GEODESY

INCLUDING

ASTRONOMICAL OBSERVATIONS, GRAVITY MEASUREMENTS, AND METHOD OF LEAST SQUARES

BY

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PREFACE

In this volume the author has attempted to produce a text-book on Geodesy adapted to a course of moderate length. The material has not been limited to what could be actually covered in the class, but much has been included for the purpose of giving the student a broader outlook and encouraging him to pursue the subject farther. Numerous references are given to the standard works.

Throughout the book the aim has been to make the underlying principles clear, and to emphasize the theory as well as the details of field work. The methods of observing and computing have been brought up to date so as to be consistent with the present practice of the Coast and Geodetic Survey.

The chapters on astronomy and least squares are included for the sake of completeness but do not pretend to be more than introductions to the standard works. The student cannot expect to master either of these subjects in a short course on geodesy, but must make a special study of each.

The author desires to acknowledge his indebtedness to those who have assisted in the preparation of this book, and especially to Professor J. W. Howard of the Massachusetts Institute of Technology for suggestions and criticism of the manuscript; to the Superintendent of the Coast and Geodetic Survey for valuable data and for the use of many photographs for illustrations; and to Messrs. C. L. Berger & Sons for the use of photographs of the pendulum apparatus and several electrotype plates. Tables XII to XVII are from electrotype plates from Breed and Hosmer's Principles and Practice of Surveying, Vol. II.

G. L. H.

CAMBRIDGE, *April*, 1919.

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GEODESY

CHAPTER I

GEODESY AND GEODETIC SURVEYING— TRIANGULATION

1. Geodesy.

Geodesy is the science which treats of investigations of the form and dimensions of the earth's surface by direct measurements. The two methods chiefly employed in determining the earth's figure are (1) the measurement of arcs on the surface, combined with the determination of the astronomical positions of points on these arcs, and (2) direct observation of the variation in the force of gravity in different parts of the earth's surface.

2. Geodetic Surveying.

Geodetic Surveying is that branch of the art of surveying which deals with such great areas that it becomes necessary to make systematic allowance for the effect of the earth's curvature. In making an accurate survey of a whole country, for example, the methods of plane surveying no longer suffice, and the whole theory of locating points and calculating their positions must be modified accordingly. Such surveys require the accurate location of points separated by long distances, to control the accuracy of subsequent surveys for details, such as coast charts and topographic maps, or for national and state boundaries. The general method employed is that of triangulation, in which the location of points is made to depend upon the measurement of horizontal angles, the distances being calculated by trigonometry instead of being measured directly. This method was first applied to the measurement of arcs on the earth's surface by Snellius of Holland in 1615.

Although we may make this distinction when defining the terms it is not necessary to separate the two in practice. It is evident that geodetic surveys must be made before accurate dimensions of the earth can be computed; and, conversely, it is true that before geodetic surveys can be calculated exactly, the earth's dimensions must be known. Hence geodetic surveys are usually conducted with a twofold purpose: (1) for collecting the scientific data of geodesy, and (2) for mapping large areas, every survey depending upon data previously determined, but also adding to or improving the data already existing. For this reason the measurements are made with greater refinement than would be necessary for practical purposes alone.

3. Triangulation.

A triangulation system consists of a network of triangles the vertices of which are marked points on the earth's surface. It is essential that the length of one side of some triangle should be measured, and also that a sufficient number of angles should be measured to make possible the calculation of all the remaining triangle sides. In addition to the measurements that are absolutely necessary for making these calculations it is important to have other measurements for the purpose of verifying the accuracy of both the calculations and the field-work.

4. Classes of Triangulation.

Triangulation is divided, somewhat arbitrarily, into three grades, called *primary*, *secondary*, and *tertiary*, the classification depending upon the purpose for which the triangulation is to be used and upon the degree of accuracy demanded. The primary system is planned and executed for the purpose of furnishing a few well-determined positions for controlling the accuracy of all dependent surveys. Since the primary is usually the only triangulation which is employed in the purely scientific problems of geodesy, the selection of the primary points will be governed in part by the requirements of any geodetic problem that it is proposed to investigate. The secondary triangulation is somewhat less accurate than the primary, and the lines are generally shorter; it is often simply a means of connecting the primary with the tertiary system. Sometimes the secondary is extended into a region which is to be surveyed but which is not covered at all by the primary triangulation, and then it becomes the controlling triangulation of the region. The tertiary triangulation furnishes points needed for filling in details on the hydrographic or topographic maps. It is of a low order of accuracy as compared with the primary, but is amply accurate for controlling the surveys for detail. These tertiary stations furnish the starting points for plane-table surveys, traverse lines, etc. All three classes of triangulation are not necessarily present in a survey unless it be a very extensive one. In surveys of minor importance there may be but one class of triangulation.

5. Length of Line.

The length of line which may be used is determined largely by the character of the country to be surveyed. In California, where the mountains are high and the atmosphere is exceptionally clear, the network of triangulation known as the "Davidson quadrilaterals" (Fig. 1) is composed of lines varying in length from 50 to over 150 miles; whereas in flat country, lines from 15 to 25 miles long are the most common. Although the progress of the triangulation is apparently more rapid when long lines are used, it is not necessarily economical to use very long sights. The time gained by having but few stations to occupy may be more than offset by the delays due to unfavorable atmospheric conditions. Furthermore, it may be necessary to introduce many additional stations in the detail surveys in order to reach all parts of the area to be mapped. The accuracy of triangulation is not appreciably lessened by using rather short lines. In planning the system an attempt should be made to use that length of line which will result in the greatest economy, taking into consideration the cost of reconnoissance, signal building, base-line measurement, and the measurement of the angles.

6. Check Bases.

It has already been stated that at least one line in a system must be measured. In order to verify the accuracy of all the measurements, it is customary to introduce additional base lines into the triangulation at intervals varying from 50 to 500 miles.

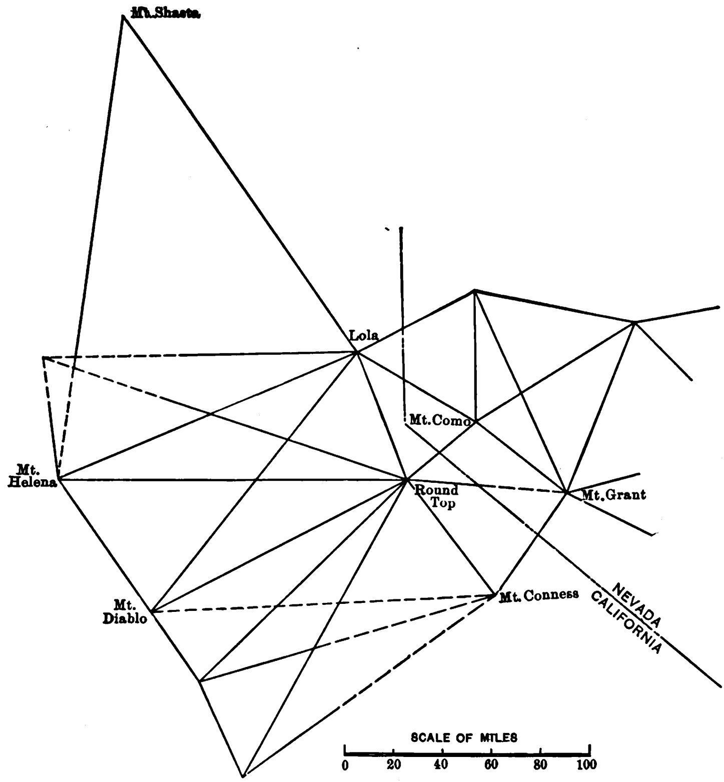


FIG. 1. Primary Triangulation in California (Davidson Quadrilaterals).

The lengths of these bases may be found by calculation of the triangles as well as by the direct measurement; this furnishes a most valuable check on the accuracy of the field work. In the triangulation of the United States Coast and Geodetic Survey the frequency with which these check bases should occur is determined by the *strength* of the chain of triangulation as found by the method given in Art. 8. The factor R_1 (Equa. [a]) between bases should be about 130 for primary work, although this may be increased to 200 if necessary.

In the triangulation of New England there are three bases: (1)

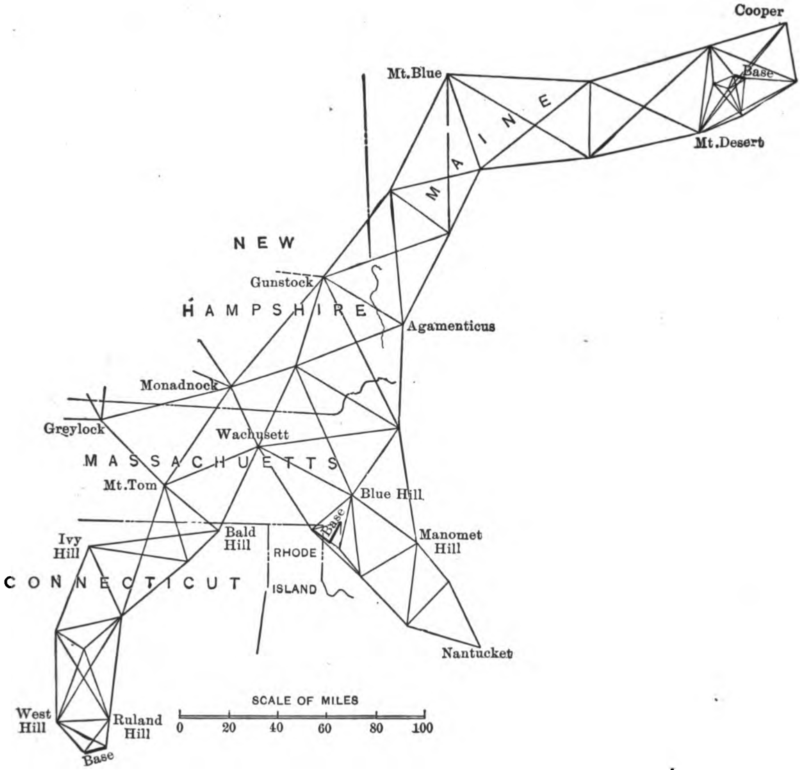


FIG. 2. Primary Triangulation of New England.

the Fire Island base, about 9 miles long, in the southern part of Long Island; (2) the Massachusetts base, about 10 miles long, near the Northeast corner of Rhode Island; and (3) the Epping base, about 5 miles long, in Maine. These base lines are shown as heavy lines in Fig. 2. The total length of the triangulation between the Epping and Fire Island bases is about 350 miles.

The accuracy with which the triangulation was executed is indicated by a comparison of the measured and computed lengths. The length of the Epping base as calculated from the Fire Island base is 0.042 meter less than the measured length; the length of Epping base calculated from the Massachusetts base is 0.136 meter less than the measured length.

7. Geometric Figure.

The geometric figure generally recognized as the best one for triangulation purposes is the quadrilateral, consisting of four stations joined by six lines, thus forming four triangles in which there are altogether eight independent angles to be measured. This figure furnishes a greater number of checks than any of the simple figures and therefore gives a good determination of length. The polygon having an interior station is also a strong figure. Figures which are more complex than these usually make the calculation troublesome and expensive, while simpler figures, like single triangles, result in diminished accuracy. In the work of the United States Coast Survey the primary triangulation is made up chiefly of complete quadrilaterals and partly of polygons having an interior station. In these figures all of the stations are supposed to be occupied with the triangulation instrument, but for secondary and tertiary triangulation some stations may be left unoccupied.

8. Strength of Figure.

In deciding which of several possible triangulation schemes should be adopted it is essential to inspect the chain of triangles with a view to ascertaining which is the strongest geometric figure, that is, which one will give the calculated length of the final line with the least error due to the shape of the triangles.

An estimate of the uncertainty in the computed side of a triangle is given by its *probable error* as found by the method of least squares. The square of the probable error (p) of a triangle side as computed through a chain of triangles is given by the equation

$$p^2 = \frac{4}{3}(d^2)\frac{N_d - N_c}{N_d}\sum[\delta_A{}^2 + \delta_A\delta_B + \delta_B{}^2],$$

in which d is the probable error of an observed direction, N_d is the number of directions observed, N_c is the number of geometric conditions that must be satisfied in the figure, and δ_A and δ_B are the differences in the log sines corresponding to a difference of 1″ in the angles A and B, A being opposite the known side and B opposite the computed side. A and B are known as the ***distance angles***. The $\sum$ indicates that the quantity in brackets is to be computed for each triangle in the chain and the sum of these numbers used in the formula. The factor $\frac{N_d - N_c}{N_d}$ depends upon the kind of figure chosen and the factor $\sum [\delta_A^2 + \delta_A\delta_B + \delta_B^2]$ depends upon the shape of the triangles of which the figure is composed; hence the product of the two is a measure of the strength of figure and is independent of the precision with which the angles themselves are measured. The strength R of any figure is therefore given by the equation

$$R = \frac{N_d - N_c}{N_d} \sum [\delta_A^2 + \delta_A\delta_B + \delta_B^2]. \qquad [a]$$

The smaller the value of this product the more favorable the geometric conditions, and the stronger the figure.

If the value of this product be computed for every possible route through the triangulation system, there will result a minimum value (R_1) for the best chain of triangles, a second best value (R_2), and a third and fourth, and so on. It will be found that the chain of triangles having the greatest influence in fixing the length of the final line is that corresponding to R_1, or the best chain. The second-best chain will have some influence, and the third and fourth correspondingly less. Hence, in choosing between two or more possible systems of triangulation which join a given base with some specified line, that route having the smallest R_1 is to be preferred, unless R_1 proves to be nearly the same for the different routes, in which case that chain having the smallest R_2 would be chosen.

As an example of the way in which the preceding method would be applied, take the case of the quadrilateral shown in Fig. 3. Assuming the base AB to be already fixed in direction, the point C is then determined by observing the new directions AC and BC. D is fixed by the directions AD and BD. In addition to these four the directions CB, CA, CD, DC, DB, DA are all observed. This gives 10 observed directions as the value of N_d.

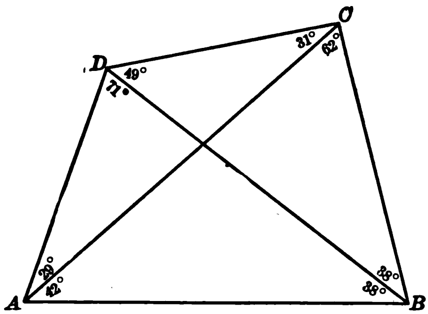


FIG. 3.

In determining the number of geometric conditions it is seen that there are four triangles, and that in each triangle the sum of the three angles must equal a fixed amount, 180° + the spherical excess of that triangle. It will be found, however, that if any three of these triangles are made to fulfill these conditions, the fourth will necessarily do so, and hence is not really independent; in other words, there are but three conditions dependent upon the closure of the triangles. In addition to these three angle conditions there is also a distance check; that is, the angles must be so related that the computed length of side CD is the same, no matter which pair of triangles is used in making the computation. The angles of the triangle may in each case add up to the correct amount, and yet the figure will not be a perfect quadrilateral unless this last condition is fulfilled. There are then, in all, four geometric conditions existing among the angles ($N_c = 4$). Therefore the factor for the completed quadrilateral is

$$\frac{N_d - N_c}{N_d} = \frac{10 - 4}{10} = 0.6.$$

In the triangle *ADB* the distance angles for computing the diagonal are *DAB* and *ADB*, that is, 71° and 71°. The difference for 1″ for 71° is 0.72 in units of the 6th decimal place. The quantity in brackets in the formula is therefore (0.52 + 0.52 + 0.52) = 1.56, or 2 to the nearest unit. In Table I these numbers are given for all combinations of angles which will occur in practice, so that this factor may be found at once by entering the table with the two distance angles. For the triangle *BDC* the distance angles for computing the side *DC* are 93° and 38°, the tabular number being 7. For this chain of triangles, then, $R_1 = 0.6 \times (2 + 7) = 5.4$. For triangle *BAC* the angles are 76° and 62°, and the number equals 2. For triangle *DCA* the angles are 120° and 29°, and the number equals 11. Therefore $R_2 = 0.6 \times 13 = 7.8$. If we compute *CD* through the triangles *ACB* and *DCB*, we find $R_3 = 15.6$. Using triangles *DBA* and *DCA*, $R_4 = 30.6$. In comparing the strength of this quadrilateral with that of any other figure, reliance would be placed mainly upon $R_1 = 5.4$ and partly upon $R_2 = 7.8$.

Following are the values of factor $\frac{N_d - N_c}{N_d}$ for several figures frequently used in triangulation: — single triangle, 0.75; quadrilateral, 0.60; quadrilateral with one station on fixed line not occupied, 0.75; quadrilateral with one station not on fixed line not occupied, 0.71; triangle with interior station, 0.60; triangle with interior station, one station on fixed line not occupied, 0.75; triangle with interior station, one station not on fixed line not occupied, 0.71; four-sided figure with interior station, 0.64; five-sided figure with interior station, 0.67; six-sided figure with interior station, 0.68. (For additional cases see *General Instructions for the Field Work of the Coast and Geodetic Survey*, 1908; or *Special Publication No.* 26.)

9. Number of Conditions in a Figure.

In determining the number of conditions in any figure it is well to proceed by plotting the figure point by point, and to write down the conditions as they arise, but it will be of assistance to have a check on the results obtained by this process. If n represents the total number of angles measured, and s the number of stations, then, since it requires two angles to locate a third point from the base line, two more to locate a fourth point from any two of these three points, and so on, the number of angles required is $2(s - 2)$; and since each additional angle gives rise to a condition, the number of conditions will equal the number of superfluous angles, or

$$N_c = n - 2(s - 2)$$
$$= n - 2s + 4.$$

For example, in a quadrilateral in which one station is unoccupied there are six angles measured, and $N_c = 6 - 8 + 4 = 2$.

The number of conditions may also be found from the equation

$$N_c = 2l - l_1 - 3s + S_u + 4,$$

where l = the total number of lines,

l_1 = the number of lines sighted in one direction only,

s = the total number of stations,

and s_u = the number of unoccupied stations.

In the preceding example this equation becomes

$$N_c = 12 - 3 - 12 + 1 + 4 = 2.$$

10. Allowable limits of R_1 and R_2.

In the primary triangulation of the United States Coast and Geodetic Survey, the extreme limits for R_1 and R_2 between base nets are 25 and 80, respectively. These are reduced to 15 and 20 if this does not increase the cost over 25 per cent. For secondary triangulation the limits for R_1 and R_2 are 50 and 150; these are reduced to 25 and 80 if the cost is not more than 25 per cent greater. For tertiary triangulation the 50 and 150 limit may be exceeded if it appears necessary. As stated in Art. 6, when R_1 has accumulated to 130 between bases, a new base line should be introduced as a check on the accuracy of the calculated lengths. If the character of the country is such that a base cannot be located at this point, R_1 may be increased to 200 if necessary.

11. Reconnoissance.

The work of planning the system is in many respects the most important part of the project and demands much experience and skill. Upon the proper selection of stations will depend very largely the accuracy of the result, as well as the cost of the work. No amount of care in the subsequent field-work will fully compensate for the adoption of an inferior scheme of triangulation. Three points in particular will have to be kept in mind in planning a survey: (1) the "strength" of the figures adopted; (2) the distribution of the points with reference to the requirements of the subsequent detail surveys; and (3) the cost of the work. In deciding which stations to adopt it is desirable to make a preliminary examination of all available data, such as maps and known elevations. If no map of the region exists, a sketch map must be made as the reconnoissance proceeds. While much information may be obtained from such maps as are available, the final decision regarding the adoption of points must rest upon an examination made in the field. All lines should be tested to see if the two stations are intervisible. This may be done by means of field glasses and heliotrope signals. In cases where the points are not intervisible, owing to intervening hills or to the curvature of the earth's surface, it will be necessary to determine approximately, by means of vertical angles or by the barometer, the elevation of the proposed stations and of as many intermediate points as may be required, and then to calculate the height to which towers will have to be built in order to render the proposed stations visible. If the height of the towers is such as to make the cost prohibitive, the line must be abandoned and another scheme of triangles substituted.

12. Calculation of Height of Observing Tower.

After determining the elevations of the stations and the intervening hills along a line, as well as the distances between them, the height of the tower required may be found by the following method: The curvature of the earth's surface causes all points to appear lower than they actually are. A hill appearing to be exactly on the level of the observer's eye is in reality higher above sea-level than the observer. The light coming from the hill to the observer's eye does not, however, travel in a straight line, but is bent, or refracted, by the atmosphere into a curve which is concave downward and is approximately circular. The result is that the object appears higher than it would if there were no refraction. The amount of the apparent change in height due to refraction is found to be only about one-seventh part of the apparent depression due to curvature. Since these two corrections always have opposite signs and have a nearly fixed relation to each other, it is sufficient in practice to calculate the correction to the difference in height due to both curvature and refraction, and to treat the combined correction as though it were due to curvature alone, since the curvature correction, being the larger, always determines which way the total correction shall be applied.

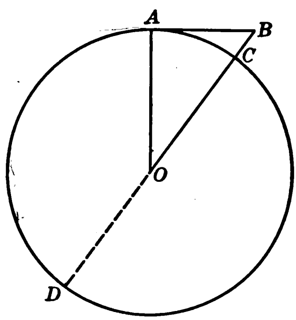


FIG. 4.

In Fig. 4, A is the position of the observer, looking in a horizontal direction toward point B. BC is the amount by which B appears lower than it really is, since A and C are both at the same elevation (sea-level).

By geometry, $BC : AB = AB : BD$

or
$$BC = \frac{\overline{AB}^2}{BD}.$$

Since BC is small compared with BD, the percentage error is small if we call $AB = AC$ and $BD =$ the diameter of the earth, whence

$$BC = \frac{(\text{dist.})^2}{\text{diameter}} \text{ (approx.)}.$$

The light from B' (Fig. 5) follows the dotted curved path which is tangent to the sight line at A. The observer therefore sees B'

at B. In order to find the relation of BB' to BC it is convenient to employ m, the *coefficient of refraction*, which is defined as the number by which the central angle AOB must be multiplied in order to obtain the angle BAB'; therefore

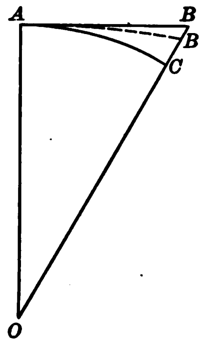


FIG. 5.

$$\text{angle of refraction} = 2 \times m \times BAC.$$

Since these angles are small, distances BB' and BC are nearly proportional to the angles themselves, hence

$$BB' : BC = BAB' : BAC$$

and $$BB' = 2\,m \times BC.$$

The net correction ($B'C = h$) is the difference between the two, that is

$$h = BC - BB'$$
$$= \frac{(\text{dist.})^2}{\text{diam.}} - 2\,m\,\frac{(\text{dist.})^2}{\text{diam.}}$$
$$= \frac{(\text{dist.})^2}{\text{diam.}}(1 - 2\,m).$$

The mean value of m is found to be about 0.070. Substituting this, and the value for the earth's diameter, and reducing h to feet, we have

$$h \text{ (in feet)} = K^2 \text{ (in miles)} \times 0.574,$$

or $$K \text{ (in miles)} = \sqrt{h \text{ (in ft.)}} \times 1.32,$$

in which K is the distance in miles. Values of h and K for distances up to 60 miles will be found in Table II.

As an example of how this formula is applied, suppose it is desired to sight from A to B (Fig. 6), and that a hill C obstructs the line. At A draw a horizontal line AD and also a curve AE parallel to sea-level. The distance from the tangent to the dotted curve at C is $\frac{K^2}{\text{diam.}}$, which for 46 miles is 1411.9 ft. Similarly, at B, $\frac{K^2}{\text{diam.}} = 4708.0$ ft. But since the ray of light from B to A

is curved, B is seen at B', or 659.2 ft. nearer to the tangent AD; similarly, C appears to be 197.7 ft. nearer the tangent line. Therefore, in deciding the question of visibility we may compute the combined correction and say at once that the curve at C is

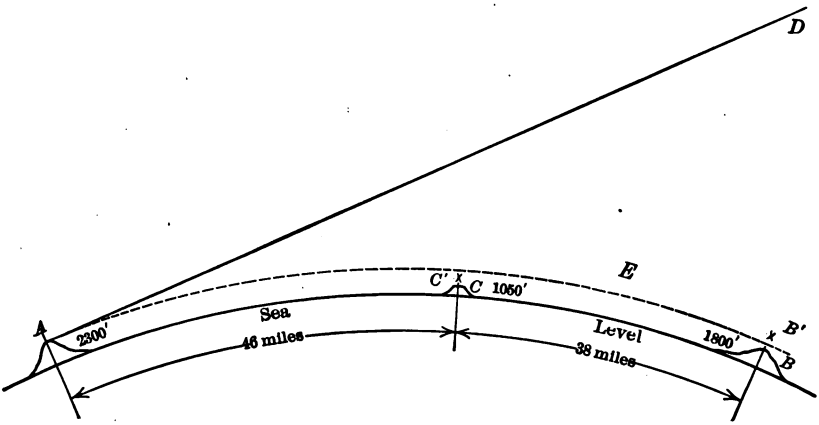


FIG. 6.

1214.2 ft. below AD, and at B is 4048.8 ft.* below AD. Adding 2300 ft. (the elevation of A) to each of these values of h, we obtain the (vertical) distances from the tangent line down to sea-level, namely 3514.2 ft. and 6348.8 ft. at C and B, respectively. Sub-

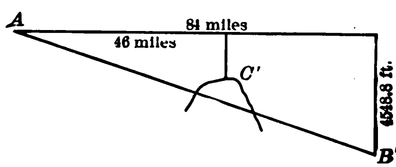


FIG. 7.

tracting the elevations of C and B, we obtain 2464.2 ft. and 4548.8 ft. as the distances of points C and D below the tangent line AD. The three points are now referred to a straight line (the tangent), and the question of visibility is determined at once by similar

* Since the table extends only to 60 miles, the value of h is first found for half the distance (42 mi.), and the result multiplied by 4.

triangles. In Fig. 7 it will be seen that the straight line from B' to A is $\frac{46}{84} \times 4548.8 = 2491.0$ ft. below the tangent (opposite C), and consequently is 26.8 ft. lower than C. Twenty-seven-foot towers would therefore barely make B' visible from A. In order to avoid the atmospheric disturbances near the ground at C the towers would really have to be carried up to a height of 40 ft. or even more. Of course the line of sight is not actually straight between A and B, as shown in the diagram; but this method of solving the problem gives the same result as though the curvature and refraction were dealt with separately and the sight lines all drawn curved.

If it were required to find the heights of towers necessary to make it possible to sight from A across a water surface to D, we should proceed as follows: Suppose the elevation of A above the water surface is 20 ft. and that of D is 10 ft. From A we may draw a line tangent to the water-level at T (Fig. 8). Knowing

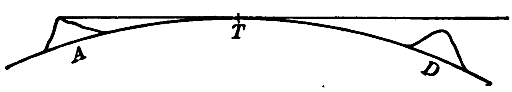


FIG. 8.

the height of A, we may find the distance AT from Table II. Subtracting this distance from AD, we find the distance TD. From this latter distance we may compute the height of the tangent line above the surface at D, and, finally, knowing the height of D, we find the distance of D below the tangent line. Now that the points are referred to a straight line, we have at once the height of tower required on D alone. If the two towers are to be of equal height, we may estimate the required height closely and then verify the result by a second computation, adding the assumed height of the tower to the elevation of A.

If it is desired to keep the line of sight at least 10 ft. above the surface at every point in order to avoid errors due to excessive refraction, we may draw a parallel curve 10 ft. above the water surface and solve the problem as before. The difference in radii of the two curves will not have an appreciable effect on the computed values of h and K.

13. Method of Marking Stations.

The importance of permanently marking a trigonometric station and connecting it with other reference marks cannot be easily overestimated, since by this means we may avoid the costly work of reproducing triangulation points which have been lost.

When the station is on ledge, the point is best marked by making a fairly deep drill hole and setting a copper bolt into it. A triangle is chiseled around the hole as an aid in identifying the point. Other drill and chisel marks should be made in the vicinity, and their distances and directions from the center mark determined; these will serve as an aid in recovering the position of the center mark in case it is lost.

If the station is on gravel or other soft material, the station mark on the surface is usually a stone or concrete post, set deep enough to be unaffected by frost action and having a drill hole or other distinguishing mark on top. There is usually also a subsurface mark, such as a second stone post, a bottle or a circular piece of earthenware, placed some distance below the surface mark, to preserve the location in case the latter is lost. The Coast and Geodetic Survey and the United States Geological Survey use cast metal discs provided with a shaft ready to place in concrete, and bearing an inscription giving the name of the organization and other information. (See Figs. 9a and 9b.)

The following description and sketch are given to illustrate a description of a triangulation station.

Triangulation Station "Beacon Rock."

The station is in the town of ————, ————, on a hill on the property of John Smith situated on the north side of the road from Bourne to Canterbury. It is reached by a trail which leaves the road at a point about 250 meters west of Smith's house. It is about 225 meters by trail to the station. The point is marked by a one-inch copper bolt set in a drill hole in the ledge and with a triangle chiseled around it, and by witness marks as shown in the accompanying sketch. The hill is somewhat wooded to the north and west, but there is a clear view in all other directions.

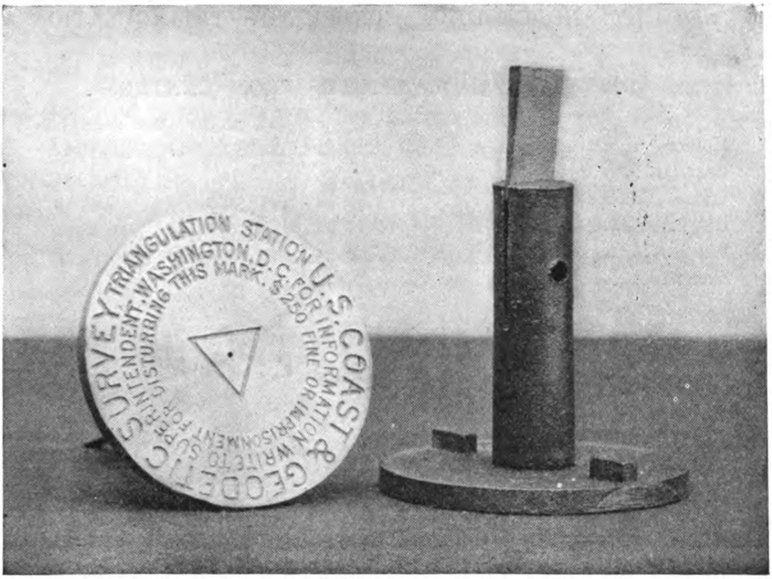


Fig. 9a. Triangulation Station Mark.
(Coast and Geodetic Survey.)

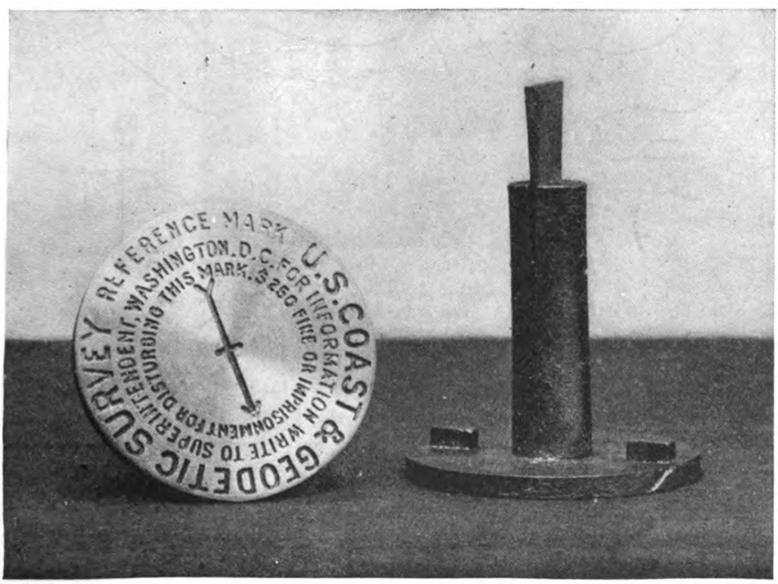


Fig. 9b. Reference Mark.
(Coast and Geodetic Survey.)

DISTANCES AND AZIMUTHS FROM CENTER

Station.	Azimuth.	Dist. to drill hole.
Holder	21° 50′	71.3 m.
Bear Hill	121° 16′	41.0 m.
Witness Mark	185° 30′	21.47 m.
Dayton	259° 10′	101.2 m.
Witness Mark	283° 05′	78.34 m.
Sheep Id.	325° 40′	....

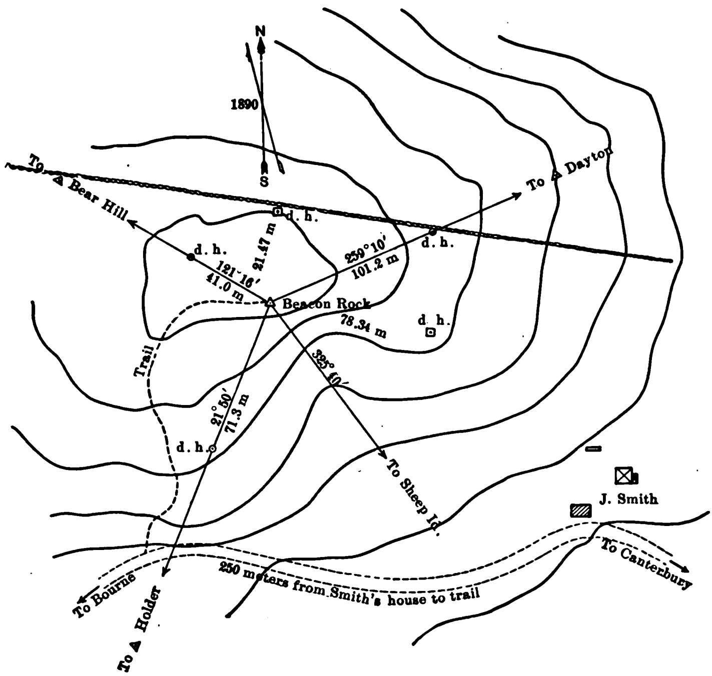


FIG. 10. Sketch of Triangulation Station.

14. Signals, Tripods.

In order that the exact position of the station may be visible to the observer when measuring the angles, a signal of some sort is erected over the station. For comparatively short lines, less than about 15 miles, the tripod signal is often sufficient. (See Fig. 11.) It is not expensive to build, saves the cost of a man to attend signal lights (as is necessary with heliotropes or acetylene lights), and permits setting the instrument over the station without removing the signal. It usually consists of a mast of 4″ × 4″ spruce, with legs of about the same size. Three horizontal braces of smaller dimensions (2″ × 3″) tie the mast to the legs, and three longer horizontal braces are nailed to the legs. If the

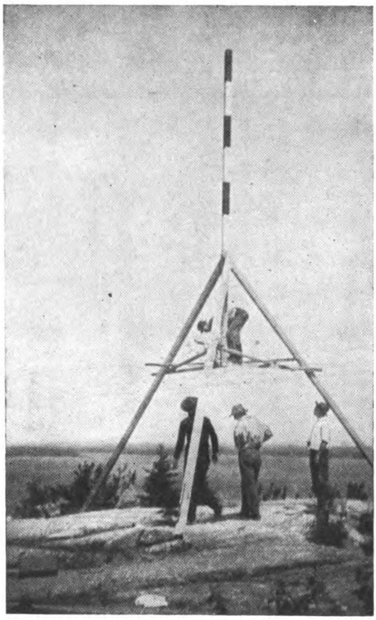


FIG. 11. Tripod Signal.

signal is very large, additional sets of braces may be put on, to give greater stiffness. The size of the mast may be increased by nailing on one-inch boards, giving a mast 6″ × 6″.

15. Heliotropes.

When sighting over longer lines it is necessary to use heliotrope signals if observing by day, and acetylene lights if observing by night. The heliotrope is simply a plane mirror with some device for pointing it so that reflected sunlight will reach the distant station. The two more common heliotropes are (1) the one in which the light is directed through two circular rings of slightly different diameters (Fig. 12), and (2) that known as the Steinheil heliotrope (Fig. 13).

The ring heliotrope consists essentially of two circular metal rings, of slightly different diameters, mounted on a frame, and a mirror mounted in line with the two rings in such a manner that it can be moved about two axes at right angles to each other. For convenience in observing distant stations these two rings and the mirror are often mounted on the barrel of a telescope. The

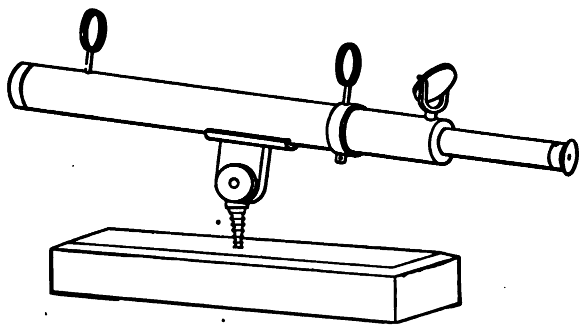


FIG. 12. Heliotrope.

rings should be so mounted that the line between the centers of the rings may be adjusted parallel to the line of sight of the telescope. In using the heliotrope the axis of the rings is pointed by means of threads which mark the center of the openings, or by means of the telescope itself after the axis of rings and the line of sight of the telescope have been made parallel. Since the sun's apparent diameter is about 0° 32′, the angle of the cone of rays reflected from the mirror is also 0° 32′. It is not necessary, therefore, to point the beam of light with great precision. If the central ray is nearly a quarter of a degree to one side of the station, there will still be some light visible to the observer at the distant station. On account of the rapidity of the sun's motion it is necessary to reset the heliotrope mirror at intervals of about one minute.

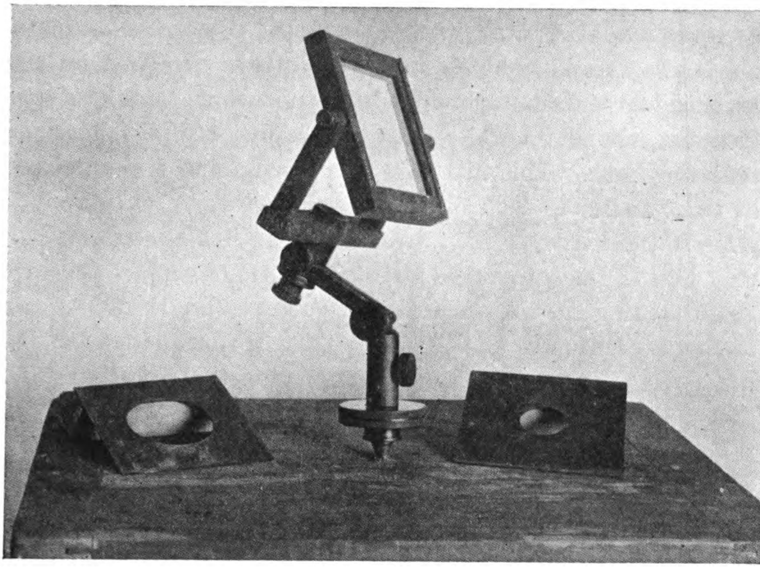


FIG. 13a. Steinheil Heliotrope.

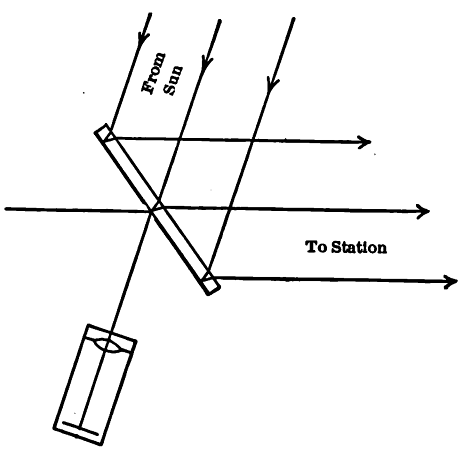


FIG. 13b.

The Steinheil heliotrope consists of a mirror with both faces ground plane and parallel and so mounted that it can be moved about two axes at right angles to each other. One of these axes is coincident with that of a cylindrical tube which contains a small biconvex lens and a white surface (usually plaster of Paris) for reflecting light. This tube may be moved about two other axes at right angles to each other. A small circular portion of the glass in the center of the mirror is left unsilvered, so that light may pass through the glass plate down into the tube.

In pointing the Steinheil heliotrope the cylindrical tube containing the lens must be pointed toward the sun, so that the light which passes through the hole in the mirror will pass through the lens, and, after reflection from the plaster surface, will again pass through the lens to the back surface of the mirror, there to be partly reflected and partly transmitted through the glass. Keeping the tube in this position, the mirror itself must be so turned that the spot of light made visible by this last reflection will appear to cover the hill or station to which the light is to be sent.

One form of heliotrope, in use by the Coast Survey, called a box heliotrope, consists of a pair of rings with a mirror mounted behind them, and with sights above the rings for pointing. A telescope is mounted to one side of and parallel to the heliotrope. The various parts remain in position in the box when in use. (Fig. 14.)

The size of mirror used in any heliotrope must be regulated according to the length of line and the atmospheric conditions. Most heliotropes are provided with some arrangement for varying the size of the opening through which the light passes. If the exposed portion of the mirror subtends an angle of about $0.''2$ the amount of light will be sufficient for average conditions. This is equivalent to making the diameter of the opening about one-half inch for each ten miles. Different atmospheric conditions will require different openings.

All heliotropes are provided with a second mirror, usually larger than the first, called the back mirror; this is to be used whenever the angle between the sun and the station is too great to permit sending the ray by a singe reflection. The back mirror is set so as to throw light onto the first mirror and the heliotrope is then adjusted to the reflection of the sun as it appears in the back mirror.

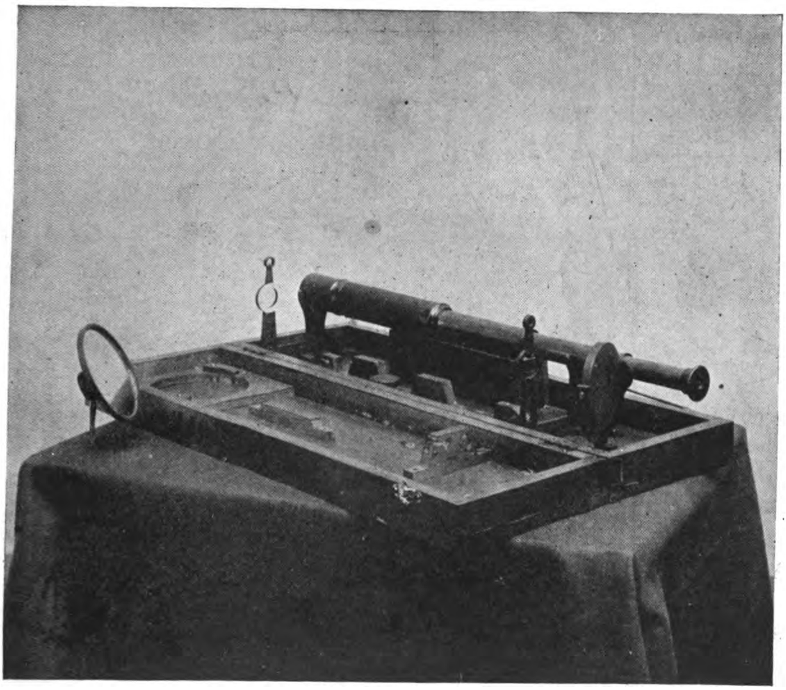


Fig. 14. Box Heliotrope.
(Coast and Geodetic Survey.)

16. Acetylene Lights.

In the triangulation along the ninety-eighth meridian, in 1902, the Coast and Geodetic Survey experimented with acetylene lights for triangulation at night. These experiments were successful, and, owing to the fact that the work could usually proceed regardless of clouds, the use of lights resulted in greater economy than observations by daylight. The lamps used at first were ordinary acetylene bicycle lamps remodeled in the instrument division of the Survey. The front door of the lamp was removed and the ordinary lens replaced by a pair of condensing lenses 5 inches in diameter. When in use the lamp is secured to the platform by means of a screw, and may be moved both in

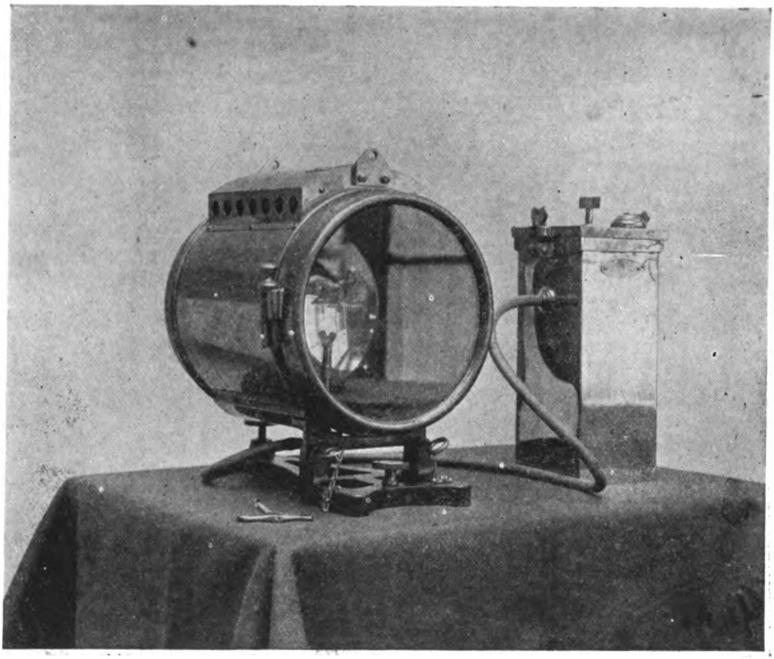


FIG. 15. Acetylene Signal Lamp.
(Coast and Geodetic Survey.)

altitude and in azimuth. A small tube is fastened to the top of the lamp for pointing it toward the observer's station. The entire outfit, including a 5-lb. can of carbide, weighs but 21½ lbs. (See *Coast and Geodetic Survey Report for* 1903, p. 824.) The recent practice of the Survey is to use automobile lamps in place of the bicycle lamps. (Fig. 15.)

17. Towers.

Where a line is obstructed by hills or woods, or where the curvature of the earth is sufficient to make the station invisible, it becomes necessary to construct towers. If there is much heavy timber about the station, placing the instrument station

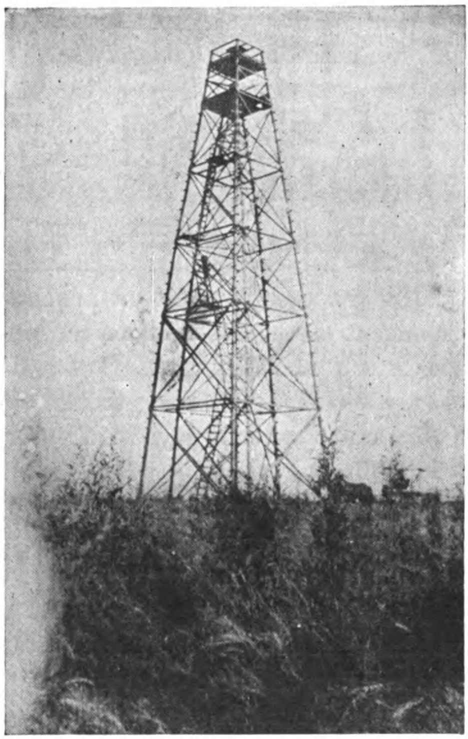


FIG. 16. Eighty-foot Tower.
(Coast and Geodetic Survey.)

on the ground may necessitate so much cutting that it will be more economical to construct a tower than to cut the timber.

The form of tower now used by the United States Coast Survey is very light and slender as compared with the older ones. This kind of tower (Fig. 16) admits of more rapid construction and can be built at a lower cost; it is sufficiently rigid to withstand all ordinary storms. The manner of framing the tower is shown in the cut (Fig. 17). When the ties are nailed on, the legs are sprung slightly into the form of a bow, thus giving additional stiffness to the structure.

One side of the inner tripod, which is to support the instrument, is first framed on the ground. This side and the third leg of the tripod are raised into position by a fall and tackle and a derrick, which may be a tree or a section of one of the legs of the outer scaffold. The derrick should be at least two-thirds the height of the piece to be raised. After the tripod is raised and all braces nailed on, it is itself used as a derrick for hoisting the two opposite frames of the outer scaffold into position. The ties and braces of the other two sides are then nailed in place. It should be observed that the inner and outer structures are entirely separate, so that the movement of the observer on the platform of the scaffold will not disturb the instrument. The legs of the tripod and the scaffold are anchored by nailing them to foot pieces set underground. The outer tower is guyed with wire as a protection against collapse in high winds.

This kind of signal saves lumber, transportation, and cost of construction; it has a small area exposed to the action of the wind; the short ties have the effect of reducing the vibration due to wind, which is troublesome in large towers; the light keeper is placed above the observer (10 ft. or so) and can operate his lights without interfering with the observations. Another advantage of these towers is that the amount of twisting due to the sun's heating is found to be exceedingly small. (For further details consult *Coast Survey Report for* 1903, p. 829.)

The United States Lake Survey now uses a tower constructed entirely of gas pipe, which has proved to be more economical than timber. It is put together in sections and hoisted as it is built. The upper part of the structure is built first and is then hoisted from the ground by means of tackles; the next section is then added on, all the work being done from the ground. This

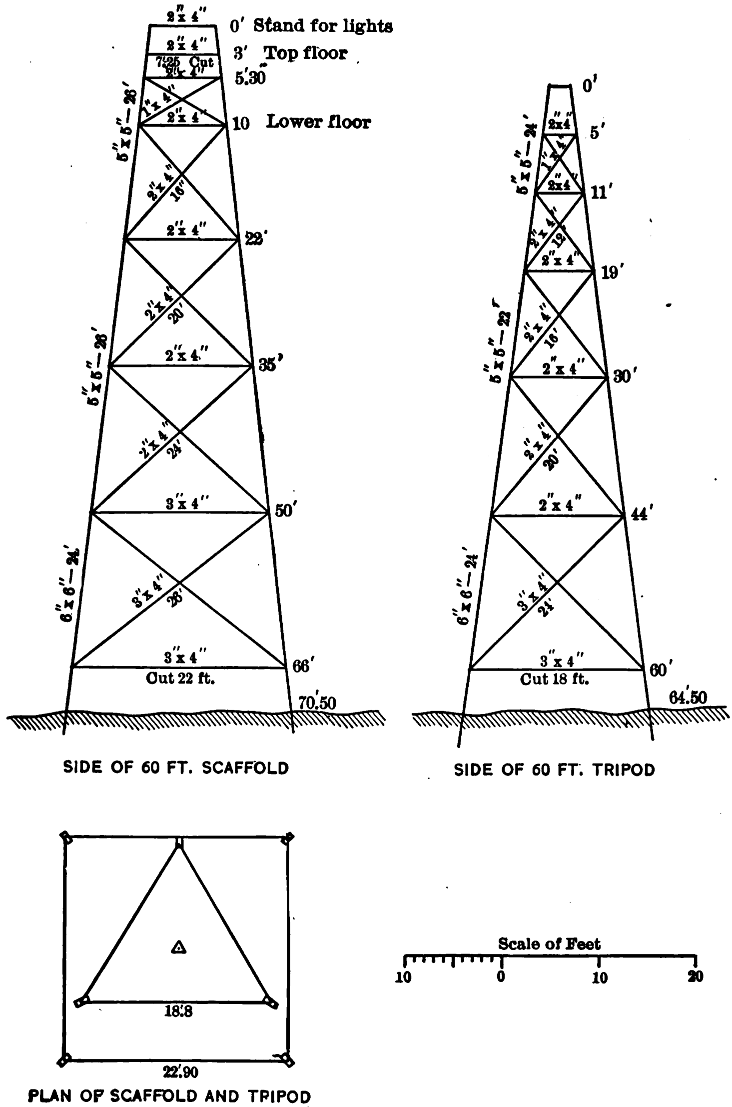


FIG. 17. Framing plan of 60-ft. Tower.

kind of tower is easy to construct, and the material is portable; the area exposed to wind is very small.

Figs. 18 and 19 illustrate small towers built of green poles cut near the station. These towers were erected to enable the observer to see over the dense growth of timber. In the tower shown in Fig. 19 standing trees, stripped of their branches, were utilized for two of the legs of the outer scaffold.

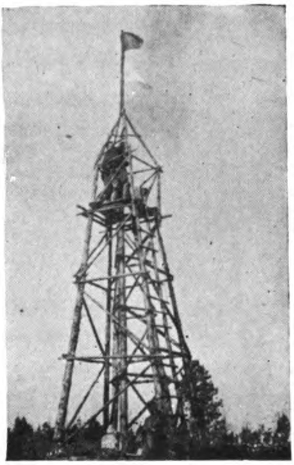


FIG. 18. Twenty-five-foot Tower built of Green Poles.

18. Reconnoissance for Base Line. With the Invar tape apparatus, to be described in Chapter II, base lines may now be measured over much rougher ground than was formerly possible, when bar apparatus was used; still it is advantageous to have the base line located in as smooth and level country as possible, provided this does not require weak triangulation to connect the base with the main scheme of triangles. The network of triangles required in making this connection should be selected with the same care and according to the same principles as was described for primary triangulation. In some cases it is found practicable to use the side of a primary triangle for the base line. For example, in the triangulation extending from Texas to California the Stanton base, which is one of the primary lines (8 miles in length), was measured directly with the tape apparatus.

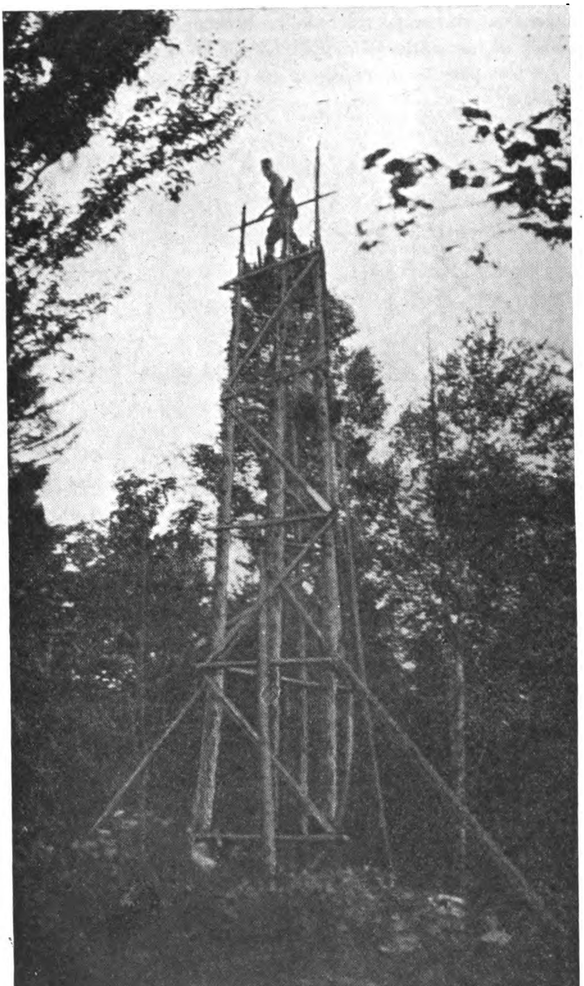


FIG. 19. Forty-foot Tower built on trees in place.

PROBLEMS

Problem 1. What is the strength of the quadrilateral having all the angles equal to 45°? In case one station on the base is not occupied with the instrument, what is the strength? If one station not on the base is unoccupied, what is the strength?

Problem 2. Compare the strength of the three figures given in Fig. 19a.

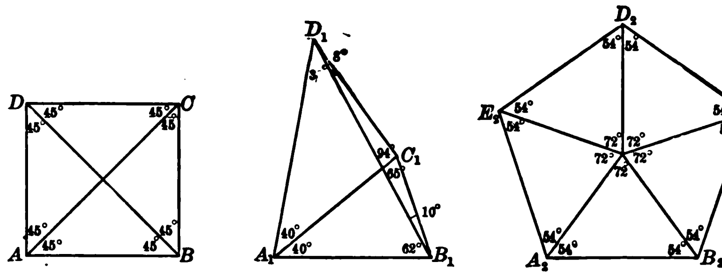


FIG. 19a.

Problem 3. Three hills A, B, and C are in a straight line. The distance from A to B is 10 miles and the distance from B to C is 15 miles. The elevations are A = 600 ft., B = 550 ft., and C = 650 ft. respectively. Compute the height of a tower to be built on C the top of which will just be visible from A.

Problem 4. Four hills A, B, C, and D are in a straight line. The elevations are A = 810 ft., B = 775 ft., C = 1030 ft, D = 1300 ft. respectively. The distances of B, C, and D from A are 8 miles, 28 miles, and 38 miles. Find the height of towers on A and D to sight over B and C with a 10-ft. clearance. The two towers are to be of the same height.

Problem 5. What angle is subtended by a six-inch mast at a distance of twelve miles?

Problem 6. If a fourteen-inch mirror is used on a heliotrope at a distance of 150 miles, what is the apparent angular diameter of the light?

CHAPTER II

BASE LINES

19. Bar Apparatus for Measuring Bases.

In nearly all the earlier base-line measurements (up to about 1885) the apparatus employed consisted of some arrangement of metal bars. Such apparatus was capable of yielding accurate results, but was cumbersome to use; consequently the base-line work was a comparatively expensive part of the survey. An account of the development of base-measuring apparatus will be found in Clarke's *Geodesy* and in Jordan's *Vermessungskunde*, Vol. III; descriptions of numerous forms used in this country will be found in the reports of the superintendent of the Coast and Geodetic Survey.

20. Steel Tapes.

Experiments with the use of steel tapes for base-line measurements were made by Jäderin at Stockholm in 1885, by the Missouri River Commission in 1886, and by Woodward on the Coast and Geodetic Survey base at Holton, Indiana, in 1891. The use of steel tapes for this purpose was attended with such success that for twenty years they were very generally used, and by 1900 they had almost wholly superseded the bar apparatus in this country.

The greatest practical difficulty encountered in the use of steel tapes for precise measurement is that of determining the true temperature of the steel when making the measurements in sunlight. The air temperature, as indicated by ordinary mercurial thermometers, is seldom the correct temperature for the tape, except during rainy weather or at night. For this reason it was found necessary to make all measurements of base-lines at night in order to secure the required accuracy.

21. Invar Tapes.

In 1906 the Coast Survey made a series of tests on six primary base-lines, using the ordinary steel tapes and also several new 50-meter tapes made of an alloy of nickel and steel called *invar*. This alloy was discovered by C. E. Guillaume, of the Interna-

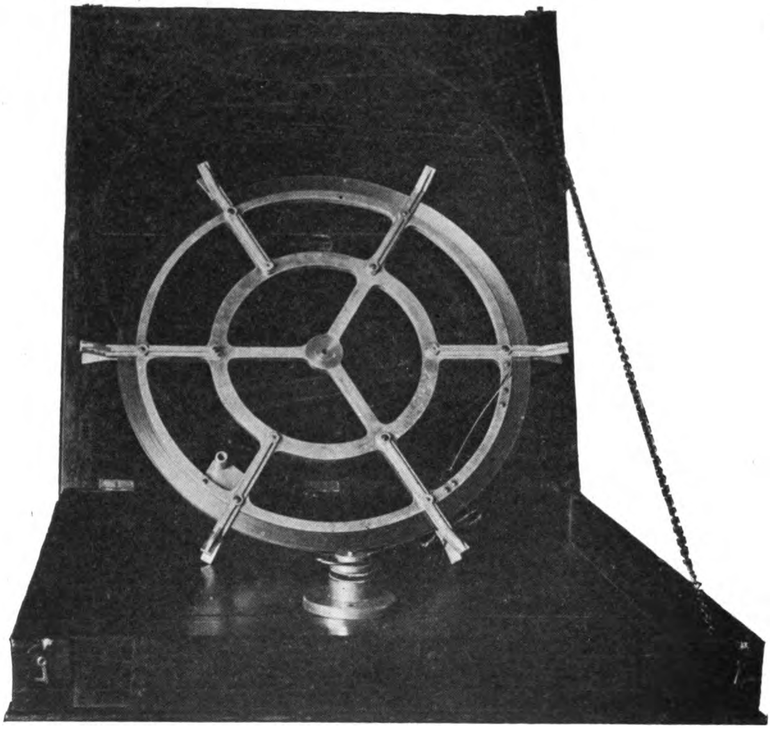


FIG. 20. Invar Tape on Reel.

tional Bureau of Weights and Measures, Paris. The tapes were made by J. H. Agar-Baugh, of London. The alloy mentioned has a very low coefficient of expansion, roughly one-twenty-fifth that of steel,* and consequently has a great advantage over steel

* The coefficient of steel is about 0.000011, that of invar is about 0.0000004, for 1° C.

for base-line measurement. The metal is more easily bent than steel, but with proper care in handling the tapes, and with the use of fairly large reels, there is little difficulty in making the measurements and in securing the required accuracy. The results of these tests on the invar tapes may be summed up as follows:

Measurements with invar tapes may be made during daylight with all the accuracy demanded in base-line work, whereas measurements with steel tapes must be made at night in order to secure the required accuracy.

In working by daylight the errors of observation are smaller and the party can make greater speed than when working at night.

On account of the small temperature coefficient of the invar tape any error due to the failure of the thermometers to indicate the true temperature of the tape has much less effect on the computed length when the measurements are made with invar than when they are made with steel.

Since it is not necessary to standardize the invar tape *in the field*, as was always done with the steel tape, the cost of measurements made with the invar is materially less than that of measurements made with steel.

The superiority of these tapes has been demonstrated by repeated trials, and they are now used almost exclusively by the Coast Survey in making base measurements.

22. Accuracy Required.

It is found that there is little, if any, advantage in measuring a base-line with a precision greater than one part in 500,000, since to do this would give the base-line a greater precision than could be maintained in the angle measurements. There is little difficulty, however, in obtaining a higher precision; the bases measured by the Coast Survey in 1906 and 1909 show a precision of one part in 2,000,000 or better. It is customary to divide bases into sections of about a kilometer in length, and to measure each section twice. If the two results show a discrepancy greater than $20^{mm} \sqrt{K}$ (K being the number of kilometers in the section), the measurements are repeated until they do agree within this limit; if the first two results agree within this limit, no additional measurements are taken. This procedure is consistent with the requirement that the base be measured with a precision of at least 1 in 500,000, but that no attempt be made to increase the precision much beyond this limit.

23. Description of Apparatus.

The invar tapes are usually about 53 meters long, with two graduations 50 meters apart. In some tapes a length of one decimeter at each end of the 50-meter length is subdivided into millimeters for convenience in reading. Intermediate points on the tape, such as the 25 meter point, are marked by single lines. The tape is about $\frac{1}{4}$ inch $\times$ $\frac{1}{50}$ inch in cross section and weighs about 25 grams per meter. This metal is softer than steel and has to be wound on a reel of at least 16 inches diameter in order to avoid permanent bends in the tape and consequent changes in length. (Fig. 20.) In use it is supported at the ends and usually at one intermediate point. The tension is applied by means of a spring balance reading to 25 grams, the tension ordinarily used being 15 kilograms. An apparatus used for applying the tension and similar to that used by the Coast Survey is shown in Fig. 21. The point of the iron bar holding the spring balance is pushed into the ground, and the upper end is moved right or left to align the tape. The adjustable clamp makes it possible to raise or lower the balance so as to bring the end of the tape to the right grade. The spring balance employed is a commercial article and is constructed to read correctly when held in a vertical position and with the weight hung on the hook. When the balance is used in a horizontal position, the true tension is greater than the indicated tension. The correction to be applied to the scale readings is found by suspending known weights on a cord passing over a pulley and secured to the hook of the balance when held in a horizontal position. The thermometers used with this apparatus are graduated to half degrees and are provided

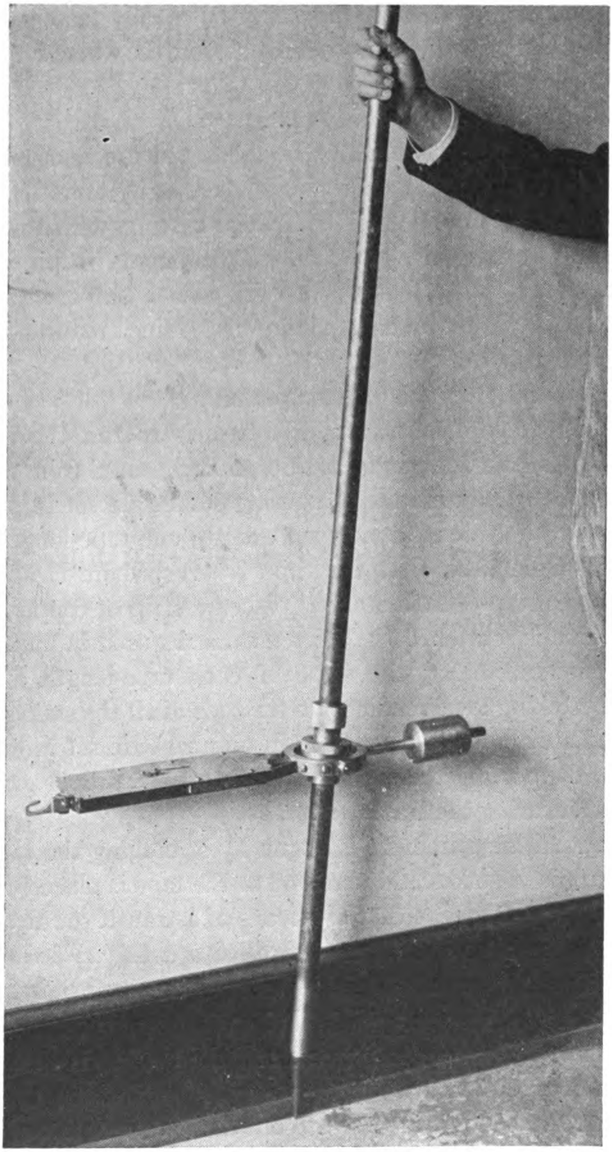


FIG. 21. Tension Apparatus.

with spring clamps so that they may be readily fastened to the tape for making readings, or removed from it when it is being carried forward.

24. Marking the Terminal Points.

The ends of the base line to be measured are marked in the same manner as triangulation points, that is, by bolts set in drill holes in stone monuments or by special castings set in concrete; the points are tied in by several measurements to prevent the position being lost. There is usually also a sub-surface mark (see Art. 13). Intermediate points on the line are often marked by stone or concrete posts.

25. Preparation for the Measurement.

The first step in measuring the base is to run the line out roughly with transit and tape and clear the ground from obstructions; at the same time the measuring stakes are set in position. These may be 4″ × 4″ stakes set exactly one tape-length apart and high enough so that the tape is everywhere clear of the ground. On top of each stake is placed a strip of copper or zinc upon which is scratched the reference marks used in making the measurements. Next, the slope of each tape-length is determined by taking level readings on the tops of all the stakes. The intermediate stakes (one or three in number) are set in line, and nails for supporting the tape are placed at the proper grade.

26. Measuring the Base.

The actual measurement is begun by stretching the tape over the first pair of stakes; the zero end of the tape is placed over the end mark of the base, either by means of a transit set at one side of the line or by a special device called a *cut-off cylinder*. The tape is aligned by means of field glasses or by a transit set on line and the tension is then applied. When the zero graduation of the tape is exactly over the end mark and the tension is correct, the position of the forward (50 meter) end is marked on the metal strip, and the temperature is read on all the thermometers. The tape is then carried forward and the process repeated until the measurement of the section is completed. If there is a short measurement at the end of the line, this may be taken with an ordinary metric steel tape graduated its whole length. Whenever it is necessary to set forward or backward on one of the metal strips in order to bring the reference mark on the millimeter scale, this fact is recorded; it is also indicated on the metal strips, which are all preserved as a part of the permanent record. Measurements of bases made in this manner can be made at the rate of about 2 kilometers per hour. If the wind is blowing, it may be found necessary to use three intermediate supports in order to maintain the required standard of accuracy. If the first two measurements of any one section of the base show a discrepancy not exceeding $20^{mm} \times \sqrt{K}$, the mean is considered as sufficiently accurate and no further measurements of this section are made.

27. Corrections to Base-Line Measurements. — Correction for Grade.

Where the slope is determined by direct leveling, the most convenient formula for computing the horizontal distance is one

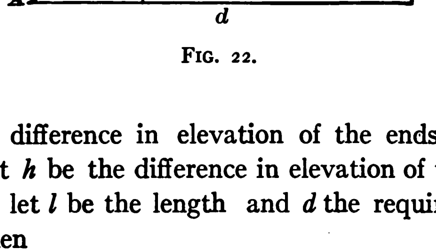


FIG. 22.

involving the difference in elevation of the ends of the tape. In Fig. 22, let h be the difference in elevation of the end points A and B, and let l be the length and d the required horizontal distance. Then

$$\text{Corr. for grade} = C_g = l - d = l - \sqrt{l^2 - h^2} = l - l\sqrt{1 - \frac{h^2}{l^2}}.$$

But $$\left(1 - \frac{h^2}{l^2}\right)^{\frac{1}{2}} = 1 - \frac{h^2}{2\,l^2} - \frac{h^4}{8\,l^4} - \cdots.$$

Therefore $$C_g = l - l\left(1 - \frac{h^2}{2\,l^2} - \frac{h^4}{8\,l^4}\cdots\right)$$
$$= \frac{h^2}{2\,l} + \frac{h^4}{8\,l^3} + \cdots. \qquad [1]$$

If the slope has been found in terms of the vertical angle α, the correction may be computed by the expression

$$C_g = 2\,l\sin^2\tfrac{1}{2}\,\alpha = l\text{ vers }\alpha. \qquad [2]$$

In good base-line work the errors in length due to errors in determining the grade should never exceed one part in one million.

28. Corrections for Alignment.

The errors in aligning a straight base-line can easily be kept so small as to be negligible. If any point is found, however, to be out of line by an amount sufficient to affect the length, the correction may be computed by Formula [1].

29. Broken Base.

Sometimes it is desirable or necessary to break a base into two parts which make a small (deflection) angle with each other. If the two sections are measured with the usual precision, and if the angle also is accurately measured, the length may be computed

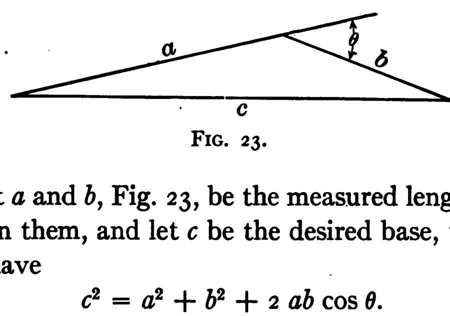


FIG. 23.

as follows: let a and b, Fig. 23, be the measured lengths, and θ the angle between them, and let c be the desired base, then from the triangle we have

$$c^2 = a^2 + b^2 + 2\,ab\cos\theta.$$

Putting for $\cos\theta$ the series $1 - \frac{\theta^2}{2} + \cdots$, there results

$$c^2 = (a + b)^2 - ab\theta^2.$$

Placing the factor $(a + b)^2$ outside the brackets and extracting the square root,

$$c = (a + b)\left[1 - \frac{ab\theta^2}{(a + b)^2}\right]^{\frac{1}{2}}$$

$$= a + b - \frac{ab\theta^2}{2\,(a + b)} + \cdots,$$

or

$$c = a + b - \frac{ab\theta^2}{2\,(a + b)}(\sin 1')^2,$$

where θ is in minutes of arc. Substituting the value of $\sin 1'$,

$$c = a + b - 0.000{,}000{,}042{,}308\,\frac{ab\theta^2}{a + b}. \qquad [3]$$

$$(\log.\ 0.000{,}000{,}042{,}308 = 2.62642 - 10).$$

30. Correction for Temperature.

The temperature correction may be computed if we know the coefficient of expansion, the actual temperature of the tape and the standard temperature, and the measured length of line. If k is the coefficient, t the observed temperature, t_0 the standard temperature, and L the measured length, then

$$\text{Temperature correction} = +kL\,(t - t_0). \qquad [4]$$

The temperature correction is often expressed as a term in the tape equation, as shown in the following article.

31. Correction for Absolute Length.

The length of the tape is usually expressed in the form of an equation, such as

$$T_{516} = 50^{m} + (12.382^{mm} \pm 0.016^{mm}) + (0.0178^{mm} \pm 0.0007^{mm})\,(t - 25^{\circ}.8\ \text{C.}), \qquad [5]$$

meaning that tape number 516 is 12.382^{mm} more than 50^{m} long at a temperature of $25^{\circ}.8$ C., and that 0.016^{mm} is the uncertainty of this determination. The quantity 0.0178 is the temperature change for 1° for a 50^{m} length, and 0.0007 is the uncertainty in this number. (The temperature coefficient for this tape is 0.000,000,356.)

According to the present practice, tapes are standardized at Washington * under exactly the same conditions, in regard to tension, temperature determination, and manner of support, as those which are to govern the field measurements. By this means all uncertainty in the absolute length and in the tension correction is kept within narrow limits.

32. Reduction of Base to Sea-Level.

In order that all triangulation lines may be referred to the same surface it is customary to employ the length of the line at sea-level between the verticals through the stations.

In Fig. 24, let B represent the measured base at elevation h above sea-level (supposed spherical), and b the length of base reduced to sea-level, R_α being the radius of curvature of the surface (see Art. 97 and Table XI). Then, since the arcs are proportional to their radii,

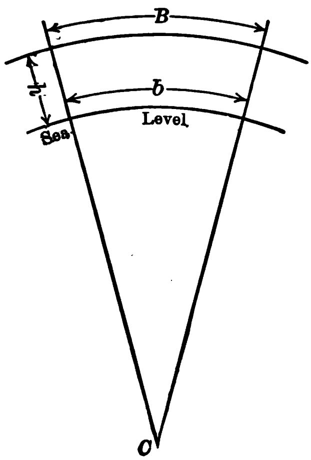


FIG. 24.

$$\frac{b}{B} = \frac{R_\alpha}{R_\alpha + h}$$

and

$$b = B\frac{1}{1 + \frac{h}{R_\alpha}}$$

$$= B\left(1 - \frac{h}{R_\alpha} + \frac{h^2}{R_\alpha^2} - \cdots\right).†$$

Therefore the reduction to sea-level is

$$b - B = -B\frac{h}{R_\alpha} + B\frac{h^2}{R_\alpha^2} - \cdots. \qquad [6]$$

If there is a great difference in elevation in different parts of the base, the line should be divided into sections and the mean value of h found for each section. Then B in the formula is taken as

* United States Bureau of Standards, Washington, D. C.

† See footnote on page 51.

the length of the section in question. The logarithm of the mean radius of curvature in latitude 45°, which may be used for short sections, is 6.80470.

Question. Is it necessary to reduce each triangulation line separately to sea-level?

33. Correction for Sag.

Between any two consecutive points of support the tape hangs in a curve known as the *catenary*, its form depending upon the weight of the tape, the tension applied, and the distance between the points of support.

In Fig. 25 let l be the horizontal distance between the supports, the two being supposed at the same level; let n be the number of

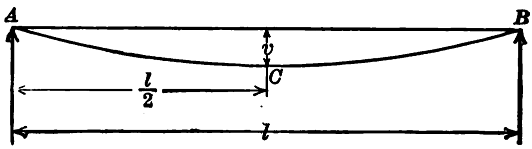


FIG. 25.

such spans in the tape-length, t the tension, and w the weight of a piece of tape of unit length. Also let v equal the (vertical) sag of the middle point of the tape below the points of support. Since the curve is really quite flat under the tension actually employed in field-work, the length of the catenary will be sensibly equal to that of a parabola whose axis is vertical and which passes through the points A, B, and C. The equation of this parabola is $x^2 = \frac{l^2}{4v} \cdot y$, and the length of curve, found by the usual method of the calculus, is $2s = l + \frac{8v^2}{3l} + \cdots$. The difference $2s - l$ between the length of curve AB and the chord AB is approximately

$$2s - l = \frac{8}{3} \times \frac{v^2}{l}. \qquad [a]$$

If we consider the forces acting on the tape at the point C, and take moments about the point of support A, we have

$$\frac{wl}{2} \times \frac{l}{4} = v \cdot t.$$

Therefore $$v = \frac{wl^2}{8t}. \qquad [b]$$

Substituting in [a] the value of v found in [b], we find that the shortening of this section of tape due to sag is

$$2s - l = \frac{8}{3l}\left(\frac{wl^2}{8t}\right)^2 = \frac{l}{24}\left(\frac{wl}{t}\right)^2.$$

For n sections, we have $nl = L$, whence

$$\text{Correction for sag} = C_s = \frac{L}{24}\left(\frac{wl}{t}\right)^2. \qquad [7]$$

34. Tension.

The modulus of elasticity of the tape due to the tension applied equals the stress divided by the strain. If a = the elongation and L the length, and if t equals the tension and S the area of the cross section, then the modulus of elasticity E is given by

$$E = \frac{t}{S} \div \frac{a}{L}$$

$$= \frac{Lt}{Sa}.$$

The elongation is

$$a = C_t = \frac{Lt}{SE}, \qquad [8]$$

where C_t is the correction for the increase in length due to tension. Evidently the difference in length due to a change from tension t_0 to tension t is $a = \frac{L}{SE}(t - t_0)$.

The value of E must be found by trial, applying known tensions and observing a directly.

To allow for slight variations in tension, such as those due to the failure of the spring balance to give the desired reading the instant the scale of the tape is read, the correction may be derived as follows:

Since the effective length of the tape depends both upon the elongation due to tension and upon the shortening due to sag, and since these both involve t, the variation may be found by differentiating the expression

$$L_1 = L + C_t - C_s$$
$$= L + \frac{L \cdot t}{SE} - \frac{L}{24}\left(\frac{wl}{t}\right)^2,$$

regarding t as the independent variable. The differentiation gives

$$dL = \frac{L}{SE}dt + \frac{L}{12} \cdot \left(\frac{wl}{t}\right)^2 \cdot \frac{dt}{t}. \qquad [9]$$

This is the correction due to small variations in t. This quantity may be found satisfactorily by actual tests, varying t by known amounts and observing the change in length directly.

It was once the practice to compare the tape with the standard when it was supported its entire length, and to calculate the sag and tension corrections to obtain the effective length when supported at a few points. The present practice of comparing the tape under the same conditions that are to exist in the field-work eliminates all uncertainty in these computed corrections.

PROBLEMS

Problem 1. Derive the equation of the parabola stated in Art. 33. Compute the length of the parabola between the points of support A and B.

Problem 2. The difference in elevation of the ends of a 50-meter tape is 7.22 ft., obtained by leveling. What is the horizontal distance?

Problem 3. A base line is broken into two sections which meet at an angle of 1° 59′ 31″.6. The lengths of the two segments are 1854.275 meters and 3940.740 meters. What is the distance between the terminal points?

Problem 4. The length of a base line is $17486^m.5800$ measured at an altitude of 34.16 meters. The latitude of the middle point of the base is 38° 36′. The azimuth of the base is 16° 54′. What is the corresponding length of the base at sea-level?

CHAPTER III

FIELD-WORK OF TRIANGULATION — MEASUREMENT OF HORIZONTAL ANGLES

35. Instruments Used in Measuring Horizontal Angles.

Instruments intended for triangulation work are of two kinds: the *direction instrument*, first designed in England by Ramsden in 1787, and the *repeating instrument*, first used in France about 1790. The former is the one chiefly used at the present time for primary triangulation; the repeating instrument, on account of its comparative lightness and simplicity, is much used on triangulation of lesser importance.

Triangulation instruments are larger than ordinary surveying transits, the diameter of the circles varying in different instruments from 8 to 30 inches. It is found, however, that small circles can be graduated so accurately that little or nothing is gained by using circles more than from 10 to 12 inches in diameter. Furthermore, the smaller circles are less affected by flexure than the larger circles. All triangulation instruments except the very smallest are built with three leveling screws and are used on solid supports, like stone piers, or on the tripods of observing towers. Small instruments intended for work of a lower grade of accuracy may be used on their own tripods.

36. The Repeating Instrument.

The repeating instrument has an upper and a lower plate arranged exactly as in the surveyor's transit, and the graduated circle is read by two or more verniers graduated to 10″ or to 5″. Verniers reading finer than 5″ are not practicable, and dependence must be placed upon the repetition principle for securing greater precision. Fig. 26 shows a repeating instrument having an 8-inch circle which is read by two verniers to 10 seconds. The

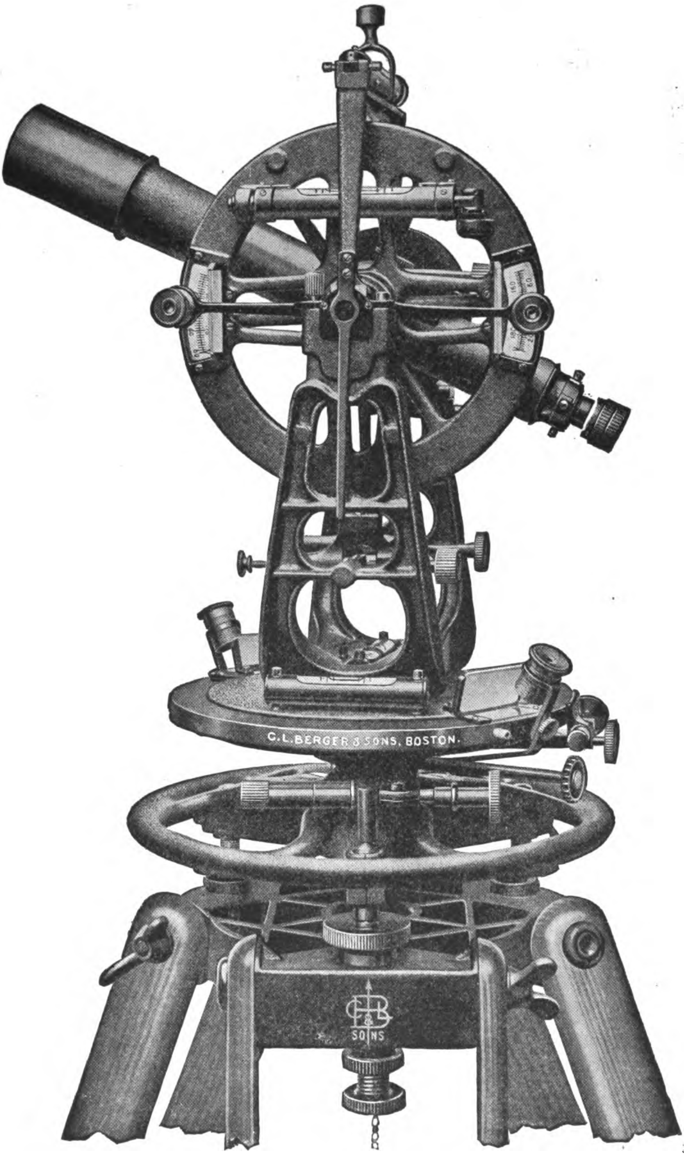


FIG. 26. Repeating Instrument.
(C. L. Berger & Sons.)

telescope of this instrument has an aperture of $1\frac{1}{2}$ inches and a magnifying power of 30. Since an instrument of this kind is likely to be used in sighting on pole signals, the cross-hairs are usually arranged in the form of an X, the pole bisecting the angle between the hairs when the pointing is made. Single vertical hairs would not be practicable except on short lines and wide signals, as the width of the ordinary hair is so great that it completely obscures the pole on long distances.

37. The Direction Instrument.

The direction instrument has but one horizontal circle, read by two or more microscopes instead of verniers. The circle can be turned about the axis and clamped in any desired position. The motion of the telescope and the microscopes is entirely independent of the motion of the circle; the latter can be shifted while the upper part of the instrument (called the *alidade*) remains clamped. It is evident that a repeater could be used as a direction instrument, but that a direction instrument could not be used for measuring angles by the repetition method. Fig. 27 shows a 12-inch theodolite with microscopes reading to seconds.

The circle of the direction instrument is usually graduated into 5′ spaces. The direction of the line of sight of the telescope is read by first noting the degrees and 5′ spaces in a small index microscope, and then accurately measuring the fractional parts of the 5′ spaces by means of the three equidistant micrometer microscopes. The micrometers can usually be read to seconds directly, and to tenths of a second by estimation. The mean of the three micrometer readings is taken as the true reading, and this is added to the reading of the index microscope to obtain the direction.

The telescope of the 12-inch theodolite used by the Coast Survey has an aperture of 2.4 inches, a focal length of 29 inches, and magnifying powers of 30, 45, and 60. The circle is graduated to 5′ and reads to seconds by means of three microscopes. A camel's-hair brush (inside the cover plate) sweeps over the graduations. The base is made very heavy, and the bearing

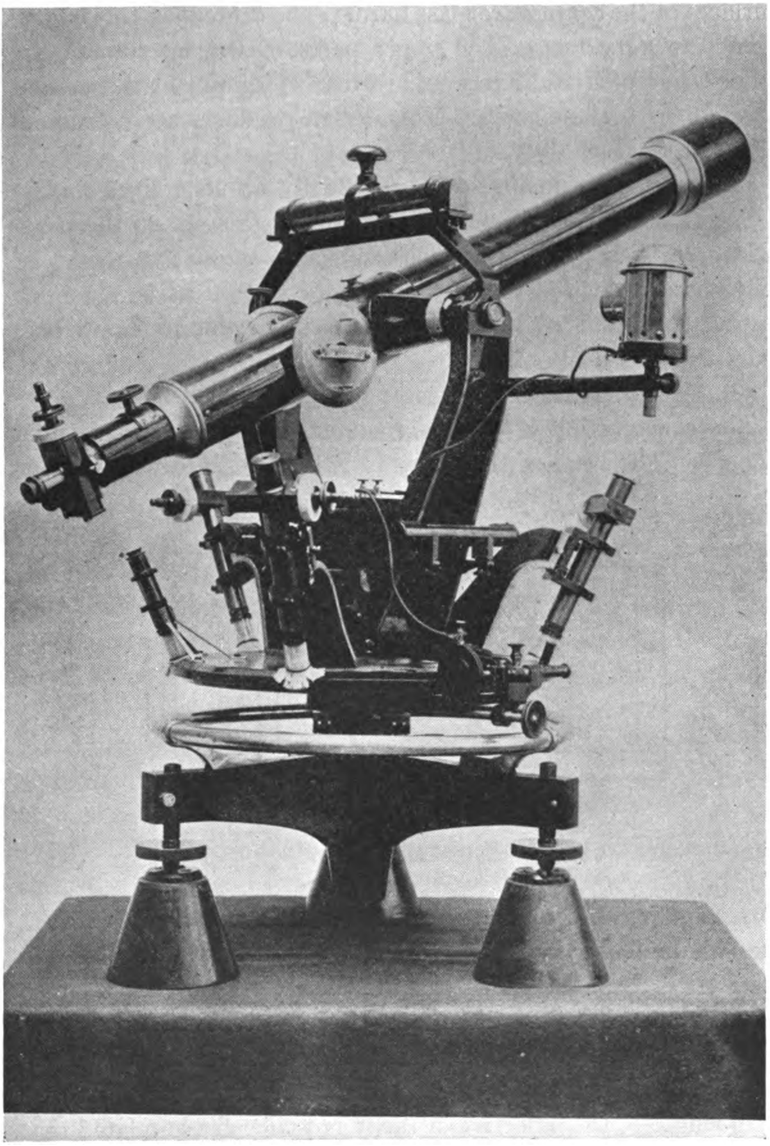


FIG. 27. Twelve-inch Theodolite.
(Coast and Geodetic Survey.)

surfaces of the centers are glass hard. The centers on this instrument are very long. The upper parts of the instrument are made chiefly of aluminum, in order to diminish the weight bearing upon the centers. This design produces an instrument of exceptional stability.

Direction instruments are used chiefly on long lines and in connection with heliotropes or lights. For this reason the cross-hairs usually consist of two vertical hairs, set so as to subtend an angle of from 10″ to 20″, and two horizontal hairs, set much farther apart and used merely to limit the portion of the vertical hairs to be used in pointing.

38. The Micrometer Microscope.

The construction of the micrometer microscope is shown in Fig. 28. The head of the screw is graduated into 60 divisions

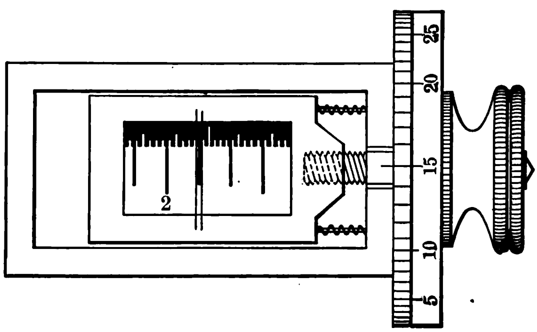


FIG. 28.

corresponding to seconds of angle. As the screw head is turned the two parallel hairs in the field of the microscope are moved in a direction parallel (tangent) to the edge of the graduated circle. The distance between these hairs is just sufficient to leave a small white space on each side of a line of graduation when it is centered between the two hairs. The pitch of the screw and the focal length of the objective of the microscope are so related that five whole turns of the screw will carry the hairs from one graduation to the next. The number of whole turns of the screw may be counted on a notched scale visible in the field of view of the microscope. The fraction of a space to be measured is that lying between the zero point of the notched (comb) scale and the graduated line last passed over by the zero point. Strictly speaking, the zero point is that position of the hairs in the zero notch at which the scale on the screw head will read exactly zero. The position of the hairs for a zero reading of the screw may be adjusted by holding fast the graduated ring on the screw head and turning the milled edge screw head which moves the hairs. The microscope inverts the image of the graduated circle so that graduations increasing in the direction of azimuths will appear to increase from left to right in the field of view of the microscope. The readings on the screw head increase as the screw is turned left-handed, and the hair lines move in the direction of decreasing graduations over the circle.

To measure the space between the zero of the microscope and the last line passed over, it is only necessary to turn the screw until the graduation in question bisects the space between the hairs, and then to read the comb scale and the scale on the screw head. This reading is to be added to the number of the graduated line, to obtain the direction as shown by this microscope. For example, if the screw is turned two revolutions (two notches) and ten divisions in order to center the $47° 05'$ mark between the hairs, the reading of this microscope is $47° 05' + 2' 10'' = 47° 07' 10''$. A complete set of readings of one direction would consist of readings of each of the three microscopes on both the preceding and the following graduations, six readings in all.

39. Run of the Micrometer.

If the microscope is perfectly adjusted with respect to the graduated circle, and if the latter is perfectly plane, then five whole revolutions of the screw should carry the hairs from one line to the next, and the reading of the screw should be the same on all lines. Since this condition is rarely fulfilled, there is ordinarily a small difference in the forward and backward readings, called the *error of run* of the micrometer.

The forward reading F is the reading taken when the threads are moved from the zero position (Fig. 29) to the preceding mark (25′ in Fig. 29a). The back reading B is the one taken on the following (30′) mark, Fig. 29b. The graduations on the screw-head decrease as the threads move from 25′ to 30′. If the

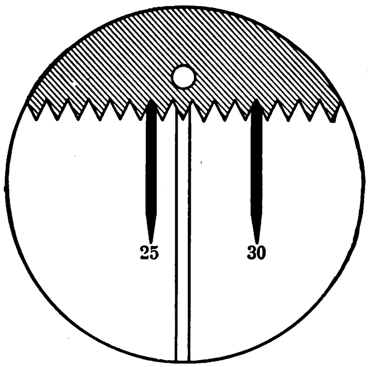


FIG. 29. Field of Micrometer Microscope.

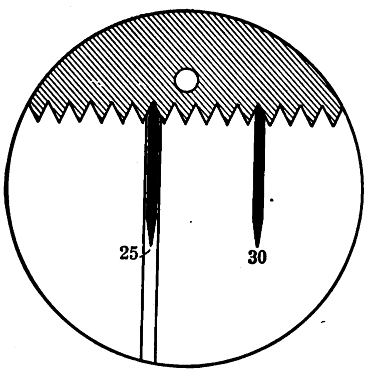


FIG. 29a. Forward Reading.

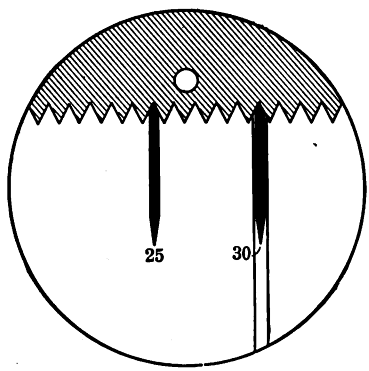


FIG. 29b. Back Reading.

micrometer screw is turned so that the threads move from its zero to the 25′ mark, then the reading F is to be added directly to the circle reading. In the figure the reading is 201° 25′ + 1′ 26.2″ = 201° 26′ 26.2″. Without assuming anything in regard to the actual value of one turn of the screw, the value may be computed by dividing the angular space between graduations by the number of turns or divisions recorded in passing from one graduation to the next. If R = the value of one revolution, then

$$R = \frac{300''}{300 + F - B} = \frac{300''}{300 + r},$$

where r is the run of the micrometer in seconds (divisions) as indicated by the differences of the forward and backward readings, positive if F is greater than B. If the screw turns more than five times in passing from the 25′ line to the 30′ line, the reading B, on the 30′ line, will be smaller than F, since the screw readings are decreasing. This makes $F - B = r$ positive. Hence the denominator of the above fraction is greater than 300, and the value of one turn is less than unity, as it should be according to the assumption.

If F is the forward reading in any given case, it must be converted into arc by multiplying it by the value of one turn, since F is simply a certain number of turns and divisions, not the true number of minutes and seconds,

Therefore $\qquad$ True reading $= F\left(\dfrac{300''}{300 + r}\right)$.

Since r is small (say 2″ or 3″), it is permissible to write

$$R = \frac{300''}{300 + r} = \frac{1}{1 + \frac{r}{300}} = 1 - \frac{r}{300} + \cdots,^*$$

and the true reading $= F\left(1 - \dfrac{r}{300} \cdots\right)$

$$= F - F\frac{r}{300}. \qquad (a)$$

This formula corrects the forward reading only, and assumes

* By actual division $\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots$. If x is small enough so that x^2 and the following term may be neglected, then $\frac{1}{1 + x} = 1 - x$.

that the bisection and reading are perfectly made. If the back reading is corrected in a similar manner, the result is

$$\text{True reading} = 300'' - (300'' - B)\left(1 - \frac{r}{300}\right).$$

The first 300″ is the space between 25′ and 30′. The factor (300″ − B) is the space between zero and the 30′ mark determined by the B reading. It should be remembered that when the micrometer is turned to the 30′ mark, the readings are decreasing; therefore the direct reading does not give this space, but 5 minus this space. Simplifying this expression we have

$$\text{True reading} = B + r - B\frac{r}{300}. \qquad (b)$$

Since there is no reason for preferring either the forward or the backward reading, the mean is used as the best value. The mean of (a) and (b) is

$$\frac{F+B}{2} + \frac{r}{2} - \frac{F+B}{2} \times \frac{r}{300}.$$

If $\frac{F+B}{2} = m$, then the correction to m, the mean of the two readings, is

$$\text{Corr.} = \frac{r}{2} - m\frac{r}{300}. \qquad [10]$$

A general table may be computed for different values of m and r, so that no special computation is necessary when correcting a direction. It is good practice to determine r from all the F and B readings, and to employ this average value when making the corrections.

40. Vertical Collimator.

In centering a signal over a station, placing a mark under a new signal, or centering the theodolite over the station mark, the Coast Survey observers sometimes employ the vertical collimator shown in Fig. 30. The instrument is adjusted by means of spirit levels, which revolve around a vertical axis like those of a transit.

A telescope may be placed in coincidence with the vertical axis of the collimator, and its line of sight adjusted to point vertically downward. With the instrument in this position the observer may obtain a point which is vertically above the center mark of the station.

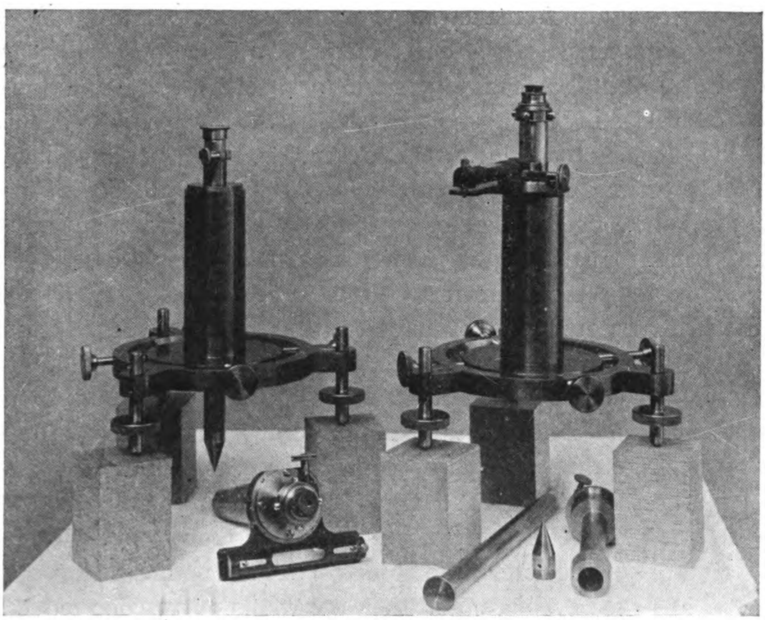


FIG. 30. Vertical Collimator.
(Coast and Geodetic Survey.)

41. Adjustments of the Theodolite.

The adjustment of the levels attached to the alidade is made by means of reversals about the vertical axis of the instrument, exactly as with the engineer's transit.

The adjustment of the stride level is tested by placing it on the horizontal axis, reading both ends of the bubble, and then reversing the level and reading again. The adjusting screws of the stride level should be turned so that the bubble moves halfway back from the second position to the first. When the stride level is so adjusted that it reads the same in either position, it is in correct adjustment, and the horizontal rotation axis may then be leveled by moving the adjustable end of the axis until the bubble is in the center of its tube. Of course the two adjustments may be made simultaneously. If desired, the stride level may be used also to make the vertical axis truly vertical.

The adjustment of the line of sight in a plane perpendicular to the horizontal axis may be made by reversals about the horizontal axis as in testing an engineer's transit; or it may be made by sighting an object, lifting the telescope out of its bearings, and, after reversing the axis, replacing it in the bearings. If the object is no longer in the line of sight, the reticle is brought halfway back from the second position toward the first.

The test of the adjustment of the microscopes is made by measuring the run of each micrometer, taking first a forward and then a back reading. In case the run of a micrometer is greater than about 3″, it should be adjusted by changing the distance from the objective to the reticle, and then moving the whole microscope so that the graduations are again in focus. If the image of the division is greater than 5 whole turns of the screw, the objective should be moved toward the eyepiece, and then the whole microscope moved away from the circle. Moving the objective away from the micrometer lines diminishes the angle between the two lines of sight corresponding to the 5 turns, and reduces the size of the image of the division (Fig. 31). It will usually require a series of trials to perfect this adjustment.

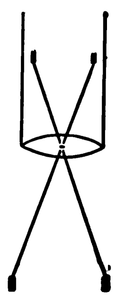


FIG. 31.

42. Effect of Errors of Adjustment on Horizontal Angles. The effect of errors due to the inclination of the horizontal axis to the horizon, and those due to the imperfect adjustment for collimation (line of sight), are not independent of each other. These errors are usually so small, however, that it is permissible to compute their effect separately, as though only one existed at one time. In Fig. 32, Z is the true zenith and Z' the point where the vertical axis of the instrument prolonged pierces the celestial sphere. S is a point whose altitude is h. Assuming that the horizontal axis makes an angle i with the horizon, and that all other errors are zero, then from the figure it will be seen that we may write

$$\frac{\sin Z'}{\sin i} = \frac{\sin HS}{\sin Z'S},$$

or, with sufficient accuracy,

$$Z = i \tan h, \qquad [11]$$

where h is the angular altitude of the point sighted.

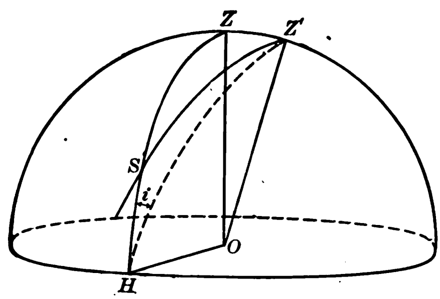


FIG. 32.

It appears, then, that for each point sighted there should be a correction to the circle reading equal to $i \tan h$. Triangulation points are usually so nearly on the horizon, and by careful attention to the leveling the error i may easily be kept so small, that there is seldom any necessity for applying the correction except for such observations as those on a circumpolar star for azimuth.

In the preceding paragraph it is assumed that the vertical axis is truly vertical, the graduated circle being horizontal, while the horizontal axis is not horizontal. If the two axes are at right angles to each other, but the vertical axis is inclined to the true vertical by a small angle i, it may be shown, by a diagram similar to Fig. 32, that the same correction applies to this case also.

The error of a horizontal direction due to an error of collimation may be computed as follows: Let the error in the sight line be represented by c; then, when the axis of collimation (Fig. 33) traces out the great circle ZN, the line of sight traces out the parallel circle SA, which is c seconds from ZN. If S be any point

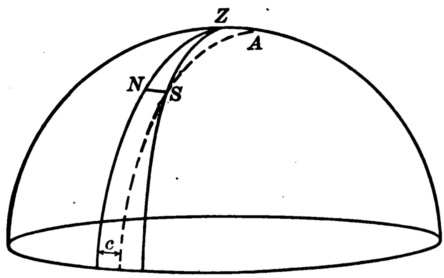


FIG. 33.

toward which the cross-hair is pointing, and if arc SN be drawn perpendicular to ZN, the error in direction, or the angle at Z, is found from the equation

$$\frac{\sin Z}{\sin N} = \frac{\sin c}{\sin ZS},$$

or, since $\angle N = 90°$, $$Z = c \sec h. \qquad [12]$$

Each direction should therefore be corrected by the quantity $c \sec h$. On account of the small value of c in a well-adjusted instrument this correction is necessarily small; furthermore, it is usually eliminated from the final result by the method employed in making the observations.

43. Method of Measuring the Angles.

In measuring angles with the repeating instrument the common practice has been to measure the angle six times, beginning with the left-hand signal of a pair and measuring toward the right, and then, after reversing the telescope both in altitude and in azimuth, to measure six times from right to left. The recent practice of the Coast Survey has been to measure first the angle itself by six repetitions, left to right, with the telescope direct, then the explement (360° minus the angle) six times, moving the alidade in the same direction as before, left to right, the telescope being reversed. This brings the vernier nearly back to the same reading as by the previous method, but it differs in the mechanical operation. If there is any systematic effect on the angle, due to the action of clamps or to drag on the centers, it is eliminated from the final result, provided such errors are the same for a large as for a small angle.

The reversal of the telescope in the preceding programs is intended to eliminate the errors of adjustment of the line of collimation and of the rotation axis of the telescope. It does not eliminate errors due to imperfect leveling. The measurement of angles in both the left-to-right and the right-to-left direction is designed to eliminate possible *twist* in the support of the instrument, upon the assumption that this twist takes place at a uniform rate.

In order to eliminate errors due to faulty graduation of the circle, the initial reading for different sets of observations may be shifted by $\frac{360^\circ}{mn}$, where m is the number of sets taken and n is the number of verniers. For example, in taking four sets with a two-vernier instrument, the vernier would be set ahead 45° each time. Errors in the graduation of the verniers may be eliminated in a similar manner by changing the vernier setting $\frac{1}{m}$th part of a circle division at the beginning of each new set. For four sets, on a 10′ graduation, the first setting might be zero, the second 45° 02′ 30″, the third 90° 05′ 00″, and the fourth 135° 07′ 30″.

With the direction instrument the method of measurement consists in first pointing the telescope at some conspicuous signal, selected as the first of the series around the horizon, and reading all the microscopes, then turning the telescope to the other signals in order and reading all the microscopes at each pointing. After the last pointing has been completed and the microscopes read, the telescope is reversed, the pivots remaining in the same bearings, and the series is repeated, the signals being sighted in the reversed order. The horizontal circle remains clamped during the entire process. The above measurements constitute a single "set." As many sets may be taken as are required to give the necessary accuracy. To eliminate systematic errors of graduation and errors of the micrometers, the circle reading is advanced for each new set, as explained later in the "Instructions for Primary Triangulation." It should be observed that the accuracy depends upon the circle's remaining undisturbed in azimuth during each set.

In making bisections, either when pointing on the signal or when reading the microscopes, the observer should proceed as rapidly as he can without making careless pointings and without danger of making mistakes. Much time spent in perfecting settings and in watching them to see if they are correct appears to reduce slightly the accidental errors of observation, but does not really increase the accuracy of the work, as shown by the final results of the triangulation. The longer the time that is permitted to intervene between pointings, the greater the opportunity for the circle to shift its position or change its temperature; and the effects of these changes are probably greater than the accidental errors of pointing and reading.

44. Program for Measuring Angles.

Various programs of observations have been devised, with a view to eliminating or reducing the errors in horizontal angles. The principal errors which have to be considered in planning the field work of the triangulation, and the methods adopted for eliminating them, are as follows:

1. *Errors due to non-adjustment of the theodolite.* These are all eliminated by the use of the instrument in the direct and reversed positions, except that due to erroneous leveling. The leveling of the plate and the horizontal axis must be carefully attended to in setting up the instrument, and must be corrected whenever it becomes necessary. This may be done at any time

between sets of angles or, if a repeater is used, at any time when the lower clamp is loose.

2. *Errors arising from imperfections of graduation.* These are practically eliminated by distributing the readings uniformly around the circle.

3. *Errors of eccentricity of the circle and alidade.* These errors are almost wholly eliminated by reading two or more equidistant microscopes or verniers.

4. *Errors due to twisting of the tripod under the action of the sun's rays.* The twist is eliminated by reversing the direction of the measurements, provided the rate of twist and the speed of measuring the angles are both uniform. The rate of twist on some towers has been found to be about one second of angle per minute of time. On the slender towers used on the 98th meridian triangulation, the twist was so small that it could not be detected by an examination of the measurements.

5. *Errors due to irregular refraction of the atmosphere and to difficult seeing.* These will be partly eliminated by taking a large number of measurements; if the results indicate that the necessary precision is not being obtained, it will be best to wait until the conditions are more favorable. This can be judged best by the "probable error" of the direction.

6. *The personal error of the observer.* The personal error is partly eliminated by measuring the angle a large number of times.

7. *Errors due to temperature and wind.* Errors due to fluctuation of the temperature of the instrument, and to vibrations caused by wind, may be reduced by shielding the instrument from the sun and wind, either by a tent or by a temporary building.

The following list of instructions to observers is taken from the Coast and Geodetic Survey Special Publication No. 19 (1914), and represents the present practice of that Survey.

1. *Instruments.* — In general, direction instruments of the highest grade should be used in triangulation of this class. Repeating theodolites are to be used only when the station to be occupied is in such a position as to be difficult of occupation

with a direction instrument or when there is doubt of the instrument support being of such a character as to insure that the movement of the observer about the instrument does not disturb it in azimuth. Such stations usually occur on lighthouses and buildings.

2. *Number of observations — Main scheme — Direction instrument.* — In making the measurements of horizontal directions measure each direction in the primary scheme 16 times, a direct and reverse reading being considered one measurement, and 16 positions of the circle are to be used, corresponding approximately to the following readings upon the initial signal:

Number.	Reading.	Number.	Reading.
	° ′ ″		° ′ ″
1	0 00 40	9	128 00 40
2	15 01 50	10	143 01 50
3	30 03 10	11	158 03 10
4	45 04 20	12	173 04 20
5	64 00 40	13	192 00 40
6	79 01 50	14	207 01 50
7	94 03 10	15	222 03 10
8	109 04 20	16	237 04 20

3. When a broken series is observed, the missing signals are to be observed later in connection with the chosen initial or with some other one, and only one, of the stations already observed in that series. With this system of observing no local adjustment is necessary. Little time should be spent in waiting for the doubtful signal to show. If it is not showing within, say, one minute of when wanted, pass to the next. A saving of time results from observing many or all of the signals in each series, provided there are no long waits for signals to show, but not otherwise.

4. *Standard of accuracy.* — In selecting the conditions under which to observe primary directions, proceed upon the assumption that the maximum speed consistent with the requirement that the closing error of a single triangle in the primary scheme shall seldom exceed three seconds, and that the average closing error shall be but little greater than one second, is what is desired rather than a greater accuracy than that indicated with slower progress. This standard of accuracy used in connection with other portions of these instructions defining the necessary strength of figures and frequency of bases will in general insure that the probable error of any base line, as computed from an adjacent base, is about 1 part in 88,000, and that the actual discrepancy between bases is always less than 1 part in 25,000.

5. *Rejections — Direction observations.* — The limit for rejection of observations upon directions in the main scheme shall be 5 seconds from the mean. No observation agreeing with the mean within this limit is to be rejected unless the rejection is made at the time of taking the observation and for some other reason than simply that the residual is large. A new observation is to be substituted for the rejected one before leaving the station, if possible without much delay.

. . . . . .

11. *Vertical measures in main scheme.* — At each station in the main scheme

vertical measures are to be made over all lines in the main scheme radiating from it. These vertical measures should be made on as many days as possible during the occupation of the station, but in no case should the occupation of the station be prolonged in order to secure such measures. Three measures, each with the telescope in both the direct and the reversed positions, on each day, are all that are required. These measures may be made at any time between 11.00 A.M. and 4.30 P.M., except that in no case should primary vertical measures be made within one hour of sunset. It is desirable, however, with a view of avoiding errors due to diurnal variation of refraction, to have a fixed habit of observing the verticals in the main scheme at a certain hour, as, for example, between 2 and 3 P.M. If the vertical measures at a station are made by the micrometric method, double zenith distance measures shall be made on at least two of the lines radiating from that station.

. . . . . .

13. *Marking of stations.* — Every station, whether it is in the main scheme or is a supplementary or intersection station, which is not in itself a permanent mark, as are lighthouses, church spires, cupolas, towers, large chimneys, sharp peaks, etc., shall be marked in a permanent manner. At least one reference mark of a permanent character shall be established not less than 10 meters from each station of the main scheme and accurately referred to it by a distance and direction. Such reference marks shall preferably be established on fence or property lines, and always in a locality chosen to avoid disturbance by cultivation, erosion, or building. It is desirable to establish such reference marks at all marked stations. At all stations where digging is feasible both underground and surface marks which are not in contact with each other shall be established. Wood is not to be used in permanent marks.

14. *Descriptions of stations.* — Descriptions shall be furnished of all marked stations. For each station which is in itself a mark, as are lighthouses, church spires, cupolas, towers, large chimneys, sharp peaks, etc., either a description must be furnished, or the records, lists of directions, and lists of positions must be made to show clearly in connection with each point by special words or phrases if necessary the exact point of the structure or object to which the horizontal and vertical measures refer. Every land section corner connected with the triangulation must be fully described. The purpose of the description is to enable one who is unfamiliar with the locality to find the exact point determined as the station and to know positively that he has found it. Nothing should be put into the description that does not serve this purpose. A sketch accompanying the description should not be used as a substitute for words. All essential facts which can be stated in words should be so stated, even though they are also shown in the sketch.

15. *Abstracts and duplicates.* — The field abstracts of horizontal directions and vertical measures are to be kept up and checked as the work progresses, and all notes as to eccentricities of signals or instrument, of height of point observed above ground, etc., which are necessary to enable the computation to be made, are to be incorporated in the abstracts. As soon as each volume of the original record has been fully abstracted and the abstracts checked, it is to be sent to the Office, the corresponding abstracts being retained by the observer. A duplicate of the description of stations is to be made. If the original descriptions of stations are

written in the record books, a copy of these descriptions compiled in a separate book may be considered the duplicate and should then be marked as such. A duplicate of the miscellaneous notes mentioned above may also be made if considered desirable. No other duplicates of the original records are to be made. Pencil originals should not be inked over.

16. *Number of observations — Main scheme — Repeating theodolite.* — If a repeating theodolite is used for observations in the main scheme, corresponding to those indicated in paragraph 2, make the observations in sets of six repetitions each. For each angle measured follow each set of six repetitions upon an angle with the telescope in the direct position immediately by a similar set of six on the explement of the angle with the telescope in the reversed position. It is not necessary to reverse the telescope during any set of six. Make the total number of sets of six repetitions on each angle ten — five directly on the angle and five on its explement. Measure only the single angles between adjacent lines of the primary scheme and the angle necessary to close the horizon. With this scheme of observing no local adjustment is necessary, except to distribute the horizon closure uniformly among the angles measured. The limit of rejection corresponding to that stated in paragraph 5 shall be for a set of six repetitions 4″ from the mean.

. . . . . .

19. *Field computations.* — The field computations are to be carried to hundredths of seconds in the angles, azimuths, latitudes, and longitudes, and to seven places in the logarithms. The field computation may be stopped with the completion of the lists of directions for all stations and objects, and the triangle side computation for the main scheme and supplementary stations, unless there are special reasons for carrying it further. The computation to this point should be kept up as closely as possible as the work progresses, to enable the observer to know that the observations are of the required degree of accuracy. No least square adjustments are to be made in the field. All of the computation, taking of means, etc., which is done in the record books and the lists of directions should be so thoroughly checked by some person other than the one who originally did it as to make it unnecessary to examine it in the Office. The initials of the person making and checking the computations in the record books and the lists of directions should be signed to the record as the computation and checking progress.

Investigations of the accumulated error in the azimuth of a chain of triangles indicate that there is a systematic tendency of the triangulation to twist in azimuth, due to unequal heating of the different parts of the theodolite by the sun. In day observations on arcs running north and south there appears to be a greater accumulated error in azimuth on the east side of the chain than on the west side. This is apparently due to the fact that the observations were made chiefly or wholly in the afternoon. Observations made at night show less difference between the two sides of a chain of triangles. The errors due to this cause may be diminished by making the instrument out of metal having a lower coefficient of expansion, such as nickel-iron, and by increasing the proportion of night observations. The unequal heating effect may also be diminished in day observations by turning the circle 180° in azimuth between sets. The following set of pointings, to be substituted for that on p. 60, is designed to accomplish this purpose.

CIRCLE READINGS FOR INITIAL DIRECTIONS.*

Position.	Telescope direct.	Telescope reversed.	Position.	Telescope direct.	Telescope reversed.
	° ′ ″	° ′ ″		° ′ ″	° ′ ″
1	0 00 40	180 00 40	9	128 00 40	308 00 40
2	195 01 50	15 01 50	10	323 01 50	143 01 50
3	30 03 10	210 03 10	11	158 03 10	338 03 10
4	225 04 20	45 04 20	12	353 04 20	173 04 20
5	64 00 40	244 00 40	13	192 00 40	12 00 40
6	259 01 50	79 01 50	14	27 01 50	207 01 50
7	94 03 10	274 03 10	15	222 03 10	42 03 10
8	289 04 20	109 04 20	16	57 04 20	237 04 20

For a method of correcting azimuths for the accumulated twist of triangulation, see page 202.

45. Time for Measuring Horizontal Angles.

It was formerly the practice to measure angles only during that part of the day when signals appear steady, that is, during the latter part of the afternoon and sometimes in the early morning. In 1902 the Coast Survey parties were instructed to observe from 3 P.M. until dark, on heliotropes, and then to continue, with the use of acetylene lights, until 11 P.M. The criterion to be used in deciding whether conditions were favorable was not the appearance of the signals themselves, but the variations of the measures of the angles. The results showed that angles can often be measured with sufficient accuracy at times when the appearance of the signals would indicate poor conditions. From the results of this season's work it became evident that night observations

* From Coast and Geodetic Survey Special Publication No. 19.

are somewhat more accurate than those made in daylight. Observing at night is also more economical than observing in the day on heliotropes, because at night the observer is less dependent upon weather conditions (see Art. 16).

46. Forms of Record.

The following are forms of record which may be used for horizontal angles of triangulation.

HORIZONTAL ANGLES. DIRECTION INSTRUMENT.

Station, Corey Hill. Date, May 21, 1907. Observer, A. N. Recorder, W. R. N. Inst. No. 31. Set No. 2.

Station observed.	Time.	Telescope.	Micro.	Circle.			Run.	Mean.	Cor. for run.	Cor'd meas.
				° ′	F.	B.				
	h m									
Blue Hill	4 30	Dir.	*A*	15 01	51.5	50.5				
			B		54.0	53.7				
			C		49.0	48.5				
					51.5	50.9	0.6	51.2		
Prospect		Dir.	*A*	138 30	20.9	20.5				
			B		22.0	21.5				
			C		18.1	18.0				
					20.3	20.0	0.3	20.2		

HORIZONTAL ANGLES. REPEATING INSTRUMENT.

Station, Corey Hill. Date, May 21, 1907. Observer, J. N. B. Instr. B. & B., No. 1567.

Station.	Time.	Tel.	Rep.	Ver. A.	B.	Mean.	Angle.	Mean.
	h m			° ′ ″	″	″	° ′ ″	° ′ ″
Blue Hill	3 20	D	0	0 00 00	00	00		
to								
Prospect	P.M.		1	123 28 10	20	15		
			6	*20 49 40	40	40	123 28 16.7	
		R	0	20 49 40	40	40		
			6	0 00 10	10	10	123 28 15.0	123 28 15.8

* *Note.* — Since the angle is over 120 degrees the *A* vernier has passed 360 degrees twice in the six repetitions. In computing the mean we divide the 720 degrees by 6 mentally and write down 12 —, then divide the 20 degrees by 6, add the whole degrees to 120, and then divide the minutes and seconds. Observe that when six repetitions are used, the remainder, when dividing the degrees by 6, gives the first figure of the minutes, i.e., 20 degrees ÷ 6 = 3 degrees in the mean, plus 2 degrees to be carried to the minutes column giving 20 minutes. Similarly in dividing the minutes by 6 the remainder is the tens place in the seconds.

47. Accuracy Required.

As stated in paragraph 4 on p. 60 the degree of accuracy required on the Coast Survey triangulation is such that the error of closure of a triangle shall seldom exceed 3″ and shall average about 1″. The following list, taken at random from a longer list in Special Publication No. 19, will indicate the degree of accuracy actually obtained in the work of the Coast Survey.

Section.	Probable error of an observed direction.	Average closing error of a triangle.	Max. cor. to direction.	Maximum closing error of a triangle.
	″	″	″	″
Nevada — California	±0.23	0.57	0.60	1.57
New England	±0.26	0.75	1.17	2.02
Eastern Oblique Arc	±0.30	0.78	0.74	2.73
Holton Base net	±0.34	0.79	0.84	2.28
Atlanta base to Dauphin Island-base	±0.36	1.10	0.84	2.69
Lampasa base to Seguin base	±0.45	1.13	1.96	3.31
Calif. — Washington Arc	±0.53	1.22	2.03	6.35

48. Reduction to Center.

In case certain lines from any station are obstructed, it may become necessary to set the instrument over a point at one side of the center, called an eccentric station, and to measure the angles at this new point. These angles are measured with the same degree of precision as though the instrument were at the center. Before such angles can be used for solving the triangles, they must be reduced to the values they would have if the instrument were placed at the center. The data necessary for the calculation include the approximate distances (D) to the points sighted, the distance from the center mark to the instrument (d), called the *eccentric distance*, and the angle at the instrument between the center mark and each of the signals sighted.

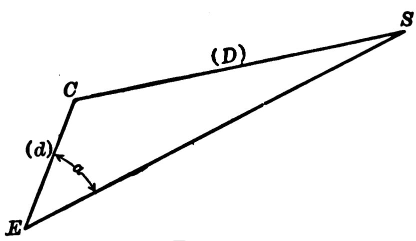


FIG. 34.

In Fig. 34, let C be the center, E the instrument, and S one of the signals. The angle $CES = \alpha$ (called the azimuth), measured right-handed from the center to the distant signals, may be calculated for each signal by combining angles already measured, provided the line EC has been connected with any one signal by means of an angle. The angle S is the change in the direction, or azimuth, of the triangulation line due to the eccentricity of the instrument station. Solving the triangle for S, we have

$$S'' = \frac{d \sin \alpha}{D \sin 1''}.* \qquad [13]$$

It should be observed that the algebraic sign of sin α shows whether the azimuth is to be increased or diminished.

The following example shows the method employed when several angles are to be reduced to center simultaneously.

EXAMPLE OF REDUCTION TO CENTER.

Harpers △

$d = 1^m.342$ log = 0.12755

Eccentric sta. No. 1. Colog sin 1″ = 5.31443

log const. = 5.44218

Measured angles: — Center to Smith's Cupola = 42° 14′ 20″, Smith's Cupola to Cotton's = 62° 33′ 10″.1, Methodist Church to Cotton's = 58° 45′ 31″.0, Cotton's to White Flag = 56° 22′ 36″.1, White Flag to Baldwin's = 43° 59′ 57″.4.

The azimuths from the center are computed, and the computation is tabulated as follows:

Station.	Smith's Cupola.	Methodist Church.	Cotton's.	White Flag.	Baldwin's.
Azimuth........	42° 14′ 20″.0	46° 01′ 59″.1	104° 47′ 30″.1	161° 10′ 06″.2	205° 10′ 03″.6
Log sin az.......	9.8275	9.8572	9.9853	9.5090	9.6286 n
Colog dist.......	6.1052	6.1025	6.0640	6.2672	6.0909
Log const........	5.4422	5.4422	5.4422	5.4422	5.4422
Log S″..........	1.3749	1.4019	1.4915	1.2184	1.1617 n
S″.............	+23″.7	+25″.2	+31″.0	+16″.5	−14″.5
Azimuth........	42° 14′ 43″.7	46° 02′ 24″.3	104° 48′ 01″.1	161° 10′ 22″.7	205° 09′ 49″.1

Reduced angles: — Smith's Cupola to Cotton's = 62° 33′ 17″.4, Methodist Church to Cotton's = 58° 45′ 36″.8, Cotton's to White Flag = 56° 22′ 21″.6, White Flag to Baldwin's = 43° 59′ 26″.4.

* To reduce the angle to seconds we should divide by arc 1″; but since arc 1″ is nearly equal to sin 1″ the result is numerically the same if we employ the latter.

If the distances are not known with sufficient accuracy at first, as might be the case where there are two eccentric stations in the same triangle, it may be necessary to obtain the reduced angle by a second approximation. After the angles have been reduced to center, as already explained, the lengths of the lines may be calculated with a greater degree of accuracy than at the beginning of the computation. By using these improved values of the distances the reduction to center may be repeated and better values of the angles obtained.

49. Phase of Signal.

If the sights are taken on pole signals, and the illumination is stronger on one side than on the other, as it usually would be in bright sunlight, the observer cannot judge the position of the center but sights the center of the part that he can see.

In this case it becomes necessary to correct the observed angles for this effect, which is known as the "correction for phase." If the shaded portion of the pole is so indistinct that it cannot be used in judging the position of the center, then the illuminated portion must be bisected. If the signal pole is cylindrical, the effect of phase on the measured angle may be calculated as follows: in Fig. 35, representing a section of the signal pole, let CS be a line pointing in the direction of the sun, and CO the line to the observer. The limits of the bright portion of the pole visible to the observer are B and D. By measuring the angle between the sun and the signal the observer obtains the angle

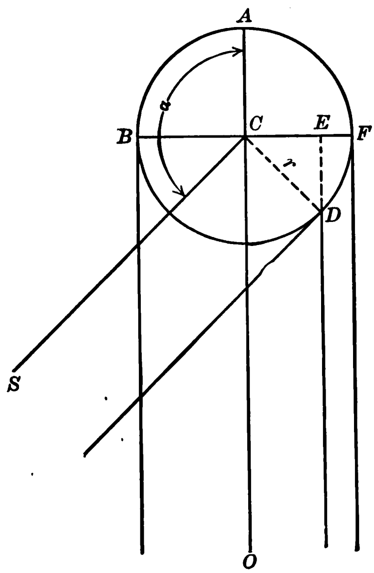


FIG. 35.

$ACS = \alpha$. The total width of the pole is $2r$, and the apparent width is

$$BE = r + r\cos(180 - \alpha).$$

The decrease in width of the object is, therefore,

$$EF = 2r - r(1 - \cos\alpha).$$

The angle at the observing station subtended by this distance is

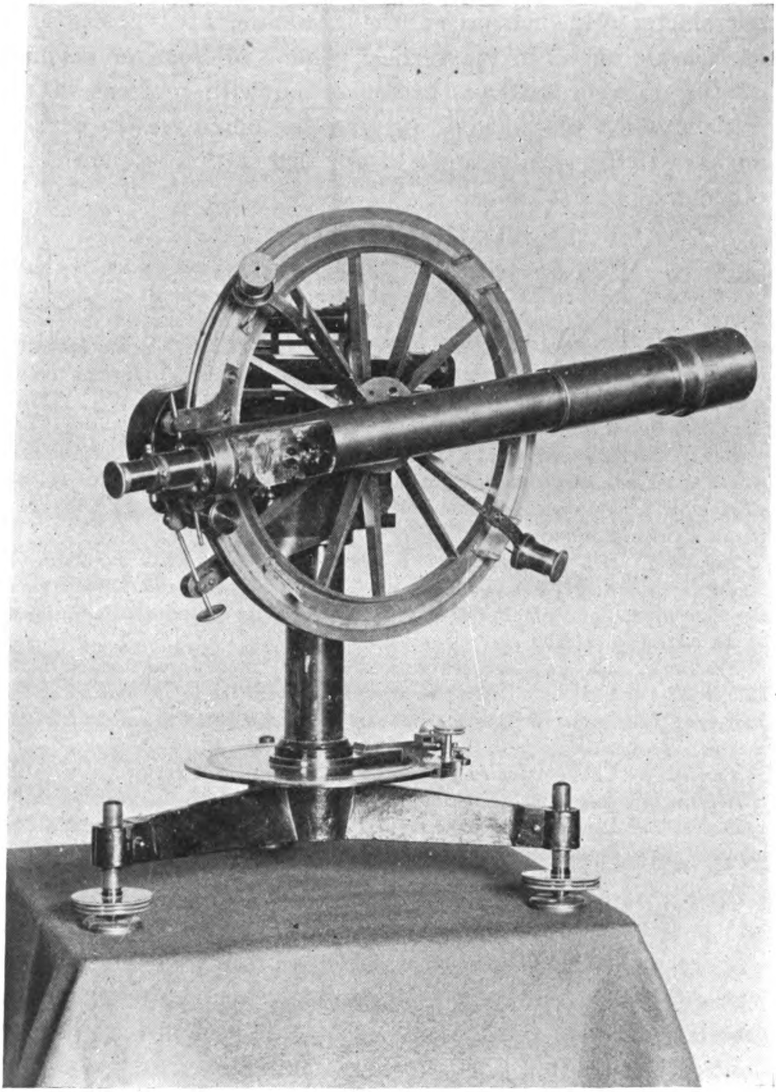
$$\frac{EF}{D \cdot \text{arc } 1''} = \frac{r(1 + \cos\alpha)}{D \cdot \text{arc } 1''}.$$

The correction to the observed direction is one-half this amount since in each case the space is bisected. The final correction is, therefore,

$$\text{Corr.} = \frac{r\cos^2 \frac{1}{2}\alpha}{D \cdot \text{arc } 1''}. \qquad [14]$$

50. Measures of Vertical Angles.

The method of determining the elevations of triangulation points will be discussed in a later chapter, but since the field-work of measuring the vertical angles is carried on in connection with the measurement of the horizontal angles, it will be briefly discussed here. The instrument used for these measurements may be a repeating-circle or a fixed circle read by microscopes. On account of the difficulty of ascertaining the exact effect of atmospheric refraction, vertical angles are subject to much greater errors than horizontal angles. A relatively small number of measures of the vertical angle, half with the instrument direct and half with it in the reversed position, is sufficient to determine the angle as closely as the uncertainty of refraction will permit. Owing to diurnal changes in the amount of the refraction, it is advisable to make the measurements between 11 A.M. and 4 P.M., because the refraction is nearly stationary during these hours. About an hour before sunset the refraction is very uncertain. In recording the angle it is essential to state exactly the height of the instrument above the station mark and also the exact point sighted, so that the angle on each line may be reduced to that of the line between the two station marks.



Vertical Circle.
(Coast and Geodetic Survey.)

The vertical angles may also be obtained by means of the micrometer in the eye-piece of the theodolite, if it is placed so as to measure angles in the vertical plane. Micrometer readings on the different stations, in connection with readings of the spirit level on the alidade, will give the differences in vertical angles. If the vertical angle of any one station is known, the others may be determined.

PROBLEMS

Problem 1. The circle of an alt-azimuth instrument is graduated into 10-minute spaces. The pitch of the micrometer screw is such that two turns are required to move the hairs from one graduation to the next. The head of the screw is divided into minutes and each minute into 10-second spaces. The forward reading (on the 260° 10′ line) is 4′ 03″; the back reading (on the 260° 20′ line) is 3′ 55″. What is the run of this micrometer? What is the correct reading?

Problem 2. The readings of a striding level on a theodolite show that the horizontal axis is inclined 1.5 divisions, the left end being higher. What error will this cause in the azimuth reading on the pole star, at an altitude of 41° 20′, if the value of one division of the level is 10″.0?

Problem 3. If a horizontal angle is measured between a mark 12° above the horizon and bearing N 45° W, and the pole star, 41° altitude, what is the error in the angle produced by an error of 8″ to the right in the (collimation) adjustment of the vertical cross-hair.

Problem 4. The angle between stations *A* and *B* is measured from station *E* and found to be 71° 10′ 19″.5. The angle from *O*, to the right, to station *A* is 110° 15′. The distance *OE* is 7.460 meters. *OA* is 17,650 meters and *OB* is 24,814 meters. Reduce the angle to the center *O*.

Problem 5. The illuminated portion of a cylindrical pole is bisected with the cross hairs of a theodolite. The angle from the sun, to the right, to this signal is 130° 40′. The diameter of the pole is 6 inches. The distance to the signal is 8100 meters. What is the correction to the observed direction for phase of the signal?

CHAPTER IV

ASTRONOMICAL OBSERVATIONS

51. Astronomical Observations — Definitions.

In every trigonometric survey, whether made for scientific purposes or for the purpose of making maps, it is essential that some of the triangulation points be located on the earth's surface by means of their astronomical coördinates. In determining the earth's size and figure by measuring arcs on the surface it is essential that the curvature be determined by means of astronomical observations. If the triangulation is used to control the accuracy of a topographical survey, the astronomical work furnishes the data necessary for correctly locating and orienting the map on the earth's surface. The astronomical data also furnish a means of detecting the accumulated twist of a chain of triangulation, and of correcting the azimuth at intervals along the line. Astronomical observations are also frequently made in order to supply data to be used in other measurements, as, for example, when rating chronometers for gravity or magnetic observations. These astronomical observations form a distinct branch of geodetic work.

It will be assumed that the student has a general knowledge of astronomy, and only such definitions will be given as are essential in viewing the subject from the standpoint of the geodesist. The astronomical observations which it is important for us to consider include the determination of the following four coördinates: (1) time, (2) longitude, (3) latitude, and (4) azimuth. Before describing the instruments and methods, we will define the following terms which are to be employed.

The *vertical* at any point on the earth's surface (OZ, Fig. 36) is the direction in which the force of gravity acts at that point.

In general it does not perfectly coincide with the normal to the spheroidal surface, and hence there is a difference between the astronomical coördinates and the geodetic coördinates. The deflection of the plumb line from the normal at any place is called the *station error*. The point vertically overhead (*Z*) is called the *zenith*. We may consider that the universe is bounded by a

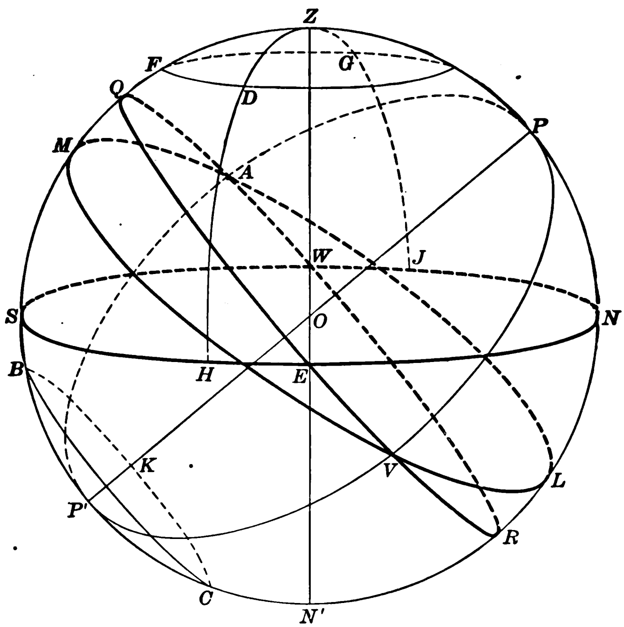


FIG. 36. The Celestial Sphere.

sphere of infinite radius, and that the zenith is the point where the vertical pierces that sphere. The *horizon* (*NEHS*) is the great circle on the celestial sphere which is everywhere 90° from the zenith. Its plane passes through the observer and is perpendicular to the vertical line. Any plane which contains the vertical line cuts from the sphere a *vertical circle* (*HDZ*).

The earth's rotation axis, prolonged, pierces the sphere in two points, called the *north celestial pole* (*P*) and the *south celestial pole* (*P'*). The great circle which is everywhere 90° from the poles is the *celestial equator* (QVR). Any plane through the axis or parallel to it cuts from the sphere an *hour circle* (PVP'). The vertical circle which passes through the celestial pole is called the *meridian* (SQZ). If the vertical does not intersect the earth's axis, the meridian plane cannot contain the axis but is parallel to it. The *prime vertical* is a vertical circle perpendicular to the meridian. The *ecliptic* is a great circle cut by the plane of the orbital motion of the earth (MVL). That point on the sphere where the ecliptic and the equator intersect, and where the sun passes (in March) from the southern to the northern hemisphere, is called the *vernal equinox*.

The *altitude* (h) of a point is its angular distance above the horizon. Its *zenith distance* (ζ) is the complement of the altitude. The *azimuth* (Z) of a point is the horizontal angle between the meridian and the point. It is usually reckoned from the south point of the horizon, right-handed, from 0° to 360°. The *declination* (δ) of a point is its angular distance north (+) or south (−) of the equator. Its *polar distance* (p) is the complement of the declination. *The hour angle* (t) of a point is the arc of the equator measured from the meridian westward to the hour circle through the point. The *right ascension* (α) is the arc of the equator measured from the vernal equinox eastward to the hour circle through the point.

The astronomical *latitude* * (ϕ) of a place is the angular distance of the zenith north or south of the equator, or, in other words, the declination of the zenith. The longitude (λ) of a place is the arc of the equator between the observer's meridian and a primary meridian, as Greenwich or Washington.

52. The Determination of Time.

The determination of time, practically considered, means the determination of the error of a chronometer on the local sidereal time at the station. The sidereal time (S) at any instant is the hour angle of the vernal equinox; it is usually expressed in hours, minutes, and seconds. From a consideration of the definitions

* For goedetic latitude see p. 123.

of sidereal time, hour angle, and right ascension it is evident that the first equals the sum of the other two; that is,

$$S = \alpha + t. \qquad [15]$$

When the star is on the meridian, t is obviously equal to zero, and we have

$$S = \alpha, \qquad [16]$$

that is, the right ascension of any star is equal to the sidereal time at the instant when that star is passing the meridian. If we note the chronometer reading when a certain star is passing the meridian, we know that the local sidereal time (or true chronometer reading) at that instant is the same as the right ascension of that star as given for that date in the Ephemeris,* and that the error of the chronometer is the difference between the two. The determination of time with a transit mounted in the plane of the meridian depends upon the foregoing principle.

53. The Portable Astronomical Transit.

The instrument chiefly used for determining time and longitude in geodetic work is the portable transit. This class of work necessitates carrying the instrument to many stations located in places which are difficult to reach; hence it should be light enough to be easily transported. The small size of the transit, however, does not necessarily imply inferior accuracy in the results; it is found by experience that comparatively small instruments, when properly handled, give results of great accuracy. Indeed, the very fact that the instrument is light is a point in its favor, for this makes it easier to reverse, and obviates certain difficulties encountered in using large instruments in observatories, for example, the error due to flexure, or those due to temporary strains caused by reversal of the instrument. The portable transit is usually mounted on a brick or concrete pier, to which the base of the instrument is firmly cemented.

The transit instrument itself consists of a telescope with a

* The American Ephemeris and Nautical Almanac, published by the Navy Department.

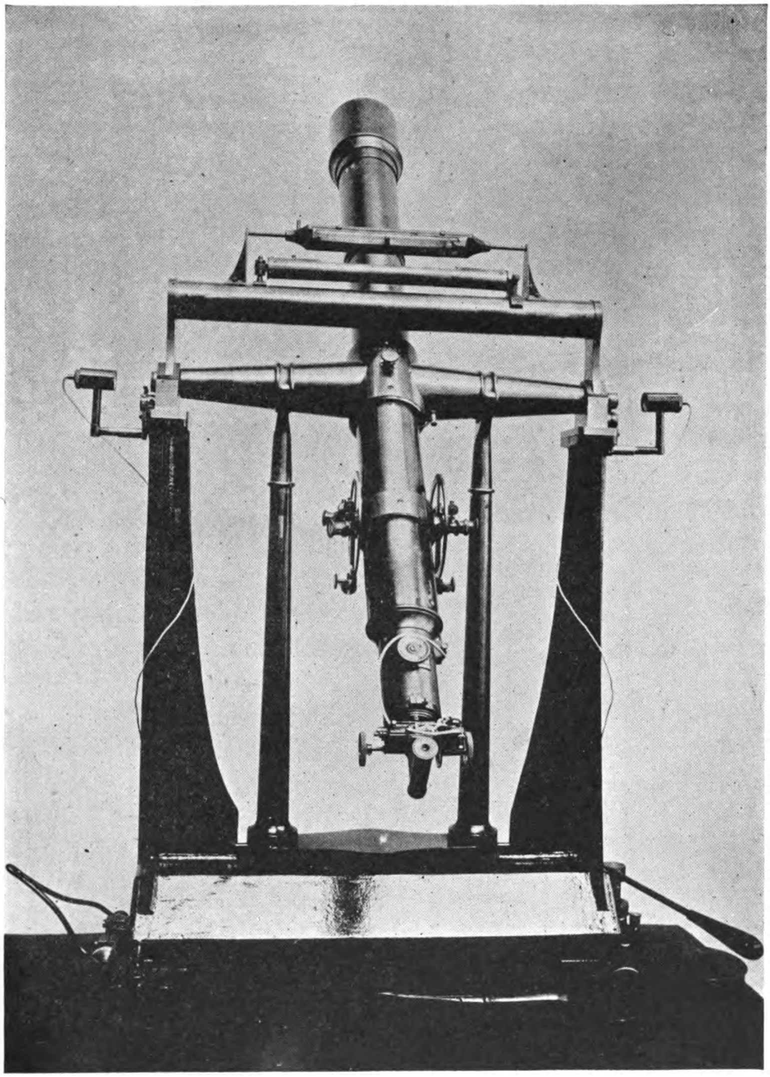


FIG. 37. Portable Transit (with transit micrometer.)
(Coast and Geodetic Survey.)

rotation axis rigidly attached at right angles to it; this axis terminates in pivots which rest in wye bearings at the upper ends of a pair of standards. A stride level is provided for measuring the inclination of the rotation axis. The axis of collimation, which is a line through the optical center of the objective and perpendicular to the rotation axis, rotates in a vertical plane when the horizontal axis is truly level. For the purpose of determining the time the instrument may be set in any vertical plane, for example, the vertical plane through a close circumpolar star; but in this country it is used almost exclusively in the plane of the meridian.

Fig. 37 shows a portable astronomical transit used for the determination of time and longitude by the Coast and Geodetic Survey. The focal length is 94 cm, the aperture 76 mm, and the magnifying power 104 diameters.

54. The Reticle.

In the old style of transit the reticle consisted of several closely spaced vertical spider threads or of lines ruled on glass, and two horizontal threads or lines to limit the portion of the vertical threads used for observations. A common arrangement of the vertical threads, when the chronograph is to be used for recording the observed time, is shown in Fig. 38, the smallest intervals corresponding to about 2.5^s of time for an equatorial star.

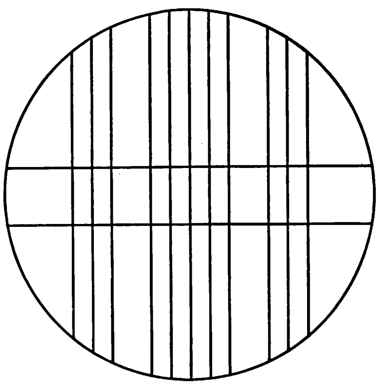


FIG. 38.

55. Transit Micrometer.

The hand-driven transit micrometer has now replaced the old style of reticle on the instruments of the United States Coast Survey. In this instrument (Fig. 39) a single vertical thread is made to traverse the field of the telescope at such a speed that it

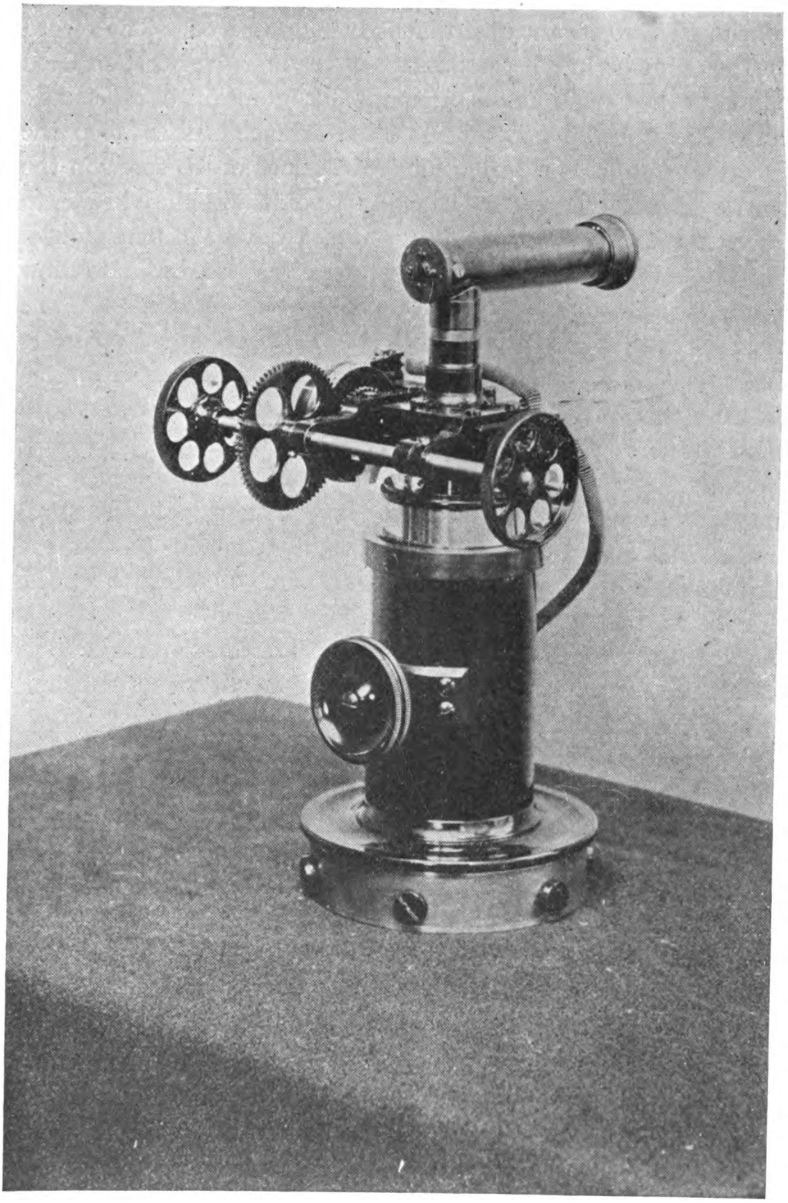


FIG. 39. The Transit Micrometer.
(Coast and Geodetic Survey.)

continually bisects the star that is being observed. The record on the chronograph of the passage of the star over certain fixed points in the field is made automatically by means of an electric circuit. An automatic cut-out is so arranged as to keep the circuit broken except during four revolutions of the screw in the central part of the field. The contact points are placed so as to record twenty observations on the star, arranged in four groups. The observer has simply to set the thread on the star and follow it until it has passed beyond the range of observation. The observer does not know exactly when the observations are being made; he simply watches the thread and the star and keeps the bisection as nearly perfect as he can. It is necessary to use both hands in order to give the thread a steady motion. The result of these observations is the same as though the observer had noted accurately the time of passage of the star over 20 vertical threads. The great advantage of the instrument is that the large personal error due to estimating times of transit over the threads is almost wholly eliminated. A further advantage is that 20 observations may be made in about ten seconds, on an equatorial star, thus permitting observations on stars culminating in quick succession.

56. Illumination.

The field of the telescope is illuminated by means of a lamp or an electric bulb which sends light through the hollow axis of the instrument to a mirror at the center of the telescope, which reflects it down the telescope tube to the reticle. The threads appear as black lines against a bright field.

57. Chronograph.

The chronograph is a registering apparatus driven by clockwork, and connected electrically with a chronometer and with either the transit micrometer or an observing key. The record is made on a sheet of paper wound around a drum which revolves once per minute. A pen fastened to the armature of an electromagnet is carried by a screw in a direction parallel to the axis of the drum. These combined motions cause the pen to draw a line spirally around the drum. When the sheet is laid flat, the record appears as a series of straight parallel lines. The chronometer breaks the circuit once per second (or once per two seconds), and this break causes the armature to move the pen to one side and make a small notch on the record. The times of passage of stars over the threads of the transit are also recorded in a similar

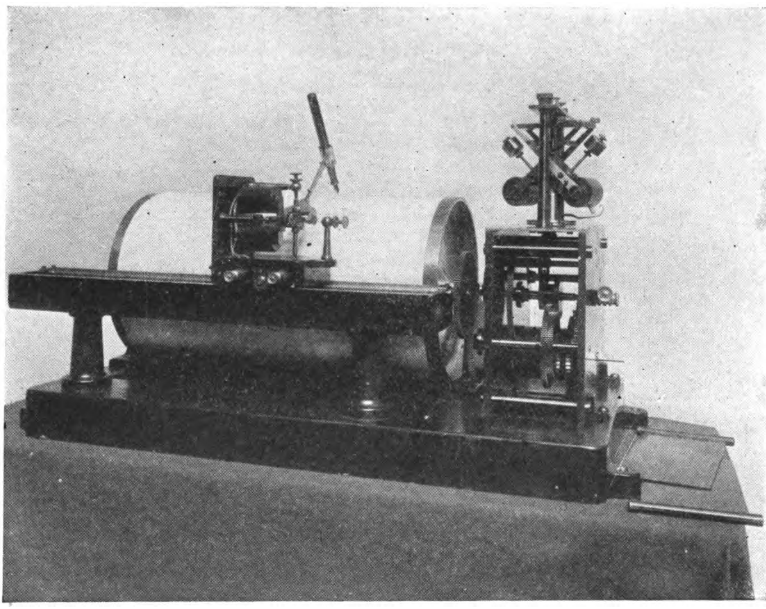


FIG. 40. Chronograph.
(Coast and Geodetic Survey.)

manner. The character of the two kinds of marks is usually dissimilar, and they may easily be distinguished. If any one of the chronometer marks on the record sheet is identified, then the chronometer time of every mark on the sheet becomes known, and the determination of the fraction of a second for each observation is simply a matter of scaling off the position of the corresponding mark. A convenient way to mark the time without disturbing the sheet is to make notches on the sheet by means

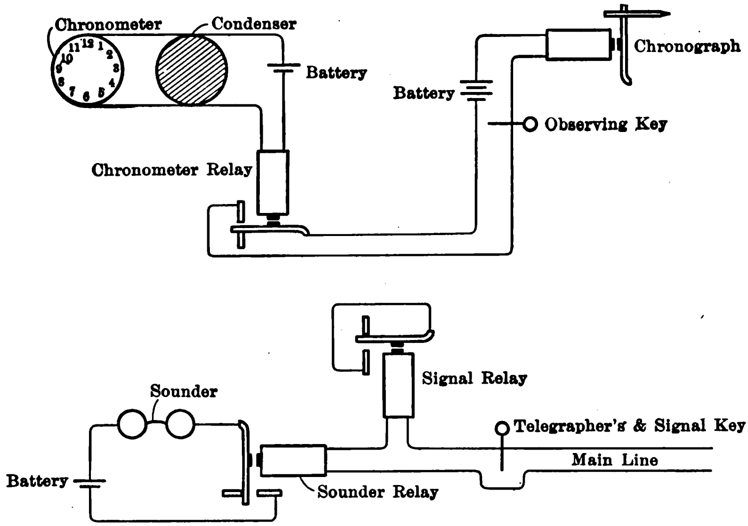


FIG. 41. Electrical Connections — Time Observations by Key Method.

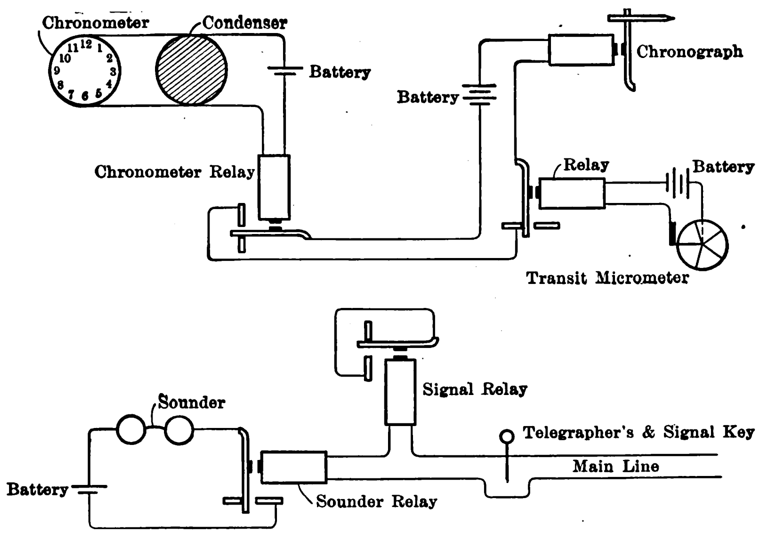


FIG. 42. Electrical Connections — Time Observations by Transit Micrometer Method.

of the observing key, the number of marks so made showing the number of some minute of the chronometer reading. The speed and the diameter of the cylinder are usually such as to make one second of time occupy a space of one centimeter. Fig. 40 shows a chronograph such as is used in longitude observations.

58. Circuits.

The arrangements of circuits for operating the chronograph are shown in Figs. 41 and 42. The chronometer is placed in a

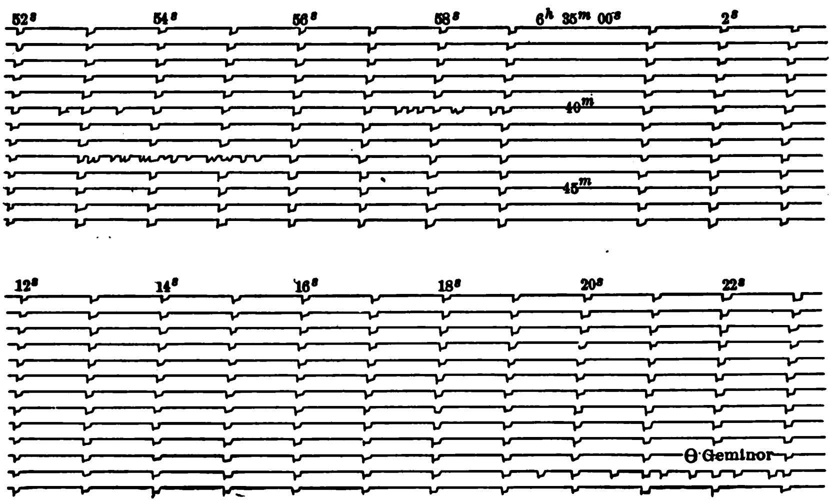


Fig. 43. Chronograph Record.

separate circuit having a battery of only one cell, in order to avoid injury to the mechanism, and operates the chronograph circuit through the points of a relay. The transit micrometer operates on the *make-circuit*, which is converted into breaks by a relay. If a key is used, it replaces the micrometer relay and breaks the circuit when the key is pressed.

Fig. 43 shows a portion of a chronograph record.

59. Adjustment of the Transit.

In placing the transit on the supporting pier before adjusting it in the meridian, the base of the instrument must be placed so nearly in the meridian that all further adjustment in azimuth may be made by the adjusting screws provided for this purpose. The foot plates should then be cemented to the pier. The telescope is focused as in an engineer's transit — first the eyepiece, then the objective. A distant terrestrial object may be used for the first trial, but the final focusing should be done at night on the stars. A difference is usually noticed between the focus required by day and that found at night when artificial light is used.

The striding level and the horizontal axis may be adjusted simultaneously by placing the level in position, reading both ends of the bubble, then reversing it, end for end, and taking another set of readings. Half the displacement of the bubble may be corrected by adjustment of the level and half by leveling the axis.

The verticality of the threads or the micrometer line is tested by rotating the telescope slightly about its horizontal axis and noting whether a fixed object remains continuously on the thread as it traverses the field of view. Adjustment is made by rotating the diaphragm or the micrometer box until this condition is fulfilled.

The collimation is adjusted by placing the middle line of the reticle or the mean position of the micrometer line as nearly as possible in the collimation axis. To test this, point the wire on some object, reverse the telescope in its supports (axis end for end), and see if the object is still sighted. If it is not, bring the wire halfway back by means of the lateral adjusting screws.

The finder circles should be tested to see if they read zero when the collimation axis is vertical. Point on some object, level the bubble, and read the circle. Reverse the telescope, point on the same object, and repeat the readings. The mean reading is the true zenith distance, and half the difference between the two readings is the error of adjustment. Set the vernier to read the true zenith distance, sight the object again, and then center the bubble by means of the adjusting screws.

To place the line of collimation in the meridian, first determine a rough chronometer correction by leveling the axis and setting the circles for the zenith distance of some star which is near the zenith and which is about to culminate. If the (sidereal) chronometer is nearly regulated to local sidereal time, the right ascension of such a star will be nearly the same as the chronometer reading. If the chronometer is not regulated at all, it may be set approximately right by calculating the sidereal time corresponding to the mean time as indicated by a watch. An error of one or two minutes will not cause great inconvenience, as all that is necessary is to identify the star and begin observing before it has passed. The time at which this star will pass the middle vertical thread must necessarily be very close to the true sidereal time (right ascension of star), because near the zenith the effect of the azimuth error on the observed time is very small. The difference between the right ascension of the star and the chronometer reading is an approximate value of the chronometer error. Using this value of the chronometer error, calculate the chronometer time when some slowly-moving (circumpolar) star will pass the meridian. When this calculated time arrives, point the middle thread or the micrometer thread on the star, using the azimuth adjustment screws. This places the instrument nearly in the meridian. A repetition of the whole process (on a different pair of stars) will give a still closer approximation.

It is not necessary or desirable to spend much time in reducing the errors of azimuth, level, and collimation to very small quantities. They should be so small as to cause no inconvenience in making the observations and in computing the results, but since they must be determined and allowed for in any case, the final result is quite as accurate if the errors themselves are not extremely small.

60. Selecting the Stars for Time Observations.

There are two general methods of selecting the stars to be used for a time determination. The older method requires observations on ten stars, five with the axis of the telescope in one position (say illumination or clamp *east*) and five with the axis reversed (illumination or clamp *west*). In each half-set one of the stars is a slow-moving one, that is, one situated near the pole. Of the remaining four stars in each half-set two should preferably be north of the zenith and two south of the zenith, and in such positions that their azimuth errors balance each other, that is, their A factors (see Art. 66) should add up to zero.

In the more modern method, used with the transit micrometer, twelve stars are employed, six in each position of the axis. None of these is near the pole, but their positions are so chosen as to make the algebraic sum of their A factors nearly equal to zero.

By the older method the error in azimuth adjustment is more accurately determined, but with a proper selection of stars the value of the azimuth correction need not be determined so accurately, because it has a relatively small effect upon the computed chronometer correction.

In preparing for observations a list of stars should first be made out, giving the name or number of each star, its magnitude, right ascension, declination, and zenith distance, together with the *star factors* depending upon its position, as explained later. The declination of the stars chosen should be such that the algebraic sum of the A factors is less than unity. It is desirable that the list contain as many stars per hour as possible, but sufficient time must be allowed for reading the stride level, reversing the instrument, making records, etc. The telescope should be reversed before each half-set. In preparing this list the zenith distance of a star is computed by the relation

$$\zeta = \phi - \delta, \qquad [17]$$

where ζ is the zenith distance (positive if south of the zenith), ϕ is the latitude, and δ is the declination (positive for stars north of the equator).

61. Making the Observations.

In beginning the observations, set the vernier of the finding circle at the zenith distance of the first star and bring the bubble to the center of its scale by moving the whole telescope. The clamp had better not be used if the telescope can be relied upon to remain in position when unclamped. When the star appears in the field, bring it between the two horizontal hairs by tapping the telescope with the finger. Set the micrometer line on the star and keep it bisected until the observations (4 turns of screw) are completed. If the instrument is not provided with a micrometer, the observer simply presses the observing key as the star passes each of the vertical threads. When the observations are made by the key method, the observer attempts to press the key as soon as possible after the star is actually bisected by the wire. In doing this he makes an error which tends to become constant as the observer gains in experience. This is known as his *personal equation*. Since the personal equation depends chiefly upon the rapidity and uniformity with which the observer is able to record his observations, rather than upon his ability to bisect the star's image, the use of the transit micrometer very nearly eliminates this error.

After half the stars in one set have been observed, the axis should be reversed, end for end, in the supports. The striding level should be read one or more times during each half-set. If the pivots are not truly circular in section, the average inclination of the axis may be found by taking level readings with the telescope set at different zenith distances, both north and south.

The striding level should be used with great care, because the level corrections may be relatively large and cannot be eliminated by the method of observing, as in case of the collimation error and, to some extent also, the azimuth error.

Following is a record of a set of observations as read from the chronograph sheet, together with the readings of the striding level. (See United States Coast and Geodetic Survey Special Publication No. 14, p. 21.)

Station, Key West. Date, Feb. 14, 1907. Instrument, transit No. 2, with transit micrometer. Observer, J. S. Hill. Recorder, J. S. Hill. Chronometer, Sidereal 1824.

Star: S. Monocer.	ψ^5 Aurigae	18 Monocer.	ζ Geminor.	ζ Geminor.	63 Aurigae
Clamp: W	W	W	W	W	W
Level: W E; *d d*; N 62.0 20.0; 17.7 59.5; +44.3 −39.5; +4.8		W E; *d d*; S 61.2 19.4; 17.7 59.6; +43.5 −40.2; +3.3		W E; *d d*; N 61.5 19.5; 17.7 59.7; +43.8 −40.2; +3.6	

Computation of level constant: Mean N + 4.20 *d*
S + 3.30
+ 3.75 × 0.039 = + 0.146 = b_W

h m 6 35			*h m* 6 39			*h m* 6 42			*h m* 6 46			*h m* 6 58			*h m* 7 04		
s	*s*	Sums	*s*	*s*	Sums	*s*	*s*	Sums	*s*	*s*	Sums	*s*	*s*	Sums	*s*	*s*	Sums
32.0	41.4	73.4	41.3	54.0	95.3	41.5	50.5	92.0	19.5	30.4	49.9	16.2	26.0	42.2	55.3	67.0	122.3
32.4	41.1	0.5	41.8	53.5	0.3	41.9	50.2	0.1	20.0	30.1	50.1	16.5	25.5	2.0	55.6	66.5	0.1
33.1	40.4	0.5	42.8	52.6	0.4	42.5	49.7	0.2	20.6	29.4	0.0	17.2	24.8	2.0	56.4	65.8	0.2
33.6	39.8	0.4	43.5	51.9	0.4	43.1	49.1	0.2	21.3	28.7	0.0	17.7	24.3	2.0	57.1	65.1	0.2
33.9	39.5	0.4	43.9	51.4	0.3	43.3	48.8	0.1	21.7	28.3	0.0	18.0	23.9	1.9	57.5	64.6	0.1
34.6	38.8	0.4	44.7	50.6	0.3	44.0	48.1	0.1	22.3	27.6	49.9	18.8	23.1	1.9	58.4	63.9	0.3
35.0	38.5	0.5	45.3	50.3	0.6	44.3	47.9	0.2	22.8	27.1	9.9	19.1	22.9	2.0	58.8	63.4	0.2
35.6	37.9	0.5	46.0	49.3	0.3	44.8	47.3	0.1	23.6	26.4	50.0	19.8	22.3	2.1	59.5	62.6	0.1
36.1	37.4	0.5	46.9	48.5	0.4	45.4	46.6	0.0	24.3	25.7	0.0	20.5	21.6	2.1	60.3	61.9	0.2
36.4	37.1	0.5	47.2	48.1	0.3	45.7	46.3	0.0	24.6	25.4	0.0	20.7	21.4	2.1	60.7	61.5	0.2
Sum 734.6			Sum 953.6			Sum 921.0			Sum 499.8			Sum 420.3			Sum 1221.9		
Mean 36.73			47.68			46.05			24.99			21.02			01.10		

	6 35	6 39	6 42	6 46	6 58	7 04
R*						
κ	− 0.02	− 0.03	− 0.02	− 0.02	− 0.02	− 0.02
Bb	+ 0.14	+ 0.19	+ 0.14	+ 0.17	+ 0.16	+ 0.18
t	6 35 36.85	6 39 47.84	6 42 46.17	6 46 25.14	6 58 21.16	7 05 01.26
α	6 35 51.85	6 40 02.92	6 43 01.21	6 46 40.17	6 58 36.16	7 05 16.28
(α − *t*)	+15.00	+15.08	+15.04	+15.03	+15.00	+15.02

* R, correction for rate, is negligible in this time set.

62. The Corrections.

The corrections that have to be applied to the mean of the observed times, to reduce it to the time corresponding to the meridian passage are those for (1) level, (2) collimation, (3) azimuth, (4) rate, and (5) diurnal aberration.

63. Level Correction.

The level correction to any observed time, Bb, is made up of the constant b, depending upon the level readings, and a factor B, depending upon the position of the star and upon the observer's latitude. If w and e are the readings of the west and east end of the level bubble in one position, and w' and e' the readings for the second position, then for the first position, the inclination of the axis of the level in terms of scale divisions is $\frac{1}{2}(w-e)$; for the second position it is $\frac{1}{2}(w'-e')$. The mean of the two is the inclination of the transit axis, free from errors of adjustment of the level. If b represents the inclination, then

$$\begin{aligned} b &= \tfrac{1}{2}\left[\tfrac{1}{2}(w-e) + \tfrac{1}{2}(w'-e')\right] \\ &= \tfrac{1}{4}\left[(w+w') - (e+e')\right]. \end{aligned}$$

If d is the value of one division of the level scale expressed in seconds of arc, then b in seconds of time is

$$b = \frac{d}{60}\left[(w+w') - (e+e')\right], \qquad [18]$$

in which the scale divisions are supposed to be numbered each way from zero; b is positive if the west end of the axis is too high. If, however, the divisions of the level are numbered continuously from one end of the tube to the other, the equation is

$$b = \frac{d}{60}\left[(w-w') + (e-e')\right], \qquad [19]$$

in which the primed letters refer to that position of the level in which the zero of the scale is west.

64. Pivot Inequality.

If the pivots are found to be unequal in diameter, then the apparent inclination as found from the level readings must be corrected by a quantity p, which is the inequality as found by a special set of readings of the level. If β_e and β_w are the inclinations as derived from the level readings, and b_e and b_w the true inclinations for the two positions of the axis,

then $$p = \frac{\beta_e - \beta_w}{4};$$

also $$b_e = \beta_e + p$$
and $$b_w = \beta_w - p. \qquad [20]$$

To determine the effect of this inclination error on the observed time of transit of any star, let S (Fig. 44) be the star observed, and let HS be the path of the vertical thread, inclined to the true

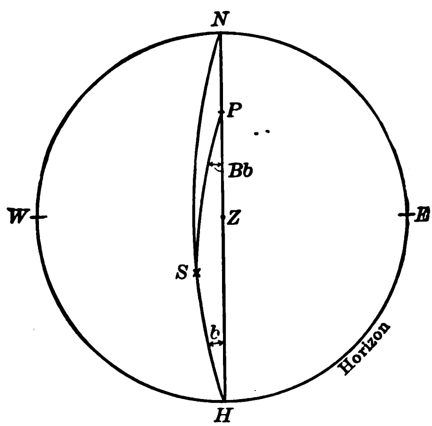


FIG. 44.

vertical at an angle b. In the triangle PHS the angle at P is the error which is to be computed. The angle at H is b; PS is the polar distance, or $90° - \delta$; HS is the altitude (nearly), or $90° - \zeta$. From the triangle PHS,

$$\frac{\sin P}{\sin H} = \frac{\sin HS}{\sin PS},$$

or $$P = b \cos \zeta \sec \delta \text{ (approx.)}$$
$$= b \cdot B. \qquad [21]$$

The factor B may be taken from Table III when the zenith distance and the declination of the star are known.

65. Collimation Correction.

The correction to the observed time is cC, c being the constant angle between the collimation axis and the ***mean thread***, expressed in seconds of time, and C the collimation factor, varying with the position of the star. The collimation constant c may be found by special observations, but is usually computed from the time observations themselves, as explained later; it is considered positive if the line of sight is east of the true position when the clamp is east.

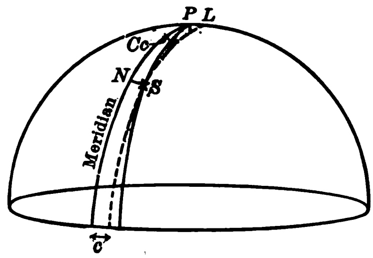


FIG. 45.

In Fig. 45, P is the pole, S the star, PN the meridian, and SL the trace of the thread all points of which are at the same distance (c) from PN. The error is the angle P. Since the angle N is 90°,

$$\sin P = \frac{\sin SN}{\sin PS} = \frac{\sin c}{\cos \delta},$$

or

$$P = c \sec \delta = cC. \qquad [22]$$

The collimation factor C will be found in Table III.

66. Azimuth Correction.

The error of setting the instrument in the meridian is measured by the constant a, the azimuth of the axis of collimation expressed in seconds of time. This constant is derived from the variations in the observations themselves. In Fig. 46, P is the pole, Z the zenith, and S the star. In the triangle PZS, P is the required correction, and $S'ZS$ is a, the azimuth error. Applying the law of sines,

$$\frac{\sin P}{\sin S'ZS} = \frac{\sin \delta}{\cos \delta},$$

or

$$P = a \sin \zeta \sec \delta$$
$$= a \cdot A. \qquad [23]$$

The azimuth factor A may be taken from Table III. The constant a is positive when the plane of the axis of collimation is east of south. A is positive for all stars except those between the zenith and the pole.

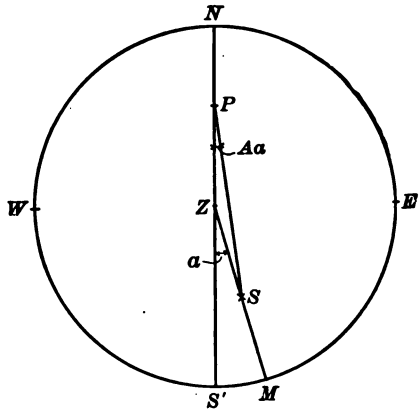


FIG. 46.

67. Rate Correction.

In order to compute these corrections it is necessary to reduce all observations of the chronometer correction to some definite epoch, for example, the mean of all the observed times, so that variations in the chronometer correction itself will not affect the determination of the transit errors. This is done by applying the correction

$$R = (t - T_o)\, r_h, \qquad [24]$$

where t is the chronometer time of transit,

T_o is the mean epoch of the set,

and r_h is the hourly rate of the chronometer, positive if losing, negative if gaining.

68. Diurnal Aberration.

The motion of the observer due to the diurnal motion of the earth makes all stars appear farther east than they actually are; in other words it apparently increases their right ascensions. The amount of the correction is expressed by the equation

$$\kappa = 0^s.021 \cos \phi \sec \delta. \qquad [25]$$

This formula may be derived as follows: the velocity of a point on the earth's equator (toward the east) is 0.288 mile per second. For any other latitude the velocity is 0.288 cos ϕ mile per second. The velocity of light is 186,000 miles per second, and the angular

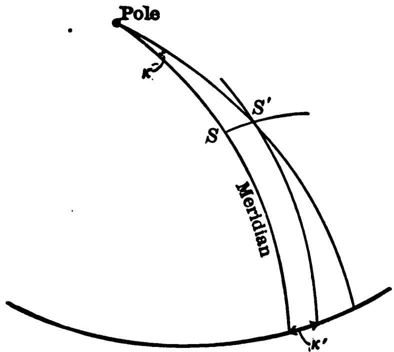


FIG. 47.

displacement (κ') of the star toward the east point of the horizon is therefore equal to $\tan^{-1}\dfrac{0.288 \cos\phi}{186{,}000}$. The effect on the observed time is the angle κ at the pole, Fig. 47. Hence

$$\frac{\sin\kappa}{\sin 90^\circ} = \frac{\sin\kappa'}{\cos\delta},$$

or

$$\kappa = 0''.319 \cos\phi \sec\delta$$
$$= 0^s.021 \cos\phi \sec\delta.$$

Values of this correction will be found in Table IV.

69. Formula for the Chronometer Correction.

The true sidereal time, or right ascension of the star, is given by the equation

$$\alpha = t + \Delta T + \kappa + R + Aa + Bb + Cc, \qquad [26]$$

in which t is the mean of the observed transits and ΔT is the chronometer correction. Since the corrections for aberration, rate, and inclination may be found directly, they are applied to t at once. If we call t_1 the value of t thus corrected, then

$$\alpha - t_1 = \Delta T + Aa + Cc,$$

or

$$\Delta T = (\alpha - t_1) - Aa - Cc. \qquad [27]$$

70. Method of Deriving Constants *a* and *c*, and the Chronometer Correction, ΔT.

The method shown in the following table is the one used when the observations are made with the transit micronometer and when the latitude is less than 50°. For greater latitudes the observations are reduced by the method of least squares.

COMPUTATION OF TIME SET.

[Station, Key West, Florida. Date, Feb. 14, 1907. Set, 2. Observer, J. S. Hill. Computor, J. S. Hill.]

Star.	Clamp.	$\alpha - t$.	δt.	C.	A.	Cc.	Aa.	$\Delta T = (\alpha - t) - Cc - Aa$.	v.
		s	s			s	s	s	s
1. S Monocer......	W	+15.00	0.00	+1.02	+0.26	+0.27	+0.02	+14.71	+0.02
2. ψ^5 Aurigae......	W	+15.08	+0.08	+1.38	−0.45	+0.36	−0.03	+14.75	−0.02
3. 18 Monocer.....	W	+15.04	+0.04	+1.01	+0.37	+0.26	+0.03	+14.75	−0.02
4. θ Geminor....	W	+15.03	+0.03	+1.21	−0.20	+0.32	−0.01	+14.72	+0.01
5. ζ Geminor.....	W	+15.00	0.00	+1.07	+0.07	+0.28	0.00	+14.72	+0.01
6. 63 Aurigae......	W	+15.02	+0.02	+1.30	−0.34	+0.34	−0.02	+14.70	+0.03
7. ι Geminor.....	E	+14.43	−0.57	−1.13	−0.07	−0.30	0.00	+14.73	0.00
8. β Can. Min....	E	+14.45	−0.55	−1.02	+0.28	−0.27	+0.01	+14.71	+0.02
9. α Can. Min....	E	+14.45	−0.55	−1.01	+0.33	−0.26	+0.01	+14.70	+0.03
10. β Geminor.....	E	+14.41	−0.59	−1.13	−0.08	−0.30	0.00	+14.71	+0.02
11. π Geminor.....	E	+14.42	−0.58	−1.21	−0.19	−0.32	−0.01	+14.75	−0.02
12. ϕ Geminor.....	E	+14.47	−0.53	−1.12	−0.05	−0.29	0.00	+14.76	−0.03

Mean $\Delta T = +14.727$

1. $3.00\,\delta t + 3.10\,c + 0.70\,a_W - 0.04 = 0$
2. $3.00\,\delta t + 3.89\,c - 0.99\,a_W - 0.13 = 0$
5. $2.12\,\delta t + 2.75\,c - 0.70\,a_W - 0.09 = 0$ (2) × 0.707
6. $5.12\,\delta t + 5.85\,c \quad - 0.13 = 0$ (1) + (5)
9. $4.71\,\delta t + 5.38\,c \quad - 0.12 = 0$ (6) × 0.920
10. $9.53\,\delta t \quad + 2.61 = 0$ (8) + (9) 11. $\delta t = -0.274^s$ from (10)

$$\Delta T = +15.00 - 0.274 = +14.726^s$$

3. $3.00\,\delta t - 3.15\,c + 0.56\,a_E + 1.63 = 0$
4. $3.00\,\delta t - 3.47\,c - 0.34\,a_E + 1.74 = 0$
7. $1.82\,\delta t - 1.91\,c + 0.34\,a_E + 0.99 = 0$ (3) × 0.607
8. $4.82\,\delta t - 5.38\,c \quad + 2.73 = 0$ (4) + (7)
12. $-1.32 \quad -5.38\,c \quad + 2.73 = 0$ from (8) 13. $c = +0.262^s$ from (12)

14. $-0.82 \quad +1.02 \quad -0.99\,a_W \quad - 0.13 = 0$ 15. $a_W = +0.071$
16. $-0.82 \quad -0.83 \quad +0.56\,a_E \quad + 1.63 = 0$ 17. $a_E = +0.036$

The serial numbers in the lower part of the table show the order of the different steps of the computation. Equation 1 is obtained by taking the terms corresponding to the three southernmost stars (that is, Nos. 1, 3, and 5), substituting the sums of these numbers in the equation $\Delta T + Cc + Aa - (\alpha - t_1) = 0$, and treating this result as though it were the equation for a single star. Equations 2, 3, and 4 are found in a similar manner. This gives four equations for the twelve stars, two for each half-set. Since there are now as many equations as there are unknowns, the quantities c, a_W, a_E, and ΔT may be found by solving these equations simultaneously. Notice that in this solution 15^s has been dropped from ΔT, and that δt is the small correction which must be added to 15^s to obtain ΔT.

The following method of deriving the constants and the chronometer correction without employing least squares is applicable when the two groups of stars have A factors which are not so nearly balanced, or where the list of observed stars consists of one slowly-moving (azimuth) star and several time stars in each half-set. This method gives, by a series of approximations, very nearly the same result that would be obtained by the method of least squares. The various steps in the computation are shown in tabular form in Fig. 48.

The formulas on which the method is based are as follows: For each star we may write an equation of the form

$$\alpha - t_1 = \Delta T + Aa + Cc. \qquad [28]$$

Then for the east and west groups we have

$$\left.\begin{aligned}(\alpha - t_1)_W &= \Delta T + A_W a_W + C_W c,\\ (\alpha - t_1)_E &= \Delta T + A_W a_E + C_E c.\end{aligned}\right\} \qquad (a)$$

Assuming at first that a_E and a_W are equal, we find an approximate value of c by subtracting the second equation from the first. Solving for c, we find

$$c = \frac{(\alpha - t_1)_W - (\alpha - t_1)_E}{C_W - C_E}.$$

In the above example,

$$c = \frac{19.25 - 19.17}{1.42 + 1.34} = +0.03.$$

TRANSIT RECORD FOR TIME.

Station, Middlesex Fells. Date, Nov. 15, 1898. Observer, G. L. H. Chronometer, Bond 541.

	π Pegasi. Clamp. *E.*	7 Lacertae. *E.*	10 Lacertae. *E.*	ι Cephei. *E.*	π Cephei. *W.*	Br. 3077. *W.*	τ Pegasi. *W.*	ι Androm. *W.*
Level	32.0 56.0	12.5	48.5 39.0	Temp.	59.0 30.0	22.0	32.0 56.0	
N	62.5 26.0	15.0	S 46.0 40.5	45° F.	N 41.0 48.0	9.0	S 61.0 28.0	
	94.5 82.0	13.75 × 0.0092	94.5 79.5		100.0 78.0	15.5 × 0.0092	93.0 84.0	
	82.0	= +0.13	79.5		28.0	= +0.14	84.0	
	12.5	$b_e = 0.13 - 0.02 = +0.11$	15.0		22.0	$b_w = 0.14 + 0.02 = +0.16$	9.0	
	h m s							
V	22 03 42.10	22 25 10.71	22 32 53.50	22 43 45.90	I 23 01 46.64	23 06 21.74	23 13 53.08	23 32 19.15
	56.79	29.95	33 09.41	44 15.82	02 34.00	43.82	14 06.64	35.84
	04 11.45	48.85	25.12	45.96	03 21.63	07 06.40	20.15	52.30
	26.10	26 08.42	41.07	45 15.85	04 08.95	28.67	33.70	33 09.52
I	40.96	27.25	57.05	45.82	V 04 56.00	51.40	47.10	26.42
	22 04 11.48	22 25 49.04	22 33 25.23	22 44 45.87	23 03 21.44	23 07 06.41	23 14 20.13	23 32 52.65
κ	−0.02	−0.02	−0.02	−0.04	−0.06	−0.03	−0.02	−0.02
Bb	+0.13	+0.17	+0.14	+0.25	+0.52	+0.28	+0.16	+0.22
Rate	−0.03	−0.01	−0.01	0.00	+0.01	+0.01	+0.01	+0.02
t	22 04 11.56	22 25 49.18	22 33 25.34	22 44 46.08	23 03 21.91	23 07 06.67	23 14 20.28	23 32 52.87
α	22 05 30.65	22 27 08.45	22 34 44.50	22 46 05.99	23 04 42.76	23 08 26.16	23 15 39.26	23 33 12.14
$\alpha - t$	+01 19.09	1 19.27	1 19.16	1 19.91	1 20.85	1 19.49	1 18.98	+ 1 19.27

$\frac{d}{60} = 0.0092$ $\quad r_h = +0.033.$ $\quad p = +0.02$

FIG. 48.

COMPUTATION OF ΔT

Star.	Pos.	$\alpha - t_1$.	C.	A.	Cc.	Aa.	ΔT. $\alpha - t_1$, $-Cc-Aa$.	v.	
π Pegasi	E	+1 − 19.09	−1.19	+0.20	−0.02	−0.15	+1 19.26	−0.05	
7 Lacertae	E	19.27	−1.55	−0.20	−0.02	+0.15	19.14	+0.07	
10 Lacertae	E	19.16	−1.28	+0.09	−0.02	−0.07	19.25	−0.04	
ι Cephei	E	19.91	−2.43	−0.96	−0.04	+0.74	19.21	0.00	$\Delta T = +1^m\ 19^s.21$ at $22^h\ 52^m$
π Cephei	W	+1 20.85	+3.82	−2.05	+0.06	+1.58	19.21	−0.01	
Br. 3077	W	19.49	+1.81	−0.44	+0.03	+0.34	19.12	+0.08	
τ Pegasi	W	18.98	+1.09	+0.36	+0.02	−0.28	19.24	−0.04	
ι Androm.	W	19.27	+1.36	−0.01	+0.02	+0.01	19.24	−0.04	
		$\alpha - t_1$.	C.	A.	Cc.	$\alpha - t_1$, $-Cc$.	Aa.	$\alpha - t_1$, $-Cc$ $-Aa$.	$c = +0.03$
1st. approx. Mean of time stars	E	1 − 19.17	−1.34	+0.03	−0.04	19.21	−0.02	+1 19.23	$a_E = -0.78$
1st. approx. Azimuth star	E	1 − 19.91	−2.43	−0.96	−0.07	19.98	+0.75	19.23	
1st. approx. Mean of time stars	W	1 − 19.25	+1.42	−0.03	+0.04	19.21	+0.02	19.19	$a_W = -0.76$
1st. approx. Azimuth star	W	1 − 20.85	+3.82	−2.05	+0.11	20.74	+1.56	19.18	
2d. approx. Mean of time stars					−0.02	19.19	−0.02	+1 19.21	$c = +0.016$
2d. approx. Azimuth star					−0.04	19.95	+0.74	19.21	$a_E = -0.768$
2d. approx. Mean of time stars					+0.02	19.23	+0.02	19.21	$a_W = -0.772$
2d. approx. Azimuth star					+0.06	20.79	+1.58	19.21	

FIG. 48.

Using this approximate value of c, the last terms in Equations (a) are computed and subtracted from $(\alpha - t_1)$ in each case, leaving the equations in the form

$$(\alpha - t_1 - C_c) = \Delta T + A_W a_W.$$

Taking each half-set separately, and also grouping the azimuth star and the time stars separately, we have for the next group

$$\left.\begin{aligned}(\alpha - t_1 - C_c) &= \Delta T + (A_W a_W)\ az,\\ (\alpha - t_1 - C_c) &= \Delta T + (A_W a_W)\ time,\end{aligned}\right\} \qquad (b)$$

and a similar pair of equations for the second position of the axis. From Equa. (b) we derive

$$a_W = \frac{(\alpha - t_1 - C_c)_{az.} - (\alpha - t_1 - C_c)_{time}}{A_{az.} - A_{time}}.$$

In the example,

$$a_E = \frac{19.98 - 19.21}{-0.96 - 0.03} = -0.78$$

and $$a_W = \frac{20.74 - 19.21}{-2.05 + 0.03} = -0.76.$$

Employing these approximate values of a_E and a_W, the Aa corrections are computed and subtracted, giving the value in the column headed $\alpha - t_1 - C_c - Aa$. For the time stars these values are 19.23 and 19.19. Since these values do not agree for the two positions of the instrument, the value of c is evidently in error. A second approximation must be made by treating the difference of these numbers (0.04) as an error in c and obtaining a correction to c by the same process that was used in finding c in the first instance, that is,

$$\text{Correction to } c = \frac{19.19 - 19.23}{1.42 + 1.34} = -0.014.$$

Hence $$c = +0.03 - 0.014 = +0.016.$$

With this improved value of c new values of a_E and a_W are computed as before. The second values are $a_E = -0.768$ and $a_W = -0.772$. Using these values, the chronometer corrections are found to agree, and hence no further approximation is necessary.

The azimuth and collimation corrections are now found for each star, as shown in the upper part of the table. The mean of the ΔT's for all the stars is the chronometer correction for the mean of the observed times. The residuals (v) are computed by subtracting ΔT for each star from the mean of the ΔT's for that group. These should add up nearly to zero.

Whenever the most accurate results are desired, the computation may be made by the method of least squares. For the details of this method see Coast and Geodetic Survey Special Publication No. 14, p. 41.

71. Accuracy of Results.

The error in the computed value of ΔT due to accidental errors alone may be kept within a few hundredths of a second. Observations made by the key method may be subject to a large constant error, the observer's personal equation, which may be several times as large as the accidental error. Observations made with the transit micrometer are nearly free from personal errors.

72. Determination of Differences in Longitude.

The determination of the difference in longitude of two stations consists in measuring the difference between the sidereal times at the two places. The method almost exclusively used for accurate longitudes in places where a telegraph line is available is that in which the times are compared by electric signals sent over the telegraph line. Wireless apparatus may be used for this purpose, but it has not as yet come into general use, probably because it is not as economical as the ordinary lines. The method used at present by the Coast Survey differs considerably from the old method, owing to the introduction of the transit micrometer.

According to the usual program each observer, provided with transit, chronometer, and chronograph, determines the local sidereal time by the method previously described; then the two chronometers are compared by means of arbitrary signals, which are sent over the telegraph line and recorded simultaneously on

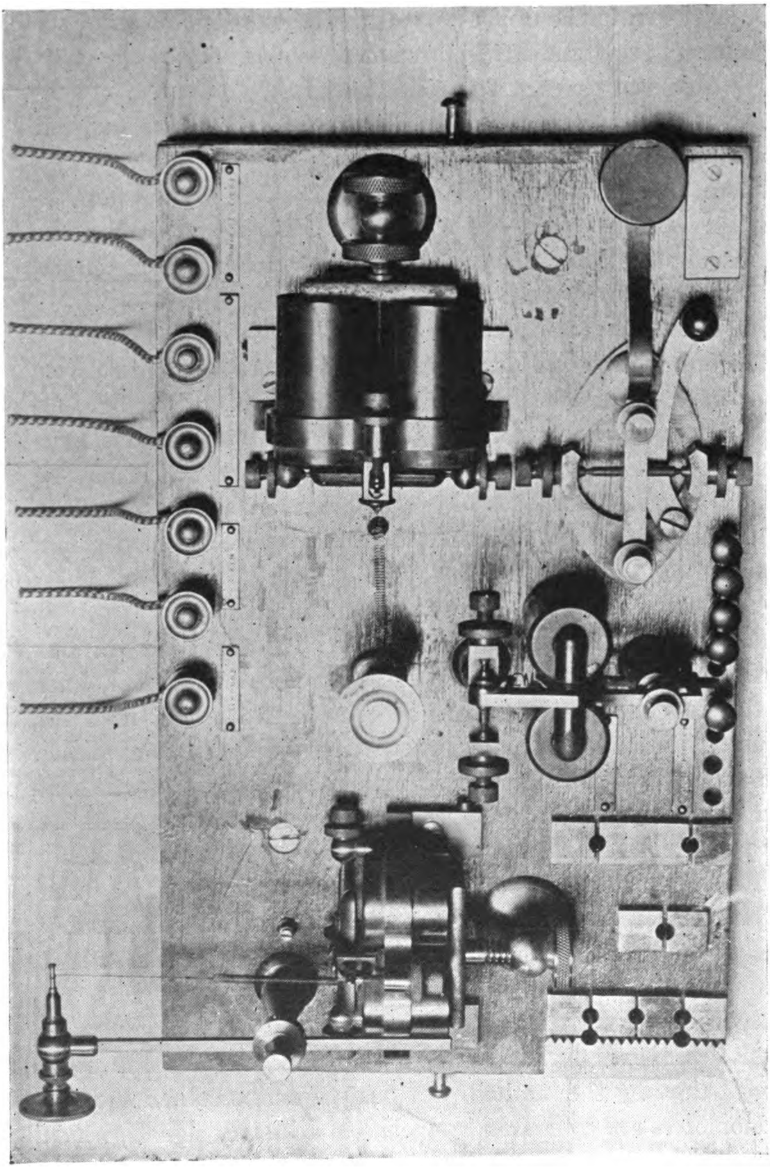


FIG. 49. Switch Board — Telegraphic Longitude.
(Coast and Geodetic Survey.)

both chronographs; and, finally, each observer again determines the local sidereal time.

According to the Coast Survey instructions (*Spec. Pub. No.* 14) each half-set should consist of from 5 to 7 stars (preferably 6), all of these to be time stars (no azimuth star). The algebraic sum of the azimuth factors (A) should be less than unity. Four half-sets are observed during an evening, and the telescope axis is reversed before each half-set. The observers do not exchange places during the occupancy of the station, as was formerly the practice. Observations on three or four nights usually give the desired accuracy.

Fig. 49 shows the switchboard and the arrangement of the electric circuits required in longitude observations. When the observer is making observations for time, the circuit is arranged as shown in Fig. 42.

Fig. 50 shows the circuit as arranged during the exchange of arbitrary signals. These signals are made by tapping the signal key in the main-line circuit. Half of these signals are sent by the eastern observer, half by the western, in order to eliminate the error due to the time of transmission of the signal. The chronometers mark the record sheets while the signals are being sent, so that the time of each signal may be read from each chronograph sheet. The difference in longitude is found from interpolated chronometer corrections.

73. Observations by Key Method.

If the transit micronometer is not used, the selection of stars must be modified so as to allow more time between observations. Since the observations will be subject to the personal errors of both observers, it is important that the observers should exchange places at the middle of the series, so that their relative personal equation will enter the latter half of the observations with its algebraic sign changed. The arrangement of the circuits is shown in Fig. 51, in which an observing key replaces the relay and circuit of the transit micrometer.

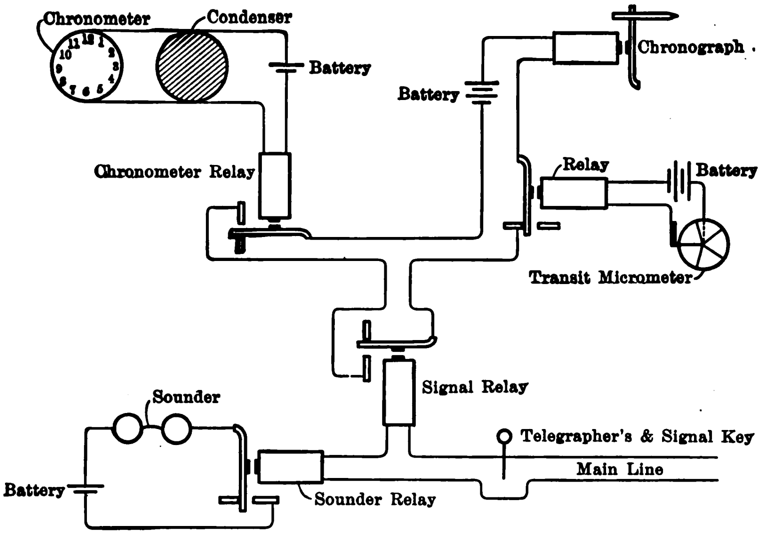


FIG. 50. Electrical Connections — Exchange of Signals — Transit Micrometer Method.

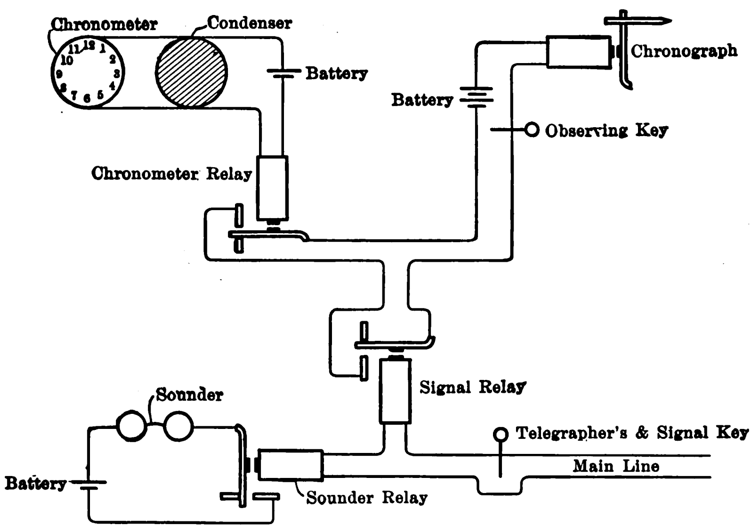


FIG. 51. Electrical Connections — Exchange of Signals — Key Method.

74. Correction for Variation of the Pole.

The periodic variation of the position of the pole affects all observations for longitude and must be allowed for by applying the corrections given in tables published annually by the International Geodetic Association. (See Art. 81, p. 106.)

75. Determination of Latitude.

The method which has been chiefly used in this country for determining astronomical latitudes for geodetic purposes is that known as Talcott's (or the Harrebow-Talcott) Method. The instrument employed is the zenith telescope, illustrated in Fig. 52. The principle involved is that of measuring, not the absolute zenith distances of stars, as is done with the meridian circle, but the small *difference* between the zenith distances of two stars which are on opposite sides of the zenith. By a proper selection of stars this difference in zenith distance may be made so small that the whole angular distance to be measured comes within the range of the eye-piece micrometer, which for most instruments is about half a degree. A sensitive spirit level attached to the telescope serves to measure any slight change in the inclination of the vertical axis of the instrument between the two observations on a pair of stars. The accuracy of the results obtained by this method is superior to that of every other field method, and compares favorably with the results obtained with the largest instruments.

The horizontal axis of the telescope is very short as compared with that of the transit instrument; small errors in the inclination of the axis, however, have very little effect upon the results; a close adjustment is therefore unnecessary. Since the instrument is used in the plane of the meridian and must be quickly turned from the north side to the south, or vice versa, the horizontal circle is provided with stops which are adjustable, so that the telescope may be quickly changed from one side of the zenith to the other. The micrometer, placed in the focal plane of the eyepiece, is set so as to permit of measuring small angles in the vertical plane. The head of the screw is graduated to read to about

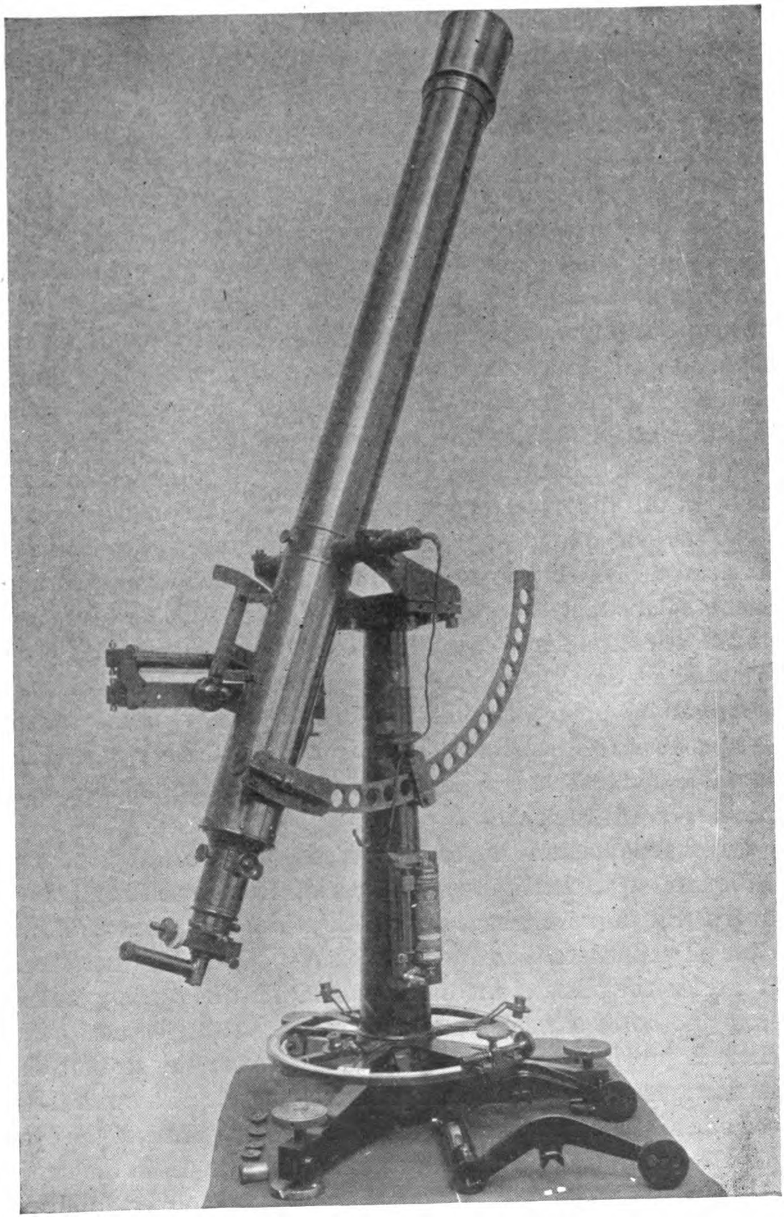


FIG. 52. Zenith Telescope.
(Coast and Geodetic Survey.)

0″.5 directly and to 0″.05 by estimation. The spirit level has an angular value of one (2^{mm}) division equal to about 1″.5.

76. Adjustments of the Zenith Telescope.

When the instrument is in perfect adjustment, the plate levels should be central in all azimuths as the telescope is turned about the vertical axis. The leveling may be perfected by use of the more sensitive latitude level. The horizontal axis must be at right angles to the vertical axis. The movable micrometer threads must be truly horizontal. They may be adjusted by a method similar to that used in adjusting the engineer's level — by swinging the telescope horizontally through a small angle and observing whether the thread remains on a fixed point. The collimation adjustment should be made in the same manner as in a transit, but is not of so great importance. Allowance must be made for the eccentricity of the telescope when making the collimation adjustment. The value of one turn of the micrometer may be determined approximately by observations upon a close circumpolar star near its elongation. The most satisfactory way, however, is to derive the value of one turn from the latitude observations themselves, by the method of least squares. The value of one division of the latitude level may be determined by means of a level trier, or it may be found by varying the inclination of the telescope and employing the eye-piece micrometer to determine the amount of this inclination by observations on a terrestrial mark.

When in use the instrument is mounted on a wooden or concrete pier. It is usually protected by a tent or other temporary shelter.

In order to make the observations, it is necessary to have a chronometer regulated to local sidereal time with an error not exceeding one second of time.

77. Selecting Stars.

The list of stars in the American Ephemeris will not ordinarily be sufficient for latitude observations, on account of the exacting nature of the conditions. It will be necessary to consult such star catalogues as Boss's *Preliminary General Catalogue of 6188 stars for the Epoch 1900*, or one of the Greenwich catalogues. In order to keep the zenith distances within the required limits, it will often be necessary to observe on stars which are much fainter than those used for time observations. The pairs of stars selected should, if possible, differ by less than 20^m in their right ascension and by less than 20′ in their declinations. The actual zenith distance of a star should not exceed 45°. Following is a specimen star list for zenith telescope observations.

OBSERVING LIST (FORM 1).

[St. Anne, Ill., June 25, 1908. Zenith telescope No. 4. $\phi = 41° 01'.3$. Search factor $= 2\phi = 82° 03'$.]

Star No. Boss catalogue.	Mag.	Right ascension.	Declination δ.	Difference between δ's.	$\Sigma\delta$ = sum of declinations.	$\Sigma\delta - 2\phi$.	$N-S=a$* $(\Sigma\delta - 2\phi)$.	Star north or south.	Setting = ½ difference of δ's.	Turns.
		h m s	° ′	° ′	° ′	′			° ′	
4327	4.5	16 55 22	82 11					N		12
4379	4.9	17 11 53	−0 21	82 32	81 50	−13	−17	S	41 16	28
4441	5.9	17 28 13	28 28					S		10
4494	5.8	17 42 04	53 50	25 22	82 18	+15	+20	N	12 41	30
4623	5.1	18 13 22	64 22					N		24
4651	5.4	18 18 45	17 47	46 35	82 09	+ 6	+ 8	S	23 18	16
4669	5.9	18 22 26	29 47					S		20
4711	5.5	18 31 52	52 17	22 30	82 04	+ 1	+ 1	N	11 15	20

* a = number of turns of the micrometer screw in one minute of arc = 1.34. The value of one turn of the micrometer screw = 44″.650.

78. Making the Observations.

In observing on a pair the finder circle is set for the mean of the two zenith distances, and the level is brought nearly to the center of the tube. If the northerly star of the pair culminates first, the telescope is set on the north side of the meridian by means of the azimuth stop. When the star enters the field, the observer bisects it with the micrometer line. If a pair of lines is used, the star is centered in the space between the two. When the star is on the meridian, as shown by the chronometer reading, the bisection of the star is perfected; the latitude level is read immediately, and then the scale of the micrometer screw. As soon as these readings are recorded, the telescope is turned to the south side of the meridian and the bubble is brought to the center, if necessary, by moving the whole telescope. In leveling the bubble the tangent screw of the setting circle must not be disturbed in any case, because the accuracy of the method depends upon preserving a fixed relation between the direction of the zero micrometer reading and the axis of the latitude level. The slightest change in the angle between these two during the observations on a pair will render the observations worthless. When the southern star appears in the field the pointing and the readings are made exactly as for the northern star.

79. Formula for the Latitude.

The principle involved in this method may be seen in Fig. 53. The latitude, *EZ*, as derived from the southern star, is

$$EZ = ES_s + S_sZ,$$

or

$$\phi = \delta_s + \zeta_s,$$

and from the northern star it is

$$EZ = ES_n - ZS_n,$$

or

$$\phi = \delta_n - \zeta_n.$$

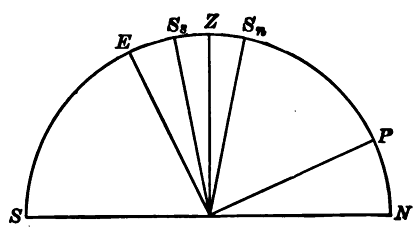


FIG. 53.

The mean of the two values of ϕ is

$$\phi = \frac{\delta_s + \delta_n}{2} + \frac{\zeta_s - \zeta_n}{2}. \qquad [29]$$

If we let

n_s and s_s = the level readings for the southerly star,
n_n and s_n = the level readings for the northerly star,
d = the angular value of one division of the level,
r_s and r_n = the refraction corrections,
M_s and M_n = the micrometer readings,

and R = the value of one turn of the micrometer, then the latitude is determined by the equation

$$\phi = \frac{1}{2}(\delta_s + \delta_n) + \frac{1}{2}(M_s - M_n) \cdot R + \frac{d}{4}\{(n_s + n_n) - (s_s + s_s)\} + \frac{1}{2}(r_s - r_n). \qquad [30]$$

This formula applies when the zero of the level scale is in the center of the tube. If the zero is at the eye-piece end of the tube, the level correction is

$$+\frac{d}{4}\{(n_s - n_n) + (s_s - s_n)\}.$$

If for any reason the observations are not made when the star is exactly on the meridian, another term must be added to the above formula; this will be of the form $+\frac{1}{2}(m_s + m_n)$ when m_s and m_n are the reductions of the measured zenith distances to the true zenith distances. (See *Special Publication No.* 14, p. 119.) For the application of least squares to the computation of latitude see Chauvenet, *Spherical and Practical Astronomy*; Hayford, *Geodetic Astronomy*; and *Coast and Geodetic Survey Special Publication No.* 14.

80. Calculation of the Declinations.

When the stars selected are not found in the Ephemeris, it will be necessary to calculate the apparent declinations for the date of the observation. Formulæ and tables for making these reductions will be found in Part II of the Ephemeris. See also *Coast and Geodetic Survey Special Publication No.* 14, p. 116.

81. Correction for Variation of the Pole.

The observed latitude may be in error by several tenths of a second, owing to the fact that the observed value necessarily refers to the position of the pole at the date of the observation, whereas the fixed value of the latitude of a place is that referred to the mean position of the pole. Fig. 54 shows the plotted positions of the pole for every 0.1 year during the period 1900.0 to 1906.0 (Jordan). The coördinates of the instantaneous pole

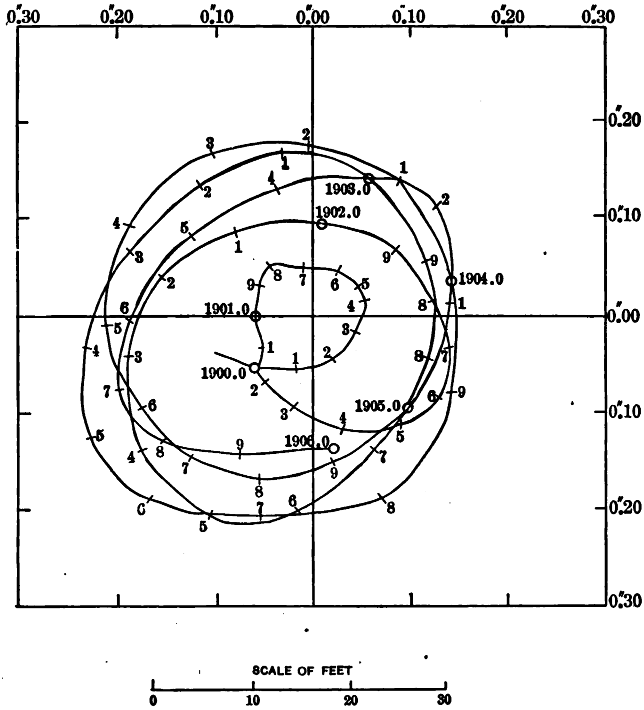


FIG. 54. Motion of the North Pole, 1900 to 1906.

and data for correcting observed values are published annually by the International Geodetic Association, and observations may be referred to the mean pole by employing these tables.

82. Reduction of the Latitude to Sea-Level.

In order that all latitudes may refer to the same level surface, they are all reduced to their values at sea-level. If we suppose a lake surface, in the northern hemisphere, to be at a great height above sea-level, then it may be shown that the northern end of this lake surface is actually nearer to the surface of the sea than is the southern end of the lake surface. If we imagine a series of such surfaces at varying heights above sea-level, it is obvious that the *vertical* is a curved line, since it must at every point be normal to the level surface passing through that point. Evidently this curved line is concave toward the earth's rotation axis. To correct an observed latitude at elevation h to the corresponding latitude at sea-level, it is necessary to apply the correction

$$\Delta\phi = -0''.052\, h \sin 2\phi, \qquad [31]$$

where h is in thousands of feet. If h is expressed in meters, the formula becomes

$$\Delta\phi = -0.000171\, h \sin 2\phi. \qquad [32]$$

(See Art. 170, p. 256.) Values of this correction will be found in Table VII. Below is an example of the form of record and computation of latitude from *Special Publication No.* 14.)

RECORD OF LATITUDE OBSERVATION.

[Station, St. Anne. Date, June 25, 1908. Chronometer, 2637. Observer, W. Bowie.]

No. of pair.	Star number Boss cat.	N or S.	Micrometer. Turns.	Micrometer. Div's.	Level. North.	Level. South.	Chronometer time of culmination.	Chronometer time of observation.	Meridian distance.	Remarks.
...	....	...	....	....	....	....	....	......	....	....
	4623	N	24	88.2	9.2	42.6				
					71.6	103.8		18 13 18		
11	4651	S	16	66.0	42.2	8.7	*	18 18 39	*	+16†
					103.2	71.0				
...	....	...	....	....	....	....	....	......	....	....

* These columns used only when star is observed off the meridian.

† This is the continuous sum, up to this pair, of the south minus the north micrometer turns.

LATITUDE COMPUTATION

Date.	Catalogue.		Micrometer.		Level.			Meridian distance.	Declination.
	Star. No.	N or S.	Reading.	Diff. Z. D.	N.	S.	Diff.		
1908.				t. d.			d.	S.	° ′ ″
June 25	4623	N	24 88.2		09.2	42.6			64 21 59.53
				−8 22.2	71.6	103.8			
	4651	S	16 66.0		42.2	08.7	−1.05		17 46 48.62
					103.2	71.0			

Sum and half sum.	Corrections.				Latitude.	Remarks.
	Micrometer.	Level.	Refraction.	Meridian.		
° ′ ″	′ ″	″	″	″	° ′ ″	
82 08 48.15 41 04 24.08	−3 03.56	−0.39	−0.06		41 01 20.07	

Value of one division of latitude level: Upper −1″.600
Lower −1.364
Mean −1.482
Value of one turn of micrometer = 44″.650

83. Accuracy of the Observed Latitude.

The latitude may be determined by this method with a probable error of from 0″.3 to 0″.4 from one pair of stars. The final value for the latitude of the station determined from as many pairs of stars as can be observed on one night may be found with an error of from 0″.05 to 0″.10 (or 5 to 10 feet). It is not considered advisable to observe the same pair of stars on several nights, as was formerly the practice, owing to the comparatively large errors in the declinations themselves. The present practice is to observe each pair but once and to observe such a number of pairs that the uncertainty of the final latitude is not greater than 0″.10.

In view of the fact that nearly every latitude is affected by a station error which may amount to several seconds, and that the real object of the observation is to determine this station error, it is better to determine a large number of latitudes with the degree of accuracy above mentioned than to attempt to diminish the error of observation and occupy but a small number of stations. This results in the practice of occupying stations but one night, unless for some reason it is apparent that the required accuracy will not be reached without additional observations.

84. Determination of Azimuth.

When determining an azimuth for the purpose of orienting a triangulation system, the observer usually has a choice of several methods, all of them capable of yielding the required accuracy, for example, (1) measuring the angles between a circumpolar star and the triangulation lines by means of the direction instrument, (2) measuring from a triangulation station to a circumpolar star with the repeating instrument, or (3) measuring from a circumpolar star to an azimuth mark with the micrometer of a transit instrument. In all determinations of azimuth it is necessary to know the local time in order to compute the azimuth of the star. This must be found by special observations, unless, as is often the case, the longitude is being determined at the same time and the chronometer correction is already known. For the purpose of orienting the primary triangulation it is necessary to determine the azimuth with an error not exceeding $0''.50$. At Laplace stations (coincident triangulation, longitude, and azimuth stations), where the accumulated twist of the chain of triangles is to be determined, it is desirable to determine the azimuth within $0''.30$ or less. It is also desirable that the instrument station and the azimuth mark should both be triangulation stations. When horizontal angles are being measured at night, the azimuth observation is made a part of the same program by including pointings on a circumpolar star with the regular series of pointings on lights at the triangulation stations. An azimuth found by this method is more accurate than one determined by means of an auxiliary point and subsequently connected with the triangulation by means of a horizontal angle measured by daylight.

On account of the slow apparent motions of stars near the pole, nearly all accurate azimuth observations are made on close circumpolars, since errors of the latitude and the time have less effect on the result than for stars farther from the pole. The stars ordinarily used for azimuth observations are shown in Fig. 55.

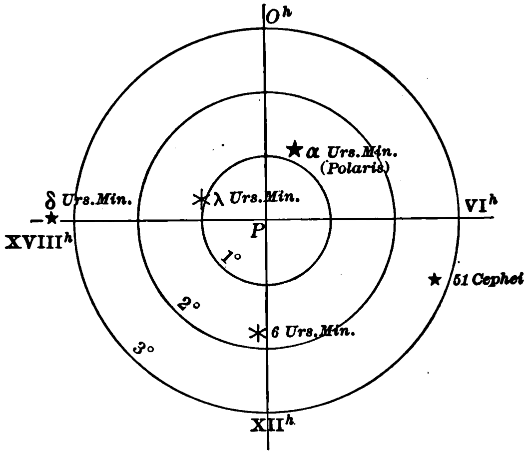


FIG. 55. Circumpolar Stars.

85. Formula for Azimuth.

In general all these methods consist in calculating the azimuth of the star at the instant of observation and combining this azimuth with the measured horizontal angle from the star to the station. The azimuth of a circumpolar star is found by the formula

$$\tan Z = -\frac{\sin t}{\cos\phi\tan\delta - \sin\phi\cos t}, \qquad [33]$$

where Z is the azimuth measured from the north toward the east, and t is the hour angle.

If Equa. [33] be divided by $\cos\phi\tan\delta$, then

$$\tan Z = -\frac{\cot\delta\sec\phi\sin t}{1 - \cot\delta\tan\phi\cos t}$$

$$= -\cot\delta\sec\phi\sin t\left(\frac{1}{1-a}\right). \qquad [34]$$

If values of $\frac{1}{1-a}$ are tabulated,* this formula will be found more convenient than Equa. [33].

86. Curvature Correction.

In computing the azimuth of the star it would evidently be inconvenient to apply the formula to each separate pointing on the star, on account of the large amount of computation. It is simpler and sufficiently accurate to calculate the azimuth of the star at the *mean* of the observed times of pointing, and then to correct the computed azimuth for the small difference between this azimuth and the mean of all the azimuths. The correction for this difference is

$$\text{Curvature Correction} = -\tan Z \frac{1}{n} \sum \frac{2 \sin^2 \frac{\tau}{2}}{\sin 1''}, \qquad [35]$$

in which n = the number of pointings
and τ = the interval of time (in seconds) between the observed time and the mean.

The sign of the correction is such that it always decreases the angle between the star and the pole. For the derivation of this formula see Hayford's *Geodetic Astronomy*, p. 213. The correction may also be written in the form $-\tan Z\ [6.73672] \frac{1}{n} \sum \tau^2$. (See Doolittle's *Practical Astronomy*, p. 537.)

87. Correction for Diurnal Aberration.

On account of the motion of the observer, due to the earth's rotation, the star is apparently displaced toward the east. The correction to the computed azimuth for the effect of this apparent displacement is given by the expression

$$\text{Corr. for Aberra.} = 0''.32 \frac{\cos Z \cos \phi}{\cos h}. \qquad [36]$$

This correction is always positive for an azimuth counted clockwise. For the derivation of this formula see Doolittle's *Practical Astronomy*, p. 530.

* For a table of values of $\log \frac{1}{1-a}$ see *Special Pub. No.* 14.

88. Level Correction.

If the horizontal axis is not level when a pointing is made on the star, the observed direction must be corrected by the following quantity:

$$\text{Lev. Corr.} = \frac{d}{4}[(w + w') - (e + e')] \tan h. \qquad [37]$$

For proof of this formula see pp. 55 and 87. If the level is graduated from one end to the other,

$$\text{Lev. corr.} = \frac{d}{4}[(w - w') + (e - e')] \tan h, \qquad [38]$$

where w and e are read before, and w' and e' are read after, the reversal of the striding level. If the azimuth mark is not near the horizon, it is necessary to apply a similar correction to the observed direction of the mark. The correction is to be added algebraically to readings which increase in a clockwise direction.

89. The Direction Method.

In observing for azimuth by this method the observations are carried out almost exactly as in measuring the angles of a triangulation, except that the chronometer is read whenever a pointing is made on the star, and level readings to determine the inclination of the axis are made just before or just after pointing on the star. The altitude of the star should be measured at least twice during the observations. In observing Polaris in connection with a number of triangulation stations, it is best to take the pointing on the star last. From twelve to sixteen sets should be made with the direction instrument, in order to secure the necessary precision.

Following is an example of the form of record and computation of an azimuth by the method of directions.

HORIZONTAL DIRECTIONS

[Station, Sears, Tex. (Triangulation Station). Observer, W. Bowie. Instrument, Theodolite 168. Date, Dec. 22, 1908.]

Position.	Objects observed.	Time.	Tel. D or R.	Mic.	Backward.			For-ward.	Mean.	Mean D and R.	Direc-tion.	Remarks.
		h m			°	′	″	″			″	
1	Morrison..	8 19	D	*A*	0	0	35	35				1 division of the
				B			41	41				striding level
				C			36	34	37.0			= 4″.194
			R	*A*	180	00	36	35				
				B			32	31				
				C			35	34	33.8	35.4	00.0	
	Buzzard...		D	*A*	53	30	43	42				
				B			41	42				
				C			34	33	39.2			
			R	*A*	233	30	39	37				
				B			34	32				
				C			38	38	36.3	37.8	02.4	
	Allen......		D	*A*	170	14	61	62				
				B			57	55				
				C			61	59	59.2			
			R	*A*	350	14	50	49				
				B			63	60				
				C			53	53	54.7	57.0	21.6	
	Polaris....		D	*A*	252	01	54	53				W E
	h m s			*B*			54	53				9.3 28.0
	1 48 35.5			*C*			51	51	52.7			27.7 9.1
	1 51 06.0											
												18.4 − 0.5 18.9
	1 49 50.8											
			R	*A*	72	01	09	09				24.9 6.3
				B			02	01				13.0 31.7
				C			10	08	06.5	29.6		
												11.9 −13.5 25.4
												− 7.0

COMPUTATION OF AZIMUTH, DIRECTION METHOD.

[Station, Sears, Tex. Chronometer, sidereal 1769. $\phi = 32° 33' 31''$
Instrument, theodolite 168. Observer, W. Bowie.]

Date, 1908, position	Dec. 22, 1	2	3	4
Chronometer reading	1 49 50.8	2 01 33.0	2 16 31.0	2 43 28.8
Chronometer correction	− 4 37.5	− 4 37.5	− 4 37.4	− 4 37.3
Sidereal time	1 45 13.3	1 56 55.5	2 11 53.6	2 38 51.5
α of Polaris	1 26 41.9	1 26 41.9	1 26 41.8	1 26 41.8
t of Polaris (time)	0 18 31.4	0 30 13.6	0 45 11.8	1 12 09.7
t of Polaris (arc)	4° 37′ 51″.0	7° 33′ 24″.0	11° 17′ 57″.0	18° 02′ 25″.5
δ of Polaris	88 49 27.4			
log cot δ	8.31224	8.31224	8.31224	8.31224
log tan ϕ	9.80517	9.80517	9.80517	9.80517
log cos t	9.99858	9.99621	9.99150	9.97811
log a (to five places)	8.11599	8.11362	8.10891	8.09552
log cot δ	8.312243	8.312243	8.312243	8.312243
log sec ϕ	0.074254	0.074254	0.074254	0.074254
log sin t	8.907064	9.118948	9.292105	9.490924
log $\frac{1}{1-a}$	0.005710	0.005673	0.005618	0.005445
log ($-$tan A) (to 6 places)	7.299271	7.511124	7.684220	7.882866
A = Azimuth of Polaris, from north*	0 06 50.8	0 11 09.2	0 16 36.9	0 26 15.0
Difference in time between D. and R.	m s 2 30	m s 2 00	m s 3 18	m s 1 38
Curvature correction	0	0	0	0
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
Altitude of Polaris = h	33 46	33 46	33 46	33 46
$\frac{d}{4}$ tan h = level factor	0.701	0.701	0.701	0.701
Inclination †	−7.0	−7.2	−7.0	−1.8
Level correction	−4.9	−5.0	−4.9	−1.3
Circle reads on Polaris	252 01 29.6	86 58 11.2	281 54 27.0	116 45 48.6
Corrected reading on Polaris	252 01 24.7	86 58 06.2	281 54 22.1	116 45 47.3
Circle reads on mark	170 14 57.0	5 15 58.2	200 17 42.4	35 18 45.4
Difference, mark — Polaris	278 13 32.3	278 17 52.0	278 23 20.3	278 32 58.1
Corrected azimuth of Polaris, from north *	0 06 50.8	0 11 09.2	0 16 36.9	0 26 15.0
	180 00 00.0	180 00 00.0	180 00 00.0	180 00 00.0
Azimuth of Allen (Clockwise from South)	98 06 41.5	98 06 42.8	98 06 43.4	98 06 43.1

To the mean result from the above computation must be applied corrections for diurnal aberration and eccentricity (if any) of Mark.

Carry times and angles to tenths of seconds only.

* Minus, if west of north.

† The values shown in this line are actually four times the inclination of the horizontal axis in terms of level divisions.

90. Method of Repetition.

In observing by the repetition method the program given on p. 57 is followed, with the addition of readings of the chronometer and the stride level, taken when the telescope is pointing at the star. The altitude of the star should be measured, if possible, but may be computed from the known time if necessary. The verniers are read only at the beginning and end of a half set, as when measuring the angles of a triangulation.

Following is an example of the form of record and computation of an azimuth by the method of repetition.

RECORD — AZIMUTH BY REPETITIONS.

[Station, Kahatchee △. State, Alabama. Date, June 6, 1898. Observer, O. B. F. Instrument, 10-inch Gambey No. 63. Star, Polaris.]

[One division striding level = 2.″67.]

Objects.	Chr. time on star.	Pos. of tel.	Repetitions.	Level readings. W. E.	Circle readings. °	′	A ″	B ″	Mean.	Angle.
	h m s									° ′ ″
Mark......		D	0		178	03	22.5	20	21.2	
Star.......	14 46 30		1	4.5 10.7 9.2 5.9						
	49 08		2							
	52 51	D	3	9.6 5.6 5.2 10.0						
	56 10	R	4	11.3 4.0 7.8 7.4						
Set No. 5..	14 59 12		5							
	15 01 55	R	6	8.7 6.6 11.9 3.4	100	16	20	20	20	72 57 50.2
	14 54 17.7			68.2 53.6 + 14.6						
Star.......	15 04 44	R	1	11.9 3.4 8.5 6.8						
	07 18		2							
	09 54	R	3	7.9 7.3 11.2 4.1						
Set No. 6..	14 15	D	4	9.0 6.1 5.9 9.6						
	16 14		5							
	15 18 24		6	5.9 9.6 9.1 6.2						
Mark......		D			177	27	00	00	00	72 51 46.7
	15 11 48.2			69.4 53.1 +16.3						

COMPUTATION — AZIMUTH BY REPETITIONS

[Kahatchee, Ala. $\phi = 33° 13' 40''.33$.]

Date, 1898, set	June 6 5	June 6 6
Chronometer reading	14 54 17.7	15 11 48.2
Chronometer correction	−31.1	−31.1
Sidereal time	14 53 46.6	15 11 17.1
α of Polaris	1 21 20.3	1 21 20.3
t of Polaris (time)	13 32 26.3	13 49 56.8
t of Polaris (arc)	203° 06′ 34″.5	207° 29′ 12″.0
δ of Polaris	88 45 46.9	
log cot δ	8.33430	8.33430
log tan ϕ	9.81629	9.81629
log cos t	9.96367*n*	9.94798*n*
log a (to five places)	8.11426*n*	8.09857*n*
log cot δ	8.334305	8.334305
log sec ϕ	0.077535	0.077535
log sin t	9.593830*n*	9.664211*n*
log $\frac{1}{1-a}$	9.994387	9.994584
log ($-$tan A) (to 6 places)	8.000057*n*	8.070635*n*
A = Azimuth of Polaris, from north *	0° 34′ 22″.8	0° 40′ 26″.8
	m s ″	m s ″
τ and $\frac{2 \sin^2 \frac{1}{2}}{\sin 1''}$	7 47.7 119.3	7 04.2 98.1
	5 09.7 52.3	4 30.2 39.8
	1 26.7 4.1	1 54.2 7.1
	1 52.3 6.9	2 26.8 11.8
	4 54.3 47.2	4 25.8 38.5
	7 37.3 114.0	6 35.8 85.4
Sum	343.8	280.7
Mean	57.3	46.8
log $\frac{1}{n} \sum \frac{2 \sin^2 \frac{1}{2}\tau}{\sin 1''}$	1.758	1.670
log (curvature corr.)	9.758	9.741
Curvature correction	−0.6	−0.6
Altitude of Polaris = h	32° 07′	
$\frac{d}{4}$ tan h = level factor	0.419	0.419
Inclination	+3.6	+4.1
Level correction	−1″.5	−1″.7
Angle, star — mark	72 57 50.2	72 51 46.7
Corrected angle	72 57 48.7	72 51 45.0
Corrected azimuth of star *	0 34 22.2	0 40 26.2
Azimuth of mark E of N	73 32 10.9	73 32 11.2
	180 00 00.0	180 00 00.0
Azimuth of mark (Clockwise from south)	253 32 10.9	253 32 11.2

To the mean result from the above computation must be applied corrections for diurnal aberration and eccentricity (if any) of Mark. Carry times and angles to tenths of seconds only.

* Minus if west of north.

91. Micrometric Method.

In employing this method it is necessary to place a mark nearly in the same vertical plane with the star at the time of the observation. For greatest accuracy, as well as for convenience, the star should be observed when near its greatest elongation. Near culmination the star's motion will carry it beyond the range of the micrometer in a comparatively short time. The small difference in azimuth between the star and the mark is to be measured with the micrometer in the eyepiece of a transit instrument. The instrument is clamped in azimuth, and the readings are taken in the following order: take five pointings on the mark; point toward the star and place the stride level in position; take three pointings on the star with their corresponding chronometer times; read and reverse the stride level; take two more pointings on the star, noting the times; read the stride level; reverse the horizontal axis of the instrument in the bearings, point the telescope at the star, and place the level in position; take three pointings on the star, with chronometer times; read the level and reverse it; take two more pointings on the star and the times; read the level; finally, take five pointings on the mark. Three such sets will be found to require from thirty to fifty minutes' time. Either the altitude or the zenith distance of the star should be read twice during the set, in order that an altitude for use in calculating the azimuth may be interpolated.

The angle given by the micrometer readings is in the plane of the line of collimation and the horizontal axis. To reduce this angle to the horizontal plane, multiply it by the secant of the altitude. Each half-set may be reduced separately. The altitude for the middle of each half set may be used for reducing to horizontal. The value of one turn of the micrometer screw may be found by observing a circumpolar star near culmination, or, better still, by measuring a small angle by means of a theodolite and then measuring this angle with the micrometer.

Following is an example of record and computation.

RECORD AND COMPUTATION — AZIMUTH BY MICROMETRIC METHOD

[Station No. 10, Mexican Boundary. Date, Oct. 13, 1892. Observer, J. F. H. Instrument, Fauth Repeating Theodolite, No. 725 (10 in.). Star, Polaris near eastern elongation.]

Circle.	Level readings. W	Level readings. E	Chronometer time.	τ.	$\frac{2\sin^2\frac{1}{2}\tau}{\sin 1''}$.	Micrometer readings — On star.	Micrometer readings — On mark.	
			h m s	m s				
E	8.0	9.9	9 06 38.0	3 58.6	31.05	18^t.379	18^t.310	$\lambda = 2^h\ 12^m$ W of
	10.	7.3	07 32.0	3 04.6	18.59	0.388	0.315	Washington
								$\phi = 31°\ 19'\ 35''$
	+18.0	−17.2	08 05.5	2 31.1	12.45	0.400	0.315	1 div. of level
	+0.8		09 13.0	1 23.6	3.82	0.424	0.311	= 3″.68
E			09 48.0	0 48.6	1.29	0.430	0.316	1 turn of mic.
								= 123″.73
						18.4042	18.3134	Means
W	9.0	9.0	9 12 01.8	1 25.2	3.96	18.100	18.290	
	7.0	10.9	12 24.7	1 48.1	6.37	0.100	0.275	
	+16.0	−19.9	12 48.3	2 11.7	9.46	0.090	0.279	
	−3.9		13 36.3	2 59.7	17.61	0.086	0.281	
W	Mean	1^d.55	13 58.1	3 21.5	22.14	0.080	0.279	
			9 10 36.6		12.67	18.0912	18.2808	Means

ζ of star at middle of first half of set = 58° 48′. cosec ζ = 1.1691. cot 58° 47′ = 0.606.
ζ of star at middle of second half of set = 58° 46′. cosec ζ = 1.1695.
$\alpha = 1^h\ 20^m\ 07^s.4$. $\delta = 88°\ 44'\ 10''.4$.

Collimation axis reads ½ (18.3134 + 18.2808)* = 18^t.2971

* In this instrument increased readings of the micrometer correspond to a movement of the line of sight toward the east when the vertical circle is to the east, and toward the west when the vertical circle is to the west.

Mark east of collimation axis 18.3134 − 18.2971 = 0 .0163 = 02″.02
Circle E., star E of collimation axis (18.4042 − 18.2971) (1.1691) = 0 .1252
Circle W., star E of collimation axis (18.2971 − 18.0912) (1.1695) = 0 .2408
Mean, star E of collimation axis = 0 .1830 = 22 .64
Mark west of star = 20 .62
Level correction (1.55) (0.92) (0.606) = −0 .86
Mark west of star, corrected = 19 .76

Mean chronometer time of observation = 21^h 10^m 36^s.6
Chronometer correction = − 2 11 28 .2
Sidereal time = 18 59 08 .4
α = 1 20 07 .4
Hour-angle, t, in time 17 39 01 .0
" in arc 264° 45′ 15″.0

log cot δ = 8.34362
log tan ϕ = 9.78436
log cos t = 8.96108 n
log a = 7.08906 n
log cot δ = 8.343618
log sec ϕ = 0.068431
log sin t = 9.998177 n
log $\frac{1}{1-a}$ = 9.999467
log (−tan A) = 8.409693 n
A = +1° 28′ 16″.91
log. 12.67 = 1.10278
log. curvature corr. = 9.51247

Curvature corr. = −0.33
Diur. Aber. corr. = +0.32
Mean azimuth of star = +1° 28′ 16″.90
Mark west of star 19 .76
Azimuth of mark, E of N = +1° 27′ 57″.14

92. Reduction to Sea-Level.

If the azimuth mark is at a high elevation, the computed azimuth must be reduced to its value at the point where the vertical through the mark intersects the sea-level. This correction in seconds is

$$+\frac{e^2 h}{2\,a \sin 1''} \cdot \cos^2 \phi \sin 2\,\alpha, \qquad [39]$$

in which h is the elevation, ϕ is the latitude, α is the azimuth, and e and a are for the Clarke Spheroid of 1866 (see Art. 102, p. 136). If h is expressed in meters, this becomes

$$+0''.000109\, h \cos^2 \phi \sin 2\,\alpha. \qquad [40]$$

$$(\log \text{ of } 0.000109 = 6.0392 - 10.)$$

If the mark is either northeast or southwest of the observing station the observed azimuth must be increased to obtain the correct azimuth; if the mark is northwest or southeast, the observed azimuth must be decreased.

Reduction to Mean Position of the Pole.

The observed azimuth must be reduced to its value corresponding to the mean position of the pole. In latitude 50° (northern United States) this correction may be as great as half a second (see p. 107).

PROBLEMS

Problem 1. What should be the linear distance between the vertical threads of a transit having a 30-inch focus in order to give $2^s.5$ intervals of time between threads for an equatorial star?

Problem 2. The following readings were taken to determine the pivot inequality of a transit. Clamp east, level direct, $w = 43.5$, $e = 34.0$; level reversed, $w = 36.7$, $e = 41.0$. Clamp west, level direct, $w = 39.1$, $e = 37.0$; level reversed, $w = 34.2$, $e = 41.8$. The value of one division of the level is $0''.75$. This level has the zero at the center and is numbered both ways. Find the pivot inequality.

If a star is observed with the transit in the position clamp east what is the level correction to the observed time of transit if $\delta = +30°$ and $\phi = +40°$?

Problem 3. If the collimation axis of a transit has a true bearing of S 0° 00′ 15″ E what is the correction to the observed time of transit of a star if $\delta = +39°$ and $\phi = +30°$?

Problem 4. If a latitude is found to be 36° 49′ 50″.261 at an altitude of 6250 feet what will this latitude be when reduced to sea-level?

Problem 5. Compute the latitude from the following zenith telescope observations.

Star No. 2125, south; chr. time $13^h\ 37^m$; micrometer $16^t.063$; level, n 83.0, s 30.0. Star No. 2141, north; chr. time, $13^h\ 43^m$; micrometer, $13^t.504$; level n 31.0, s 83.5. Eyepiece on side toward micrometer head; level zero on side opposite to eyepiece. Declination of 2125, $28°\ 34'\ 09''.80$; declination of 2141, $39°\ 00'$ 08.80. One division of latitude level = $1''.00$. One turn of micrometer = $2'.559$.

CHAPTER V

PROPERTIES OF THE SPHEROID

93. Mathematical Figure of the Earth.

In calculating the positions of survey points on the earth, it is necessary to consider these points as lying upon some mathematical surface, like the sphere or the ellipsoid, taken to represent the figure of the earth. This is accomplished by projecting the position of the station vertically downward onto the surface in question. The actual shape of the earth's surface is quite irregular and, from the nature of the problem, can only be determined approximately. But even if it could be found exactly, it would not be adapted to the purpose of computation. For this reason it is necessary to select some figure, the use of which will simplify the computation, but which will nowhere depart from the true figure by an amount sufficient to produce serious errors in the results. The figure generally adopted is the *oblate spheroid* or *ellipsoid of revolution*. Such a figure is generated by rotating an ellipse about its shorter axis. This surface approaches much nearer the actual figure of the earth than does the sphere, but perhaps not quite so near as an ellipsoid of three unequal dimensions. The latter, however, would be an inconvenient figure to use, and the gain in accuracy would be very slight.

The oblate spheroid is an ellipsoidal surface with two of its axes equal, but with the third axis, about which the figure rotates, shorter than the other two. All plane sections of such a surface are ellipses, except those cut by planes perpendicular to the rotation axis. Sections through the rotation, or polar, axis are ellipses whose major axes are the equatorial diameter, and whose minor axes are the polar diameter, of the spheroid. The nature of this surface will be understood best if we investigate first the properties of the ellipse which generates the spheroid.

94. Properties of the Ellipse.

In Fig. 56, PP' is the polar axis of the spheroid, and EE' is any one of the equatorial diameters. F is one focus of the ellipse. At M, any point on the curve, the line MA is drawn tangent to the ellipse; MH is perpendicular to the tangent, that is, normal

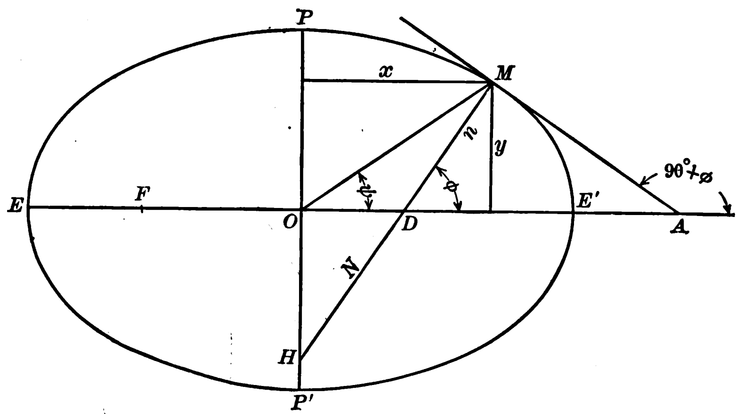


FIG. 56.

to the curve. MH is the direction that the plumb line at M is supposed to assume unless deflected by local causes, such as variations in density. The distance MH $(=N)$, terminating in the minor axis, is called the *normal*. MD $(=n)$ is the normal terminating in the major axis. The angle made by the normal with OE', that is, with the plane of the earth's equator, is the *geodetic latitude* (ϕ).* The angle made by MO with OE' is the *geocentric latitude* (ψ).

Another angle which is of importance in the geometry of the ellipse is the *eccentric angle*, or *reduced latitude*, θ. It is the angle $E'Om$, Fig. 57, in which M is any point on the ellipse, MN is

* The *astronomical latitude* is the angle made by the actual direction of gravity (plumb line) with the plane of the equator.

perpendicular to OE', and m is the point where this perpendicular cuts the circle whose center is O and radius OE'.

The equation of the ellipse whose major and minor semiaxes are a and b, referred to its own axes as coördinate axes, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

To determine the coördinates of any point M (Fig. 56), in terms of the latitude, differentiate this equation and the result is

$$\frac{y}{x} = -\frac{b^2}{a^2}\frac{dx}{dy}. \qquad (1)$$

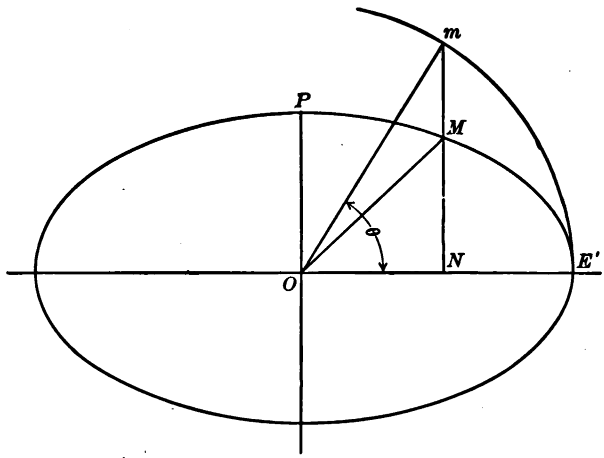


FIG. 57.

Since the tangent line to an ellipse makes an angle with the axis of X whose tangent is $\frac{dy}{dx}$,

$$\tan(90^\circ + \phi) = -\frac{dy}{dx},$$

or

$$\tan\phi = -\frac{dx}{dy}.$$

The eccentricity e is the distance from the focus to the center divided by a, that is $\frac{OF}{OE}$. From the triangle OFP it will be seen

that
$$e^2 = \frac{a^2 - b^2}{a^2},$$

or
$$\frac{b^2}{a^2} = 1 - e^2.$$

Therefore (1) may be written

$$\frac{y}{x} = (1 - e^2) \tan \phi. \qquad (2)$$

From the equation of the ellipse,

$$x^2 + \frac{y^2}{1 - e^2} = a^2. \qquad (3)$$

Squaring (2) and substituting in the result the value of y^2 from (3), we obtain *

$$x = \frac{a \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \qquad [41]$$

and
$$y = \frac{a(1 - e^2) \sin \phi}{\sqrt{1 - e^2 \sin^2 \phi}}. \qquad [42]$$

95. Radius of Curvature of the Meridian.
To find the radius of curvature of the meridian (R_m), apply the general formula

$$R_m = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

From (1)
$$\frac{dy}{dx} = -\frac{x}{y}\frac{b^2}{a^2}.$$

* The relation $1 + \tan^2 \phi = \sec^2 \phi$ is used in this transformation.

Differentiating this equation, we have

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right)$$

$$= -\frac{b^2}{a^2y^2}\left(y + \frac{x^2}{y}\cdot\frac{b^2}{a^2}\right)$$

$$= -\frac{b^4}{a^2y^3}.$$

Therefore
$$R_m = -\frac{\left[1 + \frac{x^2}{y^2}\frac{b^4}{a^4}\right]^{\frac{3}{2}}}{\frac{b^4}{a^2y^3}}$$

$$= -\frac{[a^4y^2 + b^4x^2]^{\frac{3}{2}}}{a^4b^4}$$

$$= -\frac{\left[\frac{a^6(1 - e^2)^2\sin^2\phi}{1 - e^2\sin^2\phi} + \frac{b^4a^2\cos^2\phi}{1 - e^2\sin^2\phi}\right]^{\frac{3}{2}}}{a^4b^4}.$$

Then, since
$$b^2 = a^2(1 - e^2),$$

$$R_m = -\frac{a(1 - e^2)}{(1 - e^2\sin^2\phi)^{\frac{3}{2}}}.* \qquad [43]$$

Values of log R_m will be found in Table X.

96. Radius of Curvature in the Prime Vertical.

The radius of curvature of the surface of the spheroid in a plane at right angles to the meridian may be proved to be equal to the length of the normal (N) terminating in the minor axis. If a central section be taken through a point M and perpendicular to the meridian, and the radius of curvature of this ellipse at point M be computed, it will be found to be $\rho = \frac{a(1 - e^2\sin^2\phi)^{\frac{1}{2}}}{\cos\phi \sec\psi}$. According to Meunier's theorem the radius of curvature of the normal section equals the radius of curvature of this central section divided by the cosine of the angle between the two planes,

* The negative sign indicates only the direction of bending. It is customary to regard the value of R_m as positive.

that is, by $\cos(\phi - \psi)$. Hence the radius of curvature of the prime vertical section is N.

To show this geometrically, let A and B in Fig. 58 be two points on the same parallel of latitude. The normals to the surface at A and B always intersect at H on the minor axis. Let C be a point on the prime vertical section through A, and also on the meridian of B. The normals at A and C intersect at some point K above H. K is approximately the center of curvature of the

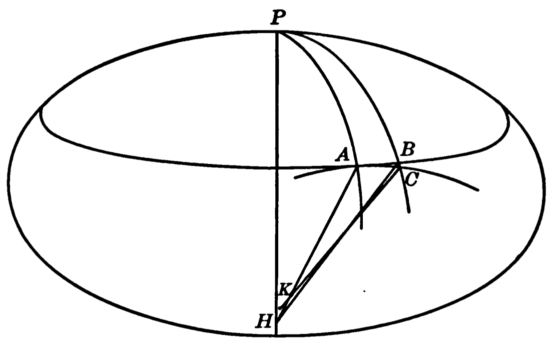


FIG. 58.

arc AC. When the meridian PBC is taken nearer to A, points A and C approach each other, the intersection of their normals approaches the true center of curvature, and the length CK approaches the true radius of curvature. But the nearer C approaches A, the nearer it approaches B. Hence CK must ultimately coincide with AH; that is, H is the point toward which the center of curvature is approaching and the normal N is the radius of curvature of the prime vertical section at A.

From Fig. 56 it is evident that

$$N = \frac{x}{\cos\phi}$$

$$= \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}}. \qquad [44]$$

Values of $\log N$ will be found in Table X.

The normal terminating in the minor axis is

$$n = \frac{y}{\sin\phi} = \frac{a(1 - e^2)}{\sqrt{1 - e^2\sin^2\phi}} = N(1 - e^2). \qquad [45]$$

The radius of the parallel of latitude ($= x$) is given by

$$R_p = N\cos\phi. \qquad [46]$$

97. **Radius of Curvature of Normal Section in any Azimuth.** Having found the radii of curvature of the two principal sections, it now remains to find a general expression for the radius of curvature in any azimuth, and it will be shown that this may be expressed in terms of the two radii already found.

The equation of the spheroid is

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{a^2} + \frac{z_1^2}{b^2} = 1,$$

or

$$b^2x_1^2 + b^2y_1^2 + a^2z_1^2 = a^2b^2. \qquad (a)$$

In Fig. 59 the Z_1-axis coincides with the polar axis of the spheroid. If M be any point on the meridian Z_1M, and MY any section cut by a plane through MH (the normal) making an angle α with the meridian, then the equation of the spheroid may be transformed so as to refer to the origin C and the new Z axis CM. Let the coördinates of any point P be x_1, y_1, z_1, and let the new coördinates be x, y, z. Then, from Fig. 59, the relation of the new coördinates to the old is given by

$$\begin{aligned} x_1 &= OG = OC + x + z\cos\phi + y\cos\alpha\sin\phi \\ &= Ne^2\cos\phi + x + z\cos\phi + y\cos\alpha\sin\phi, \\ y_1 &= y\sin\alpha, \\ z_1 &= z\sin\phi - y\cos\alpha\cos\phi. \end{aligned}$$

Substituting these values in (a),

$$b^2(Ne^2\cos\phi + x + z\cos\phi + y\cos\alpha\sin\phi)^2 + b^2y^2\sin^2\alpha + a^2(z\sin\phi - y\cos\alpha\cos\phi)^2 = a^2b^2,$$

which is the equation of the spheroid referred to the new axes. If x is made equal to zero, then P will be on the curve MY, and the equation becomes the equation of this plane section, that is,

$$b^2 (Ne^2 \cos\phi + z\cos\phi + y\cos\alpha\sin\phi)^2 + b^2y^2\sin^2\alpha$$
$$+ a^2 (z\sin\phi - y\cos\alpha\cos\phi)^2 = a^2b^2,$$

the equation of the ellipse *MY*.

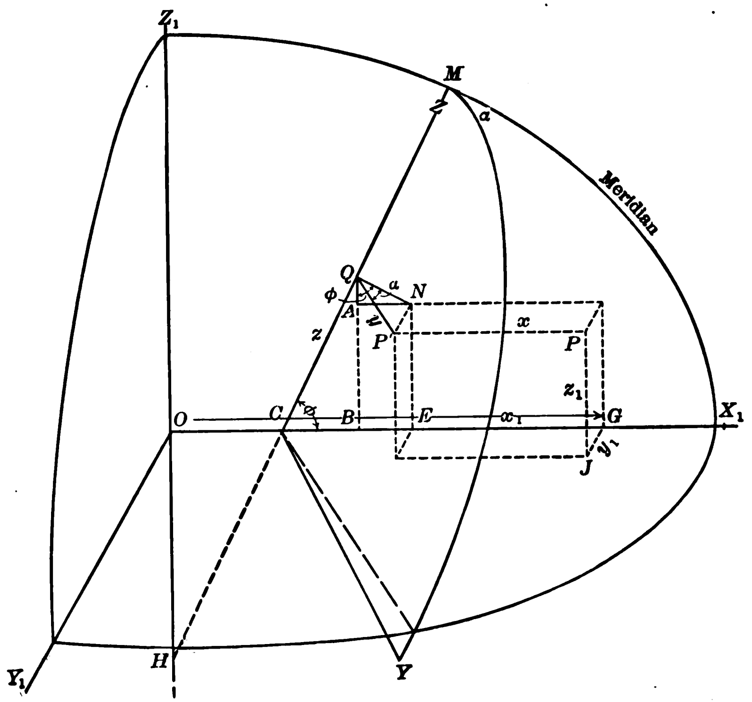


FIG. 59.

To determine the radius of curvature at *M* it is necessary to find $\frac{dz}{dy}$ and $\frac{d^2z}{dy^2}$ and to substitute these values in the general formula for radius of curvature.

Expanding the last equation, collecting terms, and dividing through by a^2,

$$y^2 [1 - e^2 (1 - \cos^2\alpha\cos^2\phi)] + z^2 (1 - e^2\cos^2\phi)$$
$$- yz (2 e^2 \cos\alpha\sin\phi\cos\phi) + 2 y (1 - e^2) \cdot N \cdot e^2 \cos\alpha\sin\phi\cos\phi$$
$$+ 2 ze^2 (1 - e^2) \cdot N \cdot \cos^2\phi = (1 - e^2)(a^2 - Ne^4\cos^2\phi),$$

or, in abbreviated form,

$$y^2A + z^2B - yzC + 2\,yD + 2\,zE = F.$$

Differentiating this equation, y being taken as the independent variable,

$$2\,yA + 2\,z\frac{dz}{dy}B - Cy\frac{dz}{dy} - Cz + D + E\frac{dz}{dy} = 0.$$

Differentiating again,

$$2\,A + 2\,B\left(z\frac{d^2z}{dy^2} + \left(\frac{dz}{dy}\right)^2\right) - C\left(y\frac{d^2z}{dy^2} + \frac{dz}{dy}\right) - C\frac{dz}{dy} + E\frac{d^2z}{dy^2} = 0,$$

$$\frac{d^2z}{dy^2}(2\,Bz - Cy + E) = -\left(2\,A + 2\,B\left(\frac{dz}{dy}\right)^2 - 2\,C\frac{dz}{dy}\right),$$

$$\frac{d^2z}{dy^2} = -\frac{2\,A + 2\,B\left(\frac{dz}{dy}\right)^2 - 2\,C\frac{dz}{dy}}{2\,Bz - Cy + E}.$$

For point M, $y = 0$ and $z = n = N(1 - e^2)$. Therefore

$$\frac{dz}{dy} =$$

$$-\frac{N(1-e^2)(2\,e^2\cos\alpha\sin\phi\cos\phi) - 2(1-e^2)Ne^2\cos\alpha\sin\phi\cos\phi}{2\,Bz - Cy + E} = 0,$$

and

$$\frac{d^2z}{dy^2} = -\frac{2\,[1 - e^2(1 - \cos^2\alpha\cos^2\phi)]}{2\,N(1 - e^2)(1 - e^2\cos^2\phi) + 2\,e^2(1 - e^2)\cos^2\phi \cdot N}$$

$$= -\frac{1 - e^2 + e^2\cos^2\alpha\cos^2\phi}{N(1 - e^2)}$$

$$= -\frac{(1 - e^2)(\sin^2\alpha + \cos^2\alpha) + e^2\cos^2\alpha(1 - \sin^2\phi)}{N(1 - e^2)}$$

$$= -\frac{(1 - e^2)\sin^2\alpha + \cos^2\alpha - \cos^2\alpha \cdot e^2\sin^2\phi}{N(1 - e^2)} \times \frac{R_m}{R_m}$$

$$= -\frac{R_m\sin^2\alpha + \dfrac{R_m}{1 - e^2}\cos^2\alpha(1 - e^2\sin^2\phi)}{NR_m}$$

$$= -\frac{R_m\sin^2\alpha + N\cos^2\alpha}{NR_m}.$$

Substituting these differential coefficients in the usual formula for radius of curvature, we have

$$R_\alpha = \frac{NR_m}{N\cos^2\alpha + R_m\sin^2\alpha}. \qquad [47]$$

If $\alpha = 0°$,

$$R_\alpha = \frac{NR_m}{N} = R_m,$$

the radius of curvature of the meridian; and if

$$\alpha = 90°,$$

then $$R_\alpha = \frac{NR_m}{R_m} = N,$$

the radius of curvature of the prime vertical.

Values of $\log R_\alpha$ for different latitudes and azimuths will be found in Table XI.

98. The Mean Value of R_α.

The mean value of R_α at any point for all azimuths from 0° to 360° may be found as follows: if the angular space about any point M be divided into a large number of small parts, each equal to $d\alpha$ and each expressed as a fraction of a radian, then the number of such parts in a radian will be $\frac{1}{d\alpha}$, and the number in a circumference will be $\frac{2\pi}{d\alpha}$. If the value of R_α be computed for each of these azimuths, then the sum of these values of R_α, divided by their number, is the mean value; that is,

$$\text{mean } R_\alpha = \int_0^{2\pi} R_\alpha \frac{d\alpha}{2\pi}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{NR_m}{N\cos^2\alpha + R_m\sin^2\alpha}\cdot d\alpha$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{NR_m}{N\cos^2\alpha + R_m\sin^2\alpha}\cdot d\alpha.$$

To integrate this quantity, substitute a new variable, $t = \tan\alpha\sqrt{\frac{R_m}{N}}$, from which $dt = \sqrt{\frac{R_m}{N}}\cdot\frac{1}{\cos^2\alpha}$. By dividing both numerator and denominator by $N\cos^2\alpha$ and factoring NR_m, the integral may be put in the form

$$\text{mean } R_\alpha = \frac{2}{\pi}\sqrt{R_mN}\int_0^{\frac{\pi}{2}}\frac{\sqrt{\frac{R_m}{N}}\cdot\frac{1}{\cos^2\alpha}\cdot d\alpha}{1+\frac{R_m}{N}\cdot\frac{\sin^2\alpha}{\cos^2\alpha}},$$

which, by substitution, becomes

$$\begin{aligned}\text{mean } R_\alpha &= \frac{2}{\pi}\sqrt{R_mN}\int_0^\infty\frac{dt}{1+t^2}\\ &= \frac{2}{\pi}\sqrt{R_mN}\,[\tan^{-1}t]_0^\infty\\ &= \frac{2}{\pi}\sqrt{R_mN}\cdot\frac{\pi}{2}\\ &= \sqrt{R_mN}. \qquad [48]\end{aligned}$$

The mean radius of curvature is, therefore, the geometric mean of the radii of curvature of the two principal sections.

99. Geometric Proofs.

Geometric proofs of the last two formulæ will be found instructive. To find R_α geometrically, imagine a tangent plane at the point M and also a parallel plane at an infinitesimal distance below M. This second plane will cut from the surface a small ellipse. It has already been shown that the radius of curvature of the prime vertical section is N. In Fig. 60 the points A, M, and B are on the circle whose radius is N and whose center is the point H on the axis. By similar triangles,

$$MC : CA = CA : CK.$$

Since MC is infinitesimal,

$$MC = \frac{a^2}{2N};$$

similarly, for a section in the meridian

$$MC = \frac{b^2}{2\,R_m};$$

and, in general, for any section,

$$MC = \frac{s^2}{2\,R_\alpha}.$$

The coördinates of the point P (Fig. 61) are

$$x = s \cdot \sin\alpha \quad \text{and} \quad y = s \cdot \cos\alpha.$$

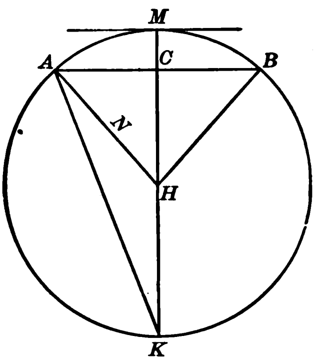


FIG. 60.

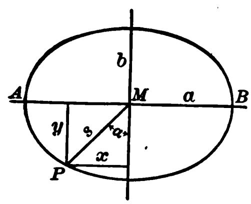


FIG. 61.

Substituting these in the general equation of the ellipse,

$$\frac{s^2 \sin^2\alpha}{a^2} + \frac{s^2 \cos^2\alpha}{b^2} = 1.$$

But, from the preceding equations,

$$\frac{s^2}{a^2} = \frac{R_\alpha}{N} \quad \text{and} \quad \frac{s^2}{b^2} = \frac{R_\alpha}{R_m};$$

hence
$$\frac{R_\alpha}{N} \cdot \sin^2\alpha + \frac{R_\alpha}{R_m} \cdot \cos^2\alpha = 1,$$

or
$$R_\alpha = \frac{NR_m}{N\cos^2\alpha + R_m \sin^2\alpha}. \qquad [47]$$

To show geometrically that the mean value of $R_\alpha = \sqrt{R_m N}$, observe that, as before,

$$\text{mean } R_\alpha = \frac{1}{2\pi}\int_0^{2\pi} R_\alpha \cdot d\alpha$$

and, from the preceding paragraph,

$$R_\alpha = \frac{R_m s^2}{b^2}.$$

Therefore $$\text{mean } R_\alpha = \frac{1}{2\pi}\int_0^{2\pi} \frac{R_m s^2}{b^2} \cdot d\alpha.$$

But $$\frac{1}{2}\int_0^{2\pi} s^2\, d\alpha = \text{area of ellipse} = \pi ab.$$

Therefore $$\text{mean } R_\alpha = \frac{1}{\pi} \times \pi ab \times \frac{R_m}{b^2}$$

$$= \frac{aR_m}{b}.$$

But $$\frac{a}{b} = \sqrt{\frac{N}{R_m}}.$$

Therefore $$\text{mean } R_\alpha = \sqrt{NR_m}. \qquad [48]$$

100. Length of an Arc of the Meridian.
Any small arc of the meridian ellipse may be regarded as an arc of a circle whose radius is R_m, the error being very small for short arcs. The length, therefore, is

$$s = R_m d\phi,$$

or, if $d\phi$ is in seconds of arc,

$$s = R_m d\phi'' \cdot \text{arc } 1''. \qquad [49]$$

If the arc is so long that the value of R_m varies appreciably, it is necessary to find s by integrating the expression

$$ds = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}} \cdot d\phi$$

between the limits ϕ_1 and ϕ_2.

If we expand the denominator by the binomial theorem, we have

$$ds = a(1 - e^2)(1 + \tfrac{3}{2}e^2\sin^2\phi + \tfrac{15}{8}e^4\sin^4\phi + \tfrac{35}{16}e^6\sin^6\phi \ldots)\,d\phi.$$

Integrating,

$$s = a(1 - e^2)\int_{\phi_1}^{\phi_2}(1 + \tfrac{3}{2}e^2\sin^2\phi + \tfrac{15}{8}e^4\sin^4\phi + \cdots)\,d\phi.$$

In order to integrate the terms of the series in parenthesis we simplify the expression by means of the following relations:

$$\begin{aligned}
\sin^2\phi &= \tfrac{1}{2} - \tfrac{1}{2}\cos 2\phi,\\
\sin^4\phi &= \tfrac{3}{8} - \tfrac{1}{2}\cos 2\phi + \tfrac{1}{8}\cos 4\phi,\\
\sin^6\phi &= \tfrac{5}{16} - \tfrac{15}{32}\cos 2\phi + \tfrac{3}{16}\cos 4\phi - \tfrac{1}{32}\cos 6\phi.
\end{aligned}$$

Integrating and substituting the limits, ϕ_1 and ϕ_2, we have

$$\begin{aligned}
s = a(1 - e^2)\{A(\phi_2 - \phi_1) &- \tfrac{1}{2}B(\sin 2\phi_2 - \sin 2\phi_1)\\
&+ \tfrac{1}{4}C(\sin 4\phi_2 - \sin 4\phi_1) \ldots\}, \qquad [50]
\end{aligned}$$

in which $A = 1.0051093$, $B = 0.0051202$, and $C = 0.0000108$.

(See Jordan's *Handbuch der Vermessungskunde*, Vol. III, p. 226; and Crandall's *Geodesy and Least Squares*, p. 163.)

101. Miscellaneous Formulas.

The following formulas, relating to the ellipse, are given here for convenience of reference.

The geocentric latitude may be found from the expression

$$\tan\psi = \frac{y}{x} = (1 - e^2)\tan\phi = \frac{b^2}{a^2}\tan\phi. \qquad [51]$$

The maximum difference between ϕ and ψ is about 0° 11′ 40″, at latitude 45°. At the equator and at the poles the difference is zero.

The reduced latitude, θ (see Art. 94, p. 123), may be found from the geodetic latitude by means of the relation

$$a\tan\theta = b\tan\phi \qquad [52]$$

which is readily proved from Fig. 57.

The compression of the spheroid, that is, the flattening at the poles, is expressed by

$$f = \frac{a - b}{a}. \qquad [53]$$

The length of a quadrant of the meridian is given by

$$q = \frac{a\pi}{2}\left(1 - \frac{e^2}{4} - \frac{3}{64}e^4 - \cdots\right).^* \qquad [54]$$

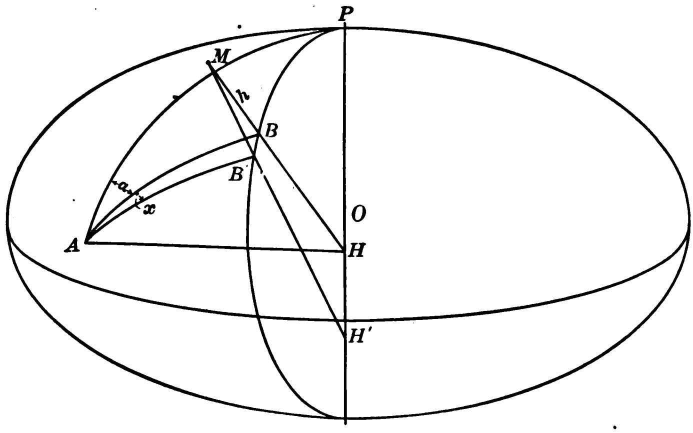


FIG. 62.

102. Effect of Height of Station on Azimuth of Line.

Since the normals drawn from two points on the surface do not in general lie in the same plane, there will be an error in the observed horizontal direction of a station, depending upon its height above the surface of the spheriod. This error may be likened to the error of sighting on an inclined range-pole; the

* From the equation for the length of a meridian arc, we have for the quadrant

$$q = a(1 - e^2)\int_0^{\frac{\pi}{2}}\left(1 + \frac{3}{4}e^2(1 - \cos 2\phi) + \frac{15}{64}e^4(3 - 4\cos 2\phi + \cos 4\phi)\right)d\phi$$

$$= a(1 - e^2)\left[\phi\left(1 + \frac{3}{4}e^2 + \frac{45}{64}e^4\right) - \frac{3}{8}e^2\sin 2\phi - \frac{15}{32}e^4\sin 2\phi + \frac{15}{256}e^4\sin 4\phi\right]_0^{\frac{\pi}{2}}$$

$$= a(1 - e^2)\left[\frac{\pi}{2}\left(1 + \frac{3}{4}e^2 + \frac{45}{64}e^4 + \cdots\right)\right]$$

$$= \frac{a\pi}{2}\left(1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \cdots\right).$$

higher up the sight is taken, the greater the error in the horizontal angle. In Fig. 62 the observer is at A and sighting at point M, which is at an elevation h above sea-level. The vertical plane of the instrument projects M down to sea-level at B on the line MH, H being the end of the normal at A. The point which is verti-

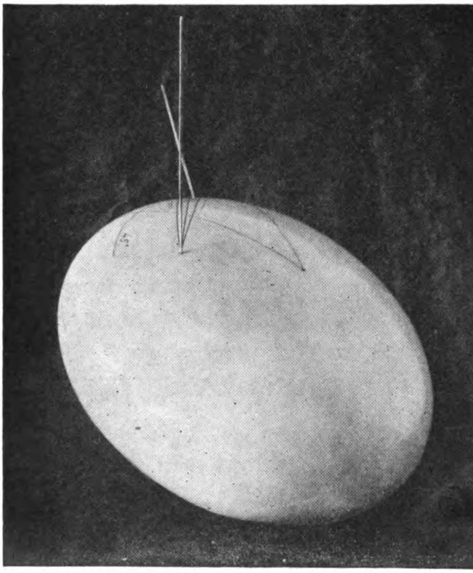


FIG. 62a. A vertical in latitude 0° and a vertical in latitude 60°; $d\lambda = 80°$; $e = 0.81$; (looking SW).

cally below M is B', as determined by the normal MH'. Denote by δ the angle HMH' or, what is nearly the same, HBH'. The angle (x) subtended by BB' at point A (the observer's position) is the correction desired. The latitude of A is ϕ, and that of M is ϕ'. In the triangle MHH'

$$\frac{\sin\delta}{\sin HH'M} = \frac{HH'}{HB + BM} = \frac{HH'}{HB} \text{ (approx.)},$$

or

$$\delta = \frac{HH'}{HB} \cdot \cos\phi',$$

where ϕ' is the latitude of B'.

Now
$$\begin{aligned} HH' &= OH' - OH \\ &= (N' - n') \sin \phi' - (N - n) \sin \phi \\ &= N'e^2 \sin \phi' - Ne^2 \sin \phi. \end{aligned}$$

Therefore
$$\begin{aligned} \delta &= \frac{\cos \phi'}{N} (N'e^2 \sin \phi' - Ne^2 \sin \phi) \\ &= e^2 \cos \phi' \left(\frac{N'}{N} \sin \phi' - \sin \phi\right), \end{aligned}$$

in which $\frac{N'}{N}$ may be put $= 1$ with small error.

Then
$$\delta = e^2 \cos \phi' \left(2 \cos \frac{1}{2} (\phi + \phi') \sin \frac{\Delta\phi}{2}\right),$$

where $\Delta\phi = \phi' - \phi$;

Then
$$\begin{aligned} \delta &= e^2 \cos^2 \phi' \times \Delta\phi \text{ (approx.)}. \\ &= e^2 \cos^2 \phi' \frac{s}{R_m} \cos \alpha \\ &= e^2 \cos^2 \phi' \cdot s \cos \alpha \cdot \frac{(1 - e^2 \sin^2 \phi')^{\frac{3}{2}}}{a (1 - e^2)} \end{aligned}$$

The factor $\frac{(1 - e^2 \sin^2 \phi')^{\frac{3}{2}}}{(1 - e^2)}$ differs but little from unity and may be considered equal to unity in this equation.

Then
$$\delta = \frac{e^2 \cdot s \cdot \cos^2 \phi' \cdot \cos \alpha}{a}. \qquad (a)$$

The linear distance $BB' = h\delta$, and the correction to the azimuth (x) at point A is given by

$$\begin{aligned} x'' &= \frac{h\delta \sin \alpha}{s \operatorname{arc} 1''} \\ &= \frac{he^2 \cos^2 \phi' \sin \alpha \cos \alpha}{a \operatorname{arc} 1''} \\ &= \frac{h}{a \operatorname{arc} 1''} \cdot e^2 \cdot \frac{1}{2} \cdot \sin 2\alpha \cdot \cos^2 \phi', \end{aligned} \qquad [55]$$

as given by Clarke (*Geodesy*, p. 112). This may be written

$$x'' = k \cdot h \cdot \sin 2\alpha \cos^2 \phi' \qquad [56]$$

where $$k = \frac{e^2}{2\,a \operatorname{arc} 1''},$$

the dimensions being in meters.

The logarithm of k is 6.03920.

When the signal is NE or SW of the observer the azimuth must be increased to obtain the correct azimuth at sea-level; if the signal is NW or SE the observed azimuth must be decreased.

If, when deriving the above equation, we place the fraction $\frac{1 - e^2 \sin^2 \phi'}{1 - e^2} = 1$, the formula for x'' should have a replaced by N.

For $\phi = 45°$, $\alpha = 45°$, and $h = 1000$ meters, the value of x'' is $0''.0547$. This is much smaller than the probable error of an observed direction (see p. 65), and is therefore negligible except for great heights. This correction has been applied to angles measured in the primary triangulation of the California and Texas arc and the California and Washington arc. It is too small to affect the triangulation of the eastern half of this country.

Questions. — What influence does the height of the observer have upon the result?

Why does the distance not enter into the formula?

Which one of the two approximations is more accurate, that giving a in the denominator, or that giving N?

103. Refraction.

Inasmuch as the refraction acts in the vertical plane at any point, and the vertical plane changes its direction as the ray proceeds along the line, it is evident that there must be some horizontal displacement of the object sighted, due to the refraction. Investigations show that this error is quite inappreciable for all lines that can actually be observed.

104. Curves on the Spheroid. The Plane Curves.

When a theodolite is set up at any point A and leveled, its vertical axis is made to coincide with the direction of the normal at A, which, except for local deflections, coincides with the direction of the force of gravity at A. If another theodolite is set up at B, in a different latitude and a different longitude, it is evident that these vertical axes are not in the same plane, since their normals (plumb lines) never intersect. The greater the latitude, the lower the point where the normal intersects the polar axis. It is clear that the line marked out on the surface of the spheroid by the line (or, rather, plane) of sight of the first theodolite is not the same as the line marked out by the vertical plane of the other theodolite. If A is southwest of B, then the curve cut by the plane of the theodolite at A is south of that cut by the plane of sight of the theodolite at B. This may be seen from the fact that both planes contain the chord AB; and since the normal at A is higher at the polar axis, the curve itself must be lower (farther south).

105. The Geodetic Line.

Another curve which holds an important place in the theory of geodesy is known as the *geodetic line*. This is the shortest line that can be drawn on the surface of the spheroid between two given points. It is not a plane curve, but has a *double curvature*. A characteristic property of the curve is that the osculating plane * at any point on the curve contains the normal to the surface at that point. In most cases the geodetic line is found to lie between the two plane curves and has a reversed curvature. Fig. 63 is a photograph of a model, the semi-axes of which are $a = 6$ inches and $b = 3.5$ inches. The two plane curves are shown and between them, with the curvature slightly exaggerated, is the geodetic line.

In order to obtain a clear conception of the nature of the geodetic line, let us imagine that a transit instrument is set at point A (Fig. 64), leveled, and then sighted at point B. Then it is moved to point B, set up, and leveled again, and a back sight is taken on A; point C is then fixed by reversing the telescope. When the sight is taken to A, the sight line traces out the plane curve BbA; and when point C is sighted, it traces out BbC. The

* The osculating plane may be considered to pass through three consecutive points of the curve. In reality it is the limiting position approached by the plane as the distance between the three points decreases indefinitely.

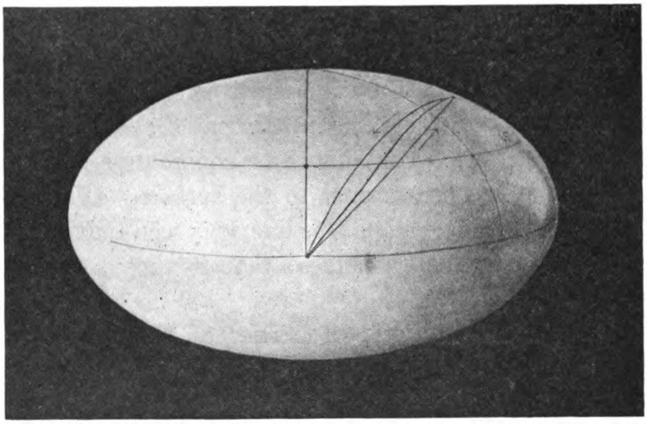


FIG. 63. Plane Curves and Geodetic Line.

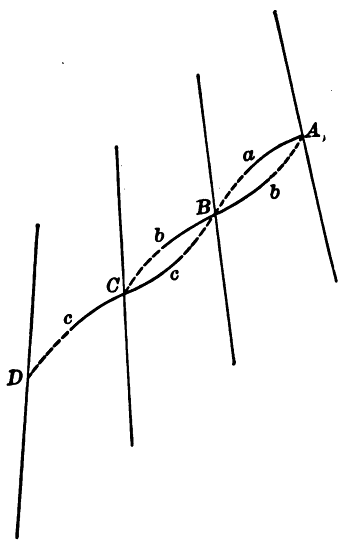


FIG. 64.

instrument is then taken to C and the process repeated. It should be observed that the (vertical) sight plane of the instrument coincides with the normal to the surface at each station. If the points A, B, C, D are imagined to approach nearer and nearer, so that AB, BC, etc., become infinitesimal elements of the curve, the plane which contains three consecutive points of the curve also contains the normal to the surface. If we imagine the instrument to move along this line, it is seen that the vertical plane of the instrument twists so that it always contains the normal.

One of the characteristic properties of the geodetic line is shown by the equation

$$R_p \sin \alpha = k, \text{ a constant} \qquad [57]$$

R_p being the radius of the parallel and α the azimuth of the geodetic line at any point. This equation may be derived analytically by the methods of the calculus of variations (see Clarke, *Geodesy*, p. 125) or by geometric construction (see Jordan, *Vermessungskunde*, Vol. III, p. 395). From this equation it will be seen that when α is a maximum (90°), $\sin \alpha = 1$ and $R_p = k$. The constant of the equation is therefore the radius of the parallel of latitude beyond which the geodetic line does not pass. When α is a minimum, R_p is a maximum, that is, $R_p = a$, the equatorial radius of the spheroid. This shows that in general a geodetic line cutting the equator at any angle α may go northward up to some (limiting) parallel of latitude $\phi°$ (corresponding to $R_p = k$), but will not pass north of this parallel. In the southern hemisphere it will reach a limit $(-\phi°)$ having the same numerical value. Such a geodetic line, when traced completely around the spheroid, will not in general return exactly on itself, but will pass the initial point on the equator in a slightly different longitude and then proceed to form another loop around the spheroid.

Except for a few particular cases the geodetic line lies between the two plane curves and divides the angle between them in the ratio of about 2 to 1, as shown in Fig. 65.

If the terminal points P and Q are in nearly the same latitude, the geodetic line may cross the plane curve.

It is important to bear in mind that the lengths of these different curves on the spheroid differ by quantities that are quite inappreciable in practice. The differences in length are far shorter than the distances by which the curves are separated at their middle points (Art. 107), and even these latter are negligible in practice. Also the angle by which the azimuth of the geodetic differs from the azimuth of the plane section is much smaller than can be measured.

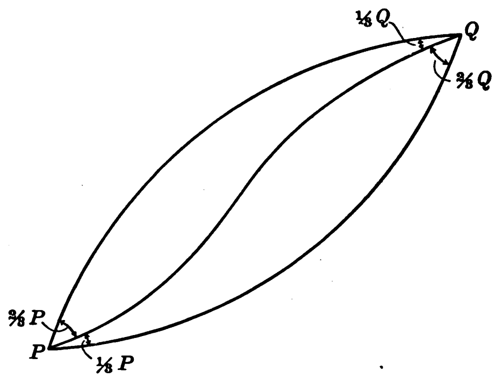


FIG. 65.

It should be noted that the geodetic line itself cannot be sighted over directly, because it is not a plane curve, and that the *geodetic triangle* can be obtained only by computation.

106. The Alignment Curve.

Another curve which may be drawn on the surface is defined in the following manner: if the theodolite be supposed to move from A to B, keeping always in line between the two points (that is, the azimuths of A and B 180° apart), and the instrument being always leveled, its path will be a curve which lies very close to the geodetic line and generally between the two plane curves. This is called the *alignment curve*.

It is possible to define other curves * between these two points.

* See Coast Survey Report for 1900, p. 369.

Such curves are of theoretical value only, since the lengths of all such lines on the earth's surface differ from each other by quantities too small to measure. The two-plane curves, however, are separated by a distance which is quite appreciable.

107. Distance between Plane Curves.

The maximum separation of the two plane curves may be computed approximately as follows: the angle (δ') between the

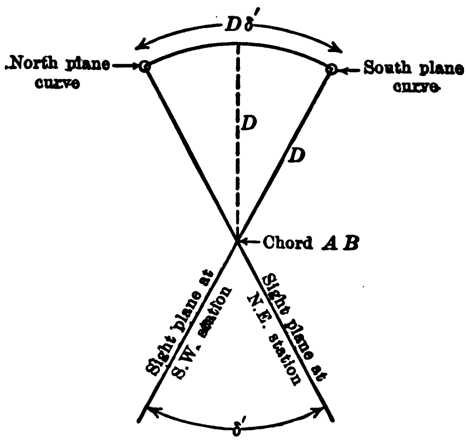


FIG. 66.

two planes is very nearly equal to the angle δ multiplied by sin α, since δ is the angle measured in the plane of the meridian, whereas the angle desired (δ', Fig. 66) is that perpendicular to the planes of sight.

Therefore $$\delta' = \frac{se^2 \cos^2 \phi \cos \alpha \sin \alpha}{N}$$

(see equation (a), p. 138).

The distance of the chord AB (Fig. 67) below the surface (D) at its middle point is given by

$$D : \frac{s}{2} = \frac{s}{2} : 2\, R_\alpha,$$

or, approximately,

$$D = \frac{s^2}{8\, N}.$$

The curves are separated at their middle points by the horizontal distance

$$D\delta' = \frac{s^2}{8N} \times \frac{se^2 \cos^2 \phi \cos \alpha \sin \alpha}{N}$$

$$= \frac{s^3}{8N^2} e^2 \cos^2 \phi \cos \alpha \sin \alpha. \qquad [58]$$

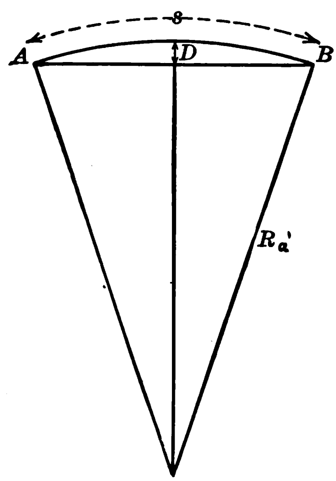


FIG. 67.

The difference in azimuth may be computed approximately by finding the angle between the two tangents to the curve drawn from one of the stations and prolonged half the distance (Fig. 68). The terminal points of these tangents will be at a distance D above the surface and will be separated by a distance $2\,D\delta'$. The angle between these two lines is nearly

$$= \frac{2\,D\delta'}{\frac{1}{2}\,s \text{ arc } 1''}$$

$$= \frac{2\,s^3}{8N^2} \cdot \frac{e^2 \cos^2 \phi \cos \alpha \sin \alpha}{\frac{1}{2}\,s \text{ arc } 1''}$$

$$= \frac{s^2}{2N^2} \cdot \frac{e^2 \cos^2 \phi \cos \alpha \sin \alpha}{\text{arc } 1''}. \qquad [59]$$

For the oblique boundary line between California and Nevada * $s = 650{,}000$ m., (400 mi.), $\phi_m = 37^\circ\ 00'$, $\alpha = 134^\circ\ 33'$; whence $D\delta' = 1.8$ meters and the difference in azimuth $= 2''.3$.

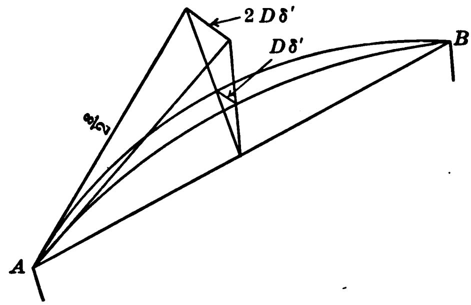


FIG. 68.

For the western boundary of Massachusetts $s = 80{,}930$ m., (50 mi.), $\phi_m = 42^\circ\ 24'$, $\alpha = 195^\circ\ 12'$; this gives $D\delta' = 0.0015$ meter and $\Delta\alpha = 0''.016$.

PROBLEMS

Problem 1. Prove by the process outlined in the first paragraph of Art. 96 that the radius of curvature of the prime vertical section of the spheroid is N, the normal terminating in the minor axis.

Problem 2. A model of the spheroid has an equatorial diameter of 12 ins. and a polar diameter of 7 ins. Compute the correction to reduce to "sea-level" the azimuth of a line in latitude 45°, the azimuth being 45° and the elevation of object being one inch above the surface of the spheroid.

Problem 3. What will be the maximum separation of two plane curves drawn on the model described in problem 2 if $s = 7.5$ ins., mean $\phi = 30^\circ$, $\alpha = 45^\circ$? (Use the approximate formula.)

* See Coast Survey Report for 1900, p. 368.

CHAPTER VI

CALCULATION OF TRIANGULATION

108. Preparation of the Data.

From the records of the field-work of the triangulation we obtain a value for each angle, supposed to be freed from the errors of the instrument, eccentricity of station, phase of signal, elevation of signal, etc. Before these angles are employed for solving the triangles, they should be examined to see if they satisfy any geometric conditions existing among them. If at any station two or more angles and their sum have been measured, then these angles must be so corrected that they exactly equal their sum. If the horizon has been *closed*, the measured angles must be adjusted so that their sum equals 360°. If the angles have been measured with different degrees of precision, as, for example, with different instruments or a different number of sets or of repetitions, the different angles should be given proper weights; and if the best possible values are desired, the angles at each station should be adjusted by the method of least squares.

After the *station adjustment*, as it is called, has been completed, the triangles must be examined to see if the sum of the three angles in each triangle fulfills the requirement that this sum shall equal 180° plus the spherical excess of the triangle. The verticals at the three triangulation stations are not parallel to each other, because the surface is curved. Consequently the sum of the angles will exceed 180° by an amount which, on a spherical surface, would be exactly proportional, and which, on a spheroidal surface, is nearly proportional to the area of the triangle.

As was shown in the preceding chapter (Art. 102), the error in the direction of an object, due to the fact that the earth is spheroidal instead of spherical, is extremely small, even when the object is several thousand meters above sea-level. Hence it follows that if the vertices of a spheroidal triangle are projected vertically onto the surface of a tangent sphere,* the errors thus produced in the horizontal angles of the triangle will be much less than the errors in the measurement of the angles, because the points on the sphere and those on the spheroid are separated by comparatively short distances. This enables us to compute spheroidal triangles as spherical triangles and greatly simplifies the computation. The lengths of the triangle sides will be practically the same on the two surfaces.

In this connection it is well to bear in mind that if the topography of the earth's surface were represented on an 18-inch globe the total variation in elevation would scarcely be greater than the thickness of a coat of varnish. The elevation of the geoid above the spheroid would be very much smaller than this, and the distance between the spheroid and the tangent sphere at any station would usually be still smaller. This will give some idea of the minuteness of the errors under discussion.

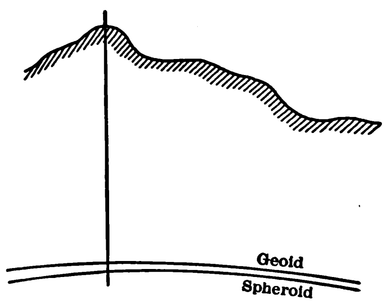


FIG. 68a.

It should be remembered that, whereas the triangulation stations themselves are at various heights above sea-level, these are all supposed to have been projected down vertically onto the spheroid before beginning the computation of the triangle. The points of which we shall speak in discussing the solution of the triangles and the geographical positions of the stations are these points on the spheroidal surface and not the original station points.

* The sphere is supposed to be tangent at the center of gravity of the triangle to be computed.

In solving triangles by the methods given below, the following approximations have been made, and it is assumed that in all cases the resulting errors are negligible.

1. The reduction to sea-level reduces the observed direction to that corresponding to the geoid (or actual surface), not the spheroid, as is assumed.

2. The effect of local deflection of the plumb line is not allowed for.

3. The effect of atmospheric refraction on the direction (horizontal refraction) is neglected.

4. The reduction of the observed direction (plane curve) to that of the geodetic, or shortest, line is omitted. There are in reality eight triangles formed by the plane curves, which are treated as if they were identical (see Art. 104).

109. Solution of a Spherical Triangle by Means of an Auxiliary Plane Triangle.

The direct solution of the triangles of a net as spherical triangles would be unnecessarily complicated. This may be avoided by employing a principle known as Legendre's Theorem, namely, that if we have a spherical triangle whose sides are short compared with the radius of the sphere, and also a plane triangle whose sides are *equal in length* to the corresponding sides of the spherical triangle, then the corresponding angles of the two triangles differ by approximately the same quantity, which is one-third of the spherical excess of the triangle.

110. Spherical Excess.

The spherical excess of a triangle is directly proportional to its area, as shown in spherical geometry. Hence, if A' is the area of any triangle, R is the radius of the sphere, S is the surface of the sphere, and e is the spherical excess of the triangle; then, since the spherical excess of the tri-rectangular triangle is $\frac{\pi}{2}$,

$$\frac{e}{\frac{\pi}{2}} = \frac{A'}{\frac{1}{8}S},$$

or
$$\frac{2e}{\pi} = \frac{2A'}{\pi R^2}.$$

Therefore
$$e = \frac{A'}{R^2}.$$

To express e in seconds of arc, divide by arc 1″, and we have

$$e'' = \frac{A'}{R^2 \operatorname{arc} 1''} = \frac{bc \sin A}{2 R^2 \operatorname{arc} 1''}, \qquad [60]$$

where b, c, and A are two sides and the included angle of the triangle, a and b being in linear units.

The sphere which is tangent to the spheroid at the center of gravity of the triangle, and which has the same average curvature, is a sphere of radius $= \sqrt{R_m N}$; whence

$$e'' = \frac{bc \sin A}{2 R_m N \operatorname{arc} 1''} = mbc \sin A. \qquad [61]$$

The quantity $\frac{1}{2 R_m N \operatorname{arc} 1''} = m$ is given for different latitudes in Table XII. The latitude to be used in finding m is the mean of the latitudes of the three vertices of the triangle.

Questions. — Is this auxiliary plane triangle the same as the chord triangle formed by joining the points by straight lines? Are the two similar in shape?

111. Proof of Legendre's Theorem.

To prove Legendre's theorem, let A', B' and C' be the angles of the spherical triangle, and A, B, and C those of the plane triangle; the sides of the plane triangle are a, b, and c, and those of the spherical triangle are $a'R$, $b'R$, and $c'R$, then, in the plane triangle,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}, \qquad (a)$$

or
$$\sin^2 A = 1 - \cos^2 A = \frac{4\,b^2c^2 - (b^2 + c^2 - a^2)^2}{4\,b^2c^2},$$

$$= \frac{2\,a^2b^2 + 2\,a^2c^2 + 2\,b^2c^2 - a^4 - b^4 - c^4}{4\,b^2c^2}. \qquad (b)$$

In the spherical triangle,

$$\cos A' = \frac{\cos a' - \cos b' \cos c'}{\sin b' \sin c'}.$$

Expanding each sine and cosine (omitting terms of higher order than the fourth),

$$\cos A' = \frac{1 - \frac{a'^2}{2} + \frac{a'^4}{24} - \left(1 - \frac{b'^2}{2} + \frac{b'^4}{24}\right)\left(1 - \frac{c'^2}{2} + \frac{c'^4}{24}\right)}{\left(b' - \frac{b'^3}{6}\right)\left(c' - \frac{c'^3}{6}\right)}$$

$$= \frac{\frac{1}{2}(-a'^2 + b'^2 + c'^2) - \frac{1}{24}(b'^4 + c'^4 - a'^4) - \frac{1}{4}b'^2c'^2}{b'c'\left[1 - \frac{1}{6}(b'^2 + c'^2)\right]}$$

$$= \left[\tfrac{1}{2}(-a'^2 + b'^2 + c'^2) - \tfrac{1}{24}(b'^4 + c'^4 - a'^4) - \tfrac{1}{4}b'^2c'^2\right]\frac{1 + \frac{1}{6}(b'^2 + c'^2)}{b'c'}$$

$$= \frac{b'^2 + c'^2 - a'^2}{2\,b'c'} - \frac{b'^4 + c'^4 - a'^4 + 6\,b'^2c'^2}{24\,b'c'}$$
$$+ \frac{-a'^2b'^2 + b'^4 + 2\,b'^2c'^2 - a'^2c'^2 + c'^4}{12\,b'c'}.$$

whence $\cos A' =$

$$\frac{b'^2 + c'^2 - a'^2}{2\,b'c'} - \frac{1}{6}\cdot\frac{2\,a'^2b'^2 + 2\,a'^2c'^2 + 2\,b'^2c'^2 - a'^4 - b'^4 - c'^4}{4\,b'c'}. \qquad (c)$$

From (*a*), (*b*), and (*c*)

$$\cos A' = \cos A - \tfrac{1}{6}\,b'c' \sin^2 A.$$

Let x be the difference between A and A'. Then

$\cos x = 1$ and $\sin x = x''$ arc $1''$ (nearly), since x is small,

and
$$\begin{aligned} \cos A' &= \cos(A + x) \\ &= \cos A - \sin A x'' \text{ arc } 1'' \\ &= \cos A - \tfrac{1}{6}\,b'c' \sin^2 A; \end{aligned}$$

that is, x'' arc $1'' \sin A = \frac{1}{6}\,b'c' \sin^2 A$.

Therefore
$$x'' = \frac{b'c' \sin A}{6 \text{ arc } 1''},$$

or, since $$b' = \frac{b}{R} \quad \text{and} \quad c' = \frac{c}{R},$$

$$x'' = \frac{bc \sin A}{6 R^2 \text{arc } 1''}. \qquad [62]$$

It will be noticed that this is one-third of the spherical excess as found in Equa. [60]. The same result would also be found for angles B and C.

112. Error of Legendre's Theorem.

The error in Legendre's theorem * as applied to the sphere may be studied by carrying out the above series so as to include terms of higher powers than the fourth. Jordan (*Vermessungskunde*) gives a numerical example showing the amount of this error in a triangle of which the side AC is about 65 miles in length; the angles are shown below:

$$\begin{aligned} A' &= 40^\circ\ 39'\ 30''.380 \\ B' &= 86\ \ \ 13\ \ \ 58\ .840 \\ C' &= 53\ \ \ 06\ \ \ 45\ .630 \\ \hline & 180^\circ\ 00'\ 14''.850 \end{aligned}$$

Denoting the spherical angles by A', B', C', and the corresponding plane angles by A, B, C, the differences are as follows, the first column containing the values derived from Legendre's theorem in its ordinary form, the second containing the smaller terms which are usually neglected.

	Approx.	Exact.
$A' - A$	4″.950018	4.950036
$B' - B$	4 .950018	4.949997
$C' - C$	4 .950018	4.950021

113. Calculation of Spheroidal Triangles as Spherical Triangles.

It is customary to assume that the differences between the spherical and spheroidal triangles are negligible when the actual points are projected down onto a tangent sphere of radius $\sqrt{R_m N}$. Clarke, in his *Geodesy*, shows the error of this assumption in the case of a triangle having a side over 200 miles long, the result being as follows:

* See Coast Survey Special Publication No. 4, p. 51.

	Spheroidal		Spherical
A'	98° 44′ 37″.0965	A	98° 44′ 37″.1899
B'	58° 16′ 46″.5994	B	58° 16′ 46″.4737
C'	23° 00′ 12″.7303	C	23° 00′ 12″.7634
e'	1′ 36″.4262	e	1′ 36″.4270

The preceding example indicates that in triangles composed of lines such as can be sighted over on the earth's surface the error involved in computing spheroidal triangles as spherical triangles is negligible in practice.

114. Calculation of the Plane Triangle.

After the spherical excess has been computed, the angles of an auxiliary plane triangle may be found by applying Legendre's theorem, that is, by deducting one-third of the spherical excess from each spherical angle. The difference between the sum of these plane angles and 180° is the error of measurement and may be distributed equally among the three angles unless a least-square adjustment is to be made. In any case this method of distributing the error may be used for a preliminary determination of the distances. The lengths of the triangle sides are now found by plane trigonometry. Since all three angles of a triangle will usually be known, the only formula that will be used, except in rare cases, is the sine formula,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

A convenient arrangement of this computation, used by the Coast and Geodetic Survey, is shown in the following table. The spherical excess of the triangle in this case is 0″.86, which gives 1″.2 as the error of closure of the triangle.

Stations.	Observed angles.	Correction.	Spherical angles.	Spherical excess.	Plane angles and distances.	Logarithms.
Blue Hill to Prospect......					22723.08 m.	4.356 4673
	° ′ ″	″	″	″	° ′ ″	
Observatory..............	61 47 18.8	0.4	18.4	0.3	61 47 18.1	0.054 9218
Blue Hill..................	35 45 15.4	0.4	15.0	0.3	35 45 14.7	9.766 6415
Prospect..................	82 27 27.9	0.4	27.5	0.3	82 27 27.2	9.996 2261
Observatory to Prospect...	180 00 02.1				15067.13	4.178 0306
Observatory to Blue Hill..					25563.20	4.407 6152

115. Second Method of Solution by Means of an Auxiliary Plane Triangle.*

Another method of solution which has been used to some extent in Europe is as follows:

Let ABC (Fig. 69) be the spherical triangle and $A'B'X$ an auxiliary plane triangle having two of its angles, α and β, equal to the corresponding angles in the spherical triangle. Evidently the third angles will not be equal.

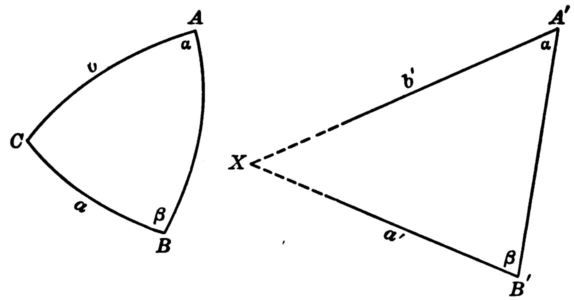


FIG. 69.

Let a' and b' in the plane triangle be the sides corresponding to a and b.

In the spherical triangle we have

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin\frac{a}{R}}{\sin\frac{b}{R}},$$

and in the plane triangle

$$\frac{\sin\alpha}{\sin\beta} = \frac{a'}{b'}$$

for all values that may be given to a' and b'; whence

$$\frac{\sin\frac{a}{R}}{\sin\frac{b}{R}} = \frac{a'}{b'} = \frac{\frac{a'}{R}}{\frac{b'}{R}}.$$

* See Jordan, *Vermessungskunde*, Vol. III, 39.

This equation is satisfied if we place

$$\frac{a'}{R} = \sin\frac{a}{R}$$

and

$$\frac{b'}{R} = \sin\frac{b}{R}.$$

The general expression for any triangle side may be written

$$\frac{s'}{R} = \sin\frac{s}{R},$$

s' being the side of an auxiliary plane triangle corresponding to the side s of the spherical triangle.

Taking logs of both members,

$$\log\frac{s'}{R} = \log\sin\frac{s}{R} = \log\left(\frac{s}{R} - \frac{s^3}{6\,R^3} + \frac{s^5}{120\,R^5}\ \cdots\right)$$

$$= \log\frac{s}{R} + \log\left(1 - \frac{s^2}{6\,R^2} + \cdots\right).$$

Now, since

$$\log(1 + x) = M\left(x - \frac{x^2}{2} + \frac{x^3}{3}\ \cdots\right)$$

(where $M = \log_e 10 = 0.4342945$, the modulus of the common logarithms), we may write

$$\log\frac{s'}{R} = \log\sin\frac{s}{R} = \log\frac{s}{R}$$

$$+ M\left(-\frac{s^2}{6\,R^2} + \frac{s^4}{120\,R^4}\ \cdots\right) - \frac{M}{2}\left(-\frac{s^2}{6\,R^2}\right)^2 \cdots$$

$$= \log\frac{s}{R} - \frac{Ms^2}{6\,R^2}.^*$$

Therefore

$$\log\frac{s}{R} - \log\frac{s'}{R} = \frac{Ms^2}{6\,R^2},$$

or

$$\log s - \log s' = \frac{Ms^2}{6\,R^2}, \qquad [63]$$

which is the correction to the log of the triangle side.

* The next term $= \frac{M}{180}\cdot\frac{s^4}{R^4} = 0.000\,000\,0001$ for a distance of 100 kilometers.

In calculating this correction, R^2 should be replaced by $R_m N$. Values of these corrections will be found in Table XIII for the argument log s.

Example.

Stations.	Spherical angles.	Distances.	Logarithms.
Blue Hill to Prospect.....	° ′ ″	22,723.08	4.356 4673
Correction................			9
s'........................			4.356 4664
Observatory..............	61 47 18.4		0.054 9215
Blue Hill..................	35 45 15.0		9.766 6423
Prospect...................	82 27 27.5		9.996 2262
s'........................			4.178 0302
Correction................			4
Observatory to Prospect..		15,067.13	4.178 0306
s'........................			4.407 6141
Correction................			11
Observatory to Blue Hill..		25,563.20	4.407 6152

Notice that after the base of the first triangle has once been reduced by subtracting the correction, the computation of the whole chain of triangles may be carried out, using the spherical angles only. It is not necessary to add the corrections to the logarithms of the computed sides until their true values are to be found.

PROBLEMS

Problem 1. Compute the area in square miles of a triangle on the earth's surface having a spherical excess of 1″, assuming that the earth is a sphere of radius 3960 miles.

Problem 2. Compute the sides of the following triangles:

	Station.	Correction to angles from figure adjustment.	Error of closure of triangle.	Corrected spherical angles.	Spherical excess.
(*a*)	Mt. Ellen	−0″.70	+0″.22	49° 36′ 36″.88	34.″33
	Tushar	+0 .98		55 56 26 .70	
	Wasatch	−0 .06		74 27 30 .75	

Wasatch to Mt. Ellen; azimuth, 333° 01′ 08″.65; back-azimuth, 153° 25′ 05″.00; dist. 123,556.70 meters; logarithm, 5.0918663. Latitude of Wasatch, 39° 06′-54″.364; longitude, 111° 27′ 11″.915.

	Station.	Correction to angles from figure adjustment.	Error of closure of triangle.	Corrected spherical angles.	Spherical excess.
(*b*)	Uncompahgre	+0″.17	+0″.65	31° 54′ 61″.57	46″.15
	Mt. Waas	−0 .10		98 16 41 .16	
	Tavaputs	+0 .58		49 48 63 .42	

Mt. Waas to Uncompahgre; azimuth, 288° 01′ 25″.71; back-azimuth, 109° 07′-06″.11; dist. 162,928.01 meters, logarithm, 5.211 9958. Latitude Mt. Waas, 38° 32′ 21″.444; longitude, 109° 13′ 38″.302.

Problem 3. Position of point *B* { lat. 39° 13′ 26″.686; long. 98° 32′ 30″.506 }

Position of point *C* { lat. 38° 51′ 50″.913; long. 98° 29′ 15″.508 }

Azimuth *B* to *C* 353° 17′ 21″.81; dist. 40232.35 meters; (log = 4.604 5754); back-azimuth 173° 19′ 24″.64.

The spherical angles are *A* 57° 53′ 14″.39 (*A* is east of *BC*.)
B 62° 23′ 31″.40
C 59° 43′ 17″.93

Compute the spherical excess and solve the triangle.

Problem 4. Position of pt. *L*; latitude 42° 26′ 13″.276, longitude 70° 55′52″.088. Distance *L* to *N*, 3012.0 meters (log = 3.478 8600). Azimuth *L* to *N*, 314° 34′ 00″; back-azimuth, 134° 35′ 03″. Position of pt. *N*, latitude 42° 25′ 04″.764, longitude 70° 54′ 18″.232. Angle at *L*, 36° 15′ 07″; at *N*, 63° 44′ 59″; at *E*, 79° 59′ 57″. (*E* is east of *LN*.) Compute the spherical excess and solve the triangle.

Problem 5. The observed angles of a triangle and their corrections as found by adjustment are as follows:

	Angle.	Corrections.
Sand Hill	40° 57′ 28″.13	−0″.35
Rutherford	54 22 59 .51	−0 .61
Miller	84 39 35 .03	−0 .44

The position of Rutherford is latitude = 37° 08′ 57″.928 N, longitude = 98° 06′ 31″.618 W. The position of Miller is latitude = 37° 02′ 20″.963 N, longitude 97° 55′ 43″.908 W. The azimuth from Miller to Rutherford = 127° 28′-17″.95; back-azimuth 307° 21′ 47″.30. Distance in meters, 20139.64; logarithm, 4.304 0518. Solve the triangle.

Problem 6. Show that the substitution of Equa. (b) p. 150 in Equa. (c) p. 151 is permissible under the assumptions made in Arts. 109 and 111.

CHAPTER VII

CALCULATION OF GEODETIC POSITIONS

116. Calculation of Geodetic Positions.

In geodetic surveys covering large areas the positions of the triangulation points are expressed by means of their latitudes and longitudes. Over limited areas a system of rectangular spherical coördinates may be used to advantage, but for such areas as have to be surveyed in this country the latitude and longitude system is preferable.

Before the latitude and longitude of one triangulation station can be calculated from the coördinates of another station, it is necessary to know the dimensions of the spheroid which is taken to represent the earth's figure, and also to fix definitely the latitude and longitude of some specified station, as well as the azimuth of the direction to some other triangulation station. This selected position and direction determine the relative position of the whole survey with respect to the adopted spheroid, and constitute what is known as the *geodetic datum.* The surveys of different countries may be computed on different spheroids or may be located inconsistently on the same spheroid. The different portions of a survey of the same country will be located inconsistently on the same spheroid until they have been connected by triangulation.

The two spheroids which have been most extensively used for geodetic surveys are (1) that computed by Bessel in 1841, and (2) that by Clarke in 1866. The Bessel spheroid was computed from data obtained chiefly on the continent of Europe, and consequently conforms closely to the curvature of that portion of the earth. This spheroid is still in general use in Europe. Clarke's spheroid of 1866 was computed from arcs distributed over a much larger portion of the earth's surface; it shows a greater amount of flattening at the poles than the Bessel spheroid, and consequently assigns a flatter curvature to the surface in the latitude of Europe and of the United States. The Bessel spheroid was employed by the Coast Survey in the earlier years. As the surveys gradually extended, the errors due to using this spheroid became more and more apparent, until finally, in 1880, it was decided to change to the Clarke spheroid. The latter conforms much more nearly to the curvature of the surface in the United States.

117. The North American Datum.*

In 1901 the United States Coast and Geodetic Survey adopted what was then called the United States Standard Datum, by assigning to the station *Meade's Ranch* the following position on the Clarke spheroid:

Latitude,	39° 13′ 26″.686
Longitude,	98° 32′ 30″.506
Azimuth to *Waldo*,	75° 28′ 14″.52

In 1913 this datum was adopted by the governments of Canada and Mexico, and it is now known as the North American Datum.

In deciding upon a geodetic datum it was necessary to consider two important points: first, the datum should be so chosen as to reduce to a minimum the labor of recomputing the geodetic positions; second, it must place the triangulation system in such a position that no serious error will occur in any part of the system. At the time this datum was selected there was a large number of triangulation points located along the Atlantic Coast. By selecting a position for *Meade's Ranch* consistent with the old datum upon which this triangulation was calculated, a large amount of recomputation was avoided. At the same time it was apparent that this also placed the triangulation very near to its theoretically best position.

* See Coast Survey Special Publication No. 24, p. 8, or Special Publication No. 19, p. 80.

118. Method of Computing Latitude and Longitude.

Assuming that the latitude and longitude of a station (A) are known, as well as the distance and azimuth to a second station (B), we will now develop the formulæ * necessary to compute the geodetic latitude and longitude of the second point In doing this we shall have to solve the differential spherical triangle formed by joining the two points with the pole.

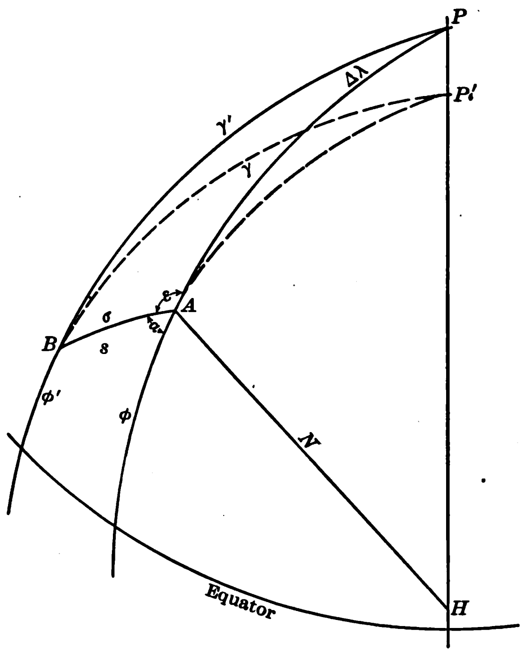


FIG. 70.

119. Difference in Latitude.

In Fig. 70, P' is the pole of the spheroid. P is the pole of a sphere tangent to the spheroid along the parallel of latitude through A. The radius of the sphere is N, and its center is at H. Let A be the known station and B the unknown station.

* These formulæ were first given by Puissant; see his Traité de Géodesie, Vol. I; see also Coast and Geodetic Survey Report for 1894, and Special Publication No. 8.

The angular distance of A from the pole is γ; the unknown distance of B is γ'; σ is the arc AB; α is the azimuth; and $\epsilon = 180° - \alpha$.

If γ' is computed by a direct solution of the spherical triangle ABP, the required precision can be reached only by the use of about ten-place logarithms. It is more convenient, and quite as accurate, for such short lines as occur in practice, to employ formulæ giving the *difference in latitude*, that is $\gamma - \gamma'$.

The formula for the direct solution of γ' in the spherical triangle is

$$\cos\gamma' = \cos\gamma\cos\sigma + \sin\gamma\sin\sigma\cos\epsilon. \qquad (a)$$

Since γ' is a function of σ, its value may be expressed as a converging series by means of Maclaurin's formula, giving

$$\gamma' = \gamma_{\sigma=0} + \frac{d\gamma}{d\sigma_{\sigma=0}}\cdot\sigma + \frac{1}{2}\cdot\frac{d^2\gamma}{d\sigma^2_{\sigma=0}}\cdot\sigma^2 + \frac{1}{6}\frac{d^3\gamma}{d\sigma^3_{\sigma=0}}\cdot\sigma^3 + \cdots. \qquad (b)$$

To evaluate the three differential coefficients, differentiate Equa. (a) three times in succession, and in each resulting equation substitute $\sigma = 0$. The results of the first two differentiations are as follows:

$$-\sin\gamma'\frac{d\gamma'}{d\sigma} = -\cos\gamma\sin\sigma + \sin\gamma\cos\sigma\cos\epsilon, \qquad (c)$$

$$-\sin\gamma'\frac{d^2\gamma'}{d\sigma^2} - \cos\gamma'\left(\frac{d\gamma'}{d\sigma}\right)^2 = -\cos\gamma\cos\sigma - \sin\gamma\sin\sigma\cos\epsilon$$
$$= -\cos\gamma', \text{(by } (a)). \qquad (d)$$

Before differentiating a third time, (d) may be written

$$\tan\gamma'\frac{d^2\gamma'}{d\sigma^2} + \left(\frac{d\gamma'}{d\sigma}\right)^2 = 1. \qquad (e)$$

Differentiating (e), we have

$$\tan\gamma'\frac{d^3\gamma'}{d\sigma^3} + \sec^2\gamma'\cdot\frac{d\gamma'}{d\sigma}\cdot\frac{d^2\gamma'}{d\sigma^2} + 2\frac{d\gamma'}{d\sigma}\cdot\frac{d^2\gamma'}{d\sigma^2} = 0. \qquad (f)$$

When $\qquad \sigma = 0, \qquad \gamma' = \gamma,$

and (c) becomes

$$-\sin\gamma\frac{d\gamma}{d\sigma} = \sin\gamma\cos\epsilon.$$

Therefore $$\frac{d\gamma}{d\sigma} = -\cos\epsilon. \qquad (g)$$

(*e*) becomes

$$\tan\gamma\frac{d^2\gamma}{d\sigma^2} + \cos^2\epsilon = 1.$$

Therefore $$\frac{d^2\gamma}{d\sigma^2} = \sin^2\epsilon\cot\gamma. \qquad (h)$$

(*f*) becomes

$$\tan\gamma\frac{d^3\gamma}{d\sigma^3} + \sec^2\gamma(-\cos\epsilon)(\sin^2\epsilon\cot\gamma) + 2(-\cos\epsilon)(\sin^2\epsilon\cot\gamma) = 0.$$

Therefore $$\begin{aligned}\frac{d^3\gamma}{d\sigma^3} &= \cos\epsilon\sin^2\epsilon\cot^2\gamma\,(2 + \sec^2\gamma)\\ &= (2\cot^2\gamma + \operatorname{cosec}^2\gamma)\sin^2\epsilon\cos\epsilon\\ &= (1 + 3\cot^2\gamma)\sin^2\epsilon\cos\epsilon. \qquad (i)\end{aligned}$$

Substituting these results, (*g*), (*h*), and (*i*) in equation (*b*), Maclaurin's series, we obtain

$$\gamma' = \gamma - \sigma\cos\epsilon + \frac{\sigma^2}{2}\cdot\sin^2\epsilon\cot\gamma + \frac{\sigma^3}{6}(1 + 3\cot^2\gamma)\sin^2\epsilon\cos\epsilon + \cdots. \qquad (j)$$

Changing to latitudes and azimuths by placing

$$\begin{aligned}\gamma' &= 90^\circ - \phi',\\ \gamma &= 90^\circ - \phi,\\ \epsilon &= 180^\circ - \alpha,\end{aligned}$$

Equation (*j*) becomes

$$\begin{aligned}\phi - \phi' = \sigma\cos\alpha &+ \frac{\sigma^2}{2}\sin^2\alpha\tan\phi\\ &- \frac{\sigma^3}{6}(1 + 3\tan^2\phi)\sin^2\alpha\cos\alpha\ \ldots \qquad (k)\end{aligned}$$

In order to transfer the coördinates of the triangulation points from the sphere to the spheroid, it should be noticed that if the radius of the sphere is N (the normal) and its center is at H (Fig. 70), and the polar axes of the sphere and spheroid coincide, then the parallels of latitude through A coincide, the spheroid being tangent to the sphere along this parallel; also, the latitude (ϕ)

will be the same for both surfaces, and the distances and azimuths of AB on the two will differ by inappreciable quantities. We may therefore put $\sigma = \frac{s}{N}$, where s is the distance in linear units.

Then (k) becomes

$$\phi - \phi' = \frac{s \cos \alpha}{N} + \frac{s^2}{2N^2} \sin^2 \alpha \tan \phi - \frac{s^3}{6N^3} \sin^2 \alpha \cos \alpha (1 + 3 \tan^2 \phi). \quad (l)$$

The difference in latitude should be measured, however, on a curve of radius R_m, since it is measured along a meridian. The linear difference in latitude is nearly the same for the two surfaces, and the angular difference in latitude will vary inversely as the radii; that is,

$$(\phi - \phi') N = \Delta\phi'' R_M \operatorname{arc} 1''. \quad (m)$$

Therefore $$\Delta\phi'' = (\phi - \phi') \frac{N}{R_M \operatorname{arc} 1''},$$

$\Delta\phi''$ being in seconds of arc on the spheroid, and R_M the radius of curvature of the meridian at the middle point between the parallels through A and B. The difference in latitude is therefore

$$-\Delta\phi'' = \frac{s \cos \alpha}{R_M \operatorname{arc} 1''} + \frac{s^2 \sin^2 \alpha \tan \phi}{2NR_M \operatorname{arc} 1''} - \frac{s^3 \sin^2 \alpha \cos \alpha (1 + 3 \tan^2 \phi)}{6N^2 R_M \operatorname{arc} 1''}. \quad (n)$$

Since the middle latitude is not known at the beginning of the computation, it is more convenient first to take out the value of R_m for the known latitude of A, giving $\delta\phi''$, and then to correct to R_M by changing $\delta\phi''$ to $\Delta\phi''$ in the inverse ratio of the radii.

Since $$\frac{\delta\phi''}{R_M} = \frac{\Delta\phi''}{R_m},$$

$$\Delta\phi'' = \delta\phi'' \frac{R_m}{R_M} = \delta\phi'' \left(1 - \frac{R_M - R_m}{R_M}\right)$$

$$= \delta\phi'' \left(1 - \frac{dR_M}{R_M}\right),$$

in which $\delta\phi'' \frac{dR_M}{R_M}$ is a correction to be subtracted from the first value.

From [43] $$R_m = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}.$$

Therefore $$dR_m = \frac{a(1 - e^2) \cdot 3 e^2 \sin \phi \cos \phi \, d\phi}{(1 - e^2 \sin \phi)^{\frac{5}{2}}}.$$

Since dR_m is half the change from the starting point to the middle point, $d\phi$ is taken as half the difference in latitude, $\delta\phi$; that is,

$$d\phi = \frac{\delta\phi \text{ arc } 1''}{2}.$$

Therefore $$\delta\phi'' \frac{dR_m}{R_m} = \frac{3 e^2 \sin \phi \cos \phi \text{ arc } 1''}{2(1 - e^2 \sin^2 \phi)} (\delta\phi'')^2$$
$$= D \cdot (\delta\phi'')^2. \qquad (o)$$

If we now put for brevity $\frac{1}{R_m \text{ arc } 1''} = B$, $\frac{\tan \phi}{2 N R_m \text{ arc } 1''} = C$, $\frac{s \cos \alpha}{R_m \text{ arc } 1''} = h$ (the first term in (n)), and $\frac{1 + 3 \tan^2 \phi}{6 N^2} = \text{E}$, then Equa. ($n$) becomes

$$-\Delta\phi'' = s \cdot B \cdot \cos \alpha + s^2 \cdot C \cdot \sin^2 \alpha + (\delta\phi'')^2 \cdot D - h \cdot s^2 \cdot E \cdot \sin^2 \alpha, \qquad [64]$$

and the new latitude is given by

$$\phi' = \phi + \Delta\phi''. \qquad [65]$$

The logarithms of the factors B, C, D, and E are given in Table XIV, p. 351, in metric units, for the Clarke spheroid of 1866.

The D term is inserted before the E term, because it is usually the larger. The E term may be omitted when $\log s$ is less than 4.23. . . . The D term may be omitted when $\log s$ is less than 2.31 . . . , and h^2 may be substituted for $(\delta\phi'')^2$ when $\log s$ is less than 4.93. . . . The fourth differential coefficient in the series may be neglected except for the very longest lines (see Coast Survey Report for 1894, p. 284).

120. Difference in Longitude.

The difference in longitude is such a small angle that we may obtain it with sufficient precision by a direct solution of the triangle *PAB*, Fig. 70, using 7-place logarithms.

Applying the law of sines,

$$\sin \Delta\lambda = \frac{\sin \sigma \sin \alpha}{\cos \phi'}.$$

The sphere on which the points are projected is that whose radius is N' and whose center is at H' corresponding to point B.

As before, let $$\sigma = \frac{s}{N'}.$$

Therefore $$\sin \Delta\lambda = \sin \frac{s}{N'} \cdot \frac{\sin \alpha}{\cos \phi'}. \qquad (p)$$

In practice it is more convenient to solve the equation in the form

$$\Delta\lambda'' \text{ arc } 1'' = \frac{s}{N'} \cdot \sin \alpha \sec \phi',$$

and then to apply corrections for the difference between the arc and sine; the equation should therefore be written

$$\Delta\lambda'' - \text{corr.}_{\log \Delta\lambda} = \frac{s}{N' \text{ arc } 1''} \cdot \sin \alpha \sec \phi' - \text{corr.}_{\log s},$$

since each side of the equation is too large by the difference between the arc and sine.

Placing $\frac{1}{N' \text{ arc } 1''} = A'$, the equation becomes

$$\Delta\lambda'' = A \cdot s \cdot \sin \alpha \sec \phi' + \text{corr.}_{\log \Delta\lambda} - \text{corr.}_{\log s} \qquad [66]$$

in which the corrections are to be applied to the logarithms. Values of $\log A'$ will be found in Table XIV, p. 351.

In Art. 115, p. 154, it was shown that

$$\log \frac{s}{R} - \log \sin \frac{s}{R} = \frac{Ms^2}{6R^2},$$

when s is the length of any line on the surface.

If $\frac{s}{R}$ is an angle expressed in seconds, then the last equation becomes

$$\log \frac{s}{R} - \log \sin \frac{s}{R} = \frac{M\left(\frac{s''}{R}\right)^2 \operatorname{arc}^2 1''}{6}.$$

Taking logs of both members,

$$\log (\text{diff. of logs}) = \log \left(\frac{M \operatorname{arc}^2 1''}{6}\right) + 2 \log \left(\frac{s''}{R}\right).$$

Applying this formula first to $\Delta\lambda''$,

$$\log (\text{diff. of logs}) = 8.2308 + 2 \log \Delta\lambda''. \qquad (q)$$

Apply the formula to $\frac{s}{N'}$, and, observing that the second term is $2 \log \frac{s}{N' \operatorname{arc} 1''}$, we have

$$\log (\text{diff. of logs.}) = 8.2308 + 2 \log s + 2 \log A' \qquad (r)$$
$$= 5.2488 ^* + 2 \log s. \qquad (s)$$

This correction is to be subtracted because arc $\frac{s}{N'}$ is greater than $\sin\left(\frac{s}{N'}\right)$.

In Table XIII the corrections are tabulated to show the values of $\log s$ and $\log \Delta\lambda''$ for the *same* log diff. The correction for $\log s$ is negative and that for $\log \Delta\lambda''$ is positive. The algebraic sum of the two corrections is to be added to $\log \Delta\lambda''$. The method of making these corrections is illustrated in the example on p. 170. The new longitude λ' is given by

$$\lambda' = \lambda + \Delta\lambda''. \qquad [67]$$

121. Forward and Back Azimuths.

Owing to the convergence of the meridians the forward and reverse azimuths of a line will not differ by exactly 180°, as in plane

* Based on the value 8.5090 for $\log A'$.

coördinates. The amount of this convergence is computed as follows:

In the triangle PAB, Fig. 70, by Napier's analogies,

$$\tan \frac{1}{2}(A + B) = \cot \frac{1}{2}\Delta\lambda \cdot \frac{\cos \frac{1}{2}(\gamma' - \gamma)}{\cos \frac{1}{2}(\gamma' + \gamma)}.$$

Substituting, and noting that $A + B + \Delta\alpha = 180°$, and that an increase in $\Delta\lambda$ causes a decrease in $\Delta\alpha$,

$$-\cot \frac{1}{2}\Delta\alpha = \cot \frac{1}{2}\Delta\lambda \cdot \frac{\cos \frac{1}{2}(\phi - \phi')}{\sin \frac{1}{2}(\phi + \phi')},$$

whence

$$-\tan \frac{1}{2}\Delta\alpha = \tan \frac{1}{2}\Delta\lambda \frac{\sin \frac{1}{2}(\phi + \phi')}{\cos \frac{1}{2}(\phi - \phi')}$$

$$= \tan \frac{1}{2}\Delta\lambda \cdot \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}.$$

Therefore

$$-\frac{\Delta\alpha}{2} = \tan^{-1}\left(\tan \frac{\Delta\lambda}{2} \cdot \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}\right).$$

Putting for $\frac{1}{2}\Delta\alpha$ the series

$$\left[\tan \frac{1}{2}\Delta\lambda \cdot \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}\right] - \frac{1}{3}\left[\tan \frac{1}{2}\Delta\lambda \cdot \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}\right]^3 + \cdots,$$

and for $\tan \frac{1}{2}\Delta\lambda$ the series

$$\frac{1}{2}\Delta\lambda + \frac{(\frac{1}{2}\Delta\lambda)^3}{3} + \cdots,$$

then

$$-\frac{1}{2}\Delta\alpha = \left[\left(\frac{1}{2}\Delta\lambda + \frac{\Delta\lambda^3}{24}\right)\frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}\right] - \frac{1}{3}\left[\left(\frac{1}{2}\Delta\lambda + \frac{\Delta\lambda^3}{24}\right)\frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}}\right]^3 + \cdots$$

$$= \frac{1}{2}\Delta\lambda \cdot \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}} + \frac{\Delta\lambda^3}{24} \cdot \frac{\sin \phi_m}{\cos \frac{\Delta\phi}{2}} - \frac{\Delta\lambda^3}{24} \cdot \frac{\sin^3 \phi_m}{\cos^3 \frac{\Delta\phi}{2}} + \cdots,$$

Multiplying by 2 and factoring out $\frac{\Delta\lambda^3}{24}$,

$$-\Delta\alpha = \Delta\lambda \frac{\sin\phi_m}{\cos\frac{1}{2}\Delta\phi} + \frac{1}{12}(\Delta\lambda)^3\left(\frac{\sin\phi_m}{\cos\frac{1}{2}\Delta\phi} - \frac{\sin^3\phi_m}{\cos^3\frac{1}{2}\Delta\phi}\right).$$

Placing $\cos\frac{1}{2}\Delta\phi = 1$ in the small term and reducing $\Delta\alpha$ and $\Delta\lambda$ to seconds of arc,

$$-\Delta\alpha'' = \Delta\lambda'' \frac{\sin\phi_m}{\cos\frac{1}{2}\Delta\phi} + \frac{1}{12}(\Delta\lambda'')^3 \sin\phi_m \cos^2\phi_m \operatorname{arc}^2 1''$$

$$= \Delta\lambda'' \sin\phi_m \sec\frac{\Delta\phi}{2} + (\Delta\lambda'')^3 \cdot F, \qquad [68]$$

in which F is an abbreviation for $\frac{1}{12}\sin\phi_m\cos^2\phi_m \operatorname{arc}^2 1''$ and is given by its log in Table XIV. This F term amounts to only $0''.01$ when $\log\Delta\lambda'' = 3.36$. . . .

The back azimuth α' is given by

$$\alpha' = \alpha + \Delta\alpha + 180°. \qquad [69]$$

In calculating the geodetic position of a point, the azimuth of the line to that point is to be found from the known azimuth of the fixed side of the triangle by using the corrected spherical angle, not the plane angle of the auxiliary triangle. The computations of ϕ' and λ' may be verified by computing the position from two sides of the triangle and noting whether the same ϕ' and λ' are obtained from the two lines. The reverse azimuths are checked by noting whether their difference equals the spherical angle at the new station. In this manner the calculation of each triangle may be made to check itself.

122. Formulæ for Computation.

For convenience of reference the working formulæ are here brought together.

$$-\Delta\phi = s\cdot B\cdot\cos\alpha + s^2\sin^2\alpha\cdot C + (\delta\phi'')^2\cdot D - h\cdot s^2\cdot\sin^2\alpha\cdot E,^* \qquad [64]$$

$$\Delta\lambda = A'.s.\sin\alpha\sec\phi' \qquad [66]$$

* The value of $-\Delta\phi$ may be made more accurate by the addition of the following term:

$$-\tfrac{1}{3}s^2\cdot k\cdot E + \tfrac{2}{3}s^2\cos^2\alpha\cdot k\cdot E + \tfrac{1}{2}s^2\cdot\cos^2\alpha\sec^2\phi\cdot A^2\cdot k\operatorname{arc}^2 1'',$$

in which $k = s^2\cdot\sin^2\alpha\cdot C$.

$$(\text{or, } \log \Delta\lambda'' = \log s + C_{\log \Delta\lambda} - C_{\log s} + \log \sin \alpha + \log A' + \log \sec \phi'),$$

$$-\Delta\alpha = \Delta\lambda'' \sin \tfrac{1}{2}(\phi + \phi') \sec \tfrac{1}{2} \Delta\phi + (\Delta\lambda'')^3 \cdot F, \qquad [68]$$

in which

$$h = s \cdot \cos \alpha \cdot B,$$

$$-\delta\phi = s \cdot \cos \alpha \cdot B + s^2 \sin^2 \alpha \cdot C - hs^2 \sin^2 \alpha \cdot E.$$

The position of the new point and the reverse azimuth are then given by

$$\phi' = \phi + \Delta\phi, \qquad [65]$$

$$\lambda' = \lambda + \Delta\lambda, \qquad [67]$$

$$\alpha' = \alpha + \Delta\alpha + 180°. \qquad [69]$$

The arrangement of the computation is illustrated by the following example. The two pages show the two computations of a position in the same triangle.

In the first page of the computation, the known station is *Waldo* and the position of *Bunker Hill* is to be found. Since the value of $\Delta\alpha$ depends upon $\Delta\lambda$ and $\Delta\lambda$ depends upon ϕ', the three parts of the solution must be carried out in the order indicated. In computing $\Delta\phi$, take out B, C, D, and E for the given latitude ϕ. The $(\delta\phi)$ used in the D term is usually taken as the algebraic sum of the first two terms of the series; if the E term is large, it should be included also. The h in the E term is the first (B) term alone. The algebraic signs of the functions of α are important and should be carefully attended to.

When computing $\Delta\lambda$, ϕ' is known and the factor $\log A'$ must be taken out for this new latitude ϕ', not for ϕ. The primes are inserted to call attention to this. To correct for the difference between the arc and the sine, enter Table XIII with $\log \Delta\lambda$ and $\log s$ as arguments. The algebraic sum of the two values of "log. diff." is the correction to be applied to $\log \Delta\lambda$. The value of $\Delta\alpha$ is found last.

The values of ϕ' and λ' are checked by noting whether the same values are obtained from the two computations. The two reverse azimuths should differ by the spherical angle at the new station, which checks the computations of $\Delta\alpha$.

α	Waldo to Meade's Ranch	255° 17′ 17″.52
$\angle$	Meade's Ranch and Bunker Hill	86 20 54 .50
α	Waldo to Bunker Hill	341 38 12 .02
$\Delta\alpha$		+4 43 .09
		180°
α'	Bunker Hill to Waldo	161 42 55 .11
	Third angle	38 08 34 .02

ϕ	39° 09′ 55″.645	Waldo	λ	98° 49′ 50″.128
$\Delta\phi$	−17 39 .209	s = 34,407.64 meters	$\Delta\lambda$	−07 29 .652
ϕ'	38 52 16 .436	Bunker Hill	λ'	98 42 20 .476

s	4.536 6549	s^2	9.07331	$(\delta\phi)^2$	6.0499	$-h$	3.0249 n
cos α	9.977 3018	sin² α	8.99674			s^2 sin² α	8.0700
B	8.510 9150	C	1.31553	D	2.3832	E	6.0871
h	3.024 8717		9.38558		8.4331		7.1820 n
1st term	1058″.9409	3d term	+0.0271			$(\Delta\lambda)^2$	7.959
2d term	0 .2429	4th term	−0.0015			F	7.872
	1059 .1838		+0.0256				5.831
3rd and 4th		s	4.536 6549				
terms	+.0256	sin α	9.498 3680 n	Arg		$\Delta\lambda$	2.652 877 n
$-\Delta\phi$	1059.2094	A'	8.509 1469	s	−21	sin ½ $(\phi + \phi')$	9.799 043
½ $(\phi + \phi')$	39° 01′ 06″.04	sec ϕ'	0.108 7088	$\Delta\lambda$	+03	sec ½ $(\Delta\phi)$	1
			2.652 8786 n	Corr.	−18		2.451 921 n
			18			$-\Delta\alpha$	−283″.09
			2.652 8768 n				
		$\Delta\lambda$	449″.652				

123. The Inverse Problem.

Not infrequently it is required to find the distance and mutual azimuths between two stations whose latitudes and longitudes are known.

If we place $x = s \sin \alpha$ and $y = s \cos \alpha$, then, from Equa. [66] and [64], we have

$$x = \frac{\Delta\lambda \cos \phi'}{A'} \qquad [70]$$

and
$$y = -\frac{1}{B}\left[\Delta\phi + Cx^2 + D(\delta\phi)^2 + E(\Delta\phi)x^2\right], \qquad [71]$$

from which
$$\tan \alpha = \frac{x}{y} = \frac{\Delta\lambda \cos \phi'}{A' \cdot h} \qquad [72]$$

and
$$\left.\begin{aligned} s &= y \sec \alpha \\ &= x \operatorname{cosec} \alpha. \end{aligned}\right\} \qquad [73]$$

α	Meade's Ranch to Waldo	75° 28′ 14″.52
$\angle$	Bunker Hill and Waldo	55 30 33 .73
α	Meade's Ranch to Bunker Hill	19 57 40 .79
$\Delta\alpha$		−06 11 .66
		180
α'	Bunker Hill to Meade's Ranch	199 51 29 .13

ϕ	39° 13′ 26″.686	Meade's Ranch	λ	98° 32′ 30″.506
$\Delta\phi$	−21 10 .250	s = 41,661.11 meters	$\Delta\lambda$	+09 49 .969
ϕ'	38 52 16 .436	Bunker Hill	λ'	98 42 20 .475

s	4.619 7308	s^2	9.23946	$(\delta\phi)^2$	6.2076	$-h$	3.1037 n
cos α	9.973 0924	sin² α	9.06649			s^2 sin² α	8.3060
B	8.510 9105	C	1.31644	D	2.3835	E	6.0882
h	3.103 7337		9.62239		8.5911		7.4979 n
1st term	+1269.795	3d term	+0.0390			$(\Delta\lambda)^2$	8.312
2d term	0.419	4th term	−0.0031			F	7.871
	+1270.214		+0.0359				6.183
3d and 4th		s	4.619 7308				
term	+0.036	sin α	9.533 2455	Arg.		$\Delta\lambda$	2.770 830
$-\Delta\phi$	+1270.250	A'	8.509 1469	s	−31	sin ½ $(\phi + \phi')$	9.799 317
		sec ϕ'	0 108 7088	$\Delta\lambda$	+06	sec ½ $(\Delta\phi)$	2
½ $(\phi + \phi')$	39° 02′ 51″.56		2.770 8320	Corr.	−25		2.570 149
			−25				
			2.770 8295			$-\Delta\alpha$	371″.66
		$\Delta\lambda$	+589″.9694				

The inverse solution may be worked out on the same printed form that is used for the direct solution, but the order of procedure is modified as follows: First, compute x by Equa. [70], then the C, D, and E terms in Equa. [71], obtaining finally y. The azimuth is then found through its tangent; the calculation of s is the final step.

124. Location of Boundaries.

Whenever it becomes necessary to establish on the ground a boundary line between two states or countries, the length of the lines and the accuracy demanded usually make it necessary to employ geodetic methods. A boundary may consist of a meridian arc, a parallel of latitude, or a great circle inclined to the meridian; or it may be a combination of these.

125. Location of Meridian.

If a boundary is a meridian arc the longitude of which is fixed by law, it is first necessary to assume approximate positions for the terminal points, and then to determine the longitude of these by direct observations. These points are then corrected in position. After the terminals have been established on the ground, the line may be run from one to the other as a random line, to be subsequently corrected if necessary. Observations on Polaris for azimuth will show the direction of the meridian. The line is then run out by backsighting and foresighting. If necessary, the direction of the meridian may be determined at intermediate points. When the second point is reached, the error in the running of the line becomes known, and the random line may be set over or re-run in the usual manner. If the boundary is long, the intermediate points may be found by triangulation instead of by direct measurement. In any case triangulation will furnish a valuable check.

126. Location of Parallel of Latitude.

In order to establish a parallel of latitude on the ground, it is necessary to assume a point as nearly as may be on the desired parallel. The exact position of this assumed point is then determined by Talcott's method, and the station moved, if necessary, to the correct position. If the difference between the observed and the desired latitude is $\Delta\phi$, the sea-level distance which the station must be moved is $s' = R_m \, \Delta\phi'' \cdot \text{arc } 1''$.

At higher elevations s' should be increased in proportion to the distance from the center of the earth (Equa. [6]). If the error in position proves to be large, it may be advisable to make another determination of the latitude, in order to avoid the effect of station errors. (See Art. 83, p. 109).

The next step is to determine the azimuth of a reference mark, by observation on Polaris, and to establish the direction of a great circle at right angles to the meridian (prime vertical). Points on the parallel are then determined by measuring offsets from the prime vertical as a reference line.

In Fig. 71 we have, in the triangle PAB,

$$PA = 90^\circ - \phi,$$
$$A = 90^\circ,$$
and
$$\tan \sigma = \tan \Delta\lambda \cos \phi,$$
or
$$\sigma = \tan^{-1} (\tan \Delta\lambda \cos \phi).$$

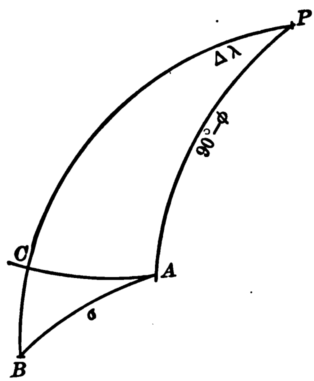


FIG. 71.

Expanding σ by the formula for $\tan^{-1} x$, p. 330, and also $\tan \Delta\lambda$ in terms of $\Delta\lambda$ by the formula for $\tan x$, p. 330, we have

$$\sigma = \Delta\lambda \cos \phi + \tfrac{1}{3} (\Delta\lambda \cos \phi)^3 \tan^2 \phi,$$
or
$$s = \sigma N = N \Delta\lambda'' \cdot \cos \phi \cdot \text{arc } 1'' + \tfrac{1}{3} N (\Delta\lambda'' \cos \phi \cdot \text{arc } 1'')^3 \tan^2 \phi, \qquad [74]$$

which gives the distance AB corresponding to any difference in longitude $\Delta\lambda$.

If in Equa. [64] we place $\alpha = 90^\circ$,

$$-\Delta\phi'' = \frac{s^2 \tan \phi}{2 N R_m \text{ arc } 1''}.$$

The offset P from the prime vertical (tangent) for any distance s from the initial point is

$$P = -\Delta\phi'' R_m \text{ arc } 1'' = \frac{s^2 \tan \phi}{2 N}. \qquad [75]$$

Since P varies as s^2, the offsets for equidistant intervals along the line may be readily calculated. The direction of the pole from any point (x) on AB is given by

$$PxA = 90^\circ + \Delta\alpha,$$

in which it is sufficiently accurate to take

$$-\Delta\alpha = \Delta\lambda \sin \phi_m. \qquad [76]$$

Since the numerical value of $\Delta\alpha$ increases directly as $\Delta\lambda$, it will be sufficient to take the increments of $\Delta\alpha$ as proportional to s.

If the arc of the parallel is a long one, it is advisable to break it into sections, and to establish a new point at the beginning of each section by direct latitude observation.

(See *United States Northern Boundary Survey*, Washington, 1878.)

127. Location of Arcs of Great Circles.

The general method of laying out arcs not coincident with the meridian is that of determining astronomically the latitudes and longitudes of the terminal points, and then running a random line between them. The direction and distance between the terminals may be found by Formulæ [70] to [73] for the inverse solution of the geodetic problem. The azimuth is determined by observation at intermediate points. The error of the random line is corrected in the usual way. For long arcs triangulation would be substituted for direct measurement.

(See Appendix 3, Coast Survey Report for 1900, "The Oblique Boundary Line between California and Nevada.")

128. Plane Coördinate Systems.

When all the points to be located in a survey are comprised within a relatively small area, such as a city or a metropolitan district, the calculations are greatly simplified by the use of plane coördinates. If there are reliable triangulation points already established within the area, these will naturally be used as a basis for the new survey, or at any rate to check the new triangulation.

In establishing a system of plane coördinates it is necessary to decide first upon the positions of the coördinate axes. These will naturally be a meridian and a great circle at right angles to it; or, more properly speaking, they will be straight lines tangent to these two circles at their point of intersection, all points being supposed to lie in the plane defined by these two lines. The origin of the system must be defined in terms of the coördinates of some specified point of the survey (geodetic datum, p. 158). Unless this is done, the origin will not be the same when derived from different points, and ambiguity will exist regarding the true position of the origin. The origin may be taken as coincident with the selected triangulation point, as in the case of the survey of Boston, Massachusetts, and Baltimore, Maryland; or it may be the intersection of a selected meridian and parallel as derived from the assigned latitude and longitude of some station. In Springfield, Massachusetts, for example, the origin is the intersection of the 42° 04′ parallel and the 72° 28′ meridian, as determined by the published latitude and longitude of the United States Armory flagpole. The direction of the meridian must be defined as making a certain angle with a specified line of the survey, preferably one which passes through the fundamental point.

The point at which the plane is tangent to the spheroid must not be confused with the (0, 0) point of the system. The former should be within the area surveyed, preferably at its center, in order to avoid large spherical errors. The latter may be taken at any convenient distance outside the area by assigning to the tangent point large values of x and y, in order to avoid negative values in the coördinates of the survey points. The tangent point is on the sphere as well as on the plane; the (0, 0) point is not necessarily on the sphere.

129. Calculation of Plane Coördinates from Latitude and Longitude.

In calculating the plane coördinates of a point, we may apply Formulæ [70] to [73] for the inverse solution of the geodetic problem, one of the points being the origin (tangent point) whose coördinates are ϕ and λ, and the other the triangulation point the coördinates of which are ϕ' and λ'. The x and y there given are the plane coördinates desired. If the coördinates of many points are to be transformed, it will prove to be more convenient to use specially prepared auxiliary tables and to modify the calculations as follows.

In Fig. 72 P is the triangulation point whose latitude and longitude are known, and whose coördinates x and y with reference to the origin O are desired. For such distances as are likely

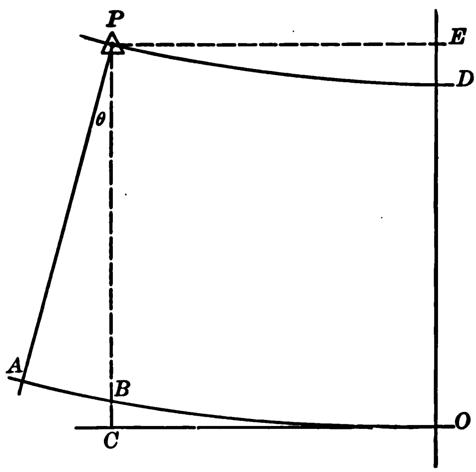


FIG. 72.

to occur in a plane system it may be assumed that $PE = PD$; that is, x equals the length of the arc of the parallel PD. The ordinate $y = PC$ may be taken as PA (the difference in latitude) plus BC * (the offset from great circle to parallel). From Formula [70],

$$x = PD = \Delta\lambda'' \cdot \frac{\cos \phi'}{A'}. \qquad [77]$$

If x is to be expressed in feet,

$$x = \Delta\lambda'' \cdot \frac{\cos \phi'}{A'} \times 3.2808\tfrac{1}{3}. \qquad [78]$$

(See Table A.)

* If P is south of the origin, the offset must be subtracted.

TABLE A. VALUES OF LOG $\frac{\cos \phi'}{A'}$ + 0.515 9842*

Distance west of origin in feet = $x = \Delta\lambda'' \times H$

Lat. ϕ'. ° ′ ″	Log H.	Lat. ϕ'. ° ′ ″	Log H.
42 10	1.876 8536	42 20	1.875 7103
30	7966	30	6530
11	7396	21	5957
30	6825	30	5383
12	6255	22	4809
30	5684	30	4235
13	5114	23	3661
30	4543	30	3086
14	3971	24	2512
30	3400	30	1937
15	2828	25	1362
30	2256	30	0787
16	1684	26	1.875 0212
30	1112	30	1.874 9636
17	1.876 0541	27	9061
30	1.875 9968	30	8485
18	9396	28	7910
30	8823	30	7334
19	8250	29	6757
30	7677	30	6181
20	1.875 7103	30	1.874 5604

P. P. ″	570	572	574	576
1	19	19	19	19
2	38	38	38	38
3	57	57	57	58
4	76	76	77	77
5	95	95	96	96
6	114	114	115	115
7	133	134	134	134
8	152	153	153	154
9	171	172	172	173
10	190	191	191	192
11	209	210	210	211
12	228	229	230	230
13	247	248	249	250
14	266	267	268	269
15	285	286	287	288
16	304	305	306	307
17	323	324	325	326
18	342	343	344	346
19	361	362	364	365
20	380	381	383	384
21	399	400	402	403
22	418	419	421	422
23	437	439	440	442
24	456	458	459	461
25	475	477	478	480
26	494	496	497	499
27	513	515	517	518
28	532	534	536	538
29	551	553	555	557
30	570	572	574	576

* This is the form adopted by the city of Springfield, Mass., for its coördinate system.

TABLE B. VALUES OF 0.515 9842 − log B

Dist. N. of Origin in Feet = $\Delta\phi'' \times K + x^2 \dfrac{\tan\phi}{2\,N}$

Dist. S. of Origin in feet = $\Delta\phi'' \times K - x^2 \dfrac{\tan\phi}{2\,N}$

Lat.	Log. *K*.	Lat.	Log. *K*.	P. P., Diff. 1′ = 12.8.			
° ′ ″		° ′ ″		″		″	
42 10	2.005 2891	42 20	2.005 3109	1	0	22	5
30	2988	30	3116	2	0	23	5
11	2994	21	3122	3	1	24	5
30	3000	30	3129	4	1	25	5
12	3006	22	3135	5	1	26	6
30	3013	30	3141	6	1	27	6
13	3019	23	3147	7	1	28	6
30	3026	30	3154	8	2	29	6
14	3032	24	3160	9	2		
30	3039	30	3167	10	2		
15	3045	25	3173	11	2		
30	3052	30	3180	12	3		
16	3058	26	3186	13	3		
30	3064	30	3193	14	3		
17	3070	27	3199	15	3		
30	3077	30	3205	16	3		
18	3083	28	3211	17	4		
30	3090	30	3218	18	4		
19	3096	29	3224	19	4		
30	3103	30	3231	20	4		
20	2.005 3109	30	2.005 3237	21	4		

The difference in latitude *PA* is converted into feet by multiplying $\Delta\phi''$ by $\dfrac{3.2808\frac{1}{3}}{B}$. (Table B.)

The offset *BC* (Formula [75]) $= \dfrac{\tan\phi}{2\,N} \times x^2$.* [79]

The factor $\dfrac{\tan\phi}{2\,N}$, in feet, may be taken from Table C which was calculated by the formula

$$\log \frac{\tan\phi}{2 \cdot N} = \log C - \log B - \log 3.2808\tfrac{1}{3}. \qquad [80]$$

* For another method of calculating this offset, see an article entitled "A Method of Transforming Latitude and Longitude into Plane Coördinates," by Sturgis H. Thorndike, *Journal Boston Society Civil Engineers*, Vol. 3, No. 7, September, 1916.

TABLE C. VALUES OF LOG $\frac{\tan \phi}{2N}$ (ft.) = log C − log B − 0.515 9842

Offset from parallel = log L + 2 log x

Lat.	Log. L.	Lat.	Log. L.	P. P. Diff. 1′ = 25.4.			
° ′ ″		° ′ ″		″		″	
42 10	2.33 460	42 20	2.33 714	1	0	24	10
30	473	30	727	2	1	25	11
11	486	21	739	3	1	26	11
30	499	30	752	4	2	27	11
12	512	22	765	5	2	28	12
30	525	30	778	6	3	29	12
13	537	23	790	7	3		
30	550	30	803	8	4		
14	562	24	815	9	4		
30	575	30	828	10	4		
15	587	25	840	11	5		
30	600	30	853	12	5		
16	612	26	865	13	6		
30	625	30	878	14	6		
17	638	27	892	15	6		
30	651	30	905	16	7		
18	663	28	917	17	7		
30	676	30	930	18	8		
19	689	29	942	19	8		
30	702	30	955	20	8		
20	2.33 714	30	2.33 967	21	9		
				22	9		
				23	10		

Example. As an illustration of how this method would be applied, let us suppose that it is desired to compute the plane coördinates of △ *Powderhorn* in a system whose origin is the dome of the State House, Boston, Massachusetts. We first compute $\Delta\phi''$ and $\Delta\lambda''$ and then apply formulæ [78], [79] and [80] as shown.

	Lat.	Long.
Powderhorn	42° 24′ 04″.683	71° 01′ 52″.006
State House	42 21 29 .596	71 03 51 .040
	2′ 35″.087	1′ 59″.034
	$\Delta\phi''$ = 155″.087	$\Delta\lambda''$ = 119″.034

log x^2 = 7.90183   log $\Delta\phi''$ = 2.190 5754   log $\Delta\lambda''$ = 2.075 6710
log L = 2.33752   log K = 2.005 3129   log H = 1.875 2422

log = 0.23935   log = 4.195 8883   log x = 3.950 9132

Offset = 1.7352 ft.   15699.59 ft. x = 8931.27 ft. East of State House.
1.74

y = 15701.33 ft. North of State House

If it is preferred to make the conversion from $\Delta\lambda$ to x always on the same parallel of latitude, that of the origin, a table may be calculated, giving the length of each minute (1′ to 10′) and each second (1″ to 60″) of arc on this parallel; the difference in longitude may be taken out, by parts, from this table. If this is done, however, it is necessary to make allowance for the convergence of the meridians between the two parallels by solving for the distance $AB = y \sin \theta$ (Fig. 72). The convergence $\theta = \Delta\lambda'' \sin \phi_m$ and its sine may be tabulated for different values of $\Delta\lambda$ and ϕ_m. If the triangulation point is north of the origin, AB is to be subtracted; if south, it is to be added.

130. Errors of a Plane System.

In order to investigate the errors of a plane coördinate system like the preceding, let us assume that a line starts from the origin o, Fig. 73, in an azimuth α, and follows the surface of a sphere of

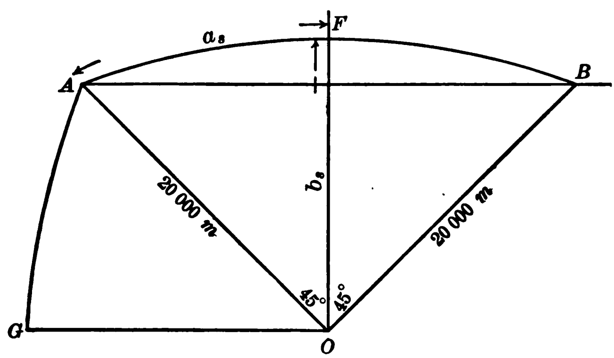


FIG. 73.

radius $\sqrt{R_m N}$ (for latitude ϕ) for a distance s meters, to point A; and that another line OA', having the same azimuth and length, lies in the plane which is tangent to the sphere at o. The point A' in the plane then represents the point A on the sphere as determined by a direct measurement from the origin. The defects of the plane system as a means of representing points on a sphere will be shown by the error in reproducing point A' by following different routes, such, for example, as traversing due north and then due west on the sphere, or due west and then due north.

If a perpendicular AF (an arc of a great circle) be let fall from

A (Fig. 73) to the meridian through o, its length will be determined by

$$\sin\frac{a}{R} = \sin\frac{s}{R}\cdot\sin\alpha,$$

where a is the perpendicular distance in meters and R is the radius of the sphere.

For the corresponding distance on the plane,

$$a = s\cdot\sin\alpha.$$

Distinguishing the plane and spherical values of a by subscripts, p and s, the difference in length may be found as follows:

$$\begin{aligned} a_p - a_s &= s\sin\alpha - R\sin^{-1}\left(\sin\alpha\sin\frac{s}{R}\right) \\ &= s\cdot\sin\alpha - R\left[\sin\alpha\left(\frac{s}{R}-\frac{s^3}{6R^3}\right)+\frac{\sin^3\alpha}{6}\left(\frac{s}{R}-\frac{s^3}{6R^3}\right)^3\right] \\ &= s\cdot\sin\alpha - \frac{Rs\sin\alpha}{R}+\frac{s^3}{6R^2}\sin\alpha-\frac{s^3}{6R^2}\sin^3\alpha+\cdots \\ &= \frac{s^3}{6R^2}\sin\alpha\cos^2\alpha+\cdots. \end{aligned}$$

Assuming that $\phi = 40°$, $\alpha =$ N 45° W, and $s =$ 20,000 meters (about 12 miles), then $a_p - a_s = 0^m.0116$. If another such line were to extend 20,000^m, N 45° E, to B, the terminal points A and B would then be $0^m.0232$ farther apart if calculated on a plane than if calculated on the sphere.*

If the survey proceeds from o northward to the point F, where the great circle from A, perpendicular to the meridian, intersects that meridian, and then westward along this great circle to A, the point A would be reached without error, if the measurements were perfect. The point computed on the plane would not agree, however, with A' as already established. The excess of the spherical distance b_s, along the meridian to the foot of the perpendicular F, over the plane distance b_p is found as follows:

* This does not refer to the chord-distance AB, but to the distance on the spherical surface.

In the spherical right triangle,

$$\tan\frac{b}{R} = \tan\frac{s}{R}\cos\alpha.$$

Then

$$b_s - b_p = R\tan^{-1}\left(\tan\frac{s}{R}\cos\alpha\right) - s\cos\alpha$$

$$= \frac{s^3\cos\alpha\sin^2\alpha}{3R^2}.$$

Assuming the same data as before, we find that in order to reach A, on the sphere, we must run N 14142.15886 meters and then W 14142.12400 meters. Since in this case $s\sin\alpha = s\cos\alpha =$ 14142.13563^m, such a traverse, when computed on the plane, gives a point 0^m.02323 N and 0^m.01163 E of point A'. A similar traverse running west to point G (Fig. 73) and then north to A would give a point 0^m.01163 S and 0^m.02323 W of point A'. The relative positions are shown (actual size) in Fig. 74.

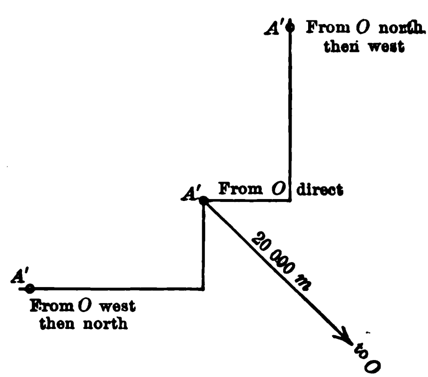


FIG. 74.

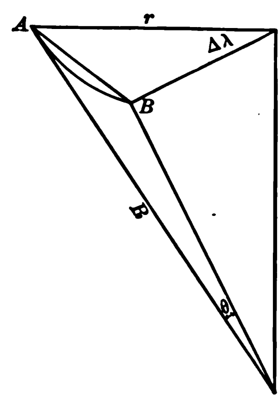


FIG. 75.

The maximum discrepancy in the traverse is then about 0^m.05, or about two inches. This would appear as an error of closure of the traverse $OFAGO$ even if there were no error whatsoever in the measurements themselves.

The difference in length between an arc of the parallel and an arc of the great circle is found as follows: In Fig. 75, $\frac{1}{2} AB = r \sin \frac{\Delta\lambda}{2} = R \sin \frac{\theta}{2}$. Replacing the sines by their series in terms of the arcs, $r\left(\frac{\Delta\lambda}{2} - \frac{\Delta\lambda^3}{48}\right) = R\left(\frac{\theta}{2} - \frac{\theta^3}{48}\right)$. The difference between $r\,\Delta\lambda$, the arc of the parallel, and $R\theta$, the arc of the great circle, is

$$r\Delta\lambda - R\theta = r\frac{\Delta\lambda^3}{24} - R\cdot\frac{\theta^3}{24}$$

$$= R\cos\phi\frac{\Delta\lambda^3}{24} - R\cdot\frac{\Delta\lambda^3\cos^3\phi}{24} \text{ (approx.)}$$

since $$\theta = \Delta\lambda\cos\phi, \text{ nearly.}$$

Therefore $$r\Delta\lambda - R\theta = \tfrac{1}{24}R\cos\phi\,\Delta\lambda^3\,(1 - \cos^2\phi)$$
$$= \tfrac{1}{24}R\,(\Delta\lambda'')^3\,.\,\text{arc}^3\,1''\cos\phi\sin^2\phi.$$

In order to compare this with the previous examples, we must put $\Delta\lambda'' = 1192''.4$, which corresponds to the distance between A and B. The error $r\,\Delta\lambda - R\theta$ is found to be $0^m.0186$ for the total arc, or $0^m.0093$ for the half arc. The difference between the length of the parallel and the x coördinate is therefore $0^m.0116 - 0^m.0093 = 0^m.0023$.

These results indicate that a plane system may be extended over an area twelve miles in radius without involving errors of computation as great as the errors of measurement, and also that the formulæ given may be used whenever it is safe to use plane coördinates.

131. Adjusting Traverses to Triangulation.

Whenever a traverse is to be run from one triangulation point to another, or if the circuit is to return to the original triangulation point, some method must be provided to allow for the effect of convergence of the meridians. The most obvious method is to refer all bearings in the traverse to the direction of the initial meridian, taking no account of true bearings at any other point of the survey. This method is subject to very small errors, far within the limit of accuracy of the field measurements, unless the area is much greater than that ordinarily covered by a traverse.

PROBLEMS

Problem 1. Calculate the latitude and longitude of point A, Problem 3, Chapter VI, from both lines, and the back azimuths AB and AC.

Problem 2. Calculate the latitude and longitude of point E, Problem 4, Chapter VI, and the back azimuths EL and EN.

Problem 3. Calculate the position of Sand Hill in Problem 5, Chapter VI.

Problem 4. What will be the error of closure of a survey which follows the circumference of a circle whose radius is 20,000 meters (on the earth's surface) if the survey is calculated as though it were on a plane, the latitude of the center being 40° N. and the measurements being exact?

CHAPTER VIII

FIGURE OF THE EARTH

132. Figure of the Earth.

The term "figure of the earth" may have various interpretations, according to the sense in which it is employed and the degree of precision with which we intend to define the earth's figure. When we say that the earth is spherical, we mean that the sphere is a rough approximation to the true figure, sufficiently close for many purposes. We adopt the sphere to represent this figure because it is a simple surface to deal with mathematically. When a closer approximation is required, we employ the spheroid, or ellipsoid of revolution. This figure is so near the truth that no closer approximation has ever been needed in practical geodetic operations, although an ellipsoid (three unequal axes) or an ovaloid (southern hemisphere the larger) may be nearer the truth. All the surfaces mentioned are *regular* mathematical surfaces, substituted for the true surface on account of their simplicity.

In defining the true figure it is necessary to distinguish between the topographical surface and that surface to which the waters of the earth tend to conform because they are free to adjust themselves perfectly to the forces acting upon them. It is this latter surface with which we are chiefly concerned in geodesy; the land surface is not referred to except in such questions as the effect of topography upon the direction and intensity of gravity. The true figure, called the *geoid*, is defined as a surface which is everywhere normal to the force of gravity, that is, an *equipotential* surface; and of all the possible surfaces of this class it is that particular one which coincides with the mean surface of the oceans of the earth. Under the continents it is the surface to which the waters of the ocean would tend to conform if allowed to flow into very narrow and shallow canals cut through the land. It is necessary to suppose these canals narrow and shallow in order that the quantity of water removed may not modify the figure over the ocean areas.

Some idea of the relation of the spheroid, the geoid, and topographical surface may be gained by an inspection of Fig. 76. It will be seen that the geoidal surface coincides with the surface of the ocean, and that it intersects the spheroid at some distance out from the shore line. The inclination of the *normal* to the

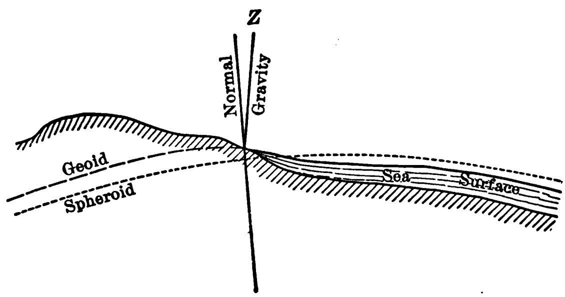


FIG. 76.

plumb line (station error) shows the angle between the two surfaces at this point.

The surface of the geoid may be represented conveniently by means of contour lines referred to the spheroid as a datum surface. In Fig. 77, which shows contours of the geoid within the limits of the United States proper, that portion of the contours shown in full lines is taken from a map published by the Coast and Geodetic Survey in "Figure of the Earth and Isostasy" (1909); the remaining portions (dotted) were sketched in by eye, following in a general way the topography of the continent. Such a map conveys no real information about the elevations of the geoid except along the full lines, but is given simply to show how the contours would be used in representing the geoid.

When we speak of the spheroid as the "figure of the earth" we mean that particular spheroid which best represents the earth as a whole, or which most closely fits some specified area. The dimensions of such a spheroid are not to be regarded as fixed, but are subject to revision with each accession of new data. Such a spheroid necessarily depends upon a large amount of data, and the calculations for fixing its dimensions are long and complicated, involving the adjustment of many observations by the method of least squares.

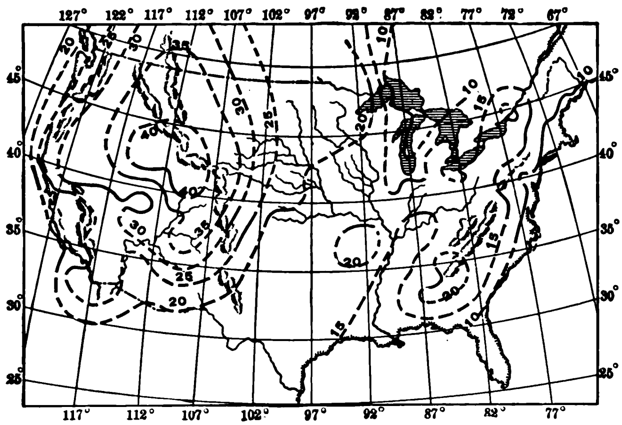


FIG. 77. Contours of the Geoid.

The principal methods of determining the spheroid are (1) by the measurement of arcs, which may be portions of meridians, of parallels, or of great circles; (2) by means of areas containing several astronomical stations rigidly connected by triangulation; and (3) by observations of the force of gravity.

133. Dimensions of the Spheroid from Two Arcs.

The simplest method by which the dimensions of the spheroid can be determined is by the measurement of two meridian arcs. The length of each arc and the latitudes of the terminal points of each must be measured. If the earth were a perfect spheroid, and if there were no errors of measurement, the two arcs would determine exactly the elements of the spheroid.

In the equation of the ellipse there are two constants to be determined, and it will be shown that the determination of the curvature of the meridian ellipse at two points will enable us to compute these constants and consequently all the other elements of the ellipse. In Fig. 78, suppose that the lengths of the two

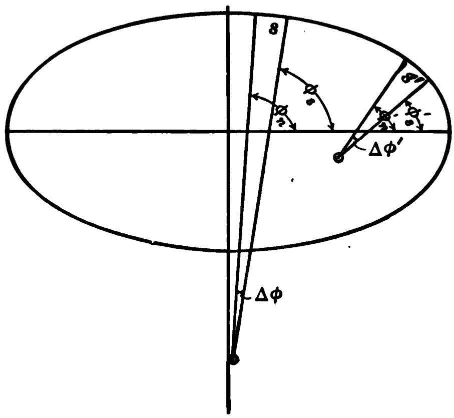


Fig. 78.

meridian arcs have been measured by triangulation and that their lengths are s and s', and that the differences of the latitudes of their terminals are $\Delta\phi$ and $\Delta\phi'$, respectively. The radii of curvature of the ellipse at the middle points of the arcs are

$$R_m = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}$$

and

$$R_m' = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi')^{\frac{3}{2}}},$$

in which ϕ and ϕ' refer to the middle points of the arcs and a and e are unknown. If the two arcs are regarded as arcs of circles whose radii are to be found, then

$$R_m = \frac{s}{\Delta\phi \text{ arc } 1''} \quad \text{and} \quad R_m' = \frac{s'}{\Delta\phi' \text{ arc } 1''}$$

are the two radii of curvature, $\Delta\phi$ being in seconds. The shorter the arcs, the less the error involved in assuming that they are circular.

Equating the two values of R_m and R_m', we have

$$\frac{s}{\Delta\phi \text{ arc } 1''} = \frac{a(1 - e^2)}{(1 - e^2 \sin^2\phi)^{\frac{3}{2}}} \tag{a}$$

and

$$\frac{s'}{\Delta\phi' \text{ arc } 1''} = \frac{a(1 - e^2)}{(1 - e^2 \sin^2\phi')^{\frac{3}{2}}}. \tag{b}$$

Dividing (*a*) by (*b*) and solving for e^2,

$$e^2 = \frac{1 - \left(\frac{s\,\Delta\phi'}{s'\,\Delta\phi}\right)^{\frac{2}{3}}}{\sin^2\phi' - \left(\frac{s\,\Delta\phi'}{s'\,\Delta\phi}\right)^{\frac{2}{3}} \sin^2\phi}. \tag{[81]}$$

Having found e^2 from Equa. [81], the equatorial radius a may be computed by substituting the value of e^2 in either (*a*) or (*b*). The value of b may then be found from the relation

$$b^2 = a^2 (1 - e^2). \tag{c}$$

The compression f is given by

$$f = \frac{a - b}{a}. \tag{[53]}$$

The length of a quadrant of the meridian may be found by applying Equa. [54], Chapter V.

In this method of determining the elements of the spheroid it should be observed that there are just enough measurements to enable us to solve the equations, and no more. All errors of measurement enter the result directly; we should not, therefore, expect to derive very accurate values from two arcs.

As an illustration of the preceding method let us take the Peruvian Arc and a portion of the Russian Arc, the data for which are as follows:

PERUVIAN ARC

Station.	Astr. lat.	Dist. in meters between the parallels of latitude.
	° ′ ″	
Tarqui Cotchesqui	S 3 04 32.068 N 0 02 31.387	344,740.5
RUSSIAN ARC (Northern End)		
	° ′ ″	
Tornea Fuglenaes	N 65 49 44.57 N 70 40′ 11.23	539,841.7

Substituting in Formulæ 81, (*a*) and (*c*), the resulting values are

$$e^2 = 0.0065473,$$
$$a = 6{,}377{,}352\ m,$$
$$b = 6{,}356{,}440\ m.$$

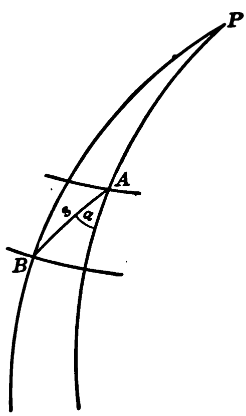


FIG. 79.

134. Oblique Arcs.

If an arc (*AB*, Fig. 79) is inclined to the meridian at a small angle, it may be utilized to determine the curvature of the meridian as follows: Referring to Equa. (*n*), Chapter VII, it is seen that the difference in latitude of the terminal points of the line is given by the series for $\Delta\phi''$. Hence the length of the meridian arc is given by $\Delta\phi''$. $R_m \cdot \text{arc } 1''$, and

$$\Delta\phi'' \cdot R_m \cdot \text{arc } 1'' = -s\cos\alpha - \frac{1}{2N}s^2\sin^2\alpha\tan\phi + \frac{1}{6N^2}s^3\sin^2\alpha\cos\alpha\,(1 + 3\tan^2\phi). \quad [82]$$

Each line of a chain of triangles may be projected onto the meridian, and its length found by this formula. The length and difference in latitude of the end points are thus found, and the projection treated as though it were a measured meridian arc.

The sum of all these short arcs may then be treated as a single arc to be combined with another similar arc in the computation of a and e.

135. Figure of the Earth from Several Arcs.

When several arcs are to be used to determine the elements of the spheroid, there are more data than are necessary for the direct solution as given in Art. 133. The arcs usually consist of several sections; that is, the latitudes of several stations along the same meridian are observed and the distances between them are determined by the triangulation. The problem is one of combining all these measurements by the method of least squares in order to obtain the most probable values of the elements. Only the outline of the method can be given here.

From Equa. [49] we have for the length of a meridian arc

$$s = \Delta\phi \cdot R_m \cdot \text{arc } 1'',$$

which is sufficiently accurate for short arcs. For long arcs a more accurate expression is necessary. Suppose that an arc consists of several sections, the latitude of the initial point being ϕ_1, the second ϕ_2, etc., and that the meridian distances between the stations are s, s_1, etc. From the first two latitudes

$$\phi_2 - \phi_1 = \frac{s}{R_m \text{ arc } 1''}, \qquad (e)$$

in which

$$\frac{1}{R_m} = \frac{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}{a(1 - e^2)}. \qquad (f)$$

Instead of finding a and e^2 directly, it is more convenient to assume approximate values of these quantities and to compute the most probable corrections. Let us assume the equations

$$a = a_o + \delta a$$

and

$$e^2 = e_o^2 + \delta e^2.$$

Let R_o be the value of R_m corresponding to e_o^2 and a_o. Expanding (f) by Taylor's theorem,

$$\frac{1}{R_m} = \frac{1}{R_o} + \frac{d\left(\frac{1}{R_m}\right)}{da} \cdot \delta a + \frac{d\left(\frac{1}{R_m}\right)}{de^2} \cdot \delta e^2 + \cdots \qquad (g)$$

Evaluating the two differential coefficients,

$$\frac{d\left(\frac{1}{R_m}\right)}{da} = -\frac{(1 - e^2\sin^2\phi)^{\frac{3}{2}}}{a^2(1 - e^2)} = \frac{1}{a^2},$$

neglecting the e^2 terms, and

$$\frac{d\left(\frac{1}{R}\right)}{de^2} = -\frac{a(1 - e^2)\cdot\frac{3}{2}\cdot(1 - e^2\sin^2\phi)^{\frac{1}{2}}\sin^2\phi - (1 - e^2\sin^2\phi)^{\frac{3}{2}}\cdot a}{a^2(1 - e^2)}$$

$$= \frac{1}{a}(1 - \tfrac{3}{2}\sin^2\phi), \text{ neglecting } e^2 \text{ terms.}$$

Substituting these values in (g),

$$\frac{1}{R} = \frac{1}{R_o} - \frac{1}{a_o^2}\cdot\delta a + \frac{1}{a_o}(1 - \tfrac{3}{2}\sin^2\phi)\cdot\delta e^2.$$

Hence (e) becomes

$$\phi_2 - \phi_1 = \frac{s}{\text{arc } 1''}\left(\frac{1}{R_o} - \frac{\delta a}{a_o^2} + (1 - \tfrac{3}{2}\sin^2\phi)\frac{\delta e^2}{a_o}\right). \qquad (h)$$

The errors in the measured latitudes are so large in comparison with the errors in the measured arcs that the lengths are considered exact and the observed latitudes are given corrections v_1, v_2, etc. Equa. (h) then becomes

$$\phi_2 + v_2 - \phi_1 - v_1 = \frac{s}{\text{arc } 1''}\left(\frac{1}{R_o} - \frac{\delta a}{a_o^2} + (1 - \tfrac{3}{2}\sin^2\phi)\frac{\delta e^2}{a_o}\right)\cdot\cdot\cdot\cdot \qquad (i)$$

In the small terms, containing δa and δe^2, the e^2 terms were omitted; that is, e^2 was placed equal to zero. This makes $R_m = a$ and $\phi_2 - \phi_1 = \frac{s}{a \text{ arc } 1''}$ in these terms.

Substituting in (i),

$$v_2 - v_1 = -(\phi_2 - \phi_1) + \frac{s}{R_o \text{ arc } 1''} - \frac{s\cdot\delta a}{a_o^2 \text{ arc } 1''} + (1 - \tfrac{3}{2}\sin^2\phi)\frac{\delta e^2}{a_o}\frac{s}{\text{arc } 1''}$$

$$= -\frac{\phi_2 - \phi_1}{a_o}\cdot\delta a + (1 - \tfrac{3}{2}\sin^2\phi)(\phi_2 - \phi_1)\cdot\delta e^2 + \frac{s}{R_o \text{ arc } 1''} - (\phi_2 - \phi_1). \quad (j)$$

If we place

$$x = \delta a,$$
$$y = \delta e^2,$$

substituting in (j), we have

$$a_1 x + b_1 y + l_1 = v_2 - v_1, \qquad (k)$$

where

$$a_1 = -\frac{\phi_2 - \phi_1}{a_o},$$
$$b_1 = (\phi_2 - \phi_1)(1 - \tfrac{3}{2}\sin^2\phi),$$
$$l_1 = \frac{s_1}{R_o \text{ arc } 1''} - (\phi_2 - \phi_1).$$

It is evident that an equation of this form (k) may be written for each section of each arc. There will be more equations than there are unknown quantities to be found. From these equations we may form a set of "normal" equations (Art. 201, p. 293), equal in number to the number of unknown quantities, that is, equal to the number of arcs plus two. The simultaneous solution of the normal equations gives the corrections δa and δe^2, and also the correction to the initial latitude of each arc.

136. Principal Determinations of the Spheroid.*

The spheroids which have been most extensively used are those of Bessel (1841) and Clarke (1866). Bessel's determination was based on the following arcs; the Peruvian, French, English, Hannoverian, Danish, Prussian, Russian, Swedish, and two Indian arcs. The resulting elements of the spheroid are generally used in Europe at the present time in geodetic surveys. They were employed in the United States up to about 1880. Clarke's spheroid (1866) was calculated from the following six arcs, the total amplitude being about 76° 35′; the French, English, Russian, South African, Indian, and the Peruvian. The Clarke spheroid is larger and flatter than Bessel's. It was adopted by the Coast and Geodetic Survey about 1880, after it became evident that the surface in this part of the globe has a flatter curva-

* For an account of the different arc measurements see *A History of the Determination of the Figure of the Earth from Arc Measurements*, by A. D. Butterfield, Worcester, 1906.

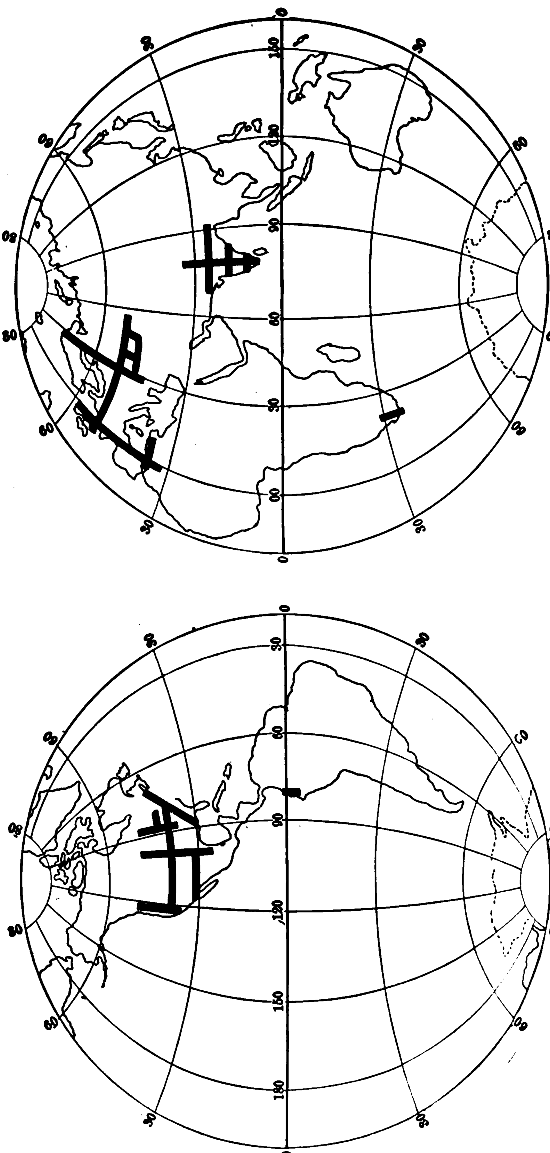


FIG. 79a. Location of Principal Arcs.

ture than that indicated by the Bessel spheroid. The semiaxes of these two spheroids are shown below, their dimensions being based on Clarke's value of the meter, namely, $1^m = 39.370113$ inches.

	a (meters).	b (meters).
Bessel (1841)	6 377 397	6 356 079
Clarke (1866)	6 378 206	6 356 584

Several other spheroids have been calculated from different groups of arcs, but have not been extensively used for geodetic purposes.

137. Geodetic Datum.

The question of where to place the spheroid with respect to the station points of a survey, and the question whether a certain spheroid properly represents the curvature of the area being surveyed, are determined by a comparison of the geodetic and astronomical positions of the survey points. As the survey progresses the geodetic latitudes and longitudes will be calculated on the surface of the adopted spheroid, starting from some assumed position of one of the triangulation stations. At the same time the positions of many of the stations will be determined astronomically. The differences in the latitudes, astronomical minus geodetic $(A - G)$, the differences in the longitudes, and the differences in the azimuths are computed for every station where the astronomical observations have been made. A study of these differences and their manner of distribution will show what corrections to the assumed position of the initial point will reduce the algebraic sum of the quantities $(A - G)$ to a minimum. If these differences were due wholly to errors in the assumed latitude and longitude of the initial point, it would be possible to reduce $\sum (A - G)$ to zero, but a part of this difference is due to local deflection of the vertical, that is, to the difference in slope of the geoidal and spheroidal surfaces. For this reason the most that can be expected is to place the spheroid so as to reduce $\sum (A - G)$ to a small quantity. The remaining values of $(A - G)$ at the different stations after a recomputation has been made, serve to indicate the slope of the geoid with reference to the spheroid.

If the reference spheroid adopted has too great a curvature, the computed latitudes will increase or decrease faster than the astronomical latitudes as the survey proceeds north or south from the initial point (Fig. 80). This was observed as the surveys in this country were gradually extended on the Bessel

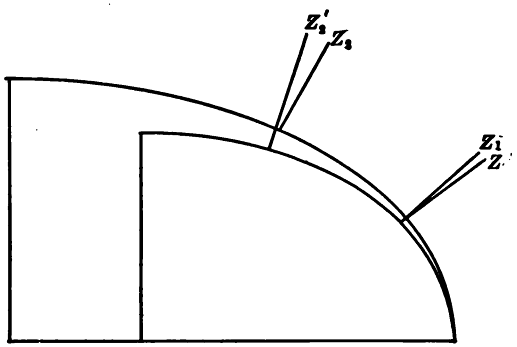


FIG. 80.

spheroid. If we consider an area instead of a meridian arc, then we see that if all the astronomical zeniths are swung inward with reference to the geodetic zeniths, the spheroid that we are using for the calculations must have too great a curvature for the area in question. If the observed latitudes are sometimes too great, sometimes too small, as we proceed along a meridian, this simply shows that the verticals are deflected locally, and that the average curvature of the surface is nearly that of the spheroid.

138. Determination of the Geoid.

The form of the geoid is determined by observing the local variations from the spheroid as a surface of reference. These deviations may be determined either from the station error (difference between astronomical and observed position) or from the observed variation in the force of gravity.

The station error at any point, or local deflection of the vertical, is a direct measure of the slope of the surface of the geoid with reference to the spheroid. The geodetic coördinates of the point are computed with reference to a line normal to the spheroid, while the astronomical coördinates are referred to the actual direction of the plumb line, which is normal to the geoidal surface.

139. Effect of Masses of Topography on the Direction of the Plumb Line.

The deflection of the plumb line by masses of topography may be computed by applying Newton's law of gravitation, that is, if m_1 and m_2 be any two masses, D the distance between them, and k a constant (to be found by experiment), then the force of attraction between m_1 and m_2 is

$$k \cdot \frac{m_1 \cdot m_2}{D^2};$$

that is, the force of attraction is proportional to the product of the masses and varies inversely as the square of the distance between them. The effect of any mass, such as a mountain, in deflecting the direction of gravity at any station may be found by combining the attraction of the mountain with the attraction of the earth regarded as a sphere. It may be shown that the attraction of a sphere at any external point is the same as though its mass were concentrated at its center. The relative attractions of the mountain and the earth upon the plumb bob at the station are as $\frac{m}{d^2}$ to $\frac{M}{R^2}$ (Fig. 81), where m is the mass of the mountain, M that of the earth, and d the distance of the mountain

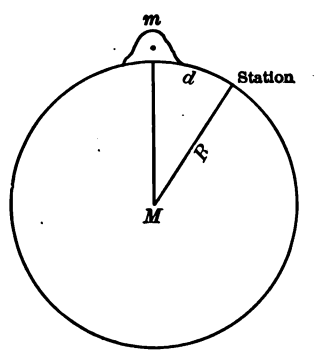


FIG. 81.

from the station. The angle D through which the plumb-line is deflected is given by

$$\tan D = \frac{mR^2}{Md^2}.$$

The earth's mass is $\frac{4}{3}\pi R^3 \times 5.58$ (the constant 5.58 being the mean density of the earth). If the mountain has a volume v and density δ, and the earth's radius be taken as 6370 kilometers, then

$$D'' = 0.00138\frac{v\delta}{d^2}, \qquad [83]$$

the dimensions being in meters and the angle in seconds.

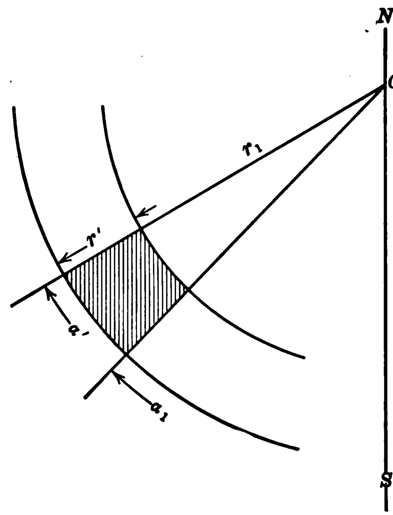


FIG. 82.

In order to take into account all of the topography about a station when computing the deflection of the plumb line, the following method may be employed (see Clarke, *Geodesy*, p. 294). The area surrounding the station is supposed to be divided into circular rings of any desired width, and the rings cut into four-sided compartments by radial lines, as in Fig. 82.

It is desirable to separate the component of the deflection in the meridian plane from that in the prime vertical. Let h be the height of the upper surface of the mass above station O; let α and r be the azimuth and horizontal distance to any particle P in the mass; and let z be its height above O and δ its density. The mass of the particle is then $\delta \cdot r \cdot d\alpha \cdot dr \cdot dz$. The attraction of the particle on O is

$$k \cdot \frac{\delta \cdot r \cdot d\alpha \cdot dr \cdot dz}{r^2 + z^2},$$

k being the gravitation constant.*

The component of this attraction in the plane of the meridian is the total attraction multiplied by the cosine of the angle between PO and SO, which is $\frac{r \cos \alpha}{\sqrt{r^2 + z^2}}$.

The total attraction of the mass in the compartment in the direction SO is

$$A = k \int_{\alpha_1}^{\alpha'} \int_{r_1}^{r'} \int_0^h \frac{\delta \cdot r^2 \cdot \cos \alpha \, d\alpha \cdot dr \cdot dz}{(r^2 + z^2)^{\frac{3}{2}}}$$

$$= k \cdot \delta \,(\sin \alpha' - \sin \alpha_1) \int_{r_1}^{r'} \int_0^h \frac{r^2 \, dr \, dz}{(r^2 + z^2)^{\frac{3}{2}}}$$

$$= k \cdot \delta \cdot h \,(\sin \alpha' - \sin \alpha_1) \int_{r_1}^{r'} \frac{dr}{\sqrt{r^2 + h^2}}$$

$$= k \cdot \delta \cdot h \,(\sin \alpha' - \sin \alpha_1) \log_e \frac{r' + \sqrt{r'^2 + h^2}}{r_1 + \sqrt{r_1^2 + h^2}}.$$

Unless h is very large, the equation may be written with sufficient accuracy

$$A = k\delta h \,(\sin \alpha' - \sin \alpha_1) \log_e \frac{r'}{r_1};$$

that is, the mass is considered to lie in the plane of the horizon of the station.

* The gravitation constant may be defined as the attraction of one unit mass on another unit mass at a unit distance away. In the C. G. S. system this is 6673×10^{-11}.

The attraction of the earth at point O, supposing it to be a sphere of radius R (3960 miles) and of density Δ, is

$$A' = k\frac{\Delta \cdot \frac{4}{3}\pi R^3}{R^2}$$
$$= k \cdot \tfrac{4}{3}\pi R\Delta.$$

The angle of deflection in the plane of the meridian is given by the ratio of attractions, that is,

$$D = \frac{\delta}{\Delta} \cdot \frac{h\,(\sin\alpha' - \sin\alpha_1)\log_e \frac{r'}{r_1}}{\frac{4}{3} \cdot \pi \cdot R}$$
$$= 12''.44\,\frac{\delta}{\Delta} \cdot h \cdot (\sin\alpha' - \sin\alpha_1)\log_e \frac{r'}{r_1}. \qquad [84]$$

The ratio of densities $\frac{\delta}{\Delta}$ may be taken as $\frac{1}{2.09}$;* $\delta = 2.67$ and $\Delta = 5.576$.

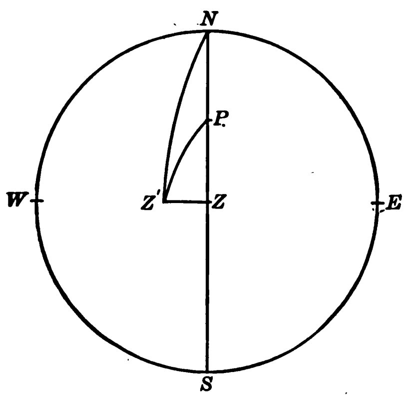


FIG. 82a.

By extending the rings outward this computation may be carried as far from the station as desired. If a compartment is very far from the station, it becomes necessary to correct for the curvature of the earth, because the mass no longer lies in the horizon of the station, as at first assumed.

* See Harkness, *The Solar Parallax and its Related Constants*, Washington, 1891.

If the angles α_1 and α' are measured from the prime vertical instead of from the meridian, the formula gives the deflection in a plane at right angles to the meridian.

By the foregoing process we may compute for any station what is called the topographic deflection. It shows what the deflection of the plumb line would be if no other forces acted upon it than those mentioned. A comparison of the values so computed with the station errors actually observed shows the former to be much larger than the latter; from which we infer that the attraction of the surface topography cannot be the only force tending to deflect the plumb line.

Laplace Points.

As stated above, it is customary to resolve the deflection of the plumb line into two components, one in the plane of the meridian and the other in the plane of the prime vertical. The meridian component is found directly by subtracting the geodetic (computed) latitude from the observed astronomic latitude. The prime vertical component must be obtained indirectly either from the astronomic and geodetic longitudes or from the astronomic and geodetic azimuths. In terms of the longitudes this component is

$$\text{p. v. component} = (\lambda_A - \lambda_G) \cos \phi_G.$$

In terms of the azimuth it is

$$\text{p. v. component} = -(\alpha_A - \alpha_G) \cot \phi_G.$$

Both of these relations may be derived from the figure (82a).

If we equate the two values for the prime vertical component we obtain

$$(\alpha_A - \alpha_G) = -(\lambda_A - \lambda_G) \sin \phi_G$$

which is known as the Laplace equation. Triangulation stations at which the astronomic longitude and azimuth have been observed are called Laplace points.

The geodetic and astronomic longitudes in the United States are subject to probable errors of less than $0''.5$. The astronomic azimuths are also determined with about the same accuracy.

The geodetic azimuths, however, as carried through the triangulation, are subject to an error about ten times as great. The triangulation may therefore be greatly strengthened by correcting the geodetic azimuths at Laplace points by means of the above equation.

The manner of correcting the geodetic azimuth is illustrated by the following example, taken from *Supplementary Investigation in 1909 of the Figure of the Earth and Isostasy.*

U. S. Standard longitude of Parkersburg	=	88° 01′ 49″.00
Astronomic " " "	=	88 01 48 .30
$A - G$ in longitude		−0″.70
$A - G$ in azimuth = (−0.70) ($-\sin \phi_0$)	=	+0 .44
Astronomic azimuth Parkersburg to Denver	=	143° 16 15 .55
True geodetic azimuth Parkersburg to Denver	=	143 16 15 .11
U. S. Standard azimuth Parkersburg to Denver	=	143 16 15 .64
Correction to U. S. Standard azimuth	=	−0″.53

140. Isostasy — Isostatic Compensation.

For many years it has been known that the estimated and observed values of the station error are not in even approximate agreement, and it has long been suspected that the explanation would be found in the fact that the densities of the material immediately beneath the surface are unequal, regions of deficient density lying beneath mountain ranges, and regions of excessive density lying beneath low areas and under the ocean bottom. It is supposed that at some depth the excess above the surface is compensated by the defect below the surface, and *vice versa.* This condition is given the name *isostasy.* It appears that the theory was first clearly stated by Major C. E. Dutton in 1889, and since that time it has been the subject of much study.

In 1909 and 1910 there were published by the Coast and Geodetic Survey the results of a very extensive investigation conducted by Professor J. F. Hayford, then Inspector of Geodetic Work and Chief of the Computing Division. The investigation was based primarily upon the computation of the topographic deflections at a large number of astronomical stations in the United States. The best topographic maps available were used for this purpose.

These computed deflections were then compared with the known (observed) deflections at these same stations as found from the triangulation and astronomical observations. In substantially all cases the computed deflection was found to exceed the observed deflection by a large amount, although the two were usually of the same algebraic sign. Computations were then made to test the theory that this condition called isostasy actually exists.

The condition known as isostasy may be stated as follows: the *mass* in any prismatic column which has for its base a unit area of the horizontal surface lying at the depth of compensation, for its edges vertical lines (lines of gravity), and for its upper limit the actual irregular surface of the earth (or the sea surface if the area in question is beneath the ocean), is the same as the mass in any other similar prismatic column having a unit area on the same surface for its base. Such prismatic columns have different heights but the same mass, and their bases are at the same depth below the geoidal (sea-level) surface.

Computations were made assuming different depths of compensation, for the purpose of finding at what depth the computed deflections (taking isostasy into account) most nearly agree with the observed deflection. It was found that the compensation was most nearly complete (more than $\frac{9}{10}$ complete) at a depth of about 122 kilometers, or about 76 miles.

It should be observed that, while the densities in the prismatic columns tend to compensate, the resultant deflection of the plumb line is not zero, for the portions of the column nearest the station have a much greater influence than the distant portions. The tendency is to throw all the zeniths outward from the continental dome, assigning to the surface a curvature which is greater than it should be. Thus, if isostasy is not taken into account, the dimensions of a spheroid computed from such data will be too small. This investigation not only included a determination of the most probable depth of compensation, and a substantial proof of the validity of the theory in so far as it applies to the

United States, but also included a determination of the most probable dimensions of the spheroid for that area. In this calculation the *area method* was employed. The dimensions of the spheroid resulting from this investigation are as follows:

$$a = 6{,}378{,}388^m \pm 18^m,$$
$$b = 6{,}356{,}909^m,$$
$$\frac{1}{f} = 297.0 \pm 0.5.$$

The general conclusions in regard to the existence of isostasy within the limits of the United States were later confirmed by the results of a similar investigation of the compensating effect upon observed values of the force of gravity determined with the pendulum.

The results of these investigations will be found in the following publications of the United States Coast Survey:

John F. Hayford, *The Figure of the Earth and Isostasy from Measurements in the United States*, 1909.

John F. Hayford, *Supplementary Investigations in* 1909 *of the Figure of the Earth and Isostasy*, 1910.

John F. Hayford and William Bowie, *The Effect of Topography and Isostatic Compensation upon the Intensity of Gravity*, Special Publication No. 10, 1912.

William Bowie, *The Effect of Topography and Isostatic Compensation upon the Intensity of Gravity*, Special Publication No. 12, 1912.

William Bowie, *Investigation of Gravity and Isostasy*, Special Publication No. 40, 1917.

PROBLEMS

Problem 1. Compute the dimensions of the spheroid from the following arcs.

Name.	Lat. of middle point.	Amplitude.	Length in feet.
	° ′ ″	° ′ ″	
Peruvian (Delambre's).......	S 1 31 .00	3 07 03.1	1 131 057
English.....................	N 52 35 45	3 57 13.1	1 442 953

Problem 2. Compute the dimensions of the spheroid from the following arcs.

Station.	Latitude.	Distance in feet.
	° ′ ″	
Formentera	- 38 39 53.17	4 509 790.84
Dunkirk	51 02 08.41	
Tarqui	S 3 04 32.07	1 131 036.3
Cotchesqui	N 0 02 31.39	

Problem 3. Lake Superior arc; latitudes, 38° 43′ 17″.22 and 48° 07′ 06″.62; dist., 1,043,974 meters. Peruvian arc; latitudes, −3° 04′ 32″.0, +0° 02′ 31.″4; dist., 344,736.8 meters. Compute a and e^2.

CHAPTER IX

GRAVITY MEASUREMENTS

141. Determination of Earth's Figure by Gravity Observations. The determination of the force of gravity by means of pendulums affords a second means of determining the earth's figure, which is entirely independent of the arc method previously discussed. In this method the force of gravity is measured at points of known latitude and longitude. From the observed variation of gravity with the latitude the polar compression may be computed. Such measurements, therefore, will give the form but not the absolute dimensions of the spheroid.

In the following discussion the term *gravity* (g) will be taken to mean the resultant obtained by combining the force of the earth's attraction due to gravitation and the centrifugal force due to the rotation of the earth.

142. Law of the Pendulum.

The relation between l, the length of a simple pendulum, P, its period of oscillation, and g, the force of gravity is given by the formula

$$P = \pi\sqrt{\frac{l}{g}} \qquad [85]$$

or, more accurately,

$$P = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{h}{8l}\right), \qquad [86]$$

where h is the height through which the point of oscillation falls during a half oscillation.

143. Relative and Absolute Determinations.

Determinations of gravity are of two kinds:

(1) *Absolute determinations*, in which both P and l are measured and from which g may be calculated; and (2) *relative determina-*

tions, in which P is measured at two stations and the ratio of the corresponding values of g at the two places becomes known. If the time of oscillation P of the same pendulum has been observed at two stations, then

$$g_1 = \frac{l\pi^2}{P_1^2}$$

and

$$g_2 = \frac{l\pi^2}{P_2^2};$$

whence

$$\frac{g_1}{g_2} = \frac{P_2^2}{P_1^2}. \qquad [87]$$

Absolute determinations of g are far more difficult than relative determinations, owing to the practical difficulties of measuring the length l with sufficient accuracy.

Relative determinations may be made with very great accuracy, since the time of oscillation may be measured in such a manner that the personal errors of the observer have but little effect on the results.

Most of the pendulum observations for geodetic purposes are now made by the relative method, and all values of g are made to depend upon some one reliable determination of the absolute value. The relative values of g in such a system, however, still remain more accurate than the computed absolute values.

144. Variation of Gravity with the Latitude.

The approximate law governing the variation of gravity with the latitude may be expressed thus:

$$g_\phi = g_e\left(1 + \frac{g_p - g_e}{g_e}\sin^2\phi_0\right), \qquad [88]$$

in which g_0, g_e, and g_p are values of g at latitude ϕ_0, at the equator, and at the pole, respectively. By means of two such equations, one for g_ϕ observed near the equator and one for g_ϕ near the pole, the two unknowns g_e and g_p may be found.

Equation [88] may be derived in a simple manner if we may neglect variations in the attraction at different parts of the surface.* Suppose the earth to be a sphere of radius r, the attraction G having the same value everywhere. Then g_ϕ, the resultant of the attraction G and the centrifugal force c, is found as follows:

At the equator the centrifugal force $= c_e = \omega^2 r$.† At the pole $c_p = 0$.

Also at the equator

$$g_e = G - c_e$$

and at the pole $$g_p = G - c_p = G;$$

whence $$g_p - g_e = c_e.$$

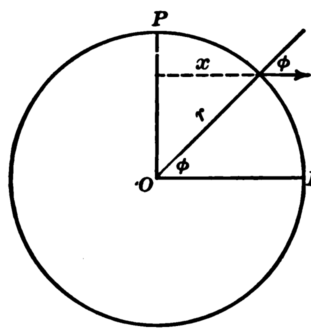


FIG. 83.

In latitude ϕ (Fig. 83) $x = r\cos\phi$ and $c_\phi = \omega^2 r\cos\phi = c_e\cos\phi$. The component of c_ϕ directly opposed to G is $c_e\cos^2\phi$ (vertically upward).

Hence $$g_\phi = G - c_e\cos^2\phi. \qquad [89]$$

* See Jordan's *Handbuch der Vermessungskunde*, Vol. III, p. 627.

† The centrifugal force may be expressed by $\frac{v^2}{r}$, where v is the velocity of a particle at the equator. The distance moved by a particle in one rotation (= 1 sidereal day = T seconds) is $2\pi r$. Hence the centrifugal force $= \left(\frac{2\pi}{T}\right)^2 r = \omega^2 r$, where ω is the angular velocity. T = 86,400 sidereal seconds = 86,164.09 mean solar seconds.

Substituting in [89] the value of G at the equator,

$$\begin{aligned} g_\phi &= g_e + c_e - c_e \cos^2 \phi \\ &= g_e + c_e \sin^2 \phi \\ &= g_e + (g_p - g_e) \sin^2 \phi; \end{aligned}$$

that is,
$$g_\phi = g_e\left(1 + \frac{g_p - g_e}{g_e} \sin^2 \phi\right). \qquad [88]$$

In order to obtain an accurate numerical expression for g_ϕ, of the same general form as the above, we may write

$$g_\phi = g_e (1 + B \sin^2 \phi)$$

and then determine the value of B which is in best agreement with all observed values of g. For such a formula Dr. Helmert * published, in 1884, the equation

$$g_0 = 978.000 (1 + 0.005310 \sin^2 \phi), \qquad [90]$$

in which g_0 is supposed to be the value at sea-level and the unit is dynes of force, or centimeters of acceleration.

This may be expressed for convenience in terms of g_0 at latitude 45°. Since $\sin^2 45° = \frac{1}{2}$,

$$g_{45} = g_e\left(1 + \frac{B}{2}\right);$$

and since
$$\begin{aligned} 2 \sin^2 \phi &= 1 - \cos 2 \phi, \\ \sin^2 \phi &= \tfrac{1}{2} - \tfrac{1}{2} \cos 2 \phi \end{aligned}$$

and
$$\begin{aligned} g_0 &= \frac{g_{45}}{1 + \frac{B}{2}}\left(1 + B\left(\frac{1}{2} - \frac{1}{2} \cos 2 \phi\right)\right) \\ &= g_{45}\left(1 - \frac{\frac{B}{2}}{1 + \frac{B}{2}} \cos 2 \phi\right), \end{aligned}$$

which becomes

$$g_0 = 980.597 (1 - 0.002648 \cos 2 \phi). \qquad [91]$$

* Helmert, *Höhere Geodäsie*, Vol. II, p. 241.

In 1901 Dr. Helmert gave the more accurate forms

$$g_0 = 978.046\,(1 + 0.005302 \sin^2 \phi - 0.000007 \sin^2 2\,\phi) \quad [92]$$

and $$g_0 = 980.632\,(1 - 0.002644 \cos 2\,\phi + 0.000007 \cos^2 2\,\phi), \quad [93]$$

in which the number 0.000007 $(= \frac{1}{4} B_4)$ is a coefficient found theoretically from assumptions regarding the internal structure of the earth.

These formulæ refer to the absolute value of g at Vienna. To refer to the "Potsdam system," to which all values of g observed in the United States are referred,* the equations must be written

$$g_0 = 978.030\,(1 + 0.005302 \sin^2 \phi - 0.000007 \sin^2 2\,\phi) \quad [94]$$

and $$g_0 = 980.616\,(1 - 0.002644 \cos 2\,\phi + 0.000007 \cos^2 2\,\phi). \quad [95]$$

In the Coast Survey Special Publication No. 12, entitled "Effect of Topography and Isostatic Compensation upon the Intensity of Gravity" (second paper) the following formula is given:

$$g_0 = 978.038\,(1 + 0.005302 \sin^2 \phi - 0.000007 \sin^2 2\,\phi), \quad [96]$$

equivalent to

$$g_0 = 980.624\,(1 - 0.002644 \cos 2\,\phi + 0.000007 \cos^2 2\,\phi),$$

which is Helmert's formula of 1901 corrected by 0.008 dyne. The constants in these equations were derived from observations in the United States only.

In Special Publication No. 40, a study is made of observations in the United States, Canada, Europe and India. The formula resulting from this investigation is

$$g_0 = 978.039\,(1 + 0.005294 \sin^2 \phi - 0.000007 \sin^2 \phi), \quad [97]$$

145. Clairaut's Theorem.

The relation between the flattening of the spheroid at the poles

* The American observations for g were referred to Greenwich (England), Paris (France), and Potsdam (Germany) by observations made in 1900 by G. R. Putnam, (see Coast Survey Report for 1901).

and the values of g_p and g_e is expressed by Clairaut's theorem, published in 1743, namely,

$$\frac{a-b}{a} = \frac{5}{2} \cdot \frac{c_e}{g_e} - \frac{g_p - g_e}{g_e}, \tag{98}$$

in which c_e is the centrifugal force at the equator. In this formula the terms of the second order have been omitted. If these terms are included, the formula becomes

$$\frac{a-b}{a} = \frac{5}{2} \cdot \frac{c_e}{g_e} - B - \left(\frac{10}{3}\left(\frac{c_e}{g_e}\right)^2 - \frac{17}{14} \cdot \frac{c_e}{g_e} \cdot B - \frac{B^2}{21} - \frac{2}{21} B_4\right), \tag{98a}$$

in which B and B_4 are coefficients to be determined from the observations (Helmert, *Höhere Geodäsie*, Vol. II, p. 83). It is by means of this equation that the form of the earth is computed from gravity observations.

146. Pendulum Apparatus.

Nearly all of the observations of gravity for geodetic purposes are made with pendulums of invariable length, by the *relative* method. The description of apparatus in the following articles will be limited to one type, the half-seconds *invariable* pendulum apparatus as designed and constructed by the United States Coast Survey. The first half-seconds invariable pendulum with electrical apparatus for determining the period appears to have been devised by Sterneck (Austria) in 1882. In 1890 T. C. Mendenhall, then Superintendent of the Coast and Geodetic Survey, designed an apparatus of this kind but differing in many details, however, from any previous design. This apparatus has been used ever since that time in substantially the same form excepting the addition of the interferometer for determining the flexure. This apparatus includes three half-second pendulums, each about 248^{mm} long and having an agate plane at the point of suspension. The agate plane rests on a knife-edge support (angle of 130°) attached to the pendulum case in which the pendulums are enclosed when they are swung. The use of the blunt angle on the knife edge and the placing of the plane (rather than the

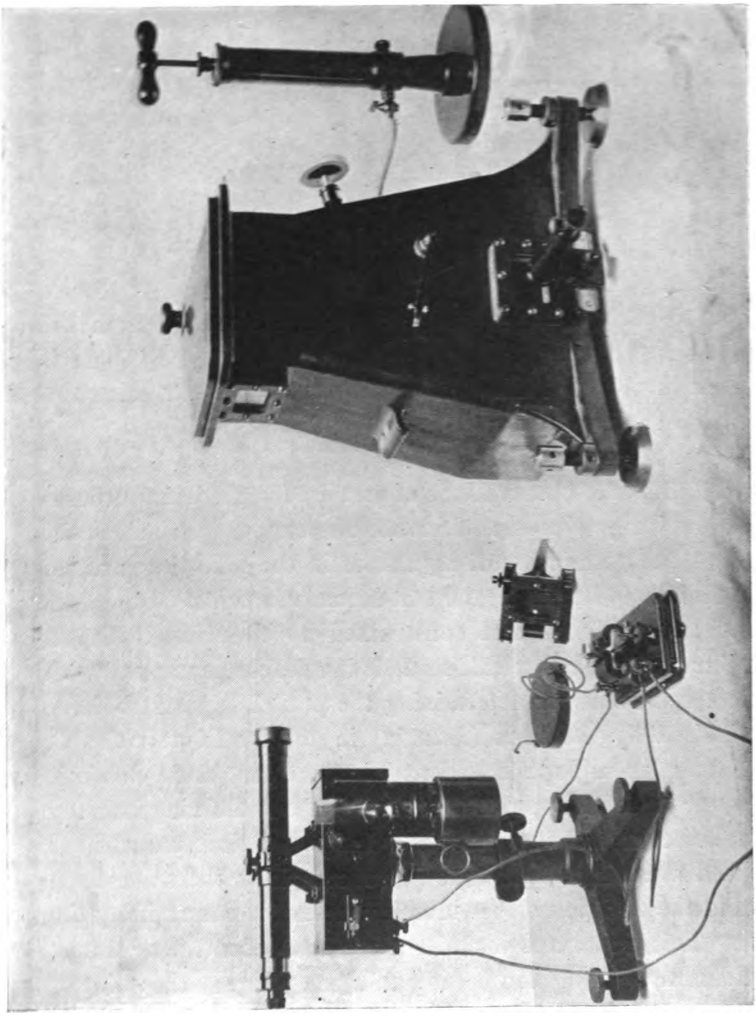


FIG. 84. Pendulum Apparatus.
(C. L. Berger and Sons.)

knife edge) on the pendulum are designed to secure greater permanence of length, upon which the accuracy of the method depends. The pendulums are made of an alloy of copper and aluminum and weigh 1200 grams each. The three are of slightly different lengths so that they will have different periods. Their

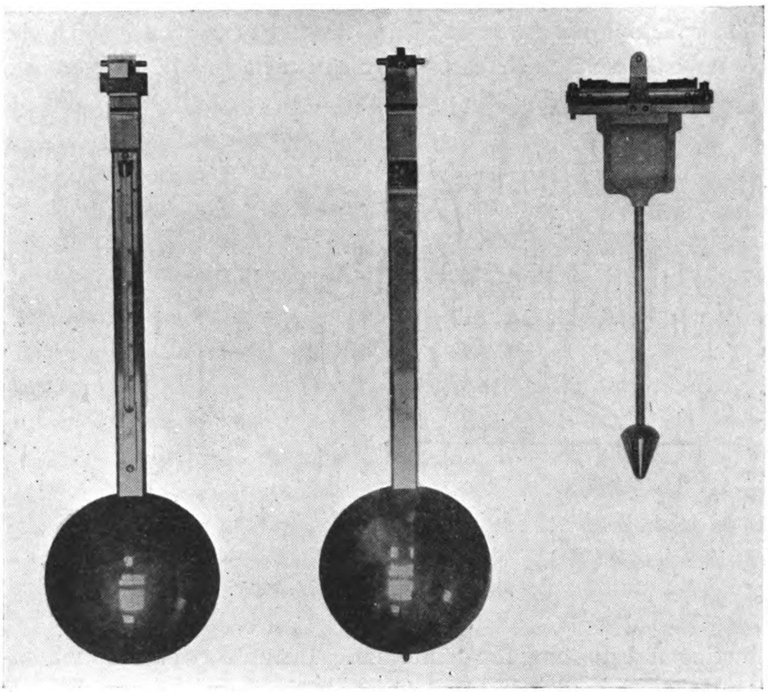


FIG. 85. Dummy Pendulum (with thermometer), Regular Pendulum, and Leveling Pendulum.
(C. L. Berger and Sons.)

form (Fig. 85) is such as to give strength and at the same time offer but little resistance to the air. In addition to the three observing pendulums there is a dummy pendulum, of the same size and shape but carrying a thermometer packed in filings of the same metal. There is also a small pendulum provided with a spirit level for leveling the knife edge. Pendulums made of invar metal are now (1919) being constructed by the instrument division of the Coast and Geodetic Survey so that it will be possible to make gravity observations on mountain peaks and other places where the control of temperature is difficult. The use of this metal will make it unnecessary to construct a "constant temperature room."

The pendulums are swung in an air-tight case from which the air may be nearly exhausted by means of a pump. Levers are provided for lowering the pendulum onto the knife edge and for

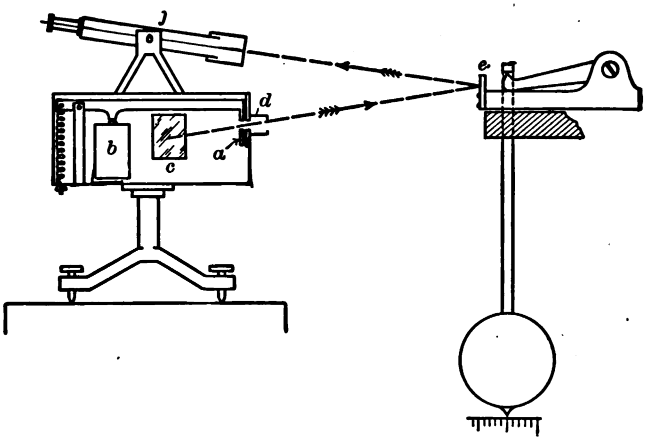


FIG. 86. Flash Apparatus.

starting and stopping the pendulum. Inside the case is a manometer tube for registering the air pressure, and also an additional thermometer. Levels are provided for leveling the case, and there is a graduated scale under the pendulum for reading the arc of oscillation. In the most recent work of the Coast Survey the pendulum receiver has been enclosed in a felt and leather case to prevent fluctuations in temperature.

The observations are made by comparing the times of oscillation of the pendulums with the half-second beats of a break-circuit (sidereal) chronometer connected electrically with the "flash apparatus" used for observing the coincidence.

The flash apparatus (Fig. 86) consists of a shutter *a* operated by the armature of an electromagnet *b* in the circuit and a mirror *c* behind the shutter which reflects light through the slit *d* to two small mirrors *e*, which reflect it into an observing telescope *f*; one of the small mirrors is attached to the pendulum and the other to the knife-edge support. In the most recent form of the flash apparatus, the observer looks down through a vertical telescope and sees the flash reflected by a prism. This arrangement is more convenient for the observer than the older form because the pendulum receiver is usually mounted on a very low support.

When the pendulum is at rest and the shutter open, a beam of light from a lamp* at one side of the apparatus strikes the mirror *c* at an angle of 45° and passes through the slit; it is reflected from both mirrors at *e* and appears to the observer as two horizontal bright slits side by side. The mirrors may be adjusted so that these slits appear to be at the same height, so as to form one continuous band. If the pendulum is set swinging, the reflected image now appears to travel up and down, while the image from the other mirror is stationary. If the shutter is closed and allowed to open only for an instant at the end of each second (or each two seconds), the observer sees that at each successive opening of the shutter the moving image has changed its position relative to the fixed image. This is due to the fact that the period of the pendulum is longer than the sidereal second and the pendulum has made slightly less than one complete (double) oscillation. By watching the flashes and noting the chronometer readings when they coincide, the observer obtains the number of seconds between two successive coincidences. During this interval the pendulum has evidently lost just one oscillation on the (half-second) beats of the chronometer. In the interval between two successive coincidences the pendulum has made one less than twice as many oscillations as

* An electric bulb placed inside the flash box is now used instead of the oil lamp.

the chronometer has beat seconds. During the interval between any two coincidences the number of oscillations is twice the number of seconds (s) less the number of coincidence intervals (n). Hence the time of one oscillation (P) is given by

$$P = \frac{s}{2s - n}. \qquad [99]$$

An examination of this formula will show that an error in noting the times of coincidence produces a relatively small error in P, and for this reason the method is almost independent of the observer's errors.

On account of the variation of g (and consequently of P) with the latitude of the station, it is necessary to use a mean-time chronometer at stations situated near the pole, because the period of the pendulum approaches so closely to the sidereal half-second that the coincidence intervals are inconveniently long. In case a mean-time chronometer is used, the formula becomes

$$P = \frac{s}{2s + n}. \qquad [100]$$

147. Apparatus for Determining Flexure of Support.

Observations with pendulums mounted on a very flexible support show plainly that when a pendulum is set swinging, it communicates motion to the case and the support and sets them oscillating, and this oscillation in turn affects the observed period of the pendulum. The apparatus now used to measure the effect of this flexure is one which operates on the principle of the interferometer.* This is an optical device (Fig. 87) consisting of a lamp and lens arranged so as to furnish a beam of sodium light; a glass plate arranged so as to separate the beam of light into two parts, one of which is transmitted, the other reflected; two mirrors, one in the path of each beam of light; and a telescope for observing the image. When the different parts of the appara-

* A description of the interferometer will be found in the Coast Survey Report for 1910.

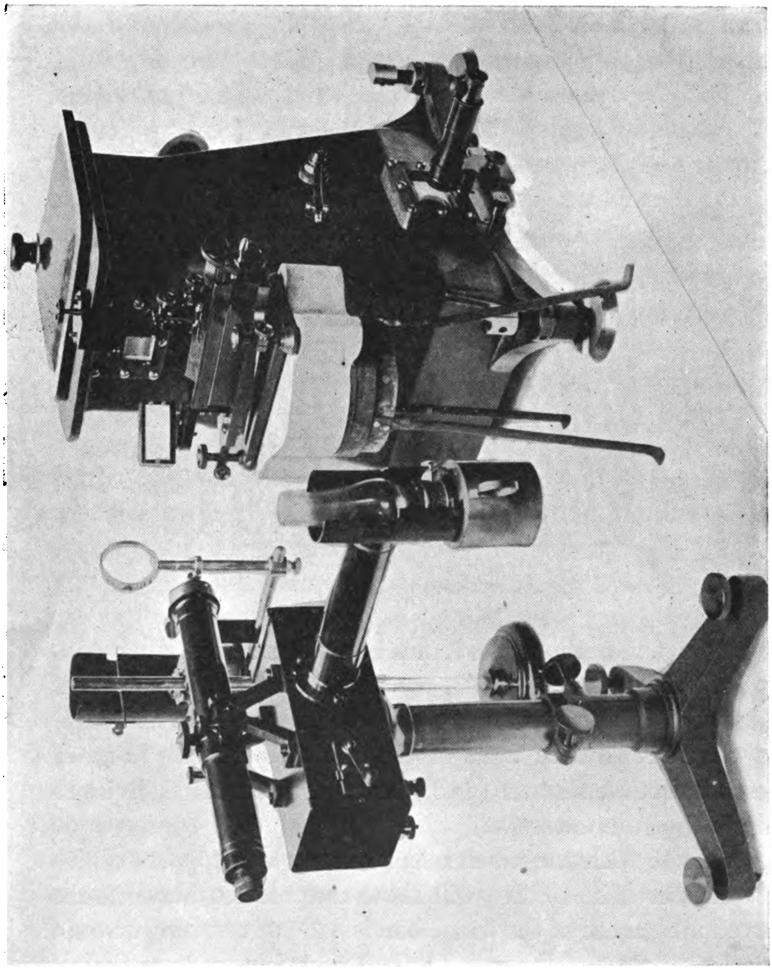


FIG. 87. Interferometer (showing relative position). During flexure observations the interferometer is on an independent support.
(C. L. Berger and Sons.)

tus are properly adjusted, dark and light bands will appear in the field of the telescope, owing to interference of the sodium-light waves of the two beams. One of the mirrors is mounted on the pendulum receiver, while the rest of the apparatus is on an independent support in front of it. When the pendulum is set swinging, it sets the case in motion, and this in turn moves the mirror, causing a slight variation in the length of the path of one of the beams of light. This causes the interference bands to shift back and forth; the amount of shift may be estimated by observing the motion of the bands over a cross-hair or a scale in the field of the telescope. It is usually observed by noting the

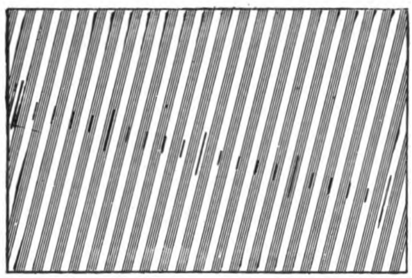


FIG. 88.

scale readings of both edges of some band in each of its two positions (before and after shifting). The movement of the edges of a band divided by the width of the band (in scale divisions) gives the movement in units of the width of a band. Fig. 88 represents the interference (dark) bands and the scale divisions in the field of the telescope.

Tests made with the pendulum mounted on supports of different degrees of flexibility will show the relation between the observed movement of the fringe bands and the resulting error in the period of the pendulum. In the Coast Survey tests the results showed that a movement equal to the width of one band produced a change of 173 in P in units of the seventh decimal place. This is more conveniently expressed as follows: 0.01 F

produces a change of 1.73 in P, where F is the width of a band. This constant was determined with the pendulum swinging through an arc of 5^{mm} on the scale, and all observed flexures must be reduced to this arc before correcting P.

148. Methods of Observing.

The receiver should be mounted on a solid support such as a cement or brick pier, the foot screws cemented to the pier, and the instrument sheltered as in case of astronomical observations. It is important that the instrument should be so sheltered that the temperature will not fluctuate rapidly. The apparatus should be leveled by means of the spirit level on the outside of the case and then the knife edge should be leveled by means of the leveling pendulum. In moving the pendulums great care should be used to protect them from injury and to prevent any foreign matter from adhering to them. The accuracy of the results will depend upon the permanency of length, and any injury due to fall, or change of period due to change in the mass, will affect the period and vitiate the results. The pendulums should not be touched with the hands, but should be lifted by means of a special hook made for this purpose. The flash apparatus, chronometer, and interferometer should be placed upon supports that are entirely independent of the pendulum support.

Various programs of observing have been tried, but the following has been chiefly used by observers of the Coast Survey. Each of the three pendulums is swung first in the direct and then in the reversed position, making six swings each of eight hours' duration. The error of the chronometer is obtained by star-transit observations (Arts. 52–71) made just before the beginning and at the end of the series. The following table will indicate more clearly the order of operations.

Star Observations	9–10 P.M.
Start Pendulum No. 1	10 P.M.
Reverse No. 1	6 A.M.
Start No. 2	2 P.M.
Reverse No. 2	10 P.M.
Start No. 3	6 A.M.
Reverse No. 3	2 P.M.
Star Observations	9 P.M.
Stop Pendulum No. 3	after star observations

If star observations are lost at the end of the set, the swings are continued until star observations are obtained. At the beginning and end of each swing several coincidences are observed. At the end of each swing several more are observed. Very little time is lost between swings, so that they are almost continuous between star observations. For this reason the variations in the rate of the chronometer are almost entirely eliminated from the mean result of all the swings.

Since 1913 the Coast Survey observers have obtained the chronometer corrections from the Naval Observatory time signals instead of by direct observations. This results in a great saving of time and cost. Another change in the regular program, recently introduced, is to swing the pendulums for twelve hours instead of eight, and in the direct position only, instead of direct and reversed.

After a pendulum is placed in position on its support, the case closed, and the air exhausted until the pressure is about 60^{mm}, the observer lowers the pendulum until it rests upon the knife edge, starts it swinging through an arc of about $0° 53'$, and notes the arc on the scale. To observe coincidences, the observer switches in the chronometer and the flash apparatus and then watches the flashes to see when they are approaching coincidence. As the two approach he notes the hours, minutes, and seconds on the chronometer when the advancing edge of the moving flash touches the first edge of the fixed flash. A few seconds later he notes when the receding edge of the moving flash touches the second edge of the fixed flash. The mean of the two gives the true time of coincidence of centers more accurately than it could be observed directly. Such observations are made on several successive coincidences, the flash moving alternately upward and downward. By combining the *up* and the *down* observations, errors of adjustment are eliminated. After a few of these have been recorded, the observer cuts out the chronometer and leaves the pendulum swinging for a period of nearly eight hours. Immediately after the observations for coincidences are completed, the temperatures are read on the two thermometers, and the pressure is read on the manometer tube. At the end of the eight-hour period the observer again observes a few coincidences as well as the arc (now diminished to about $0° 20'$), the pressure, and the temperatures. It is not necessary that he continue observing throughout the whole eight-hour period, because the few observations already referred to make it possible to estimate correctly the number of coincidences which must have occurred between the observed times. It is customary to take the observations with two or more chronometers as a check.

This description applies to the 8-hour program outlined above. If the pendulums are swung for a 12-hour period it is necessary to start each pendulum with a somewhat larger arc ($1°27'$) in order that it may have a sufficient amplitude at the end of 12 hours to enable the observer to read the coincidences of the flash conveniently and accurately.

It is desirable that the temperature of the apparatus be kept as nearly uniform as possible, and that there be little vibration. In order to allow the pendulum time to assume the temperature of the receiver the next pendulum to be swung is placed inside the case before it is used in the observations. While the case is still in position the observer must place the interferometer in position and observe the movement of the interference bands while the pendulum is swinging.

149. Calculation of Period.

After the observations are complete and the time observations and the chronometer rates are computed, the time of one oscillation for each pendulum in each position is found as follows: divide the total number of seconds in an 8^h interval by the number of seconds found for one coincidence interval (see example), to obtain the number of intervals that have occurred during the swing. Since this must be a whole number, there will be no difficulty in determining it correctly. Then reverse the process, dividing the total interval by the number of coincidence intervals, to obtain the accurate value of the number of seconds (s) in one coincidence interval. The uncorrected period of the pendulum is found by

$$P = \frac{s}{2s - 1} \qquad [101]$$

for a sidereal chronometer, Table G, or

$$P = \frac{s}{2s + 1} \qquad [102]$$

for a mean-time chronometer.

150. Corrections.

This period must then be corrected to reduce it to its value at assumed standard conditions, namely,

Infinitesimal arc,
Temperature 15° C.,
Pressure 60^{mm} at 0° C.,
True sidereal time, and
Inflexible support.

The correction to reduce P to its value for an infinitesimal arc is

$$-\frac{PM}{32} \cdot \frac{\sin(\phi + \phi')\sin(\phi - \phi')}{\log\sin\phi - \log\sin\phi'}; \qquad [103]$$

a formula given by Borda, in which P = the period, M = the modulus of the common system of logarithms, and ϕ and ϕ' = the initial and final arcs.

The temperature correction is

$$\alpha P(15° - T°), \qquad [104]$$

$T°$ being the observed temperature centigrade and α the coefficient to be found by trial. (α = +0.000 008 34).

The pressure correction is

$$K\left[60^{mm} - \frac{Pr}{1 + 0.00367\,T°}\right], \qquad [105]$$

in which Pr = observed pressure in *mm*,
$T°$ = temperature centigrade,
and K = coefficient to be found by trial.

The constant 0.00367 is the coefficient of expansion of air for 1° C.

The rate correction is given by the expression

$$+ 0.000011574\, RP, \qquad [106]$$

where R = daily rate of chronometer on sidereal time, + when losing and − when gaining. The coefficient is the reciprocal of the number of seconds in one day.

The flexure correction is computed by dividing the observed movement of the fringe band (in scale divisions) by the width of a band and then reducing this to an arc of 5^{mm} by dividing by the observed arc and multiplying by 5. The result is the displacement for a 5^{mm} arc in terms of the width of a band. This displacement, multiplied by the coefficient (173 mentioned before), gives the correction to be subtracted from P.

TABLE D. — REDUCTION OF SCALE READING IN MILLIMETERS TO MINUTES OF ARC

Scale.	1.0 mm.	2.0 mm.	3.0 mm.	4.0 mm.	5.0 mm.
mm.	′	′	′	′	′
0.0	12	23	35	46	58
0.1	13	24	36	48	59
0.2	14	26	37	49	60
0.3	15	27	38	50	61
0.4	16	28	39	51	63
0.5	17	29	41	52	64
0.6	19	30	42	53	65
0.7	20	31	43	55	66
0.8	21	32	44	56	67
0.9	22	34	45	57	68

TABLE E. ARC CORRECTIONS (ALWAYS SUBTRACTIVE) FOR HALF-SECOND PENDULUMS

Arc at Beginning

Arc at end.	90′.	85′.	80′.	75′.	70′.	65′.	60′.	55′.	50′.	45′.	40′.	35′.	30′.	25′.	20′.
′															
5															
10	12.0	11.0	10.0	9.0	8.1	7.3	6.5	5.8	5.0	4.3	3.6	3.0	2.4	1.9	1.4
15	14.4	13.3	12.2	11.1	10.0	9.0	8.0	7.2	6.3	5.4	4.6	3.9	3.2		
20	16.9	15.6	14.3	13.0	11.8	10.7	9.6	8.6	7.6	6.6	5.7	4.9	4.1		
25	19.3	17.8	16.4	15.0	13.7	12.4	11.2	10.1	9.0	8.0	6.9				
30	21.7	20.1	18.5	17.0	15.6	14.2	12.9	11.6	10.4	9.2	8.1				
35	24.1	22.4	20.7	19.2	17.6	16.1	14.6	13.2	11.8						
40	26.5	24.7	22.9	21.2	19.5	17.9	16.3	14.8	13.3						
45	29.0	27.1	25.2	23.4	21.6	19.9	18.2								
50	31.5	29.4	27.4	25.5	23.6	21.8	20.0								
55	34.1	32.0	29.8	27.8	25.8										
60	36.7	34.4	32.2	30.0	27.9										
65	39.4	37.0	34.6												
70	42.1	39.6	37.1												
75	44.9														
80	47.7														
85															
90															

In practice it is convenient to combine Tables D and E into a single table computed for such intervals that little interpolation is necessary.

TABLE F. — CORRECTION FOR PRESSURE

Temp. C.	50 mm.	55 mm.	60 mm.	65 mm.	70 mm.	75 mm.	80 mm.	85 mm.	90 mm.
0	+10	+5	0	−5	−10	−15	−20	−25	−30
1	+10	+5	0	−5	−10	−15	−20	−25	−30
2	+10	+5	0	−5	− 9	−14	−19	−24	29
3	11	6	+1	4	9	14	19	24	29
4	11	6	+1	4	9	14	19	24	29
5	11	6	+1	4	9	14	19	24	28
6	11	6	+1	4	9	14	19	24	28
7	11	6	2	3	8	13	18	23	28
8	11	6	2	3	8	13	18	23	27
9	12	7	2	3	8	13	17	22	27
10	12	7	2	3	8	13	17	22	27
11	12	7	2	3	7	12	17	21	26
12	12	7	2	2	7	12	17	21	26
13	12	7	3	2	7	12	17	21	26
14	12	8	3	2	7	11	16	21	26
15	13	8	3	2	6	11	16	20	26
16	13	8	3	2	6	11	16	20	25
17	13	8	4	1	6	11	15	20	25
18	13	8	4	1	6	10	15	20	24
19	13	9	4	−1	5	10	15	20	24
20	13	9	4	−1	5	10	15	20	24
21	14	9	4	−1	5	10	14	19	24
22	14	9	4	−1	5	10	14	19	23
23	14	9	5	0	5	9	14	19	23
24	14	9	5	0	4	9	14	18	23
25	14	10	5	0	4	9	13	18	22
26	14	10	5	+1	4	9	13	18	22
27	14	10	5	+1	4	8	13	17	22
28	+15	+10	+6	+1	− 4	− 8	−13	−17	22
29	+15	+10	+6	+1	− 3	−18	−12	−17	−21
30	+15	+10	+6	+1	− 3	− 8	−12	−17	−21

Body of table gives corrections (in 7th decimal place of seconds) to period of half seconds pendulum.

TABLE G. — PERIODS OF QUARTER METER PENDULUM

Note: To obtain period to 7th decimal place, prefix .50 or .500 to figures in the table. Body of table gives

0	2200	2300	2400	2500	2600	2700	2800	2900	3000	3100
0	11,390	10,893	10,438	10,020	9634	9276	8944	8636	8347	8078
1	84	89	34	16	30	73	41	33	44	75
2	79	84	30	12	26	70	38	30	42	72
3	74	79	25	08	23	66	35	27	39	70
4	69	74	21	04	19	63	32	24	36	67
5	11,364	10,870	10,417	10,000	9615	9259	8929	8621	8333	8064
6	58	65	12	9996	12	56	25	18	30	62
7	53	60	08	92	08	52	22	15	28	59
8	48	55	04	88	04	49	19	12	25	57
9	43	51	10,399	84	01	46	16	09	22	54
10	11,338	10,846	10,395	9980	9597	9242	8913	8606	8320	8052
11	33	41	91	76	93	39	10	03	17	49
12	28	37	86	72	90	35	06	00	14	46
13	22	32	82	68	86	32	03	8597	11	44
14	17	27	78	64	82	28	00	94	08	41
15	11,312	10,822	10,373	9960	9578	9225	8897	8591	8306	8039
16	07	18	69	56	75	22	94	88	03	36
17	02	13	65	52	71	18	91	85	00	33
18	11,297	08	61	48	68	15	87	82	8297	31
19	92	04	56	44	64	12	84	79	95	28
20	11,287	10,799	10,352	9940	9560	9208	8881	8576	8292	8026
21	82	94	48	36	57	05	78	73	89	23
22	76	90	43	32	53	01	75	70	86	20
23	72	85	39	28	49	9198	72	68	84	18
24	66	80	35	25	46	95	68	64	81	15
25	11,261	10,776	10,331	9921	9542	9191	8865	8562	8278	8013
26	56	71	26	17	38	88	62	59	75	10
27	51	67	22	13	35	84	59	56	73	08
28	46	62	18	09	31	81	56	53	70	05
29	41	57	14	05	27	78	53	50	67	03
30	11,236	10,753	10,309	9901	9524	9174	8850	8547	8264	8000
31	31	48	05	9897	20	71	46	44	62	7997
32	26	44	01	93	17	68	43	41	59	95
33	21	39	10,297	89	13	64	40	38	56	92
34	16	34	92	85	09	61	37	35	53	90
35	11,211	10,730	10,288	9881	9506	9158	8834	8532	8251	7987

WHEN PENDULUM IS SLOWER THAN CHRONOMETER

Top and left-hand arguments combined give interval s = ten coincidence intervals.
t = period in seconds.

3200	3300	3400	3500	3600	3700	3800	3900	4000	4100	4200	0
7825	7587	7364	7153	6954	6766	6588	6418	6258	6105	5960	0
22	85	62	51	52	64	86	17	56	04	58	1
20	83	59	49	50	62	84	15	55	02	57	2
17	80	57	47	48	60	82	14	53	01	55	3
15	78	55	45	46	59	81	12	52	6099	54	4
7812	7576	7353	7143	6944	6757	6579	6410	6250	6098	5952	5
10	74	51	41	42	55	77	09	48	96	51	6
08	71	49	39	41	53	76	07	47	95	50	7
05	69	46	37	39	51	74	05	45	93	48	8
03	67	44	35	37	49	72	04	44	92	47	9
7800	7564	7342	7133	6935	6748	6570	6402	6242	6090	5945	10
7798	62	40	31	33	46	69	00	41	89	44	11
96	60	38	29	31	44	67	6399	39	87	42	12
93	58	36	27	29	42	65	97	38	86	41	13
91	55	34	25	27	40	63	96	36	84	40	14
7788	7553	7331	7123	6925	6738	6562	6394	6234	6083	5938	15
86	51	29	21	23	37	60	92	33	81	37	16
83	48	27	18	21	35	58	91	31	80	35	17
81	46	25	16	19	33	56	89	30	78	34	18
78	44	23	14	18	31	55	87	28	77	33	19
7776	7542	7321	7112	6916	6730	6553	6386	6227	6075	5931	20
74	39	19	10	14	28	51	84	25	74	30	21
71	37	16	08	12	26	50	82	24	72	28	22
69	35	14	06	10	24	48	81	22	71	27	23
66	32	12	04	08	22	46	79	20	70	26	24
7764	7530	7310	7102	6906	6720	6544	6378	6219	6068	5924	25
62	28	08	00	04	19	43	76	17	66	23	26
59	26	06	7098	02	17	41	74	16	65	21	27
57	23	04	96	00	15	39	73	14	64	20	28
7754	7521	01	94	6898	13	38	71	13	62	19	29
7752	7519	7299	7092	6897	6711	6536	6369	6211	6061	5917	30
50	16	97	90	95	10	34	68	10	59	16	31
47	14	95	88	93	08	32	66	08	58	14	32
45	12	93	86	91	06	31	64	07	56	13	33
42	10	91	84	89	04	29	63	05	55	12	34
7740	7508	7289	7082	6887	6702	6527	6361	6204	6053	5910	35

TABLE G (*Con.*).—PERIODS OF QUARTER METER PENDU-

Note: To obtain period to 7th decimal place, prefix .50 or .500 to figures in the table.
Body of table gives

0	2200	2300	2400	2500	2600	2700	2800	2900	3000	3100
36	06	25	84	78	02	54	31	30	48	85
37	01	20	80	74	9498	51	28	27	45	82
38	11,196	16	75	70	95	48	24	24	43	80
39	91	11	71	66	91	44	21	21	40	77
40	11,186	10,707	10,267	9862	9488	9141	8818	8518	8237	7974
41	81	02	63	58	84	38	15	15	34	72
42	76	10,698	58	54	81	34	12	12	32	69
43	71	93	54	50	77	31	09	09	29	67
44	66	88	50	46	73	27	06	06	26	64
45	11,161	10,684	10,246	9842	9470	9124	8803	8503	8224	7962
46	56	79	42	39	66	21	00	00	21	59
47	51	75	38	35	62	18	8797	8498	18	57
48	46	70	33	31	59	14	94	95	16	54
49	41	66	29	27	55	11	90	92	13	52
50	11,136	10,661	10,225	9823	9452	9108	8787	8489	8210	7949
51	31	56	21	19	48	04	84	86	08	47
52	26	52	17	16	45	01	81	83	05	44
53	21	47	12	12	41	9098	78	80	02	42
54	16	43	08	08	38	94	75	78	8199	39
55	11,111	10,638	10,204	9804	9434	9091	8772	8475	8197	7936
56	06	34	10,200	9800	30	88	69	72	94	34
57	01	29	10,196	9796	27	84	66	69	91	32
58	11,096	25	92	92	23	81	63	66	89	29
59	91	20	88	88	20	78	60	63	86	26
60	11,086	10,616	10,183	9785	9416	9074	8757	8460	8183	7924
61	82	11	79	81	13	71	54	57	81	21
62	77	07	75	77	09	68	51	54	78	19
63	72	02	71	73	06	65	47	52	75	16
64	67	10,598	67	69	02	61	44	49	73	14
65	11,062	10,593	10,163	9766	9398	9058	8741	8446	8170	7911
66	57	89	59	62	95	55	38	43	67	09
67	52	84	54	58	92	51	35	40	65	06
68	47	80	50	54	88	48	32	37	62	04
69	42	75	46	50	84	45	29	34	59	01
70	11,038	10,571	10,142	9747	9381	9042	8726	8432	8157	7899

LUM WHEN PENDULUM IS SLOWER THAN CHRONOMETER

Top and left-hand arguments combined give interval s = ten coincidence intervals.
t = period in seconds.

3200	3300	3400	3500	3600	3700	3800	3900	4000	4100	4200	0
38	05	86	80	85	01	26	60	02	52	09	36
35	03	84	78	83	6699	24	58	00	50	07	37
33	01	82	76	81	97	22	56	6199	49	06	38
30	7498	80	74	80	95	21	55	97	47	05	39
7728	7496	7278	7072	6878	6693	6519	6353	6196	6046	5903	40
26	94	76	70	76	92	17	52	94	44	02	41
23	92	74	68	74	90	16	50	93	43	00	42
21	90	72	66	72	88	14	48	91	42	5899	43
18	87	70	64	70	86	12	47	90	40	98	44
7716	7485	7267	7062	6868	6684	6510	6345	6188	6039	5896	45
14	83	65	60	66	83	09	44	87	37	95	46
11	80	63	58	64	81	07	42	85	36	93	47
09	78	61	56	62	79	05	40	84	34	92	48
06	76	59	54	61	77	04	39	82	33	91	49
7704	7474	7257	7052	6859	6676	6502	6337	6180	6031	5889	50
02	72	55	50	57	74	00	36	79	30	88	51
7699	69	53	48	55	72	6499	34	77	28	86	52
97	67	51	46	53	70	97	32	76	27	85	53
95	65	48	44	51	69	95	31	74	26	84	54
7692	7463	7246	7042	6849	6667	6494	6329	6173	6024	5882	55
90	60	44	40	47	65	92	28	71	23	81	56
88	58	42	38	46	63	90	26	70	21	80	57
85	56	40	36	44	61	88	24	68	20	78	58
83	54	38	34	42	60	87	23	67	18	77	59
7680	7452	7236	7032	6840	6658	6485	6321	6165	6017	5875	60
78	49	34	30	38	56	83	20	64	15	74	61
76	47	32	28	36	54	82	18	62	14	73	62
73	45	30	26	34	52	80	16	61	12	71	63
71	43	28	24	32	51	78	15	59	11	70	64
7669	7440	7225	7022	6831	6649	6477	6313	6158	6010	5868	65
66	38	23	20	29	47	75	12	56	08	67	66
64	36	21	18	27	45	73	10	55	07	66	67
62	34	19	17	25	44	72	08	53	05	64	68
59	32	17	15	23	42	70	07	52	04	63	69
7657	7429	7215	7013	6821	6640	6468	6305	6150	6002	5862	70

TABLE G (*Con.*). — PERIODS OF QUARTER METER PENDU-

NOTE: To obtain period to 7th decimal place, prefix .50 or .500 to figures in the table.

Body of table gives

0	2200	2300	2400	2500	2600	2700	2800	2900	3000	3100
71	33	66	38	43	77	38	23	29	54	96
72	28	62	34	39	74	35	20	26	51	94
73	23	57	30	35	70	32	17	23	49	91
74	18	53	26	31	67	29	14	20	46	89
75	11,013	10,548	10,122	9728	9363	9025	8711	8418	8143	7886
76	08	44	17	24	60	22	08	15	41	84
77	04	40	13	20	56	19	05	12	38	81
78	10,999	35	09	16	53	16	02	09	35	79
79	94	31	05	12	49	12	8699	06	33	76
80	10,989	10,526	10,101	9709	9346	9009	8696	8403	8130	7874
81	84	22	10,097	05	42	06	93	01	28	72
82	79	18	93	01	39	02	90	8398	25	69
83	74	13	89	9697	35	8999	87	95	22	67
84	70	09	85	94	32	96	84	92	20	64
85	10,965	10,504	10,081	9690	9328	8993	8681	8389	8117	7862
86	60	10,500	77	86	25	90	78	86	14	59
87	55	10,495	73	82	21	86	75	84	12	57
88	51	91	68	79	18	83	72	81	09	54
89	46	87	64	75	14	80	69	78	06	52
90	10,941	10,482	10,060	9671	9311	8977	8665	8375	8104	7849
91	36	78	56	68	08	74	62	72	01	47
92	31	73	52	64	04	70	60	70	8098	44
93	27	69	48	60	01	67	56	67	96	42
94	22	65	44	56	9297	64	54	64	93	40
95	10,917	10,460	10,040	9653	9294	8961	8650	8361	8091	7837
96	12	56	36	49	90	57	48	58	88	34
97	08	52	32	45	87	54	44	56	85	32
98	03	47	28	41	83	51	42	53	83	30
99	10,898	43	24	38	80	48	39	50	80	27
100	10,893	10,438	10,020	9634	9276	8944	8636	8347	8078	7825

LUM WHEN PENDULUM IS SLOWER THAN CHRONOMETER

Top and left-hand arguments combined give interval s = ten coincidence intervals.
t = period in seconds.

3200	3300	3400	3500	3600	3700	3800	3900	4000	4100	4200	0
55	27	13	11	19	38	67	04	49	01	60	71
52	25	11	09	18	37	65	02	47	00	59	72
50	23	09	07	16	35	63	00	46	5998	58	73
48	21	07	05	14	33	61	6299	44	97	56	74
7645	7418	7205	7003	6812	6631	6460	6297	6142	5995	5855	75
43	16	02	01	10	30	58	96	41	94	53	76
41	14	00	6999	08	28	57	94	40	92	52	77
38	12	7198	97	06	26	55	92	38	91	51	78
36	10	96	95	05	24	53	91	36	89	49	79
7634	7407	7194	6993	6803	6622	6452	6289	6135	5988	5848	80
31	05	92	91	01	21	50	88	34	87	47	81
29	03	90	89	6799	19	48	86	32	85	45	82
27	01	88	87	97	17	47	85	30	84	44	83
24	7399	86	85	95	16	45	83	29	82	42	84
7622	7396	7184	6983	6794	6614	6443	6281	6128	5981	5841	85
20	94	82	81	92	12	42	80	26	80	40	86
17	92	80	79	90	10	40	78	24	78	38	87
15	90	78	77	88	09	38	77	23	77	37	88
13	88	76	75	86	07	37	75	22	75	36	89
7610	7386	7174	6974	6784	6605	6435	6274	6120	5974	5834	90
08	83	72	72	82	03	33	72	18	72	33	91
06	81	70	70	81	02	32	70	17	71	32	92
03	79	67	68	79	6600	30	69	16	69	30	93
01	77	65	66	77	6598	28	67	14	68	29	94
7599	7375	7163	6964	6775	6596	6427	6266	6112	5967	5828	95
96	72	61	62	73	95	25	64	11	65	26	96
94	70	59	60	71	93	23	62	10	64	25	97
92	68	57	58	70	91	22	61	08	62	23	98
90	66	55	56	68	89	20	59	06	61	22	99
7587	7364	7153	6954	6766	6588	6418	6258	6105	5960	5821	100

151. Form of Record of Pendulum Observations.

Following is a specimen record of a single swing made with "Apparatus B," belonging to the Coast Survey.

Station: Sawah Loento, Sumatra. Date: May 7, 1901.
Observer: G. L. H. Chronometer: Bond 541 (sid.)

Pendulum *B* 4, Direct, on Knife edge *I*

Observed coincidences.	Pressure.	Temperature.	Arc.
h *m* *s*	*mm.*	(*C*).	*mm.*
D 9 59 03			
U 10 02 12	27.5		
D 05 11	27.5	22°.6	4.5 = 52′
U 08 18	55.0		
D 11 12			
U 14 19			
D 4 54 42			
U 58 12	28.0		
D 5 00 43	28.0	28.8	0.9 = 10′
U 04 08	56.0		
D 06 42			
U 10 06			

55.5			25.70
4.2		Ther. error	−.30
51.3 at 0° C.			25°.40

Total interval (mean) $6^h\ 55^m\ 43^s = 24{,}943^s$.
Approximate length of coincidence interval $= 3^m\ 01^s = 181^s$.
Number of coincidence intervals = 138.
Length of one coincidence interval = 180.75.
Period (uncorrected) = 0.5013869.

Uncorrected Period	0.5013869
Corr. for Arc	−5
" " Temp.	−436
" " Press.	+9
" " Rate (No. 541)	+128
" " Flexure	−6
Corrected Period =	0.5013559

152. Calculation of g.

After the period has been corrected for instrumental errors, the value of gravity (g) may be found by comparing the period (P) with that of the same pendulum at some point where the value of g is known, say at Washington. If the value at Washington is g_w, then

$$g = \frac{P^2_w}{P^2} \cdot g_w. \qquad [107]$$

Evidently it is of the greatest importance that the period should not change during a series of observations made for the purpose of comparing P at different stations. The pendulum should be swung at frequent intervals at the base station, to test its invariability; in any case it should be swung at the beginning and end of every series.

Example. Suppose that the mean corrected period of a set of pendulums at a station is 0.5012480, and at Washington, the base station, is 0.5007248, and that g_w is taken as 980.111 dynes. Then, by formula [107], g = 978.066 dynes.

153. Reduction to Sea-Level.

The value of gravity found in the manner just described is the value at the station, assuming the length of the pendulum to be invariable and the chronometer correction to be correct. In comparing values at different stations, however, it is essential to reduce the observed value to the value at sea-level. A formula long used for this purpose is one devised by Bouguer when reducing observations made along the Peruvian arc in 1749. This formula is

$$dg = + \frac{2gH}{r}\left(1 - \frac{3}{4} \cdot \frac{\delta}{\Delta}\right), \qquad [108]$$

in which H is the elevation of the station above sea-level,
r is the radius of the earth,
δ is the density at the surface,
and Δ is the mean density of the earth.

The first term of this formula allows for the decrease in gravity due to height alone; the second term, for the increase in attraction due to the topography beneath the station.

The correction for height of station is derived from the law of gravitation, namely that the force of attraction varies inversely as the square of the distance; whence

$$\frac{g_0}{g} = \frac{(r+H)^2}{r^2} = \left(1 + \frac{H}{r}\right)^2 = \left(1 + \frac{2H}{r} \ldots\right).$$

Therefore $$g_0 = g\left(1 + \frac{2H}{r}\right). \qquad [109]$$

The correction for topography is based upon the assumption that it is due to the attraction of a cylinder whose axis is vertical and whose height is small compared with its width. The attraction on a unit mass at the station is shown by Helmert (*Höhe. Geodäsie*, Vol. II, pp. 142 and 164) to be

$$\Delta g = 2\pi k\delta H. \qquad (a)$$

The attraction of the sphere on the same mass is

$$g = k\frac{M}{r^2} = k \times \frac{4}{3}\pi r\Delta. \qquad (b)$$

Dividing (*a*) by (*b*) and multiplying by *g*,

$$\Delta g = g \cdot \frac{3}{2} \cdot \frac{\delta}{\Delta} \cdot \frac{H}{r}. \qquad [110]$$

Adding both corrections ([109] and [110]) and remembering that the two are of opposite sign,

$$g_0 = g + g\frac{2H}{r} - g\frac{3}{2}\cdot\frac{\delta}{\Delta}\cdot\frac{H}{r}$$

$$= g\left(1 + \frac{2H}{r}\left(1 - \frac{3}{4}\cdot\frac{\delta}{\Delta}\right)\right).* \qquad [111]$$

Another method of reduction which has been much used is to omit the last term of Bouguer's formula, and correcting for height only. In this case the correction to *g* is

$$\text{Corr.} = +\frac{2H}{r}g, \qquad [112]$$

or $$\text{Corr.} = +0.0003086\, H \text{ (meters)}. \qquad [112a]$$

* See also Clarke, *Geodesy*, p. 325. For an additional term for irregularity in topography see Coast Survey Report for 1894, p. 22.

This method was introduced because the former method showed large disagreement between observed and computed values. The second, or "free-air," method showed better agreements, indicating a compensation due to variations of density beneath.

The method employed by Professor Hayford in the Coast Survey investigation shows that still better agreement is obtained by the introduction of the assumption of isostasy. The results corrected by this method show a close general agreement, but in certain localities there is evidence that the isostatic adjustment is imperfect — for example, near Seattle in the United States and at certain places near the Himalayas in India.

154. Calculation of the Compression.

By employing a large number of observed values of g the most probable values of the constants g_e and g_p may be found. From these data the compression may be derived by applying Clairaut's formula,

$$\frac{a-b}{a} = \frac{5}{2} \cdot \frac{c_e}{g_e} - \frac{g_p - g_e}{g_e}. \qquad [98]$$

The value of c_e is $\left(\frac{2\pi}{T}\right)^2 \cdot a$, where $T = 86164.09$ seconds and a is the equatorial radius. Using Clarke's value of a, the resulting value of c_e is found to be

$$c_e = 0.033916,$$

and using for g_e the value 978.038,* we obtain

$$\frac{c_e}{g_e} = \frac{1}{288.37} = 0.0034678.$$

Then for the compression, we have

$$\frac{a-b}{a} = \frac{1}{297.1}.$$

If the more accurate form [98*a*] of Clairaut's equation is employed, the result is

$$\frac{a-b}{a} = \frac{1}{298.2}.$$

* See Coast Survey Special Publication No. 12.

By studying a large number of gravity observations in all parts of the world Helmert obtained the value

$$\frac{a-b}{a}=\frac{1}{298.3 \pm 0.7}. \qquad [113]$$

In the publication entitled *Effect of Topography and Isostatic Compensation upon the Intensity of Gravity* the authors give

$$\frac{a-b}{a}=\frac{1}{298.4 \pm 1.5}. \qquad [114]$$

In the most recent report on gravity work (Coast Survey Special Publication No. 40, 1917), the compression calculated from the observations in the United States, Canada, Europe and India is

$$\frac{a-b}{a}=\frac{1}{297.4}. \qquad [115]$$

By employing Equa. [88] the value of g may be computed for each station on the assumption that the earth is a spheroid. A comparison at each station of the observed and computed values of gravity indicates to what extent the geoid departs from the spheroid at each point.

PROBLEMS

Problem 1. Compute $\frac{a-b}{a}$ from the following data:

Station.	g_0.	Latitude.
		° ′ ″
Umanak, Greenland.............	982.595	+70 40 29
Sawah Loento, Sumatra..........	978.057	−00 41 40

Problem 2. If the coincidence intervals are 5^m during an 8-hour swing, what will be the error in P due to an error of 1^s in noting the time of a coincidence?

CHAPTER X

PRECISE LEVELING — TRIGONOMETRIC LEVELING

155. Precise Leveling.

The term *precise leveling* is applied to the operation of determining differences in elevation of successive points on the earth's surface with instruments and methods which, though similar to those used in ordinary leveling, are more refined and capable of yielding a much higher degree of precision. In order to secure the greatest possible accuracy, it is necessary to modify our conception of the nature of a level surface and to introduce certain corrections which are ordinarily negligible. It should be observed that since the line of sight of the instrument is always theoretically perpendicular to the direction of gravity at each station, it lies in a plane which is tangent to the geoid, not to the spheroid. In tracing out a level line by means of the spirit level we are following the curvature of the geoidal surface.

The term *precise leveling* has for many years been applied to all leveling of a fairly high degree of precision, but there have been various limits of precision prescribed by the different organizations carrying on the work. The accuracy obtainable has been so greatly increased through recent developments in instruments and methods that in 1912 a new class of leveling, known as *leveling of high precision*, was established by the International Geodetic Association; it is to include every line, set of lines, or net, which is run twice in opposite directions, on different dates, and whose errors, both accidental and systematic, computed in accordance with formulas stated in the resolution,* do not exceed $\pm 1^{mm}$ per kilometer for the probable accidental error and $\pm 0.2^{mm}$ per kilometer for the probable systematic error.

* See Coast Survey Special Publication No. 18, p. 88. See also Report of International Geodetic Association for 1912.

Many different instruments have been used in the past for precise leveling, some of the "wye" type and some of the "dumpy" type. All precise levels, however, have certain

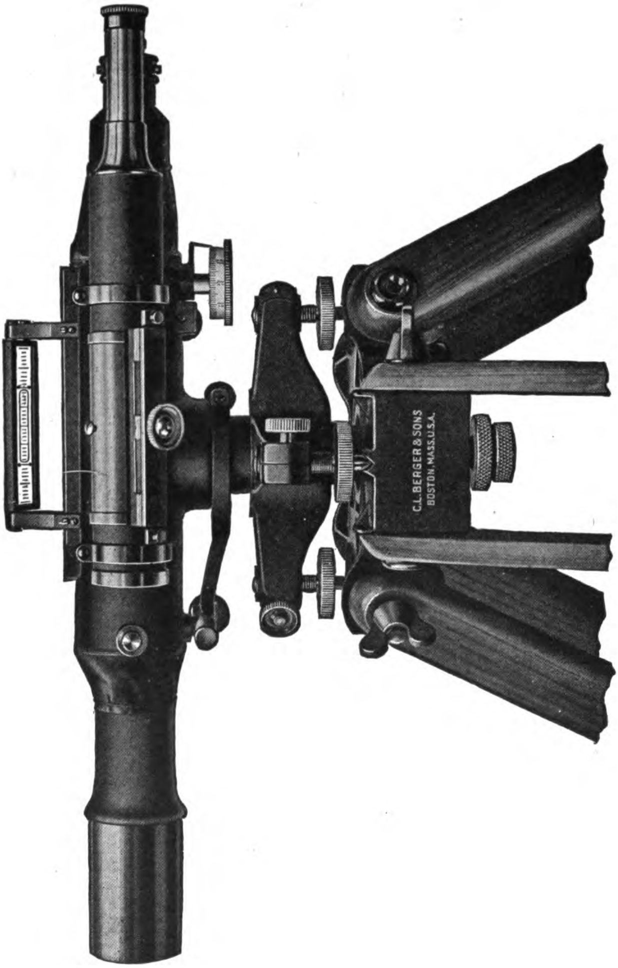


FIG. 89. Precise Level.
(C. L. Berger and Sons.)

characteristics in common: namely, (1) a telescope of high magnifying power, mounted on a heavy tripod: (2) a sensitive spirit level; (3) a slow-motion screw for centering the bubble; (4) stadia wires for determining the length of sight; and (5) a mirror or other optical device for viewing the bubble from the eye end of the telescope. Before the year 1899 the precise leveling of the United States Coast Survey was done with a wye level and target rods. The target was not set exactly on the level of the instrument, but

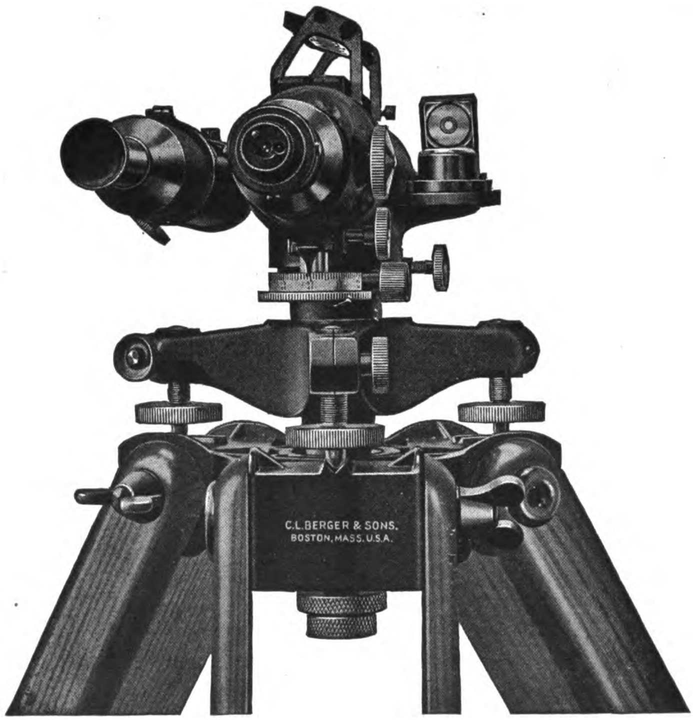


FIG. 89a. Precise Level.
(C. L. Berger and Sons.)

was set approximately, and corrections to this approximate reading were determined, using the micrometer screw to measure the small vertical angles. Since 1899 * a dumpy level of new design has been substituted for the wye level, the self-reading rod

* For a discussion of this change in methods see Coast Survey Report for 1899, p. 8, and for a description of the new instrument see Coast Survey Report for 1900, p. 521, and for 1903, p. 200.

adopted, and the micrometer screw used only for centering the bubble. This new instrument and method have been adopted by several other branches of the government service.

156. Instrument.

The new instrument, sometimes called the *prism level*, is designed to reduce, so far as possible, any errors arising from unequal heating of the different portions of the instrument. (Fig. 89.) The telescope barrel is made of an alloy of iron and nickel having a low coefficient of expansion (0.000004 per 1° C.). The level vial is set into the telescope tube as low as possible without interfering with the cone of rays from the object glass. This diminishes the effect of differential expansion of the parts supporting the level. At one side of the telescope is another (similar) tube containing a pair of prisms which, together with a mirror mounted above the telescope, enable the observer to view the ends of the bubble with the left eye at the same time that he looks at the rod with the right eye. The arrangement of mirror and prisms is such that there is no parallax caused by the glass in the level or the mirror. The instrument is provided with the usual small levels for the approximate leveling of the base.

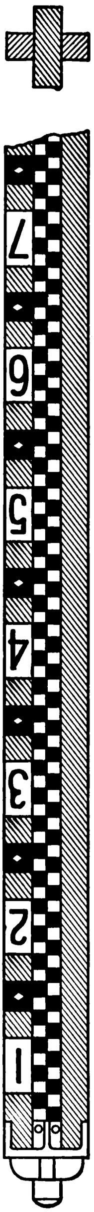


Fig. 90. Rod for Precise Leveling.

157. Rods.

The rods used are of the non-extensible type, graduated to centimeters and marked so that they may be read directly by the observer through the telescope, the millimeters being estimated. (Fig. 90.) The rods are in the form of a cross (in section); they are treated with paraffin to make them proof against moisture. Metal plugs are inserted three meters apart for verifying the length of the rod. Each rod has a spirit level attached, to show when it is vertical, and also a thermometer, which is read at each sight.

158. Turning Points.

Foot-pins are carried by all leveling parties, to be used when other turning points are not available. These are about one foot long, with a depression at the top in which to hold the rod. A rope run through a small hole is provided for pulling up the pin. Most of the leveling of the Coast Survey is carried along railroad lines, and the top of a rail is the usual turning point.

159. Adjustments.

The adjustments of the level are nearly the same as those of the ordinary dumpy level. The rough levels are adjusted so as to remain in the center when the telescope is revolved about the vertical axis. The axis of the long bubble tube is adjusted parallel to the line of sight of the telescope whenever it is much in error. This adjustment is tested each day by taking four readings, like those used in the "peg" method, except that the shorter sights are 10 meters in length and the longer sights are of the usual length, (say 100^m). From these four readings a factor C is computed, which is the ratio of the correction for any reading to the corresponding rod interval. The difference in the sums of the foresight and backsight at any set-up is to be multiplied by this factor C.

To find an expression for C, call n_1 and n_2 the rod readings for the nearer sights, and d_1 and d_2 the rod readings for the distant sights, s_1 and s_2 the nearer stadia intervals, and S_1 and S_2 the distant stadia intervals, the subscripts referring to the first and second instrument positions. Then the true difference in elevation from the first set-up is

$$(n_1 + Cs_1) - (d_1 + CS_1),$$

and for the second set-up,

$$(d_2 + CS_2) - (n_2 + Cs_2).$$

Equating and solving for C,

$$C = \frac{(n_1 + n_2) - (d_1 + d_2)}{(S_1 + S_2) - (s_1 + s_2)}. \qquad [116]$$

C is + if the line of sight is inclined downward.

Below is table showing a determination of C (from Coast Survey Report for 1903).

DETERMINATION OF C. 8.20 A.M., AUGUST 28, 1900

(Left-hand page.)				(Right-hand page.)			
Number of station.	Thread reading, backsight.	Mean.	Thread interval.	Rod.	Thread reading, foresight.	Mean.	Thread interval.
	1515		13		0357		105
A	1528	1528.3	14	W	0462	0461.7	104
	1542		27		0566		209
	2252		105		1276		
B	2357	2357.0	105	W	1288	1288.3	12
	2462		210		1301		13
		0461.7	419			1528.3	25
		2818.7	52			2816.6	
Corr. for curv. and ref.		−0.8	367			2817.9	
		2817.9			367)−1.3 (−0.004=C		

If the value of C is less than 0.005, the instrument should not be adjusted. If between 0.005 and 0.010, the observer is advised not to adjust. If over 0.010, the adjustment should be made. The adjustment is made by moving the level rather than the cross-hair ring, to avoid moving the line of sight from the optical axis.

160. Method of Observing.*

It is customary to use two rods, the one that is held for a fore-sight on a certain turning point being kept at the same turning point for a back sight. The instrument is set up and leveled,

* The General Instructions for Precise Leveling will be found in Coast Survey Special Publication No. 22, p. 29.

and all three hairs are read on the back rod, the level being kept central at each reading. As soon as possible thereafter the three hairs are read in a similar manner on the forward rod. The readings are estimated to millimeters. The temperature on the rod thermometer is read at the same time. The level should be shaded from the sun in order to avoid unequal heating of its parts. In selecting instrument and rod points, the observer must keep the difference in length of the forward and backward sight less than 10 meters on any one set-up and less than 20 meters for the accumulated difference at any time. The readings of the upper and lower (stadia) wires enable the recorder to determine the difference in distance at each set-up. The maximum length of sight allowable is 150^m, a distance reached only under exceptionally favorable conditions. At odd-numbered stations the back sight is taken first; at even-numbered stations the fore sight is taken first. This results in the same rod being read first each time.

Lines between bench marks are divided into sections of from one to two kilometers each. Each of these sections is run forward and backward. If the two differences in elevation so determined are found to differ by more than $4^{mm}\sqrt{K}$ (K = kilometers), both runnings must be repeated until such a check is obtained. Lines may be run with such care that it is seldom necessary to repeat, but the maximum economy appears to be reached when from 5 to 15 per cent of the sections have to be rerun.

On page 244 is a set of notes used in leveling with this instrument (Coast Survey Report, 1903).

The most recent practice is to record the readings directly on adding machines carried with the leveling outfit. This results in a saving of time and in avoiding many mistakes in recording and adding.

161. Computing the Results.

In computing the results of precise leveling, corrections are applied for the nonadjustment of the level, for curvature and refraction, for error in length of rod, for error due to temperature of rod, and for the orthometric correction. The curvature and refraction corrections are usually taken from tables (Coast Survey Report, 1903). The length of rod is tested at the office at the beginning and end of the season, and variations during the season are tested in the field by means of a steel tape. The temperature correction is derived from tables, the argument being the observed temperatures.

SPIRIT LEVELING

(Left-hand page.)					(Right-hand page.)				
Date: August 29, 1900. Sun: *C*. Forward. ~~Backward.~~ (Strike out one word.)					From B.M.: 68. To B.M.: *G* Wind: S.T.				
No. of station.	Thread reading, back-sight.	Mean.	Thread interval.	Sum of intervals.	Rod and temp.	Thread reading fore-sight.	Mean.	Thread interval.	Sum of intervals.
43	0674		99		V	2683		99	
	0773	0773.0	99		38	2782	2782.3	100	
	0872		198			2882		199	
	0925		106		W	2415		103	
44	1031	1030.3	104		35	2518	2518.0	103	
	1135		210	408		2621		206	405
	0484		98		V	2510		96	
45	0582	0582.3	99		35	2606	2606.0	96	
	0681		197	605		2702		192	597
	0398		97		W	2859		96	
46	0495	0495.0	97		34	2955	2954.7	95	
	0592		194	799		3050		191	788
	1027		26		V	1006		29	
47	1053	1053.3	27		34	1035	1034.7	28	
	1080		53	852		1063		57	845
							11895.7		
		3933.9					−7961.8		
							2 : 25 P.M.		

162. Bench Marks.

The bench marks used in precise leveling are of various types. Wherever it is practicable, the metallic plates shown in Fig. 91 are used to mark the points, but nearly all of the kinds of bench marks which are used by engineers are used also in this class of work. The distance between benches is not allowed to exceed 15 kilometers; every 100 kilometer section should have at least 20 bench marks, a good average distance being 2.5 kilometers. In cities the old bench marks are often utilized for the precise levels.

163. Sources of Error.

The sources of error which it is particularly necessary to study in this class of work are (1) unequal effects of temperature changes in the instrument, (2) gradual rising or settling of the instrument or rods, (3) variations in refraction of the air, (4) unequal lengths of sights, (5) errors in length and temperature of rod, and (6) convergence of level surfaces.

TABLE H. — TOTAL CORRECTION FOR CURVATURE AND REFRACTION

Distance.	Correction to rod reading.	Distance.	Correction to rod reading.
m. *m.*	*mm.*	*m.*	*mm.*
0 to 27	0.0	160	−1.8
28 to 47	−0.1	170	−2.1
48 to 60	−0.2	180	−2.3
61 to 72	−0.3	190	−2.6
73 to 81	−0.4	200	−2.8
82 to 90	−0.5	210	−3.0
91 to 98	−0.6	220	−3.3
99 to 105	−0.7	230	−3.7
106 to 112	−0.8	240	−4.0
113 to 118	−0.9	250	−4.3
119 to 124	−1.0	260	−4.7
125 to 130	−1.1	270	−5.0
131 to 136	−1.2	280	−5.4
137 to 141	−1.3	290	−5.8
142 to 146	−1.4	300	−6.2
147 to 150	−1.5		

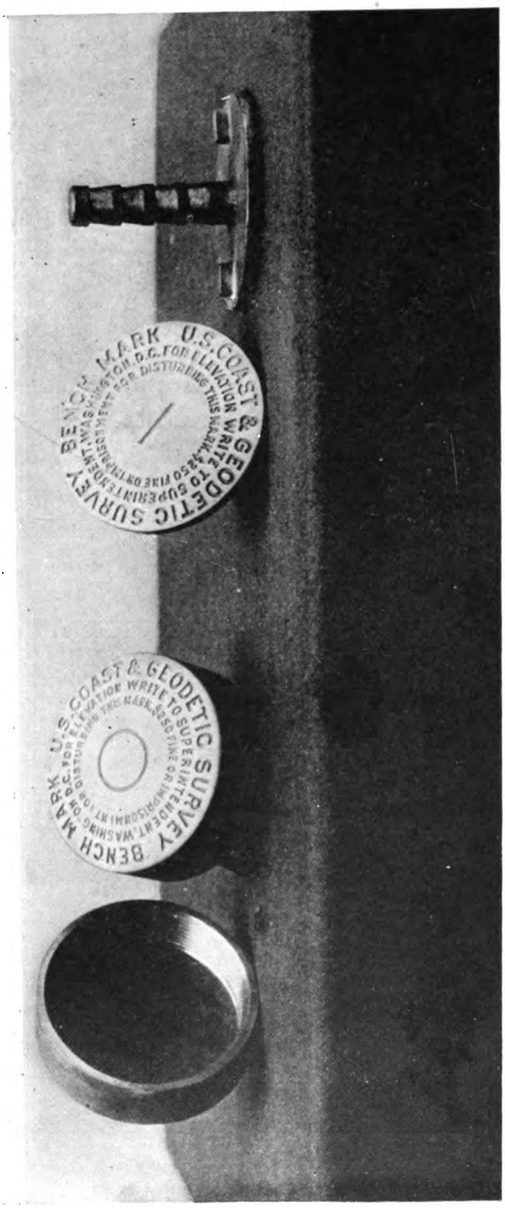


FIG. 91. Bench Marks.
(Coast and Geodetic Survey.)

TABLE I. — DIFFERENTIAL CORRECTION FOR CURVATURE AND REFRACTION

Mean length of sight in rod interval in millimeters.	Difference of sights in rod interval in millimeters.																												
	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	4	44	46	48	50	52	54	56	58
10	.0	.0	.0	.0	.0																								
20	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0																			
30	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0														
40	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0									
50	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0				
60	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1
70	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1
80	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1
90	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
100	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
110	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
120	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
130	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
140	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
150	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
160	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1
170	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2
180	.0	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2
190	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2
200	.0	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2
210	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2
220	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2
230	.0	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2
240	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2
250	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2
260	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2
270	.0	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2
280	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2
290	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3
300	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3
310	.0	.0	.0	.0	.0	.1	.1	.1	.1	1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3
320	.0	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3
330	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3
340	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3
350	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3
360	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3	.3
400	.0	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3	.3	.3	.4
440	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3	.3	.3	.4	.4	.4
480	.0	.0	.0	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3	.4	.4	.4	.4	.4	.4
520	.0	.0	.1	.1	.1	.1	.1	.1	.1	.2	.2	.2	.2	.2	.2	.3	.3	.3	.3	.3	.3	.3	.4	.4	.4	.4	.4	.4	.4

TABLE J. — CORRECTION FOR TEMPERATURE (IN MILLIMETERS)

Temp. C.	Difference of elevation in meters.													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1
2	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
3	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2
4	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2
5	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.3	0.3
6	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.3
7	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.4	0.4
8	0.0	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.3	0.4	0.4	0.4	0.4
9	0.0	0.1	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.4	0.4	0.5	0.5
10	0.0	0.1	0.1	0.2	0.2	0.2	0.3	0.3	0.4	0.4	0.4	0.5	0.5	0.6
11	0.0	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.4	0.5	0.5	0.6	0.6
12	0.0	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.5	0.6	0.6	0.7
13	0.0	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.5	0.6	0.6	0.7	0.7
14	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.7	0.8
15	0.1	0.1	0.2	0.2	0.3	0.4	0.4	0.5	0.5	0.6	0.7	0.7	0.8	0.8
16	0.1	0.1	0.2	0.3	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.8	0.8	0.9
17	0.1	0.1	0.2	0.3	0.3	0.4	0.5	0.5	0.6	0.7	0.8	0.8	0.9	0.9
18	0.1	0.1	0.2	0.3	0.4	0.4	0.5	0.6	0.6	0.7	0.8	0.9	0.9	1.0
19	0.1	0.2	0.2	0.3	0.4	0.5	0.5	0.6	0.7	0.8	0.8	0.9	1.0	1.1
20	0.1	0.2	0.2	0.3	0.4	0.5	0.6	0.6	0.7	0.8	0.9	1.0	1.0	1.1
21	0.1	0.2	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.8	0.9	1.0	1.1	1.2
22	0.1	0.2	0.3	0.4	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.1	1.2
23	0.1	0.2	0.3	0.4	0.5	0.6	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3
24	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.2	1.3
25	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
26	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.5
27	0.1	0.2	0.3	0.4	0.5	0.6	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
28	0.1	0.2	0.3	0.4	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.6
29	0.1	0.2	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.3	1.4	1.5	1.6
30	0.1	0.2	0.4	0.5	0.6	0.7	0.8	1.0	1.1	1.2	1.3	1.4	1.6	1.7
31	0.1	0.2	0.4	0.5	0.6	0.7	0.9	1.0	1.1	1.2	1.4	1.5	1.6	1.7
32	0.1	0.3	0.4	0.5	0.6	0.8	0.9	1.0	1.2	1.3	1.4	1.5	1.7	1.8
33	0.1	0.3	0.4	0.5	0.7	0.8	0.9	1.1	1.2	1.3	1.4	1.6	1.7	1.8
34	0.1	0.3	0.4	0.5	0.7	0.8	1.0	1.1	1.2	1.4	1.5	1.6	1.8	1.9
35	0.1	0.3	0.4	0.6	0.7	0.8	1.0	1.1	1.3	1.4	1.5	1.7	1.8	2.0
36	0.1	0.3	0.4	0.6	0.7	0.9	1.0	1.2	1.3	1.4	1.6	1.7	1.9	2.0
37	0.1	0.3	0.4	0.6	0.7	0.9	1.0	1.2	1.3	1.5	1.6	1.8	1.9	2.1
38	0.1	0.3	0.5	0.6	0.8	0.9	1.1	1.2	1.4	1.5	1.7	1.8	2.0	2.1
39	0.2	0.3	0.5	0.6	0.8	0.9	1.1	1.2	1.4	1.6	1.7	1.9	2.0	2.2
40	0.2	0.3	0.5	0.6	0.8	1.0	1.1	1.3	1.4	1.6	1.8	1.9	2.1	2.2
41	0.2	0.3	0.5	0.7	0.8	1.0	1.1	1.3	1.5	1.6	1.8	2.0	2.1	2.3
42	0.2	0.3	0.5	0.7	0.8	1.0	1.2	1.3	1.5	1.7	1.8	2.0	2.2	2.3
43	0.2	0.3	0.5	0.7	0.9	1.0	1.2	1.4	1.5	1.7	1.9	2.1	2.2	2.4
44	0.2	0.3	0.5	0.7	0.9	1.1	1.2	1.4	1.6	1.8	1.9	2.1	2.3	2.5
45	0.2	0.3	0.5	0.7	0.9	1.1	1.3	1.4	1.6	1.8	2.0	2.2	2.3	2.5

164. Datum.

The datum for precise levels is mean sea-level, or the surface of the geoid, as found from tidal observations. This is assumed to be correctly given by the mean of the several "annual means" as derived from tidal observations for sea-level. The heights of the tide are recorded automatically on a self-registering gauge. (See Cut.) The vertical motion of the float is reduced (the ratio

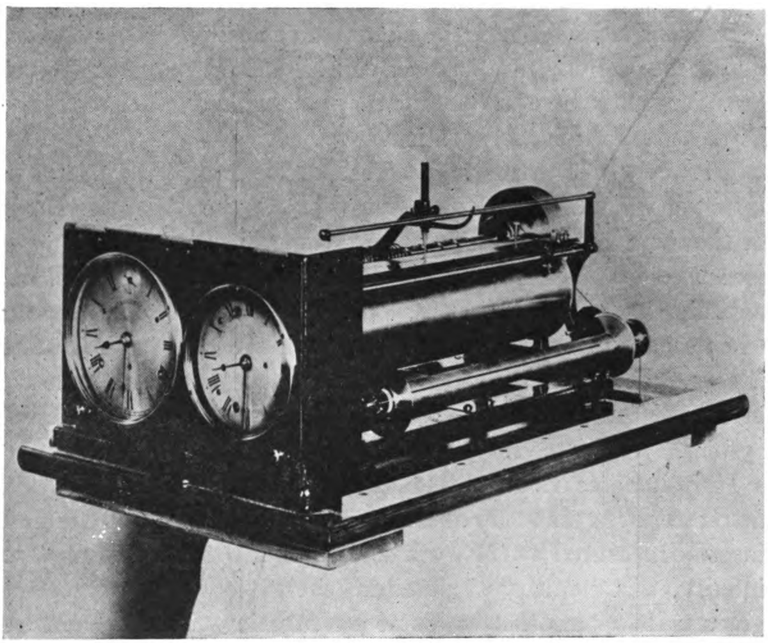


FIG. 91A. Self-Registering Tide Gauge.
(Coast and Geodetic Survey.)

depending upon the range of tide) by passing the connecting wire and cord over a series of pulleys, and is communicated to a recording pencil which marks on a sheet of paper passing over a revolving drum. The drum is revolved at a uniform rate by clock mechanism. The height of the water is referred to a bench mark in the vicinity. Observations of the tide should be extended over a period of at least one year in order to determine sea-level with sufficient precision for this class of leveling. In the tidal records at some stations there appear to be small systematic variations in the annual means extending over periods of several years; but, taking the records as a whole, the variations do not seem to follow any particular law, and they are treated as accidental. (See Coast Survey Special Publication No. 26.)

165. Potential.

In order to investigate the nature of the *orthometric correction*, due to the convergence of level surfaces, it will be necessary to consider first some of the elementary mechanical principles of the earth's gravitation and rotation.

Whenever two attracting bodies are separated, work is done upon them and energy is stored up; that is, the potential energy of the system is increased. The change in potential energy is measured by the amount of work done. When the bodies are an infinite distance apart, the potential energy is a maximum; when the bodies are in contact, the potential energy of the system is zero. If the masses are free to move, they will always move in such a direction as to *diminish* the potential energy of the system.

If we imagine a unit mass placed at any point P in space and attracted by a mass M, and if the potential energy of the unit mass be measured by the work done upon it to move it from P to infinity, this quantity of potential energy is a property of the given point P; in other words, it is a function of the coördinates of P. It is called the *potential* at that point. It is not necessary that there should actually be a unit mass at the point, but the conditions are such that *if* a unit mass were placed at P, it would have this amount of potential energy. It should be observed that the increase of potential energy is measured by the *fall* in potential.

166. The Potential Function.

If an attracting body M be divided into small elements, and the mass Δm of each element be divided by its distance from a point P, the limit of the sum of all these fractions, as the elements are made smaller, is called the value at P of the potential function due to M, or simply the potential of P. Calling this function V, then

$$V = \text{limit}_{m=0} \sum \frac{\Delta m}{r}, \qquad [117]$$

or, if Δm is of density δ and has the coördinates x', y', z', and P has the coördinates x, y, z, then

$$V = \int\int\int \frac{\delta \cdot dx' \cdot dy' \cdot dz'}{[(x' - x)^2 + (y' - y)^2 + (z' - z)^2]^{\frac{1}{2}}}. \qquad [118]$$

The integration over the entire mass gives the value of the potential function at P.*

167. The Potential Function as a Measure of Work Done. The amount of work required to move a unit mass (concentrated at a point) from a point P_1 to another point P_2, by any path (Fig. 92), against the attraction of a mass M, is equal to the fall in potential $V_1 - V_2$, where V_1 and V_2 are the values of the potential function at the points P_1 and P_2. To show this, let r_1 and r_2 be the distances from the center of M to the points P_1 and P_2.

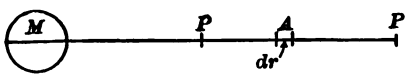


FIG. 92.

The work done in moving a unit mass through a small space dr equals the force $\left(\frac{1}{r^2}\right)$ times the space dr. But the force $\left(\frac{1}{r^2}\right)$ at any point A equals $-\frac{dV}{dr}$ at that point, since V (for a unit mass) $= \frac{1}{r}$. Hence the work

$$= -\int_{r_1}^{r_2} \frac{dV}{dr} \cdot dr = V_1 - V_2; \qquad [119]$$

that is, the work done equals the fall in potential.

* See Peirce, *Theory of the Newtonian Potential Function.*

If the point P_2 is moved to an infinite distance, V_2 become zero, and the potential at P_1 then equals the work done in moving the unit mass from P_1 to infinity; or it is the work done by it in moving from infinity to the point P_1.

168. Equipotential Surfaces.

A level surface, or an equipotential surface, is one having at every point the same gravity potential. It is everywhere perpendicular to the direction of gravity.* The mean surface of the ocean is such a surface. The surface of any lake is also an equipotential surface. From the proof given in the preceding article it is evident that if there are two such equipotential surfaces, the difference in potential is the work done upon a unit mass in moving it from one surface to the other. This difference in potential is independent of any particular points on the surfaces and of the path followed in passing from one to the other; for example, the work done in raising a unit mass from sea-level to the south end of a lake is the same as the work done in raising a unit mass from sea-level to the north end of the lake. Since the work done is the force (w) times the distance (dh) through which it acts, it is evident that $w \times dh$ is a constant between two level surfaces. Also, since g varies as the weight (force), $g \times dh$ is a constant between these two surfaces.

The force of gravity is less at the equator (Art. 144) than at the poles, on account of the action of the centrifugal force. Hence we should expect to find that a given level surface is farther from sea-level at the equator than it is at a point nearer the pole. If several such surfaces be drawn (Fig. 93), they will be seen to converge toward the pole. They are all parallel to each other at the equator and at the poles, and have their greatest difference in direction at $\phi = 45°$.

Since g is about one-half of one per cent less at the equator than

* It may be proved that if there is a resultant force at a point in space due to attracting masses, this force acts in the direction of the normal to the equipotential surface through the point (see Peirce, *Theory of the Newtonian Potential Function*, p. 38). It should be kept in mind that the "force of gravity" is the resultant of the force of attraction and the centrifugal force.

at the pole, the height h between surfaces is about one-half of one per cent greater at the equator. Hence, if a level surface were 1000 meters above the sea-surface at the equator, it would be only 995 meters above sea-level at the pole. A surface at half the elevation would converge (very nearly) half as much. In the line of levels run from San Diego to Seattle the convergence was found to be about $1\frac{1}{4}$ meters, showing that at high elevations this error is by no means a negligible one in precise leveling.

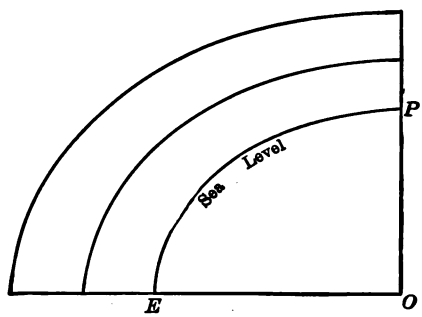


FIG. 93.

It is evident that if a series of bench marks is established along a meridian (in the northern hemisphere), and all are placed at the same elevation, using the ordinary methods, those at the northern end of the line lie nearer to sea-level than those at the southern end of the line. It becomes necessary, then, to revise the definition of *elevation*.

If the ordinary definition of elevation is retained, and no allowance made for convergence of level surfaces, then different results for the elevation of a point will be obtained, according to which path is followed. If we measure vertically upward from A to B (Fig. 94), and then level by means of the water surface BC, we obtain a greater height for point C than we should if we leveled by water from A to D and then measured vertically upward from D to C. If a correction is applied, however, to allow for the convergences of these surfaces, the result is that different portions of the lake surface have different elevations, which is apparently absurd if the true nature of the level surface is not understood. In order to avoid this apparent difficulty another method sometimes employed is to number all the surfaces with a serial number (called the Dynamic Number), so that all points on the same surface will have their elevation expressed by the same number. This number is defined as the work required to raise one kilogram from sea-level to the given surface, the unit being the kilogram-

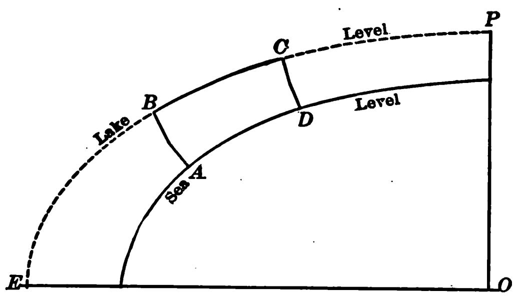


FIG. 94.

meter at sea-level in latitude 45°. The United States Coast Survey has adopted the method of applying to ordinary elevations the correction for convergence, called the *Orthometric Correction.* The Standard Elevations of the Coast Survey in Special Publication No. 18 are given by the *Orthometric Elevation.*

169. The Orthometric Correction.

Let W be the work (in absolute units) required to raise a unit mass from sea-level to a point at elevation h, and let H be the dynamic number of the surface through the point, defined by the quotient $W \div g_{45}$, where g_{45} is the value of g at sea-level in latitude 45° (Equa. [96a], p. 210). Then, since $g \times dh$ is constant for two level surfaces separated by height dh,

$$W = \int_0^h g\, dh = g_{45} \int_0^h (1 - 0.002644 \cos 2\phi \ . \ . \ .) \, dh$$

in which the integration takes place along the curved vertical.

Integrating, $W = g_{45}\Big[(1 - 0.002644 \cos 2\phi) h \ldots \Big]_0^h$. [120]

The dynamic number

$$= H = \frac{W}{g_{45}} = h(1 - 0.002644 \cos 2\phi \ldots). \qquad [121]$$

To find the correction to the elevation due to a change in the latitude, differentiate the last equation with respect to ϕ as the independent variable, and we obtain

$$0 = dh - 0.002644(-2h \sin 2\phi\, d\phi + \cos 2\phi\, dh \ldots)$$
$$= dh(1 - 0.002644 \cos 2\phi) + 0.005288\, h \sin 2\phi\, d\phi,$$

and $$dh = -\frac{0.005288\, h \sin 2\phi\, d\phi}{1 - 0.002644 \cos 2\phi} \qquad [122]$$

$$= -(0.005288\, h \sin 2\phi)(1 + 0.002644 \cos 2\phi \ldots)\, d\phi \text{ arc } 1',^{*} \qquad [123]$$

the factor arc 1′ being introduced to reduce $d\phi$ to minutes of arc.

A more definite idea of the magnitude of this correction may be gained from the following example. Assuming that the elevation of Lake Michigan is 177 meters at Chicago, latitude 41° 53′, what is the elevation of the lake at Milwaukee, in latitude 43° 03′? In the formula, $h = 177^m$, $d\phi = 70'$, and $\phi = 42° 28'$; the computed values of dh is -0.0190^m, and the lake level at Milwaukee is therefore 176.9810 meters. Tables for computing the orthometric correction will be found in Coast Survey Special Publication No. 18, pp. 54–56.

The relation between the dynamic numbers and the orthometric elevations is illustrated in the following table, which is an extract from the special publication just mentioned.

Station.	Latitude.	Orth. elev meters.	Dyn. number.
	° ′		
Smithland, La.	30 55	14.7729	14.7545
Meridian, Miss.	32 22	104.9494	104.8292
Amblersburg, W. Va.	39 23	494.9221	494.6287
Summit, Cal.	34 20	1165.4345	1164.1008
Riordan, Ariz.	35 13	2216.5452	2213.8112

* For additional terms, neglected in the above formula, see Coast and Geodetic Survey Special Publication No. 18, p. 49. See also Ch. Lallemand, *Nivellement de Haute Precision, Encyclopédie des Travaux Publics, Paris*, 1912.

170. The Curved Vertical.

In view of what has been said regarding the change in the direction of level surfaces with an increase in elevation, it is clear that the vertical line is curved, being concave toward the pole, and therefore that any observation for latitude made at a point above sea-level is referred, not to the true normal to the surface at sea-level, but to the direction of that portion of the vertical which is at the elevation (*h*) of the station. In order to determine the amount of the correction to reduce the observed latitude

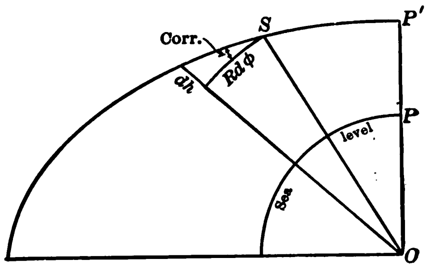


FIG. 95.

to its value at sea-level, refer again to Equa. [122], p. 255. An inspection will show that the denominator of this fraction is usually not far from unity; and since the correction desired is itself quite small, we may assume

$$dh = -0.005288\, h \sin 2\phi\, d\phi. \qquad [124]$$

The correction to the observed latitude is the difference in the slope of the two surfaces (sea-level and the level of station) measured in the plane of the meridian. From Fig. 95 it is seen that the angle between the level surface through S and a surface parallel to sea-level drawn through S is $dh \div Rd\phi$. But, by Equa. [124],

$$\frac{dh}{R\,d\phi} = -\frac{0.005288\, h \sin 2\phi}{R}.$$

Reducing this to seconds of arc,

$$\frac{dh}{R\,d\phi} = -\frac{0.005288\, h \sin 2\phi}{R \text{ arc } 1''}.$$

Since R arc 1″ = 101.3 feet (very nearly), the correction to the latitude may be written

$$- 0''.0522\, h \sin 2\,\phi, \qquad [125]$$

where h is in *thousands of feet;* or, if h is in meters, the correction is

$$- 0.000171\, h \sin 2\,\phi. \qquad [126]$$

171. Trigonometric Leveling.

The method of measuring the vertical angles between triangulation stations has already been described in the chapter on field-work. From the field note-book we have the several measures of the angles, the height of the instrument, and also of the point sighted in each case above the station marks. The elevation of one station above sea-level is assumed to be known, and that of the other is to be computed. Before this can be done, the angle must be reduced to the value it would have if the instrument and the point sighted were coincident with the station marks.

172. Reduction to Station Mark.

From the diagram (Fig. 96) it is evident that if i is the height of the instrument at A, and o that of the object sighted at B, and

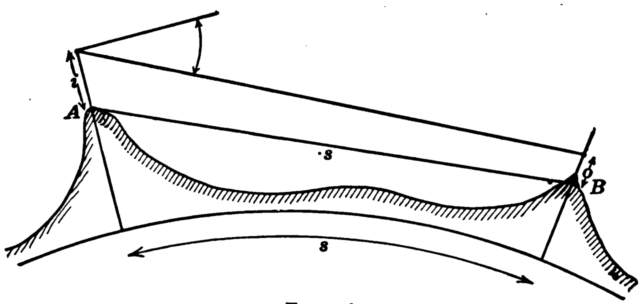


FIG. 96.

s the distance between stations, obtained from the triangulation, then the correction to the vertical angle at A is

$$\text{Corr.} = - \frac{i - o}{s \text{ arc } 1''}. \qquad [127]$$

Four places in the logarithms are sufficient in computing this correction.

This reduction need be made only in case of reciprocal observations, that is, observations of the vertical angle from both ends of the line. In case of observations from one station only, the quantity $i - o$, in meters, can be applied directly to the computed difference in elevation.

When a sight is taken from one station P_1 to another station P_2, the verticals of the two stations do not (in general) intersect, because they lie in different planes. If we imagine a plane which is parallel to both verticals, and then project both verticals onto this plane, we obtain the result shown in Fig. 97.

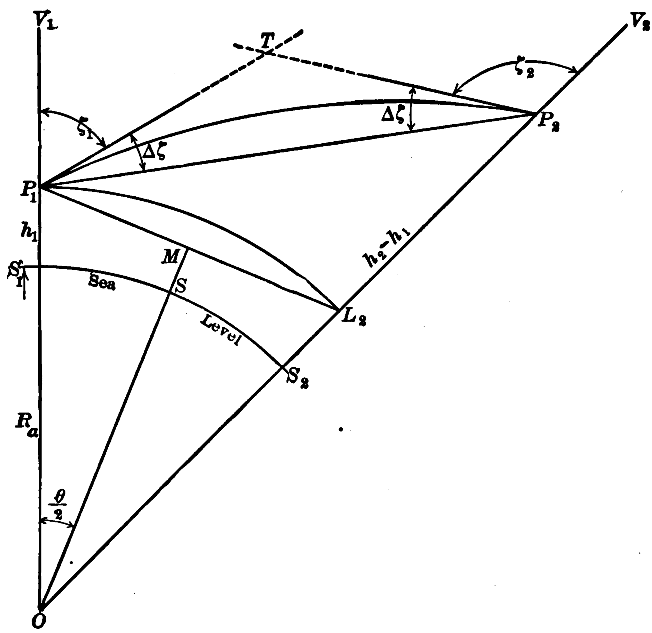


FIG. 97.

173. Reciprocal Observations of Zenith Distances.

In Fig. 97, P_1 and P_2 represent the two instrument stations; their elevations above sea-level are $P_1S_1 = h_1$ and $P_2S_2 = h_2$. The ray of light is assumed to take the form of a circular curve, whose radius is determined by the coefficient used in the calculation. The two measured zenith distances are ζ_1 and ζ_2.

The angle of refraction is $\Delta\zeta = TP_1P_2 = TP_2P_1 = m\,\theta$, where m is the coefficient of refraction, and θ the central angle P_1OP_2. The radius of curvature of the section S_1S_2 is R_α, approximately equal to OS_1, or to OS_2.

The quantity to be computed is the difference in elevation $h_2 - h_1$, which may be found by solving the triangle $P_1P_2L_2$.*

In the triangle $P_1P_2L_2$, $P_2L_2 = h_2 - h_1$, the desired difference in elevation; P_1L_2 is the chord joining the two verticals at the level surface through P_1. Observing that $P_1M = (R_\alpha + h_1) \sin\frac{\theta}{2}$, and $P_1L_2 = 2\,(R_\alpha + h_1) \sin\frac{\theta}{2}$, we have, by applying the law of sines,

$$h_2 - h_1 = 2\,(R_\alpha + h_1) \sin\frac{\theta}{2} \times \frac{\sin(P_2P_1L_2)}{\sin(P_1P_2L_2)}. \qquad (e)$$

But in the triangle $P_1L_2P_2$

$$\begin{aligned} P_2P_1L_2 &= P_1L_2O - P_1P_2L_2 \\ &= \left(90^\circ - \frac{\theta}{2}\right) - (180^\circ - \zeta_2 - \Delta\zeta) \\ &= -90^\circ - \frac{\theta}{2} + \zeta_2 + \Delta\zeta. \qquad (f) \end{aligned}$$

Also,

$$\begin{aligned} P_2P_1L_2 &= 180^\circ - \left[\zeta_1 + \Delta\zeta + 90^\circ - \frac{\theta}{2}\right] \\ &= 90^\circ - \zeta_1 - \Delta\zeta + \frac{\theta}{2}. \qquad (g) \end{aligned}$$

Adding (f) and (g) and dividing by 2,

$$P_2P_1L_2 = \frac{\zeta_2 - \zeta_1}{2}, \qquad (h)$$

In the triangle $P_1P_2P_2$

$$\begin{aligned} P_1P_2L_2 &= 180^\circ - [\theta + (180^\circ - \zeta_1 - \Delta\zeta)] \\ &= -\theta + \zeta_1 + \Delta\zeta. \qquad (i) \end{aligned}$$

* The following formulæ are those adopted by the Coast and Geodetic Survey in 1915 (see Special Publications Nos. 26 and 28).

Also, $$P_1P_2L_2 = 180^\circ - \zeta_2 - \Delta\zeta. \qquad (j)$$

Adding (i) and (j) and dividing by 2,

$$P_1P_2L_2 = 90^\circ - \left(\frac{\theta}{2} + \frac{\zeta_2 - \zeta_1}{2}\right). \qquad (k)$$

Substituting (h) and (k) in (e),

$$h_2 - h_1 = 2\,(R_\alpha + h_1)\sin\frac{\theta}{2}\,\frac{\sin\left(\frac{\zeta_2 - \zeta_1}{2}\right)}{\cos\left(\frac{\theta}{2} + \frac{\zeta_2 - \zeta_1}{2}\right)}. \qquad (l)$$

Expanding the denominator and dividing the numerator and denominator by $\cos\left(\frac{\zeta_2 - \zeta_1}{2}\right)\cos\frac{\theta}{2}$, we obtain

$$h_2 - h_1 = \frac{2\,(R_\alpha + h_1)\tan\frac{\theta}{2}\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)}{1 - \tan\frac{\theta}{2}\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)}.$$

Expanding $\tan\frac{\theta}{2}$ in series (see p. 330), retaining but two terms of the series, and putting $\theta = \frac{s}{R_\alpha}$,

$$h_2 - h_1 = \left(1 + \frac{h_1}{R_\alpha}\right)s\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)\left(1 + \frac{s^2}{12\,R_\alpha^2}\right)\left[1 + \frac{s\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)}{2\,R_\alpha}\right] \qquad [128]$$

$$= s\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right)A \cdot B \cdot C, \qquad [129]$$

in which

$$A = 1 + \frac{h_1}{R_\alpha},$$

the correction for elevation of the station of known elevation,

$$B = 1 + \frac{s}{2\,R_\alpha} \cdot \tan\left(\frac{\zeta_2 - \zeta_1}{2}\right),$$

the correction for the difference in elevation,

and $$C = 1 + \frac{s^2}{12 R_\alpha^2},$$

the correction for distance.

The logarithms of A, B, and C are given in Tables K, L, and M, for the arguments h_1, $\log\left[s \tan \frac{\zeta_2 - \zeta_1}{2}\right]$, and $\log s$, respectively.

174. When only one Zenith Distance is Observed.

From (g) and (h) we have

$$P_2P_1L_2 = \frac{\zeta_2 - \zeta_1}{2} = 90° - \zeta_1 - \Delta\zeta + \frac{\theta}{2}.$$

The refraction angle is $\Delta\zeta = m\theta$, where m is the coefficient, to be obtained from the best obtainable values, and which is approximately equal to 0.071; substituting $m\theta$ in the above equation we have

$$\frac{\zeta_2 - \zeta_1}{2} = 90° - \zeta_1 + (0.5 - m)\,\theta,$$

and $$\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right) = \tan\,(90° + (0.5 - m)\,\theta'' \operatorname{arc} 1'' - \zeta_1)$$

$$= \tan\left[90° + (0.5 - m)\frac{s}{R_\alpha \sin 1''} - \zeta_1\right]$$

since $$\theta'' = \frac{s}{R_\alpha \sin 1''}.$$

Putting this s term $= k$, we have

$$\tan\left(\frac{\zeta_2 - \zeta_1}{2}\right) = \tan\,[90° + k - \zeta_1]. \qquad (n)$$

Substituting in [129] from (n),

$$h_2 - h_1 = s \tan\,[90° + k - \zeta_1]\, A \cdot B \cdot C, \qquad [130]$$

in which A, B, and C have the same meaning as before, except that B is given for the argument $\log\,[s \tan\,(90° + k - \zeta_1)]$.

Example. Zenith Distance of Mt. Blue from Farmington, 87° 07′ 18″.8; distance, 15,519 meters; $m = 0.071$; instrument 2.20 meters above station mark; point sighted 4.40 meters above station mark; elevation of Farmington, 181.20 meters.

	0.5				
m	0.071				
$(0.5-m)$	0.429	log = 9.6325			
		log s = 4.1909			
		colog R_α sin 1″ = 8.5092			
		2.3326	90° 00′ 00″		
log R_α	= 6.8052	K = 215″.1 =	03′ 35″.1		
" sin 1″	= 4.6856		ζ = 87° 07′ 18 .8		
	1.4908		+ 2° 56′ 16″.3	tan =	8.71029
				log s =	4.19086
					2.90115
				A	1
				B	3
				C	0
					2.90119

	796.51	meters
Red. to Sta.	2.20	"
Diff. Eleva.	794.31	"
Elev. Farmington	181.20	"
Elev. Mt. Blue	975.51	"

TABLE K*

k_1.	Log A, units of fifth place of decimals.	k_1.	Log A, units of fifth place of decimals.	k_1.	Log A, units of fifth place of decimals.	k_1.	Log A, units of fifth place of decimals.
Meters.		Meters.		Meters.		Meters.	
0		1541		3156		4770	
	0		11		22		33
73		1688		3303		4917	
	1		12		23		34
220		1835		3449		5064	
	2		13		24		35
367		1982		3596		5211	
	3		14		25		36
514		2128		3743		5357	
	4		15		26		37
661		2275		3890		5504	
	5		16		27		38
807		2422		4036		5651	
	6		17		28		39
954		2569		4183		5798	
	7		18		29		40
1101		2715		4330		5945	
	8		19		30		41
1248		2862		4477		6091	
	9		20		31		
1394		3009		4624			
	10		21		32		
1541		3156		4770			

* In these tables log R_α is taken as 6.80444, the mean radius in latitude 40° on the Clarke Spheroid of 1866.

Table K gives the values of log A, the correction factor for the elevation of the known station, by showing the limiting values of the elevation h_1, between which log A may be taken as 0, 1, 2, 3, etc., units of the fifth place of decimals. Log A is positive, except in the very rare case where h_1 corresponds to a point below mean sea-level.

TABLE L

Log s tan $\frac{1}{2}$ $(\zeta_2 - \zeta_1)$ or log s tan $(90° + k - \zeta_1)$ · (s in meters.)	Log B, units of fifth place of decimals.	Log s tan $\frac{1}{2}$ $(\zeta_2 - \zeta_1)$ or log s tan $(90° + k - \zeta_1)$ · (s in meters.)	Log B units of fifth place of decimals.	Log s tan $\frac{1}{2}$ $(\zeta_2 - \zeta_1)$ or log s tan $(90° + k - \zeta_1)$ · (s in meters.)	Log B units of fifth place of decimals.
	0				
2.167		3.397		3.685	
	1		9		17
2.644		3.445		3.711	
	2		10		18
2.866		3.489		3.735	
	3		11		19
3.011		3.528		3.758	
	4		12		10
3.121		3.565		3.779	
	5		13		21
3.208		3.598		3.800	
	6		14		22
3.281		3.629		3.820	
	7		15		23
3.343		3.658		3.839	
	8		16		24
3.397		3.685		3.857	

Table L gives the values of log B, the correction factor for approximate difference of elevation by showing the limiting values of log $[s \tan \frac{1}{2} (\zeta_2 - \zeta_1)]$ or log $[s \tan (90° + k - \zeta_1)]$ between which log B may be taken as 0, 1, 2, 3, etc., units of the fifth place of decimals. Log B has the same sign as the angle $\frac{1}{2} (\zeta_2 - \zeta_1)$ or $90° + k - \zeta_1$; for example, if log $[s \tan \frac{1}{2} (\zeta_2 - \zeta_1)]$ lies between 3.565 and 3.598 and $\frac{1}{2} (\zeta_2 - \zeta_1)$ is positive, log B = +0.00013, but if $\frac{1}{2} (\zeta_2 - \zeta_1)$ is negative then log B = −0.00013, *i.e.*, 9.99987 − 10, the former way of writing being usually more convenient in practice.

TABLE M

Log s (s in meters).	Log C, units of fifth place of decimals.	Log s (s in meters).	Log C, units of fifth place of decimals.
0.000		5.297	
	0		4
4.875		5.352	
	1		5
5.113		5.395	
	2		6
5.224		5.432	
	3		7
5.297		5.463	

Table M gives the value of log C, the correction factor for distance between stations, by showing the limiting values of log s between which log C may be taken as 0, 1, 2, 3, etc., units of the fifth place of decimals. Log C is always positive.

PROBLEMS

Problem 1. Calculate the orthometric correction for a line extending 2° northward from a point in latitude 45° N at an elevation of 1000 meters.

Problem 2. Compute the correction for reducing to sea-level a latitude observed at an elevation of one mile in latitude 45° N.

Problem 3. Vertical angle from S to B, $+2° 24' 58''.94$. Vertical angle from B to S, $-2° 35' 34''.20$. Elevation of S = 108.87 meters; distance, 23,931.6 meters; log R_0, 6.8052. Compute the elevation of B.

CHAPTER XI

MAP PROJECTIONS

175. Map Projections.

Whenever we attempt to represent a spherical or a spheroidal surface on a plane some distortion necessarily results, no matter how small may be the area in question. The problem to be solved in constructing topographic or hydrographic maps is to find a method which will minimize this distortion under the existing conditions. The number of projections which have been devised is very great; for the description and the mathematical discussion of the properties of these projections the reader is referred to such works as Thomas Craig's *Treatise on Projections, United States Coast and Geodetic Survey*, 1882; *The Coast and Geodetic Survey Report*, 1880; C. L. H. Max Jurisch, *Map Projections*, Cape Town, 1890; G. James Morrison, *Maps, Their Uses and Construction*, London, 1902; and A. R. Hinks, *Map Projections*, Cambridge, 1912.

In this chapter we shall consider only those projections which are used for such maps and charts as are of importance in geodetic surveys and in navigation.

176. Simple Conic Projection.

In this projection the map is conceived to be drawn on the surface of a right circular cone which is tangent to the sphere or the spheroid along the middle parallel of latitude. The apex of the cone lies in the prolongation of the axis of the spheroid. From Fig. 98 it is evident that the distance TA from the apex to the parallel through A is equal to $N \cot \phi$. If the cone is developed on a plane surface we shall have a sector whose center is T and whose radius is $N \cot \phi$. (Fig. 99.) All other parallels of latitude on the map will be circles drawn about the same center T, and all meridians will be represented by straight lines passing through T. The spacing between the parallels of latitude is obtained by laying off distances along the central meridian which are proportional to the distances between the same parallels on the spheroid. The position of the meridians is found by subdividing the middle parallel into spaces which are proportional to the lengths of the arcs of the

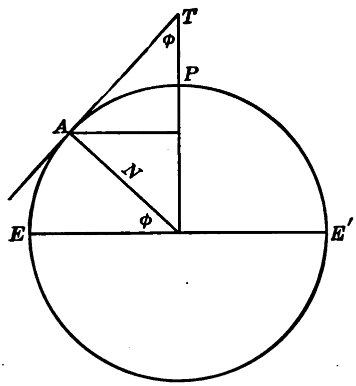


FIG. 98.

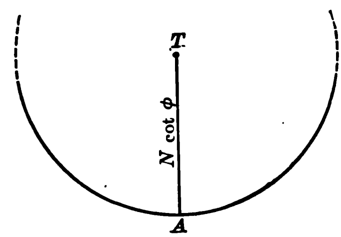


FIG. 99.

same parallel on the spheroid. Straight lines are then drawn from the center T through these points of sub-division. Any meridian or any parallel may be assumed for the central meridian and middle parallel of the map. It is evident from the above that this is not a true projection, that is, the points are not those that would be obtained by projecting from the center of the sphere onto the cone. If the scale of the map is such that the position of the center T cannot be represented on the paper, the curves may be laid off by plotting certain points by means of their rectangular coördinates as described later under the polyconic projection.

It is evident that the meridians and parallels of a conic projection intersect at right angles in all parts of the map, as they do on the sphere. The scale of the map is not correct, however, except along the middle parallel. For a map having a great extension in the longitude and but little in the latitude, the conic projection is fairly accurate. Fig. 100 shows a completed conic projection covering the area of the United States.

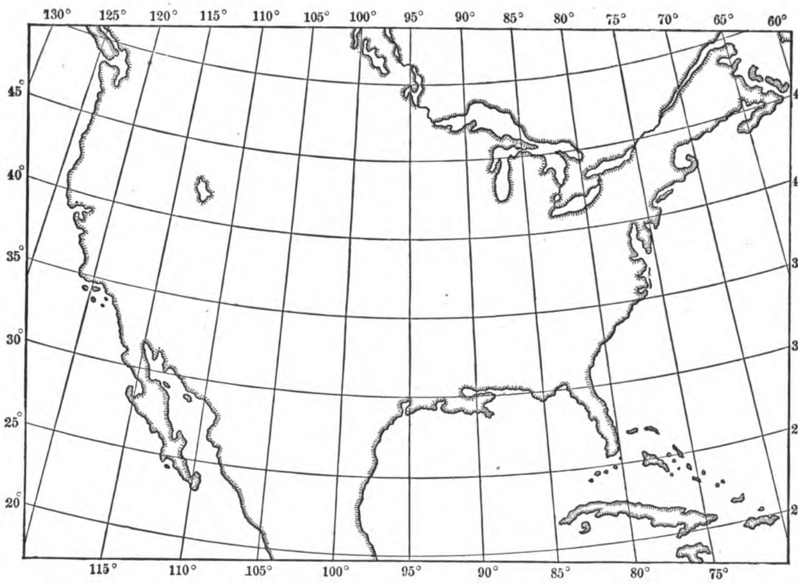


FIG. 100. Simple Conic Projection.

177. Bonne's Projection.

This projection is a modification of the simple conic and meets the objection that the scale of the latter becomes inaccurate as the distance from the middle parallel increases. The parallels of latitude are concentric circles as before, but *each* parallel is sub-divided into spaces which are proportional to the corresponding spaces on that parallel on the spheroid. The central meridian and all parallels are therefore correctly sub-divided. The meridians are obtained by joining the points of sub-division on the parallels. The meridians in this projection are all curved, except the central one, and they intersect the parallels nearly, but not quite, at right angles (Fig. 101). The distortion in this projection is very small, and for small areas it is practically a perfect projection. It has been much used in Europe.

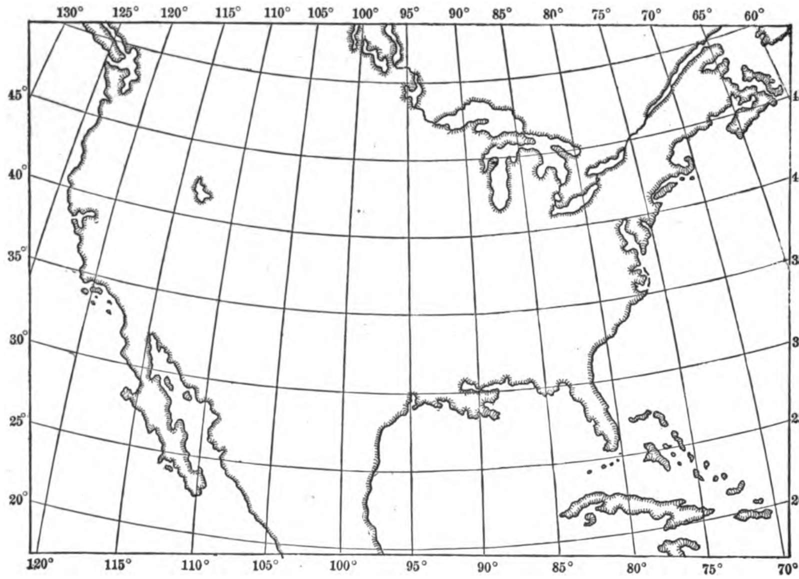


FIG. 101. Bonne's Projection.

178. The Polyconic Projection.

The idea of using several cones, or the polyconic projection, is due to Mr. F. R. Hassler, the first superintendent of the Coast Survey. Each parallel of latitude shown on the map is developed on a cone tangent along that parallel. The radius (TA) for any parallel (latitude ϕ) is $N \cot \phi$; and the angle between two elements of the cone when developed is approximately $\theta = (d\lambda) \sin \phi$, as will be evident from Fig. 102.

In constructing the map the degrees of latitude are laid off along the central meridian, the spacing corresponding to the distances on the spheroid. The points where the meridians intersect the parallels are plotted from their rectangular co-ordinates, the coördinate axes being in each case the central meridian and a line at right angles to it drawn through the latitude in question. The coördinates themselves are found as follows: In Fig. 103, let A be the intersection of some meridian and parallel which are to be drawn on the map. Then the

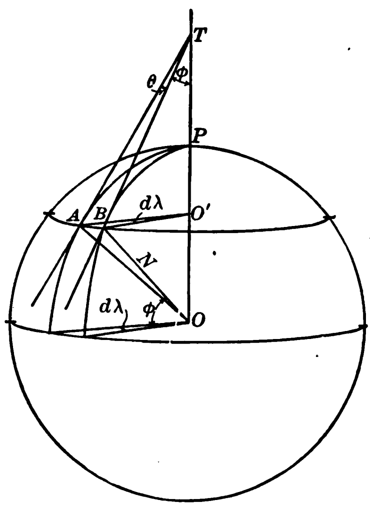


FIG. 102.

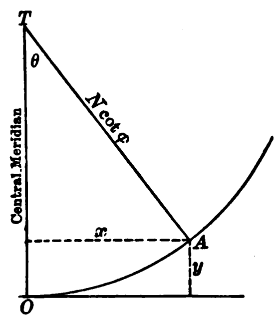


FIG. 103.

radius $TA = N \cot \phi$ may be computed from the known latitude of A, and the angle θ may be computed from the known difference in longitude between O and A by the equation $\theta = (d\lambda) \sin \phi$. Then for x and y we have

$$x = TA \sin \theta = N \cot \phi \sin (d\lambda \sin \phi) \qquad [131]$$

and

$$y = TA \operatorname{vers} \theta = \frac{x}{\sin \theta} \operatorname{vers} \theta$$

$$= x \tan \frac{\theta}{2}$$

$$= x \tan \tfrac{1}{2} (d\lambda \sin \phi) \qquad [132]$$

Values of these numbers will be found in Tables XVI and XVII.

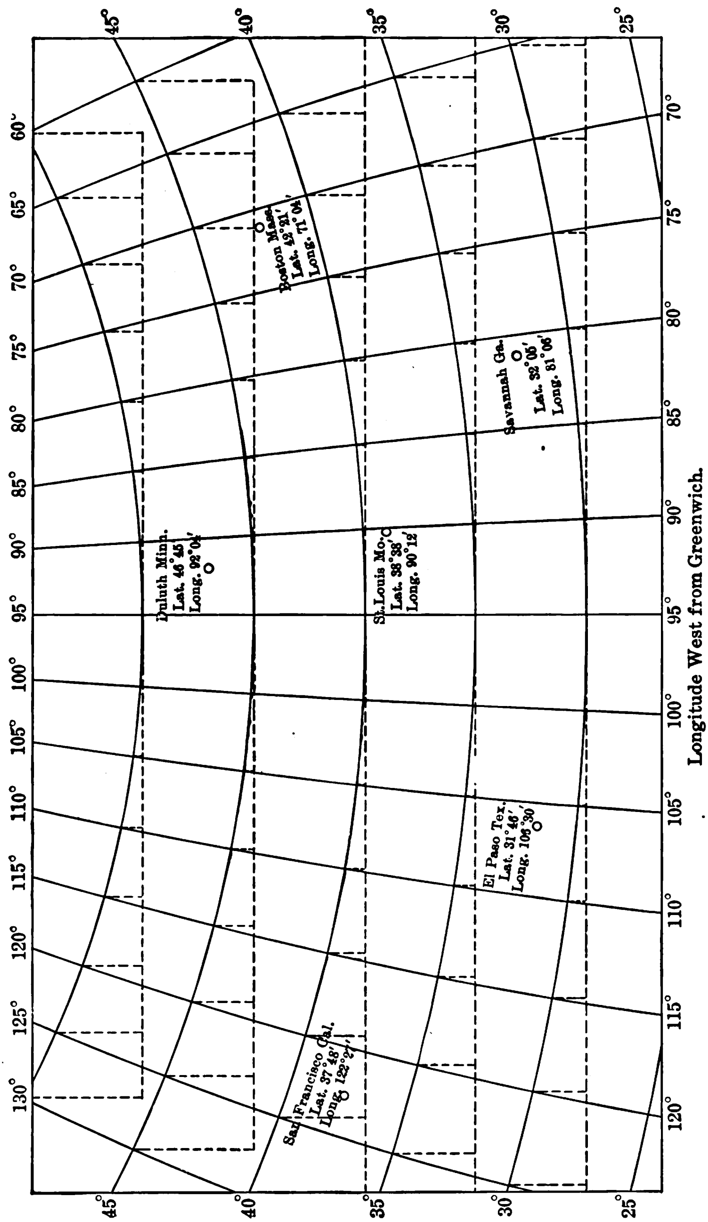


FIG. 104. Polyconic Projection.

It is evident that the parallels and meridians do not intersect at right angles except at the central meridian. The meridian and parallels are both curved, as in Bonne's projection, but since the lower parallels are flatter there is a separation of the parallels which becomes more marked toward the east and west margins of the map. For this reason this map becomes less and less accurate as the longitude is extended. In mapping areas which extend principally north and south, it is superior to other projections. It is in general use in the United States for Government maps. Fig. 104 shows a polyconic projection covering the area of the United States.

There is one disadvantage in the Polyconic and the Bonne's projections, namely, that if two maps of adjoining areas are to be placed side by side they cannot be placed exactly in contact because the limiting (common) meridian curves in opposite directions on the two maps. In the simple conic and in the Lambert projection, to be described in the next article, the meridians are straight and this difficulty does not exist.

179. Lambert's Projection.

The Lambert projection having two standard parallels was invented about the middle of the eighteenth century, but has recently been brought into prominence through its use in the French battle maps. The fundamental notion is that of a cone tangent along the middle parallel of the map, the radius of this parallel (on the map) being $N \cot \phi$, and the angle between the central meridian and any other meridian being $(d\lambda) \sin \phi$. This would give a map in which one parallel, and only one, is correctly divided. We may, however, modify the projection so as to have two standard (correct) parallels. This is done by reducing the scale (multiplying by a constant) and is practically equivalent to employing a cone which cuts the spheroid in the two standard parallels.

The other parallels are so spaced that the scale of the map is the same for all azimuths at any one place, that is, the scale along a meridian is the same as the scale in an east and west plane. A projection having this property is said to be "conformal." It may be proved that this condition is true if the spacing between parallels is $\beta + \frac{\beta^3}{6\,\rho_0^{\,2}}$, where β is the arc of the meridian between parallels on the original tangent cone measured from the parallel of contact, and ρ_0 is the mean radius of curvature of the spheroid at a point on this tangent parallel. Since the projection is conformal, all lines on the map cut each other at the same angles as do the corresponding lines on the spheroid. There is a tendency, therefore, for small figures to have the same shape on the map that they have on the earth's surface. The scale of this map is correct on the two standard parallels. Between these two parallels the scale is a little too small and outside these parallels the scale is too large. The error is not serious, however, if the standard parallels are chosen, as is usual, one sixth and five sixths the length of the meridian arc to be shown. Fig. 105 shows a Lambert projection.

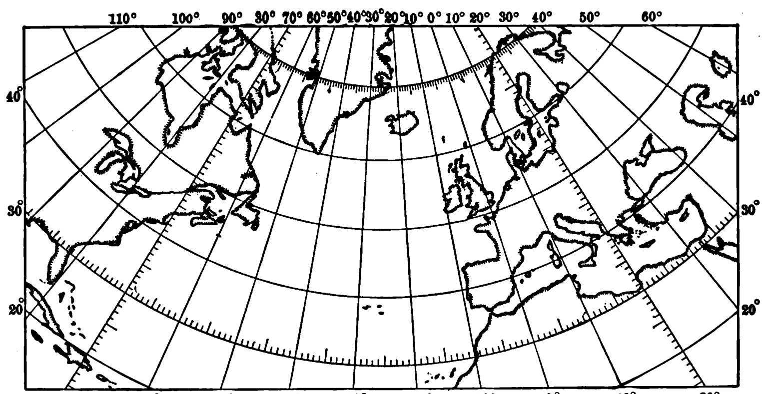


FIG. 105. Lambert Projection.

This projection may be extended indefinitely in an east and west direction without error. The error becomes greater and greater as the map is extended to the north and south. In this respect it is just the contrary of the Polyconic Projection.

For a complete description of this projection, together with tables for projecting maps, see United States Coast Survey Special Publications 47 and 52.

180. The Gnomonic Projection.

In the gnomonic, or central, projection the projecting point is at the center of the sphere and the plane of the map is tangent to the sphere at some selected point. Every plane through the center cuts the sphere in a great circle and cuts the map in a straight line; hence every great circle is represented by a straight line and every straight line on the map must represent a great circle.

Fig. 106 shows the Atlantic Ocean projected on a plane tangent at $\phi = 30°$ N and $\lambda = 30°$ W.

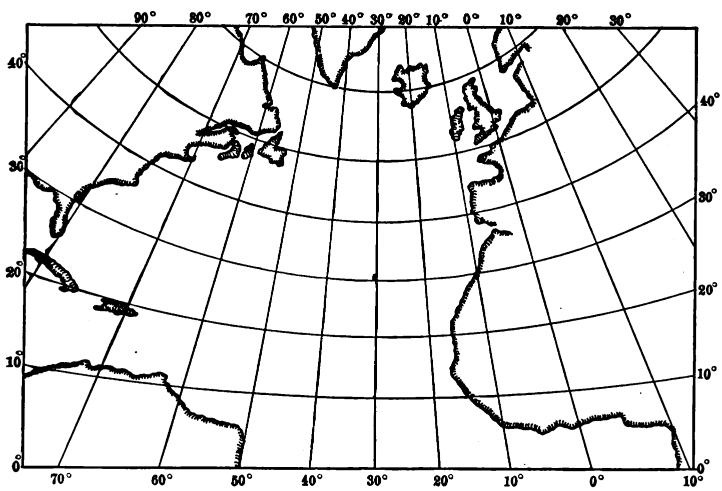


FIG. 106. Gnomonic Projection or Great-circle Chart.

The meridians and the equator are of course represented by straight lines. The parallels of latitude are conic sections, in this case hyperbolas. The parallels are best constructed by employing the equations of the curves and plotting points by means of coördinates.

The gnomonic projection is used almost exclusively for determining the positions of great circles for the purposes of navigation. By joining any two places by a straight line the great-circle (or shortest) track is at once shown. The latitudes and longitudes of any number of points on this track may be read off the chart and, if desired, may be transferred to any other chart and the curve sketched in. The point where the great circle approaches most nearly to the pole is found at once by drawing from the pole a line perpendicular to the track. The foot of this perpendicular is the vertex, or point of highest latitude.

181. Cylindrical Projection.

If a cylinder is circumscribed about a sphere so as to be tangent along the equator, and if points be projected onto the cylinder by straight lines from the center, the cylinder, when developed will give a map in which the meridians and parallels are all straight lines, the relative distances between points being approximately correct near the equator but distorted in high latitudes. The meridians will all be parallel to each other. The parallels of latitude will be parallel to each other and will be spaced wider and wider apart as the latitude increases. Evidently the scale of the map is different for different latitudes. It is also true that at any point the scale along a meridian is not the same as the scale along a parallel. Such a projection is of no practical value, but it aids in understanding the Mercator chart which is described in the next article.

182. Mercator's Projection.

A modification of the above projection, known as Mercator's, consists in so spacing the parallels of latitude that the relation between increments of latitude and longitude on the chart is the same as the relation between increments of latitude and longitude at the corresponding point on the earth's surface, or approximately, 1′ lat. on chart: 1′ long. on chart = 1′ lat. on spheroid: 1′ long. on spheroid. If this relation is preserved, it will be found that any line of constant bearing (*loxodrome* or *rhumb line*) will be represented by a straight line on the chart.

In Fig. 107 let AB on the earth's surface be represented by $A'B'$ on the chart (actual size). In order that the two lines may have the same bearing it is necessary that

$$\frac{dy}{dx} = \frac{AC}{CB} = \frac{R_m\, d\phi}{R_p\, d\lambda}$$

or

$$dy = \frac{dx}{R_p\, d\lambda} \cdot R_m\, d\phi. \qquad (a)$$

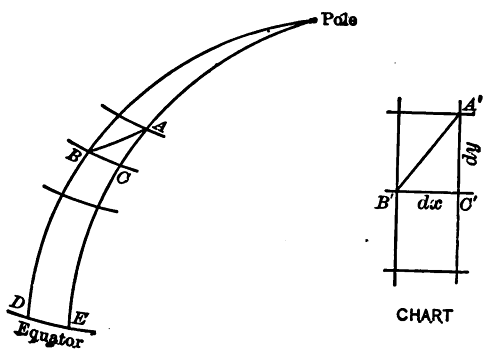


FIG. 107.

In other words, since the longitude has been expanded $\left(\text{in the ratio } \frac{a}{R_p}\right)$ by the method of constructing the chart, it is necessary to expand the latitudes in the same ratio in order to preserve the scale and give AB the same bearing. Now since dx is represented as large as the corresponding arc on the equator, we have

$$\frac{dx}{R_p\, d\lambda} = \frac{a\, d\lambda}{R_p\, d\lambda} = \frac{a}{R_p}.$$

Substituting in (a), we obtain

$$dy = \frac{a}{R_p} \cdot R_m\, d\phi$$

or, since $R_p = N \cos\phi$

$$dy = \frac{R_m}{N\cos\phi} \cdot a\, d\phi$$

$$= \frac{a\,(1 - e^2)}{\cos\phi\,(1 - e^2 \sin^2\phi)} \cdot d\phi$$

$$\therefore \quad y = a \int_0^\phi \frac{(1 - e^2)}{\cos\phi\,(1 - e^2 \sin^2\phi)} \cdot d\phi.$$

Multiplying e^2 by $\sin^2\phi + \cos^2\phi$, the integral may be separated into two, giving, after multiplying numerator and denominator by $\cos\phi$,

$$y = a \int_0^\phi \frac{\cos\phi\, d\phi}{\cos^2\phi} \cdot - ae \int_0^\phi \frac{e\cos\phi\, d\phi}{1 - e^2 \sin^2\phi}$$

$$= \frac{a}{M}\left[\frac{1}{2}\log\frac{1 + \sin\phi}{1 - \sin\phi} - \frac{1}{2}e \log\frac{1 + e\sin\phi}{1 - e\sin\phi}\right]_0^\phi$$

where $M = 0.4342945$, the modulus of the common logarithms.

Employing the formulæ,

$$\log\frac{1 + x}{1 - x} = 2\left(x + \frac{x^3}{3} + \cdots\right)$$

and

$$\frac{1 + \sin x}{1 - \sin x} = \tan\left(45^\circ + \frac{x}{2}\right)$$

the equation may be expressed

$$y = \frac{a}{M}\left[\log\tan\left(45^\circ + \frac{\phi}{2}\right)\right]_0^\phi - ae\left[e\sin\phi + \frac{(e\sin\phi)^3}{3} + \cdots\right]_0^\phi \quad [133]$$

in which y is in the same linear units as a.

In order to express y in nautical miles or minutes of arc on the equator * it is necessary to multiply by $\frac{60 \times 180}{a\pi}$, giving,

* The Nautical Mile contains 6080.20 ft.; this is not identical with the number of feet in one minute of arc on the earth's equator. For a discussion of this matter, see Appendix 12, Coast Survey Report for 1881.

$$y = 7915.705 \log \tan\left(45^\circ + \frac{\phi}{2}\right) - 3437.7\left(e^2 \sin^2\phi + \frac{e^4 \sin^3\phi}{3}\right), \qquad [134]$$

or

$$y = 7915.705 \log \tan\left(45^\circ + \frac{\phi}{2}\right) - 22'.945 \sin\phi - 0.051 \sin^3\phi. \qquad [135]$$

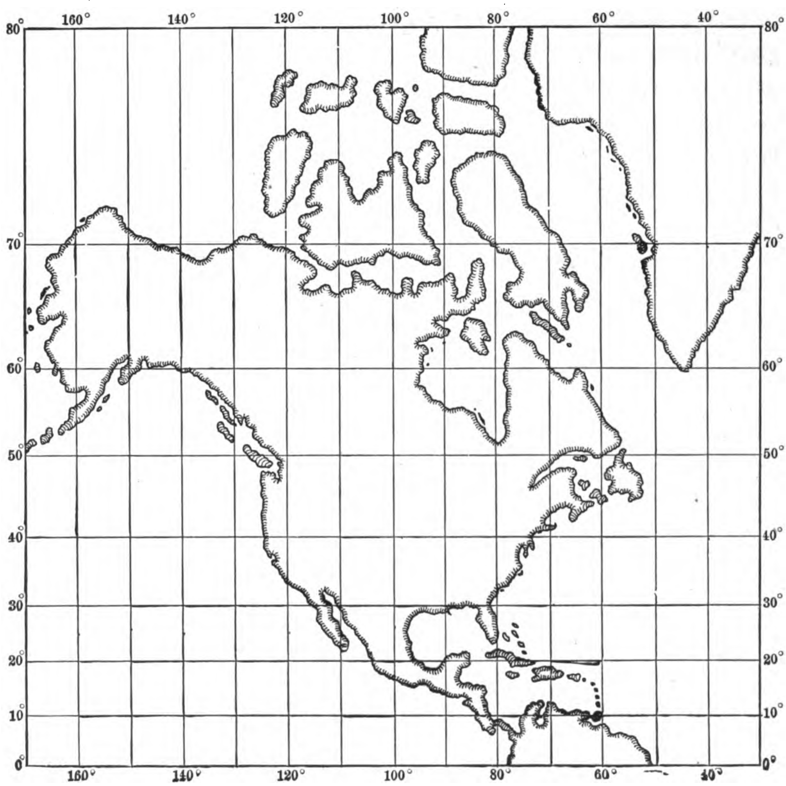


FIG. 108. Mercator Chart.

Also $\qquad x = 60 \times \lambda^\circ, \qquad [136]$

the unit being the nautical mile. Values of y, called *meridional parts*, will be found in works on navigation.

This chart is much used by navigators because it possesses the property that the bearing of any point B from a point A as measured on the chart is the same as that bearing on which a vessel must sail continuously to go from *A* to *B*. The track cuts all meridians on the globe at the same angle, just as a straight line on the chart cuts all meridians at the same angle. This track is not the shortest one between *A* and *B*, but for ordinary distances the length differs but little from that of the great-circle track. In following a great-circle track the navigator transfers to the Mercator chart a few points on the great-circle obtained from his great-circle chart, by means of their latitudes and longitudes and then sails on the rhumb lines between consecutive plotted points. Fig. 108 shows a Mercator chart.

183. Rectangular Spherical Coördinates.

A system of rectangular spherical coördinates, used in Europe, consists in referring all points to two great circles through some selected origin, one of them being the meridian, the other the prime vertical. Within small areas these coördinates are practically the same as rectangular plane coördinates. When the area is so great that the effect of curvature becomes appreciable, small corrections are introduced, so that the form of the plane coördinates is retained without loss of accuracy. Such a system is very convenient when connecting detail surveys with the triangulation, particularly for local surveyors who may not be familiar with geodetic methods of calculating latitudes and longitudes. The method is not well adapted to mapping very large areas. (See Crandall's *Geodesy*, p. 187.)

CHAPTER XII

APPLICATION OF METHOD OF LEAST SQUARES TO THE ADJUSTMENT OF TRIANGULATION

184. Errors of Observation.

Whenever an observer attempts to determine the values of any unknown quantities, he at once discovers a limit to the precision with which he can make a single measurement. In order to secure greater precision in his final result than can be obtained by a single measurement, he resorts to the expedient of making additional measurements, either under the same conditions or under different conditions. Under these circumstances it will be observed that the results are discordant and that the same numerical result almost never occurs twice.* The question at once arises, then, What are the best values of the unknown quantities which it is possible to obtain from these measurements?

The method of least squares has for its main objects (1) the determination of the best values which it is possible to obtain from a given set of measurements, and (2) the determination of the degree of dependence which can be placed upon these values, or, in other words, the relative worth of different determinations; (3) it also enables us to trace to their sources the various errors affecting the measurements and consequently to increase the accuracy of the result by a proper modification of the methods and instruments used. The method is founded

* This is only true, however, when the observer is taking each reading with the utmost possible refinement. If, for example, angles are read only to the nearest degree, the result will always be the same no matter how many times the measurement may be repeated; but if read to seconds and fractions, they will in general all be different.

upon the mathematical theory of probability, and upon the assumption that those values of the unknowns which are rendered most probable are the best that can be obtained from the measurements.

185. Probability.

If an event can happen in a ways and fail in b ways, and all of these ways are equally likely to occur, the probability that the event will happen in any one trial is expressed by the fraction $\frac{a}{a+b}$, and the probability that it will fail is expressed by $\frac{b}{a+b}$. Since it must either happen or fail, the sum of the two probabilities represents a certainty. This sum is $\frac{a}{a+b} + \frac{b}{a+b} = 1$. Therefore the probability of the happening of an event is represented by some number lying between 0 and 1, the larger the fraction the greater the probability of its happening. For example, a die may fall so that any one of its six faces is uppermost, and all of these six possibilities are equally likely to occur; the probability of any one of its faces being up is $\frac{1}{6}$.

186. Compound Events.

If a certain event can happen in a ways and fail in b ways, and if a second, independent, event can happen in a' ways and fail in b' ways, and all are equally likely to occur, then the total number of ways in which the events can take place together is $(a + b)(a' + b')$. The number of ways in which both can happen is aa', and the probability of its happening is $\frac{aa'}{(a+b)(a'+b')}$. For example, the probability of double six being thrown with a pair of dice is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$. It is evident that the probability of the simultaneous occurrence of two events is the product of the probabilities of the occurrence of the component events. In a similar way it may be shown that the probability of the simultaneous occurrence of any number of independent events is the product of their separate probabilities; that is, if P_1, P_2, P_3 . . . are the probabilities of the occurrence of any number of independent events, the probability of their simultaneous occurrence is

$$P = P_1 \times P_2 \times P_3 \ . \ . \ . \ , \qquad [137]$$

187. Errors of Measurement — Classes of Errors.

Every measurement of a quantity is subject to error, of which the following kinds may be distinguished.

1. Constant Errors.
2. Systematic Errors.
3. Accidental Errors.

188. Constant Errors.

A constant error has the same effect upon all observations in the same series of measurements. For instance, if a steel tape is 0.01 ft. too long, this error affects every 100 ft. measurement in just the same way.

189. Systematic Errors.

A systematic error is one of which the algebraic sign and the magnitude bear a fixed relation to some condition. For example, if the measurements with the tape are made at different temperatures, the error resulting from this variation of temperature is systematic and may be computed if the temperatures and the coefficient of expansion are known.

190. Accidental Errors.

Accidental errors are not constant from observation to observation; they are just as likely to be positive as negative; in general they follow the exponential law of error, as will be explained later (Art. 197). The error of placing a mark opposite to the end graduation of the tape is of this class.

191. Comparison of Errors.

There is in reality no fixed boundary between the accidental and the systematic errors. Every accidental error has some cause, and if the cause were perfectly understood and the amount and sign could be determined, it would cease to be an accidental error, but would be classed as systematic. On the other hand, errors which are either constant or systematic may be brought into the accidental class, or at least made to partially obey the law of accidental error, by so varying the conditions, instruments, etc., that the sign of the error is frequently reversed. If a tape has 0.01 ft. uncertainty in length, this produces a constant error, in the result of a measurement. If, however, we use several different tapes, each with an uncertainty of 0.01 ft., this error may be positive or negative in any one case. In the long run these different errors tend to compensate each other like accidental errors.

In the class of systematic errors would be placed such errors as those due to changes in temperature, light, and moisture, or change in the adjustments of instruments. These errors may be computed and allowed for as soon as we know the law governing their action, or they may be partially eliminated by varying conditions under which the measurements are made.

Under the constant class comes the observer's error, which tends to become constant with increased experience in observing. This error may be allowed for as soon as its magnitude and sign have been determined, or it may be eliminated by the method of observation. Certain errors in the instrument may have a constant effect on the result; these may be dealt with in the same manner as the personal error. It should be noticed that after the constant error or the systematic error has been eliminated, there still remains a small error due to the fact that the magnitude of the constant error itself was not perfectly determined or that its elimination was imperfect. This remaining error must be regarded as an error of the accidental class, since its magnitude is unknown and it is just as likely to be positive as negative.

Under accidental errors are included all those which are supposed to be small and just as likely to be positive as negative. They are due to numerous unknown causes, each error being in reality the algebraic sum of many smaller errors. Under this class may be noted errors in pointing with a telescope, errors in reading scales and estimating fractions of scale divisions, and undetected variations in all of the conditions governing systematic errors.

192. Mistakes.

These are not errors, but they must be considered in connection with the discussion of accuracy of observations. They include such cases as reading one figure for another, as a 6 for a 0, or reading a scale in the wrong direction, as reading 46° for 34°.

193. Adjustment of Observations.

When the number of measurements is just sufficient to determine the quantities desired, then there is but one possible solution, and the results must be accepted as the true values. When additional measurements are made for the purpose of increasing the accuracy of the results, this gives rise to discrepancies among the different measurements of the same quantities, since each is subject to errors. The method of least squares enables us to compute those values which are rendered most probable by the existence of the observations and in view of the discrepancies noted; it cannot, however, tell us anything about the existence of constant errors, unless new observations made under different conditions reveal new discrepancies. For example, if a pendulum is swung and certain small variations in the last decimal place of the period are noticed, these may be regarded as due to small errors in the running of the chronometer and to accidental errors of observing; but if the pendulum case be mounted on a support whose flexibility is very much greater than that of the first, and larger variations are now observed, it becomes apparent that an error of the systematic class is affecting all our observations, though it does not appear at all in the first observations, because all the measurements were affected alike. An investigation of the law governing this error, and the determination of its magnitude and sign, enable us to correct the result for such part of the error as we are able to determine. There remains in the result, however, an accidental error, namely, the error in the measurement of the flexure correction.

194. Arithmetical Mean.

The formulæ employed in adjusting observations are usually made to depend upon the axiom that if a number of observations be made directly upon the same quantity, all made under the same conditions and with the same care, the most probable value of the quantity sought is the arithmetical mean of all the separate results; that is, if the results of the observations are $M_1, M_2, M_3, \ldots M_n$, the most probable value of the quantity, M_0, is given by

$$M_0 = \frac{M_1 + M_2 + \cdots M_n}{n} = \frac{\sum M}{n}. \qquad [138]$$

It is to be carefully noted that this is not the true value, M, but simply the most probable value under the circumstances; if additional measurements be made, M_0 changes correspondingly in value, because we know more about its real value than we did at first.

195. Errors and Residuals.

It now becomes necessary to distinguish between *errors* and *residuals*. The *error* is the difference between any measured value and the true value. Its magnitude can never be known, because the true value can never be known. The *residual* is the difference between a measured value and the most probable value. This is a quantity which may be computed for any set of observations. In a set of very accurate observations which are free from constant and systematic errors the residual is a close approximation to the true error. It may be shown that for the case of direct observations the algebraic sum of the residuals is zero; that is, if we compute $v_1 = M_1 - M_0$, $v_2 = M_2 - M_0$, etc., then $\sum v = 0$, where $v_1, v_2 \ldots$ are the residuals.

196. Weights.

In case the measurements are of different degrees of reliability, they are given different *weights*. The weight of an observation may be regarded as the number of times the observation is repeated and the same numerical result obtained.

It expresses the relative worth of different measured values. Weights are purely relative and may be computed on any base desired. To say that two measurements have weights 2 and 1 respectively, is the same as saying that they have weights $\frac{1}{2}$ and $\frac{1}{4}$. From the above definition it is apparent that the *weighted mean* is expressed by

$$M_0 = \frac{p_1M_1 + p_2M_2 + \cdots}{\sum p} = \frac{\sum pM}{\sum p}; \qquad [139]$$

that is, the weighted mean is found by multiplying each observation by its weight, adding the results, and dividing by the sum of the weights.

Multiplying an observation (M_1) by its weight (p_1) is the same as taking p_1 observations each equal in value to M_1.

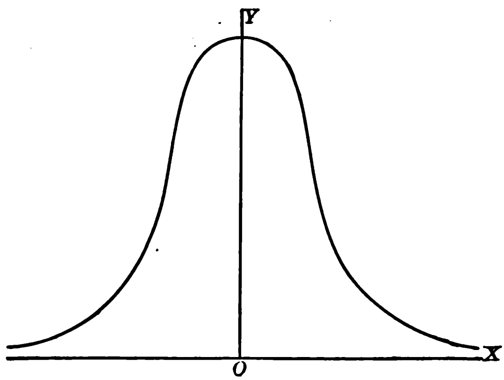


FIG. 109.

197. Distribution of Accidental Errors.

An inspection of the results of a large number of measurements will show that

(1) + and − errors are equally numerous.
(2) Small errors are much more numerous than large ones.
(3) Very large errors seldom occur.

The curve which expresses the law of variation of such errors will be of the form shown in Fig. 109. In accordance with (1) the curve is symmetrical; in accordance with (2) its maximum is at the axis of Y; from (3) it is evident that the curve cuts the axis of X at some distance from O.

The manner in which observations are affected by accidental errors is shown by the "shot apparatus" shown in Fig. 110. A

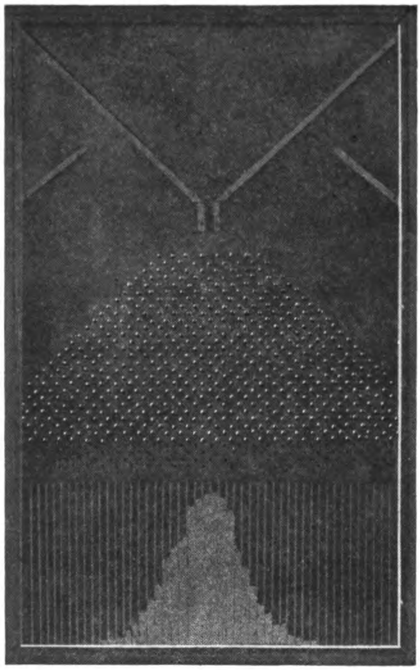


FIG. 110. "Shot Apparatus."

large number of small shot, representing observations, are allowed to drop through an opening in the middle of the case. If there were no obstructions the shot would fall directly into the central (vertical) compartment. Between the opening and the vertical compartments a number of pegs are interposed, each representing a source of error or deflection of the shot from its natural course. The shot are therefore diverted somewhat from a straight course and arrange themselves in the different columns in the manner shown. The curve joining the tops of the columns is seen to resemble closely the "curve of error."

In order to obtain a formula expressing the law of error we suppose the curve asymptotic to the axis of X, and write the equation of the curve in the general form

$$y = f(x), \qquad [140]$$

where x represents the magnitude of an error and y the frequency with which this error occurs on a large number of measurements; f represents some unknown function of x. It is necessary to assume that the number of observations is very large; otherwise the supposed balancing of + and − errors will be imperfect. The true error x can never be known, but the distribution of the residuals about the most probable value will evidently follow the same general law, so we may write also

$$y = f(v) \qquad [141]$$

as the law to which the residuals must conform. This equation also expresses the probability of the occurrence of a residual v.

If we let the total area between the curve and the axis of X be represented by unity, then the probability that a certain residual will fall between the limits v and $v + dv$ will be represented by the area included between the curve, the X axis, and the two ordinates at v and $v + dv$, since in the long run the number in a given column will be proportional to the probability expressed by the ordinate at that point, that is,

$$y\,dv = f(v)\,dv. \qquad [142]$$

If we suppose n observations of equal weight, giving the results $M_1, M_2, \ldots M_n$, to be made on any functions of the unknowns $z_1, z_2, \ldots, z_n$, giving the residuals $v_1, v_2, \ldots, v_n$, then the probability of the occurrence of these residuals is $f(v_1)\,dv$, $f(v_2)\,dv \ldots f(v_n)\,dv$. The probability of the simultaneous occurrence of these residuals is the product of the separate probabilities, that is,

$$P = f(v_1)\,dv \times f(v_2)\,dv \times \ldots f(v_n)\,dv. \qquad [143]$$

or, taking logs of both members of the equation,

$$\log P = \log f(v_1) + \log f(v_2) + \cdots \log f(v_n) + n \times \log dv.$$

The results desired for z_1, z_2, etc., are those for which the probability of the occurrence of $v_1, v_2, \ldots$ is a maximum. Therefore P must be a maximum. To find the conditions for this maximum, differentiate $\log P$ with respect to each variable, $z_1, z_2, \ldots$, and place the results equal to zero. This gives

$$\left.\begin{array}{l}
\dfrac{\partial \log P}{\partial z_1} = \dfrac{1}{f(v_1)} \cdot \dfrac{\partial f(v_1)}{\partial z_1} + \cdots \dfrac{1}{f(v_n)} \cdot \dfrac{\partial f(v_n)}{\partial z_1} = 0. \\
\dfrac{\partial \log P}{\partial z_2} = \dfrac{1}{f(v_2)} \cdot \dfrac{\partial f(v_1)}{\partial z_2} + \cdots \dfrac{1}{f(v_n)} \cdot \dfrac{\partial f(v_n)}{\partial z_2}. \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}\right\} \qquad [144]$$

But we observe that

$$\frac{\partial f(v)}{\partial z} = f'(v)\frac{\partial v}{\partial z}, \qquad [145]$$

in which f' represents some new function of v.

For brevity place $$\frac{f'(v)}{f(v)} = F(v). \qquad [146]$$

Then Equa. [144] become

$$\left.\begin{array}{l}
F(v_1)\dfrac{\partial v_1}{\partial z_1} + F(v_2)\dfrac{\partial v_2}{\partial z_1} + \cdots F(v_n)\dfrac{\partial v_n}{\partial z_1}. \\
F(v_1)\dfrac{\partial v_1}{\partial v_2} + F(v_2)\dfrac{\partial v_2}{\partial v_2} + \cdots F(v_n)\dfrac{\partial v_n}{\partial v_2}. \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}\right\} \qquad [147]$$

These equations contain all the unknown quantities (z) and are equal in number to the number of unknown quantities.

Hence, if the form of the function F were known, the solution of these equations would give the most probable values of z_1, z_2, etc.

The above equations, being perfectly general, hold true for all cases, so they must hold true for any special case. The form of F determined for the special case must therefore be the form of this function for all cases.

Consider n direct observations of equal weight on one unknown quantity z_1, the results of the measurements being M_1, M_2, . . . M_n, and the residuals being denoted by v_1, v_2, . . . v_n. The most probable value of z_1 is given by

$$z_1 = M_1 - v_1 = M_2 - v_2 = \ldots M_n - v_n.$$

Differentiating with respect to z_1,

$$1 = -\frac{\partial v_1}{\partial z_1} = -\frac{\partial v_2}{\partial z_1} = \cdot \cdot \cdot = -\frac{\partial v_n}{\partial z_1}. \qquad (a)$$

Substituting these values in Equa. [147], we obtain

$$F(v_1) + F(v_2) + \cdot \cdot \cdot + F(v_n) = 0. \qquad (b)$$

But in this special case (Art. 195),

$$v_1 + v_2 + \cdot \cdot \cdot + v_n = 0. \qquad (c)$$

Hence, if both Equa. (b) and (c) are true. F must signify multiplication by a constant; that is,

$$F(v) = cv. \qquad [148]$$

Substituting in Equa. [146] and [145],

$$\frac{\partial f(v)}{\partial z} = f(v) \cdot cv \frac{\partial v}{\partial z},$$

and

$$\frac{1}{f(v)} \cdot \frac{\partial f(v)}{\partial z} = cv \frac{\partial v}{\partial z}.$$

Integrating both members,

$$\log f(v) = \tfrac{1}{2} cv^2 + c'.$$

Therefore

$$f(v) = e^{(\frac{1}{2} cv^2 + c')}$$
$$= ke^{\frac{1}{2} cv^2}.$$

Substituting this in the equation of the curve of error $(y = f(v))$, we have

$$y = ke^{\frac{1}{2} cv^2}.$$

In reality y decreases as v increases; the exponent of e is therefore negative, and, since the constants may be combined, we have

$$y = ke^{-h^2v^2}, \qquad [149]$$

in which h^2 and k are constants depending upon the character of the observations. This equation expresses the law in accordance with which the residuals must be distributed in order to give a maximum value of P. If we replace v by x, the equation also shows the law governing the distribution of the actual errors.

It is important to note that the law governing the distribution of accidental errors holds true *in the long run;* in order to have a close agreement of the theory with the results actually observed it is essential that the number of observations should be very large. With a limited number of observations we should expect that the residuals would follow the law only approximately.

198. Computation of Most Probable Value.

From Equa. [143] we have seen that

$$P = f(v_1) \times f(v_2) \ . \ . \ . \times f(v_n)\ (dv)^n = \text{a maximum.} \qquad [150]$$

Applying Equa. [149], this becomes

$$P = k^n e^{-h^2 (v_1^2+v_2^2+ \cdots v_n^2)}\ (dv)^n = \text{a maximum.} \qquad [151]$$

It is evident that P is a maximum when

$$v_1^2 + v_2^2 + \cdots v_n^2 = \text{a minimum,} \qquad [152]$$

that is, when the sum of the *squares* of the residuals has its *least* value.

Equa. [147] express the conditions necessary to make P a maximum or to make the sum of the squares of the residuals a minimum. Since the function F means multiplication by a constant, Equa. [147] become

$$\left.\begin{array}{l}
v_1 \dfrac{\partial v_1}{\partial z_1} + v_2 \dfrac{\partial v_2}{\partial z_1} + \cdots v_n \dfrac{\partial v_n}{\partial z_1} = 0. \\
v_1 \dfrac{\partial v_1}{\partial z_2} + v_2 \dfrac{\partial v_2}{\partial z_2} + \cdots v_n \dfrac{\partial v_n}{\partial z_2} = 0. \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
v_1 \dfrac{\partial v_1}{\partial z_q} + v_2 \dfrac{\partial v_2}{\partial z_q} + \cdots v_n \dfrac{\partial v_n}{\partial z_q} = 0.
\end{array}\right\} \qquad [153]$$

These equations are equal in number to the number, q, of unknown quantities, and their simultaneous solution gives the most probable values of the unknown quantities. They are usually called *Normal Equations.*

199. Weighted Observations.

If the observations are of different weights, each observation equation should be used (Art. 196) the number of times denoted by its weight. Hence, in forming the normal equations we should multiply each observation equation by the coefficient of the unknown *and by the weight of the equation.* The normal equations in this case are as follows:

$$\left.\begin{array}{l} p_1 v_1 \dfrac{\partial v_1}{\partial z_1} + p_2 v_2 \dfrac{\partial v_2}{\partial z_1} + \cdots = 0. \\ p_1 v_1 \dfrac{\partial v_1}{\partial z_2} + p_2 v_2 \dfrac{\partial v_2}{\partial z_2} + \cdots = 0. \\ \cdots\cdots\cdots\cdots = 0. \end{array}\right\} \quad [154]$$

This same result will be obtained if we first multiply each observation equation by the square root of its weight. This shows that multiplying a set of equations by the square roots of their weights reduces them all to observations of weight unity (equal weights).

200. Relation between h and p.

If the n observations have weights p_1, p_2, . . . , and the constant h is h_1, h_2, . . . for these observations, then

$$\begin{aligned} P &= k_1 e^{-h_1^2 v_1^2} \cdot k_2 e^{-h_2^2 v_2^2} \cdots \\ &= k_1 k_2 \cdots k_n e^{-(h_1^2 v_1^2 + h_2^2 v_2^2 \cdots)} (dv)^n, \end{aligned} \quad [155]$$

and $\quad h_1^2 v_1^2 + h_2^2 v_2^2 + \cdots$ is to be a minimum. [156]

The conditions for this minimum are

$$\left.\begin{array}{l} h_1^2 v_1 \dfrac{\partial v_1}{\partial z_1} + h_2^2 v_2 \dfrac{\partial v_2}{\partial z_2} + \cdots = 0. \\ h_1 v_1 \dfrac{\partial v_1}{\partial z_1} + h_2^2 v_2 \dfrac{\partial v_2}{\partial z_2} + \cdots = 0. \\ \cdots\cdots\cdots\cdots \end{array}\right\} \quad [157]$$

Equas. [154] and [157] express the same conditions.

Hence $$p_1 : p_2 : \ldots = h_1^2 : h_2^2 : \ldots, \qquad [158]$$

showing that the weight of an observation varies as the square of the constant h for the observation. Consequently the more accurate the observation the greater the value of h.

Example. As an illustration of the manner of applying these equations to the computation of the most probable values of the unknowns, suppose that at a triangulation station O (Fig. 111), the angles have been measured as shown.

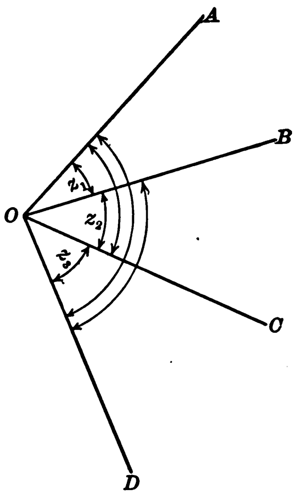


FIG. 111.

Denoting the most probable values of these angles by z_1, z_2, and z_3, the measurements are given by the following equations:

$$\begin{aligned} z_1 &= 31^\circ\ 10'\ 17''.0, \\ z_2 &= 40\ \ 50\ \ 10\ .0, \\ z_3 &= 42\ \ 10\ \ 19\ .7, \\ z_1 + z_2 &= 72\ \ 00\ \ 26\ .0, \\ z_1 + z_2 + z_3 &= 114\ \ 10\ \ 46\ .0, \\ z_2 + z_3 &= 83\ \ 00\ \ 30\ .2. \end{aligned}$$

Denoting by v_1, v_2, etc., the residuals of the different measurements, these may be written

$$\begin{aligned} z_1 - 31^\circ\ 10'\ 17''.0 &= v_1, \\ z_2 - 40\ \ 50\ \ 10\ .0 &= v_2, \\ z_3 - 42\ \ 10\ \ 19\ .7 &= v_3, \\ z_1 + z_2 - 72\ \ 00\ \ 26\ .0 &= v_4, \\ z_1 + z_2 + z_3 - 114\ \ 10\ \ 46\ .0 &= v_5, \\ z_2 + z_3 - 83\ \ 00\ \ 30\ .2 &= v_6, \end{aligned}$$

which are called *observation equations*.

If we apply equations (153), differentiating each v with respect to the three unknown quantities in succession, we obtain the *normal equations*.

$$\begin{aligned} 3 z_1 + 2 z_2 + z_3 - 217^\circ\ 21'\ 29''.0 &= 0, \\ 2 z_1 + 4 z_2 + 2 z_3 - 310\ \ 01\ \ 52\ .2 &= 0, \\ z_1 + 2 z_2 + 3 z_3 - 239\ \ 21\ \ 35\ .9 &= 0. \end{aligned}$$

Solving these simultaneously, we obtain

$$\begin{aligned} z_1 &= 31^\circ\ 10'\ 16''.45, \\ z_2 &= 40\ \ 50\ \ 09\ .875, \\ z_3 &= 42\ \ 10\ \ 19\ .90. \end{aligned}$$

These are the most probable values of the angles.

201. Formation of the Normal Equations.

It should be observed that since the observation equations are linear in this case, the differential coefficients are equal to the numerical coefficients. Hence, to form the normal equations we may proceed as follows: *For each unknown, form a normal equation by multiplying each observation equation by the numerical coefficient of the unknown in that equation, adding these results and placing the sum equal to zero.* This rule is simply a statement in words of what is expressed in Formula [153] as applied to linear equations. If the observations are of different weights, the only change in the above rule is that each observation equation is multiplied by its weight as well as by the coefficient of the unknown.

In regard to the observation equations it should be understood that they are not like ordinary equations. They are often written, however, with zero in place of the v in the right hand member. Observation equations cannot be multiplied by any number or combined with each other (except when forming normal equations); for if this is done, the *weight* of the observation is thereby changed.

202. Solution by Means of Corrections.

If the independent terms * in the observation equations are large, it will often save labor in the calculations if we place the unknown quantity Z_1 equal to an approximate value M_1 plus a correction z_1, $Z_2 = M_2 + z_2$, etc. Substituting these values in the original observation equations, we obtain a new set of equations in terms of the corrections and in which the independent terms will be small. By forming normal equations and solving as before, we find the most probable values of the corrections. Adding these corrections to the approximate values, we find the most probable values of the unknown quantities themselves.

* The independent term in any equation is that term which does not contain any of the unknowns.

Example. In the example just solved, suppose we assume for the approximate values the results of the direct measurements, and let z_1, z_2, etc., represent the most probable corrections. Then the observation equations become

$$\begin{aligned} z_1 &= 0, \\ z_2 &= 0, \\ z_3 &= 0, \\ z_1 + z_2 + 1''.0 &= 0, \\ z_1 + z_2 + z_3 + 0\ .7 &= 0, \\ z_2 + z_3 - 0\ .5 &= 0. \end{aligned}$$

Forming the normal equations as before, we have

$$\begin{aligned} 3\,z_1 + 2\,z_2 + \ \ z_3 + 1''.7 &= 0, \\ 2\,z_1 + 4\,z_2 + 2\,z_3 + 1\ .2 &= 0, \\ z_1 + 2\,z_2 + 3\,z_3 + 0\ .2 &= 0. \end{aligned}$$

The solution of these equations gives

$$\begin{aligned} z_1 &= -0''.55, \\ z_2 &= -0\ .125, \\ z_3 &= +0\ .20, \end{aligned}$$

which, added to the values observed directly, give the same results as before.

203. Conditioned Observations.

If the quantities sought are not independent of each other, but are subject to certain conditions, the solution must be modified accordingly. Each observation gives rise to an observation equation, and each condition may be expressed by a condition equation. The solution may be effected by eliminating, between the two sets of equations, as many unknowns as there are equations of condition. From the remaining equations we may form the normal equations and solve for the most probable values of the unknowns. Substituting these values back in the original condition equations, we obtain the remaining unknowns.

Example. The three angles of a triangle are $A = 61° 07' 52''.00$, $B = 76° 50' 54''.00$, and $C = 42° 01' 12''.15$. The spherical excess is $02''.11$. The weights assigned to the measured angles are 3, 2, and 2, respectively. These angles are subject to the fixed relation $A + B + C = 180° 00' 02''.11$.

Letting v_1, v_2, v_3 be the most probable corrections to the observed values, the observation equations are

$$\begin{aligned} v_1 &= v_1, && \text{wt. } 3 \\ v_2 &= v_2, && \text{“ } 2 \\ v_3 &= v_3, && \text{“ } 2 \end{aligned}$$

and the condition equation is

$$v_1 + v_2 + v_3 - 3''.96 = 0. \qquad (d)$$

Eliminating v_3, there remain

$$v_1 = v_1, \qquad \text{wt. } 3$$
$$v_2 = v_2, \qquad \text{" } 2$$
$$v_3 = -v_1 - v_2 + 3''.96 \qquad \text{" } 2$$

Forming the normal equations and solving,

$$v_1 = +0''.99,$$
$$v_2 = +1\ .485.$$

Substituting these values in equation (d),

$$v_3 = +1''.485.$$

These corrections, added to the measured angles, give the adjusted angles, as follows:

$$A = 61^\circ\ 07'\ 52''.99,$$
$$B = 76\ \ 50\ \ 55\ .48,$$
$$C = 42\ \ 01\ \ 13\ .64.$$

Notice that the discrepancy is distributed inversely as the weights. This will always be the case when each unknown is directly observed, and there is but one equation of condition; that is, the correction to the first is

$$\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{2}+\frac{1}{2}} \times +3''.96 = +0''.99,$$

and the correction to the second is

$$\frac{\frac{1}{2}}{\frac{1}{3}+\frac{1}{2}+\frac{1}{2}} \times +3''.96 = +1''.485.$$

The correction to the third is the same as the correction to the second.

204. Adjustment of Triangulation.

The adjustment of the angles of a triangulation net naturally divides itself into two parts: (1) the adjustment for the discrepancies arising at each station, and (2) the adjustment of the figure as a whole. According to theory these should all be adjusted simultaneously in order to obtain the most probable values of the angles. The usual practice, however, is to deal with the two separately. The local, or station, adjustment is made first if the method of observing is such that a local adjustment is required. If the observations are made in accordance with the program given in Art. 44 (sec. 2, Coast Survey instructions), no station adjustment is necessary. If the angles are measured by the repetition method and the horizon is closed, the error is distributed in inverse proportion to the weights (see Art. 203). If there are conditions existing among the angles, due to measuring sums of the different single angles, the adjustment may be effected by expressing these as condition equations and then forming normal equations and solving, as in the example, p. 294.

This method of making the local adjustment first is justified, not only on the ground of saving labor, but also because of the well-known fact that the most serious errors are those due to eccentricity of signal and instrument, phase of signal, refraction, etc., which do not appear to any large extent in the local adjustment but which do appear in the figure adjustment. If we compute the precision of angles from the discrepancies noted at each station, and then estimate from these values the error of closure to be expected in the triangle, we find that these are smaller than the errors of closure actually occurring, showing the presence of constant errors, which do not appear in the local adjustment.

205. Conditions in a Triangulation.

The geometric conditions connecting the angles in a net are of two classes: (1) those which express the relation among the angles of a triangle or other figure, and (2) those which express the relation existing among the sides of the figure. If we plot, for example, a quadrilateral figure, starting from one side as fixed, we shall find that if the sum of the angles in three of the triangles equals their theoretical sums, all sums in the other triangles will also (necessarily) equal their theoretical amounts, namely, $180° + e''$. This shows that of all the possible angle equations which might be written for this figure only three are really independent.

In order to determine the number of angle equations in any net, let s be the total number of stations, s_u the number of stations not occupied, l the total number of lines in the figure, and l_1 the number of lines sighted over in one direction only; then the number of angle equations in the figure is

$$l - l_1 - s + s_u + 1. \qquad [159]$$

In a triangle it is necessary that all stations should be occupied and that all lines should be sighted over in both directions, in order to have one angle equation, that is,

$$l - s + 1 = 3 - 3 + 1 = 1.$$

If a new station is added, it must be occupied and the two lines sighted over in both directions, in order to yield a new angle equation. If this is done, the quantity $l - s$ is increased by $2 - 1 = 1$. If a line is drawn between two stations already located, l is increased by 1 and there is a new angle equation corresponding. For each new line sighted in one direction only, l is increased by 1 and l_1 is increased by 1, so that the total is unchanged.

The number of side equations in a net may be estimated as follows: Starting with one line as fixed, it is evidently necessary to have two more sides in order to fix a third point. Hence, in order to plot a figure, we must have at least $2(s - 2)$ lines in addition to the base, that is, $2s - 3$ lines in all. Any additional lines used must conform to those already used, in order to give a perfect figure; hence the number of conditions giving rise to side equations will equal the number of superfluous lines, that is, $l - 2s + 3$, where l is the total number of lines and s is the number of stations. It should be observed that while the side equation is primarily a relation among the sides, it is also a relation among the sines of the angles, and this fact enables us to adjust the figure by altering the angles.

206. Adjustment of a Quadrilateral.

For any quadrilateral figure in which all of the (eight) angles have been measured there may be found three equations which express the condition that the triangles must all "close." There are more than three equations which may be formed; but if any three of these equations are satisfied, the others necessarily follow and hence are not independent. There will also be one side equation expressing the condition that the length of a side (AB), when computed from the opposite side (CD), is exactly the same, no matter which pair of triangles is employed in the computation.

In selecting the three-angle equations we may take any three triangles and write an equation for each expressing the condition that the sum of the three angles equals $180° + e''$. It is advantageous in this case to avoid triangles having small angles. In selecting the side equation it is well, however, to select one involving small angles, so as to give large coefficients of the corrections. If the angle equations were also chosen so as to involve the small angles, the solution would be likely to prove unstable, on account of the equality of some of the coefficients.

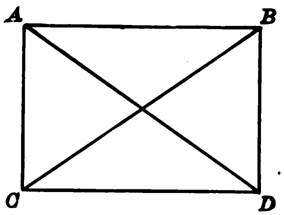


FIG. 112.

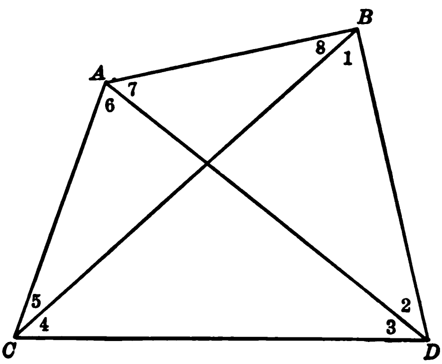


FIG. 113.

A convenient method of writing a side equation is to select some point, called the *pole*, and write the three directions from it to the other stations in the order of azimuths. For example, taking the pole at A, Fig. 112, write first

$$AB \cdot AD \cdot AC.$$

Then from this write the ratios

$$\frac{AB}{AD} \cdot \frac{AD}{AC} \cdot \frac{AC}{AB},$$

the method of forming which is evident. If we now replace each line by the sine of the angle opposite to it in the triangle which is indicated by the fraction, and place the whole equal to unity, we have

$$\frac{\sin ADB}{\sin ABD} \times \frac{\sin ACD}{\sin ADC} \times \frac{\sin ABC}{\sin ACB} = 1. \qquad [160]$$

It may be shown, by solving the different triangles and eliminating the sides, that this equation expresses the condition that the length of AB as computed from CD is the same no matter which route is followed in the computation.

Problem. Prove by a direct solution of the triangles in Fig. 112 that Equation [160] is true.

Designating the angles by means of the numbers shown in Fig. 113, the equation becomes

$$\frac{\sin 2 \sin (4 + 5) \sin 8}{\sin (1 + 8) \sin 3 \sin 5} = 1. \qquad [161]$$

Before this equation can be used, however, it is practically necessary to reduce it to linear form, since an application of Equa. [153] to any but linear equations would be complicated.

Suppose our equation to be put in the general form

$$\frac{\sin (M_1 + v_1)}{\sin (M_2 + v_2)} \times \frac{\sin (M_3 + v_3)}{\sin (M_4 + v_4)} \cdots = 1, \qquad [162]$$

in which the angle is written as an approximate value M plus a small correction v. Taking logs of both members and then applying Taylor's theorem, we have, neglecting squares and higher powers,

$$\log \sin M_1 + \frac{\partial}{\partial M_1} (\log \sin M_1)\, v_1 + \cdots$$
$$- \left(\log \sin M_2 + \frac{\partial}{\partial M_2} (\log \sin M_2)\, v_2 + \cdots \right) = 0. \qquad [163]$$

The quantity $\frac{\partial}{\partial M_1} (\log \sin M_1)$ is the variation per 1″ in a table of log sines, the correction v being in seconds. Hence, placing $\delta_1 = \frac{\partial}{\partial M_1}(\log \sin M_1)$, etc., we have

$$\delta_1 v_1 - \delta_2 v_2 + \delta_3 v_3 - \delta_4 v_4 + \cdots + \log \sin M_1 - \log \sin M_2 + \cdots = 0. \qquad [164]$$

The algebraic sum of the log sines represents the amount by which they fail to satisfy the condition equation. Placing this sum equal to l, the side equation given above becomes

$$\delta_2 v_2 + \delta_{4+5} v_{4+5} + \delta_8 v_8 - (\delta_{1+8} v_{1+8} + \delta_3 v_3 + \delta_6 v_6) - l = 0. \qquad [165]$$

Example. Let us suppose that the measured angles are (Fig. 113),

1.	61° 07′ 52″.00
2.	38 28 34 .90
3.	38 22 19 .10
4.	42 01 12 .15
5.	29 14 32 .85
6.	70 21 59 .20
7.	49 26 21 .85
8.	30 57 07 .10

These angles are supposed to have been adjusted for local conditions.

To form the angle equations, take the triangles ABD, ADC, and ABC for which the values of the spherical excess are 1″.36, 1″.77 and 1″.02, respectively. The computation is shown in tabular form as follows:

1 + 8	92° 04′ 59″.10
2	38 28 34 .90
7	49 26 21 .85
	179 59 55 .85
	180 00 01 .36
	+5″.51
3	38° 22′ 19″.10
4 + 5	71 15 45 .00
6	70 21 59 .20
	180 00 03 .30
	180 00 01 .77
	−1″.53
5	29° 14′ 32″.85
6	70 21 59 .20
7	49 26 21 .85
8	30 57 07 .10
	180 00 01 .00
	180 00 01 .02
	+0″.02

This gives for the three angle equations

$$(1 + 8) + 2 + 7 = 180° \ 00' \ 01''.36,$$
$$3 + (4 + 5) + 6 = 180 \ \ 00 \ \ 01 \ .77,$$
$$5 + 6 + 7 + 8 \ = 180 \ \ 00 \ \ 01 \ .02,$$

or, written as corrections,

$$v_{1+8} + v_2 + v_7 - 5.51 \quad = 0,$$
$$v_3 + v_{4+5} + v_6 + 1.53 \quad = 0,$$
$$v_5 + v_6 + v_7 + v_8 - 0.02 = 0.$$

To form the side equation, take the pole at A. Then we have

$$\frac{AB}{AD} \cdot \frac{AD}{AC} \cdot \frac{AC}{AB},$$

giving

$$\frac{\sin 2}{\sin (1 + 8)} \cdot \frac{\sin (4 + 5)}{\sin 3} \cdot \frac{\sin 8}{\sin 5} = 1,$$

or

$$\log \sin 2 + \log \sin (4+5) + \log \sin 8 - \log \sin (1+8) - \log \sin 3 - \log \sin 5 = 0.$$

The computation of the constant term of this equation is given in the following table. The log sines of those angles appearing in the numerator, together with their diff. for 1″ (in units of the 6th place of decimals) are placed in the left-hand column, and those in the denominator are placed in the right-hand column. The constant l is the difference in the sums of the log sines.

Angle.	log sine (+).	Diff. 1″.	Angle.	log sine (−).	Diff. 1″.
2	9.7939242	+2.65	1+8	9.9997129	−0.08
4+5	9.9763501	+0.72	3	9.7929268	+2.66
8	9.7112329	+3.51	5	9.6888702	+3.76
	9.4815072			9.4815099	
				72	
				−27	

Therefore $\qquad l = -2.7.$

The side equation becomes

$$2.65\, v_2 + 0.72\, v_{4+5} + 3.51\, v_8 + 0.08\, v_{1+8} - 2.66\, v_3 - 3.76\, v_5 - 2.7 = 0.$$

Since the observations are direct, all of the observation equations take the form

$$v_{1+8} = v_{1+8}.$$
$$v_2 \quad = v_2.$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The eight observation equations and the four condition equations are now written, and we are ready to adjust the quadrilateral.

207. Solution by Direct Elimination.

If we select for the four independent unknowns v_2, v_3, v_5, and v_8, and express the four conditions in terms of these, we have

$$
\begin{aligned}
v_{1+8} &= 3.015\,v_2 - 5.282\,v_3 - 4.751\,v_5 + 6.609\,v_8 + 0.2351,\\
v_2 &= v_2,\\
v_3 &= v_3,\\
v_{4+5} &= -4.015\,v_2 + 4.282\,v_3 + 5.751\,v_5 - 5.609\,v_8 + 3.725,\\
v_5 &= v_5,\\
v_6 &= +4.015\,v_2 - 5.282\,v_3 - 5.751\,v_5 + 5.609\,v_8 - 5.255,\\
v_7 &= -4.015\,v_2 + 5.282\,v_3 + 4.751\,v_5 - 6.609\,v_8 + 5.275,\\
v_8 &= v_8.
\end{aligned}
$$

From these we form the following normal equations (Art. 198, Equa. [153]):

v_2.	v_3.	v_5.	v_8.	Const.
+58.450	− 75.534	− 79.581	+ 91.501	−56.527
−75.534	+102.036	+105.193	−123.474	+70.329
−79.581	+105.193	+112.292	−127.388	+75.589
+91.501	−123.474	−127.388	+151.282	−83.678

The simultaneous solution of these equations will give the most probable values of the corrections.

208. Gauss's Method of Substitution.

In solving a large number of equations simultaneously it is convenient to use some definite system of eliminating the unknowns, in order to avoid labor and the danger of mistakes. Let us suppose that the observation equations are of the form

$$
\begin{aligned}
a_1x + b_1y + c_1z + l_1 &= v_1,\\
a_2x + b_2y + c_2z + l_2 &= v_2,\\
\cdots\cdots&\cdots
\end{aligned}
$$

and that the normal equations are represented by

$$
\left.
\begin{aligned}
[aa]\,x + [ab]\,y + [ac]\,z + [al] &= 0,\\
[ab]\,x + [bb]\,y + [bc]\,z + [bl] &= 0,\\
[ac]\,x + [bc]\,y + [cc]\,z + [cl] &= 0,
\end{aligned}
\right\} \qquad [166]
$$

in which the brackets indicate the sum of all the terms found by multiplying the numerical coefficients according to the rule on p. 293.

If the first normal equation be divided by $[aa]$ and solved for x, the result is

$$x = -\frac{[ab]}{[aa]}y - \frac{[ac]}{[aa]}z - \frac{[al]}{[aa]}.$$

Substituting this in the second equation, we have

$$\left([bb] - \frac{[ab]}{[aa]}[ab]\right)y + \left([bc] - \frac{[ac]}{[aa]}[ab]\right)z + \left([bl] - \frac{[al]}{[aa]}[ab]\right) = 0. \quad [167]$$

This is usually abbreviated

$$[bb \cdot 1]\,y + [bc \cdot 1]\,z + [bl \cdot 1] = 0. \quad [168]$$

Substituting this in the third equation, we have

$$[bc \cdot 1]\,y + [cc \cdot 1]\,z + [cl \cdot 1] = 0. \quad [169]$$

These two equations, [168] and [169], are called the "first reduced normal equations."

Solving [168] for y,

$$y = -\frac{[bc \cdot 1]}{[bb \cdot 1]}z - \frac{[bl \cdot 1]}{[bb \cdot 1]};$$

whence

$$[cc \cdot 2]\,z + [cl \cdot 2] = 0, \quad [170]$$

in which

$$[cc \cdot 2] = [cc \cdot 1] - \frac{[bc \cdot 1]}{[bb \cdot 1]}[bc \cdot 1]$$

and

$$[cl \cdot 2] = [cl \cdot 1] - \frac{[bc \cdot 1]}{[bb \cdot 1]}[bl \cdot 1].$$

The solution of [170] gives the value of z. By substituting this in [168] and [169] the value of y may be found. Finally, from [166] the value of x may be found.

An inspection of [166] will show that all coefficients below and to the left of a diagonal drawn from the x term of the first equation to the z term of the third equation are duplicates of the others. These may be omitted in writing the equations.

Solving the equations in Art. 207, p. 302, by the method of substitution just described, we obtain the following results:

$$v_2 = +1.80,$$
$$v_3 = -0.19,$$
$$v_5 = -0.07,$$
$$v_8 = -0.75.$$

Substituting these values in the condition equations, p. 301, we find for the remaining unknowns,

$$v_{1+8} = +2.05,$$
$$v_7 = +1.67,$$
$$v_{4+5} = -0.51,$$
$$v_6 = -0.83.$$

The final angles are as follows:

1.	61°	07′	54.″	80
2.	38	28	36.	70
3.	38	22	18.	91
4.	42	01	11.	71
5.	29	14	32.	78
6.	70	21	58.	37
7.	49	26	23.	52
8.	30	57	06.	35

The above is an example of a rather unstable solution of normal equations. It requires a relatively large number of significant figures to give the corrections to two places of decimals.

209. Checks on the Solution.

In practice it would not be advisable to proceed in the solution of a large number of equations without some safeguard against mistakes of computation. A valuable check consists in adding to the normal equations an extra term which is merely the sum of all the coefficients of v_1, v_2, etc., and treating this term like any other term of the equation. This is illustrated later in the example on p. 313.

210. Method of Correlatives.

When there are many condition equations, the method of substitution is likely to prove laborious. If, as is usually the case in triangulation, the observations are direct and equal in number to the number of unknowns, the "Method of Correlatives" will be found preferable. By this method we eliminate one unknown for each condition equation, employing for this purpose the method of undetermined multipliers.

Suppose that we have made m direct observations, $M_1, M_2, \ldots,$ M_m, of m different quantities, of which the most probable values are

$$z_1 = M_1 + v_1, \quad z_2 = M_2 + v_2, \quad \ldots, \quad z_m = M_m + v_m.$$

Let these m unknowns be connected by the following n conditions equations:

$$\left.\begin{array}{l} a_1v_1 + a_2v_2 \;.\;.\;.\; a_mv_m + l_1 = 0, \\ b_1v_1 + b_2v_2 \;.\;.\;.\; b_mv_m + l_2 = 0, \\ .\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \end{array}\right\} \qquad [171]$$

the a's being the coefficients in the first equation, the b's those of the second, etc. The quantities l_1, l_2, etc., represent the amounts by which the observations fail to satisfy the condition equations. If the original condition equations are not linear in form, they must be made so by a method similar to that given on p. 299.

Since the most probable values of the v's are to be found, we must have

$$v_1^2 + v_2^2 + \cdots = \text{a minimum}, \qquad [172]$$

or

$$v_1\,dv_1 + v_2\,dv_2 + \cdots = 0 \qquad [172]$$

for all possible values of dv_1, dv_2, etc.

Hence it must hold true for the equations

$$\begin{array}{l} a_1\,dv_1 + a_2\,dv_2 + \cdots = 0, \\ b_1\,dv_1 + b_2\,dv_2 + \cdots = 0, \\ .\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \end{array} \qquad [173]$$

obtained by differentiating [171]. The number of these equations is n. The number of terms in [172] is m, m being greater than n. Let the first equation in [173] be multiplied by k_1, the second by k_2, etc., and Equa. [172] by -1. The products are then added, giving

$$(a_1k_1 + b_1k_1 + \cdots - v_1)\,dv_1 + (a_2k_2 + b_2k_2 + \cdots - v_2)\,dv_2 + \cdots = 0. \qquad [174]$$

The k's are to be so determined that this equation will hold true. This equation will be satisfied if the coefficient of each differential in it is placed equal to zero, that is, if

$$\left.\begin{array}{l} k_1a_1 + k_2b_1 + \cdots k_nl_1 = v_1, \\ k_1a_2 + k_2b_2 + \cdots k_nl_2 = v_2, \end{array}\right\} \qquad [175]$$

$$\cdots\cdots\cdots\cdots$$

Substituting these values of v_1, v_2, etc., from Equa. [175] in Equa. [171], we obtain

$$\left.\begin{array}{l} k_1\,[aa] + k_2\,[ab] + \cdots k_n\,[al] + l_1 = 0, \\ k_1\,[ab] + k_2\,[bb] + \cdots k_n\,[bl] + l_2 = 0, \end{array}\right\} \qquad [176]$$

$$\cdots\cdots\cdots\cdots\cdots$$

The solution of these equations gives the values of k_1, k_2, k_3, etc., which are the correlatives of the condition equations. By substituting these values in Equa. [176] the v's are found. Since the form of Equa. [176] is the same as that of normal equations, it is evident that they may be solved by the method of substitution.

In case the observations are of different weight, the minimum equation would be

$$p_1v_1^2 + p_2v_2^2 + \cdots p_mv_m^2 = \text{a minimum}, \qquad [177]$$

and the other equations would be modified accordingly.

Example. As an illustration of the method of correlatives we will use the same quadrilateral that was adjusted by the method of direct elimination. The observation equations are eight in number and all of the form

$$\begin{array}{l} v_{1+8} = v_{1+8}, \\ v_2 \;\;\;= v_2, \\ \cdots\cdots \end{array}$$

The four condition equations are

$$\text{Angle equations} \left\{ \begin{array}{l} v_{1+8} + v_2 + v_7 - 5.51 \;\;= 0, \\ v_3 + v_{4+5} + v_6 + 1.53 \;\;= 0, \\ v_5 + v_6 + v_7 + v_8 - 0.02 = 0. \end{array}\right.$$

Side equation,

$$2.65\,v_2 + 0.72\,v_{4+5} + 3.51\,v_8 + 0.08\,v_{1+8} - 2.66\,v_3 - 3.76\,v_5 - 2.70 = 0.$$

The coefficients of the corrections are then tabulated as follows:

v.	*a.*	*b.*	*c.*	*d.*
v_{1+8}	+1			+0.08
v_2	+1			+2.65
v_7	+1		+1	
v_3		+1		−2.66
v_{4+5}		+1		+0.72
v_6		+1	+1	
v_5			+1	−3.76
v_8			+1	+3.51

Substituting these in the angle equations,

$$\begin{array}{lll} (1+8) & = +2.058 - 0.094 \times 0.08 & = +2.05 \\ 2 & = +2.058 - 0.094 \times 2.65 & = +1.81 \\ 7 & = +2.058 - 0.406 & = \underline{+1.65} \\ & & +5.51 \quad \text{Check} \end{array}$$

$$\begin{array}{lll} 3 & = -0.435 + 2.66 \times -0.094 & = -0.19 \\ (4+5) & = -0.435 + 0.72 \times -0.094 & = -0.50 \\ 6 & = -0.435 - 0.406 & = \underline{-0.84} \\ & & -1.53 \quad \text{Check} \end{array}$$

$$\begin{array}{lll} 5 & = -0.4064 - 3.76 \times -0.094 & = -0.05 \\ 6 & = -0.435 - 0.406 & = -0.84 \\ 7 & = +2.058 - 4.06 & = +1.65 \\ 8 & = 0.406 + 3.51 \times -0.094 & = \underline{-0.74} \\ & & +0.02 \quad \text{Check} \end{array}$$

The coefficients of the correlative Equa. [176] are tabulated as follows:

	aa.	*ab.*	*ac.*	*ad.*	*bb.*	*bc.*	*bd.*	*cc.*	*cd.*	*dd.*
v_{1+8}	+1			+0.08						0.0064
v_2	+1			+2.65						7.0225
v_7	+1		+1					+1		
v_3					+1		−2.66			7.0756
v_{4+5}					+1		+0.72			0.5184
v_6					+1	+1		+1		
v_5								+1	−3.76	14.1376
v_8								+1	+3.51	12.3201
Sum	+3		+1	+2.73	+3	+1	−1.94	+4	−0.25	+41.0806

The correlative equations are therefore

$$
\begin{array}{ll}
\text{I.} & 3k_1 + 0 + k_3 + 2.73\,k_4 - 5.51 = 0 \\
\text{II.} & 0 + 3k_2 + k_3 - 1.94\,k_4 + 1.53 = 0 \\
\text{III.} & k_1 + k_2 + 4k_3 - 0.25\,k_4 - 0.02 = 0 \\
\text{IV.} & +2.73\,k_1 - 1.94\,k_2 - 0.25\,k_3 + 41.0806\,k_4 - 2.7 = 0
\end{array}
$$

The solution of these equations gives for the correlatives,

$$
\begin{aligned}
k_1 &= +2.058, \\
k_2 &= -0.435, \\
k_3 &= -0.406, \\
k_4 &= -0.094.
\end{aligned}
$$

Applying these equations to the measured angles, we obtain the final angles.

	Measured Angles.	Correction.	Seconds Corrected.	
(1 + 8)	92° 04′ 59″.10	+ 2″.05	05′ 01″.15	
2	38 28 34 .90	+ 1 .81	36 .71	
7	49 26 21 .85	+ 1 .65	23 .50	
			01 .36	
			e'' = 01 .36	
			00 .00	Check
3	38° 22′ 19″.10	− 0″.19	18″.91	
(4 + 5)	71 15 45 .00	− 0 .50	44 .50	
6	70 21 59 .20	− 0 .84	58 .36	
			01″.77	
			e'' = 01 .77	
			00 .00	Check
5	29° 14′ 32″.85	− 0″.05	32″.80	
6	70 21 59 .20	− 0 .84	58 .36	
7	49 26 21 .85	+ 1 .65	23 .50	
8	30 57 07 .10	− 0 .74	06 .36	
			01 .02	
			e'' = 01 .02	
			00 .00	Check

The check of the side equations is as follows:

$$
\begin{array}{rl}
2.65 \times 1.81 = & 4.80 \\
0.72 \times -0.50 = & -0.36 \\
3.51 \times -0.74 = & -2.60 \\
\hline
& +1.84 \\
& -0.85 \\
\hline
& +2.69
\end{array}
\qquad
\begin{array}{rl}
-0.08 \times 2.05 = & -0.16 \\
+2.66 \times -0.19 = & -0.49 \\
+3.76 \times -0.05 = & -0.20 \\
\hline
& -0.85
\end{array}
$$

(Should equal 2.70.)

If the sums of the log sines are again computed (see p. 301), using the corrected seconds, they will be found to equal 9.481 5090 for both columns.

211. Method of Directions.

The method of correcting the directions instead of the angles is particularly applicable when the measurements have been taken by the method of directions, Art. 43. In the United States Coast Survey office it is the usual practice to employ this method of adjusting, whether the observations were made by the direction method or by the method of repetition.

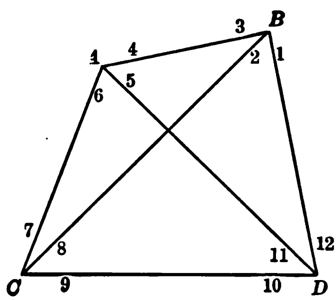


FIG. 114.

In the quadrilateral adjusted in Arts. 208–210, let us denote the directions by the numbers 1 to 12 (Fig. 114) and the corrections to those directions by the same numbers, (1), (2), etc., enclosed in parentheses. Each angle is expressed as the difference of two directions; that is, the angle $-4+5$ means the angle between the directions marked 4 and 5. The four condition equations are the same as before except as to the change in notation.

$$\text{Angle Equations}\begin{cases} -(4)+(5)-(1)+(3)-(11)+(12)-5.51=0. \\ -(5)+(6)-(7)+(9)-(10)+(11)+1.53=0. \\ -(2)+(3)-(7)+(8)-(4)+(6)-0.02=0. \end{cases}$$

Side equation,

$$-5.31\,(11)+2.65\,(12)+3.04\,(7)+0.72\,(9)-3.51\,(2)$$
$$+3.59\,(3)-0.08\,(1)+2.66\,(10)-3.76\,(8)-2.7=0.$$

If CD were a fixed line obtained by a previous adjustment, the corrections (9) and (10) would be omitted. The angle equations could be simplified in this case by selecting two equations which involve angles depending upon those two directions.

The first table for the coefficients of the corrections is given below.

Direction.	*a.*	*b.*	*c.*	*d.*
1	−1			+0.08
2			−1	−3.51
3	+1		+1	+3.59
4	−1		−1	
5	+1	−1	+1	
6		+1		
7		−1	−1	+3.04
8			+1	−3.76
9		+1		+0.72
10		−1		+2.66
11	−1	+1		−5.31
12	+1			+2.65

The remainder of the work, that is, the calculation of coefficients $\sum aa$, $\sum ab$, etc., and the solution of the numerical equations, is carried out as in the preceding example (Art. 210). The solution of the normal equations gives the corrections to the directions. The correction to any angle is the difference of the corrections to the directions of its sides.

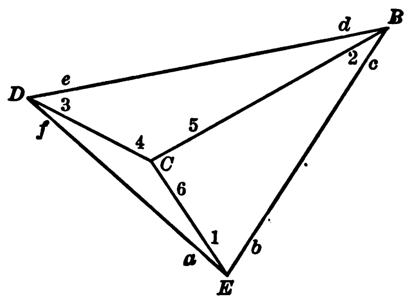


FIG. 115.

212. Adjusting New Triangulation to Points already Adjusted. In the quadrilateral shown in Fig. 115 the triangle *BDE* is supposed to have been previously adjusted. Point *C* is determined by the directions 1, 2, and 3 in connection with the directions along the sides of the fixed triangle, and also by directions 4, 5, and 6. The directions to be found are 1, 2, 3, 4, 5, and 6. The directions as taken from the field-notes are as follows:

Point sighted.	Direction after local adjustment.	Corrected seconds.
	At *C*	
	° ′ ″	
D	0 00 00.00	
B	123 49 24.97	
E	207 52 33.50	
	At *D*	
	° ′ ″	″
A	0 00 00.00	00.67
C	296 57 55.83	
E	311 12 14.48	12.69
B	258 27 57.39	57.18
	At *E*	
	° ′ ″	″
D	0 00 00.00	01.32
C	13 38 27.54	
B	81 28 43.98	43.05
	At *B*	
	° ′ ″	″
F	0 00 00.00	01.06
E	122 32 11.29	12.56
C	150 38 41.62	
D	168 19 14.81	15.48

In taking directions from this table, the corrected seconds should be used whenever an adjustment has been made.

The number of angle equations in the figure is $l - s + 1$, or $6 - 4 + 1 = 3$. The number of side equations is $l - 2s + 3$, or $6 - 8 + 3 = 1$. Since, however, the exterior triangle is already adjusted, there will be but two angle equations needed in the adjustment. For these two angle equations take the triangles *DCE* and *BEC*;

then $\quad -(a) + (1) - (3) + (f) - (6) + (4) = 0$

and $\quad -(1) + (b) - (5) + (6) - (c) + (2) = 0.$

But since the exterior lines are not to be changed, (*a*), (*f*), (*b*), and (*c*) are all zero.

The absolute terms in the angle equations are found as follows:

$-(a)+(1)$	13° 38′ 26″.22
$-(3)+(f)$	14 14 16 .86
$-(6)+(4)$	152 07 26 .50
	180 00 09 .58
	180 00 00 .02
	−09″.56

$-(1)+(b)$	67° 59′ 15″.51
$-(5)+(6)$	84 03 08 .53
$-(c)+(2)$	28 06 29 .06
	179 59 53 .10
	180 00 00 .08
	+6″.98

For the side equation take the pole at *C*.

$$\frac{\sin(-(2)+(d))}{\sin(-(e)+(3))}\cdot\frac{\sin(-(1)+(b))}{\sin(-(c)+(2))}\cdot\frac{\sin(-(3)+(f))}{\sin(-(a)+(1))}=1.$$

Tabulating the log sines,

		log sin (+)	dif. 1″
$-(2)+(d)$	17° 40′ 33″.86	9.4823521	+66.1
$-(1)+(b)$	67 50 15 .51	9.9666666	+ 8.6
$-(3)+(f)$	14 14 16 .86	9.3908478	+83.0
		8.8398665	

		log sin (−)	
$-(e)+(3)$	38° 29′ 58″.65	9.7941460	+26.5
$-(c)+(2)$	28 06 29 .06	9.6731464	+39.5
$-(a)+(1)$	13 38 26 .22	9.3726010	+86.7
		8.8398934	
		8665	
	constant =	−269	

The side equation is therefore

$$+6.61\times-(2)+0.86\times-(1)+8.30\times-(3)$$
$$-2.65\times(3)-3.95\times(2)-8.67\times(1)-26.9=0.$$

Carrying out the same process as outlined in Art. 210, we have the following:

TABLE OF COEFFICIENTS.

Direction.	*a*.	*b*.	*c*.	Sum.	*aa*.	*ab*.	*ac*.	*as*.	*bb*.	*bc*.	*bs*.	*cc*.	*cs*.
1	+1	−1	− 9.53	− 9.53	+1	−1	− 9.53	− 9.53	+1	+ 9.53	+9.53	90.8209	90.8209
2		+1	−10.56	− 9.56					+1	−10.56	−9.56	111.5136	100.9536
3	−1		−10.95	−11.95	+1		+10.95	+11.95				119.9025	130.8525
4	+1			+ 1	+1			+ 1					
5		−1		− 1					+1		+1		
6	−1	+1		0	+1	−1			+1				
Total....					+4	−2	+1.42	+3.42	+4	−1.03	+0.97	322.2370	322.6270

From these sums we derive the correlative equations.

CORRELATIVE EQUATIONS

Number.	k_1.	k_2.	k_3.	Const.	Check.	Sum.
1	+4	−2	+ 1.42	+ 9.56	+ 12.98	+ 3.42
2		+4	− 1.03	− 6.98	− 6.01	+ 0.97
3			+322.24	−26.9	+295.73	+322.63

It should be observed that the "constant" terms are taken directly from the condition equations. The "sum" term contains the sum of the coefficients of the *k*'s. The "check" term is the algebraic sum of the constant and sum terms. The solution is given in detail in the following table: The different operations are indicated in the left-hand column. The factors by which the equations are multiplied are in the right-hand column.

					Factor
2	+4	−1.03	−6.98	−6.01	
$I \times -\frac{-2}{4}$	−1	+0.71	+4.78	+6.49	$+\frac{1}{2}$
II	+3	−0.32	−2.20	+0.48	

3	+322.24	−26.9	+295.73	
$I \times -\frac{1.42}{4}$	− 0.50	− 3.39	− 4.61	−0.355
$II \times -\frac{0.32}{3}$	− 0.03	− 0.24	+ 0.05	
III	+321.71	−30.53	+291.17	

The preceding table is an abbreviated form of the method of substitution explained in Art. 208.

The correlatives are found as follows:

	I.	II.	III.	
Const.	+9.56	−2.20	−30.53	$k_3 = \frac{30.53}{321.70} = +0.0949$
k_3	+0.135	−0.03		$k_2 = \frac{2.23}{3} = +0.7433$
k_2	−1.487	−2.230		$k_1 = \frac{8.208}{4} = -2.052$
k_1	+8.208			

Calculating the corrections for the correlatives,

	1.	2.	3.	4.	5.	6.
k_1	−2.052		+2.052	−2.052		+2.052
k_2	−0.743	+0.743			−0.743	+0.743
k_3	−0.904	−1.002	−1.039			
	−3.699	−0.262	+1.013	−2.052	−0.743	+2.795

Applying these corrections to the directions, we have the final adjusted values

Dir. No.	Observed directions.	Correction.	Corrected seconds.
	° ′ ″	″	″
4	0 00 00.00	−2.05	57.95
5	123 49 24.97	−0.74	24.23
6	207 52 33.50	+2.80	36.30
1	13 38 27.54	−3.70	23.84
2	150 38 41.62	−0.26	41.36
3	296 57 55.83	+1.01	56.84

213. The Precision Measures.

Referring to the equation of the curve of error, Art. 197,

$$y = ke^{-h^2x^2}, \qquad [149]$$

we see that there are two constants to be determined for any particular set of observations. These two constants are not independent, however, as will be shown. The total area between the curve and the X axis was taken equal to unity; therefore

$$k\int_{-\infty}^{\infty} e^{-h^2x^2}\,dx = 1,$$

or

$$k\int_{0}^{\infty} e^{-h^2x^2}\,dx = \tfrac{1}{2},$$

from which

$$\int_{0}^{\infty} e^{-h^2x^2}\,h\,dx = \frac{h}{2k}.$$

In order to integrate this expression let $t = hx$ and $dt = h\,dx$.

Then

$$\int_{0}^{\infty} e^{-t^2}\,dt = \int_{0}^{\infty} e^{-h^2x^2}\,h\,dx.$$

Multiplying this equation by

$$\int_{0}^{\infty} e^{-t^2}\,dt = \int_{0}^{\infty} e^{-h^2}\,dh,$$

we have

$$\left[\int_{0}^{\infty} e^{-t^2}\,dt\right]^2 = \int_{0}^{\infty}\int_{0}^{\infty} e^{-h^2(1+x^2)}\,h\,dx\,dh$$

$$= \int_{0}^{\infty} -\frac{1}{2(1+x^2)}\,dx\int_{0}^{\infty} e^{-h^2(1+x^2)}\,(-2h)(1+x^2)\,dh$$

$$= \frac{1}{2}\int_{0}^{\infty}\frac{dx}{1+x^2} = \frac{1}{2}\Big[\tan^{-1}x\Big]_{0}^{\infty} = \frac{\pi}{4}.$$

Therefore

$$\int_{0}^{\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

and

$$\frac{\sqrt{\pi}}{2} = \frac{h}{2k},$$

or

$$k = \frac{h}{\sqrt{\pi}}, \qquad [178]$$

which shows the relation between the two constants.

The equation of the curve of error may now be written

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2x^2}, \qquad [179]$$

214. The Average Error.

The average error (η) is the arithmetical mean of the errors, all taken with the same sign. To derive an expression for the average error, we see from equation (142) that $f(x)\,dx$ is the probability that an observation will fall between the limits x and $x + dx$; that is, it represents the proportion of all the errors that will probably fall within these limits. Hence, if n observations are made, the number in this strip will be $n f(x)\,dx$. The sum of all the observations will be

$$n \int_{-\infty}^{\infty} x f(x)\,dx,$$

or

$$2n \int_{0}^{\infty} x f(x)\,dx.$$

The average error equals the sum of the errors divided by the number, that is,

$$\eta = 2 \int_0^{\infty} x f(x)\,dx$$

$$= \frac{2h}{\sqrt{\pi}} \int_0^{\infty} e^{-h^2x^2} x\,dx$$

$$= -\frac{1}{h\sqrt{\pi}} \int_0^{\infty} e^{-h^2x^2} (-2h^2x)\,dx$$

$$= \frac{1}{h\sqrt{\pi}}. \qquad [180]$$

215. The Mean Square Error.

The mean square error (μ) of an observation is the square root of the arithmetical mean of the squares of the errors. Since the number of errors between x and $x + dx$ is $n f(x)\,dx$, the sum of the squares of these errors is

$$nx^2 f(x)\,dx.$$

The sum of the squares of all the errors is

$$n\int_{-\infty}^{\infty} x^2 f(x)\,dx.$$

Therefore
$$\mu^2 = \frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-h^2x^2}x^2\,dx. \qquad (d)$$

But
$$\frac{h}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-h^2x^2}\,dx = 1, \quad \text{or} \quad \int_{-\infty}^{\infty} e^{-h^2x^2}\,dx = \frac{\sqrt{\pi}}{h}.$$

If we differentiate this with respect to h as the independent variable, we obtain

$$-2h\int_{-\infty}^{\infty} e^{-h^2x^2}x^2\,dx = -\frac{\sqrt{\pi}}{h^2}. \qquad (e)$$

Substituting (e) in (d),

$$\mu = \frac{1}{h\sqrt{2}}. \qquad [181]$$

216. The Probable Error.

The probable error (r) of an observation is an error such that one half the errors of the series are greater than it and the other half are less than it; that is, the probability of making an error greater than r is just equal to the probability of making an error less than r.

The probability that an error of an observation will fall between the limits x and $x + dx$ is $f(x)\,dx$. The probability that the error will fall between the limits $+r$ and $-r$ is given by

$$P = \int_{-r}^{+r} f(x)\,dx = \frac{2h}{\sqrt{\pi}}\int_{0}^{+r} e^{-h^2x^2}\,dx = \frac{1}{2},$$

by the definition.

To integrate, let $t = hx$, and $dt = h\,dx$,

Then
$$\frac{2}{\sqrt{\pi}}\int_{0}^{hr} e^{-t^2}\,dt = \frac{1}{2}.$$

If we evaluate this integral for assumed values of hr and then interpolate for the value of hr corresponding to $\frac{1}{2}$, we find it to be 0.47694.

Therefore
$$r = \frac{0.47694}{h}. \qquad [182]$$

All the precision measures have now been expressed in terms of h, and it is evident that,

$$r = 0.8453\,\eta \qquad [183]$$
$$= 0.6745\,\mu. \qquad [184]$$

The mean square error (μ) is the largest, and the probable error (r) is the smallest, of the three precision measures.

Any one of the three precision measures may be used to compare the relative accuracy of different series of observations, provided the different series are made under the same conditions, so as to be affected by the same constant errors. In Europe the mean square error has been used more than the probable error; in the United States the probable error is generally employed. There are some advantages, however, in the use of the average error (η). Theoretically it is slightly less accurate than either of the others; but inasmuch as the quantity itself is an estimate of an uncertainty in measurement, this objection is not a serious one. The value of η lies between the values of μ and r. The method of computing η is simpler, as will be shown later, than the computation of either μ or r.

Since in Equa. [158] it was shown that p varies as h^2, it follows that

$$p \propto \frac{1}{\eta^2} \propto \frac{1}{\mu^2} \propto \frac{1}{r^2} \propto h^2; \qquad [185]$$

that is, the weights of the different observations on a quantity vary inversely as the squares of the precision measures.

If μ is the precision measure of a direct observation of weight 1, and μ_0 is the precision measure of the mean, then since the weight of the mean is n, the number of observations,

$$\mu_0 = \frac{\mu}{\sqrt{n}}. \qquad (f)$$

217. Computation of the Precision Measures.

Direct Observations of Equal Weight. To find μ, the mean square error of an observation, suppose that we have n direct observations of equal weight made on a quantity M, and that the results are M_1, M_2, . . ., and that M_0 is the most probable value. Let the errors be x_1, x_2, . . . and the residuals v_1, v_2, ...

Then in this case the residuals are

$$v_1 = M_1 - M_0,$$
$$v_2 = M_2 - M_0.$$
$$\cdots\cdots\cdots$$

and
$$\mu = \sqrt{\frac{\sum x^2}{n}}.$$

If M_0 were the true value of M, the residuals would be the same as the true errors, and in that case

$$\mu = \sqrt{\frac{\sum v^2}{n}}. \qquad [186]$$

But in any limited number of observations this is not sufficiently exact. To obtain a more accurate expression, place

$$M_0 + x_0 = M;$$

then

$$x_1 = M_1 - (M_0 + x_0) = v_1 - x_0,$$
$$x_2 = M_2 - (M_0 + x_0) = v_2 - x_0,$$
$$\cdots\cdots\cdots\cdots\cdots$$

Squaring, adding, and dividing by n,

$$\frac{\sum x^2}{n} = \mu^2 = \frac{1}{n}\left(\sum v^2 - 2\,x_0 \sum v + n x_0^2\right).$$

Since $\sum v = 0$, Art. 195, this reduces to

$$\frac{\sum x^2}{n} = \frac{\sum v^2}{n} + x_0^2.$$

The real value of x_0 is unknown; it may be taken as approximately equal to the mean square error of M_0, which, from Equa. (f), is

$$\mu_0 = \frac{\mu}{\sqrt{n}}; \qquad [187]$$

whence

$$\mu^2 = \frac{\sum v^2}{n} + \frac{\mu^2}{n}.$$

Therefore

$$\mu = \sqrt{\frac{\sum v^2}{n - 1}}. \qquad [188]$$

To find μ_0, the mean square error of the mean value, we have, by Equa. (f),

$$\mu_0 = \sqrt{\frac{\sum v^2}{n(n-1)}}. \qquad [189]$$

From Equa. [184],

$$r = 0.6745 \sqrt{\frac{\sum v^2}{n - 1}} \qquad [190]$$

and

$$r_0 = 0.6745 \sqrt{\frac{\sum v^2}{n(n-1)}}. \qquad [191]$$

To find the average error (η) of a single observation, we see that, from Equa. [188],

$$\sum v^2 = (n-1)\frac{\sum x^2}{n}.$$

On the average the values of these residuals will be

$$v_1 = \sqrt{\frac{n-1}{n}} \cdot x_1,$$

$$v_2 = \sqrt{\frac{n-1}{n}} \cdot x_2.$$

. . . .

Adding and dividing by n,

$$\frac{\sum v}{n} = \sqrt{\frac{n-1}{n}} \cdot \frac{\sum x}{n} = \sqrt{\frac{n-1}{n}} \cdot \eta.$$

Therefore
$$\eta = \frac{\sum v}{\sqrt{n(n-1)}}, \qquad [192]$$

and
$$\eta_0 = \frac{\sum v}{n\sqrt{n-1}}. \qquad [193]$$

The probable error is sometimes computed from the average error in order to avoid computing the squares of the residuals. From Equa. [183],

$$r = \frac{0.8453 \sum v}{\sqrt{n(n-1)}}, \qquad [194]$$

and
$$r_0 = \frac{0.8453 \sum v}{n\sqrt{n-1}}. \qquad [195]$$

Evidently the mean error may also be computed from η.

218. Observations of Unequal Weights.

If the observations have unequal weights, let p_1, p_2, etc., be the weights; then

$$\mu_0 = \frac{\mu}{\sqrt{\sum p}}, \qquad \mu_1 = \frac{\mu}{\sqrt{p_1}}, \quad \text{etc.}$$

By Art. 199, if each observation is multiplied by the square root of its weight, the observations are all reduced to weight unity. The residuals are therefore

$$v_1\sqrt{p_1}, \qquad v_2\sqrt{p_2}, \quad \text{etc.}$$

Applying Formulæ [188] to [195] to these residuals, we have

$$\mu = \sqrt{\frac{\sum pv^2}{n-1}}. \qquad [196]$$

$$\mu_1 = \sqrt{\frac{\sum pv^2}{p_1(n-1)}}. \qquad [197]$$

$$\mu_0 = \sqrt{\frac{\sum pv^2}{\sum p\,(n-1)}}. \qquad [198]$$

$$r = 0.6745\sqrt{\frac{\sum pv^2}{n-1}}. \qquad [199]$$

$$r_1 = 0.6745\sqrt{\frac{\sum pv^2}{p_1(n-1)}}. \qquad [200]$$

$$r_0 = 0.6745\sqrt{\frac{\sum pv^2}{\sum p(n-1)}}. \qquad [201]$$

Also,

$$\eta = \frac{\sum v\sqrt{p}}{\sqrt{n(n-1)}}, \qquad [202]$$

$$\eta_1 = \frac{\sum v\sqrt{p}}{\sqrt{p_1(n-1)}}, \qquad [203]$$

$$\eta_0 = \frac{\sum v\sqrt{p}}{\sqrt{\sum p(n-1)}}, \qquad [204]$$

from which

$$r = 0.8453\,\eta, \qquad [205]$$

$$r_1 = 0.8453\,\eta_1, \qquad [206]$$

$$r_0 = 0.8453\,\eta_0. \qquad [207]$$

219. Precision of Functions of the Observed Quantities. Suppose that a quantity M is defined by

$$M = M_1 + M_2,$$

where M_1 and M_2 are independent and are observed directly. Let the mean square error (m.s.e.) of M_1 be μ_1, and let that of M_2 be μ_2, the m.s.e. of the function M being denoted by μ_F. If we suppose the errors in the determination of M_1 to be x_1', x_1'', x_1''', . . . , and those of M_2 to be x_2', x_2'', x_2''', . . . , then the real errors of M, computed from the separate observations on M_1 and M_2, will be

$$x_1' \pm x_2', \quad x_1'' \pm x_2'', \ldots,$$

and $$\mu_F^2 = \frac{(x_1' \pm x_2')^2 + (x_1'' + x_2'')^2 + \cdots}{n}$$

$$= \frac{\sum x_1^2 + 2\sum x_1x_2 + \sum x_2^2}{n}.$$

But the $\sum x_1x_2$ terms will cancel out, because in the long run there will be as many $+$ as $-$ products x_1x_2 of the same magnitude.

Therefore $$\mu_F^2 = \mu_1^2 + \mu_2^2. \qquad [208]$$

From Equas. [183] and [184] it is evident that

$$r_F^2 = r_1^2 + r_2^2 \qquad [209]$$

and $$\eta_F^2 = \eta_1^2 + \eta_2^2. \qquad [210]$$

Let us suppose that the function is defined by

$$M = a_1M_1,$$

where a_1 is a constant; then the real errors of M will be

$$a_1x_1', \quad a_1x_1'', \quad a_1x_1''', \ldots,$$

and $$\mu_F^2 = \frac{a_1^2\sum x_1^2}{n} = a_1^2\mu_1^2,$$

or $$\mu_F = a_1\mu_1. \qquad [211]$$

By combining [208] with [211] it is clear that if

$$M = a_1M_1 + a_2M_2 + a_3M_3 + \cdots,$$

then $$\mu_F^2 = \sum a^2\mu^2, \qquad [212]$$

$$r_F^2 = \sum a^2r^2, \qquad [213]$$

$$\eta_F^2 = \sum a^2\eta^2. \qquad [214]$$

Suppose that the function is of the general form indicated by

$$M = f(M_1, M_2, M_3, \ldots). \qquad (g)$$

Let $M_1 = a_1 + m_1$, $M_2 = a_2 + m_2$, etc., in which a_1 is a close approximation to M_1, a_2 is a close approximation to M_2, and m_1 and m_2 are small corrections such that their squares may be neglected. We may regard m_1 and m_2, etc., as containing the real errors of M_1, M_2, ..., and μ_1, μ_2, ... may be considered as the mean square errors of m_1, m_2, etc. Substituting in (g), we have

$$M = f\left((a_1 + m_1), (a_2 + m_2) \ldots\right).$$

Expanding this function by Taylor's theorem and denoting $f(a_1, a_2, \ldots)$ by M',

$$M = M' + m_1 \frac{\partial M'}{\partial a_1} + m_2 \frac{\partial M'}{\partial a_2} + \cdots, \tag{h}$$

in which the terms containing the squares and higher powers of $m_1, m_2, \ldots$ have been omitted. Then the m.s.e. of M is the same as the m.s.e. of the terms in (h).

By Equa. [212], this is

$$\mu_F^2 = \mu_1^2 \left[\frac{\partial M'}{\partial a_1}\right]^2 + \mu_2^2 \left[\frac{\partial M'}{\partial a_2}\right]^2 + \cdots,$$

or, with sufficient accuracy,

$$\mu_F^2 = \mu_1^2 \left[\frac{\partial M}{\partial M_1}\right]^2 + \mu_2^2 \left[\frac{\partial M}{\partial M_2}\right]^2 + \cdots. \qquad [215]$$

Similarly,

$$r_F^2 = r_1^2 \left[\frac{\partial M}{\partial M_1}\right]^2 + r_2^2 \left[\frac{\partial M}{\partial M_2}\right]^2 + \cdots, \qquad [216]$$

and

$$\eta_F^2 = \eta_1^2 \left[\frac{\partial M}{\partial M_1}\right]^2 + \eta_2^2 \left[\frac{\partial M}{\partial M_2}\right]^2 + \cdots. \qquad [217]$$

It should be observed that in the preceding cases the unknowns are supposed to be independent of each other. If the quantities M_1, M_2, etc., are functions of the same variable, a different procedure is necessary.

Also, in case the unknowns are subject to any number of conditions, the computation of the precision measure of any function must be so modified as to take into account the effect of these conditions.

220. Indirect Observations.

The computation of the precision of the adjusted values in the case of indirect observations is more complicated than in the case of direct observations, because it is necessary to know the weight of each of the unknowns, and this can only be found by the solution of equations similar to the normal equations.

It may be shown that if there are n observations on q unknowns, ther

$$\mu = \sqrt{\frac{\sum v^2}{n - q}}, \qquad [218]$$

where μ is the m.s.e. of an observation of weight unity.

If p_z is the weight of an unknown, then the m.s.e. of this unknown is

$$\mu_z = \frac{\mu}{\sqrt{p_z}} = \sqrt{\frac{\sum v^2}{p_z(n - q)}}. \qquad [219]$$

Similarly,

$$r = 0.6745 \sqrt{\frac{\sum v^2}{n - q}}, \qquad [220]$$

$$r_z = 0.6745 \sqrt{\frac{\sum v^2}{p_z(n - q)}}, \qquad [221]$$

and

$$\eta = \frac{\sum v}{\sqrt{n\,(n - q)}}, \qquad [222]$$

$$\eta_z = \frac{\sum v}{\sqrt{p_z n\,(n - q)}}. \qquad [223]$$

221. Caution in the Application of Least Squares.

In applying the preceding principles it should be kept in mind that the ordinary adjustment by the method of least squares deals with the accidental errors only and can tell us nothing about the constant or systematic errors which may affect the results of observation. The "probable error" may therefore be far from the true error because such constant errors are present. We should think of the precision measures as indicating the deviation of the result from the mean result of a large number of such observations, rather than its deviation from the true value. It is usually true that the constant or the systematic errors are far more serious than the accidental errors; the observer should be continually on the watch for constant errors which may affect his result. So long as the conditions under which a measurement is made remain exactly the same the systematic errors are likely to be the same and are therefore not observed. The presence of such errors is most likely to be observed when the conditions are varied as much as possible. If observations are made at different temperatures, or under different conditions of illumination, or with different instruments, the variations of the results are usually greater than when the conditions are not changed. These variations indicate the presence of systematic errors and often enable the observer to estimate their magnitude.

The computation of the most probable value improves the result with respect to the accidental errors, but leaves the more serious form of error untouched. The futility of multiplying observations and adjusting them for the purpose of removing the small accidental errors, and at the same time failing to remove the large constant error, may be illustrated by the results obtained by a marksman who holds his rifle steadily and places all his shots in a small group, but whose rifle sights are so far out of alignment that his shots all strike far from the bull's-eye. Of what use is the large number of shots under those circumstances? An adjustment of his results by least squares would correspond to an attempt to find the center of his group of shots, and would tell nothing about the distance from the bull's-eye. A study of the causes of the error so that he could make an adjustment of his sights would accomplish more toward hitting the mark than an infinite number of shots find under the original conditions. Of course the comparison is quite untrue in one respect; the marksman knows where his mark is, while the observer can never know the true value of the quantity he is measuring.

While the method of least squares may not show directly the presence of constant errors, a study of the precision of the results, and a knowledge of the law governing the behavior of accidental errors, may enable the observer to detect the presence of constant error, or at least to decide whether it is probably present, and consequently to so modify his methods of observing as to reduce the effect of such constant error. Variations in the result which are greater than the error of observation shown by the precision measures is likely to mean that systematic error is present. This tracing of errors to their sources, and the consequent modification of instruments and methods, may constitute the most important application of least squares.

REFERENCES

Following are a few references to extended works on the subject of Least Squares.

BARTLETT, The Method of Least Squares (an Introductory Treatise).
CHAUVENET, Treatise on the Method of Least Squares. (Theory — Applications to Astronomy.)
CRANDALL, Geodesy and Least Squares. (Applications to Geodesy.)
MERRIMAN, Treatise on the Method of Least Squares.
UNITED STATES COAST AND GEODETIC SURVEY, Special Publication No. 28. (Practice of the United States Coast and Geodetic Survey.)
WRIGHT AND HAYFORD, Adjustment of Observations. (Applications to Geodesy.)

PROBLEMS

Problem 1. The following angles are measured at station O.

AOB =	31° 10′ 15″.6	weight (1)
BOC =	19 21 17 .4	" (1)
AOC =	50 31 33 .5	" (2)
COD =	38 50 16 .0	" (2)
BOD =	58 11 32 .0	" (1)
AOD =	89 21 51 .5	" (1)

Adjust the angles.

Problem 2. The angles of a triangle are as follows:

A	53° 53′ 38″.94	wt. (3)
B	79 22 56 .17	" (4)
C	46 43 29 .27	" (2)

The spherical excess is 2″.83.
Adjust the triangle.

Problem 3. The angles of a quadrilateral are as follows, the numbers corresponding to those in Fig. 113. The weights are all unity. The spherical excess may be neglected.

1.	23° 31′ 12″.5
2.	37 01 22 .5
3.	67 35 38 .3
4.	51 51 26 .7
5.	29 56 50 .0
6.	30 35 33 .2
7.	72 37 35 .0
8.	46 49 47 .5

The sum angles are

8 + 1	70° 21′ 05″.0
2 + 3	104 37 00. 0
4 + 5	81 48 20 .8
6 + 7	103 13 08 .4

Adjust the quadrilateral.

FORMULÆ AND TABLES

FORMULÆ

SERIES

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots.$$

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots.$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots.$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots.$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}.$$

BINOMIAL THEOREM

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{\lfloor 2} a^{m-2}b^2 + \cdots.$$

MACLAUREN'S THEOREM

$$f(x) = f(0) + \frac{x}{1} f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \cdots.$$

TAYLOR'S THEOREM

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \cdots.$$

LOGARITHMIC SERIES

$$\log(1+x) = M\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.\right)$$

$$\log(1-x) = -M\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots.\right)$$

OTHER SERIES

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots.$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots.$$

ELLIPSE AND SPHEROID

$$e^2 = \frac{a^2 - b^2}{a^2}.$$

$$f = \frac{a - b}{a}.$$

$$R_m = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \phi)^{\frac{3}{2}}}.$$

$$N = \frac{a}{(1 - e^2 \sin^2 \phi)^{\frac{1}{2}}}.$$

$$R_\alpha = \frac{NR_m}{N \cos^2 \alpha + R_m \sin^2 \alpha}.$$

$$\text{Mean radius} = \rho = \sqrt{NR_m}.$$

CONSTANTS

$\log_{10} x = M \log_e x.$

M = modulus of system of common logarithms

= 0.434 2945.

$\log M$ = 9.637 7843.

π = 3.141 592 65. log = 0.497 1499.

$\frac{180}{\pi}$ = 57.29577. log = 1.758 1226.

$\frac{180° \times 60'}{\pi}$ = 3437.747. log = 3.536 2739

$\frac{180° \times 60' \times 60''}{\pi}$ = 206 264.8. log = 5.314 4251.

$= \frac{1}{\text{arc } 1''} = \frac{1}{\sin 1''} = \frac{1}{\tan 1''}.$ (Approx.)

arc 1″ = 0.000 004 848 137. log = 4.685 5749.

$\frac{1}{\text{arc } 1''}$ = 206 264.806 = number of seconds in the radian.

arc 1″ = about 0.3 inch at distance of one mile.

CLARKE SPHEROID (1866)

a = 6 378 206.4 meters. log = 6.804 6985.

b = 6 356 583.8 meters. log = 6.803 2238.

(Clarke's value of meter, 3.280 8693 feet.)

a = 6 378 276.5 legal meters. log = 6.804 7033.

b = 6 356 653.7 legal meters. log = 6.803 2285.

(U. S. legal meter, 39.37 inches or 3.280 8333 feet.)

COAST SURVEY SPHEROID (1909)

$a = 6\,378\,388 \pm 18$ meters.

$\frac{1}{f} = 297.0 \pm 0.5.$

$b = 6\,356\,909$ meters.

RELATION BETWEEN UNITS OF LENGTH

(Legal) Meters in one foot	= 0.304 8006.	log = 9.484 0158.
Feet in one (legal) meter	= 3.280 8333.	log = 0.515 9842.
Inches in one (legal) meter	= 38.37.	

TABLE I.—TABLE FOR DETERMINING RELATIVE STRENGTH OF FIGURES IN TRIANGULATION

°	10°	12°	14°	16°	18°	20°	22°	24°	26°	28°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
10	428	359																					
12	359	295	253																				
14	315	253	214	187																			
16	284	225	187	162	143																		
18	262	204	168	143	126	113																	
20	245	189	153	130	113	100	91																
22	232	177	142	11	103	91	81	74															
24	221	167	134	111	95	83	74	67	61														
26	213	160	126	104	89	77	68	61	56	51													
28	206	153	120	99	83	72	63	57	51	47	43												
30	199	148	115	94	79	68	59	53	48	43	40	33											
35	188	137	106	85	71	60	52	46	41	37	33	27	23										
40	179	129	99	79	65	54	47	41	36	32	29	23	19	16									
45	172	124	93	74	60	50	43	37	32	28	25	20	16	13	11								
50	167	119	89	70	57	47	39	34	29	26	23	18	14	11	9	8							
55	162	115	86	67	54	44	37	32	27	24	21	16	12	10	8	7	5						
60	159	112	83	64	51	42	35	30	25	22	19	14	11	9	7	5	4	4					
65	155	109	80	62	49	40	33	28	24	21	18	13	10	7	6	5	4	3	2				
70	152	106	78	60	48	38	32	27	23	19	17	12	9	7	5	4	3	2	2	1			
75	150	104	76	58	46	37	30	25	21	18	16	11	8	6	4	3	2	2	1	1	1		
80	147	102	74	57	45	36	29	24	20	17	15	10	7	5	4	3	2	1	1	1	0	0	
85	145	100	73	55	43	34	28	23	19	16	14	10	7	5	3	2	2	1	1	0	0	0	0
90	143	98	71	54	42	33	27	22	19	16	13	9	6	4	3	2	1	1	1	0	0	0	0
95	140	96	70	53	41	32	26	22	18	15	13	9	6	4	3	2	1	1	0	0	0	0	
100	138	95	68	51	40	31	25	21	17	14	12	8	6	4	3	2	1	1	0	0	0		
105	136	93	67	50	39	30	25	20	17	14	12	8	5	4	2	2	1	1	0	0			
110	134	91	65	49	38	30	24	19	16	13	11	7	5	3	2	2	1	1	1				
115	132	89	64	48	37	29	23	19	15	13	11	7	5	3	2	2	1	1					
120	129	88	62	46	36	28	22	18	15	12	10	7	5	3	2	2	1						
125	127	86	61	45	35	27	22	18	14	12	10	7	5	4	3	2							
130	125	84	59	44	34	26	21	17	14	12	10	7	5	4	3								
135	122	82	58	43	33	26	21	17	14	12	10	7	5	4									
140	119	80	56	42	32	25	20	17	14	12	10	8	6										
145	116	77	55	41	32	25	21	17	15	13	11	9											
150	112	75	54	40	32	26	21	18	16	15	13												
152	111	75	53	40	32	26	22	19	17	16													
154	110	74	53	41	33	27	23	21	19														
156	108	74	54	42	34	28	25	22															
158	107	74	54	43	35	30	27																
160	107	74	56	45	38	33																	
162	107	76	59	48	42																		
164	109	79	63	54																			
166	113	86	71																				
168	122	98																					
170	143																						

TABLE II. — CORRECTION FOR EARTH'S CURVATURE AND REFRACTION

Dist.	Corr.	Dist.	Corr.	Dist.	Corr.
Miles.	Feet.	Miles.	Feet.	Miles.	Feet.
1	0.6	21	253.1	41	964.7
2	2.3	22	277.7	42	1012.2
3	5.2	23	303.6	43	1061.0
4	9.2	24	330.5	44	1111.0
5	14.4	25	358.6	45	1162.0
6	20.6	26	388.0	46	1214.2
7	28.1	27	418.3	47	1267.7
8	36.7	28	449.9	48	1322.1
9	46.4	29	482.6	49	1377.7
10	57.4	30	516.4	50	1434.6
11	69.4	31	551.4	51	1492.5
12	82.7	32	587.6	52	1551.6
13	97.0	33	624.9	53	1611.9
14	112.5	34	663.3	54	1673.3
15	129.1	35	703.0	55	1735.8
16	146.9	36	743.7	56	1799.6
17	165.8	37	785.6	57	1864.4
18	185.9	38	828.6	58	1930.4
19	207.2	39	872.8	59	1997.5
20	229.5	40	918.1	60	2065.8

TABLE III. — SHORT TABLE OF FACTORS FOR REDUCTION OF TRANSIT OBSERVATIONS

Top Argument = Star's Declination (δ).
Side Argument = Star's Zenith Distance (ζ).

[For factor *A* use left-hand argument. For factor *B* use right-hand argument. For factor *C* use bottom line.]

ζ	0°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°	65°	70°	ζ
1°	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.03	0.03	0.03	0.04	0.05	89°
5	0.09	0.09	0.09	0.09	0.10	0.10	0.11	0.11	0.12	0.13	0.15	0.17	0.21	0.25	85
10	0.17	0.18	0.18	0.19	0.19	0.20	0.21	0.23	0.25	0.27	0.30	0.35	0.41	0.51	80
15	0.26	0.26	0.27	0.28	0.29	0.30	0.32	0.34	0.37	0.40	0.45	0.52	0.61	0.76	75
20	0.34	0.35	0.35	0.36	0.38	0.40	0.42	0.45	0.48	0.53	0.60	0.68	0.81	1.00	70
25	0.42	0.43	0.44	0.45	0.47	0.49	0.52	0.55	0.60	0.66	0.74	0.85	1.00	1.24	65
30	0.50	0.51	0.52	0.53	0.55	0.58	0.61	0.65	0.71	0.78	0.87	1.00	1.18	1.46	60
35	0.57	0.58	0.59	0.61	0.63	0.66	0.70	0.75	0.81	0.89	1.00	1.15	1.36	1.68	55
40	0.64	0.65	0.67	0.68	0.71	0.74	0.78	0.84	0.91	1.00	1.12	1.29	1.52	1.88	50
45	0.71	0.72	0.73	0.75	0.78	0.82	0.86	0.92	1.00	1.10	1.23	1.41	1.67	2.07	45
50	0.77	0.78	0.79	0.82	0.85	0.89	0.94	1.00	1.08	1.19	1.34	1.53	1.81	2.24	40
55	0.82	0.83	0.85	0.87	0.90	0.95	1.00	1.07	1.16	1.27	1.43	1.64	1.94	2.40	35
60	0.87	0.88	0.90	0.92	0.96	1.00	1.06	1.13	1.22	1.35	1.51	1.73	2.05	2.53	30
65	0.91	0.92	0.94	0.96	1.00	1.05	1.11	1.18	1.28	1.41	1.58	1.81	2.14	2.65	25
70	0.94	0.95	0.97	1.00	1.04	1.09	1.15	1.23	1.33	1.46	1.64	1.88	2.22	2.75	20
75	0.97	0.98	1.00	1.03	1.07	1.12	1.18	1.26	1.37	1.50	1.68	1.93	2.29	2.82	15
80	0.98	1.00	1.02	1.05	1.09	1.14	1.20	1.29	1.39	1.53	1.72	1.97	2.33	2.88	10
85	1.00	1.01	1.03	1.06	1.10	1.15	1.22	1.30	1.41	1.55	1.74	1.99	2.36	2.91	5
90	1.00	1.02	1.04	1.06	1.10	1.15	1.22	1.31	1.41	1.56	1.74	2.00	2.37	2.92	0

TABLE IV. — DIURNAL ABERRATION (κ)

Latitude = φ.	Declination = δ.										
	0°	10°	20°	30°	40°	50°	60°	70°	75°	80°	85°
°	s	s	s	s	s	s	s	s	s	s	s
0	0.02	0.02	0.02	0.02	0.03	0.03	0.04	0.06	0.08	0.12	0.24
10	0.02	0.02	0.02	0.02	0.03	0.03	0.04	0.06	0.08	0.12	0.24
20	0.02	0.02	0.02	0.02	0.03	0.03	0.04	0.06	0.08	0.11	0.23
30	0.02	0.02	0.02	0.02	0.02	0.03	0.04	0.05	0.07	0.10	0.21
40	0.02	0.02	0.02	0.02	0.02	0.03	0.03	0.05	0.06	0.09	0.18
50	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.04	0.05	0.08	0.15
60	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.03	0.04	0.06	0.12
70	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.03	0.04	0.08
80	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01	0.02	0.04

TABLE V. — CORRECTION TO LATITUDE FOR DIFFERENTIAL REFRACTION = $\frac{1}{2}(r - r')$.

[The sign of the correction is the same as that of the micrometer difference.]

One-half diff. of zenith distances.	Zenith distance.							
	0°	10°	20°	25°	30°	35°	40°	45°
′	″	″	″	″	″	″	″	″
0.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.5	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02
1.0	0.02	0.02	0.02	0.02	0.02	0.03	0.03	0.03
1.5	0.03	0.03	0.03	0.03	0.03	0.04	0.04	0.05
2.0	0.03	0.03	0.04	0.04	0.04	0.05	0.06	0.07
2.5	0.04	0.04	0.05	0.05	0.06	0.06	0.07	0.08
3.0	0.05	0.05	0.06	0.06	0.07	0.08	0.09	0.10
3.5	0.06	0.06	0.07	0.07	0.08	0.09	0.10	0.12
4.0	0.07	0.07	0.08	0.08	0.09	0.10	0.11	0.13
4.5	0.08	0.08	0.09	0.09	0.10	0.11	0.13	0.15
5.0	0.08	0.09	0.10	0.10	0.11	0.13	0.14	0.17
5.5	0.09	0.10	0.10	0.11	0.12	0.14	0.16	0.18
6.0	0.10	0.10	0.11	0.12	0.13	0.15	0.17	0.20
6.5	0.11	0.11	0.12	0.13	0.14	0.16	0.19	0.22
7.0	0.12	0.12	0.13	0.14	0.16	0.18	0.20	0.23
7.5	0.13	0.13	0.14	0.15	0.17	0.19	0.21	0.25
8.0	0.13	0.14	0.15	0.16	0.18	0.20	0.23	0.27
8.5	0.14	0.15	0.16	0.17	0.19	0.21	0.24	0.29
9.0	0.15	0.16	0.17	0.18	0.20	0.23	0.26	0.30
9.5	0.16	0.16	0.18	0.19	0.21	0.24	0.27	0.32
10.0	0.17	0.17	0.19	0.20	0.22	0.25	0.29	0.34
10.5	0.18	0.18	0.20	0.21	0.23	0.26	0.30	0.35
11.0	0.18	0.19	0.21	0.22	0.25	0.28	0.31	0.37
11.5	0.19	0.20	0.22	0.23	0.26	0.29	0.33	0.39
12.0	0.20	0.21	0.23	0.25	0.27	0.30	0.34	0.40
12.5	0.21	0.22	0.24	0.26	0.28	0.31	0.36	0.42
13.0	0.22	0.22	0.25	0.27	0.29	0.33	0.37	0.44
13.5	0.23	0.23	0.26	0.28	0.30	0.34	0.39	0.45
14.0	0.23	0.24	0.27	0.29	0.31	0.35	0.40	0.47
14.5	0.24	0.25	0.28	0.30	0.32	0.36	0.41	0.49
15.0	0.25	0.26	0.29	0.31	0.34	0.38	0.43	0.50
15.5	0.26	0.27	0.29	0.32	0.35	0.39	0.44	0.52
16.0	0.27	0.28	0.30	0.33	0.36	0.40	0.46	0.54
16.5	0.28	0.29	0.31	0.34	0.37	0.41	0.47	0.55
17.0	0.29	0.29	0.32	0.35	0.38	0.43	0.49	0.57
17.5	0.29	0.30	0.33	0.36	0.39	0.44	0.50	0.59
18.0	0.30	0.31	0.34	0.37	0.40	0.45	0.51	0.60
18.5	0.31	0.32	0.35	0.38	0.41	0.46	0.53	0.62
19.0	0.32	0.33	0.36	0.39	0.43	0.48	0.54	0.64
19.5	0.33	0.34	0.37	0.40	0.44	0.49	0.56	0.65
20.0	0.34	0.35	0.38	0.41	0.45	0.50	0.57	0.67

TABLE VI. — CORRECTION TO LATITUDE FOR REDUCTION TO MERIDIAN

[Star off the meridian but instrument in the meridian. The sign of the correction to the latitude is positive except for stars south of the equator and subpolars.]

δ	10^s	15^s	20^s	22^s	24^s	26^s	28^s	30^s	32^s	34^s	36^s	38^s	δ
°	″	″	″	″	″	″	″	″	″	″	″	″	°
1										0.01	0.01	0.01	89
2					0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	88
3			0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.02	87
4			0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.03	86
5		0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.03	0.03	0.03	85
6		0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.03	0.04	0.04	84
7		0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.03	0.04	0.04	0.05	83
8		0.01	0.02	0.02	0.02	0.03	0.03	0.03	0.04	0.04	0.05	0.05	82
9		0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.04	0.05	0.05	0.06	81
10		0.01	0.02	0.02	0.03	0.03	0.04	0.04	0.05	0.05	0.06	0.07	80
12	0.01	0.01	0.02	0.03	0.03	0.04	0.05	0.05	0.06	0.06	0.07	0.08	78
14	0.01	0.01	0.03	0.03	0.04	0.04	0.05	0.06	0.07	0.07	0.08	0.09	76
16	0.01	0.02	0.03	0.03	0.04	0.05	0.06	0.07	0.07	0.08	0.09	0.10	74
18	0.01	0.02	0.03	0.04	0.05	0.05	0.06	0.07	0.08	0.09	0.10	0.12	72
20	0.01	0.02	0.04	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.13	70
22	0.01	0.02	0.04	0.05	0.05	0.06	0.07	0.09	0.10	0.11	0.12	0.14	68
24	0.01	0.02	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.12	0.13	0.15	66
26	0.01	0.02	0.04	0.05	0.06	0.07	0.08	0.10	0.11	0.12	0.14	0.15	64
28	0.01	0.03	0.05	0.05	0.07	0.08	0.09	0.10	0.12	0.13	0.15	0.16	62
30	0.01	0.03	0.05	0.06	0.07	0.08	0.09	0.11	0.12	0.14	0.15	0.17	60
32	0.01	0.03	0.05	0.06	0.07	0.08	0.10	0.11	0.13	0.14	0.16	0.18	58
34	0.01	0.03	0.05	0.06	0.07	0.09	0.10	0.11	0.13	0.15	0.16	0.18	56
36	0.01	0.03	0.05	0.06	0.07	0.09	0.10	0.12	0.13	0.15	0.17	0.19	54
38	0.01	0.03	0.05	0.06	0.08	0.09	0.10	0.12	0.13	0.15	0.17	0.19	52
40	0.01	0.03	0.05	0.07	0.08	0.09	0.11	0.12	0.14	0.16	0.17	0.19	50
45	0.01	0.03	0.05	0.07	0.08	0.09	0.11	0.12	0.14	0.16	0.18	0.20	45

δ	40^s	42^s	44^s	46^s	48^s	50^s	52^s	54^s	56^s	58^s	60^s	δ
°	″	″	″	″	″	″	″	″	″	″	″	°
1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	89
2	0.02	0.02	0.02	0.02	0.02	0.02	0.03	0.03	0.03	0.03	0.03	88
3	0.02	0.03	0.03	0.03	0.03	0.04	0.04	0.04	0.04	0.05	0.05	87
4	0.03	0.03	0.04	0.04	0.04	0.05	0.05	0.06	0.06	0.06	0.07	86
5	0.04	0.04	0.05	0.05	0.05	0.06	0.06	0.07	0.07	0.08	0.09	85
6	0.05	0.05	0.06	0.06	0.07	0.07	0.08	0.08	0.09	0.10	0.10	84
7	0.05	0.06	0.06	0.07	0.08	0.08	0.09	0.10	0.10	0.11	0.12	83
8	0.06	0.07	0.07	0.08	0.09	0.09	0.10	0.11	0.12	0.13	0.14	82
9	0.07	0.07	0.08	0.09	0.10	0.11	0.11	0.12	0.13	0.14	0.15	81
10	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	80
12	0.09	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.19	0.20	78
14	0.10	0.11	0.12	0.14	0.15	0.16	0.17	0.19	0.20	0.22	0.23	76
16	0.12	0.13	0.14	0.15	0.17	0.18	0.20	0.21	0.23	0.24	0.26	74
18	0.13	0.14	0.16	0.17	0.18	0.20	0.22	0.23	0.25	0.27	0.29	72
20	0.14	0.15	0.17	0.19	0.20	0.22	0.24	0.26	0.28	0.29	0.32	70
22	0.15	0.17	0.18	0.20	0.22	0.24	0.26	0.28	0.30	0.32	0.34	68
24	0.16	0.18	0.20	0.21	0.23	0.25	0.27	0.29	0.32	0.34	0.36	66
26	0.17	0.19	0.21	0.23	0.25	0.27	0.29	0.31	0.34	0.36	0.39	64
28	0.18	0.20	0.22	0.24	0.26	0.28	0.31	0.33	0.35	0.38	0.41	62
30	0.19	0.21	0.23	0.25	0.27	0.30	0.32	0.34	0.37	0.40	0.42	60
32	0.20	0.22	0.24	0.26	0.28	0.31	0.33	0.36	0.39	0.41	0.44	58
34	0.20	0.22	0.24	0.27	0.29	0.32	0.34	0.37	0.40	0.42	0.45	56
36	0.21	0.23	0.25	0.28	0.30	0.32	0.35	0.38	0.41	0.44	0.47	54
38	0.21	0.23	0.26	0.28	0.30	0.33	0.36	0.39	0.41	0.44	0.48	52
40	0.21	0.24	0.26	0.28	0.31	0.34	0.36	0.39	0.42	0.45	0.48	50
45	0.22	0.24	0.26	0.29	0.31	0.34	0.37	0.40	0.43	0.46	0.49	45

TABLE VII.—REDUCTION OF LATITUDE TO SEA LEVEL

[The correction is negative in every case.]

h \ φ		5° / 85°	10° / 80°	15° / 75°	20° / 70°	25° / 65°	30° / 60°	35° / 55°	40° / 50°	45°
Feet.	Meters.	"	"	"	"	"	"	"	"	"
100	30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01
200	61	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.01
300	91	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02
400	122	0.00	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.02
500	152	0.00	0.01	0.01	0.02	0.02	0.02	0.02	0.03	0.03
600	183	0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.03	0.03
700	213	0.01	0.01	0.02	0.02	0.03	0.03	0.03	0.04	0.04
800	244	0.01	0.01	0.02	0.03	0.03	0.04	0.04	0.04	0.04
900	274	0.01	0.02	0.02	0.03	0.04	0.04	0.04	0.05	0.05
1000	305	0.01	0.02	0.03	0.03	0.04	0.05	0.05	0.05	0.05
1100	335	0.01	0.02	0.03	0.04	0.04	0.05	0.05	0.06	0.06
1200	366	0.01	0.02	0.03	0.04	0.05	0.05	0.06	0.06	0.06
1300	396	0.01	0.02	0.03	0.04	0.05	0.06	0.06	0.07	0.07
1400	427	0.01	0.02	0.04	0.05	0.06	0.06	0.07	0.07	0.07
1500	457	0.01	0.03	0.04	0.05	0.06	0.07	0.07	0.08	0.08
1600	488	0.01	0.03	0.04	0.05	0.06	0.07	0.08	0.08	0.08
1700	518	0.02	0.03	0.04	0.06	0.07	0.08	0.08	0.09	0.09
1800	549	0.02	0.03	0.05	0.06	0.07	0.08	0.09	0.09	0.09
1900	579	0.02	0.03	0.05	0.06	0.08	0.09	0.09	0.10	0.10
2000	610	0.02	0.04	0.05	0.07	0.08	0.09	0.10	0.10	0.10
2100	640	0.02	0.04	0.05	0.07	0.08	0.09	0.10	0.11	0.11
2200	671	0.02	0.04	0.06	0.07	0.09	0.10	0.11	0.11	0.11
2300	701	0.02	0.04	0.06	0.08	0.09	0.10	0.11	0.12	0.12
2400	732	0.02	0.04	0.06	0.08	0.10	0.11	0.12	0.12	0.13
2500	762	0.02	0.04	0.07	0.08	0.10	0.11	0.12	0.13	0.13
2600	792	0.02	0.05	0.07	0.09	0.10	0.12	0.13	0.13	0.14
2700	823	0.02	0.05	0.07	0.09	0.11	0.12	0.13	0.14	0.14
2800	853	0.03	0.05	0.07	0.09	0.11	0.13	0.14	0.14	0.15
2900	884	0.03	0.05	0.08	0.10	0.12	0.13	0.14	0.15	0.15
3000	914	0.03	0.05	0.08	0.10	0.12	0.14	0.15	0.15	0.16
3100	945	0.03	0.06	0.08	0.10	0.12	0.14	0.15	0.16	0.16
3200	975	0.03	0.06	0.08	0.11	0.13	0.14	0.16	0.16	0.17
3300	1006	0.03	0.06	0.09	0.11	0.13	0.15	0.16	0.17	0.17
3400	1036	0.03	0.06	0.09	0.11	0.12	0.15	0.17	0.17	0.18
3500	1067	0.03	0.06	0.09	0.12	0.14	0.16	0.17	0.18	0.18

TABLE VII (*Con.*).—REDUCTION OF LATITUDE TO SEA LEVEL

[The correction is negative in every case.]

h (Feet.)	h (Meters.)	φ 5° / 85°	10° / 80°	15° / 75°	20° / 70°	25° / 65°	30° / 60°	35° / 55°	40° / 50°	45°
		"	"	"	"	"	"	"	"	"
3600	1097	0.03	0.06	0.09	0.12	0.14	0.16	0.18	0.18	0.19
3700	1128	0.03	0.07	0.10	0.12	0.15	0.17	0.18	0.19	0.19
3800	1158	0.03	0.07	0.10	0.13	0.15	0.17	0.19	0.20	0.20
3900	1189	0.04	0.07	0.10	0.13	0.16	0.18	0.19	0.20	0.20
4000	1219	0.04	0.07	0.10	0.13	0.16	0.18	0.20	0.21	0.21
4100	1250	0.04	0.07	0.11	0.14	0.16	0.19	0.20	0.21	0.21
4200	1280	0.04	0.07	0.11	0.14	0.17	0.19	0.21	0.22	0.22
4300	1311	0.04	0.08	0.11	0.14	0.17	0.19	0.21	0.22	0.22
4400	1341	0.04	0.08	0.11	0.15	0.18	0.20	0.22	0.23	0.23
4500	1372	0.04	0.08	0.12	0.15	0.18	0.20	0.22	0.23	0.23
4600	1402	0.04	0.08	0.12	0.15	0.18	0.21	0.23	0.24	0.24
4700	1433	0.04	0.08	0.12	0.16	0.19	0.21	0.23	0.24	0.24
4800	1463	0.04	0.09	0.13	0.16	0.19	0.22	0.24	0.25	0.25
4900	1494	0.04	0.09	0.13	0.16	0.20	0.22	0.24	0.25	0.26
5000	1524	0.05	0.09	0.13	0.17	0.20	0.23	0.24	0.26	0.26
5100	1554	0.05	0.09	0.13	0.17	0.20	0.23	0.25	0.26	0.27
5200	1585	0.05	0.09	0.14	0.17	0.21	0.23	0.25	0.27	0.27
5300	1615	0.05	0.09	0.14	0.18	0.21	0.24	0.26	0.27	0.28
5400	1646	0.05	0.10	0.14	0.18	0.22	0.24	0.26	0.28	0.28
5500	1676	0.05	0.10	0.14	0.18	0.22	0.25	0.27	0.28	0.29
5600	1707	0.05	0.10	0.15	0.19	0.22	0.25	0.27	0.29	0.29
5700	1737	0.05	0.10	0.15	0.19	0.23	0.26	0.28	0.29	0.30
5800	1768	0.05	0.10	0.15	0.19	0.23	0.26	0.28	0.30	0.30
5900	1798	0.05	0.11	0.15	0.20	0.24	0.27	0.29	0.30	0.31
6000	1829	0.05	0.11	0.16	0.20	0.24	0.27	0.29	0.31	0.31
6100	1859	0.06	0.11	0.16	0.20	0.24	0.28	0.30	0.31	0.32
6200	1890	0.06	0.11	0.16	0.21	0.25	0.28	0.30	0.32	0.32
6300	1920	0.06	0.11	0.16	0.21	0.25	0.28	0.31	0.32	0.33
6400	1951	0.06	0.11	0.17	0.21	0.26	0.29	0.31	0.33	0.33
6500	1981	0.06	0.12	0.17	0.22	0.26	0.29	0.32	0.33	0.34
6600	2012	0.06	0.12	0.17	0.22	0.26	0.30	0.32	0.34	0.34
6700	2042	0.06	0.12	0.17	0.22	0.27	0.30	0.33	0.34	0.35
6800	2073	0.06	0.12	0.18	0.23	0.27	0.31	0.33	0.35	0.35
6900	2103	0.06	0.12	0.18	0.23	0.28	0.31	0.34	0.35	0.36
7000	2134	0.06	0.12	0.18	0.23	0.28	0.32	0.34	0.36	0.36

TABLE VII (*Con.*).—REDUCTION OF LATITUDE TO SEA LEVEL

[The correction is negative in every case.]

h	φ	5° / 85°	10° / 80°	15° / 75°	20° / 70°	25° / 65°	30° / 60°	35° / 55°	40° / 50°	45°
Feet.	Meters.	"	"	"	"	"	"	"	"	"
7100	2164	0.06	0.13	0.19	0.24	0.28	0.32	0.35	0.36	0.37
7200	2195	0.07	0.13	0.19	0.24	0.29	0.33	0.35	0.37	0.38
7300	2225	0.07	0.13	0.19	0.24	0.29	0.33	0.36	0.37	0.38
7400	2256	0.07	0.13	0.19	0.25	0.30	0.33	0.36	0.38	0.39
7500	2286	0.07	0.13	0.20	0.25	0.30	0.34	0.37	0.38	0.39
7600	2316	0.07	0.14	0.20	0.25	0.30	0.34	0.37	0.39	0.40
7700	2347	0.07	0.14	0.20	0.26	0.31	0.35	0.38	0.40	0.40
7800	2377	0.07	0.14	0.20	0.26	0.31	0.35	0.38	0.40	0.41
7900	2408	0.07	0.14	0.21	0.26	0.32	0.36	0.39	0.41	0.41
8000	2438	0.07	0.14	0.21	0.27	0.32	0.36	0.39	0.41	0.42
8100	2469	0.07	0.14	0.21	0.27	0.32	0.37	0.40	0.42	0.42
8200	2499	0.07	0.15	0.21	0.27	0.33	0.37	0.40	0.42	0.43
8300	2530	0.08	0.15	0.22	0.28	0.33	0.37	0.41	0.43	0.43
8400	2560	0.08	0.15	0.22	0.28	0.34	0.38	0.41	0.43	0.44
8500	2591	0.08	0.15	0.22	0.28	0.34	0.38	0.42	0.44	0.44
8600	2621	0.08	0.15	0.22	0.29	0.34	0.39	0.42	0.44	0.45
8700	2652	0.08	0.16	0.23	0.29	0.35	0.39	0.43	0.45	0.45
8800	2682	0.08	0.16	0.23	0.29	0.35	0.40	0.43	0.45	0.46
8900	2713	0.08	0.16	0.23	0.30	0.36	0.40	0.44	0.46	0.46
9000	2743	0.08	0.16	0.23	0.30	0.36	0.41	0.44	0.46	0.47
9100	2774	0.08	0.16	0.24	0.30	0.36	0.41	0.45	0.47	0.47
9200	2804	0.08	0.16	0.24	0.31	0.37	0.42	0.45	0.47	0.48
9300	2835	0.08	0.17	0.24	0.31	0.37	0.42	0.46	0.48	0.48
9400	2865	0.09	0.17	0.24	0.31	0.38	0.42	0.46	0.48	0.49
9500	2896	0.09	0.17	0.25	0.32	0.38	0.43	0.47	0.49	0.50
9600	2926	0.09	0.17	0.25	0.32	0.38	0.43	0.47	0.49	0.50
9700	2957	0.09	0.17	0.25	0.32	0.39	0.44	0.48	0.50	0.51
9800	2987	0.09	0.17	0.26	0.33	0.39	0.44	0.48	0.50	0.51
9900	3018	0.09	0.18	0.26	0.33	0.40	0.45	0.48	0.51	0.52
10000	3048	0.09	0.18	0.26	0.33	0.40	0.45	0.49	0.51	0.52

TABLE VIII.—FOR CONVERTING SIDEREAL INTO MEAN SOLAR TIME

[Increase in Sun's Right Ascension in Sidereal h. m. s.]

Mean Time = Sidereal Time − C'.

Sid. Hrs.	Corr.	Sid. Min.	Corr.	Sid. Min.	Corr.	Sid. Sec.	Corr.	Sid. Sec.	Corr.
	m s		s		s		s		s
1	0 9.830	1	0.164	31	5.079	1	0.003	31	0.085
2	0 19.659	2	0.328	32	5.242	2	0.005	32	0.087
3	0 29.489	3	0.491	33	5.406	3	0.008	33	0.090
4	0 39.318	4	0.655	34	5.570	4	0.011	34	0.093
5	0 49.148	5	0.819	35	5.734	5	0.014	35	0.096
6	0 58.977	6	0.983	36	5.898	6	0.016	36	0.098
7	1 8.807	7	1.147	37	6.062	7	0.019	37	0.101
8	1 18.636	8	1.311	38	6.225	8	0.022	38	0.104
9	1 28.466	9	1.474	39	6.389	9	0.025	39	0.106
10	1 38.296	10	1.638	40	6.553	10	0.027	40	0.109
11	1 48.125	11	1.802	41	6.717	11	0.030	41	0.112
12	1 57.955	12	1.966	42	6.881	12	0.033	42	0.115
13	2 7.784	13	2.130	43	7.045	13	0.035	43	0.117
14	2 17.614	14	2.294	44	7.208	14	0.038	44	0.120
15	2 27.443	15	2.457	45	7.372	15	0.041	45	0.123
16	2 37.273	16	2.621	46	7.536	16	0.044	46	0.126
17	2 47.102	17	2.785	47	7.700	17	0.046	47	0.128
18	2 56.932	18	2.949	48	7.864	18	0.049	48	0.131
19	3 6.762	19	3.113	49	8.027	19	0.052	49	0.134
20	3 16.591	20	3.277	50	8.191	20	0.055	50	0.137
21	3 26.421	21	3.440	51	8.355	21	0.057	51	0.139
22	3 36.250	22	3.604	52	8.519	22	0.060	52	0.142
23	3 46.080	23	3.768	53	8.683	23	0.063	53	0.145
24	3 55.909	24	3.932	54	8.847	24	0.066	54	0.147
		25	4.096	55	9.010	25	0.068	55	0.150
		26	4.259	56	9.174	26	0.071	56	0.153
		27	4.423	57	9.338	27	0.074	57	0.156
		28	4.587	58	9.502	28	0.076	58	0.158
		29	4.751	59	9.666	29	0.079	59	0.161
		30	4.915	60	9.830	30	0.082	60	0.164

TABLE IX.—FOR CONVERTING MEAN SOLAR INTO SIDEREAL TIME

[Increase in Sun's Right Ascension in Solar h. m. s.]

Sidereal Time = Mean Time + C.

Mean Hrs.	Corr.	Mean Min.	Corr.	Mean Min.	Corr.	Mean Sec.	Corr.	Mean Sec.	Corr.
	m s		s		s		s		s
1	0 9.856	1	0.164	31	5.093	1	0.003	31	0.085
2	0 19.713	2	0.329	32	5.257	2	0.005	32	0.088
3	0 29.569	3	0.493	33	5.421	3	0.008	33	0.090
4	0 39.426	4	0.657	34	5.585	4	0.011	34	0.093
5	0 49.282	5	0.821	35	5.750	5	0.014	35	0.096
6	0 59.139	6	0.986	36	5.914	6	0.016	36	0.099
7	1 8.995	7	1.150	37	6.078	7	0.019	37	0.101
8	1 18.852	8	1.314	38	6.242	8	0.022	38	0.104
9	1 28.708	9	1.478	39	6.407	9	0.025	39	0.107
10	1 38.565	10	1.643	40	6.571	10	0.027	40	0.110
11	1 48.421	11	1.807	41	6.735	11	0.030	41	0.112
12	1 58.278	12	1.971	42	6.900	12	0.033	42	0.115
13	2 8.134	13	2.136	43	7.064	13	0.036	43	0.118
14	2 17.991	14	2.300	44	7.228	14	0.038	44	0.120
15	2 27.847	15	2.464	45	7.392	15	0.041	45	0.123
16	2 37.704	16	2.628	46	7.557	16	0.044	46	0.126
17	2 47.560	17	2.793	47	7.721	17	0.047	47	0.129
18	2 57.417	18	2.957	48	7.885	18	0.049	48	0.131
19	3 7.273	19	3.121	49	8.049	19	0.052	49	0.134
20	3 17.129	20	3.285	50	8.214	20	0.055	50	0.137
21	3 26.986	21	3.450	51	8.378	21	0.057	51	0.140
22	3 36.842	22	3.614	52	8.542	22	0.060	52	0.142
23	3 46.699	23	3.778	53	8.707	23	0.063	53	0.145
24	3 56.555	24	3.943	54	8.871	24	0.066	54	0.148
		25	4.107	55	9.035	25	0.068	55	0.151
		26	4.271	56	9.199	26	0.071	56	0.153
		27	4.435	57	9.364	27	0.074	57	0.156
		28	4.600	58	9.528	28	0.077	58	0.160
		29	4.764	59	9.692	29	0.079	59	0.162
		30	4.928	60	9.856	30	0.082	60	0.164

TABLE X. — LENGTHS OF ARCS OF THE PARALLEL AND THE MERIDIAN AND LOGS OF N AND R_m

[Metric Units.]

Latitude.		Parallel. Value of 1°.	Meridian. Value of 1°.	Log N.	Log R_m.
°	′	Meters.	Meters.		
0	00	111,321	110,567.2	6.8046985	6.8017489
	30	1,361	567.3	6987	7493
1	00	1,304	567.6	6990	7502
	30	1,283	568.0	6996	7519
2	00	1,253	568.6	7003	7543
	30	1,215	569.4	7012	7573
3	00	1,169	570.3	7025	7610
	30	1,114	571.4	7040	7654
4	00	1,051	572.7	7057	7704
	30	110,980	574.1	7076	7761
5	00	110,900	110,575.8	6.8047097	6.8017824
	30	0,812	577.6	7120	7894
6	00	0,715	579.5	7146	7971
	30	0,610	581.6	7174	8054
7	00	0,497	583.9	7203	8144
	30	0,375	586.4	7235	8240
8	00	0,245	589.0	7270	8343
	30	0,106	591.8	7307	8452
9	00	109,959	594.7	7345	8568
	30	9,804	597.8	7385	8690
10	00	109,641	110,601.1	6.8047428	6.8018819
	30	9,469	604.5	7474	8954
11	00	9,289	608.1	7520	9094
	30	9,101	611.9	7570	9241
12	00	108,904	615.8	7620	9395
	30	8,699	619.8	7673	9555
13	00	8,486	624.1	7729	9720
	30	8,265	628.4	7786	9892
14	00	8,036	633.0	7845	6.8020070
	30	107,798	637.6	7907	0254
15	00	107,553	110,642.5	6.8047970	6.8020443
	30	7,299	647.5	8035	0639
16	00	7,036	652.6	8102	0839
	30	6,766	657.8	8171	1047
17	00	6,487	663.3	8242	1258
	30	6,201	668.8	8315	1477
18	00	5,906	674.5	8389	1701
	30	5,604	680.4	8465	1930
19	00	5,294	686.3	8544	2165
	30	4,975	692.4	8624	2404
20	00	104,649	110,698.7	6.8048705	6.8022649
	30	4,314	705.1	8789	2900
21	00	3,972	711.6	8874	3155
	30	3,622	718.2	8960	3415
22	00	3,264	725.0	9049	3680
	30	2,898	731.8	9139	3950

TABLE X (*Con.*) — LENGTHS OF ARCS OF THE PARALLEL AND THE MERIDIAN AND LOGS OF N AND R_m

[Metric Units.]

Latitude.		Parallel. Value of 1°.	Meridian. Value of 1°.	Log N.	Log R_m.
°	′	Meters.	Meters.		
23	00	102,524	110,738.8	6.8029231	6.8044225
	30	2,143	746.0	9323	4504
24	00	1,754	753.2	9418	4788
	30	1,357	760.6	9514	5077
25	00	100,952	110,768.0	6.8049612	6.8025370
	30	0,539	775.6	9711	5667
26	00	0,119	783.3	9812	5968
	30	99,692	791.1	9914	6274
27	00	9,257	799.0	6.8050017	6584
	30	8,814	807.0	0121	6897
28	00	8,364	815.1	0227	7215
	30	7,906	823.3	0334	7536
29	00	7,441	831.6	0443	7862
	30	6,968	840.0	0552	8190
30	00	96,488	110,848.5	6.8050663	6.8028522
	30	6,001	857.0	0774	8857
31	00	95,506	865.7	0888	9197
	30	5,004	874.4	1002	9539
32	00	4,495	883.2	1117	9883
	30	3,979	892.1	1233	6.8030231
33	00	3,455	901.1	1350	0582
	30	2,925	910.1	1468	0935
34	00	2,387	919.2	1586	1292
	30	1,842	928.3	1706	1651
35	00	91,290	110,937.6	6.8051826	6.8032012
	30	0,731	946.9	1947	2375
36	00	0,166	956.2	2069	2741
	30	89,593	965.6	2192	3109
37	00	9,014	975.1	2315	3479
	30	8,428	984.5	2439	3850
38	00	7,835	994.1	2564	4224
	30	7,235	111,003.7	2689	4599
39	00	6,629	013.3	2814	4976
	30	6,016	023.0	2940	5354
40	00	85,396	111,032.7	6.8053067	6.8035734
	30	4,770	042.4	3194	6115
41	00	4,137	052.2	3321	6496
	30	3,498	061.9	3448	6878
42	00	2,853	071.7	3576	7262
	30	2,201	081.6	3704	7646
43	00	1,543	091.4	3832	8031
	30	0,879	101.3	3960	8416
44	00	80,208	111.1	4089	8802
	30	79,532	121.0	4218	9188
45	00	78,849	111,130.9	6.8054347	6.8039574
	30	8,160	140.8	4476	9960

TABLE X (*Con.*).—LENGTHS OF ARCS OF THE PARALLEL AND THE MERIDIAN AND LOGS OF N AND R_m

[Metric Units.]

Latitude.		Parallel. Value of 1°.	Meridian. Value of 1°.	Log N.	Log R_m.
°	′	Meters.	Meters.		
46	00	77,466	111,150.6	6.8054604	6.8040346
	30	6,765	160.5	4732	0731
47	00	6,058	170.4	4861	1117
	30	5,346	180.2	4989	1502
48	00	4,628	190.1	5118	1887
	30	3,904	199.9	5246	2270
49	00	3,174	209.7	5373	2653
	30	2,439	219.5	5500	3034
50	00	71,698	111,229.3	6.8055628	6.8043416
	30	0,952	239.0	5754	3796
51	00	0,200	248.7	5880	4175
	30	69,443	258.3	6006	4552
52	00	8,680	268.0	6131	4928
	30	7,913	277.6	6256	5302
53	00	7,140	287.1	6380	5674
	30	6,361	296.6	6504	6044
54	00	5,578	306.0	6627	6413
	30	4,790	315.4	6749	6779
55	00	63,996	111,324.8	6.8056870	6.8047144
	30	3,198	334.0	6991	7506
56	00	2,395	343.3	7111	7866
	30	1,587	352.4	7230	8223
57	00	0,774	361.5	7348	8578
	30	59,957	370.5	7465	8929
58	00	9,135	379.5	7582	9279
	30	8,309	388.4	7697	9624
59	00	7,478	397.2	7811	9968
	30	6,642	405.9	7924	6.8050307
60	00	55,802	111,414.5	6.8058037	6.8050644
	30	4,958	423.1	8148	0977
61	00	4,110	431.5	8258	1307
	30	3,257	439.9	8366	1633
62	00	2,400	448.2	8474	1956
	30	1,540	456.4	8580	2274
63	00	0,675	464.4	8685	2590
	30	49,806	472.4	8789	2900
64	00	8,934	480.3	8891	3208
	30	8,057	488.1	8992	3510
65	00	47,177	111,495.7	6.8059092	6.8053809
	30	6,294	503.3	9190	4103
66	00	5,407	510.7	9287	4393
	30	4,516	518.0	9382	4678
67	00	43,622	525.3	9475	4959
	30	2,724	532.3	9567	5235
68	00	1,823	539.3	9658	5506
	30	0,919	546.2	9747	5772

TABLE X (*Con.*).—LENGTHS OF ARCS OF THE PARALLEL AND THE MERIDIAN AND LOGS OF N AND R_m

[Metric Units.]

Latitude.		Parallel. Value of 1°.	Meridian. Value of 1°.	Log N.	Log R_m.
°	′	Meters.	Meters.		
69	00	40,012	111,552.9	6.8069834	6.8056034
	30	39,102	559.5	9919	6290
70	00	38,188	111,565.9	6.8060003	6.8056542
	30	7,272	572.2	0085	6788
71	00	6,353	578.4	0165	7029
	30	5,421	584.5	0244	7264
72	00	4,506	590.4	0321	7495
	30	3,578	596.2	0396	7719
73	00	2,648	601.8	0468	7938
	30	1,716	607.3	0539	8153
74	00	0,781	612.7	0608	8361
	30	29,843	617.9	0676	8563
75	00	28,903	111,622.9	6.8060742	6.8058759
	30	7,961	627.8	0805	8950
76	00	7,017	632.6	0867	9135
	30	6,071	637.1	0927	9314
77	00	5,123	641.6	0984	9487
	30	4,172	645.9	1040	9653
78	00	3,220	650.0	1093	9814
	30	2,266	653.9	1145	9968
79	00	1,311	657.8	1195	6.8060118
	30	20,353	661.4	1242	0258
80	00	19,394	111,664.9	6.8061287	6.8060394
	30	8,434	668.2	1330	0523
81	00	7,472	671.4	1371	0646
	30	6,509	674.4	1409	0763
82	00	5,545	677.2	1446	0873
	30	4,579	679.9	1480	0976
83	00	3,612	682.4	1513	1074
	30	2,644	684.7	1544	1163
84	00	1,675	686.9	1571	1248
	30	10,706	688.9	1597	1325
85	00	9,735	111,690.7	6.8061620	6.8061395
	30	8,764	692.3	1642	1459
86	00	7,792	693.8	1661	1517
	30	6,819	695.1	1678	1567
87	00	5,846	696.2	1692	1611
	30	4,872	697.2	1705	1648
88	00	3,898	697.9	1715	1679
	30	2,924	698.6	1723	1702
89	00	1,949	699.0	1728	1719
	30	975	699.3	1731	1729
90	00	0	111,699.3	6.8061733	6.8061733

TABLE XI. — TABLE OF LOGARITHMS OF RADII OF CURVATURE OF THE EARTH'S SURFACE IN METERS FOR VARIOUS LATITUDES AND AZIMUTHS

[Based upon Clarke's Ellipsoid of Rotation (1866).]

Azimuth.	0° lat.	1° lat.	2° lat.	3° lat.	4° lat.	5° lat.	6° lat.
° Meridian.	6.80175	6.80175	6.80175	6.80176	6.80177	6.80178	6.80180
5	177	177	178	178	179	180	182
10	184	184	184	185	186	187	188
15	195	195	195	196	197	198	199
20	209	209	210	210	211	212	214
25	227	228	228	228	229	230	232
30	248	249	249	250	250	251	252
35	272	272	272	273	273	274	276
40	296	297	297	297	298	299	300
45	322	322	322	323	324	324	325
50	348	348	348	348	349	350	351
55	373	373	373	373	374	374	375
60	396	396	396	396	397	398	398
65	417	417	417	418	418	418	419
70	435	435	436	436	436	437	437
75	450	450	450	450	451	451	452
80	461	461	461	461	462	462	463
85	468	468	468	468	468	469	469
90	470	470	470	470	471	471	472

Azimuth.	6° lat.	7° lat.	8° lat.	9° lat.	10° lat.	11° lat.	12° lat.
° Meridian.	6.80180	6.80181	6.80183	6.80186	6.80188	6.80191	6.80194
5	182	184	186	188	190	193	196
10	188	190	192	194	197	200	202
15	199	201	203	205	207	210	213
20	214	215	217	219	222	224	227
25	232	233	235	237	239	242	244
30	252	254	256	257	260	262	264
35	276	277	278	280	282	284	287
40	300	301	303	304	306	308	310
45	325	326	328	329	331	333	335
50	351	352	353	354	356	358	359
55	375	376	377	379	380	382	383
60	398	399	400	401	403	404	406
65	419	420	421	422	423	424	426
70	437	438	439	440	441	442	443
75	452	452	453	454	455	456	457
80	463	463	464	465	466	467	468
85	469	470	470	471	472	473	474
90	472	472	473	474	474	475	476

TABLE XI (*Con.*). — TABLE OF LOGARITHMS OF RADII OF CURVATURE OF THE EARTH'S SURFACE IN METERS FOR VARIOUS LATITUDES AND AZIMUTHS

[Based upon Clarke's Ellipsoid of Rotation (1866).]

Azimuth.	12° lat.	13° lat.	14° lat.	15° lat.	16° lat.	17° lat.	18° lat.
°							
Meridian.	6.80194	6.80197	6.80201	6.80204	6.80208	6.80213	6.80217
5	196	199	203	206	210	215	219
10	202	206	209	213	217	221	225
15	213	216	219	223	227	231	235
20	227	230	233	236	240	244	248
25	244	247	250	254	257	261	265
30	264	267	270	273	276	280	284
35	287	289	292	295	298	301	305
40	310	313	315	318	321	324	327
45	335	337	339	342	344	347	350
50	359	361	364	366	368	371	373
55	383	385	387	389	391	394	396
60	406	407	409	411	413	415	417
65	426	427	429	430	432	434	436
70	443	444	446	447	449	451	453
75	457	458	460	461	463	464	466
80	468	469	470	471	473	474	476
85	474	475	476	478	479	480	482
90	476	477	478	480	481	482	484

Azimuth.	18° lat.	19° lat.	20° lat.	21° lat.	22° lat.	23° lat.	24° lat.
°							
Meridian.	6.80217	6.80222	6.80226	6.80232	6.80237	6.80242	6.80248
5	219	224	228	234	239	244	250
10	225	230	234	239	244	250	255
15	235	239	244	249	254	259	264
20	248	252	257	262	266	271	277
25	265	269	273	277	282	287	292
30	284	287	292	296	300	305	309
35	305	308	312	316	320	324	329
40	327	330	334	338	341	345	350
45	350	353	357	360	364	367	371
50	373	376	379	382	386	389	392
55	396	398	401	404	407	410	413
60	417	419	422	424	427	430	432
65	436	438	440	443	445	448	450
70	453	454	456	459	461	463	465
75	466	468	470	472	473	476	478
80	476	478	479	481	483	485	487
85	482	483	485	487	489	490	492
90	484	485	487	489	490	492	494

TABLE XI (*Con.*). — TABLE OF LOGARITHMS OF RADII OF CURVATURE OF THE EARTH'S SURFACE IN METERS FOR VARIOUS LATITUDES AND AZIMUTHS

[Based upon Clarke's Ellipsoid of Rotation (1866).]

Azimuth.	24° lat.	25° lat.	26° lat.	27° lat.	28° lat.	29° lat.	30° lat.
°							
Meridian.	6.80248	6.80254	6.80260	6.80266	6.80272	6.80279	6.80285
5	250	256	262	268	274	280	287
10	255	261	267	273	279	285	292
15	264	270	276	282	288	294	300
20	277	282	288	293	299	305	311
25	292	297	302	308	313	319	325
30	309	314	319	324	330	335	340
35	329	333	338	343	348	353	358
40	350	354	358	362	367	372	377
45	371	375	379	383	387	391	396
50	392	396	399	403	407	411	415
55	413	416	420	423	426	430	434
60	432	435	438	442	445	448	451
65	450	453	455	458	461	464	467
70	465	468	470	473	475	478	481
75	478	480	482	484	487	489	492
80	487	489	491	493	495	498	500
85	492	494	496	498	501	503	505
90	494	496	498	500	502	504	507

Azimuth.	30° lat.	31° lat.	32° lat.	33° lat.	34° lat.	35° lat.	36° lat.
°							
Meridian.	6.80285	6.80292	6.80299	6.80306	6.80313	6.80320	6.80327
5	287	294	300	307	314	322	329
10	292	298	305	312	319	326	333
15	300	306	313	320	326	333	340
20	311	317	324	330	337	343	350
25	325	331	337	343	349	355	362
30	340	346	352	358	364	370	376
35	358	363	369	374	380	385	391
40	377	382	386	392	397	402	407
45	396	400	405	410	414	419	424
50	415	419	423	428	432	436	441
55	434	437	441	445	449	453	457
60	451	455	458	462	465	469	472
65	467	470	473	476	480	483	486
70	481	484	486	489	492	495	498
75	492	494	497	500	502	505	508
80	500	502	505	507	510	512	515
85	505	507	510	512	514	517	519
90	507	509	511	514	516	518	521

TABLE XIII.—CORRECTION TO LONGITUDE FOR DIFFERENCE BETWEEN ARC AND SINE

log s (−).	log difference.	log dλ (+)	log s (−).	log difference.	log dλ (+).
3.876	0.000 0001	2.385	4.871	0.000 0098	3.380
4.026	02	2.535	4.882	103	3.391
4.114	03	2.623	4.892	108	3.401
4.177	04	2.686	4.903	114	3.412
4.225	05	2.734	4.913	119	3.422
4.265	06	2.774	4.922	124	3.431
4.298	07	2.807	4.932	130	3.441
4.327	08	2.836	4.941	136	3.450
4.353	09	2.862	4.950	142	3.459
4.376	10	2.885	4.959	147	3.468
4.396	11	2.905	4.968	153	3.477
4.415	12	2.924	4.976	160	3.485
4.433	13	2.942	4.985	166	3.494
4.449	14	2.958	4.993	172	3.502
4.464	15	2.973	5.002	179	3.511
4.478	16	2.987	5.010	186	3.519
4.491	17	3.000	5.017	192	3.526
4.503	18	3.012	5.025	199	3.534
4.526	20	3.035	5.033	206	3.542
4.548	23	3.057	5.040	213	3.549
4.570	25	3.079	5.047	221	3.556
4.591	27	3.100	5.054	228	3.563
4.612	30	3.121	5.062	236	3.571
4.631	33	3.140	5.068	243	3.577
4.649	36	3.158	5.075	251	3.584
4.667	39	3.176	5.082	259	3.591
4.684	42	3.193	5.088	267	3.597
4.701	45	3.210	5.095	275	3.604
4.716	48	3.225	5.102	284	3.611
4.732	52	3.241	5.108	292	3.617
4.746	56	3.255	5.114	300	3.623
4.761	59	3.270	5.120	309	3.629
4.774	63	3.283	5.126	318	3.635
4.788	67	3.297	5.132	327	3.641
4.801	71	3.310	5.138	336	3.647
4.813	75	3.322	5.144	345	3.653
4.825	80	3.334	5.150	354	3.659
4.834	84	3.343	5.156	364	3.665
4.849	89	3.358	5.161	373	3.670
4.860	94	3.369	5.167	383	3.676

TABLE XIV.—LOGARITHMS OF FACTORS FOR COMPUTING GEODETIC POSITIONS

Lat.	Log *A*	Log *B*	Log *C*	Log *D*	Log *E*
° ′	−10	−10	−10	−10	−20
18 00	8.509 5862	8.512 2550	0.91816	2.1606	5.7317
10	5836	2474	0.92243	2.1641	5.7337
20	5811	2397	0.92667	2.1675	5.7358
30	5785	2320	0.93088	2.1709	5.7379
40	5759	2243	0.93505	2.1742	5.7400
50	5733	2165	0.93919	2.1775	5.7422
19 00	5707	2086	0.94330	2.1808	5.7443
10	5681	2006	0.94737	2.1840	5.7464
20	5654	1927	0.95142	2.1872	5.7486
30	5627	1847	0.95544	2.1903	5.7508
40	5600	1766	0.95943	2.1934	5.7530
50	5573	1684	0.96339	2.1965	5.7552
20 00	5546	1602	0.96733	2.1996	5.7574
10	5518	1519	0.97123	2.2026	5.7597
20	5490	1435	0.97511	2.2055	5.7619
30	5462	1351	0.97896	2.2084	5.7642
40	5434	1267	0.98279	2.2113	5.7664
50	5406	1182	0.98659	2.2142	5.7688
21 00	5377	1096	0.99037	2.2170	5.7711
10	5348	1010	0.99412	2.2198	5.7734
20	5320	0924	0.99785	2.2226	5.7757
30	5290	0836	1.00156	2.2253	5.7780
40	5261	0748	1.00524	2.2280	5.7804
50	5232	0660	1.00890	2.2307	5.7828
22 00	5202	0571	1.01253	2.2333	5.7851
10	5172	0481	1.01615	2.2359	5.7875
20	5142	0391	1.01974	2.2385	5.7899
30	5112	0301	1.02331	2.2411	5.7924
40	5082	0210	1.02686	2.2436	5.7948
50	5051	0118	1.03039	2.2461	5.7972
23 00	5020	8.512 0026	1.03390	2.2485	5.7997
10	4990	8.511 9934	1.03739	2.2510	5.8021
20	4959	9840	1.04086	2.2534	5.8046
30	4927	9747	1.04431	2.2557	5.8071
40	4896	9653	1.04775	2.2581	5.8096
50	4865	9558	1.05116	2.2604	5.8121
24 00	4833	9463	1.05456	2.2627	5.8146
10	4801	9367	1.05794	2.2650	5.8172
20	4769	9271	1.06130	2.2672	5.8197
30	4737	9174	1.06464	2.2694	5.8223
40	4704	9077	1.06797	2.2716	5.8249
50	4672	8979	1.07128	2.2738	5.8274
60	8.509 4639	8.511 8881	1.07457	2.2759	5.8300

TABLE XIV (*Continued*)

Lat. ° ′	Log *A*	Log *B*	Log *C*	Log *D*	Log *E*
25 00	8.509 4639	8.511 8881	1.07457	2.2759	5.8300
10	4606	8783	1.07785	2.2780	5.8326
20	4573	8684	1.08111	2.2801	5.8352
30	4540	8584	1.08435	2.2822	5.8379
40	4507	8484	1.08758	2.2842	5.8405
50	4473	8383	1.09080	2.2862	5.8431
26 00	4439	8283	1.09400	2.2882	5.8458
10	4406	8181	1.09718	2.2902	5.8485
20	4372	8079	1.10036	2.2922	5.8512
30	4337	7977	1.10351	2.2941	5.8539
40	4303	7874	1.10666	2.2960	5.8566
50	4269	7771	1.10979	2.2978	5.8593
27 00	4234	7667	1.11290	2.2997	5.8620
10	4200	7563	1.11600	2.3015	5.8647
20	4165	7458	1.11909	2.3033	5.8675
30	4130	7353	1.12217	2.3051	5.8702
40	4094	7248	1.12523	2.3069	5.8730
50	4059	7142	1.12829	2.3086	5.8757
28 00	4024	7036	1.13132	2.3104	5.8785
10	3988	6929	1.13435	2.3121	5.8813
20	3952	6822	1.13737	2.3137	5.8841
30	3917	6714	1.14037	2.3154	5.8870
40	3881	6607	1.14337	2.3170	5.8898
50	3845	6498	1.14635	2.3187	5.8926
29 00	3808	6389	1.14932	2.3203	5.8955
10	3772	6280	1.15228	2.3218	5.8983
20	3735	6171	1.15522	2.3234	5.9012
30	3699	6061	1.15816	2.3249	5.9041
40	3662	5950	1.16109	2.3264	5.9069
50	3625	5840	1.16401	2.3279	5.9098
30 00	3588	5729	1.16692	2.3294	5.9127
10	3551	5617	1.16981	2.3309	5.9157
20	3514	5505	1.17270	2.3323	5.9186
30	3476	5393	1.17558	2.3337	5.9215
40	3439	5281	1.17845	2.3351	5.9245
50	3401	5168	1.18131	2.3365	5.9274
31 00	3363	5054	1.18416	2.3379	5.9304
10	3325	4941	1.18700	2.3392	5.9334
20	3287	4827	1.18983	2.3405	5.9363
30	3249	4713	1.19266	2.3418	5.9393
40	3211	4598	1.19548	2.3431	5.9423
50	3173	4483	1.19828	2.3444	5.9453
60	8.509 3134	8.511 4368	1.20108	2.3456	5.9484

TABLE XIV (*Continued*)

Lat.	Log *A*	Log *B*	Log *C*	Log *D*	Log *E*
° ′					
32 00	8.509 3134	8.511 4368	1.20108	2.3456	5.9484
10	3096	4252	1.20387	2.3469	5.9514
20	3057	4136	1.20666	2.3481	5.9544
30	3018	4020	1.20944	2.3493	5.9575
40	2980	3903	1.21220	2.3504	5.9605
50	2940	3786	1.21496	2.3516	5.9636
33 00	2901	3669	1.21772	2.3527	5.9667
10	2862	3551	1.22047	2.3539	5.9698
20	2823	3433	1.22321	2.3550	5.9729
30	2784	3315	1.22594	2.3561	5.9760
40	2744	3197	1.22866	2.3571	5.9791
50	2704	3078	1.23138	2.3582	5.9822
34 00	2665	2959	1.23409	2.3592	5.9853
10	2625	2840	1.23680	2.3602	5.9885
20	2585	2720	1.23950	2.3612	5.9916
30	2545	2600	1.24219	2.3622	5.9948
40	2505	2480	1.24488	2.3632	5.9980
50	2465	2360	1.24756	2.3642	6.0011
35 00	2425	2239	1.25024	2.3651	6.0043
10	2384	2118	1.25291	2.3660	6.0075
20	2344	1997	1.25557	2.3669	6.0107
30	2304	1875	1.25823	2.3678	6.0140
40	2263	1754	1.26088	2.3687	6.0172
50	2222	1632	1.26353	2.3695	6.0204
36 00	2182	1510	1.26617	2.3704	6.0237
10	2141	1387	1.26881	2.3712	6.0269
20	2100	1265	1.27145	2.3720	6.0302
30	2059	1142	1.27407	2.3728	6.0334
40	2018	1019	1.27670	2.3735	6.0367
50	1977	0895	1.27932	2.3743	6.0400
37 00	1936	0772	1.28193	2.3750	6.0433
10	1895	0648	1.28454	2.3758	6.0466
20	1853	0524	1.28715	2.3765	6.0499
30	1812	0400	1.28975	2.3772	6.0533
40	1771	0276	1.29234	2.3779	6.0566
50	1729	0151	1.29494	2.3785	6.0600
38 00	1687	8.511 0027	1.29753	2.3792	6.0633
10	1646	8.510 9902	1.30011	2.3798	6.0667
20	1604	9777	1.30269	2.3804	6.0701
30	1562	9652	1.30527	2.3810	6.0734
40	1521	9526	1.30785	2.3816	6.0768
50	1479	9401	1.31042	2.3822	6.0802
60	8.509 1437	8.510 9275	1.31299	2.3827	6.0836

TABLE XIV (*Continued*)

Lat. ° ′	Log *A*	Log *B*	Log *C*	Log *D*	Log *E*
39 00	8.509 1437	8.510 9275	1.31299	2.3827	6.0836
10	1395	9149	1.31555	2.3832	6.0871
20	1353	9023	1.31811	2.3838	6.0905
30	1311	8897	1.32067	2.3843	6.0939
40	1269	8771	1.32323	2.3848	6.0974
50	1227	8644	1.32578	2.3852	6.1008
40 00	1184	8517	1.32833	2.3857	6.1043
10	1142	8391	1.33088	2.3861	6.1078
20	1100	8264	1.33342	2.3866	6.1113
30	1057	8137	1.33596	2.3870	6.1148
40	1015	8010	1.33850	2.3874	6.1183
50	0973	7883	1.34104	2.3878	6.1218
41 00	0930	7755	1.34358	2.3882	6.1253
10	0888	7628	1.34611	2.3885	6.1289
20	0845	7500	1.34864	2.3889	6.1324
30	0803	7373	1.35117	2.3892	6.1360
40	0760	7245	1.35370	2.3895	6.1395
50	0718	7117	1.35623	2.3898	6.1431
42 00	0675	6989	1.35875	2.3901	6.1467
10	0632	6861	1.36127	2.3903	6.1503
20	0590	6733	1.36379	2.3906	6.1539
30	0547	6605	1.36631	2.3908	6.1575
40	0504	6477	1.36883	2.3910	6.1612
50	0461	6348	1.37135	2.3913	6.1648
43 00	0419	6220	1.37386	2.3914	6.1684
10	0376	6092	1.37638	2.3916	6.1721
20	0333	5963	1.37889	2.3918	6.1758
30	0290	5835	1.38141	2.3919	6.1795
40	0247	5706	1.38392	2.3921	6.1831
50	0204	5578	1.38643	2.3922	6.1868
44 00	0162	5449	1.38894	2.3923	6.1905
10	0119	5320	1.39145	2.3924	6.1943
20	0076	5192	1.39396	2.3925	6.1980
30	8.5090033	5063	1.39648	2.3925	6.2017
40	8.5089990	4935	1.39898	2.3926	6.2055
50	9947	4806	1.40149	2.3926	6.2092
45 00	9904	4677	1.40400	2.3926	6.2130
10	9861	4548	1.40651	2.3926	6.2168
20	9818	4420	1.40902	2.3926	6.2206
30	9776	4291	1.41153	2.3926	6.2244
40	9733	4162	1.41404	2.3925	6.2283
50	9689	4034	1.41655	2.3925	6.2321
60	8.508 9647	8.510 3905	1.41906	2.3924	6.2359

TABLE XIV (*Continued*)

Lat.	Log *A*	Log *B*	Log *C*	Log *D*	Log *E*
° ′					
46 00	8.508 9647	8.510 3905	1.41906	2.3924	6.2359
10	9604	3776	1.42157	2.3923	6.2398
20	9561	3648	1.42409	2.3922	6.2436
30	9518	3519	1.42660	2.3921	6.2475
40	9475	3391	1.42911	2.3920	6.2514
50	9433	3262	1.43163	2.3918	6.2553
47 00	9390	3134	1.43414	2.3917	6.2592
10	9347	3005	1.43666	2.3915	6.2632
20	9304	2877	1.43917	2.3913	6.2671
30	9261	2749	1.44169	2.3911	6.2710
40	9219	2621	1.44421	2.3909	6.2750
50	9176	2493	1.44673	2.3906	6.2790
48 00	9133	2364	1.44926	2.3904	6.2830
10	9091	2236	1.45178	2.3901	6.2870
20	9048	2108	1.45431	2.3898	6.2910
30	9005	1981	1.45683	2.3895	6.2950
40	8963	1853	1.45937	2.3892	6.2990
50	8920	1725	1.46190	2.3889	6.3031
49 00	8878	1598	1.46443	2.3886	6.3071
10	8835	1470	1.46696	2.3882	6.3112
20	8793	1343	1.46950	2.3878	6.3153
30	8750	1216	1.47204	2.3875	6.3194
40	8708	1088	1.47459	2.3871	6.3235
50	8666	0962	1.47713	2.3866	6.3276
50 00	8623	0835	1.47968	2.3862	6.3318
10	8581	0708	1.48223	2.3858	6.3359
20	8539	0581	1.48478	2.3853	6.3401
30	8497	0455	1.48734	2.3848	6.3443
40	8455	0328	1.48989	2.3843	6.3485
50	8413	0202	1.49246	2.3838	6.3527
51 00	8371	8.510 0076	1.49502	2.3833	6.3569
10	8329	8.509 9950	1.49759	2.3828	6.3612
20	8287	9825	1.50016	2.3822	6.3654
30	8245	9699	1.50273	2.3817	6.3697
40	8203	9574	1.50531	2.3811	6.3740
50	8161	9448	1.50789	2.3805	6.3782
52 00	8120	9323	1.51048	2.3799	6.3826
10	8078	9198	1.51307	2.3792	6.3869
20	8036	9074	1.51566	2.3786	6.3912
30	7995	8949	1.51826	2.3779	6.3956
40	7953	8825	1.52086	2.3773	6.4000
50	7912	8701	1.52347	2.3766	6.4043
53 00	7871	8577	1.52608	2.3759	6.4088
10	7829	8453	1.52869	2.3751	6.4132
20	7788	8329	1.53131	2.3744	6.4176
30	7747	8206	1.53393	2.3736	6.4221
40	7706	8083	1.53656	2.3729	6.4265
50	7665	7960	1.53919	2.3721	6.4310
60	8.508 7624	8.509 7838	1.54183	2.3713	6.4355

TABLE XV.—MERIDIONAL DISTANCE IN METERS FROM WHOLE DEGREE PARALLEL.

(Art. 339, p. 355.)

Lat.	Minutes from Whole Degree Parallel									
	1′	2′	3′	4′	5′	6′	7′	8′	9′	10′
25°	1846.1	3692.3	5538.4	7384.6	9230.7	11076.9	12923.0	14769.2	16615.4	18461.5
26	1846.4	3692.8	5539.2	7385.6	9232.0	11078.4	12924.8	14771.2	16617.7	18464.1
27	1846.7	3693.3	5540.0	7386.6	9233.3	11080.0	12926.7	14773.3	16620.0	18466.7
28	1846.9	3693.8	5540.8	7387.7	9234.6	11081.6	12928.5	14775.5	16622.5	18469.4
29	1847.2	3694.4	5541.6	7388.8	9236.0	11083.2	12930.5	14777.7	16624.9	18472.2
30	1847.5	3695.0	5542.4	7389.9	9237.4	11084.9	12932.4	14779.9	16627 4	18475.0
31	1847.8	3695.5	5543.3	7391.1	9238.9	11086.7	12934.4	14782.2	16630.0	18477.9
32	1848.1	3696.1	5544.2	7392.3	9240.3	11088.4	12936.5	14784.6	16632.7	18480.8
33	1848.4	3696.7	5545.1	7393.4	9241.8	11090.2	12938.6	14787.0	16635.4	18483 8
34	1848.7	3697.3	5546.0	7394.6	9243.3	11092.0	12940.7	14789.4	16638.1	18486.8
35	1849.0	3697.9	5546.9	7395.9	9244.9	11093.9	12942.8	14791.8	16640.8	18489.9
36	1849.3	3698.5	5547.8	7397.1	9246.4	11095.7	12945.0	14794.3	16643.6	18493.0
37	1849.6	3699.2	5548.8	7398.4	9248.0	11097.6	12947.2	14796.8	16646.5	18496.1
38	1849.9	3699.8	5549.7	7399.6	9249.6	11099.5	12949.4	14799.4	16649.3	18499.3
39	1850.2	3700.5	5550.7	7400.9	9251.2	11101.4	12951.7	14801.9	16652.2	18502.5
40	1850.5	3701.1	5551.7	7402.2	9252.8	11103.4	12953.9	14804.5	16655.1	18505.7
41	1850.9	3701.7	5552.6	7403.5	9254.4	11105.3	12956.2	14807.1	16658.0	18509.0
42	1851.2	3702.4	5553.6	7404.8	9256.0	11107.3	12958.5	14809.7	16661.0	18512.2
43	1851.5	3703.1	5554.6	7406.1	9257.7	11109.2	12960.8	14812.4	16663.9	18515.5
44	1851.9	3703.7	5555.6	7407.4	9259.3	11111.2	12963.1	14815.0	16666.9	18518.8
45	1852.2	3704.4	5556.6	7408.8	9261.0	11113.2	12965.4	14817.6	16669.9	18522.1
46	1852.5	3705.0	5557.6	7410.1	9262.6	11115.2	12967.7	14820.3	16672.8	18525.4
47	1852.8	3705.7	5558.5	7411.4	9264.3	11117.1	12970.0	14822.9	16675.8	18528.7
48	1853.2	3706.3	5559.5	7412.7	9265.9	11119.1	12972.3	14825.5	16678.7	18531.9
49	1853.5	3707.0	5560.5	7414.0	9267.5	11121.1	12974.6	14828.1	16681.7	18535.2
50	1853.8	3707.7	5561.5	7415.3	9269.2	11123.0	12976.9	14830.7	16684.6	18538.5

TABLE XVI.—COÖRDINATES OF CURVATURE (METERS)

Long.	26° X	26° Y	27° X	27° Y	28° X	28° Y	29° X	29° Y
1′	1668.7	0.1	1654.3	0.1	1639.4	0.1	1624.0	0.1
2	3337.3	0.4	3308.5	0.4	3278.8	0.4	3248.0	0.5
3	5006.0	1.0	4962.8	1.0	4918.2	1.0	4872.0	1.0
4	6674.6	1.7	6617.1	1.7	6557.6	1.8	6496.1	1.8
5	8343.3	2.7	8271.4	2.7	8197.0	2.8	8120.1	2.9
6	10011.9	3.8	9925.7	3.9	9836.4	4.0	9744.1	4.1
7	11680.6	5.2	11579.9	5.4	11475.7	5.5	11368.1	5.6
8	13349.2	6.8	13234.2	7.0	13115.1	7.2	12992.1	7.3
9	15017.9	8.6	14888.5	8.8	14754.5	9.1	14616.1	9.3
10	16686.6	10.6	16542.8	10.9	16393.9	11.2	16240.1	11.5

Long.	30° X	30° Y	31° X	31° Y	32° X	32° Y	33° X	33° Y
1′	1608.1	0.1	1591.8	0.1	1574.9	0.1	1557.6	0.1
2	3216.3	0.5	3183.5	0.5	3149.8	0.5	3115.2	0.5
3	4824.4	1.1	4775.3	1.1	4724.8	1.1	4672.8	1.1
4	6432.6	1.9	6367.1	1.9	6299.7	1.9	6230.3	2.0
5	8040.7	2.9	7958.9	3.0	7874.6	3.0	7787.9	3.1
6	9648.8	4.2	9550.6	4.3	9449.5	4.4	9345.5	4.4
7	11257.0	5.7	11142.4	5.8	11024.4	6.0	10903.1	6.0
8	12865.1	7.5	12734.2	7.6	12599.4	7.8	12460.7	7.9
9	14473.2	9.5	14325.9	9.7	14174.3	9.8	14018.3	10.0
10	16081.4	11.7	15917.7	11.9	15749.2	12.1	15575.9	12.3

Long.	34° X	34° Y	35° X	35° Y	36° X	36° Y	37° X	37° Y
1′	1539.8	0.1	1521.5	0.1	1502.8	0.1	1483.6	0.1
2	3079.6	0.5	3043.0	0.5	3005.5	0.5	2967.1	0.5
3	4619.3	1.1	4564.5	1.1	4508.3	1.2	4450.7	1.2
4	6159.1	2.0	6086.0	2.0	6011.1	2.1	5934.2	2.1
5	7698.9	3.1	7607.5	3.2	7513.8	3.2	7417.8	3.3
6	9238.7	4.5	9129.0	4.6	9016.6	4.6	8901.4	4.7
7	10778.5	6.1	10650.5	6.2	10519.3	6.3	10384.9	6.4
8	12318.3	8.0	12172.0	8.1	12022.1	8.2	11868.5	8.3
9	13858.0	10.1	13693.5	10.3	13524.8	10.4	13352.1	10.5
10	15397.9	12.5	15215.0	12.7	15027.6	12.8	14835.6	13.0

TABLE XVI (*Con.*).—COÖRDINATES OF CURVATURE (METERS)

Long.	Latitudes.							
	38°		39°		40°		41°	
	X	*Y*	*X*	*Y*	*X*	*Y*	*X*	*Y*
1′	1463.9	0.1	1443.8	0.1	1423.3	0.1	1402.3	0.1
2	2927.8	0.5	2887.6	0.5	2846.5	0.5	2804.6	0.5
3	4391.7	1.2	4331.4	1.2	4269.8	1.2	4206.9	1.2
4	5855.6	2.1	5775.2	2.1	5693.0	2.1	5609.2	2.1
5	7319.6	3.3	7219.0	3.3	7116.3	3.3	7011.5	3.3
6	8783.5	4.7	8662.9	4.8	8539.6	4.8	8413.7	4.8
7	10247.4	6.4	10106.7	6.5	9962.8	6.5	9816.0	6.6
8	11711.3	8.4	11550.5	8.5	11386.1	8.5	11218.3	8.6
9	13175.2	10.6	12994.3	10.7	12809.3	10.8	12620.6	10.8
10	14639.1	13.1	14438.1	13.2	14232.6	13.3	14022.9	13.4

Long.	42°		43°		44°		45°	
1′	1380.9	0.1	1359.1	0.1	1336.8	0.1	1314.1	0.1
2	2761.8	0.5	2718.1	0.5	2673.6	0.5	2628.3	0.5
3	4142.7	1.2	4077.2	1.2	4010.4	1.2	3942.5	1.2
4	5523.5	2.2	5436.2	2.2	5347.2	2.2	5256.6	2.2
5	6904.4	3.4	6795.3	3.4	6684.0	3.4	6570.8	3.4
6	8285.3	4.8	8154.3	4.9	8020.8	4.9	7884.9	4.9
7	9666.2	6.6	9513.4	6.6	9357.7	6.6	9199.1	6.6
8	11047.1	8.6	10872.4	8.6	10694.5	8.6	10513.2	8.6
9	12428.0	10.9	12231.5	10.9	12031.3	10.9	11827.4	10.9
10	13808.8	13.4	13590.5	13.5	13368.1	13.5	13141.5	13.5

Long.	46°		47°		48°		49°	
1′	1291.1	0.1	1267.6	0.1	1243.8	0.1	1219.6	0.1
2	2582.2	0.5	2535.3	0.5	2487.6	0.5	2439.1	0.5
3	3873.3	1.2	3802.9	1.2	3731.4	1.2	3658.7	1.2
4	5164.4	2.2	5070.5	2.2	4975.2	2.1	4878.3	2.1
5	6455.5	3.4	6338.2	3.4	6219.0	3.3	6097.9	3.3
6	7746.6	4.9	7605.8	4.8	7462.8	4.8	7317.5	4.8
7	9037.6	6.6	8873.5	6.6	8706.6	6.6	8537.0	6.6
8	10328.7	8.6	10141.1	8.6	9950.4	8.6	9756.6	8.6
9	11619.8	10.9	11408.7	10.9	11194.2	10.9	10976.2	10.8
10	12910.9	13.5	12676.4	13.5	12437.9	13.4	12195.8	13.4

TABLE XVII. — COÖRDINATES OF CURVATURE (METERS)

Long.	Latitudes.					
	25°		30°		35°	
	X	*Y*	*X*	*Y*	*X*	*Y*
5°	504 645	9 307	482 288	10 523	456 261	11 421
10	1 008 603	37 215	963 658	42 074	911 379	45 656
15	1 511 190	83 685	1 443 193	94 591	1 364 214	102 619
20	2 011 722	148 656	1 919 982	167 977	1 813 632	182 168
25	2 509 518	232 038	2 393 116	262 089	2 258 507	284 102
30	3 003 900	333 718	2 861 694	376 749	2 697 724	408 168
Long.	40°		45°		50°	
5°	426 757	11 972	393 996	12 160	358 224	11 978
10	852 171	47 852	786 492	48 594	714 847	47 859
15	1 274 904	107 525	1 175 994	109 162	1 068 277	107 482
20	1 693 628	190 805	1 561 019	193 635	1 416 934	190 581
25	2 107 023	297 430	1 940 103	301 690	1 759 262	296 785
30	2 513 790	427 063	2 311 802	432 918	2 093 731	425 619

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