	.85	.09	.07	.15	.26	.41	.58	.30	.85
		.20	.01	.03	.05	.08	.12	.18	.21
60	.85	.09	.03	.07	.12	.18	.26	.36	.47
		.20		.01	.02	.04	.05	.07	.09

A glance at the table will show that with the powers in common use viz: about 20 for a 1-inch aperture, 25 for a 1 1/4, 30 to 40, for a 1 1/2, 60 for a 2-inch, the brightness is from 10% to 25% for Jordan's value of which is full large for sunny weather; while it is only from 2% to 5% for Chauvenet's value, which is none too large for work in thick woods near nightfall, or on dark November days. This serious loss of brightness at times when most needed, due to the failure of the aperture of the telescope to respond, like that of the eye, to variations in illumination, can be met by using an eyepiece of lower power in dull weather.

It should be noted that the ratio of the brightness of the sky and all

*It is stated by Nolan in the Telescope that about 7% is lost by each lens, one-half of this being reflected back from the outer surface and the other half from the inner surface as it passes through.

Experiments at the University give m about 60% for the older telescopes with terrestrial eyepieces.

other objects seen in the telescope remains constant whatever the loss. For this reason the loss is not very noticeable until quite large.

In looking at a fixed star, the more perfect the telescope, the more nearly will the image appear as a bright point, regardless of the power; the brightness will therefore increase directly with the intensity, there being no magnification. The brightness of the field will however reduce as G^2, as the area of the field from which the light comes is reduced in that ratio. This is why fixed stars can be seen in the day-time with telescopes of small apertures and large powers when they are invisible to the naked eye, the darkened field allowing them to show through as at night; also why faint stars can be seen at night which would be invisible with the same telescope and a lower power.

On the other hand, faint nebulae, tails of comets, etc., which have nearly the same degree of brightness as the sky, become invisible under high powers, because although the ratio remains constant, the difference in brightness soon becomes too small to be distinguished by the eye.

30. FIELD OF VIEW. It is customary to limit the focal plane, by a circular diaphram to about 0.5 f_1 on account of the difficulty of securing good images with an eyepiece of larger field.

From Fig. 24, since the image of each object is on the line joining the object with the optical center,

$$\gamma \tan 1^\circ = \frac{0.5\ f_1}{f'} = \frac{0.5}{G} \quad \text{by (9)}$$

But $\tan 1^\circ = 0.017$,

$$\gamma = \frac{30}{G} \text{(approx.)} \qquad (18)$$

e.g., Mag. power	G =	10	20	30	40	60
Field of view,	=	3°	1° 30′	1° 00′	0° 45′	0° 30′

As the field becomes small, the eyepiece is often made movable in order to include a greater range in one direction, either altitude or azimuth, by moving it with a tangent screw, the simultaneous field being as above.

Draw the diagonal lines AC and EF, and join their intersections with the focal plane a and b with the optical center O. All the rays coming through the object glass from any point on aO will pass through the focus a, and all reach the eyepiece, those from C passing just to the limit at A. Similarly for bO.

∴ the angle aOb. or λ = the <u>bright field</u>, or field for total light.

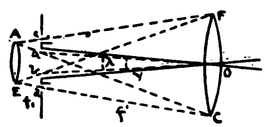


Fig. 24

From this field out the intensity and brightness both diminish, and they would reach zero at cOd were the field not restricted to γ by the diaphragm. Since γ is about equal (not much larger than λ) objects should retain their brightness nearly or quite to the edge of the field.

In order to take in the whole extent of this field the eye must be placed at the point in which the axes of the extreme pencils, diverging from the center of the object glass, meet, the axis of the telescope after emergence. The position of the eye is therefore at the focus of the eyepiece which is conjugate to the center of the object glass. The telescope tube is prolonged to this point and furnished with an eye stop.

31. SPHERICAL AND CHROMATIC ABERRATION. The simple telescope described above would be satisfactory only for very low powers. For with spherical surfaces, the only ones which can be conveniently ground, the rays from near the border of the lens are brought to a focus nearer than those passing through the central portion; the distance along the axis between these foci is called the <u>spherical aberration</u>. It is reduced for a given aperture by increasing the focal length of the lens, as a less portion of the sphere is used. Again, the different colors have different indices of refraction as seen from the spectrum, the violet coming to a focus nearest the lens and the red the farthest; the distance along the axis between these foci is called the <u>chromatic aberration</u>.

To obviate these difficulties, the object glass is usually composed of two simple lenses, see Fig. 25, an outer double convex one of crown glass having a low dispersive or spectrum forming power, and an inner double concave one of flint glass having a high dispersive power but with flatter curvature. The dispersive powers can thus be made equal for any two colors of the spectrum by a proper relation between the focal lengths, rendering the combination nearly achromatic, while the sharper curvature of the convex lens leaves a residual of converging refractive power which can be rendered nearly aplanatic, or free from spherical aberration, by giving proper radii of curvature to the four surfaces. The two adjoining surfaces usually have the same curvature; they are sometimes united by Canada balsam to prevent the loss of light by reflection from the inner surfaces; sometimes cemented around the outside to prevent the entrance of moisture: and sometimes they are held in place by the cell simply.

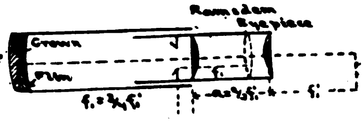


Fig. 25.

The grinding of the lenses and the first polishings are extremely simple. The finishing of a fine object glass requires great skill and patience on the part of the optician, as the effect of every flaw in the glass and defect in the grinding must be counteracted by polishing, here and there each of the four surfaces, with the finger alone or with a little of the finest rouge and water, until after repeated tests the desired degree of perfection is attained.

With two lenses thus adjusted to each other it is evident that their relative positions in the cell cannot be disturbed without injury.

32. EYEPIECES. The correction for the eyepiece is usually made by using two separate lenses of the same kind of glass placed at such a distance apart that the colored rays produced by the first lens shall fall at different angles of incidence upon the second and become recombined. The two lenses may be treated like a single one with the equivalent focal length, as found from Optics.

$$f_1 = \frac{f_1' f_1''}{f_1' + f_1'' - a} \qquad (19)$$

where f_1', f_1'', are the focal lengths of the separate lenses, and a is the distance between them.

The Huygenian, or negative eyepiece, is one of the best when cross-hairs are not required. It consists of two plano convex lenses, Fig. 26, with the plane sides towards the eye, the focal length of the farther or field glass being 3 times that of the nearer or eye-glass. They are placed about half the sum of the focal lengths apart. The field glass receives the converging rays from the object glass before they have reached the focus, and brings them to a focus between the lenses. Cross-hairs are often placed at the focus to define certain portions of the field, as in the sextant telescope, but not for accurate measurements, since the cross-hairs will be distorted, seen through the eyeglass only, while the object will not be seen through the corrected combination.

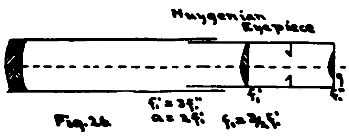


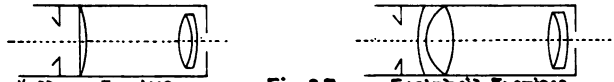
Fig. 26

Airy replaces the plano convex field glass by a concavo-convex, increasing the flatness of the field.

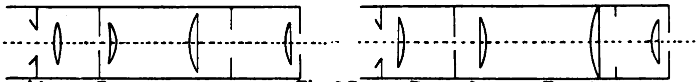
The Ramsden, Fig. 25, is the form most commonly used when accurate measurements with cross hairs or micrometer are required.

It is a positive eyepiece, i.e., it receives the diverging rays from the object glass after they have passed the focus. The two plano-convex lenses have their convex sides turned towards each other; they have the same focal length, and are placed two-thirds the focal length apart, giving by (19) an equivalent focus of 3/4 that of one of the lenses. The Kellner and the Steinheil are modifications of the Ramsden which are coming into favor on account of the greater flatness of the field or freedom from spherical aberration.

In the former, the eyeglass is an acromatic combination and in the latter both are acromatic; see Fig. 27. The former has the larger field. None of these eyepieces invert the image, and as the object glass inverts, the objects all appear inverted.



Kellner Eyepiece. Fig. 27. Steinheil Eyepiece



Airy Eyepiece Fig. 28. Fraunhofer Eyepiece

The terrestial eyepiece consists of four lenses, the object being to invert the image so that objects seen through the telescope appear erect Quite an appreciable loss of light results from the two extra lenses (at least 14% as estimated by Nolan) and a serious shortening of the focal length of the object glass for a given length of telescope which increases the difficulty of securing a flat field. Two combinations are shown. the Airy and the Fraunhofer.

Diagonal eyepieces. For convenience in looking at very high objects, a mirror of polished speculum metal is placed between the two lenses of the eyepiece, at an ∠ of 45°, so that the light emerges ⊥ to the telescope tube. This erects the object (reverses the image) in altitude but not in azimuth. For objects near the zenith, a longer ⊥ tube is desirable, and this is secured by placing the mirror between the central lenses of the terrestial eye-piece; which then inverts the object in altitude and leaves it erect in azimuth.

Instead of the speculum mirror, a glass isoceles right angled triangular prism can be used with less loss of light.

33. CROSS HAIRS. Since with the telescope, the image of any point is at the intersection of the focal plane with a line through the point and optical center of the object glass, this optical center may be taken as a fixed point for all lines of sight. The intersection of a horizontal and vertical hair placed in the focal plane (it should be in the optical axis) will give a second fixed point. The line joining them, called the line of collimation, is taken for the direction of the telescope; its greater precision is due to the magnifying power and increased light of the instrument. In pointing, the eyepiece is first focussed upon the cross hairs and then the object glass upon the object; the focal plane of the object glass is thus brought to coincide with that of the cross hairs, so that the latter will remain fixed upon the object as the eye is moved from side to side behind the eyepiece.

The first is for the eye of the observer, and this focus does not need to be disturbed when once properly made; the second is for the distance to the object, which requires change with each new distance. Spider lines are usually used for cross hairs. Some prefer to have them spun directly by a spider as needed, others to take them from cocoons. They should be opaque, cylindrical, free from dust, and so small as compatible with distinct visibility. Platinum wires are used by some instrument makers as being more opaque and less liable to stretch with age.

. The requisite fineness is obtained by coating with silver, drawing down the wire and afterwards removing the silver by nitric acid.

A glass diaphragm with etched lines is sometimes used in place of cross hairs, with perhaps some advantage as to permanence of position but with the disadvantage of loss of light, and the magnification of all dust on the glass unless thick and the cross hair side inclosed in a sealed box. The reticule of wires consists of one horizontal and one vertical for the ordinary surveying instruments. Sometimes stadia wires are added For geodetic work the vertical wire should be replaced by an X for greater accuracy in bisecting pole signals. For Astronomical work, several horizontal and vertical hairs are used, either equidistant or arranged in groups symmetrically with reference to the center. The linear distance between the wires can be computed from the focal length of the object glass as measured on the outside of the tube to the cross hair diaphragm, and laid off with a micrometer. Or better and more accurately, by using a micrometer microscope as an eye-piece and measuring the distance subtended by the divisions of a rod at a measured distance; from this distance the required distance between wires is readily computed and laid off by the micrometer. Allowance must of course be made for the change in focal length for parallel rays. The angular distance can be deter - mined from astronomical observation, or directly from circle readings .

34. TESTS OF TELESCOPE. To test for spherical aberration, reduce the effective area of the object glass about one-half by a ring of black paper and focus upon a well defined point. Then remove the ring of pa - per and cover the other half of the object glass, the distance the latter must be moved in or out, for distinct vision, which should be small if any, is an index of the spherical aberration.

To test for definition, focus upon small clear print at a distance of 20 to 100 feet, depending upon the magnifying power, and note if the print is as sharp and well defined as when viewed with the naked eye at a distance of 8 to 10 inches. Poor definition may be due to spherical aberration, or to inaccurate curvature, or to variable density or non centering of the lenses.

To test for centering, or for the coincidence of the optical axes of the different lenses, fix a white paper disk about one-eighth inch in di - ameter with sharp outline, in the center of a black surface, and look at it when placed in a good light at a distance of 30 to 40 feet. If the image of the disk, when a little out of focus is surrounded on all sides by a uniform haze, the centering is good.

Astronomical objects are sometimes preferred for testing as follows: the correction for spherical aberration is well made when the image of a star, under atmospheric favorable conditions appears as a small well defined point or round disk. Having this in the best focus, the slightest motion of the object glass out or in should enlarge the image, it remaining cir - cular if the lens is symmetrical throughout, while in the most perfect telescopes the image will enlarge to several concentric rings (circular) of light before disappearing. An imperfect unsymmetrical lens, will give distorted rings, or only a confused mass of irregularly col - ored light. If the glass is not homogeneous, bright stars will show "wings" which it is impossible to remove by perfection of figure or adjustment. The defective portion can be found by covering up different portions of the object glass and testing.

The correction for chromatic aberration is well made, when after focussing on a bright object as the moon or Jupiter, pushing in the eyepiece slowly will give a ring of purple and pulling it out, one of pale green, thus showing that the extreme colors of the spectrum, red and violet have been corrected.

The flatness of the field depends mainly upon the correction for the spherical aberration of the eyepiece . It can be tested by drawing a square some 6 to 8 inches on a side, with heavy black lines upon white paper, and looking at it when flat and at such a distance as to nearly fill the field of view. If the lines appear perfectly straight the field is flat. A telescope may distort the image appreciably without introducing any error in ordinary work, but it is objectionable for stadia work and inadmissable when measurements are to be taken in the field with a micrometer eyepiece.

The object glass should be mounted so that its optical axis coincides with the axis of the telescope tube. The object glass slide should be parallel to this same line, and the vertical plane of collimation should contain it when adjusted perpendicular th the telescope axis.

The rear end of the object glass slide is sometimes supported by an adjustable collar for ease in meeting the above requirements, but with first class workmanship it is usually considered unnecessary, while it adds an element of instability . The accuracy of workmanship can be appreciated by remembering that 10 seconds of arc will subtend only .000049 of an inch for a focal length of 10 inches.

The object glass slide is tested by placing the vertical wire in ad - justment for distant objects, (slide drawn in) and then testing the adjustment for near ones (object glass slide pushed out). This is of more importance for ordinary instruments than for geodetic and astronomical ones where the precaution is taken to not disturb the slide or focus of the object glass between sights which are combined on the supposition of a fixed line of collimation. This is possible for sights over 1 1/2 miles long no matter what the inequality, while it is not for short sights unless they are nearly equal.

The horizontal line of collimation is not restricted as closely as the vertical, so that if it is adjusted parallel to the object glass slide the deviation from the optical axis of the object glass or from the axis of the telescope will have no appreciable effect.

35. LEVEL TUBES. These for accurate work are accurately ground with emery on a revolving arbor which has been turned so as to give the desired curvature. The tube is slowly rotated about its axis so as to distribute the grinding uniformly around the circumference. The surface is then polished, the tube filled and tested on a level tester for uniform curvature by noting if equal angular changes will give a uniform motion of the bubble. For delicate levels, the defects found after this rough grinding must be corrected, requiring repeated trials and much skill and patience.

The upper inner surface, when completed, must be highly polished to render the friction of the bubble as small and uniform as possible.

The tube should be of uniform bore and thickness and of hard glass.

The liquid used for filling is usually alcohol for the more common levels, alcohol with a little ether added for fluidity for more sensitive ones, and sulphuric ether, with possibly a little chloroform for the most sensitive ones.

For delicate levels a chamber is added at one end so that the bubble can always be used at about its normal length for greater convenience and accuracy; a change of length with the temperature changing the zero if the curvature or size at one end differs from that at the other while a short bubble is more sluggish and its position of rest more effected by friction and by local defects of the tube than a long one. The best results will be obtained with the length used by the maker in testing the tube. The tube should not be directly held in rigid metallic supports on account of the danger of distortion from pressure due to changes of temperature. The support should be at two points only and with rings of cork or other yielding material which will give sufficient stability.

A very sensitive level should be inclosed in a glass box or tube so as to form a closed air space, to diminish local distortion from sudden changes of temperature.

The value of a division should be determined for different portions of the the tube to test uniformity, and at different temperatures to determine the temperature coefficient if any.

An appreciable coefficient will usually denote a cramping of the tube by the supports.

36. GRADUATED CIRCLES. The process of graduating a circle is essentially one of copying the divisions of another circle. The circle to be copied is usually some 3 feet or more in diameter, in which the graduated errors have been carefully determined. This is mounted and well centered on a heavy axis firmly supported in the graduating engine. The new circle is placed upon the old, and centered. One method of centering is by allowing the vertical arm of a sensitive level to rest against the inner surface of the hollow axis as both circles rotate. The level is radial and pivoted at the upper end of the vertical arm to the fixed frame above so that any eccentricity as the circle rotates will move the vertical arm radially and thus change the level.

The lines are made by a tool having an automatic cut in a radial direction, the circle being turned division by division as read by a microscope fixed above the large circle or fed automatically by a worm gear acting on the circumference of the circle. In the latter case the gear is adjusted by careful test until equal motions of the worm wheel will rotate the circle through equal angles. This done, the work proceeds automatically with but little hand labor. During this work the temperature must be kept very constant in order to avoid distortion from unequal expansion.

With a ten-inch circle, an error of 0.0001 of an inch in a division or in centering will give an error of 0.0001 ÷ 5 sin 1″ = 4.1 seconds; showing the extreme accuracy necessary in centering and in graduating a circle which is to be read to tenths of seconds.

Five-minute spaces are usually the finest cut upon large circles, and 10, 20 or 30-minute spaces are the smallest upon smaller circles. Intermediate readings are taken with verniers or micrometer microscopes. The vernier is too well known to need a description here.

For an illustrated description of the new dividing engine used by Fauth & Co. of Washington, see Zeit. fur. Inst. 1894, p. 84. See also U.S.C. & G. R. 1879, App. 12.

37. MICROMETER MICROSCOPES. These are usually used in place of verniers when readings finer than about 5″ are required. Cross hairs are attached to a frame which is moved through a box perpendicular to the microscope tube by an accurate micrometer screw working against spiral springs, as shown in Fig. 29.

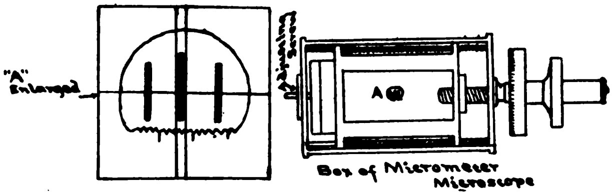


Fig. 29.

If the microscope has a flat field and the screw a uniform pitch, the apparent motion of the cross hairs across the limb, will be proportional to the turns of the screw, giving an accurate means of subdividing the spaces on the limb. A common division of the limb is into 5 minute spaces, the objective being placed at such a distance that 5 turns of the screw will move the wires over one space; each turn will then give a minute, marked by a tooth on the comb in the edge of the field, as shown, while seconds can be read from the head of the screw by dividing it into 60 equal parts.

Two parallel hairs are usually used, placed far enough apart so that when brought over a division a bright line will show on each side between the hair and scratch; the equality in width of these light lines being judged more accurately than the bisection of a scratch by a single hair

To take a reading, the micrometer screw is turned with the increasing numbers on the head, moving the hairs from zero of the comb back to the first division of the limb to the right (apparent left), the number of teeth passed and the reading of the head giving the minutes and seconds from the division to the zero. Usually the motion of the screw is reversed, turning against the graduation on the head, until the hairs bisect the division to the left of the zero. Only the reading on the head is noted and this should differ but slightly from the first if the microscope is adjusted so that 5 complete turns cover an average space.

It is often thought desirable to make the bisection with the positive motion upon the screw, rather than with the return motion from the spring, to avoid the lost motion. The observer however can work more accurately if free to move the hairs either way to perfect a bisection, than if he can only move them in one direction, turning back and moving up a second time if he passes the scratch.

The lost motion will be extremely small if the micrometer is in good condition. A test of the nearness with which a bisection can be duplicated by each method will decide which should be used in a given case. The probable error of a single bisection should be about 0″.2.

38. THE RUN OF THE MICROMETER: The micrometer is adjusted, as stated in §37, so that the nominal number of turns, usually 5, will move the hairs over a 5-minute space. This can only be approximately realized owing to the imperfections of the micrometer and graduated circle, the inaccuracies of bisection and reading, and the disturbance due to changes in temperature.

The correction for run is made in several different ways by different observers, while many equally good observers regard it as a refinement which it is a waste of time to attempt to make.

The method given by R.D.Cutts, Asst. U.S.C.& G. Survey, in App.9 Report for 1882, appears to be one of the most reasonable. A mean of the first and second readings is taken which averages the errors of bisection and graduation for the two scratches. The differences between the means of the first readings and those of the second for each reading taken in observing angles at the station are entered in a column and added and the mean taken for the average run of the micrometer. The error in pitch of the screw, due to the lack of adjustment, is distributed proportionally to the length.

Let a be the first reading, b, the second reading; r, the average run of the micrometer, positive when the first readings average greater than the second.

$$\text{Correction to a} = \frac{-r}{300''} a$$

$$\text{Correction to b} = \frac{r}{300''} (300'' - b)$$

$$\text{The mean , m} = \frac{a+b}{2}$$

$$\text{Correction to m} = \frac{r}{300''} (300 - (a+b))\frac{1}{2}$$

$$\text{Correction to m} = \frac{r}{2} - m\frac{r}{300} \qquad (20)$$

This correction has the same sign as r($= \Sigma[a - b] \div n$) for m < 2' 30", and the opposite sign for m > 2' 30".

In the record book, the mean of the first micrometer readings is taken, also that of the second, for each reading of the circle, the difference is put in the r column and the mean in the m column; after the average r has been found, the correction for each m is taken from the Table II (computed from (20)) and applied to m with its proper sign, giving the corrected readings. For an example, see The Form of Record Book §48. See also the Run of the Micrometer by George Davidson, in U.S.C.& G.S.Report for 1884, App.8.

39. ERRORS OF GRADUATED CIRCLES These may be due to an eccentricity of the upper motion or inner axis with reference to the center of the graduation, or they may be due to errors in the division lines themselves The error due to the plane of the circle not being horizontal when the axis of the upper motion is vertical as indicated by the levels remaining in the center during rotation, is so small in an instrument in which the limb will remain flush with the vernier, or the micrometer microscopes in focus during rotation, that it can be neglected.

The error due to eccentricity is of more importance with instruments for ordinary surveying work than with those for geodetic or astronomical work, for with the latter all the microscopes or verniers are used in making a reading, and it can be readily shown that the mean of any number of equidistant verniers is free from eccentricity.

Let G be the center of the graduated circle, G' ,the center of the axis for the upper motion; EE' the line joining the centers; z' the angle AGE, made up of the index reading z and the micrometer readings A,B,C; and e (π-measure) the eccentricity GG'.

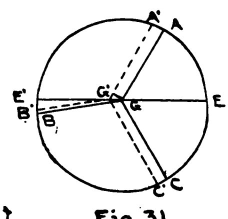


Fig. 31.

For 2 micrometers 180° apart.

From the 1st. $z' = z + A - e \sin z'$

" " 2nd. $z' = z + B - e \sin(180 + z')$

$= z + B + e \sin z'$

Mean value, $z' = z + 1/2(A+B)$, which is free from eccentricity.

For 3 micrometers 120° apart

From the 1st., $z' = z + A - e \sin z$

2nd., $z' = z + B - e \sin(120 + z')$

3rd., $z' = z + B - e \sin(240 + z')$

By 8], $\sin(120 + z') + \sin(240 + z') = 2\sin(180 + z')\cos 60°$

$$= -2\sin z' \times 1/2$$

$$= -\sin z'$$

∴ Mean value, $z' = z + 1/3(A + B + C)$, which is free from eccentricity.

Similarly it can be shown that the mean of any number of equidistant micrometers will be free from eccentricity.

Some instrument makers put in radial abutting capstan head screws between the circle and hollow axis which supports the upper motion so that the eccentricity can be adjusted out before the plate is screwed fast to the flange of the axis.

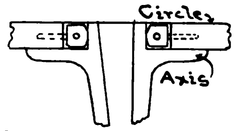


Fig.32.

The graduation errors proper are divided into accidental and periodic. The former follow the law of errors of observation given in Least Squares, hence their effect is diminished as the square root of the number of lines used.
The latter occur at regular intervals according to some law, and may therefore be expressed as functions of the reading itself. The sum of all the corrections for periodic error, including those for eccentricity, must have the general form

$$\psi(z) = u' \sin(z + U') + u'' \sin(2z + U'') + u''' \sin(3z + U''') + \text{etc.} \quad (21)$$

where $\psi(z)$ denotes the correction to the angle z and u', U', u'', U'', etc., are constants. The shorter the period of any error, the higher is the multiple of z in the term representing it.

Chauvenet, Astronomy, Vol. II, p.52, shows what terms are eliminated by taking the mean of a number of equidistant microscopes and how to determine the constants for a given circle by taking equidistant readings around the circumference. R.S.Woodward, Report, Chief of Engrs. U.S.A, 1879, Part III, App. M.M., p.1974, takes up the terms not eliminated by means of a number of equidistant microscopes and finds their effects upon a measured angle. He shows that if the distance between verniers be divided by the number of repetitions of the angle, and the circle be moved forward by this quotient each time so that the initial readings be evenly distributed over the space between two microscopes, nearly all the terms will be eliminated from the mean. Also that the remaining terms tend to add up to zero or eliminate as the number of observations increases so that the effect may be neglected with a large number of observations.

In applying the formulas to some of the Lake Surveys insts., Pri.Tri. U.S.L. Survey, 1882, he finds periodic errors ranging from 1″.7 to 2″.

In Saegmuller's Price List for 1901, p.7, are given the comparisons of 0° 10′ spaces

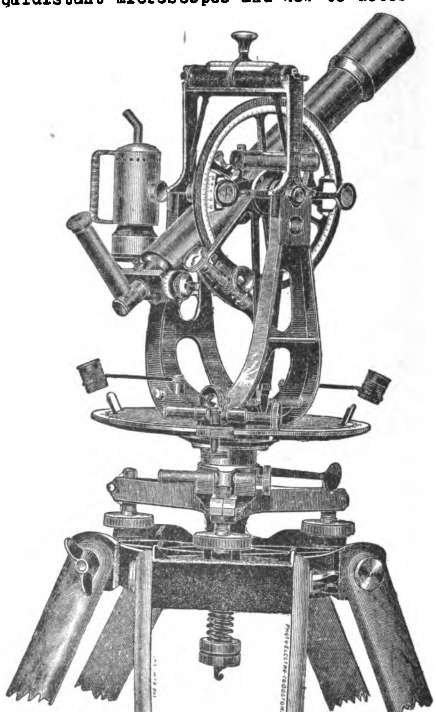


Fig.33. Alt.-Azimuth.
As made by Buff & Berger.

10° apart around the circle for 6 ,8-inch circles made for the Geol. Survey. The greatest discrepancy is 1".58. He claims that using his engine automatically the errors will be from 2" to 3", while if corrected settings are made for the main divisions no line will be out more than 1".

40 REPEATING AND DIRECTION INSTRUMENTS. The component parts have been quite fully described in the preceeding paragraphs, and the French repeating circle in §24. An 8-inch repeating instrument (having a micrometer eyepiece) reading to 10" by verniers, as made by C.L.Berger, is shown in Fig.33

The circle is from 8 to 12 inches for primary Δ'n and from 5 to 9 inches for secondary and tertiary. The power of the telescope varies from about 60 to 20 with a diameter of object glass from about 2 1/4 to 1 1/4 inches. Two verniers or microscopes are common and the upper and lower motions are the same as with the ordinary transit.

To repeat an angle, the upper motion is set at the desired initial reading and the telescope pointed on the left hand object by the lower motion; it is then pointed on the right-hand object by the upper motion, back to the left-hand by the lower and to the right-hand by the upper, etc., until the desired number of repetitions has been reached.

A U.S.C.Survey direction instrument is shown in Fig. 34. . The only essential difference between this and the repeating instrument is in the removal of the tangent screw for the lower motion which prevents the use of the ordinary method of repeating angles; the object being to add to the stability of the circle.

Sometimes the lower motion is wholly removed so that the circle can only be rotated by motion below the leveling screws, but this arrangement is less convenient. Rather larger circles are used than for repeating instruments for the same class of work, 15 to 18-inch circles being common, with about 8 inches as a minimum. Micrometer microscopes are used in place of verniers, 3 for the larger and 2 for the smaller circles.

The telescope can be made to transit, as shown in Fig.33 in which case a vertical circle is added large enough to measure vertical angles. Many observers, however, prefer short standards for greater stability which requires that the telescope be taken out of the Y's for reversal, and often that vertical angles be measured with another instrument.

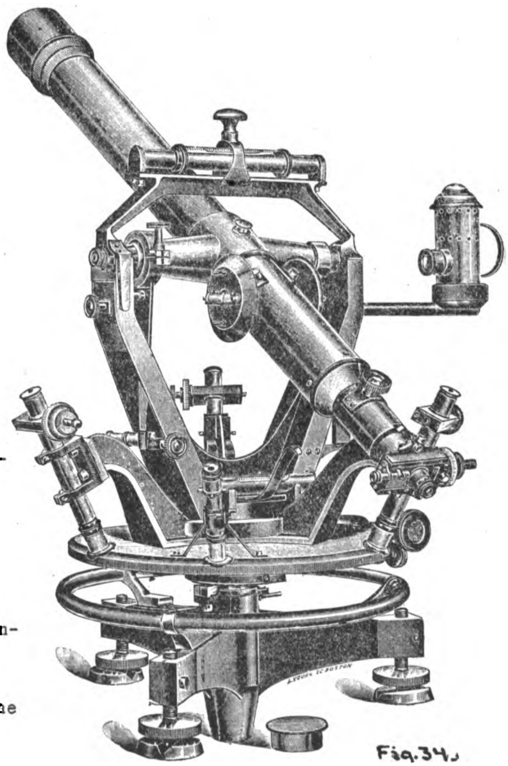


Fig.34.

12-inch Coast Survey Theodolite.

41. ADJUSTMENTS. Plate levels perpendicular to the vertical axis. These are adjusted as usual.

Line of collimation perpendicular to telescope axis when focussed for parallel rays. Sight to a well defined distant point and clamp the horizontal motions. Reverse the telescope by carefully lifting it from the Y's and changing the ends of the axis. Adjust until the point is covered by the cross hair, in both positions of the telescope.

Horizontality of telescope axis. This can be adjusted by means of the striding level more accurately than by the method used for smaller instruments.

Index error of vertical circle. Take a reading with telescope direct and another with telescope reversed upon a well defined point with bubble of reference level in the center, or the readings corrected for the out of level. Half the sum of the readings will give the true vertical angle, and half the difference the index error.

Accuracy of adjustment is of less importance than with the smaller instruments used in ordinary surveying, because the observations are arranged to eliminate errors of adjustment. Thus if the line of collimation is not ⊥ to the axis, it will describe a cone as the telescope rotates; so that in plunging up or down through a distant signal the line will not follow the vertical through the signal but will cut the plane through the vertical perpendicular to the great circle through the points in an hyperbola having its vertex at the height of the instrument and its axis horizontal.

The horizontal angle measured is then from a point at a distance x, see Fig. 35, to the left of the section. Upon reversal the measurement will be taken from a point x' to the right. But if the collimation error has remained constant and the axis is horizontal, x' will equal x and the error of collimation will be eliminated by taking the mean.

Fig. 35.

If the telescope axis is not horizontal when the plate levels are in the center, the line through the distant signal will not be vertical but inclined, referring the horizontal angle to a point at a distance x to the left, as in Fig. 35. Upon reversal, the plate levels remaining in the center, the error will be the same but in the opposite direction. The mean will eliminate the error as before.

42. DETERMINATION OF INSTRUMENTAL CONSTANTS. Value of 1^d of level. Set up the instrument on a firm support where it will be protected from sudden changes of temperature, and place the level on the telescope with the two tubes parallel. If the tube is chambered, take a bubble of about normal length. Move it by means of the vertical tangent screw from one end of the tube to the other back and forth, setting at regular intervals in seconds and reading both ends of the bubble.

If the circle cannot be read closely enough rod readings at a distance of 103.1 feet will give 2" per .001 foot on the rod.

Value of 1^d of micrometer eyepiece. If the screw is horizontal (which can be tested by noting if motion of the screw changes the altitude of the horizontal hair) put the micrometer at a given reading and sight to a well defined point by the upper motion and read the circle; turn the micrometer, say 5 turns, and bring the hairs upon the same point by the upper motion, then read the circle; continue the process until the desired accuracy has been secured.

The difference in the circle readings divided by the number of turns will give the value of one turn for the different parts of the screw.

If the screw is vertical, the same method may be employed with the vertical circle if it is suitable.

A more accurate method involving more labor is by means of following a circumpolar star near upper culmination for the horizontal screw or near elongation for the vertical screw with the circle clamped, depending upon the observed time intervals for the angles as described in Chauvenet's or Doolittle's Astronomy in connection with the zenith telescope.

Wire intervals. These may be determined by the methods given for 1^d of the micrometer.

The circle can be investigated by the methods referred to §39, while the methods for the telescope have already been given.

43. THE METHOD OF DIRECTION OBSERVATIONS IN HORIZONTAL ANGLES. This is the most common method in this country with a direction instrument. A reference line is taken, which may be the signal most easily seen under varying atmospheric conditions, or a mark set for the purpose at a sufficient distance to avoid changing focus (not less than 1 1/2 miles).

The signals are sighted in order around the horizon in the direction of the graduation, beginning with the reference line, and the micrometers read for each; the telescope is then reversed, not changing the ends of the axis in the y's if it has to be taken out for reversal, and the signals are sighted in the reverse order around the horizon, ending with the mark. This forms a set, and as many are taken as required.

The first signal each time should be approached with the telescope from the same direction as for the others in the half set so that the tendency of the circle to be dragged around by the friction of the upper motion will be taken up before the first reading. Before each set the circle is shifted so that the readings for each single object are uniformly divided over the whole circle. In order to eliminate periodic error, as pointed out in §39, the circle should be shifted each time approximately 360° ÷ mn, where n is the number of sets, and m the number of equidistant microscopes. If the instrument is in good adjustment, it will not be necessary to reverse the telescope in the middle of each set provided that the observations are equally divided between the two positions.

Sometimes the sweep of the horizon includes the reference line at the end of the first half of the series and at the beginning of the second, especially if many stations are included in the series. This serves to detect instability of the circle.

If the instrument has no lower motion it is inconvenient to shift the circle after each set. The Coast Survey practice in such cases is to choose either 5 or 7 positions, equidistant 360° ÷ 5 or 360° ÷ 7, and take an equal number of sets in each position; such that the total shall give the required accuracy.

In setting upon the reference line, the zero of the micrometer should be advanced 1/n of the smallest division of the limb each time, in order to distribute the micrometer readings uniformly over the space. This will give a uniform division of the readings upon each of the other objects sighted, so that the average of the micrometer readings upon each object will be nearly the same, and the correction for error of runs for each angle will disappear.

The objections to this method of observing angles are thus stated in the N.Y.S.Sur.Report for 1887 by Mr. Wilson. "An objection to the method of directions is that it is very difficult, practically impossible indeed, to secure full sets upon ordinary points where the highest degree of precision is desirable and where broken sets are decidedly objectionable. In addition to this drawback to the method, another and very serious one arises from the length of time consumed in taking readings and bisections to several distant primary stations.

When the theodolite is supported upon a high tower, as is frequently the case, the entire instrument is continually twisting in azimuth as the tower is subjected to the heat of the sun's rays. It is therefore of great importance that the intervals between sights should be as short as possible and that the two series in each set should be taken in about the same space of time. Frequently however, one-half of a set may be taken in five minutes, while the other may require ten or fifteen". The broken sets are afterwards filled up by new sets including the missing stations and the reference line.

44. THE METHOD OF SIMPLE ANGLE MEASUREMENT. In this the number of points in each series is reduced to the smallest possible number, or two. The angle between each signal and the reference line, or the angles between adjacent signals, can be measured independently. Or, the measurements

can be so arranged that between n_1 stations $n_1(n_1-1) \div 2$ angles will be measured; starting with the first station as a reference line and swinging to the right to each of the others will give $n_1 - 1$ angles, Fig. 36, then from the second to each of the others to the right (not including the first) $n_1 - 2$ angles; then from the third; etc.; to the $n_1 - 1$ from which only one angle is measured.

The sum of the series = first term plus last term, multiplied by one-half the number of terms, = $[(n_1 - 1) + 1](n_1 - 1) \div 2 = n_1(n_1 - 1) \div 2$, as stated above. This gives the same number of pointings, $(n_1 - 1)$, upon each signal. Each angle is repeated the same number of times, and this number is taken large enough to give the required accuracy

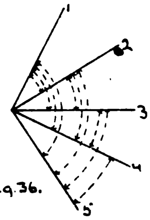


Fig. 36.

To eliminate periodic error, the initial reading for each repetition of an angle is increased by $360° \div mn$, as in §43, m being the number of microscopes and n the number of repetitions of the angle. To reduce the effect of accidental circle errors, Schreiber, Zeit. fur Vermess. p. p. 209 - 240, 1878, divides the distance between initial readings for the different repetitions of an angle $(360 \div mn)$ by the number of angles, $n_1 - 1$, to be measured from the first reference station, and increases the initial reading for each new angle by this amount, starting from zero.

The initial readings for the angles measured from the other stations as initial lines, are taken from the first set, using one each time which has not already been used with either of the lines forming the angle. An example of the settings at a station where 6 signals are sighted may be seen in N.Y. S. Report, 1887, p. 145.

This method requires the same number of pointings and readings as the preceding for two stations, 4/3 as many for 3 stations, 6/4 as many for 4 stations, etc., provided the visibility of the signals will allow of always taking full sets by the first method. For long lines, as in primary triangulation, these ratios will be less owing to imperfect sets by the first method, while if the delays in waiting for signals to show in order to complete sets are taken into account, the advantages will often be with this method.

Another advantage of this method is that angles can be measured whenever two signals are visible, provided atmospheric conditions are favorable, allowing more time to be utilized while in the field, and each signal to be sighted when under the most favorable conditions as to illumination and steadiness.

45. THE METHOD OF REPETITIONS. The impression is quite general that this method will not give as good results as those with a direction instrument described above, but the method has been a favorite one with many most excellent observers, and the results obtained have fully justified their preference. When the upper motion is always rotated in the same direction, errors due to twist of observing stand, drag of circle by friction of upper motion, travel of clamps, etc., are not eliminated by reversing the telescope, and the resulting angles will usually be too small, although sometimes too large. This is obviated by taking one-half the repetitions upon the angle, and the other half upon its explement, always swinging from left to right with the upper motion. Errors which tend to make the angle too small will thus also tend to make the explement too small, or the angle derived from it too large.

On the N.Y.S. Survey the practice was to take three repetitions of the same angle with telescope direct, reading the circle at beginning and end; then three repetitions of the explement with telescope reversed, still swinging the upper motion with the graduation, which is equivalent to "unwinding the circle, i.e., the third repetition will bring the reading back nearly to the initial one. The explement thus only enters in the direction of the swing for the upper motion, and not in the figures recorded. They took 6 sets of 6 repetitions each for an angle, and the results with only an 8-inch circle were as satisfactory on primary work as with a direction instrument.

The angle from a reference line around to each signal can be measured; the required angles then resulting as sums or differences of the measured ones without the labor of station adjustment;or the angles may be measured as shown in Fig.36. The initial readings for the different sets of an angle should differ by 360°/mn as usual,while if the angles are measured as in Fig.36 the additional precaution can be taken of having no two readings alike upon the same signal.

46. CONDITIONS FAVORABLE FOR OBSERVING. To support the instrument tripod,or stand, three solid posts are set in the ground vertically some two feet with tops level, one for each tripod leg,and well tied together and braced by nailing on boards. The dirt is then tamped around the posts and the center often filled with stone. When an elevated observing stand is used,see §20 the tripod or inner tower supports the instrument directly without the tripod,and the outer tower the observer.

In all cases the height of the instrument should be such that the observer can look through the telescope when standing erect comfortably

Some observers use a more or less portable observatory for the protection of the instrument from sun and air currents while observing,but the more common practice is to use a tent for primary and secondary work,and an umbrella or other simple shelter for tertiary. The tents used on the N.Y.S. Survey were octagonal for ground stations and square for elevated observing stands,both 8 feet in diameter,with walls 6 feet high,and made of 8 oz.duck. They are supported by 8 poles,one in the center of each side for the square tent. The wall is in one piece,supported at the top by small pockets which slip over the tops of the poles,with a flap one foot wide at the bottom to tack to the floor to shut out the wind and dust,and a triangular shaped door large enough to admit instrument boxes as well as the observers. The top is in one piece ,held up in the center a foot above the eaves by a rope attached to a small thimble sewed on the outside,with flaps about a foot wide at the eaves which are strapped to the walls. Guy ropes extend from near the tops of the poles to pegs if on the ground,or to the railings or other parts of the observing stand if elevated. Floor space is better economized by placing the tent eccentric over the station on account of storing instrument boxes,etc. Care should be taken not to obstruct lines of sight by tent poles.

The walls can be lowered a foot for observing,or a window, one foot wide can be cut around the tent at the height of the eye or telescope and covered by a flap on each side when not in use.

Tower sheets of 8 oz. duck are sometimes used on two sides of an elevated observing tower to protect the inner or instrument stand from the wind to prevent vibration,or from the sun to prevent station twist,the exposed stand having a tendency to rotate in azimuth with the sun during a bright sunny day and to return at night.

The best time for observing is on a day when the sky is overcast;next to this is a calm,pleasant,late afternoon; evenings from about an hour after sunset until about midnight are also favorable.

The hours for observing upon the U.S. C. & G.Survey are in the summer season,from sunrise until 8 a.m. and from 4 p.m. until sundown. Vertical angles are measured from 12 m. to 1 p.m. and in the afternoon until within an hour of sundown.

Lines of sight passing close to the surface are most disturbed by heat wave and other atmospheric disturbances,producing the appearance in the telescope often described as "boiling". Lines over furnaces and cities are objectionable,while those over bodies of water are not usually so clear as those over land; high lines are least affected by atmospheric disturbances.

The readings for an angle should be distributed over different days or divided between forenoon and afternoon,to equalize the effects of lateral refraction,side illumination of signals,etc. No readings should be taken under any improper conditions of the atmosphere,as shown chiefly by the appearance of the signals. The instrument should be handled with a light touch and with a certain degree of rapidity, yet in completing a pointing it should be done carefully and deliberately, without worry or bias as to the result, watching the signal long enough to be certain that it is really in the line of collimation and not temporarily there due to parallax or a sudden change of refraction either lateral or vertical.

47. ACCURACY OF RESULTS. The limiting error adopted by the U.S.C. & G. Survey in closing triangles, is 3 seconds for primary triangles, 6 for secondary, and 12 for tertiary. The average errors in closing are of course very much less.

For secondary work, the range of values for an angle is given by Gen. Cutts, a Coast Survey authority, at from 5 to 6 seconds, and the probable error as found by comparing the separate values with the mean, not over 0.8 second. These values are given to aid the observer in judging of the accuracy of his results while still in the field.

On the N.Y.S. Survey the observing party took the precaution to adjust the observations at a station while still in the field, in order that extra sets could be taken, or defective ones repeated, in case some of the directions did not show sufficient accuracy. The limit for the mean square error of a direction was placed at 0".5 for primary work, and 1".0 for secondary and tertiary.

48 FORMS FOR RECORD.

Form of Record for Repeating Instruments.

Station , Date , Observer , Recorder , Inst.

Station	Time	Tel.	Rep.	A	B	C	Mean	Angle	Mean of D and R	Remarks
Mt Equinox	7h 05m		0	30° 04' 13"	04' 35"	04' 52"	04' 23".3	70° 26' 18".2		
Greylock		D	3	241 23 10	23 35	23 12	23 19.0		70° 26' 16".7	
								17.0		
		R	3	20 04 20	04 24	04 40	04 28			

Form of Record for Direction Instruments.

Station , Date , Observer , Recdr. , Inst. Position IV,

Series and Number	Object Observed	Time	Tel.	Mic.	Circle ° '	Circle "	Circle "	Dif. r	Mean	Cor. for Run	Cor'd. Mean	Remarks
IV	Azimuth	-		A	271 31	49.0	50.2					
21	Mark	8h 43m	D	B		55.0	55.0					
	2 st.			C		60.2	62.0					
						54.7	55.7	-1.0	55.2			
22	Manomet	47	D	A	35 26	48.4	49.2					
	Hill			B		50.5	52.8					
	m.r.; m.r.;			C		56.0	58.0					
	st					51.6	53.0	-1.4	52.3			

49. PHASE. In bisecting a bright, reflecting cylindrical signal, seen against a dark ground in sunlight, the apparent center will usually be on one side of the true one, owing to phase.

Let r = radius of the cylinder; α = the angle between sun and signal (measured at the observing station at the time of the observations); D = the distance to the station, β = the correction to the angle in seconds.

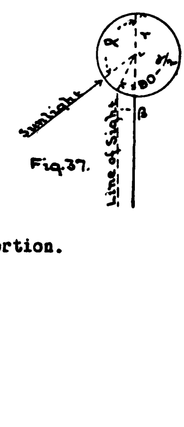


Fig. 37.

(a) Pointing made upon the bright reflecting line.

$$\sin\beta = r\sin(90-1/2\alpha) \div D \text{ , or}$$

$$\beta = \frac{r\cos 1/2\alpha}{D\sin 1''} \qquad (22)$$

β being so small that $\sin\beta = \beta\sin 1''$

(b) Pointing made by bisecting the illuminated portion.

Bisect the angle FCG, or 2S, subtending the illuminated portion as seen by the observer at C, by CE for the line of sight. Then FCO = S + β; OCG = S − β.

$$\sin(S+\beta) = (S+\beta) \times \sin 1'' = r \div D.$$

$$\sin(S-\beta) = (S-\beta)\sin 1'' = r\sin(\alpha - 90)/D = r\cos\alpha / D.$$

Subtracting,

$$2\beta \sin 1'' = (1+\cos\alpha) r/D = 2\cos^2 1/2\alpha\, r/D, \text{ by } 11)$$

$$\beta = \frac{r\cos^2 1/2\alpha}{D\sin 1''} \qquad (23)$$

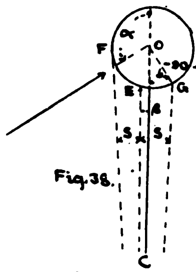


Fig. 38.

50. ECCENTRICITY. The signals during the measurement of angles should be carefully watched, and if at any time found out of center the amount and direction with reference to one of the sides should be measured and the date noted. By plotting this data to a large scale and laying off the lines to the other stations with a protractor, any ⊥ can be scaled with sufficient accuracy.

If e = ⊥ distance from the signal to the line joining the stations,

$$\text{Correction for eccentricity} = \frac{e}{D\sin 1''} \qquad (24)$$

which will apply to each line whether the eccentricity be that of signal or instrument. A sufficiently accurate value of D can be found by solving the triangles with the approximate angles. When the instrument is set up at an appreciable distance from the station point the following formula is often used:

Let C be the station; E the instrument; AEB the measured angle; ACB the required one. Measure also CEB, and CE = a; and find D and D′ by an approximate solution of the triangles.

ACB = AEB + EBC − EAC

$$\text{But } EBC = \frac{a\sin CEB}{D'\sin 1''};\quad EAC = \frac{a\sin CEA}{D\sin 1''}$$

EBC and EAC being so small that for their sines we can use the angles in seconds into sin 1″.

Substituting,

$$ACB = AEB + \frac{a\sin CEB}{D'.\sin 1''} - \frac{a\sin CEA}{D\sin 1''} \qquad (25)$$

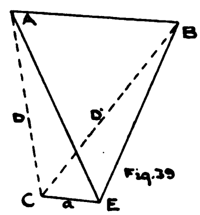


Fig. 39

BASE LINES.

51. BASE LINE SITES. Primary bases are from 3 to 11 miles long, and are placed from 200 to 600 miles apart; secondary from 2 to 3 miles, and from 50 to 150 miles apart; tertiary from 1/2 to 1 1/2 miles, and from 25 to 40 miles apart.

They should be so arranged that the sides of all important triangles can be checked from a second base. If the country is very flat, the base can be placed anywhere to fit the main triangulation, but if rough it may have to first be selected and the triangulation fitted to it.

The scheme for connection must be worked up for each particular case. The small length of base in comparison with the distances computed from it, has led in the attempt to measure accurately, to forms of primary base apparatus which require a line to be graded longitudinally to slopes of not more than 5° or 6° for a width of 10 or 12 feet . greater elevations being overcome by vertical offsets.

52. EARLY FORMS OF BASE APPARATUS. Wooden rods were at first mainly used. A set consisted of 3 or 4 rods, which were placed end to end beginning at the end of the base, the rear one was then moved forward and placed in contact with the front one, etc. Abandoned at Hounslow Heath, Eng. Ord. Sur., on account of changes of length due to moisture, and glass rods substituted.

Borda Apparatus. Fig. 40. 4 base bars; 2 toises (= 3.898^m) long each of 2 flat strips, upper of copper, lower of platinum, fastened together at rear end; difference in expansion measured at front end by graduated scale on copper and vernier on platinum at B, from which temperature or change in length inferred. "Contact" by slide C, read by microscope D. Sheltered by board cover above the bar.

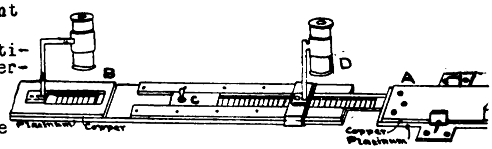


Fig. 40.

Struve Apparatus. Iron rod wrapped in cloth and raw cotton. Mercurial thermometer near each end with bulb let into body of bar. Contacts by contact lever of Fig. 41, a spring yielding as the contact end is pushed back by the next bar until the arm reads zero on the scale.

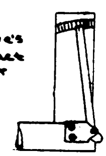


Fig. 41.

Offsets to the ground made with a transit at right angles and 25 feet distant; the position being held over night by a slide and cube on the top of an iron pin driven 2 feet into the ground.

Bessel Apparatus. Fig. 42. Components iron and zinc forming a metallic thermometer like Borda's. Expansion, and contact by slim glass wedge between the knife edges at A and B, the wedge ordinates increasing by 0.01 Paris line, = .0089 inch.

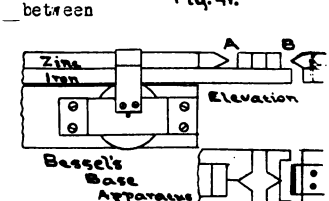


Fig. 42.

Colby Apparatus. The components brass and iron are used to compensate for temperature, and not to measure expansion as with the Borda and Bessel. The bars are placed side by side and fastened at the center as shown. The microscopic dots, a, a′ on the

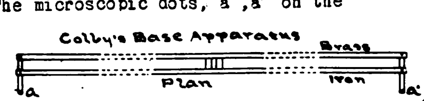
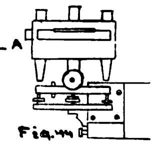


Fig. 43.

compensating levers remain fixed for equal changes of temperature in the two rods. These dots are on the side of the case so that the microscopes of Fig. 44 can be placed over them, one over its dot directly, the other over the dot of the other bar by pushing the bar back for "contact". The axis in Fig. 44 serves as a telescope tube for transfers to the ground, its verticality being indicated by the attached level. The telescope shown at A serves to align the microscope case. The upper plate connecting the microscopes is brass, the lower iron, compensating the distance between the dots, a, a'. The bar is 10 feet long and the microscopes 6 inches apart.



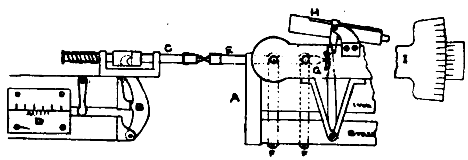
Ex. 1 Find the units of the Borda scale, Fig. 40, such that an increase of one in differential expansion shall indicate an expansion of 1^{μ} per meter for the measuring component. Length for dif. expansion assumed= 3 .8.^{m}

Ex. 2. Find the error in the computed length of the Bessel (2 toise) base apparatus due to a difference of 1° in the temperature of the two components.

Ex. 3. Find the length of the compensating levers of Fig. 43, for a distance of 3 inches between the two components.

53. BACHE-WURDEMAN APPARATUS. (See C.S.R, 1873, App. 12) Length 6^{m}

As seen in the Fig. the two component bars are rigidly attached at the rear end to the block A, and supported by rollers; while the front ends are connected by a compensating lever B. The contact rod C projects through the end of the case, while the Borda scale D can be read through a window in the side.



The contact rod E at the rear end is held in position by the || levers F, F, pivotted at the bottom of the brass.

Its inner end knife edge rests against the cylindrical surface G. By bringing the base bar back through the case with a tangent screw, the contact rod resting against the rear bar, G, is forced forward, bringing the bubble of the contact level H to the center for contact. When in this position the axis of the cylinder G is the axis of the level sector I, so that inclining the bar for slopes does not disturb the contact distances or level so long as the level sector tube remains horizontal.

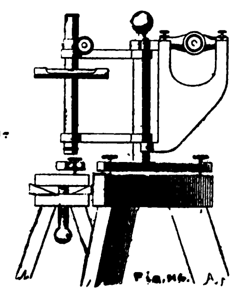
The cross sections of the Borda components are so arranged that, while the two have equal absorbing surfaces, their masses are inversely as their specific heats, allowance being made for their different conducting powers.

Both surfaces are varnished to give equal absorbing power, and the whole is protected by a double spar shaped tin case painted white to prevent rapid changes of temperature.

The heads of the supporting metallic tripods are adjustable vertically, laterally, and longitudinally, the motions for the rear one being controlled by rods running to the contact man at the rear of the bar. Each tripod leg is adjustable by rack and pinion and by foot screw.

The end of a bar is transferred by a transit at right angles

54. PORRO APPARATUS. In this a return is made to the method of measurement with chain and pins, the base bar taking the place of the chain, and 4 microscopes with very firm supports, that of the pins. As originally designed the rod was made of fir, varnished and encased in a copper tube; but as soon modified, the fir was replaced by 2 metals, forming a Borda thermometer.



The microscope, Fig. 46, has 2 objectives, one for plumbing over a point on the ground, and the other for sighting at the bar, a cap with a central opening shutting off the light which does not pass through both when looking at the bar.

The telescope of the rear stand is used for alignment by sighting along the line at an offset target and then aligning the front stand, a scale taking the place of the front telescope axis.

55. U.S.L.S. REPSOLD APPARATUS. See Pri. Tri. U.S.L. Survey, p. 138. This is of the Porro type. The components, steel and zinc are placed side by side in a 4-inch iron tube; they are fastened at the center and are free to expand each way upon rollers; their ends are cut away to the neutral axes and graduated platinum plates attached. In measuring the micrometer microscope is set upon the zero of the steel bar for contact and a reading taken upon the nearest division of the zinc for temperature.

The tube stands are placed at the ends of the bar or tube, so that the front for the first position becomes without disturbance the rear for the second position, etc. The tube is lengthened by a bracket at each end, the rear one resting on a knob in the center of the tube stand head, the front one carrying 2 rollers, one V-shaped, which rest on tracks on the tube stand head.

The microscope stand is placed opposite the tube stand, a long bracket supporting the microscope over the end of the bar.

The bar is aligned by a telescope on the tube and its inclination measured with a level sector.

To set a microscope over the starting point, the tube stand head is removed and a telescope tube placed over the rock crystal knob marking the point, the end fitting accurately. The tube is made vertical by an attached level tube and the microscope set on the zero of a horizontal scale at the top; a direct and reverse reading eliminating any index error of the scale. The tube is then removed and the end of the base bar brought under the microscope.

56. IBANEZ APPARATUS. (Engrg. News, March 1884, p. 133). This is an outgrowth of experience in Europe with the complicated forms due to the use of the Borda thermometer for temperature or compensation.

The bar is a 4^m 110^{mm}, iron ⊥-bar without case or cover except a large observing tent. Marks are engraved on small platinum disks at points 0.5 apart; while 4 mercurial thermometers with bulbs encased in iron filings are attached.

Underground monuments are set in advance dividing the base line into day's works, and no transfers to the ground are allowed at other points. Dependence is placed upon rapid continuous work (160^m per hour, Aarberger Base) between these points, and the use of a shelter tent for freedom from errors due to instability and to temperature changes

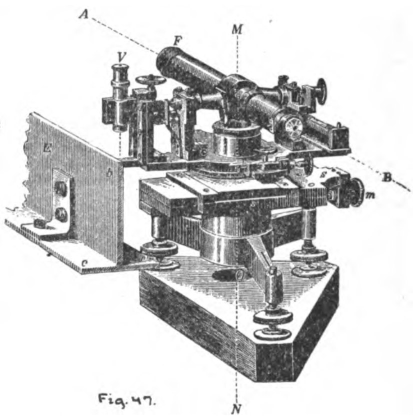


Fig. 47.

In starting, the telescope F is replaced by one having its axis near the object end so that it can be made vertical and set over the monument at N; F is returned and sighted to a target on the line at A; the next microscope stand is set up 4^m ahead and a target at M, taking the place of the telescope axis, is brought into line by sighting through F; its aligning telescope is replaced and sighted to the target ahead; the bar E is then brought under the microscopes V, the dot b at the rear end being accurately bisected, while the front microscope is moved longitudinally on the slide S to bisect the dot at the front end; the 3rd. stand is set up like the 2nd. and the bar moved forward. When a monument is reached a stand is set over it as in starting, the bar put in position and a 1/2^m scale used to measure the distance from the microscope to a 1/2^m dot on the bar.

57. U-S.C.S.SECONDARY APPARATUS. (Rep.1880,App.17) The construction is clearly shown in Fig.48 from Saegmuller's Catalogue. The measuring rod is steel 4^m or 5^m long. The outside tubes are zinc,one fastened to steel at the rear with its Borda scale at the front,the other at the front with scale at the rear. Each scale is read by a magnifying glass at the top of the case. B is the tangent screw working against the springs C at the front end for contacts with the slide D. The mercurial thermometer E is attached to the case and its bulb is not in contact with the bar. The case is a pine joist about 8″ × 8″. The tripods are mainly of wood;the cross bars can be clamped to the standards at any height.

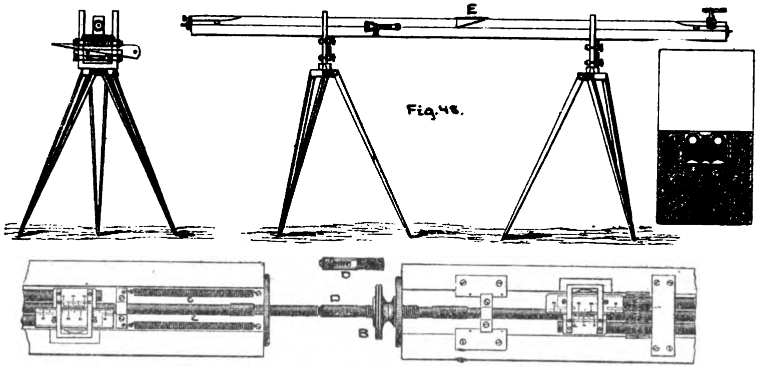


Fig.48.

With the College bars the Borda readings have been abandoned as unsatisfactory;the case has been covered with hair felt and canvas;and the thermometer has been replaced by two near the quarter points with their bulbs in close contact with the steel bar and surrounded by iron filings.

58.U.S.C.S. GRIDIRON COMPENSATING APPARATUS.(Rep.1882,App.7) The expansion of the steel is balanced by that of the zinc for equal temperature changes for the two components. The details of the secondary apparatus, § 57, are elaborated for the case, contacts, and tripods

Ex.1. Find the lengths of the components of Fig.49 for a base bar 5^m long.

Ex.2. Sketch the construction and find the lengths for a brass and steel combination 6^m long.

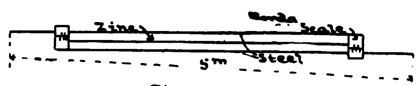


Fig 49

59. U.S.C.S. DUPLEX APPARATUS. (Rep.1897,App.11). As seen in Fig.50, there are 2 separate bars with contact slides,a steel tube and a brass one. They are placed 1 1/8″ apart in a brass tube,which can be rotated 180° about its axis in an outside usupporting tube. In use,double con-

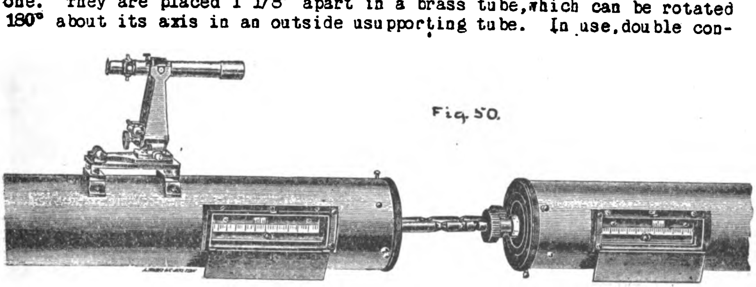


Fig. 50.

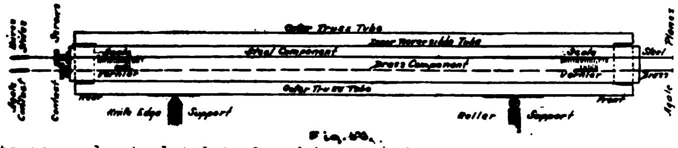


Fig. 50.

tacts are made, steel to steel and brass to brass, the accumulated differential expansion showing itself by the movement of one rod upon the other as noted by reading the vernier and scale at each end at the beginning and end of the measurement of the section.

About 2 reversals, or rotations of the tube, are required per day, arranged symmetrically as to rising and falling temperature and so as to have the same number of bars placed in each position.

The outer tube is covered with felt and canvas, and the bars are used under a portable tent drawn by a team as the work proceeds. A speed of 40 5-meter bars an hour is claimed to be easily maintained.

30. STANDARDS OF LENGTH. All measurements of the Coast Survey have been referred to one of the 12 original iron meter bars standardized in 1799 by the French Committee in terms of the toise which had served as a standard unit in measuring the meridional arcs of France and Peru. In Nov. 1899 the Government received 3 platinum iridium bars of the Prototype meter standardized by the International Bureau at Paris, and from early in 1900 these have referred the Coast Survey standard to the International.

The length of the iron bar is now taken

$$= 1^m + 0.2^\mu \pm 0.6^\mu$$

as the result of recent comparisons, instead of

$$= 1^m - 0.4^\mu \text{ ,as given in 1799.}$$

In App. 6 of the Report for 1893 it is stated that no legal standard of weight or length was adopted by Congress until July 1866 (a Troughton 82 inch scale had been used by the Treasury Dept. as a standard in collecting duties, etc.) when the metric system was legalized and the weights and measures in common use were defined in terms of the metric units, giving,

$$1 \text{ yard} = \frac{3600}{3937} \text{ meters}; \quad 1 \text{ pound} = \frac{1}{2.2046} \text{ kg.} \qquad (28)$$

As a result the Survey now uses $1^m = 3.2808\ 1/3$ feet, instead of $1^m = 3.280869$ ft. as formerly.

Standards are divided into <u>line measures</u> and <u>end measures</u>; with the former the length is between the end surfaces, with the latter, between lines or points near the ends.

31. COMPARATORS. In comparing two end measures they are placed between parallel planes or spherical surfaces, first one and then the other, and the change in position of one or both planes measured for the difference in length of the two bars. With the old Saxon pyrometer of the C.S., one plane A was fixed on the top of the masonry pier, while the other, B, was supported from a casting at the top of the second pier, being pushed towards the first by a spring and held back by a delicate chain C wound around the vertical cylinder D actuated by a weaker spring.

In placing the bar between A and B, the first spring insures contact and the second tension in the chain, giving a fixed position of the cylinder for a fixed length; this is noted by reading through the telescope E a division of the scale F reflected from a mirror on the cylinder. When the other bar is inserted, the cylinder has a different position, and another division is read.

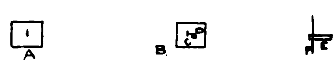


Fig. 51

The contact level comparator is more convenient, especially for field.

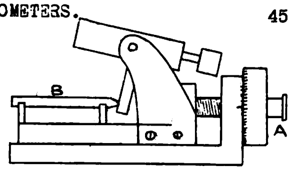


Fig. 52

comparisons. A contact level is used at each end to make certain that the bar touches without undue pressure. The micrometer screw A is turned forcing the small rod B against the arm of the level until the bubble reaches the center.

The College field comparator has the base bar contact slide in place of the contact level.

To find the length of a 6m bar, 6 1m bars are each compared with the standard; they are then placed end to end compared with the 6m bar.

In comparing line measures, micrometer microscopes are mounted on piers, or on a rigid frame if changes in distance are frequently required, and the difference in length obtained in terms of the screw.

For commensurate units the aliquoit parts are marked off on the longer bars and comparisons made with the shorter one, the results being added as with end measures.

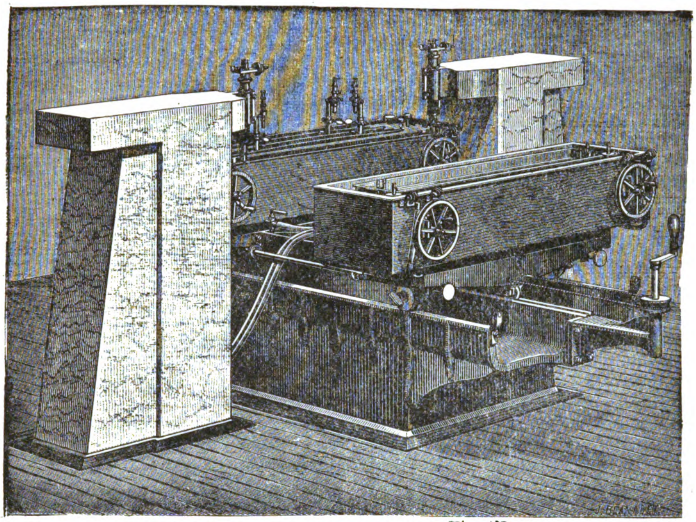


Fig. 53.

Fig. 53. shows a 1m comparator used by the International Bureau. The 2 tanks are for determining coefficients of expansion, one bar being heated by circulating warm water through the pipes, while the other remains at a constant temperature. The microscopes shown are for reading the thermometers near the bars.

The micrometer microscopes of the College line-measure comparator can be placed at any distance apart from 4″ to 4m.

62. MERCURIAL THERMOMETERS. Thermometers are divided into standard and auxiliary; the scales of the former include both the boiling and the freezing point of water which allows of their being studied and standardized each one independently; the scales of the latter do not contain both of these fixed points and they can only be standardized by comparison with some other thermometer.

With glass, as with tempered steel, zinc and its alloys, and some other substances, the volume changes lag behind the temperature changes, giving rise to <u>residual expansion</u>. This is especially apparent in the variations of the zero point, the volume of the bulb at the temperature of melting ice depending for some time upon the previous temperature of the thermometer. The depression of the zero, due to the slowness of the bulb in contracting is produced much more rapidly for a given change of temperature than the elevation due to slowness in expanding; the rapidity of both movements increases as the temperature is raised. Special high melting point glasses (the <u>verre dur</u> of the French, and the <u>Jena</u> of the Germans) are made which have much less residual expansion than the crystal glass commonly used.

When a thermometer of verre dur glass is heated from ordinary temperature to 100°, the stable condition is reached in a few minutes; when cooled more than one-half of the residual expansion remains after 24 hours, and the stable condition is only reached after several weeks. With crystal, the stable condition at 100° is reached in about an hour, while months are required after cooling.

For the accurate determination of temperature, read the thermometer, then plunge into melting ice and read; the difference will give the temperature above 0° referred to the fundamental interval 0° to 100°, the 100° point having been found by referring to 0° in the same way. Small bulbs are often blown in the tubes of standard thermometers to allow of the 0° and 100° points without too long a tube.

The scales of the best thermometers are scales of equal parts etched on the stems.

The tube is <u>calibrated</u> by breaking off columns of mercury of different lengths and noting the length in scale divisions as they are moved from end to end of the tube (a small bulb at the top is necessary for this work).

The 100° point is computed from the observed temperature in steam under a given barometric pressure, and the 0° point by melting ice immediately after. This gives the fundamental interval which is to be divided into 100 equal parts for the Cent. scale. The calibration corrections refer these equal volume parts to the scale divisions, so that the scale divisions can be expressed in degrees. A perfect tube and scale within the errors of observation is thus secured and residual expansion can be eliminated in use. These corrected temperatures (including a correction for pressure on the bulb) are called mercurial thermometer temperatures, and they are usually accepted as standard, assuming the expansion of mercury in glass to be proportional to the temperature.

The International Bureau has adopted the hydrogen scale as standard, and by comparing the mercurial thermometer readings with the corresponding pressures of a constant volume of hydrogen, by Mariottes law, they have derived correction tables for different kinds of glass. The Coast Survey has also adopted the hydrogen scale.

The corrections for verre dur glass are as follows:

t	Cor.	t	Cor.	t	Cor.	t	Cor.	t	Cor.	t	Cor.
-25°	+0.233	0°	0.000	+25°	-0.095	+45°	-0.106	+65°	-0.082	+85°	-0.038
20	.172	+ 5	-0.028	30	.102	50	.103	70	.072	90	.026
15	.119	10	.052	35	.106	55	.097	75	.062	95	.013
10	.073	15	.070	40	.107	60	.090	80	.050	100	.000
5	.034	20	.085								

See Thermométrie de Précision, by Guillaume, Paris, 1889.

63 LENGTH OF APPARATUS. From what has been given in §§60-62 the method of finding the length of a base bar is evident. All the comparisons except field comparisons are made in a room so protected that the daily range of temperature is small; thermometers are placed in contact with the bars and a few readings at a time are taken quickly before the heat of the body causes a local disturbance of the temperature of the bars, the latter being protected by a case or cover. With bars of the same material the actual temperature need not be known very closely, but the exact difference is essential.

Since the probable error in bisecting a line with a micrometer microscope under favorable conditions is given in §25 as 0.25μ upon the retina; $\frac{0.25}{.075\times 25} = 0.13\mu$ upon the scale, and 0°.010 changes the length of a steel bar 0.12μ per meter, attention should be given to securing good temperature conditions, and to avoid the accumulation of constant errors. This will require changing the order of the readings, the positions of the bars etc.,for the different sets.

The determination of the coefficient of expansion requires great care on account of the difficulty of getting all parts of the warm bar at the same temperature and keeping it constant long enough to read the thermometers and micrometer microscopes. The bar is usually immersed in water or glycerine,while its companion is surrounded by melting ice. In Fig.53 the water is heated by a gas jet at a distance and circulated through the pipes shown; circulation in the tank is secured by turning the wheels shown at the ends. Readings are taken through the water.

The comparison of incommensurate units,e.g.,the foot and meter,requires great care and labor.

Comparison of line measures with end measures.

Field comparisons are very desirable ,in order to detect any change in length due to disturbance in transportation,and also to find the actual length of the bar as compared with the computed ,when exposed to sun , wind,and rapid changes of temperature.

Ex.1 To find the length of secondary bar No.1,the following comparisons with standard No.2 and data,are given (C.S.R., 1868).

Length of standard bar No. 2 at 32° F.	5.99998233
One division of the scale of pyrometer	0.00000174
Coef. of expansion for F. scale	0.00000641
Thermometer attached to standard ,too high	-0°.7
" " " rod	0.0

Standard Thermo.	No. 2 Div.	Rod Thermo.	No.1 Div.
77°.3	21	76°.0	- 10
78.0	15	76.4	+ 41
78.5	18	77.0	+ 55
77.93	18	76.47	28.67
-0.70		77.23	18.00
77.23		+.76	10.67

Computation.

0.76 × 0.00000641 × 6 = + 0.00002923

10.67 × 0.00000174 = + 0.00001857

At 77°.23 no. 1 longer than standard 0.00004780

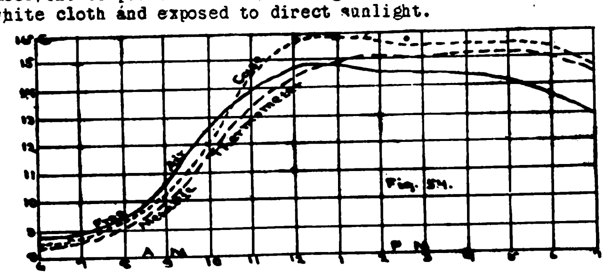
At 77 .23 standard No.2 6.00172188

At 77 .23 rod no.1 6.00176968

At 75 " " 1 6.00168391

Field comparisons are very desirable ,in order to detect any change in length due to disturbance in transportation,and also to find the actual length of the bar as compared with the computed,when exposed to sun, wind and rapid changes of temperature.

64. DEFECTS AND DIFFICULTIES. It is very difficult to find the temperature of a bar and its consequent length under field conditions. The Colby apparatus (§52) after being used in England was taken to India and a large number of bases measured,but the compensation could not be relied upon and mercurial thermometers were substituted. The Bessel apparatus (§52) gave as the mean of 2 day's observations at the Göttingen base in Aug. 1880,the temperatures shown in Fig.54. The case was wood ,covered with white cloth and exposed to direct sunlight.



The Bache-Wurdeman 15-ft. bar No. 1 made for the U.S.L. Survey gave a length at 10 P.M. Aug. 5, 1873, 0.00470 shorter than at 9 A.M., as stated p. 86 of the Report, it having been exposed to direct sunlight during the day. This would correspond to a difference of 1.3 F. between the 2 components. In 1875, at the Buffalo base, its mean length for the 11 days of comparison was 0.00230 greater at 1 p.m. than at 8 a.m., the comparisons being made in a tent. The standard was kept in melting ice.. One kilometer of the U.S.C.S. Holton base was measured with a bar in melting ice; but base lines having an uncertainty of less than one-millionth of the length are often measured with both the Borda and mercurial thermometer apparatus, and with both micrometer microscope and contact slide contacts.

All defining lines and surfaces should be in the neutral axis of the bar to prevent changes in length due to changes in stress of outer fiber by slight changes in the points of support.

Ex. 1. With the Bessel apparatus, §52, find the difference in temperature between the two components required to introduce an error of one millionth the length of the iron bar in its computed length.

Ex. 2. Find the error in the observed temperature of the iron bar for the same error in length.

Ex. 3. With the Colby apparatus, §52, find the effect upon the distance between the end dots due to a difference of 1° between the two components.

Ex. 4. Compare zinc-steel and brass-steel Borda thermometers with mercurial in the effect upon the length of the measuring bar of a difference of 1° between the two components, or the component and the mercurial thermometer.

65. FIELD WORK. The surface having been properly prepared, monuments set and signals erected, points should be fixed in line from 1/4 to 1/2 mile apart, so that the bars can be accurately aligned during measurement. Errors of alignment are cumulative.

A preliminary measurement is usually made with steel tape or wire; and on the U.S.C. S. a stub and tack is left every 30 bars to serve as a check in counting the bars. When a wire is used, a length of say 30 bars is measured off, the bars removed, the wire suspended over the line under a given tension and points plumbed up, the wire notched and the temperature noted. The wire is then moved forward, placed under the same tension, the rear notch brought over the front point, and a new point marked under the front notch; etc.

The method of final measurement varies with the form of apparatus. A large force is required. From 1/4 to 1 mile is measured per day. Many bases are divided into segments and twice measured in order to find the probable error of measurement.

The bars are seldom leveled, it being less work to correct for inclination as given by the level sector. Observing tents are often used.

The C.S. secondary bars require the following outfit: 2 transits, 1 for alignment (unless the alignment telescope is attached to the bar as in Fig. 48), one for transfer for end of bar to ground; level instrument for adjustment of level sector; steel tape; ax; stakes and tacks. Also 7 men; one for contacts; 1 for alignment; 1 for notes, reading inclinations, etc.; 2 to move and adjust bars ; 2 to set up trestles; 1 to bring up instruments, drive stakes, etc.

The left hand page of the record book is ruled in columns giving in order; time; number of bar, counting from the beginning; name or number of the bar; inclination; temperature; Borda thermometer, rear and front. The name of the base, date, and the name of the person in charge should be given at the top of the page. The temperature should be taken every 10th. bar or oftener if changing rapidly, and the time noted; also at the beginning and end of each day's work, and at any time when there is any delay.

66., TAPE MEASUREMENTS. Some years ago M. Jaderin introduced a method of measurement with tapes for which he claimed an accuracy of 1/1000000, even when the work was done in sun and wind. He used 2 tapes, one steel the other brass, each 25ᵐ long, the ends resting upon portable tripods serving as pins to mark the tape lengths, while under a fixed tension applied by a spring balance. The differential expansion of the steel and brass is relied upon for the temperature correction of the steel.

In using a long tape, 300′ to 500′, slim stakes are driven 50 ft. apart with their front faces in line, and marking posts at the ends. The posts are cut at such a height that a straight grade between can be marked on the faces of the stakes. Hooks or wire nails in the stakes support the tape on grade; while straining posts each 2 feet from the marking posts allow of a fixed tension by spring balance or bent lever and weight without disturbing the marking posts. It is usually better to read between fixed marks on the posts than to attempt to lay off an exact tape length. The tape length is sensitive to tension and to temperature, requiring a constant pull for each length and work at night or on a cloudy day when 3 or 4 thermometers distributed at stakes along the tape lengths will give a close approximation to temperature. Two to four contacts with thermometer readings for each position of the tape, letteng off the tension after each will increase the accuracy with but little increase of labor.

The length of the tape when suspended, can best be found by measuring a line which has been measured with a base bar.

The coefficient of expansion can be found by placing firm monuments a tape length apart and noting the reading on the tape at different temperatures.

At the Mass. Inst. of Tech. a thermophone, on the principle of a Wheatstone bridge, is used for temperature. The tape is paralleled by a German silver and by a copper wire as shown. The current from the battery C divides at A, part passing through the tape and wire to B and a part through the fine wire AEB. If the resistance from F to B differs from that from E to B current will flow through the copper wire operating the circuit breaker D. The arm at E is moved over the dial to equalize resistances. As the temperature increases the tape resistance increases (the German silver resistance being only slightly affected) requiring a new position for E. The dial is graduated under favorable conditions, and the temperature of the tape can then be read under field conditions.

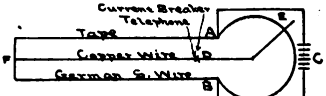


Fig. 55.

Extensive experiments were made in connection with the measurement of the Holton base (§64) and it was found that the inaccuracy of a tape base line could be reduced to less than 1/1,000,000, but at about the same cost as with bars.

In triangulation for bridge spans, or for other work where a fair degree of accuracy is required, a 100 ft. tape between tacks in hubs high enough to allow of swinging freely under a constant spring balance tension will give good results. Hubs 50 or 75 feet apart would give greater accuracy but with more labor.

67. CORRECTION FORMULAS. The length of a base is made up of the following terms: (a) the normal length of a bar into the number of times each has been applied: (b) the amount which the last bar overran or fell short of the end of the base: (c) the amount by which the true length of each bar, corrected for its <u>mean temperature</u> during measurement, differs from the normal length, into the number of bars: (d) the sum of the corrections due to contacts (in those forms only in which the distance between consecutive positions is not the exact length of the bar): (e) the sum of the corrections for inclination, both vertically and horizontally.

<u>Temperature</u>. The coefficient of expansion is constant between the limits usually used, 32° to 100°F, so that the correction can be applied to the mean temperature. With zinc, however, a term must be added involving the square of the temperature.

<u>Inclination</u>. Let a = the difference in height of the two ends of the bar or tape: b = the inclined length: b′ = the reduced length: x = the correction.

If the inclination angle i, is given

$$b' = b\cos i\ ;\quad x = b - b' = b(1-\cos i),\ \text{or}$$

$$x = 2b\sin^2 i/2 \qquad (27)$$

(27) is best used by forming a table for each minute within the limits the inclination, using the normal length of the bar.

If the average length differs sensibly from the nominal, the total correction for the base can be changed in the ratio, actual mean length to nominal length.

For tape work, where a is given by level,

$$b'^2 = (b'+x)^2 - a^2 = b'^2 + 2b'x + x^2 - a^2,\ \text{or}$$

$$x = a^2/(2b'+x) = a^2/2b \quad \text{(nearly)} \qquad (28)$$

68. REDUCTION TO SEA LEVEL: Base lines are usually reduced to sea level so that all the computed triangle sides will be arcs of the spheroid whose surface is that of the sea produced under the land.

Let B' = the reduced horizontal length of the base at an average height h: B = the sea level length: y = the correction; R_z = radius of curvature of the plane section through the base (see Table V).

Then since arcs are to each other as their radii,

$$\frac{B}{B'} = \frac{R_z}{R_z+h}\ ;\ B = \frac{B'R_z}{R_z+h}\quad ;\quad y = B' - B = B' - \frac{B'R_z}{R_z+h},\ \text{or}$$

$$y = \frac{B'h}{R_z+h} \qquad (28)$$

Unless h is large, or extreme accuracy is desired the h of the denominator may be omitted.

69. ACCURACY OF RESULTS. This can be inferred: (1) from remeasurements in segments; (2) by dividing into segments and connecting the different ones by triangulation: (3) by computing the errors from all known sources and adding. (3) in connection with (1) is the most satisfactory.

The principal sources of error for the C.S. secondary bars are: (a) in the length of the bar as found by the office comparisons. (b) in the temperature as inferred from the thermometers. (c) Instability of tripods. (d) Backward pressure of contact spring. (e) Inclination horizontally and vertically. (f) Contacts, and transfers to the ground.

In the most accurate work, the probable error from all sources is about 1/1,000,000 of the length. It diminishes slightly with the length. The same expenditure in short bases placed near together will usually give triangle sides more accurately than long ones far apart.

CHAPTER V.

TRIGONOMETRIC AND PRECISE LEVELING.

TRIGONOMETRIC LEVELING.

70. OBSERVATIONS. The zenith distances, or vertical angles, are usually measured when the station is occupied for horizontal angles. This may be done with a vertical circle, or differences may be obtained with a micrometer eyepiece.

The height above the station mark of the telescope and of each point sighted at should be measured. A line of spirit levels is usually run from tide water, or from a station of known height above tide, to one of the stations.

Refraction is least and nearest constant during the middle of the day, and greatest and most variable at night and morning. The best time for observing is thus usually from 9 a.m. to 3 p.m., and the worst at sunrise and sunset. Simultaneous observations at the two stations will give the best results. If not simultaneous they should be distributed over several days to get an average value for refraction. Instrumental errors are small in comparison with those from refraction, so that with good instruments it is not necessary to eliminate errors of graduation by shifting the position of the circle.

The following form of record is taken from the U.S.C.& G.Survey.

Station , Date , Observer , Recdr. , Inst.

Object observed	Time	D/R	Level E	Level O	Circle °	Circle '	a	b	Mean "	Level Cor. "	Corrected Reading	Zenith Distance	Remarks
Spear Helio 6.15 ft above Bolt	6h 35m p.m.	D	17	9	163	20	34.6	34.0					
							48.4	48.8					
							41.5	41.5	83.2	+5.4	163° 21' 28.7		
		R	17	5.5	342	40	33.8	34.0				90° 20' 0.0	
							61.0	59.9					
							47.4	47.0	94.4	-5.7	342 41 28.7		
		D	14	11.5	163	20	39.2	38.7				3.2	
							55.0	54.3					
							47.1	46.5	93.6	+1.6	163 21 36.2		
Buffalo Trip. Hd. 26.1 ft.													

Telescope 7.10 ft. above Bolt. 1d of Level = 1".35. 1d of Microm. = 2".

71. DIFFERENCES OF HEIGHT FROM OBSERVED ZENITH DISTANCES. Let δ, δ', be the measured zenith distances, corrected for difference of height above station mark of telescope and object sighted at; h, h'., the heights of the stations above mean tide; k., the horizontal distance in meters at sea level, R_z, its radius of curvature, and C, its central angle: $= k/R_z$: m_1 m_2, coefficients of refraction: mC, angle of refraction. Assuming the angle between the tangent T_1 A and the chord AB, Fig. 56, proportional to the distance is equivalent to assuming the line of sight an arc of a circle, though the actual curvature is irregular.

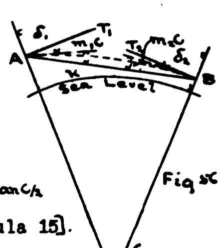


Fig 56.

(a). Non simultaneous observations: By Formula 28],

$$h_2 - h_1 = (h_2 + h_1 + 2R_z) \frac{\tan[(A-B)/2]}{\tan[(A+B)/2]}$$

But, $A = 180° - \delta_1 - m_1 C$; $B = 180° - \delta_2 - m_2 C$,

giving $(A-B)/2 = (\delta_2 - \delta_1)/2 + C(m_2 - m_1)/2$

From the $\triangle$ ABC, $(A+B)/2 = 90° - C/2$

Substituting,

$$h_2 - h_1 = \tan[(\delta_2 - \delta_1)/2] + C(m_2 - m_1)/2\ (h_2 + h_1 + 2R_z)\tan C/2$$

C/2 being small, $\tan C/2 = k/2R_z + k^3/24 R_z^3$, by Formula 15].

$C(m_2 - m_1)/2$ being very small, Formula 5] gives,

$$\tan((\delta_2 - \delta_1)/2 + C(m_2 - m_1)/2) = \tan(\delta_2 - \delta_1)/2 + k(m_2 - m_1)/2 R_z.$$

Substituting,

$$h_2 - h_1 = ((k \tan(\delta_2 - \delta_1)/2 + (m_2 - m_1)k^2/2R_z)(1 + (h_2 + h_1)/2R_z + k^2/12R_z^2) \quad (59)$$

(b). Simultaneous observations; $m_1 = m_2$, giving,

$$h_2 - h_1 = k \tan((\delta_2 - \delta_1)/2)(1 + (h_2 + h_1)/2R_z + k^2/12R_z^2) \quad (60)$$

which is the formula used on the U.S.C.& G.S.

(c). Zenith distance at one station only.

$A = 180° - \delta_1 = m_1 C$, as before. $B = \delta_1 + m_1 C - C$; giving $(A-B)/2 = 90° - (\delta_1 + (m_1 - .5)C)$.

Substituting.

$h_2 - h_1 = k \cot\ (\delta_1 + (m_1 - .5)K/R_2 \sin 1'') \left(1 + (h_2 + h_1)/2R_2 + k^2/12R_2^2\right)$,(31)

By calling the second factor unity and expanding the cot by Formula 5],(31) can be reduced to another form which is sometimes given.

$$\frac{1}{\tan(\delta_1 + (m_1 - .5)C)} = \left(1 - (m_1 - .5)C \tan\delta\right)\left(\tan\delta_1 + (m_1 - .5)C\right)^{-1} =$$

$$= \cot\delta_1 + (.5 - m)C + (.5 - m)\cot^2\delta_1 C, \text{by Formula 32]}$$

Substituting in (31),

$$h_2 - h_1 = k\cot\delta_1 + (.5 - m_1)K^2/R_2 + (.5 - m_1)(k^2/R_2)\cot^2\delta_1 \quad (32)$$

If the line is sighted from the other end, a second value will be obtained, and the weighted mean will give the required result.

72. COEFFICIENT OF REFRACTION. From Fig. 56,

$$\delta_1 + m_1 C + \delta_2 + m_2 C = 180° + C, \text{or}$$

$$m_1 + m_2 = (180° - (\delta_1 + \delta_2))(R_2/k)\sin 1' + 1 \quad (33)$$

The refraction coefficients are thus indeterminate from any number of reciprocal observations, since two unknowns are introduced for each equation. If the observations are simultaneous, m_1 is usually assumed equal to m_2. each line will give a value for m, and the average for the whole area can thus be found by taking the weighted mean. Thus

$$m = (180° - (\delta_1 + \delta_2))(R_2/2k)\sin 1'' + 1/2 \quad (34)$$

If not simultaneous, the coefficient for the lines radiating from each station may be taken the same, so that in a system of l lines joining p points, there would be p unknown coefficients with l observation equations of the form (33). If l > p, the coefficients would be found by a least squares adjustment. If the weight of each δ be taken proportional to the number of observations, n, then by Part 1, §3 and (13), (33) would have a weight w given by

$$1/w = R_2^2 \sin^2 1''/\ (1/n_1 + 1/n_2)/k^2 \quad (35)$$

that is the weight would be proportional to $n_1 n_2 k^2/(n_1 + n_2)$

Bessel assigns weights by the arbitary formula,

$$n_1 n_2 \sqrt{k}/(n_1 + n_2)$$

on the ground that errors arising from variations in m are of more importance than those from errors in δ.

The average value of m as found by the U.S.C & G.Survey is:

Across parts of the sea, near the coast.	0.078
Between primary stations	0.071
In the interior of the country, about	0.065

Clarke, Geodesy. p.281, gives the range in India from -0.09 to +1.21.

73. OBSERVED ANGLE OF ELEVATION IN SECONDS. If α = the elevation angle (supposed small) = 90° - δ, (31) becomes $h_2 - h_1 = k\tan\alpha + (.5 - m)k^2/R_2$

$$= k\alpha\tan 1'' + (.5 - m)k^2/R_2$$

Substituting for R_2 and m average values,

$(.5 - m)k^2/R$ = 0.000 000,0687 k^2 ,log const. = 2.82413

tan 1″ = 0.000 00485 " " = 4.68574

giving in metric units, α in seconds.

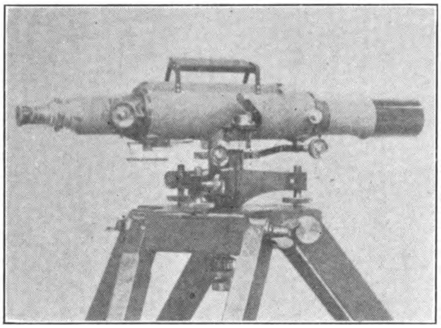
$$h_2 - h_1 = 0.000\ 004\ 85k\alpha + 0.000\ 000\ 0687\ k^2 \quad (36)$$

For k and $h_2 - h_1$ in feet, the last term becomes 0.000 0000202 k^2.

It is claimed by the U.S.C.& G.S. that for α<5° and k < 15 miles, (36) will give results within the uncertainty of refraction.

74. REDUCTION FOR DIFFERENCE IN HEIGHT OF TELESCOPE AND OBJECT ABOVE STATION MARK. Let s be the difference. Then from Fig.57,

$$\frac{\sin x}{\sin ABD} = \frac{s}{AD} = \frac{s_1}{AB}\frac{\sin BDA}{\sin ABD}$$



C.E.) Fig. 61. The Coast and Geodetic Survey Level of 1900.

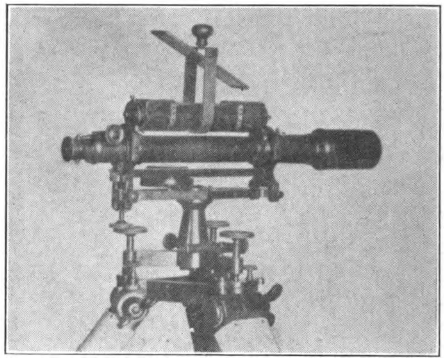


Fig. 58, Kern Precise Level.

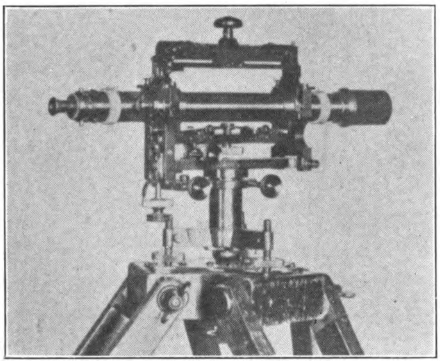
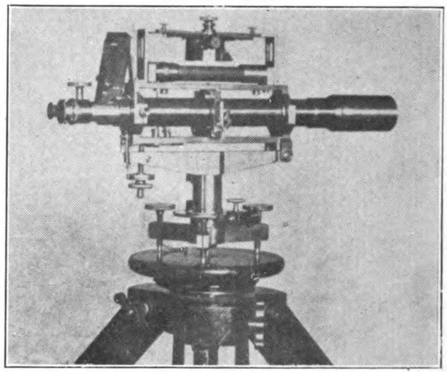


Fig. 60, Massachusetts State Survey Level.



see p53. Fig. 59 French Government Level (From Proc. Am. Soc.

$$\sin ADB = \sin(\delta_1 + m_1C - C)$$
$$\sin EDA = \sin(\delta_1 + m_1C - x - C)$$
$$\sin AED = \sin(90^\circ + C/2)$$

Substituting,

$$\sin x = \frac{s}{AE}\,\frac{\sin(\delta_1 + m_1C - C)\sin(\delta_1 + m_1C - C - x)}{\sin(90^\circ + C/2)}$$

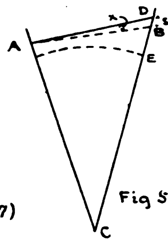


Fig 57.

The denominator is nearly unity, x is small, mC–C is small for short distances, and $\delta_1 = 90^\circ$ nearly for long distances.

Hence,

$$\left.\begin{array}{l}\text{short distances, } x'' = s \sin^2\delta/(k \sin 1'') \\ \text{long distances, } x'' = s/(k \sin 1'')\end{array}\right\} \qquad (37)$$

75. **ZENITH DISTANCE OF SEA HORIZON.** The line AB, Fig. 58, will be tangent to the sea level surface at B, giving in the right angled triangle ABC,

$$R_z + h_1 = R_z / \cos C$$

or, $h_1 = R_z (1 - \cos C)/\cos C$, by Formula 11] ,

$$= R_z 2(\sin^2 C/2)/\cos C = 2R_z((\sin^2 C/2)/\sin C)\sin C/\cos C, \text{ by Formula 10]},$$
$$= R_z((\sin C/2)/\cos C/2)\sin C/\cos C = R_z \tan C/2 \tan C$$
$$= (R_z/2)\tan^2 C, \text{ nearly}$$
$$\delta_1 + m_1C = 90^\circ + C, \text{ or } C = (\delta_1 - 90^\circ)/(1 - m_1)$$

substituting,

$$h_1 = (R_z/2)/(1-m_1)^2 \tan^2(\delta_1 - 90^\circ) \qquad (38)$$

76. **INSTRUMENTS.** Precise spirit, or geodetic, leveling is distinguished from ordinary spirit leveling by the use of better instruments and methods and more care in observing.

Some of the more common instruments in use are shown.

In Figs. 58, 59, 60, the level is used as a striding level giving greater facility of adjustment for both level tube and collimation, and opportunity to eliminate both errors by reversals in observing. The rear Y can be raised or lowered by a micrometer screw, giving a delicate means of releveling when pointing at the rod. In Fig. 60, this slight releveling cannot offset the H.I. of the instrument as with the others.

In Fig. 61, the level tube is dropped into the telescope tube down to the cone of sight rays, in order to diminish the lack of parallelism of the 2 tubes due to locally heating either end of the instrument, thus sacrificing the striding level. The two tubes are cast from an iron-nickel alloy having a coefficient of expansion = C.000,004 (Cent.), about 1/5 that of brass. The motion with micrometer screw is retained.

In Figs. 59 and 61, the mirror for reflecting the bubble to the observer at the eye end is replaced by a system of prisms which eliminates parallax by giving vertical sight rays upon both ends of the bubble.

Fig. 59 has a quick leveling ball and socket tripod head which is very stable.

The focussing side of the telescope should be long and well fitted to preserve parallelism with the line of collimation when sighting at different distances.

Buff & Berger have a more recent type of Fig. 60 in which the level tube is placed on top as a striding level with a mirror above as in Fig. 58 rather than at the side. The power is 50, with 2".1 level divisions.

The principal instrumental constants are

Fig.	Focal length	Diam. of objective.	Power	Stadia ratio.	Two m.m. div. of level.
58	14" 1/2	1 1/2 in.	50	1/231	1.7 to 3".4
59	14	1.4	25	1/100 or 1/200	8.3
60	15	1 1/2	35	1/100	6.9 to 8
61	16	1.7	43 & 32	1/333	< 2

It will be noted that these values do not differ materially from those for ordinary levels, except in the sensitiveness of the level tube and in the magnifying power.

77. RODS. Both target and speaking non-extensible rods are used.

The Kern or Swiss rod is shown in Fig. 62. This is used by the U.S. Engrs. Corps with the Kern level. The smallest graduations are centimeters, while readings are estimated to millimeters.
The French rod is shown in Fig. 63. It has a line graduation to 2^{mm} printed upon paper and pasted to the rod. The rod is rather flexible. To determine changes in length due to changes in temperature and moisture an iron and a brass bar are inserted side by side near the center line and fastened to the base plate, while at the top a scale is attached to the brass and one to the wood, each being read by an index on the iron. The brass scale is so graduated that each division represents an expan-

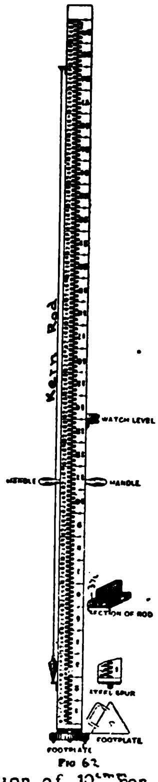


Fig 62.

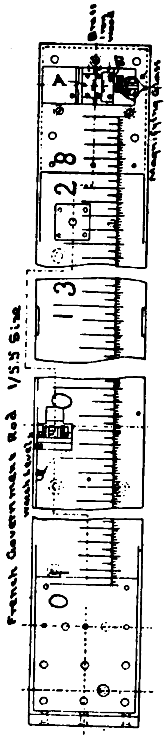


Fig 63.

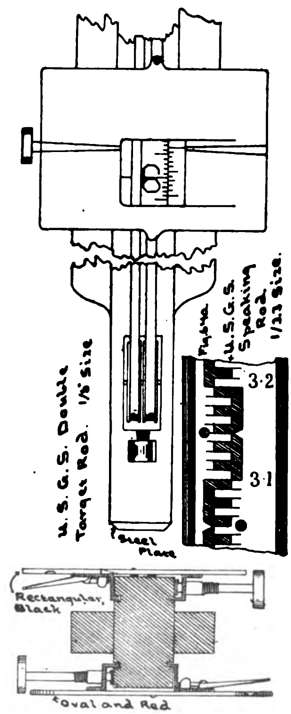


Fig. 64.

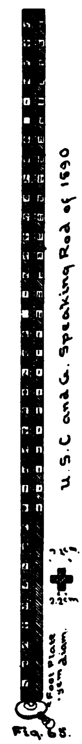


Fig. 65.

sion of 10^{cm} per meter of the iron bar: each division of the scale on the wood gives an expansion of 10^{cm} per meter of the wood. The sum of the two readings (A and B) will thus give the total change in length of the wooden rod.

The U.S.G.S. rod shown in Fig. 64 is a double target rod made by W. & L. E. Gurley of white pine impregnated with boiling paraffine to a depth of 1/8". It is graduated on both sides and each has 2 targets, one oval and red, the other rectangular and black. The targets are handled by endless tapes as shown, the length of the rod being a little over 10 ft. The steel base shoe has an area 1/2 of a square inch.

The two targets are for use on "double rodded lines," where two sets of turning points and two sets of notes are carried through with one instrument, the instrument man setting the rear and front rod targets as usual for the first set of T.P's, then the front and rear rod targets of the other faces for the second set; afterwards checking both target readings as the instrument and rear rod are moved forward.
The U.S.G.S. speaking rod is shown at Fig. 64 a. The unit is 0.2 ft., divided and read to fifths, or to .004 foot. The notes are kept on the 2-ft. basis to correspond, requiring all derived elevations to be doub-

bled. The shaded portion is red, the other portions black, on a white ground.

The U.S.C.& G.S. rod is shown in Fig. 65. The centimeter graduations are on the edge 2.2^{cm} wide.

The center of the bell metal foot is in the plane of the graduation. Silver faced plugs are placed 1^m apart and the distances between them checked by steel tape for field comparisons. A thermometer is attached for temperature, and a disk level for plumbing as with the others. The pine is soaked in boiling paraffine for its entire thickness which increases the weight, does away with moisture changes and does not appreciably affect the coefficient of expansion.

78. U.S. C. ENGRS. METHOD. The instrument is leveled and pointed at the rear rod; both ends of the bubble are read, the 3 wires and the level again, for the backsight. Similarly for the front sight. The length of sight is limited to 100^m, and the difference between front and back sight to 10^m. A heavy canvas umbrella is used to protect from the sun, or sometimes a tent if the weather is windy.

Each rod reading is corrected for observed inclination by the formula

$$\text{Correction} = 4\ id(1m \tan 1''/4) = 4\ i\ d\ A \qquad (39)$$

where $4\,i = E + E' - (O + O')$, the sum of the two eye end minus the sum of the two object end readings of the level; d = stadia interval; l = value of 1^d of level; m = stadia constant; A = 1m tan $1''/4$, a constant.

The difference in elevation between two B.Ms. is corrected by the formula.

$$\text{B.M. Cor.} = (d_1 - d_2)\ cm \tan 1'' \qquad (40)$$

where d_1 = sum of stadia intervals for back sights; d_2 = sum of stadia intervals for front sights; c = inclination of line of collimation in seconds when the bubble is in the center (+ if object end low). c must include inequality of pivots, level error and collimation error.

The level tube is adjusted until within 2 divisions, and the collimation until the mean of the

The level tube is adjusted until within 2 divisions, and the collimation until the mean of the 3 wires for direct and reverse position upon a rod at a distance of 50^m do not differ more than 2.5^{mm}. Readings are then taken every morning, and at other times when there is reason to suspect disturbances, for the level tube and collimation errors to use in (40). By keeping the sums of the stadia intervals, as in the record shown, these can be made equal in closing on a B.M. so that the correction (40) will disappear.

Steel pins are frequently used for turning points instead of the foot plate of Fig. 62.

FORM OF RECORD

Date		Locality / Direction					Observer / Recorder			Level No. / Tube No.			
BACK SIGHT							FRONT SIGHT						
Wire readings	Means	Wire interval	Bubble E	Bubble O	Rod	Remarks	Wire reading	Means	Wire intervals	Bubble E	Bubble O	Rod	Remarks
1009		175	11.5	11.5	10	T.B.M.	1686		187	11.5	11.5	13	P.B.M.
1184	1185.0	178				30	1873	1874.2	190.5				20
1362		353					2063.5		377.5				

79. FRENCH GOV. METHOD. In this method the bubble is kept in the center when sighting; the 3 wires are read on the back sight and also on the front sight; the level is reversed, the telescope rotated 180° about the line of collimation, and the note keeper reads the middle hair on the front rod and then upon the back rod. These reversals tend to eliminate the error of level and of collimation and those portions of the errors of refraction and instability which are proportional to time. If the discrepancy between the first and last readings exceeds a certain amount, both sets are repeated.

The longest sight is limited to 100^m and the greatest difference to 10^m. Wooden hubs are used for turning points, and the same ones are intended to be used on the return line between each two bench marks.

The only corrections required are for rod errors, but since these include scale errors (paper scales), they have to be made separately for each set up, taking into account the change in length of the rods as shown by comparisons with the enclosed steel bar.

The change of observers adds to cost in requiring a good observer for note keeper, and it adds to delay and instability in changing men, especially if the cross hairs have to be re-focussed on account of the change.

FORM OF RECORD

Back Sight					Front Sight									
Stadia interval	Stadia threads	Center thread	Difference first and second		Center thread	Stadia threads	Stadia interval	Rod corrections +	Rod corrections −	No. of rod	B.M. or turning point	Remarks	Atmosph. conditions
Forward		109580			427555			5	340	2	B.M.8	Began 7h 15m a.m.	Rain
367	2036					0825	365						
		+ 2403			− 1131			40			*	Rod 1	
366	2769		− 2	+2		1556	365					A = 66.5; B = 36	
		− 2405			+ 1193					1	T.P. 9	A+B = 102.5	
371	0315					1504	357					Rod 2	
		+ 0686			− 1861				5			A = 87; B = 35	
372	1058		+ 3	−4		2218	357					A+B = 122	
		− 0683			+ 1857								
Correc. D+e>0		171350	Totals		488575	Correc. D+e<0		45	345	2	78-II		Sun
		2D = − 317225				−15m8913		e = −300					
		D = − 158613											

Rod readings to mm.; rod corrections to dmm.

80. U.S.G.S. METHOD. For double rodded lines and the double target rods of Fig.64, the rear rodman holds on the T.P. of line A and clamps his red target when covered by the cross hair; the front rodman then holds on the next T.P. of line A and clamps his red target at the proper height; he then holds on the T.P. of line B and clamps his black target, the rear rodman then holds on the rear T.P. of line B and and clamps his black target.

Separate notes are kept for the two lines (claimed to be equivalent to having been run in opposite directions): while the instrument man checks all 4 rod readings as he and the rear rodman move forward.

The bubble is kept in the center when sighting. Steel pegs are preferred for T.P's. The level is adjusted daily, or oftener when necessary.

Attention is called to the fact that the length of sight should be kept so nearly constant that the focus of the telescope will not require changing for front or back sight during the day, and that if it should require changing on account of grades or atmospheric disturbance requiring shorter sights, then the level should be readjusted for the new position of the slide. It does not appear, however, that this restriction is enforced nor does it appear necessary with a well made precise level telescope. With target rods the rodman is usually required to keep a separate set of notes.

81. U.S.C.& G.S. METHOD. In the old method (Report 1879, App. 15) the Vienna or Stampfer level, slightly modified, was used. Its general construction is like the Kern. The rod is a non extensible pine rod graduated to centimeters on the front edge of the + as a speaking rod, and on a brass scale on the side of the front portion for the target. The target is moved by a chain similar to the tape of Fig.64.

FORM OF RECORD.

Back Sight								Front Sight							
No. of Station	Telescope	Level	Micrometer Horizon	Micrometer Target	Dist. wire	Edges of target	Rod read and temp.	No. of station	Telescope	Level	Micrometer Horizon	Micrometer Target	Dist. wire	Edges of target	Rod reading and temp.
Running on Road			Muddy.	Rod F on		B.M. No.61.		Weather Cloudy.				Brisk Wind.			Rod E
	I	R	17.102	17.113					I	D	17.109	17.104			39°
9		D	.126	.117				9		R	.097	.102			
	E	D	.127	.107		0.860				R	.095	.098		1.332	
		R	.117	.111	2.240	0.810	0.833			D	.107	.092	2.710	1.282	1.3072
			17.118 −0.6	17.112	140.5	0.835					17.101 −0.3	17.098	140.3	1.307	

To take a reading: a. The bubble is brought near the center and the target clamped to correspond, the bubble is then accurately centered, and the micrometer screw of the rear Y read: the target is bisected by turning the micrometer screw and the screw again read; b. The level is reversed, the bubble brought to the center, and the target bisected, and both screw readings taken. c. The telescope is rotated 180° about the optical axis, the bubble brought to the center, and the target bisected, and both screw readings recorded. d. The level is made direct, the bubble brought to the center, and the target bisected, and both screw readings recorded.

The stadia hairs and the edges of the target are then read by the levelman; while the target and the rod thermometer are read by the rodman. Having the value of 1^d of the micrometer screw, and the distance to the rod, the rod correction for each of the 4 readings can be computed by a formula similar to (39); the average of the 4 added to the target reading will give the corrected rod reading.

The method of double rodding, is in use, as also that of running a single line through and checking back.

In the new method introduced in 1899, and slightly modified in 1900 to adapt it to the new level, Fig. 61, the bubble is kept in the center while reading the 3 wires to millimeters on the speaking rod; the front and back sight readings are so taken that the time interval between shall be small; at odd stations the back sight is taken first, and at even stations the front sight; the difference between front and back sight distances is limited to 10^m; the diffenence between sums of front and back sight distances between any 2 B.M's. to 20^m; greatest length of sight 150^m.

The check line is usually run in the opposite direction from the direct, and under different atmospheric conditions, e.g., one in the forenoon the other in the afternoon:

$$\text{a difference} > 4^{mm}\sqrt{\text{distance in kilometers}}$$

between adjacent B.M's. calls for the rerunning of both lines until 2 values are obtained within the limit.

The rodman reads the rod thermometer each time, and a temperature correction is applied.

The error of collimation is determined each day by using a front sight reading (after completing a set up) with a new back sight reading about 10^m behind the level; then setting up about 10^m behind the front rod and reading both rods again. The correction constant, C = correction/(stadia interval) is found by

$$C = \frac{(\text{sum of near rod readings}) - (\text{sum of distant rod readings})}{(\text{sum of distant stadia intervals}) - (\text{sum of near stadia intervals})} \qquad (41)$$

No adjustment is made unless C > 0.005.

Correction is made for curvature and refraction, and for level when the stadia intervals differ for front and back sights; also for length of rod.

FORM OF RECORD

Number of Station	Thread Reading Back Sight	Mean	Thread Interval	Sum of Intervals	Rod and Temp.	Thread Reading Fore Sight	Mean	Thread Interval	Sum of Intervals
Time 2 PM									
43	0 674		99		V	2 683		99	
	0 773	0 773	99		38	2 782	2782.3	100	
	0 872		198			2 882		199	
44	0 925		106		W	2 415		103	
	1 031	1030.3	104		35	2 518	2518.0	103	
	1 135		210	408		2 621		206	405

Name and temp. of rear rod given.

The corrections between B.M's. are summed from tables or slide rule and entered on the computation sheet separately.

82. INEQUALITY OF PIVOTS. The level is set up on a pier or other firm support where it is protected from air currents and from sudden changes of temperature and the bubble brought to the center. The telescope is changed end for end in the Y's. and the bubble read without reversal. The out of level, if any, must be twice (within the errors of observation) the inequality of pivots referred to the supporting Y's., or 4 times the error referred to the telescope axis on the basis of circular collars.

The observations should be repeated until the desired accuracy is secured.

Fauth Precise Level, May 16, 1890. Landers, Wadsworth } Obs

Eye end of Telescope	D or R	Level East	Level West	East end high		Eye end large
W	D	41.4	6.5	-1.05	-1.27	
	R	9.	44	-1.50		0^d91
E	D	7.2	42.	.4	.55	
	R	43.2	8.2	.7		1.18
W	D	40.8	5.8	-1.7	-1.82	
	R	9.5	44.4	-1.95		.75
E	D	7.8	42.8	-.3	-.32	
	R	42.8	7.9	-.35		.80
W	D	40.5	5.5	-2.	-1.92	
	R	9.3	44.4	-1.85		1.06
E	D	7.6	42.6	-.1	.2	
	R	43	8.	.5		1.21
W	D	40.4	5.4	-2.1	-2.22	
	R	9.8	44.9	-2.35		1.18
E	D	7.7	42.7	-.2	.15	
	R	43.0	8.0	.5		1.00
W	D	40.8	5.8	-1.7	-1.85	
	R	9.5	44.5	-2.		1.03
E	D	7.5	42.5	0.	.2	
	R	42.9	7.9	.4		Mean = 1.01

Level graduated from E to O and with 25 for Center.
Value 1^d of Level = 3".8.

Referred to telescope axis, Eye end large = 1.01 × 3.8/2 = 1".92. Correction to rod reading negative.

Ex.1 If the collars are 10" apart and the angle made by the sides of the Y supports and level legs are 90°, find the inequality of the collars in inches for the value 1".92 given above.

83. ROD CORRECTION. For the paraffined rods and those where a brass scale is used, the temperature at which the rod is standard can be found by comparison with a standard. A table of double entry can then be made out or a slide rule used for the correction for any observed temperature and rod reading, it being the product of the temperature increment, the rod reading, and the coefficient of expansion, and positive when the rod is too long or the reading too small.

For the Kern rod which changes length with moisture as well as with temperature the actual error per unit length can be determined from day to day by comparison with a standard tape and the corresponding correction applied if appreciable.

For the French rod the paper scales require correction for scale errors, and the wooden rods corrections for length, and for changes in length as denoted by the A and B readings. This is accomplished by comparing the rods with a standard and at the same time reading the scales A and B. The scale corrections are platted on cross section paper as ordinates with rod readings for abscissas and the correction curve drawn. An equalizing line is also drawn through the origin, which separates the correction into 2 parts, one proportional to the rod reading and the other a local scale correction. It is assumed that only the first is affected by a change in the length of the rod

To obtain the corrections graphically, the straight line correction, say 150^{dmm} per 1^m for A + B = 135, is laid off on a vertical from D, Fig. 66, to a scale 15/1. An oblique line is drawn through D and these corrections projected upon it by horizontals, and the corresponding rod corrections marked. If the rod should expand or contract 1^{dmm} per meter, the inclination

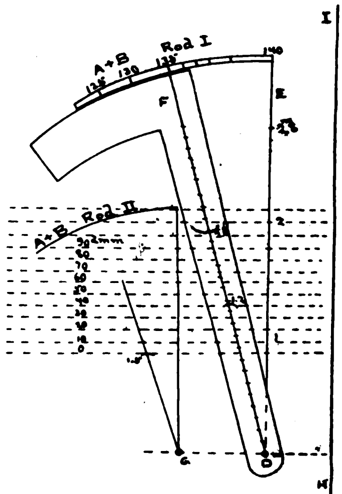


Fig. 66

of DF can be changed so that the projected length corresponding to the 1st. meter shall be 1^{dmm} longer or shorter than before, when the corrections will all project into their new values.

The cosines of the new inclinations of DF for values changing by 1^{dmm} will thus differ by unity for the radius 150. ∴ describe an arc with D as a center, lay off the different angles found from the cosines starting from the vertical and mark the corresponding numbers for A + B, starting with the highest expected in the field work Then with the scale corrections as radii and the corresponding points on DF as centers describe arcs. Horizontal tangents to these arcs will give constant values to these projected scale errors, while the straight line correction will depend upon the A and B setting.

The corrections for the other rod are placed on the same sheet with the center at G. A celluloid sheet is ruled with 5^{dmm} lines, to the scale 15/1, and kept in position by the strip HI.

To take out a correction for a set up; set each arm to the correct A + B; slide the celluloid until the zero coincides with rod II reading and read the scale for rod I reading. Thus if $(A + B)_1 = 135$; $(A + B)_2 = 118$; the correction for a back sight reading of 2.0 on I and a front sight reading of 1.5 on rod II would $= + 102^{dmm}$.

84. ACCURACY AND COST OF RESULTS. The authors of Lever des Plans et Nivellement estimate the probable error for a set up with the French Gov. level for sights 75^m long as follows:

1. Error of level. The eye can detect a difference of $1/2^{mm}$ in the readings of the ends of the bubble with the 3^{mm} divisions on the tube. This gives a probable inequality of about 2^{dmm}, or a probable out of level of 1^{dmm}. This would give the same uncertainty for a rod reading at a distance equal the radius of curvature, or 50^m, or 1.5^{dmm} at 75^m.

2. Error of estimation. With a power of 25, the centimeters of the rod at 75^m appear of the same size as millimeters at 0.3^m Under these conditions tenths can be easily estimated with a probable error about 0.33^{dmm}, giving 3.3^{dmm} when referred back to the rod.

3. Errors due to temperature changes. Experience has shown these to be as great as No. 2.

Combining, the total for a reading,

$$r = \sqrt{(1.5)^2 + (3.3)^2 + (3.3)^2} = 5^{dmm}$$

For a set up, T.P. to T.P.,

$$r'_{\cdot} = \sqrt{r^2 + r^2} = r\sqrt{2}$$

With 75^m sights there are 6 2/3 set ups per 1^k, while with the 4 observed differences between each pair of T P's would give the resulting probable error per 1^k,

$$r_k = r\sqrt{2 \times 3.66/4} = 9^{dmm}$$

which agrees with the results found for the fundamental French lines.

The above supposes all constant or systematic errors eliminated by the methods of observation or by applying computed corrections.

The principal constant errors recognized are:

1. The variation of gravity with latitude. This results in making the distance between 2 level surfaces vary inversely with g, the work required to raise a unit mass from one to the other, or hg, being constant. The observed difference in height of 2 points would thus depend on the height of the line of levels run between them. Heights above sea level obtained by direct measurement are called orthometric, obtained on the basis of work done in raising a unit mass, dynamic; the differences are usually within the errors of observation, but in rugged, country they may be greater. For full discussion see Helmert Höhere Geodesie, or Lever des Plans....

2. Variations of refraction with height of line of sight, with character of ground surface over which the line passes, and with the time of day. In ascending or descending long grades this becomes cumulative and may easily exceed the accidental errors unless short sights are taken.

3. Change in height of instrument or T.P. due to settlement or springing up of ground. This has long been one of the reasons assigned for greater discrepancies between lines run in opposite directions as compared with those run in the same direction.

4. Change in collimation and level error due to heating the end of the telescope nearest the sun. This is the principal reason assigned by the Coast Survey for the change in method introduced in 1899.

In Proc. Am. Soc. C. Engrs, Vol. 26, p 888, the prob. error per kilometer is given for some 1200 miles of U.S.C.& G. levels averaging 1.07^{mm}, and for some 1500 miles of U.S. Engr. Corps levels averaging 0.63^{mm}. These apparently are from circuit closures.

In checking forward and back between benches the limit = $4^{mm}\sqrt{\text{kilometers}}$ as already stated.

The cost is estimated by D. Molitor (Pro. A.S.C.E. 26, p. 897) at $24. per mile for a double line with permanent bench marks about 0.6 mile apart.

On p. 1160 it is stated by Hayford that the total cost of the 1899 work of the C.S. was $13.55 per mile.

Seven minutes per station is given as about the average time for the same (C.S.) work with a record of 111 stations in 9^h 20^m on June 20, with 40^m to 80^m sights, and of 10.3 miles July 14 in 7.4 hours with 80^m to 110^m sights.

85. DATUM. Mean sea level is the ultimate datum to which all land levels should be referred. It can be obtained approximately from the mean of two consecutive high tides, and the intermediate low tide. For more accurate results, a permanent bench mark and a tide gage should be established and readings taken for a semi-lunation, or longer.

The zero of the tide gage should be occasionally referred to the B.M. to guard against disturbance.

The yearly means of six year's observations at Sandy Hook, with a self recording gage, gave a mean which has a probable error of 0.031 feet; the lowest mean 1876, being 0.168 below, and the highest, 1878, 0.177 feet above.

CHAPTER VI.

TOPOGRAPHIC AND HYDROGRAPHIC SURVEYING.

86. TOPOGRAPHIC-SURVEYING. The problem is usually to collect the greatest possible amount of reliable information for a given expenditure which shall at the same time bring out the characteristics of the entire area with a detail proportioned to their relative importance and the objects in view.

While the methods are mainly those of ordinary surveying, the young topographer soon learns to distinguish the difference in accuracy and detail required for an exploration survey and a survey of valuable property for the proper study of proposed improvements. In exploration surveys, check points are obtained by observations for latitude and longitude; in more detailed surveys covering considerable areas the best results are obtained by starting from triangulation points, only a few miles apart, whose positions are known both horizontally and vertically.

Method with transit and stadia; plane table and stadia; preparation of plane table sheet; n-point problem: Colvin's lake meander; barometric heights; aneroid profile; Ashburner's method with aneroid; photographic methods; sketching. Only such details should be taken as will show when plotted to scale. Small distances which can be estimated as closely as they can be plotted need not be measured. On the other hand mistakes, omissions, inaccuracies, etc., which are not noticed by the inexperienced who have been over the ground, show themselves when the map is put to use, or are often picked out, and the map condemned by some old resident who is familiar with the particular locality.

87. HYDROGRAPHIC SURVEYING. River Surveys. For the best results a triangulation should first be extended along the river valley, and convenient points established for the detailed survey. Otherwise, points can be fixed by latitude and longitude observations. For a small stream a traverse line can be run along shore, the width can be found by direct measurement, by stadia, or by bearings from two stations on one shore

to a point on the opposite shore. If the banks are impassible the meander line can be run on the water, using a boat, the distances being obtained with a long chain or wire, or by stadia.

Depths, cross-sections, character of the bottom, velocity of the current, volume of water, rate of its surface slope, and high and low water marks are often important.

For a navigable stream, the traverse line may be run with a steamer which may be steered by a compass or by 2 points in line ahead. The direction should be changed quickly so that the course will be made up of a series of straight lines. Distances along the line may be measured with the log, anchored log, or buoy and nipper. Bearings should be taken to side objects by an observer on deck from two or more positions, and the time of each noted. The sketch must, of course, be kept up as the vessel proceeds. If some distant prominent object can be sighted frequently it will serve as a check on the bearings. Soundings may be taken with a common lead, unless specimens of the bottom are required.

Two boats can be used in place of the steamer. The distance between them may be found by the angle subtended at one by a mast of known height at the other.

If triangulation points have been established, the boat's position can be tied to them as often as desired by the N-point problem, or by taking cuts to it at a given signal, with transits at 2 or more stations. If approximate latitudes and longitudes are the only checks, only rough work can be expected.

In all field work the day's notes should be carefully looked over at night, and plotted if the work is to be plotted, so that all mistakes and obscure parts can receive attention while the notes are fresh and the parties still in the field: also the better to lay out the remaining work with reference to that already done.

<u>Lake, harbor, sea coast, surveys</u>. General methods; methods of locating soundings. A tide gage should be established and records kept so that all shallow soundings can be reduced to low water. The position of the channel; character of the bottom; depths; and for approaches to harbors, views of the shore as seen from different points with "ranges" and angles between prominent objects; are usually required.

Lead with tallow for specimens of the bottom, Sand's specimen cup. Brook's specimen cup. Ericsson's lead. American method, 32-pound shot, not recovered. A wire is used for the line in very deep soundings, and the instant of striking bottom is determined by the change in rate of descent. Miller-Cassella thermometers for deep water temperature.

88. FIELD COMMUNICATIONS. With several parties in the field, it is sometimes very convenient to be able to communicate with each other.

The Morse telegraphic alphabet is usually employed. For long distances the heliotrope is used for flashes, the parties having orders to watch for signals at a certain hour each day. For short distances a flag is used.

C H A P T E R V I I.
FIGURE OF THE EARTH.

89. MERIDIAN SECTION, COORDINATES OF POINT. In reducing geodetic data the earth is usually assumed to be an ellipse of revolution. The dimensions given in Table I best satisfy the degree measurements which had been made up to the time when they were derived.

In the meridian section, Fig. 67, through M: MH = N: MG = n: MGD = geographic latitude = L: MCD = geocentric latitude = L_1: R_m = radius of curvature of the meridian: x and y = coordinates: a and b = semi-axes.

The equation of the ellipse is

$$x^2/a^2 + y^2/b^2 = 1$$

or $$b^2x^2 + a^2y^2 = a^2b^2 \qquad (a)$$

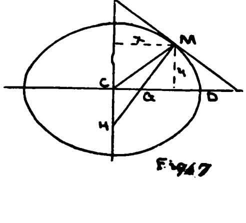


Fig 67

Differentiating,

$$2x\,dx/a^2 + 2y\,dy/b^2 = 0$$

or, $$dy/dx = -x\,b^2/(y\,a^2) \qquad (b)$$

From the differential triangle, Fig. 67,

$dy/dx = -\cot L = -\cos L/\sin L$ (c)

Equating (a) and (b),

$b^2x/(a^2y) = \cos L/\sin L$, or $b^2x^2/(a^2y^2) = a^2\cos^2 L/(b^2\sin^2 L)$ (d)

From the definition of eccentricity, $b^2 = a^2(1 - e^2)$

Substituting in (a),

$$x^2(1-e^2) + y^2 = a^2(1-e^2) \qquad (e)$$

From (d), $x^2(1-e^2)^2\sin^2 L - y^2\cos^2 L = 0$ (f)

Multiply (e) by $\cos^2 L$ and add to (f),

$$x^2(1-e^2)(\cos^2 L + \sin^2 L - e^2\sin^2 L) = a^2(1-e^2)\cos^2 L$$

$$x^2 = a^2\cos^2 L/(1-e^2\sin^2 L) \qquad (42)$$

Multiply (e) by $(1-e^2)\sin^2 L$ and subtract (f),

$$y^2 = a^2(1-e^2)^2\sin^2 L/(1-e^2\sin^2 L) \qquad (43)$$

Putting $1-e^2\sin^2 L = r^2$

$$x = a\cos L/r \quad y = a(1-e^2)\sin L/r \qquad (44)$$

90. PRINCIPAL RADII OF CURVATURE. Since arcs subtending the same angle are to each other as their radii, the radius of curvature of the meridian

$$R_m = ds/dL = -(1/\sin L)(dx/dL)$$

From (44), $dx/dL = (-ar\sin L + ar^{-1}e^2\sin L\cos^2 L)/r^2$

Substituting, $= -a(1-e^2)\sin L/r^3$

$$R_m = a(1-e^2)/r^3 = a(1-e^2)/(1-e^2\sin^2 L)^{3/2} \qquad (45)$$

The section by a plane through the normal MH and ⊥ to the meridian is called the prime vertical. It is tangent to the parallel of latitude at M and its center of motion, or of curvature, is on the axis at H as the point M moves past the meridian plane. ∴ from Fig. 67 and (44),

Radius of curvature of prime vertical = normal ending at minor axis,

$$N = x/\cos L = a/r \qquad (46)$$

Dividing (46) by (45),

$$N/R_m = r^2/(1-e^2) \qquad (47)$$

This ratio is often of value as indicating the deviation of the surface at any point from that of a sphere.

For L = 0°	N/R_m =	1.0067	L = 45°	N/R_m =	1.0034
15		1.0058	60		1.0017
30		1.0050	90		1.0000

The geometrical mean of N and R_m is taken for the mean radius of curvature at the point, i.e.,

Mean radius of curvature, $R = \sqrt{N R_m}$ (48)

Radius of parallel,

$$R_p = x = a\cos L/r = N\cos L \qquad (49)$$

Normal ending at major axis,

$$n = y/\sin L = a(1-e^2)/r \qquad (50)$$

Geocentric latitude, Fig. 67

$$\tan L_1 = y/x = (1-e^2)\tan L \qquad (51)$$

$L-L_1$ varies from 0° at the equator to 11'40" in latitude 45° and 0° again at the pole

91. RADIUS OF CURVATURE FOR A GIVEN AZIMUTH. A plane through the normal MG cuts out an ellipse. Its equation is found by expressing the coordinates of a point in the equation of the surface in terms of the co-

ordinates of the curve. The equation of the surface is

$$a^2 z'^2 + b^2(x'^2 + y'^2) = a^2 b^2 \qquad (a)$$

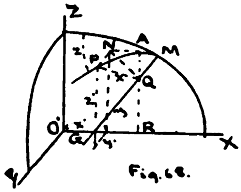


Fig. 68.

For the point P,

$x' = OG + GR - NA = Ne^2 \cos L + y \cos L - x \cos z \sin L$

$y' = PN \quad = x \sin z$

$z' = RQ + QA \quad = y \sin L + x \cos z \cos L$

substituting in (a) and using for b, $a\sqrt{1 - e^2}$

$$x^2(1-e^2(1-\cos^2 z \cos^2 L)) + y^2(1-e^2\cos^2 L) + xy(2e^2 \sin L \cos L \cos z) -$$

$$2x(1-e^2)Ne^2 \cos L \sin L \cos z + y\, 2e^2(1-e^2)N \cos^2 L = (1-e^2)(a^2 - N^2e^4\cos^2 L)$$

or, $A x^2 + By^2 + Cxy + Dx + Ey = F$

By Formula 35) $R_z = -(1 + dy^2/dx^2)^{3/2} dx^2/d^2y$

$dy/dx = -(2Ax + Cy + D)/(2By + Cx + E)$, $d^2y/dx^2 = -(2A + 2B\,dy^2/dx^2 + 2C(dy/dx))/(2By + Cx + E)$

For point M, $x = 0$, $y = n = (1-e^2)N$

$dy/dx = 0$, $d^2y/dx^2 = (1-e^2(1-\cos^2 z \cos^2 L))/((1-e^2\cos^2 L)(1-e^2)N + e^2\cos^2 L(1-e^2)N)$

$$= ((1-e^2)(\sin^2 z + \cos^2 z) + e^2\cos^2 z(1-\sin^2 L)/N(1-e^2).(R_m/R_m)$$

$$= -(R_m \sin^2 z + N\cos^2 z)/N R_m$$

$$\text{or, } R_z = N R_m/(N\cos^2 z + R_m \sin^2 z) \qquad (52)$$

If $z = 0$, $R_0 = NR_m/N = R_m$, the radius of curvature of the meridian.

$z = 90$, $R_{90} = N R_m/R_m = N$, the radius of curvature of the prime vertical.

The geometrical derivation of R_z is simpler.

In Fig.69 draw a tangent plane at M and a parallel plane at the infintesimal distance c from it. The latter will cut an ellipse as shown in plan. The 3 points B M B'. are consecutive points in the prime vertical or 3 points in the circle with radius N. Similarly for the meridian with radius of curvature R_m. Hence if a'. and b'. denote the semi-axes of the ellipse through B B'. and s the semi-diameter making the angle z with the meridian,

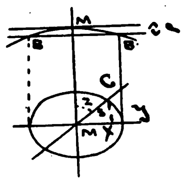


Fig. 69

$$a'^2/(2N) = c = b'^2/2R_m) = s^2/(2R_z)$$

$$a'^2 = s^2 N/R_z \qquad b'^2 = s^2 R_m/R_z \ \ldots\ldots \quad (a)$$

The coordinates of C are, $x = s \sin z$ and $y = s \cos z$,
Substituting in the equation of the ellipse,

$$s^2\sin^2 z/a'^2 + s^2\cos^2 z/b'^2 = 1$$

From (a) $R_z \sin^2 z/N + R_z \cos^2 z/R_m = 1$

$$\text{or} \qquad R_z = NR_m/(R_m \sin^2 z + N\cos^2 z) \qquad (53)$$

Table V. is computed from (53)

92. LENGTH OF MERIDIAN ARC. Since R_m changes slowly with L, for arcs of 1° to 2°,

$$ds = R_m d L'' \sin 1'' \qquad (54)$$

where dL'' is in seconds, and R_m is for the middle latitude.

For long arcs (54) must be integrated. Substituting the value of R_m.

$$ds = a(1 - e^2)/(1 - e^2\sin^2 L)^{3/2} dL \qquad \text{By Formula 32]}.$$

$$ds = a(1-e^2)(1 + (3/2)e^2\sin^2 L + (15/8)e^4\sin^4 L + (35/16)e^6\sin^6 L)dL$$

$$s = a(1-e^2)\int_{L_1}^{L_2}(1 + (3/2)e^2\sin^2 L + (15/8)e^4\sin^4 L + (35/16)e^6\sin^6 L)dL$$

By Formulas 11] and 12].

$$\sin^2 L = (1/2)(1 - \cos 2L)$$

$\sin^4 L = (1/8)(3-4\cos 2L + \cos 4L)$

By Formula 9], $\sin^6 L = \sin^2 L \sin^4 L = (1/32)(10-15\cos 2L+6\cos 4L-\cos 6L)$

Substituting and putting

$$A = 1+(3/4)e^2+(45/64)e^4+(175/256)e^6\ldots = 1.0051093 \quad \text{Log.} = 0.0022133$$
$$B = (3/4)e^2+(15/16)e^4+(525/512)e^6\ldots = 0.0051202 \quad = 7.709287$$
$$C = (15/64)e^4+(105/256)e^6 = 0.0000108 \quad = 5.03342$$
$$D = (35/512)e^6 \quad = 2.328$$

$$s = a(1-e^2)\int_{L'}^{L''}(A-B\cos 2L + C\cos 4L-D\cos 6L\ldots)dL$$
$$= a(1-e^2)(AL-(1/2)B\sin 2L+(1/4)C\sin 4L-(1/6)D\sin 6L\ldots)_{L'}^{L''}$$

Substituting the limits, and putting $L''-L'=\alpha$, $L''+L'=\beta$, we have by Formula 8],

$$s=(1.4895389)\alpha''-(4.511036)\sin\alpha\cos\beta+(1.53414)\sin 2\alpha\cos 2\beta$$
$$-(8.651)\sin 3\alpha\cos 3\beta \qquad (55)$$

where s is in meters and the numbers in parentheses are the logs of the constant factors.

Equation (55) is correct for 7 decimal places. If more are desired, the next term for A is $(11025/16384)e^8$; for B, $(2205/2048)e^8$; for C, $(2205/4096)e^8$; for D, $(315/2048)e^8$, while an E term is added $= (315/16384)e^8$.

93. AREAS ON THE ELLIPSOID. Dividing the surface into frustrums of cones by the parallels: width $= R_m dL$; circumference $= 2\pi N\cos L$. Differential area,

$$dA = 2\pi N R_m \cos L\, dL \qquad (a)$$

Substituting for N and R_m from (45) and (46), with b^2 for $a^2(1-e^2)$,

$$dA = 2\pi b^2 \cos L\, dL/(1-e^2\sin^2 L)^2$$

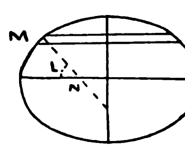


Fig. 70.

By Formula 32],

$$(1-e^2\sin^2 L)^{-2} = 1+2e^2\sin^2 L+3e^4\sin^4 L+4e^6\sin^6 L +5e^8\sin^8 L$$

Substituting, the expression to be integrated becomes

$$\int \cos L \sin^n L\, dL = (1/(n+1))\sin^{n+1} L$$

which gives, $A_{L'}^{L''} = 2b^2\pi(\sin L + (2/3)e^2\sin^3 L + (3/5)e^4\sin^5 L +(4/7)e^6\sin^7 L+\ )_{L'}^{L''}$

$$A_{L'}^{L''} = 2b^2\pi(\sin L''-\sin L' + (2/3)e^2(\sin^3 L'' - \sin^3 L') +$$
$$(3/5)e^4(\sin^5 L''-\sin^5 L') + (4/7)e^6(\sin^7 L'' - \sin^7 L')+\ ) \qquad (56)$$

To put in convenient form for computation,

$$\sin^3 L = (3/4)\sin L - (1/4)\sin 3L$$
$$\sin^5 L = (5/8)\sin L - (5/16)\sin 3L + (1/16)\sin 5L$$
$$\sin^7 L = (35/64)\sin L - (21/64)\sin 3L + (7/64)\sin 5L -(1/64)\sin 7L$$

Substituting in (56), we have by Formula 8], with $L''-L' = 2\gamma$ and $L''+L' = 2\delta$,

$$A_{L'}^{L''} = 4b^2\pi(B\sin\gamma\cos\delta - C\sin 3\gamma\cos 3\delta+D\sin 5\gamma\cos 5\delta-E\sin 7\gamma\cos 7\delta..) \quad (57)$$

where $B=1+(1/2)e^2+(3/8)e^4+(5/16)e^6+(35/128)e^8 =1.0034016$ Log=0.0014743

$C= (1/6)e^2+(3/16)e^4+(3/16)e^6+(35/192)e^8=0.0011283$ 7.05565

$D= (3/80)e^4+(1/16)e^6+(5/64)e^8 =0.0000017$ 4.2304

$E= (1/112)e^6+(5/256)e^8=0.0000000$

$F= (5/2304)e^8=0.0000000$

If (57) be divided by 360 and $L''-L' = 2\gamma = 1°$, the area for 1° square,

$$G=(b^2\pi/90)(B\sin 30'\cos\delta-C\sin 1°30'\cos 3\delta+D\sin 2°30'\cos 5\delta$$
$$-E\sin 3°30'\cos 7\delta..) \qquad (58)$$

The values of B, C, etc., should be carried to more than 7 places for accurate results with the Clarke ellipsoid, although the above values are carried as far as the data will warrant when applied to the earth.

94. SPHERICAL EXCESS. In Legendre's theorem it is proved that in a spherical triangle whose sides are short compared with the radius R of the sphere and a plane triangle with sides of equal length the corresponding angles differ by the same quantity which is one-third the spherical excess. Let A, B, C, = the angles of the spherical and A', B', C', those of the plane triangle: a, b, c, (π-measure) and a', b', c', the corresponding sides; aR = a', bR

$= b'$; $cR = c'$.

Plane triangle. By Formula 21], $\cos A' = ((b^2+c^2-a^2)/2bc)(R^2/R^2)$ (a)

By Form.1], $\sin^2 A' = 1-\cos^2 A' = ((4b^2c^2-(b^2+c^2-a^2)^2)/4b^2c^2)(R^4/R^4)$

or, $\sin^2 A' = (2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)/4b^2c^2$ (b)

Spherical triangle. By Form.27], $\cos A = (\cos a - \cos b \cos c)/\sin b \sin c$

By F.13] and 14]. $= (1-a^2/2+a^4/24-(1-b^2/2+b^4/24)(1-c^2/2+c^4/24))$
$/(b-b^3/6)(c-c^3/6)$

$= ((-a^2+b^2+c^2)/2-(b^4+c^4-a^4)/24-b^2c^2/4)/(bc(1-(b^2+c^2)/6)$

$= ((-a^2+b^2+c^2)/2-(b^4+c^4-a^4)/24-b^2c^2/4)(1+(b^2+c^2)/6)/bc$

$= ((b^2+c^2-a^2)/2+(b^4+c^4+2b^2c^2-a^2c^2-a^2b^2)/12)/bc$
$-(b^4+c^4-a^4+6b^2c^2)/24bc$. (c)

From (a) and (c), $\cos A = \cos A' - (1/6)bc \sin^2 A'$ (d)

Since b and c are very small, the difference between A and A′ must be small. Putting this difference = x, $\cos x = 1$, $\sin x = x'' \sin 1''$

$\cos A = \cos(A'+x)$, by Form.4], $= \cos A' - x'' \sin 1'' \sin A'$; by (c),

$= \cos A' - (1/6) bc \sin^2 A$

or $x'' = bc \sin A'/6 \sin 1''$ (e)

where b and c are in π-measure. If b and c are in units of length on the sphere of radius R, $x'' = bc \sin A'/6 R^2 \sin 1''$ = area of triangle$/3R^2 \sin 1''$

The same can be found for B−B′ and C−C′.

Since the areas of spherical triangles are to each other as their spherical excesses, we have from the trirectangular triangle, excess 90°,

Spherical excess in seconds, s, = area 2 90° 3600/πR^2 = $.5bc \sin A'/R^2 \sin 1''$

Comparing this with (e),

Spherical excess, s, = $3x = .5bc \sin A'/R^2 \sin 1''$ (59)

or, $s = m\, bc \sin A''$ (60)

where $m = .5/R^2 \sin 1''$, $= .5NR_m \sin 1''$ by (48), and is given in Table VI in metric units.

95. EFFECT OF HEIGHT UPON HORIZONTAL ANGLES. The observer at A, Fig.71, at sea level sights upon M at the height h above at B. The vertical plane of collimation at A projects M to B on the line drawn to H_a where the normal at A meets the axis, while the true projection is at B′ on the normal MH_L to the surface at M. This makes an error x in the horizontal angle at A due to the height h.

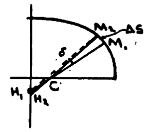


Fig.72.

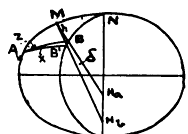


Fig.71.

First to find the angle between the two projecting lines at M. In Fig. 72 let C be the intersection of the normals at M_1 and M_2 both in the same meridian. If Δs is small C will also be the center of curvature for the arc M_1M_2, and

$$\Delta s = R_m \Delta L \qquad \text{(a)}$$

If M_1C be produced to meet the axis at H_1, and the reduced difference in latitude $M_1H_1M_2$ be called $\Delta L'$ ($M_1H_1 = N$)

$$\Delta s = N\Delta L' \qquad \text{(b)}$$

From (a) and (b), $\Delta L/\Delta L' = N/R_m$

But, $\delta = \Delta L - \Delta L' = \Delta L(1 - \Delta L'/\Delta L) = \Delta L\,(1 - R_m/N)$

From the values of R_m and N, (45) and (46),

$R_m/N = (1-e^2)/(1-e^2\sin^2 L)$

Substituting, $\delta = \Delta L\,(1-((1-e^2)/(1-e^2\sin^2 L))$

$= \Delta L\, e^2\cos^2 L/(1-e^2\sin^2 L)$

From Fig.71; $R_m\Delta L = -k \cos z$, nearly (lat. = dist. × cos of bearing)

$$\therefore \quad \delta = -k\, e^2 \cos^2 L \cos z/(1-e^2 \sin^2 L)\, R_m$$

$$= -ke^2 \cos^2 L \cos z/(1-e^2)N$$

$$BB' = h\delta = -h\, k\, e^2 \cos^2 L \cos z/\,(1-e^2)N$$

The corresponding horizontal angle error at A in π-measure,

$$x = -BB' \sin z/k = h\, e^2 \cos^2 L \sin z \cos z/(1-e^2)\, N$$

$$x'' = h\, e^2 \cos^2 L \sin z \cos z/(1-e^2)N \sin 1'' \qquad (61)$$

This will be a maximum for $z = 45°$. If L also $= 45°$,

$$x'' = .000055\, h \qquad (62)$$

where h is in meters.

For a height of 1000^m this would give 0."05. The probable error in the value of a primary angle is seldom less than 0."25, so that the above correction would be negligible except for very high altitudes.

96. TRIANGLE SIDE COMPUTATIONS. The triangles of a triangulation are strictly spheroidal, but by §95 the 3 vertices of a triangle can be projected down to sea level by lines drawn to the center of a sphere tangent to the ellipsoid at the center of gravity of the triangle and having $\sqrt{NR_m}$ for radius, only affecting the horizontal angles within the limits of the errors of observation.

The sides of these projected triangles have the same lengths, within the errors of measurement, upon the tangent spheres as upon the ellipsoid.

The triangles can thus be considered spherical, and by Legendre's theorem, computed as plane by subtracting one-third the spherical excess from each spherical angle.

In simple systems, and where the greatest accuracy is not desired, if the sum of the observed angles in any triangle does not equal 180° + s, or the sum of those about a point 360°, the error is distributed equally among the angles, or sometimes inversely as the number of repetitions.

But in complicated systems, or where extreme accuracy is desired, the errors are distributed by least squares.

The following is a convenient form for computation.

Base $O_w O_e$ = 6410.66 ft.

$O_w O_e$	3.8069028			
Sin A'_w	9.9907935			
	3.8161093	....		3.8161093
Sin O'_w	9.8593280	Sin O'_e		9.9156182
	3.6754373			3.7317275
$A_w O_e$ = 4736.28		$O_w A_w$ = 5391.72		

	3.6754373		
Sin A^2_e	9.8135605		
	3.8618768	....	3.8618768
Sin O^2_e	9.9999957	Sin A^2_w	9.8819486
	3.8618725		3.7438254
$A_w A_e$ = 7275.66		$O_w A_e$ = 5544.03	

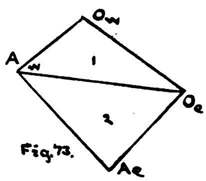


Fig. 73.

CHAPTER VIII.
GEODETIC POSITIONS.

97. DIFFERENCE OF LATITUDE. It is usual to find the latitude and the longitude of one or more of the triangulation stations by astronomical observation, as also the azimuth of one or more of the sides, and from this data to compute the positions of the other sides.

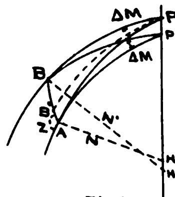


Fig.74.

In Fig.74, P′ is the pole of the ellipsoid and P that of a tangent sphere. The latitude of A and the azimuth z and distance K to B are given.

Since k is always small, its subtending angle being usually $<1°$, we have by Maclaurin's theorem, Formula 33],

$$L' = f(m) = L + (dL/dm)m + (d^2L/dm^2)m^2/2 + (d^3L/dm^3)m^3/6 +$$

In the differential triangle PAB′, Formula 27],

$$\cos PB' = \cos PA \cos AB' + \sin AP \sin AB' \cos PAB', \text{ or}$$

$$\sin(L + dL) = \sin L \cos dm - \cos L \sin dm \cos z$$

Expanding the first member,

$$\sin L + dL \cos L = \sin L - dm \cos L \cos z; \text{or } dL/dm = -\cos z \quad \text{(a)}$$

$$d^2L/dm^2 = (-d \cos z/dz)(dz/dm) = \sin z(dz/dm) \quad \text{(b)}$$

Formula 26],

$$\cot PB'A = (\sin AB' \cot PA - \cos AB' \cos PAB')/\sin PAB'$$

$$\cot(z+dz) = (dm \tan L + \cos z)/\sin z$$

$$\sin z(\cos z - dz \sin z) = (\sin z + dz \cos z)(dm \tan L + \cos z)$$

$$\sin z \cos z - dz \sin^2 z = dm \sin z \tan L + \sin z \cos z + dz \cos^2 z)$$

$$dz/dm = -\sin z \tan L. \quad \text{(c)}$$

From (b), $d^2L/dm^2 = -\sin^2 z \tan L$ (d)

$$d^3L/dm^3 = d(-\sin^2 z \tan L)/dm$$

$$= -2 \sin z \cos z \tan L(dz/dm) - \sin^2 z \sec^2 L(dL/dm)$$

$$= 2 \sin^2 z \cos z \tan^2 L + \sin^2 z \cos z(1 + \tan^2 L) \text{ by (a) and (c)}$$

$$= \sin^2 z \cos z(1 + 3 \tan^2 L)$$

substituting in 33],

$$L' - L = -m \cos z - (m^2/2)\sin^2 z \tan L + (m^3/6)\sin^2 z \cos z(1+3\tan^2 L) \quad (63)$$

where $L'-L$ and m are in π measure.

For radius N, $m = K/N$ and,

$$L' - L = -(K/N)\cos z - (K^2/2N^2)\sin^2 z \tan L + (k^3/6N^3)\sin^2 z \cos z(1+3\tan^2 L)$$

If the center of the sphere is taken at Ha it will be tangent to the ellipsoid at A so that L will be the same for both, as also k and z. The linear difference in latitude will therefore be the same for each surface, i.e., $(L'-L)N = \Delta L \sin 1'', R_m$, or $\Delta L = (L'-L)N/R_m \sin 1''$ (e), where ΔL = difference in latitude in seconds for the ellipsoid, and R_m is for the middle latitude.

Substituting,

$$-\Delta L = (K/R_m \sin 1'')\cos z + (K^2/2NR_m \sin 1'')\sin^2 z \tan L - (k^3/6N^2R_m \sin 1'') \times \sin^2 z \cos z(1 + 3\tan^2 L) \quad (64)$$

It is inconvenient to look out R_m for the middle latitude which is at first unknown. If R_L is used the resulting difference in latitude δL will be changed in inverse ratio to the radius, by (e), i.e.,

$$\Delta L : \delta L :: R_L : R_m \quad \text{or,}$$

$$\Delta L = \delta L(R_L/R_m) = \delta L(1 - (R_L - R_m)/R_m) = \delta L(1 - dR_m/R_m)$$

i.e., the true value can be found by subtracting $\delta L dR_m/R_m$ from the approx- value.

From (45), $R_m = a(1 - e^2)/(1 - e^2 \sin^2 L)^{3/2}$

$dR_m = a(1-e^2)\ 3\ e^2 \sin L \cos L\ dL/(1-e^2\sin^2 L)^{5/2}$

Since dR_m is the change from the starting point to the middle latitude, $dL/\sin 1'' = \delta L/2$. $\therefore \delta L\ dR_m/R_m = 3\ e^2 \sin L \cos L \sin 1''(\delta L)^2/2(1-e^2\sin^2 L)$

Placing $D = 3\ e^2 \sin L \cos L \sin 1''/2(1 - e \sin L)$,

$$\text{The corrective term} = (\delta L)^2 D \qquad (65)$$

If $B = 1/R_m \sin 1''$; $C = \tan L/2N\ R_m \sin 1''$; h = 1st. term of (64).

which reduces the 3rd. to $h\ k^2 \sin^2 z(1 + 3\tan^2 L)/6N^2$

With $E = (1 + 3\tan^2 L)/6N^2$, (64) finally becomes,

$$-\Delta L = k B \cos z + k^2 C \sin^2 z + (\delta L)^2 D - h k^2 E \sin^2 z \qquad (66)$$

B, C, D and E are given in Table IV, the unit being the meter.

For secondary triangulation the 4th. term can usually be omitted.

98. DIFFERENCE IN LONGITUDE. By Formula 26,

$$\sin \Delta M = \sin m \sin z / \cos L'$$

Referring to a sphere tangent at B, its center at H_2, z, L', k and ΔM are the same as for the ellipsoid, while $m = k/N'$.

$$\sin \Delta M = k \sin z/N' . \cos L' \qquad (67)$$

It is more convenient to assume

$$\Delta M = AK \sin z/ \cos L' \qquad (68)$$

where $A = 1/N' \sin 1''$, and correct for the difference between arc and sine.

Formula 13], $\sin x = x - x^3/6 \ldots = x\ (1 - x^2/6)$

Formula 37], $\log x - \log \sin x = M\ x^2/6$, where M = modulus of the common system of logs.

$$\log(\log x - \log \sin x) = \log(M\ x''^2/6 \sin^2 1'') = 8.2308 + 2 \log x$$

$$\therefore \quad \text{for } \Delta M, \ \log(\log \text{difference}) = 8.2308 + 2 \log \Delta M'' \qquad (69)$$

For $m = k/N' \sin 1''$, using an average value (8.5090) for $\log 1/N' . \sin 1''$ or $\log A$,

$$\log(\log \text{difference}) = 8.2308 + 2 \log k + 2 \log A = 5.2438 + 2 \log k \quad (70)$$

Placing $8.2308 + 2 \log \Delta M'' = 5.2438 + 2 \log k$

$$\log k - \log \Delta M'' = 1.4910 \text{ for the same log difference} \qquad (71)$$

The correction for log K is − and for log ΔM = +. The values are given in Table VIII.

99. CONVERGENCE OF MERIDIANS. Formula 23],

$$\tan (A + B)/2 = \cot (C/2) \cos ((a - b)/2)\ / \cos((a + b)/2))$$

Substituting, Fig. 74,

$$\cot (\Delta z/2) = \cot(\Delta M/2) \cos((L - L')/2)/(\sin((L + L')/2))$$

$$\text{or,} \quad \tan (\Delta z/2) = \tan(\Delta M/2) \sin((L + L')/2)/(\cos((L - L')/2)) \qquad (72)$$

Formulas 12] and 15],

$$\Delta z/2 = \tan(\Delta z/2) - (1/3)\tan^3(\Delta z/2); \ \tan(\Delta M/2) = (\Delta M/2) + (\Delta M/2)^3/3$$

substituting in (72),

$$+\Delta z = \Delta M \sin L_m / \cos \Delta L + (2/3)(\Delta M/2)^3 (\sin L_m \cos \Delta L - \sin^3 L_m/\cos^3 \Delta L)$$

or with Δz and ΔM in seconds, with $\cos \Delta L = 1$ in the corrective terms,

$$\left.\begin{aligned} -\Delta z'' &= \Delta M'' \sin L_m / \cos \Delta L + (1/12)(\Delta M'')^3 \sin L_m \cos^2 L_m \sin^2 1'' \\ &= \Delta M'' \sin L_m / \cos \Delta L + (\Delta M'')^3\ F \end{aligned}\right\} (73)$$

where $F = (1/12) \sin L_m \cos^2 L_m \sin^2 1''$, tabulated in Table IV.

The inverse azimuth,

$$z' = 180° + z - \Delta z \qquad (74)$$

For forms of computation see U.S.C. & G. Report 1894, p. 287. The adjusted spherical angles must be taken and not the plane ones used in computing the triangle sides. For each triangle, starting from the known side, the latitude and longitude of the required point must be the same computed from each of the two sides, while the inverse azimuths of these two sides must differ by the third angle, thus checking the work.

100. POLYCONIC MAP PROJECTION. This projection is the one most generally used in platting geodetic and topographic surveys. It supposes each parallel of latitude to be developed upon its own cone, the vertex of which is on the axis at its intersection with the tangent to the meridian at the parallel.

The side of the tangent cone, or radius of the developed parallel, Fig.75,

$$r = N \cot L \qquad (75)$$

If an arc of the parallel subtend the angle ΔM before development, and after development, θ

$$\theta = \Delta M\, R_p/r = \Delta M\, N \cos L/N \cot L = \Delta M \sin L \qquad (76)$$

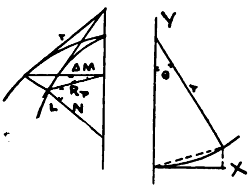


Fig. 75.

The radii of the developed parallels are so great that the parallels are plotted by coordinates.

$$x = r \sin\theta = N \cot L \sin(\Delta M \sin L)$$
$$y = x \tan\theta/2 = x \tan(\sin L\, \Delta M/2) \qquad (77)$$

In platting, a central meridian is drawn as a straight line upon the map, and the true distances between parallels are laid off from Table IX. Perpendiculars, by describing arcs with a compass, are carefully drawn, through these points for the X-axes of the parallels. The x coordinates are then laid off on each for the different longitudes (77) from the Table. Perpendiculars are drawn through these points and the y coordinates laid off from the Table. The meridians join the points of the same longitude, and the parallels those of the same latitude.

A glance at Fig.75 will show that, starting from the pole where the radius of the developed parallel is zero, the radius increases more rapidly than the distance from the pole, becoming infinity at the equator; the developed parallels will then not be concentric circles but the distances between them will increase with the longitude from the central meridian; distances in latitude will then be stretched out as we leave the central meridian, distorting the map since the longitude scale is constant.

The triangulation stations must then be plotted by latitude and longitude, interpolating between the nearest meridians and parallels, and using the triangle sides for checks only.

101. MERCATOR MAP PROJECTION. This projection is used by navigators on account of the facility in obtaining directions for constant bearing sailing. A tangent cylinder is drawn at the equator; the meridional planes are produced to meet the cylinder in elements, and the cylinder is then developed. The meridians thus become parallel straight lines at distances apart equal to the true distances at the equator. This enlarges the scale in longitude in the ratio a/R_p.

To preserve local bearings the latitude scale must be increased in the same ratio; the loxodrome or curve of constant bearing at sea thus becomes a straight line with the same bearing on the map.

To find a sailing course between any two points, the navigator joins them with a straight line on the map, measures the angle made with a meridian and allows for the magnetic variation.

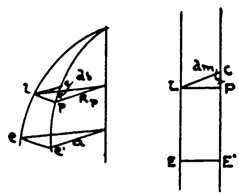


Fig. 76.

In the differential triangles LCP, lcp,

$$dm/ds = LP/lp = ee'/lp = a/R_p$$

Substituting for $ds = R_m\, dL$ and for $R_p = N \cos L$

$$dm = (aR_m/N \cos L)dL \qquad (77)$$

Substituting the values of R_m and N and integrating between the limits L_1 and L_2 will give the distance on the map between the corresponding parallels.

102. LOCATION OF GREAT ARCS. If the two extremities of the line are given, the latitude and longitude of each is accurately determined by observation. The azimuth and length of the line can then be found by (64) and (68) retaining only two terms of (64) thus,

$$\Delta L = -k \cos z/R_m \sin 1'' - k^2 \sin^2 z \tan L/2NR_m \sin 1''; \quad \Delta M = k \sin z/N' \cos L' \sin 1''$$

solving for $k \sin z$, $k \sin z = \Delta M N' \cos L' \sin 1''$

substituting for $k^2 \sin^2 z$ and solving for $k \cos z$,

$$k \cos z = -R_m \Delta L \sin 1'' - N'^2 \Delta M^2 \tan L \cos^2 L \sin^2 1''/N2$$

$$\cot z = -R_m \Delta L/N' \Delta M \cos L' - \Delta M N' \tan L \cos L' \sin 1''/2N$$
$$k = N' \Delta M \cos L' \sin 1''/\sin z \qquad (78)$$

If k is large it may be necessary to employ several triangles in locating it, or to test the direction by an observed azimuth at an intermediate point.

103. LOCATION OF PARALLELS. First to find any point A of the parallel, a station A′ as near the parallel as may be is occupied and its latitude determined; the difference between it and that of the parallel gives the arc dL to move either north or south on the meridian to reach the parallel, or in distance at sea level,

$$k = R_m \, dL'' \sin 1'' \qquad (79)$$

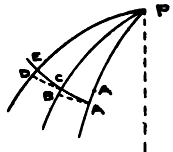


Fig.77.

If dL is large the latitude of A should be determined by a new set of observations on account of the danger of station error.

Having one point A, the parallel can be determined by offsets from the prime vertical AB.

[In the Δ PAB, Formula 25].

$\tan m = \tan \Delta M \cos L$. Formulas 17] and 15],

$$m = \tan \Delta M \cos L - (1/3)\tan^3 \Delta M \cos^3 L = \Delta M \cos L + (1/3)(\Delta M \cos L)^3 \tan^2 L$$

$$k = mN = N \sin 1'' \Delta M'' \cos L + (1/3)N(\sin 1'' \Delta M \cos L)^3 \tan^2 L \qquad (80)$$

Placing $z = 90°$ in (64) for the prime vertical,

$$-\Delta L'' = k^2 \tan L / 2N R_m \sin 1''$$

$$\therefore \quad BC = -\Delta L'' R_m \sin 1'' = k^2 \tan L / 2N \qquad (81)$$

Since BC varies as k^2, if AB, or k, be divided into n equal parts, the ordinates to the parallel will be

$$(1/n)^2 BC, \quad (2/n)^2 BC, \quad (3/n)^2 BC, \cdots \qquad (82)$$

The direction angle, $PBA = 90° - \Delta z \qquad (83)$

while those of the n − 1 ordinates, assuming k to increase proportionately to ΔM, will be,

$$90° - \Delta z/n, \quad 90° - 2\Delta z/n, \quad 90° - 3\Delta z/n \cdots \qquad (84)$$

If the parallel to be located is long ΔM should be divided into sections, and each one located from a new prime vertical to avoid long offsets.

Errors of direction may be prevented from accumulating, and station errors may be detected, by observations for azimuth and latitude at the beginning of each new prime vertical.

In locating the 49th. parallel west of the Lake of the Woods, (U.S. Northern Boundary Survey, Washington, 1878) astronomical observations for latitude and azimuth were taken at points about 20 miles apart, and the prime verticals were ranged through with transits. Each offset was made up of ; the reduction from the prime vertical to the parallel, increasing as the square of the distance from the astronomical station to the parallel, constant between stations; the difference betweet the observed, and computed latitude of the closing point made up of the station and observing errors in latitude and azimuth, and the aligning error, and taken proportional to the distance. The probable error in the position of a latitude station was about 4 feet, and in prolongong a 20-mile line, about 10 seconds.

Example 1. Required the data for locating the 42nd. parallel between N.Y. and Pa. from the Delaware River (approx. longitude 1° 30′ E) to the west end of the state (approx. longitude 2° 54′ W) total distance 4° 24′ longitude.

Dividing into three equal parts, we have $\Delta M = 1° 28'$

$5280'' = \Delta M$ $\qquad$ $L = 42°$

Log sin 1″	4.6855749	log(sin 1″ ΔM cos L)³	4.83785
ΔM	3.7226339	N	6.80536
cos L	9.8710735	1/3	9.52288
	8.2792823	tan² L	9.90887
N	6.8053577	2nd term= 11.9	1.07496
1st term=121517.8	5.0846400	121517.8	
		k = 121529.7 meters.	
k²	10.1693648	ΔM	3.72263
tan L	9.9544374	sin L	9.82557
	10.1238022	3533″.0	3.54820
2N	7.1063877	− 58′ 53″	
CB = 1040.9m	3.0174145	90°	
		z = 89° 01′ 07″	

The ordinates and direction angles for intermediate points can be found by (92) and (94).

104. PARALLELS BY SOLAR COMPASS. If $\Delta L = 0$ in (78)

$$\cot z = -(1/2)\Delta M \tan L \cos L \sin 1''; \quad k = N\Delta M \cos L \sin 1''$$

Substituting,

$$\cot z = -k \tan L/2N = (\Delta z/2) \tan 1'' \qquad (85)$$

The first instrument point being upon the parallel, the solar will give the meridian, from which z can be turned off and the next instrument point placed upon the parallel; etc.

The difference in length, d, between the north and south lines of a township will be the distance k′ between them into the convergence in seconds, times tan 1″.

$$d = k' \Delta z \tan 1'' \qquad (86)$$

For long distances the difference should be found by computing the arc of the parallel for each latitude and subtracting.

105. RECTANGULAR SPHERICAL COORDINATES. In Europe the positions of triangulation points have been found more convenient for use by local surveyors when expressed as coordinates than as latitudes and longitudes. In the rectangular system the meridian for the survey is drawn through the origin O and a great circle ⊥ to it through the required point A. The coordinates of A are x and y, and of B, x′ and y′, positive to the north and east.

The bearing or direction angle α is the angle made, not with the meridian through A, but with the arc AP parallel with the initial meridian (the parallel arc AP being ⊥ to the great circle through the poles QQ′).

To find the coordinates and direction angle at B from those at A. In the triangle A B Q the 3 sides are known as also the angles at $Q (= (x'-x)/R)$ and $A (= 90 - \alpha)$.

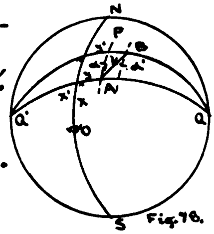


Fig. 78.

$$\therefore \text{ for } y', \text{ Form. 27], } \cos BQ = \cos AB \cos AQ + \sin AB \sin AQ \cos A \qquad (87)$$

$$\sin(y'/R) = \cos(k/R)\sin(y/R) + \sin(k/R)\cos(y/R)\sin\alpha_A$$

$$\text{For } x', \text{ Form. 28], } \sin Q = \sin AB \sin A/\sin BQ$$

$$\sin((x'-x)/R) = \sin(k/R)\cos\alpha/\cos(y'/R) \qquad (88)$$

$$\text{For } \alpha', \text{ Form. 28], } \tan((A+B)/2) = \cot(Q/2)\cos((AQ - BQ)/2)/\cos((AQ+BQ)/2)$$

$$\cot((\alpha-\alpha')/2) = \cot((x'-x)/2R)\cos((y'-y)/2R)/\sin((y'+y)/2R)$$

$$\tan((\alpha-\alpha')/2) = \tan((x'-x)/2R)\sin((y'+y)/2R)/\cos((y'-y)/2R) \qquad (89)$$

Replacing the functions of the small angles by the developments in series, (87) becomes,

$$y' - y'^3/6R^2 = (1-k^2/2R^2)(y - y^3/6R^2) + (k - k^3/6R^2)(1 - y^2/2R^2)\sin\alpha$$

$$= y(1 - k^2/2R^2 - y^2/6R^2) + k\sin\alpha(1 - k^2/6R^2 - y^2/2R^2)$$

since y'^3 has a large divisor, the approximate value, $y + k\sin\alpha$, found by neglecting all terms containing $1/R^2$ can be used, giving,

$$y' - (y + k\sin\alpha)^3/6R^2 = y + k\sin\alpha + y(-k^2/2R^2 - y^2/6R^2) + k\sin\alpha(-k^2/6R^2 - y^2/2R^2)$$

$$y' = y + k\sin\alpha - (3k^2y - 3k^2y\sin^2\alpha + k^3\sin\alpha - k^3\sin^3\alpha)/6R^2$$

$$y' = y + k\sin\alpha - (k^2y\cos^2\alpha)/2R^2 - (k^3\sin\alpha\cos^2\alpha)/6R^2 \qquad (90)$$

From (88),

$$(x'-x) - (x'-x)^3/6R^2 = (k - k^3/6R^2)\cos\alpha/(1 - y'^2/2R^2) = k\cos\alpha(1 - k^2/6R^2 + y'^2/2R^2)$$

For a first approximation, $x' - x = k\cos\alpha$

substituting,

$$x' - x = (k\cos\alpha)^3/6R^2 + k\cos\alpha - k^3\cos\alpha/6R^2 + k\,y'^2\cos\alpha/2R^2$$

$$\text{or, } x' = x + k\cos\alpha + ky'^2\cos\alpha/2R^2 - k^3\cos\alpha\sin^2\alpha/6R^2 \qquad (91)$$

$$\text{From (89), } (\alpha - \alpha') = (x'-x)(y'+y)/2R^2 \qquad (92)$$

substituting for y', $y + k\sin\alpha$

$$\alpha - \alpha' = (x'-x)y/R^2 + (x'-x)k\sin\alpha/2R^2 \qquad (93)$$

If $k\sin\alpha = n$, and $k\cos\alpha = m$, (90), (91) and (92) become,

$$\left.\begin{aligned} y' &= y + n - m^2y/2R^2 - m^2n/6R^2 \\ x' &= x + m + my'^2/2R^2 - mn^2/6R^2 \\ \alpha - \alpha' &= my/R^2\sin 1'' + mn/2R^2\sin 1'', \text{ or } = m(y+y')/2R^2\sin 1'' \end{aligned}\right\} \qquad (94)$$

For R^2 use $N\,R_m$ from Table III for the given latitude.
The terms containing $1/R^2$ in the values of y' and x' are the small corrections to the values which would be found for plane coordinates.

106. MAPPING SPHERICAL COORDINATES. In mapping, the ordinates y are laid off ⊥ to the central meridian, which enlarges the latitude scale away from the meridian.

From (90), $k\sin\alpha = (y'-y) + (x'-x)^2 y/2R^2 + (x'-x)^2(y'-y)/6R^2$

From (91), $k\cos\alpha = (x'-x) - (x'-x)y'^2/2R^2 + (x'-x)(y'-y)/6R^2$

Squaring and adding,

$$k^2 = ((y'-y) + (x'-x)^2 y/2R^2 + (x'-x)^2(y'-y)/6R^2)^2$$
$$+ ((x'-x) - (x'-x)y'^2/2R^2 + (x'-x)(y'-y)^2/6R^2)^2$$
$$= k_0^2 + (x'-x)^2(y'-y)y/R^2 + (x'-x)^2(y'-y)^2/3R^2$$
$$- (x'-x)^2 y'^2/R^2 + (x'-x)^2(y'-y)^2/3R^2$$
$$= k_0^2 + ((x'-x)^2/3R^2)(3y(y'-y) + 2(y'-y)^2 - 3y'^2)$$
$$= k_0^2 - ((x'-x)^2/3R^2)(y^2 + yy' + y'^2)$$
$$= k_0^2(1 - (\cos^2\alpha/3R^2)(y^2 + yy' + y'^2))$$
$$k = k_0(1 - (\cos^2\alpha/6R^2)(y^2 + yy' + y'^2)) \qquad (95)$$

where k_0 is the value for plane coordinates.
Putting the map magnification = G,

$$G = k_0/k = 1 + (y^2 + yy' + y'^2)\cos^2\alpha/6R^2 \qquad (96)$$

For short lines $y = y'$ nearly, giving

$$G = 1 + y^2\cos^2\alpha/2R^2 \qquad (97)$$

This becomes unity for $\alpha = 90°$, the map giving true differences of longitude, and a maximum of

$$G = 1 + y^2/2R^2 \text{ for } \alpha = 0$$

CHAPTER IX.

DETERMINATION OF THE DIMENSIONS OF THE ELLIPSOID.

107. THE MERIDIAN FROM TWO LATITUDE DEGREE MEASUREMENTS. These arcs may be on the same meridian, or on different ones if the earth is assumed to be an ellipsoid of rotation. The arc s is measured, as also the latitudes of its extremities for each case. If

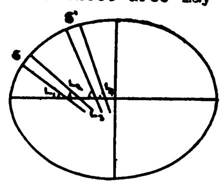


Fig. 79.

$L_2 - L_1 = \Delta L$; $(L_2 + L_1)/2 = L$; $L_4 - L_3 = \Delta L'$; $(L_4 + L_3)/2 = L'$;

$$s = \Delta L\,R_m \sin 1''; \quad s' = \Delta L'\,R'_m \sin 1'' \qquad (98)$$

Dividing by (45),

$$(s\,\Delta L'/s'\,\Delta L)^{2/3} = (1 - e^2\sin^2 L')/(1 - e^2\sin^2 L) = q^2$$
$$\therefore \quad e^2 = (1 - q^2)/(\sin^2 L' - q^2\sin^2 L) \qquad (99)$$

Since $R_m = a(1 - e^2)/(1 - e^2\sin^2 L)^{3/2}$, $R_m = c((1 - e^2)^{3/2}/(1 - e^2\sin^2 L)^{3/2})$

Substituting in (98),

$$\left.\begin{aligned} c &= s(1 - e^2\sin^2 L)^{3/2}/\Delta L \sin 1''(1 - e^2)^{3/2} \\ c &= s'(1 - e^2\sin^2 L')^{3/2}/\Delta L' \sin 1''(1 - e^2)^{3/2} \end{aligned}\right\} \qquad (100)$$

Semi-minor axis, $b = c(1 - e^2)$; Semi-major axis, $a = c\sqrt{1 - e^2}$ (101)

The entire quadrant can be found from (55) if desired.

108. REDUCTION OF A MEASURED ARC TO THE MERIDIAN. The arc is supposed to make only a small angle with the meridian.
From (64),

$$s = \Delta L\,R_m \sin 1'' = -k\cos z - (k^2\sin^2 z\tan L)/2N$$
$$+ k^3\sin^2 z\cos z\,(1 + 3\tan^2 L)/6N^2 \qquad (102)$$

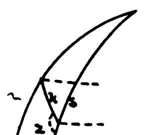


Fig. 80.

The second term of the second member is small so that an approximate value can be used for N.
For a chain of triangles, this equation can be applied to side after side until the whole length of the chain has been projected.

109. THE MERIDIAN FROM SEVERAL LATITUDE DEGREE MEASUREMENTS. This involves the formation of observation equations between the observed latitudes and the projected, or directly measured, meridional arcs. The simplest relation is (98) which can be used for a ΔL of several degrees on account of the probable error of a latitude determination, some 0.04″ or 4 feet, aside from the station error.
For longer arcs a correction for (98) will be required.

From (54), $ds = R_m dL = a(1 - e^2)dL/(1 - e^2\sin^2 L)^{3/2}$

$= a(1 - e^2)dL(1 + (3/2)e^2\sin^2 L)$, neglecting terms above e^2

But $\sin^2 L = 1/2 - (1/2)\cos 2L$

$\therefore \quad ds = a(1 - e^2)dL(1 + (3/4)e^2 - (3/4)e^2\cos 2L)$

$s = a(1 - e^2)((1 + (3/4)e^2)(L'' - L') - (3/8)e^2(\sin 2L'' - \sin 2L'))$

Form. 8), $\quad s = a(1 - e^2)((1 + (3/4)e^2)\Delta L - (3/4)e^2\sin\Delta L\cos 2L)$

For $\sin\Delta L$ use $\Delta L - (\Delta L)^3/6$,

$$s = a\Delta L(1 - e^2)(1 + (3/4)e^2 - (3/4)e^2\cos 2L + (1/8)e^2(\Delta L)^2\cos 2L) \qquad (103)$$

Expanding R_m, $\quad R_m = a(1 - e^2)(1 + (3/2)e^2\sin^2 L)$

$= a(1 - e^2)(1 + (3/4)e^2 - (3/4)e^2\cos 2L)$

$\therefore$ the approximate value by (98) for s_1,

$s_1 = \Delta L\, R_m = a\Delta L(1 - e^2)(1 + (3/4)e^2 - (3/4)e^2\cos 2L)$

Subtracting this from the true value (103) will give the correction δs to apply to the approximate value, or

$s - s_1 = \delta s = a\,\Delta L(1 - e^2)(e^2/8)(\Delta L)^2\cos 2L$, or ΔL in seconds,

$$\delta s = a(e^2/8)(\Delta L/\sin 1'')^3\cos 2L \qquad (104)$$

The correction for $\Delta L = 1°$ reduces to $-0^m.028$ in latitude 0°; $-0^m.014$ for L = 30°; $0^m.000$ for L = 45°; $+0^m.014$ for L = 60°; $+0^m.028$ for L = 90°.

Jordan gives the following data:

Latitude Degree Measurements in Europe

	Station	Latitude L	ΔL	Meridian arc s
	Formentera	L_1 = 38° 39′ 56.1″		
	Barcelona	L_2 = 41 22 47.9	2° 42′ 51.8″	301 354
French	Carcassonne	L_3 = 43 12 54.3	4 32 58.2	505 137
	Pantheon	L_4 = 48 50 49.4	10 10 53.3	1 131 050
	Dunkirk	L_5 = 51 2 8.8	12 22 12.7	1 374 572
	Dunnose	L_6 = 50 37 7.6		
	Greenwich	L_7 = 51 28 39.0	0 51 31.4	95 620
English	Arburyhill	L_8 = 52 13 28.0	1 36 20.4	178 720
	Clifton	L_9 = 53 27 31.1	2 50 23.5	315 892
	Göttingen	L_{10} = 51 18 47.8		
Hanover.	Altona	L_{11} = 53 32 45.8	2 0 57.5	224 458
	Trunz	L_{12} = 54 13 11.5		
Prussian	Königsberg	L_{13} = 54 42 50.5	0 29 30.0	54 985
	Memel	L_{14} = 55 43 40.4	1 30 28.9	187 982
	Belin	L_{15} = 52 2 40.9		
	Jakobstadt	L_{16} = 56 30 4.6	4 27 23.7	498 114
Russian	Dorpat	L_{17} = 58 22 47.3	6 20 6.4	705 209
	Hochland	L_{18} = 60 5 9.8	8 2 28.9	895 315
	Malörn	L_{19} = 65 31 30.3		
Swedish	Pahtawara	L_{20} = 67 8 49.8	1 37 19.5	180 828

The first 2 latitudes are connected by the equation,

$$L_2 - L_1 = s/R_m\sin 1'' - \delta s/R_m\sin 1'' \qquad (a)$$

where $\quad 1/R_m = (1 - e^2\sin^2 L)^{3/2}/a(1 - e^2) \qquad (b)$

Since a and e^2 are unknown, or required quantities, we substitute for them approximate values with corrections, $a = a_0 + \delta a \quad e^2 = e_0^2 + \delta e^2$ and expand by Maclaurin's theorem

$$1/R_m = 1/R_0 + (d(1/R_m)/d(\delta a))\delta a + (d(1/R_m)/d(\delta e^2))\delta e^2 \qquad (c)$$

But $\quad d(1/R_m)/d\delta a = ((1 - e^2\sin^2 L)^{3/2}/(1 - e^2))(-1/a_0^2)$, $= -1/a_0^2$ by neglecting all terms containing e^2.

$d(1/R_m)/d(\delta e^2) = (d(1/R_m)/d\delta e)(d(\delta e)/d(\delta e^2))$

$= (1/a)(((1-e_0^2)(-3e_0\sin^2 L) + (1-e_0^2\sin^2 L)^{3/2}2e_0)/(1-e_0^2)^2\,2e_0)$

$= (1/a)(1 - (3/2)\sin^2 L)$, by neglecting e^2 terms.

Substituting in (c),

$1/R_m = 1/R_0 - (1/a_0^2)\delta a + (1 - (3/2)\sin^2 L)\delta e^2/a_0$

$\therefore$ (a) becomes

$L_2 = L_1 + s/R_0\sin 1'' - s(1/a_0^2\sin 1'')\delta a$

$s(1 - (3/2)\sin^2 L)\delta e^2/a_0\sin 1'' - \delta s/R_0\sin 1''$

The value of δs is given in (104).

Considering the meridional arcs perfect or constants in comparison with the observed latitudes, with corrections v, affected by station errors the observation equations, (21) Part I, become, with

$$L_1 + v_1 = V_1;\ L_2 + v_2 = V_2 \ldots,$$

$$V_1 - L_1 = v_1 \qquad\qquad V_2 - L_2 = v_2$$

But $V_2 = V_1 + s/R_0 \sin 1'' - s(1/a_0^2 \sin 1'')\delta a + s(1 - (3/2)\sin^2 L)(1/a_0 \sin 1'')\delta e^2 - \delta s/R_0 \sin 1''$

Substituting,

$$v_1 + L_1 - L_2 + s/R_0 \sin 1'' - s(1/a_0^2 \sin 1'')\delta a + s(1 - (3/2)\sin^2 L)(1/a_0 \sin 1'')\delta e^2 - \delta s/R_0 \sin 1'' = v_2 \quad (105)$$

I.e., $v_1 + a_2 x + b_2 y + l_2 = v_2$

where $1000\, s(1/a_0^2 \sin 1'') = a_2$; $\delta a/1000 = x$; $s(1 - (3/2)\sin^2 L)(1/1000 a_0 \sin 1'') = b_2$; $1000\delta e^2 = y$; $L_1 - L_2 + s/R_0 \sin 1'' - a_0 e_0^2 (\Delta L)^3 \cos 2L / 8R_0 \sin^4 1'' = l_2$ (d)

Equations (22) Part I, thus become, as given by Jordan with a_0 = 6 377 397.2; $\log a_0$ = 6.804 6435; e_0^2 = 0.006 674 372; $\log e_0^2$ = 7.824 4104; R_0 = the corresponding value of R_m for the different latitudes by (b).

$$
\begin{array}{llllll}
v_1 & & & & & = v_1 \\
v_1 & - 1.53x & + 3.71y & - 0.2'' & & = v_2 \\
v_1 & - 2.57x & + 5.83y & - 1.4 & & = v_3 \\
v_1 & - 5.75x & + 10.36y & - 2.1 & & = v_4 \\
v_1 & - 6.98x & + 11.31y & + 1.2 & & = v_5 \\
v_6 & & & & & = v_6 \\
v_6 & - 0.48x & + 0.29y & + 3.2 & & = v_7 \\
v_6 & - 0.91x & + 0.48y & + 3.2 & & = v_8 \\
v_6 & - 1.80x & + 0.69y & - 1.9 & & = v_9 \\
v_{10} & & & & & = v_{10} \\
v_{10} & - 1.14x & + 0.40y & + 5.0 & & = v_{11} \\
v_{12} & & & & & = v_{12} \\
v_{12} & - 0.28x & + 0.01y & - 0.5 & & = v_{13} \\
v_{12} & - 0.85x & - 0.03y & + 3.3 & & = v_{14} \\
v_{15} & & & & & = v_{15} \\
v_{15} & - 2.52x & + 0.19y & + 3.8 & & = v_{16} \\
v_{15} & - 3.58x & - 0.27y & + 0.7 & & = v_{17} \\
v_{15} & - 4.54x & - 0.94y & + 2.3 & & = v_{18} \\
v_{19} & & & & & = v_{19} \\
v_{19} & - 0.92x & - 1.51y & - 1.1 & & = v_{20}
\end{array}
\qquad (e)
$$

Forming the normal equations as usual,

$$
\begin{array}{l}
+5v_1 - 18.83x + 31.21y - 2.50 = 0 \\
+4v_6 - 2.99x + 1.46y + 4.50 = 0 \\
+2v_{10} - 1.14x + 0.40y + 5.00 = 0 \\
+3v_{12} - 1.13x - 0.02y + 2.80 = 0 \\
+4v_{15} - 10.64x - 1.03y + 8.60 = 0 \\
+2v_{19} - 0.92x - 1.51y - 1.10 = 0 \\
-18.83v_1 - 2.99v_6 - 1.14v_{10} - 1.13v_{12} - 10.64v_{15} - 0.92v_{19} + 137.07x - 155.11y - 23.18 = 0 \\
+31.21v_1 + 1.46v_6 + 0.40v_{10} - 0.02v_{12} - 1.03v_{15} - 1.51v_{19} - 155.11x + 287.21y - 14.03 = 0
\end{array}
$$

Expressing the v in each of the first 6 equations in terms of the other quantities and substituting in the last 2, we find $x = +0.4023$ $y = +0.2347$. Substituting in (d),

$$\delta a = 1000\,x = +402.^m3 \qquad \delta e^2 = 0.001\,y = +0.000\,2347$$

$a = a_0 + \delta a$ = 6 377 397.2 + 402.3 = 6 377 800 $e^2 = e_0^2 + \delta e^2$ = 0.006 6744 + 0.000 2347 = 0.006 9091.

Substituting the values of x and y in the observation equations (e) the v's are readily found, from which $[v^2] = 52$.

∴ (31), Part I, $\epsilon = \sqrt{[v^2]/(n-m)} = \sqrt{52/12} = 2.''1$ for the m.s.e. of a latitude determination referred to the ellipsoid. This is very much greater than the m.s.e. of a latitude determination showing that an ellipsoid of revolution will not fit the data without large station errors or local deviations of the plumb line.

The v's for each group, i.e., French, English, etc. foot up zero within 0.''01.

110. THE ELLIPSOID FROM A DEGREE MEASUREMENT OBLIQUE TO THE MERIDIAN. The latitude and azimuth are observed at each end of the line, as also the difference in longitude and the distance.

Each observation would give an equation of the form

$f(X, Y, Z, \quad) - M_1 = v_1$ where the required quantities are the most probable values for the observed $L_1, L_2, \Delta M, Z_1, Z_2$, k, and ρ and e^2 for the ellipsoid. Denoting the corrections to the observed or assumed values by δ, we have for the initial latitude

$$f_1(L_1 + \delta L_1) - L_1 = v_1, \quad \text{or } \delta L_1 + 0 = v_1 \qquad (a)$$

For L_2, $f_2(L_1+\delta L_1, c_0+\delta c, e_0'^2+\delta e'^2) - L_2 = v_2$

$$f_2(L_1, c_0, e_0'^2) - L_2 + (df_2/dL_1)\delta L_1 + (df_2/dc)\delta c + (df_2/de'^2)\delta e'^2 = v_2 \quad (b)$$

The quantity $f_2(L_1, c_0, e_0'^2)$, the computed value of L_2, can be found by (64). Place this computed value less $L_2 = l_2$. For the differential coefficients only the first term of the second member of (64) need be used, i.e.,

$$f_2(L_1+\delta L_1, c_0+\delta c, e_0'^2+\delta e_0'^2) = L_1 - k\cos z/R_m$$
$$= L_1 - k\cos z_1 V^3/c$$

$(df_2/dL_1)\delta L_1 = \delta L_1$ $(df_2/dc)\delta c = (k\cos V^3/c^2)\delta c$; $(df_2/de'^2)\delta e'^2 = (-3k/2c)\cos z_1 \times \cos^2 L_1 V_1 \delta e'^2$

Collecting results,

$$\delta L_1 + (k\cos z_1 V_1^3/c_0^2)\delta c - (3k/2c_0)\cos z_1 \cos^2 L_1)\delta e'^2 + l_2 = v_2 \quad (c)$$

For ΔM, Place l_3 = computed value by (68) less the observed value, while for the differential formula use $k\sin z/N_2'\cos L_2$, i.e.,

$$f_3(c_0+\delta c, e_0'^2+\delta e_0'^2) = k\sin z/N'\cos L' = k\sin z_1 V_1'/c\cos L_2$$

$(df_3/dc)\delta c = -(k\sin z_1 V_1'/c^2\cos L_2)\delta c$ $(df_3/de'^2)\delta e'^2 = (k\sin z_1\cos^2 L_2/2cV')\delta e'^2$

$$\therefore\ -(k\sin z_1 V'/c_0^2\cos L_2)\delta c + (k\sin z_1\cos^2 L_2/2c_0V')\delta e'^2 + l_3 = v_3 \quad (d)$$

For azimuth, $f_4(z_1+\delta z_1) - z_1 = v_4$, or $\delta z_1 + 0 = v_4$ (e)

For z_2, $f_5(z_1+\delta z_1, c_0+\delta c, e_0'^2+\delta e'^2) - z_2 = v_5$.

$$f_5(\ ) \text{ by } (73) = z_1 + 180°'' + \Delta M\sin L_m = z_1 + 180° + k\sin z_1\tan L/N'$$
$$= z_1 + 180° + k\sin z_1\tan L_1 V'/c$$

$(df_5/dz_1)\delta z_1 = \delta z_1$ $(df_5/dc)\delta c = (-k\sin z_1\tan L_1 V'/c^2)\delta c$ $(df_5/de'^2)\delta e'^2 = (k\sin z_1\sin L_1\cos L_1/2cV')\delta e'^2$

$$\therefore\ \delta z_1 - (k\sin z_1\tan L_1 V'/c_0^2)\delta c + (k\sin z_1\sin L_1\cos L_1/2c_0V')\delta e'^2 + l_5 = v_5 \quad (f)$$

Collecting equations (a) to (f) and denoting the coefficients of δc and $\delta e'^2$ by a and b,

$$\begin{array}{llll} \delta L_1 & & l_1 & = v_1 \\ \delta L_1 & a_2\delta c + b_2\delta e'^2 & + l_2 & = v_2 \\ & a_3\delta c + b_3\delta e'^2 & + l_3 & = v_3 \\ \delta z_1 & & l_4 & = v_4 \\ \delta z_1 & a_5\delta c + b_5\delta e'^2 & + l_5 & = v_5. \end{array} \qquad (106)$$

Weights can be introduced if desired.

If k is large or poorly measured so that its m.s.e. is appreciable in comparison with those for L, ΔM, and z, another equation should be added.

$$f_6(c_0+\delta c, e_0'^2+\delta e'^2) - k = v_6$$

From (78), $f_6(\ ) = N'\Delta M\cos L_2/\sin z_1 = c\cos L_2\Delta M/V'_2\sin z_1$

$(df_6/dc)\delta c = (\Delta M\cos L_2/V_2'\sin z_1)\delta c$ $(df_6/de'^2)\delta e'^2 = -(c\,\Delta M\cos^3 L_2/2V'^3\sin z_1)\delta e'^2$

$\therefore$ $a_6\delta c + b_6\delta e'^2 + l_6 = v_6$ is the equation to be added to (106).

If a second line starts from the initial station and its azimuth is computed from the observations which gave z_1, there would be added to (106)

$$\begin{array}{ll} \delta L_1 + a_7\delta c + b_7\delta e'^2 + l_7 = v_7 & \text{from } L_3 \\ a_8\delta c + b_8\delta e'^2 + l_8 = v_8 & \text{"}\ \Delta M_{1\text{-}3} \\ \delta z_1 + a_9\delta c + b_9\delta e'^2 + l_9 = v_9 & \text{"}\ z_3 \\ a_{10}\delta c + b_{10}\, e'^2 + l_{10} = v_{10} & \text{if } k_{1\text{-}3} \text{ is considered.} \end{array}$$

The distance k can be greater than a triangle side by solving for an approximate z by by (78); computing through the chain of triangles with two angles and the included side given each time to find the third angle and the second side; calling the change in direction of k at each intersection 180°. z_1, z_2 and k as found for the total distance can then be corrected for the error in closure at B by adding x to k and dividing y by k sin1″ for the correction to z.

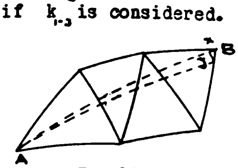


Fig. 81.

For the more general treatment for an astronomical geodetic net, taking into account station error in its effect upon latitude, longitude and azimuth, see Helmert's Höheren Geodäsie.

TABLE I. Formulas and Constants.

$\sin^2 x + \cos^2 x = 1$ 1)

$\tan x = 1/\cot x = \sin x/\cos x = \sqrt{\sec^2 x - 1}$ 2)

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$ 3)

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ 4)

$\tan(x \pm y) = (\tan x \pm \tan y)/(1 \mp \tan x \tan y)$ 5)

$\cos(180° - y) = -\cos y; \quad \sin(180° + y) = -\sin y$ 6)

For small angles, $\sin x = \tan x = x'' \sin 1'' = x''$ arc $1''$ 7)

$\sin x \pm \sin y = 2 \sin((x \pm y)/2) \cos((x \mp y)/2)$ 8)

$\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$ 9)

$\sin 2x = 2 \sin x \cos x$ 10)

$2 \cos^2 x/2 = 1 + \cos x$ 11)

$2 \sin^2 x/2 = 1 - \cos x$ 12)

$\sin x = x - x^3/3! + x^5/5! - x^7/7! + x^9/9! -$ 13)

$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8!$ 14)

$\tan x = x + x^3/3 + 2x^5/15 + 17x^7/315 + 62x^9/2835 + 1382x^{11}/155925$ 15)

$\text{arc} \sin x = x + x^3/3! + 3x^5/40 + 5x^7/112 + 35x^9/1152 + 63x^{11}/2816$ 16)

$\text{arc} \tan x = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - x^{11}/11 +$ 17)

In the last 5 equations x is in π-measure. Should x be given in seconds multiply by sin 1″

Plane Oblique Triangles.

$\sin A / a = \sin B / b = \sin C / c$ 18)

$a^2 = b^2 + c^2 - 2 b c \cos A$ 19)

$\tan((A - B)/2) = ((a - b)/(a + b)) \tan((A + B)/2)$ 20)

Area triangle $= 1/2\, b\, c \sin A$ 21)

Spherical Oblique Triangles.

$\sin A / \sin a = \sin B / \sin b = \sin C / \sin c$ 23)

$\cot B = (\sin c \cot b - \cos c \cos A)/\sin A$ 25)

$\cos a = \cos b \cos c + \sin b \sin c \cos A$ 27)

$\tan((A + B)/2) = \cot(C/2) \cos((a - b)/2)/\cos((a + b)/2)$ 28)

Spherical Right Triangle.

$\tan A \sin b = \tan a$ 25)

Binomial Theorem.

$(a + b)^m = a^m + m a^{m-1} b + (m(m - 1)/2!) a^{m-2} b^2 +$ 32)

Maclaurin's Theorem.

$u' = f(x) = (u)_{x=0} + (du/dx)_{x=0}(x/1!) + (\ldots\ldots + (d^n u/dx^n)(x^n/n!)$ 33)

Taylor's Theorem.

$u' = f(x + y) = u + (du/dx)(y/1!) + (d^2u/dx^2)(y^2/2!) +$
$(d^n u/dx^n)/(y^n/n!)$ 34)

Radius of Curvature.

$R = -(1 + dy^2/dx^2)(dx^2/d^2y)$ 35)

A.R.Clarke of the English Ordnance Survey, gives the following values for the ellipsoid of revolution as found from the various degree measurements. These values were adopted by the U.S.C.& Geodetic Survey in 1875 and the following tables which involve the ellipsoid are based upon this data.

Semi-major axis,	$a = 6378206.^m4$	log	6.8046985
semi-minor axis	$b = 6356583.8$	"	6.803 2238
eccentricity squared	$e^2 = 0.006768658$		7.830 5028

One meter = 39.37 inches (Act of Congress).

The following formulas are in use.

$e = (a^2 - b^2)^{1/2}/a \qquad e' = (a^2 - b^2)^{1/2}/b$

$a = c\sqrt{1 - e^2} = c/\sqrt{1 + e'^2} \qquad b = c(1 - e^2) = c/(1 + e'^2)$

$r^2 = 1 + e^2 \sin^2 L \qquad V^2 = 1 + e'^2 \cos^2 L = r^2/(1 - e^2)$

The following approximate values are given for the coefficients of expansion for 1° F, the unit being 1/1 000 000 of the length.

Glass	4.7	Iron	6.5
Platinum	4.8	Brass	10.2
Steel	6.2	Zinc	16.1

$\log(1+x) = M(x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \quad)$ 36)

$\log(1-x) = -M(x + x^2/2 + x^3/3 + x^4/4 + x^5/5 + \quad)$ 37)

Table II. Corrections for run of the micrometer.

Corrections same sign as r for $m < 2'30''$. Opposite signs for $m > 2'30''$

r	a = 0'						a = 1'						a = 2'				r
d	00"	10"	20"	30"	40"	50"	00"	10"	20"	30"	40"	50"	00"	10"	20"	30"	d
0.1	.05	.05	.04	.04	.04	.03	.03	.03	.02	.02	.02	.01	.01	.01	.00	.00	0.1
0.2	.10	.09	.09	.08	.07	.07	.06	.05	.05	.04	.03	.03	.02	.01	.01	.00	0.2
0.3	.15	.14	.13	.12	.11	.10	.09	.08	.07	.06	.05	.04	.03	.02	.01	.00	0.3
0.4	.20	.19	.17	.16	.15	.13	.12	.11	.09	.08	.07	.05	.04	.03	.01	.00	0.4
0.5	.25	.23	.22	.20	.18	.17	.15	.13	.12	.10	.08	.07	.05	.03	.02	.00	0.5
0.6	.30	.28	.26	.24	.22	.20	.18	.16	.14	.12	.10	.08	.06	.04	.02	.00	0.6
0.7	.35	.33	.30	.28	.26	.23	.21	.19	.16	.14	.12	.09	.07	.04	.02	.00	0.7
0.8	.40	.37	.35	.32	.29	.27	.24	.21	.19	.16	.13	.11	.08	.05	.03	.00	0.8
0.9	.45	.42	.39	.36	.33	.30	.27	.24	.21	.18	.15	.12	.09	.06	.03	.00	0.9
1.0	.50	.47	.43	.40	.37	.33	.30	.27	.23	.20	.17	.13	.10	.07	.03	.00	1.0
1.1	.55	.51	.48	.44	.40	.37	.33	.29	.26	.22	.18	.15	.11	.07	.04	.00	1.1
1.2	.60	.56	.52	.48	.44	.40	.36	.32	.28	.24	.20	.16	.12	.08	.04	.00	1.2
1.3	.65	.61	.56	.52	.48	.43	.39	.35	.30	.26	.22	.17	.13	.09	.04	.00	1.3
1.4	.70	.65	.60	.56	.51	.47	.42	.37	.33	.28	.23	.19	.14	.09	.05	.00	1.4
1.5	.75	.70	.65	.60	.55	.50	.45	.40	.35	.30	.25	.20	.15	.10	.05	.00	1.5
1.6	.80	.75	.69	.64	.59	.53	.48	.43	.37	.32	.27	.21	.16	.11	.05	.00	1.6
1.7	.85	.79	.74	.68	.62	.57	.51	.45	.40	.34	.28	.23	.17	.11	.06	.00	1.7
1.8	.90	.84	.78	.72	.66	.60	.54	.48	.42	.36	.30	.24	.18	.12	.06	.00	1.8
1.9	.95	.89	.82	.76	.70	.63	.57	.51	.44	.38	.32	.25	.19	.13	.06	.00	1.9
2.0	1.00	.93	.87	.80	.73	.67	.60	.53	.47	.40	.33	.27	.20	.13	.07	.00	2.0
2.1	1.05	.98	.91	.84	.77	.70	.63	.56	.49	.42	.35	.28	.21	.14	.07	.00	2.1
2.2	1.10	1.03	.95	.88	.81	.73	.66	.59	.51	.44	.37	.29	.22	.15	.07	.00	2.2
2.3	1.15	1.07	1.00	.92	.84	.77	.69	.61	.54	.46	.38	.31	.23	.15	.08	.00	2.3
2.4	1.20	1.12	1.04	.96	.88	.80	.72	.64	.56	.48	.40	.32	.24	.16	.08	.00	2.4
2.5	1.25	1.17	1.08	1.00	.92	.83	.75	.67	.58	.50	.42	.33	.25	.17	.08	.00	2.5
2.6	130	1.21	1.13	1.04	.95	.87	.78	.69	.61	.52	.43	.35	.26	.17	.09	.00	2.6
2.7	135	1.26	1.17	1.08	.99	.90	.81	.72	.63	.54	.45	.36	.27	.18	.09	.00	2.7
2.8	140	1.31	1.21	1.12	1.03	.93	.84	.75	.65	.56	.47	.37	.28	.19	.09	.00	2.8
2.9	145	135	1.26	1.16	1.06	.97	.87	.77	.68	.58	.48	.39	.29	.19	.10	.00	2.9
3.0	1.50	140	1.30	1.20	1.10	1.00	.90	.80	.70	.60	.50	.40	.30	.20	.10	.00	3.0
r	60"	50"	40"	30"	20"	10"	60"	50"	40"	30"	20"	10"	60"	50"	40"	30"	r
	a = 4'						a = 3'						a = 2'				

Table III. (Metric Units)

Lat.	1° of paral.	1° of Merid.*	Log. N	Log R_M	Lat.	1° of paral.	1° of Merid	Log. N	Log R_M
0° 00'	111321	110567.2	6.8046985	6.8017489	12° 00'	108904	110615.8	6.8047620	6.8019395
30	1316		6986	7492	30	8699		7674	9555
1 00	1304	567.6	6990	7502	13 00	8486	624.1	7729	9720
30	1283		6995	7519	30	8265		7786	9892
2 00	1253	568.6	7003	7543	14 00	8036	633.0	7845	6.8020070
30	1215		7013	7573	30	7798		7906	0253
3 00	1169	570.3	7025	7610	15 00	7553	642.5	7970	0443
30	1114		7040	7653	30	7299		8035	0638
4 00	1051	572.7	7057	7704	16 00	7036	652.6	8102	0839
30	0980		7076	7761	30	6766		8171	1046
5 00	110900	110575.8	6.8047097	6.8017824	17 00	106487	110663.3	6.8048241	6.8021258
30	0812		7120	7894	30	6201		8314	1476
6 00	0715	579.5	7146	7971	18 00	5906	674.5	8389	1701
30	0610		7173	8054	30	5604		8466	1931
7 00	0497	583.9	7203	8144	19 00	5294	686.3	8544	2165
30	0375		7235	8240	30	4975		8624	2404
8 00	0245	589.0	7270	8343	20 00	4649	698.7	8705	2649
30	0106		7306	8452	30	4314		8789	2900
9 00	9959	594.7	7345	8568	21 00	3972	711.6	8874	3155
30	9804		7385	8690	30	3622		8961	3415
10 00	109641	110601.1	6.8047428	6.8018819	22 00	103264	110725.0	6.8049049	6.8023680
30	9469		7473	8954	30	2898		9139	3950
11 00	9289	608.1	7520	9094	23 00	2524	738.8	9231	4225
30	9101		7569	9242	30	2143		9324	4504

* These quantities express the number of meters contained within an arc of which the degree of latitude named is the middle; thus, the quantity 110601.1 opposite latitude 10° is the number of meters between latitude 9°30' and latitude 10°30'.

Table III. (Metric Units)

Lat.	1° of paral.	1° of Merid.	Log N	Log R_M	Lat.	1° of paral.	1° of Merid.	Log N	Log R_M
24 00	101754	110753.2	6.8049418	6.8024788	57 30	59957		6.8057465	6.8048930
30	1357		9514	5077	58 00	9135	111379.5	7582	9279
25 00	100952	110768.0	6.8049612	6.8025370	30	8309		7697	9625
30	0539		9711	5667	59 00	7478	397.2	7811	9968
26 00	0119	783.3	9812	5968	30	6642		7925	6.8050308
30	99692		9914	6274	60 00	5802	414.5	8037	0644
27 00	9257	799.0	6.8050017	6584	30	4958		8148	0977
30	8814		0121	6898	61 00	4110	431.5	8258	1307
28 00	8364	815.1	0227	7215	30	3257		8367	1633
30	7906		0334	7537	62 00	2400	448.2	8474	1956
29 00	7441	831.6	0443	7862	30	1540		8580	2275
30	6968		0552	8190	63 00	0675	464.4	8685	2590
30 00	96488	110848.5	6.8050663	6.8028522	30	49806		6.8058789	6.8052901
30	6001		0775	8858	64 00	8934	111480.3	8891	3208
31 00	5506	865.7	0888	9197	30	8057		8992	3510
30	5004		1002	9538	65 00	7177	495.7	9092	3809
32 00	4495	883.2	1117	9883	30	6294		9190	4103
30	3979		1233	6.8030231	66 00	5407	510.7	9287	4393
33 00	3455	901.1	1350	0582	30	4516		9382	4676
30	2925		1467	0936	67 00	3622	525.3	9475	4959
34 00	2387	919.2	1586	1292	30	2724		9567	5235
30	1842		1706	1651	68 00	1823	539.3	9658	5506
35 00	91290	110937.6	6.8051826	6.8032012	30	40919		6.8059747	6.8055773
30	0731		1947	2376	69 00	0012	111552.9	9834	6034
36 00	0166	956.2	2069	2741	30	39102		9919	6291
30	89593		2192	3109	70 00	8188	565.9	6.8060003	6542
37 00	9014	975.1	2315	3479	30	7272		0085	6789
30	8428		2439	3851	71 00	6353	578.4	0165	7029
38 00	7835	994.1	2564	4224	30	5431		0244	7265
30	7235		2689	4599	72 00	4506	590.4	0321	7495
39 00	6629	111013.3	2814	4976	30	3578		0395	7719
30	6016		2940	5354	73 00	2648	601.8	0468	7938
40 00	85396	111032.7	6.8053067	6.8035734	30	31716		6.8060539	6.8058152
30	4770		3194	6114	74 00	0781	111612.7	0608	8361
41 00	4137	052.2	3321	6496	30	29843		0676	8563
30	3498		3448	6878	75 00	8903	622.9	0742	8759
42 00	2853	071.7	3576	7262	30	7961		0805	8950
30	2201		3704	7646	76 00	7017	632.6	0867	9135
43 00	1543	091.4	3832	8031	30	6071		0927	9314
30	0879		3961	8416	77 00	5123	641.6	0984	9487
44 00	0208	111.1	4089	8802	30	4172		1040	9653
30	79532		4218	9188	78 00	3220	650.0	1093	9814
45 00	78849	111130.9	6.8054347	6.8039574	30	22266		6.8061145	6.8059969
30	8160		4475	9960	79 00	1311	111657.8	1195	6.8060118
46 00	7466	150.6	4604	6.8040346	30	0353		1242	0259
30	6765		4733	0732	80 00	19394	664.9	1287	0394
47 00	6058	170.4	4861	1117	30	8434		1330	0524
30	5346		4990	1502	81 00	7472	671.4	1371	0646
48 00	4628	190.1	5118	1887	30	6509		1410	0763
30	3904		5246	2270	82 00	5545	677.2	1446	0873
49 00	3174	209.7	5373	2653	30	4579		1481	0977
30	2439		5501	3035	83 00	3612	682.4	1513	1074
50 00	71698	111229.3	6.8055628	6.8043416	30	12644		6.8061543	6.8061164
30	0952		5754	3796	84 00	1675	111686.9	1571	1248
51 00	0200	248.7	5880	4175	30	0706		1597	1325
30	69443		6006	4552	85 00	9735	690.7	1620	1395
52 00	8680	268.0	6131	4928	30	8764		1642	1460
30	7913		6256	5302	86 00	7792	693.8	1661	1517
53 00	7140	287.1	6380	5674	30	6819		1677	1566
30	6361		6504	6045	87 00	5846	696.2	1692	1610
54 00	5578	306.0	6627	6413	30	4872		1705	1648
30	4790		6749	6780	88 00	3898	697.9	1715	1679
55 00	63996	111324.8	6.8056870	6.8047144	30	2924		6.8061723	6.8061702
30	3198		6991	7506	89 00	1949	111699.0	1728	1719
56 00	2395	343.3	7111	7866	30	975		1732	1729
30	1587		7230	8223	90 00	0	699.3	1733	1733
57 00	0774	361.5	7348	8578					

Table IV. Logarithms of Factors A.B.C.D.E.F

Lat.		Log. A diff 1" = − 0.04	Log. B diff 1" = − 0.13	Log. C diff 1" = + 0.70	Log. D diff 1" = + 0.06	Log. E diff 1" = + 0.03	Log. F diff 10' = + 3.0
° 18	' 00	$\bar{8}$.509 5862	$\bar{8}$.5122550	$\bar{0}$.91816	$\bar{2}$.1606	$\bar{5}$.7317	$\bar{7}$.738
	30	8.509 5785	8.5122320	0.93088	2.1709	5.7379	
19	00	8.509 5707	8.5122086	0.94330	2.1808	5.7443	7.756
	30	8.509 5627	8.5121847	0.95544	2.1903	5.7508	
20	00	8.509 5546	8.5121602	0.96733	2.1996	5.7574	7.772
	30	8.509 5462	8.512 1351	0.97896	2.2084	5.7642	
21	00	8.509 5377	8.512 1096	0.99037	2.2170	5.7711	7.787
	30	8.509 5290	8.5120836	1.00156	2.2253	5.7780	
22	00	8.509 5202	8.512 0571	1.01253	2.2333	5.7851	7.800
	30	8.509 5112	8.5120301	1.02331	2.2411	5.7924	
23	00	8.509 5020	8.5120026	1.03390	2.2485	5.7997	7.812
	30	8.509 4927	8.5119747	1.04431	2.2557	5.8071	
24	00	8.509 4833	8.5119463	1.05456	2.2627	5.8146	7.823
	30	8.509 4737	8.511 9174	1.06464	2.2694	5.8223	
25	00	8.509 4639	8.511 8881	1.07457	2.2759	5.8300	7.832
	30	8.509 4540	8.511 8584	1.08435	2.2822	5.8379	
26	00	8.509 4439	8.511 8283	1.09400	2.2882	5.8458	7.841
	30	8.509 4337	8.511 7977	1.10351	2.2941	5.8539	
27	00	8.509 4234	8.511 7667	1.11290	2.2997	5.8620	7.849
	30	8.509 4130	8.511 7353	1.12217	2.3051	5.8702	
28	00	8.5094024	8.511 7036	1.13132	2.3104	5.8785	7.855
	30	8.5093917	8.511 6714	1.14037	2.3154	5.8870	
29	00	8.5093808	8.511 6389	1.14932	2.3203	5.8955	7.861
	30	8.5093699	8.511 6061	1.15816	2.3249	5.9041	
30	00	8.5093588	8.511 5729	1.16692	2.3294	5.9127	7.866
	30	8.509 3476	8.511 5393	1.17558	2.3337	5.9215	
31	00	8.509 3363	8.511 5054	1.18410	2.3379	5.9304	7.870
	30	8.5093249	8.5114713	1.19266	2.3418	5.9393	
32	00	8.5093134	8.5114368	1.20108	2.3456	5.9484	7.873
	30	8.5093018	8.5114020	1.20944	2.3493	5.9575	
33	00	8.509 2901	8.511 3669	1.21772	2.3527	5.9667	7.875
	30	8.509 2784	8.511 3315	1.22594	2.3561	5.9760	
34	00	8.5092665	8.511 2959	1.23409	2.3592	5.9853	7.877
	30	8.5092545	8.511 2600	1.24219	2.3622	5.9948	
35	00	8.5092425	8.611 2239	1.25024	2.3651	6.0043	7.877
	30	8.5092304	8.511 1875	1.25823	2.3678	6.0140	
36	00	8.5092182	8.5111510	1.26617	2.3704	6.0237	7.877
	30	8.5092059	8.5111142	1.27407	2.3728	6.0334	
37	00	8.5091936	8.5110772	1.28193	2.3750	6.0433	7.876
	30	8.5091812	8.5110400	1.28975	2.3772	6.0533	
38	00	8.5091687	8.5110027	1.29753	2.3792	6.0633	7.874
	30	8.5091562	8.5109652	1.30527	2.3810	6.0734	
39	00	8.5091437	8.510.9275	1.31299	2.3827	6.0836	7.872
	30	8.5091311	8.5108897	1.32067	2.3843	6.0939	
40	00	8.5091184	8.5108517	1.32833	2.3857	6.1043	7.869
	30	8.509 1057	8.5108137	1.33596	2.3870	6.1148	
41	00	8.5090930	8.5107755	1.34358	2.3882	6.1253	7.864
	30	8.5090803	8.5107373	1.35117	2.3892	6.1360	
42	00	8.5090675	8.5106989	1.35875	2.3901	6.1467	7.860
	30	8.5090547	8.5106605	1.36631	2.3908	6.1575	
43	00	8.5090419	8.5106220	1.37386	2.3914	6.1684	7.854
	30	8.5090290	8.510 5835	1.38141	2.3919	6.1795	
44	00	8.5090162	8.5105449	1.38894	2.3923	6.1905	7.848
	30	8.5090033	8.5105063	1.39648	2.3925	6.2017	
45	00	8.5089904	8.5104677	1.40400	2.3926	6.2130	7.840
	30	8.5089776	8.5104291	1.41153	2.3926	6.2244	
46	00	8.5089647	8.5103905	1.41906	2.3924	6.2359	7.832
	30	8.5089518	8.5103519	1.42660	2.3921	6.2475	
47	00	8.5089390	8.5103134	1.43414	2.3917	6.2592	7.824
	30	8.5089261	8.5102749	1.44169	2.3911	6.2710	

Table IV. Logarithms of Factors A, B, C, D, E, F.

Lat. °	'	Log A diff 1" = −0.07	Log B diff 1" = −0.21	Log C diff 1" = +0.42	Log D diff 1" = −0.00	Log E diff 1" = +0.07	Log F diff 10' = −1.7
48	00	$\bar{8}$.5089133	$\bar{8}$.5102364	$\bar{1}$.44926	$\bar{2}$.3904	$\bar{6}$.2830	$\bar{7}$.814
	30	8.5089005	8.5101981	1.45683	2.3895	6.2950	
49	00	8.5088878	8.5101598	1.46443	2.3886	6.3071	7.804
	30	8.5088750	8.5101216	1.47204	2.3875	6.3194	
50	00	8.5088623	8.5100835	1.47968	2.3862	6.3318	7.792
	30	8.5088497	8.5100455	1.48734	2.3848	6.3443	
51	00	8.5088371	8.5100076	1.49502	2.3833	6.3569	7.780
	30	8.5088245	8.5099699	1.50273	2.3817	6.3697	
52	00	8.5088120	8.5099323	1.51048	2.3799	6.3826	7.767
	30	8.5087995	8.5098949	1.51826	2.3779	6.3956	
53	00	8.5087871	8.5098577	1.52608	2.3759	6.4088	7.753
	30	8.5087747	8.5098206	1.53393	2.3736	6.4221	
54	00	8.5087624	8.5097838	1.54183	2.3713	6.4355	7.738
	30	8.5087502	8.5097471	1.54977	2.3688	6.4491	
55	00	8.5087381	8.5097107	1.55777	2.3661	6.4629	7.723
	30	8.5087260	8.5096745	1.56581	2.3633	6.4768	
56	00	8.5087140	8.5096385	1.57391	2.3603	6.4909	7.706
	30	8.5087021	8.5096028	1.58207	2.3572	6.5052	
57	00	8.5086903	8.5095673	1.59028	2.3539	6.5196	7.688
	30	8.5086786	8.5095321	1.59857	2.3505	6.5342	
58	00	8.5086669	8.5094972	1.60692	2.3469	6.5490	7.669
	30	8.5086554	8.5094626	1.61534	2.3432	6.5640	
59	00	8.5086440	8.5094283	1.62384	2.3392	6.5792	7.649
	30	8.5086326	8.5093943	1.63242	2.3351	6.5946	
60	00	8.5086214	8.5093607	1.64109	2.3309	6.6102	7.627
	30	8.5086103	8.5093274	1.64984	2.3264	6.6261	
61	00	8.5085993	8.5092944	1.65869	2.3218	6.6422	7.605
	30	8.5085884	8.5092618	1.66763	2.3170	6.6585	
62	00	8.5085777	8.5092295	1.67668	2.3120	6.6750	7.581
	30	8.5085671	8.5091976	1.68583	2.3068	6.6919	
63	00	8.5085566	8.5091661	1.69510	2.3014	6.7089	7.556
	30	8.5085462	8.5091350	1.70449	2.2958	6.7263	
64	00	8.5085360	8.5091043	1.71400	2.2901	6.7440	7.529
	30	8.5085259	8.5090741	1.72365	2.2840	6.7619	
65	00	8.5085159	8.5090442	1.73343	2.2778	6.7802	7.501
	30	8.5085061	8.5090148	1.74336	2.2714	6.7988	
66	00	8.5084964	8.5089858	1.75344	2.2647	6.8177	7.471
	30	8.5084869	8.5089573	1.76369	2.2578	6.8370	
67	00	8.5084776	8.5089292	1.77410	2.2506	6.8567	7.440
	30	8.5084684	8.5089016	1.78469	2.2431	6.8768	
68	00	8.5084593	8.5088745	1.79547	2.2354	6.8972	7.406
	30	8.5084504	8.5088478	1.80645	2.2275	6.9181	
69	00	8.5084417	8.5088217	1.81763	2.2192	6.9395	7.371
	30	8.5084332	8.5087960	1.82904	2.2107	6.9613	
70	00	8.5084248	8.5087709	1.84068	2.2018	6.9836	7.333
	30	8.5084166	8.5087462	1.85256	2.1926	7.0064	
71	00	8.5084086	8.5087222	1.86470	2.1831	7.0298	7.293
	30	8.5084007	8.5086986	1.87712	2.1732	7.0538	
72	00	8.5083930	8.5086756	1.88984	2.1630	7.0784	7.250

Table VIII. Corrections to Longitude for difference in arc and sine.

Log s(−)	Log difference	Log Δλ(+)	Log s(−)	Log difference	Log Δλ(+)	Log s(−)	Log difference	Log Δλ(+)
3.876	0.0000001	2.385	4.526	0.0000020	3.035	4.732	0.0000052	3.241
4.026	02	2.535	4.548	23	3.057	4.746	56	3.255
4.114	03	2.623	4.570	25	3.079	4.761	59	3.270
4.177	04	2.686	4.591	27	3.100	4.774	63	3.283
4.265	06	2.774	4.612	30	3.121	4.788	67	3.297
4.327	08	2.836	4.631	33	3.140	4.801	71	3.310
4.376	10	2.885	4.649	36	3.158	4.813	75	3.322
4.415	12	2.924	4.667	39	3.176	4.825	80	3.334
4.449	14	2.958	4.684	42	3.193	4.834	84	3.343
4.478	16	2.987	4.701	45	3.210	4.849	89	3.358
4.503	18	3.012	4.716	48	3.225	4.860	94	3.369

Table V. Logarithms of Rz.

Az.	Latitude								
	22°	25°	30°	35°	40°	45°	50°	55°	60°
0°	6.80237	6.80254	6.80285	6.80320	6.80357	6.80396	6.80434	6.80471	6.80506
10	244	261	292	326	363	400	438	474	509
20	266	282	311	343	378	413	448	483	515
30	300	314	340	370	401	433	465	496	525
40	341	354	377	402	429	457	485	512	537
50	386	396	415	436	459	482	506	528	550
60	427	435	451	469	487	506	526	544	562
70	461	468	481	495	510	526	542	567	572
80	483	489	500	512	525	539	553	566	578
90	490	496	507	518	531	544	556	569	580

Table VI. Logarithms of m.

Lat.	Log.m	Lat.	Log.m	Lat.	Log.m	Lat.	Log.m	Lat.	Log.m
° ′		° ′		° ′		° ′		° ′	
18 00	1.40639	30 00	1.40548	42 00	1.40431	54 00	1.40309	66 00	1.40203
20 00	1.40626	32 00	1.40530	44 00	1.40411	56 00	1.40290	68 00	1.40188
22 00	1.40612	34 00	1.40511	46 00	1.40390	58 00	1.40271	70 00	1.40174
24 00	1.40597	36 00	1.40491	48 00	1.40369	60 00	1.40253	72 00	1.40161
26 00	1.40582	38 00	1.40472	50 00	1.40349	62 00	1.40235		
28 00	1.40565	40 00	1.40452	52 00	1.40329	64 00	1.40219		

Table VII. Probability of error between the limits 0 and ± Δ/ε√2

t	θt	t	θt	t	θt	t	θt	t	θt	t	θt
0.00	0.0000	0.45	0.4755	0.90	0.7969	1.35	0.9438	1.80	0.9891	2.25	0.9985
0.05	0.0564	0.50	0.5205	0.95	0.8209	1.40	0.9523	1.85	0.9911	2.30	0.9989
0.10	0.1125	0.55	0.5633	1.00	0.8427	1.45	0.9597	1.90	0.9928	2.35	0.9991
0.15	0.1680	0.60	0.6039	1.05	0.8624	1.50	0.9661	1.95	0.9942	2.40	0.9993
0.20	0.2227	0.65	0.6420	1.10	0.8802	1.55	0.9716	2.00	0.9953	2.45	0.9995
0.25	0.2763	0.70	0.6778	1.15	0.8961	1.60	0.9763	2.05	0.9963	2.50	0.9996
0.30	0.3286	0.75	0.7112	1.20	0.9103	1.65	0.9804	2.10	0.9970	2.55	0.9997
0.35	0.3794	0.80	0.7421	1.25	0.9229	1.70	0.9838	2.15	0.9976		
0.40	0.4284	0.85	0.7707	1.30	0.9340	1.75	0.9867	2.20	0.9981		

Table IX. Corrections for inclination for four-meter bar.

	0°	1°	2°	3°	4°		0°	1°	2°	3°	4°
′	Cor.	Cor.	Cor.	Cor.	Cor.	′	Cor.	Cor.	Cor.	Cor.	Cor.
00	0.00000	0.00061	0.00244	0.00548	0.00974	31	0.00016	0.00140	0.00386	0.00753	0.01242
01	00	63	248	554	983	32	17	143	391	760	1251
02	00	65	252	560	991	33	18	146	396	768	1261
03	00	67	256	567	999	34	20	150	401	775	1270
04	00	69	260	573	1007	35	21	153	407	782	1279
05	00	71	264	579	1015	36	22	156	412	789	1288
06	01	74	269	585	1024	37	23	159	417	797	1298
07	01	76	273	592	1032	38	24	163	422	804	1307
08	01	78	277	598	1040	39	26	166	428	811	1317
09	01	81	282	604	1049	40	27	169	433	819	1326
10	02	83	286	611	1057	41	28	173	439	826	1336
11	02	85	290	617	1066	42	30	176	444	834	1345
12	02	88	295	624	1074	43	31	180	450	841	1355
13	03	90	299	630	1083	44	33	183	455	849	1364
14	03	93	304	637	1091	45	34	187	461	856	1374
15	04	95	308	643	1100	46	36	190	466	864	1383
16	04	98	313	650	1109	47	37	194	472	872	1393
17	05	100	318	657	1117	48	39	197	478	879	1403
18	05	103	322	663	1126	49	41	201	483	887	1413
19	06	106	327	670	1135	50	42	205	489	895	1422
20	07	108	332	677	1143	51	44	208	495	903	1432
21	07	111	336	684	1152	52	46	212	501	911	1442
22	08	114	341	690	1161	53	48	216	506	918	1452
23	09	117	346	697	1170	54	49	220	512	926	1462
24	10	119	351	704	1179	55	51	224	518	934	1472
25	11	122	356	711	1188	56	53	228	524	942	1482
26	11	125	361	718	1197	57	55	232	530	950	1492
27	12	128	366	725	1206	58	57	236	536	968	1502
28	13	131	371	732	1215	59	59	240	542	966	1512
29	14	134	376	739	1224	60	61	244	548	974	1522
30	15	137	381	746	1233						

Table X. Polyconic Projections (Metric units) ➛

Lat.	Meridional dist. from even degree parallels.	Abscissas of Developed Parallel. 5' Long	10' Long.	15' Long	20' Long.	25' Long.	30' Long.	Ordinates of developed parallel Long Int.	Y
37° 00'	-------	7417.8	14835.6	22253.4	29671.2	37089.0	44506.7	5'	3.3
10	18496.1	7401.6	14803.2	22204.8	29606.4	37008.0	44409.6	10	13.0
20	36992.7	7385.3	14770.6	22155.9	29541.2	36926.5	44311.8	15	29.2
30	55489.9	7369.0	14738.0	22107.0	29476.0	36845.0	44214.0	20	51.9
40	73987.6	7352.6	14705.2	22057.8	29410.4	36763.0	44115.6	25	81.2
50	92485.8	7336.1	14672.2	22008.3	29344.4	36680.5	44016.6	30	116.9
38 00	-------	7319.6	14639.1	21958.6	29278.2	36597.6	43917.1	5	3.3
10	18499.3	7303.0	14606.0	21909.0	29212.0	36515.0	43818.0	10	13.1
20	36999.1	7286.3	14572.6	21858.9	29145.2	36431.5	43717.8	15	29.5
30	55499.4	7269.6	14539.2	21808.8	29078.4	36348.0	43617.6	20	52.4
40	74000.3	7252.8	14505.6	21758.4	29011.2	36264.0	43516.8	25	81.9
50	92501.8	7236.0	14472.0	21708.0	28944.0	36180.0	43416.0	30	118.0
39 00	-------	7219.0	14438.1	21657.1	28876.2	36095.1	43314.1	5	3.3
10	18502.5	7202.1	14404.2	21606.3	28808.4	36010.5	43212.6	10	13.2
20	37005.5	7185.1	14370.2	21555.3	28740.4	35925.5	43110.6	15	29.7
30	55509.1	7168.0	14336.0	21504.0	28672.0	35840.0	43008.0	20	52.9
40	74013.2	7150.8	14301.6	21452.4	28603.2	35754.0	42904.8	25	82.6
50	92517.8	7133.6	14267.2	21400.8	28534.4	35668.0	42801.6	30	118.9
40 00	-------	7116.3	14232.6	21349.0	28465.3	35581.6	42697.8	5	3.3
10	18505.7	7099.0	14198.0	21297.0	28396.0	35495.0	42594.0	10	13.3
20	37012.0	7081.6	14163.2	21244.8	28326.4	35408.0	42489.6	15	29.9
30	55518.8	7064.2	14128.4	21192.6	28256.8	35321.0	42385.2	20	53.2
40	74026.1	7046.7	14093.4	21140.1	28186.8	35233.5	42280.2	25	83.2
50	92534.0	7029.1	14058.2	21087.3	28116.4	35145.5	42174.6	30	119.8
41 00	-------	7011.5	14022.9	21034.3	28045.7	35057.1	42068.5	5	3.3
10	18509.0	6993.8	13987.6	20981.4	27975.2	34969.0	41962.8	10	13.4
20	37018.5	6976.0	13952.0	20928.0	27904.0	34880.0	41856.0	15	30.1
30	55528.5	6958.2	13916.4	20874.6	27832.8	34791.0	41749.2	20	53.5
40	74039.1	6940.3	13880.6	20820.9	27761.2	34701.5	41641.8	25	83.6
50	92550.3	6922.4	13844.8	20767.2	27689.6	34612.0	41534.4	30	120.4
42 00	------	6904.4	13808.8	20713.2	27617.6	34522.0	41426.3	5	3.4
10	18512.2	6886.4	13772.8	20659.2	27545.6	34432.0	41318.4	10	13.4
20	37025.0	6868.3	13736.6	20604.9	27473.2	34341.5	41209.8	15	30.2
30	55538.3	6850.1	13700.2	20550.3	27400.4	34250.5	41100.6	20	53.8
40	74052.2	6831.9	13663.8	20495.7	27327.6	34159.5	40991.4	25	84.0
50	92566.6	6813.6	13627.2	20440.8	27254.4	34068.0	40881.6	30	120.9
43 00	------	6795.3	13590.5	20385.8	27181.0	33976.2	40771.4	5	3.4
10	18515.5	6776.9	13553.8	20330.7	27107.6	33884.5	40661.4	10	13.5
20	37031.6	6758.4	13516.8	20275.2	27033.6	33792.0	40550.4	15	30.3
30	55548.2	6739.9	13479.8	20219.7	26959.6	33699.5	40439.4	20	53.9
40	74065.3	6721.3	13442.6	20163.9	26885.2	33606.5	40327.8	25	84.3
50	92583.0	6702.7	13405.4	20108.1	26810.8	33513.5	40216.2	30	121.3
44 00	------	6684.0	13368.1	20052.1	26736.1	33420.1	40104.0	5	3.4
10	18518.8	6665.3	13330.6	19995.9	26661.2	33326.5	39991.8	10	13.5
20	37038.1	6646.5	13293.0	19939.5	26586.0	33232.5	39879.0	15	30.4
30	55558.0	6627.7	13255.4	19883.1	26510.8	33138.5	39766.2	20	54.0
40	74078.4	6608.7	13217.4	19826.1	26434.8	33043.5	39652.2	25	84.4
50	92599.5	6589.8	13179.6	19769.4	26359.2	32949.0	39538.8	30	121.5
45 00	-------	6570.8	13141.5	19712.3	26283.0	32853.7	39424.3	5	3.4
10	18522.2	6551.7	13103.4	19655.1	26206.8	32758.5	39310.2	10	13.5
20	37044.7	6532.5	13065.0	19597.5	26130.0	32662.5	39195.0	15	30.4
30	55567.9	6513.4	13026.8	19540.2	26053.6	32567.0	39080.4	20	54.1
40	74091.7	6494.1	12988.2	19482.3	25976.4	32470.5	38964.6	25	84.5
50	92615.9	6474.8	12949.6	19424.4	25899.2	32374.0	38848.8	30	121.6
46 00	-------	6455.5	12910.9	19366.4	25821.8	32277.2	38732.6	5	3.4
10	18525.4	6436.1	12872.2	19308.3	25744.4	32180.5	38616.6	10	13.5
20	37051.3	6416.6	12833.2	19249.8	25666.4	32083.0	38499.6	15	30.4
30	55577.8	6397.1	12794.2	19191.3	25588.4	31985.5	38382.6	20	54.0
40	74104.8	6377.5	12755.0	19132.5	25510.0	31887.5	38265.0	25	84.4
50	92632.4	6357.9	12715.8	19073.7	25431.6	31789.5	38147.4	30	121.6