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This book is intended for students taking a first-year course in Astronomy in the Universities and for all those interested in the subject who feel the need for a more solid foundation than the many descriptive books can provide. Now that such large numbers of our young men in the naval and air forces are required to have some knowledge of Astronomy in its application to Navigation the book will be of value to them as an introduction to the Service manuals in which greater emphasis is naturally laid on more technical matters than on a complete exposition of the foundations of Astronomy.

The mathematical treatment throughout is of an elementary nature and the reader's mathematical attainments need not go beyond a knowledge of the simple trigonometrical functions. Numerous examples designed to illustrate points of theoretical or practical interest are provided.

While the book was going through the press, the Astronomer Royal (Dr. H. Spencer Jones) announced the final result for the value of the solar parallax. Fortunately it was possible to bring the book up to date in this very important matter.

The author, formerly Instructor Lieutenant in the Royal Navy, and joint author of the Admiralty Manual of Navigation (1922), is now Regius Professor of Astronomy in the University of Glasgow.

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FOUNDATIONS OF ASTRONOMY

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PREFACE

THIS book is intended for students taking a first-year course in Astronomy in the Universities and for all those interested in the subject who feel the need for a more solid foundation than the many descriptive books can provide. Now that such large numbers of our young men in the naval and air forces are required to have some knowledge of Astronomy in its application to Navigation, I hope that the book will be of value to them as an introduction to the Service manuals in which greater emphasis is naturally laid on more technical matters than on a complete exposition of the foundations of Astronomy.

The mathematical treatment throughout is of an elementary nature and the reader's mathematical attainments need not go beyond a knowledge of the simple trigonometrical functions. Certain sections marked by an asterisk can be omitted on a first reading.

The book is provided with numerous examples designed to illustrate points of theoretical or practical interest; the majority are of a straight-forward nature. In working out the examples the student is recommended to draw diagrams, whenever possible, appropriate to the problems concerned. The examples are intended to be worked out with the usual 4-figure logarithmic tables; it may be remarked in this connection that slight discrepancies in some of the answers (as compared with those printed at the end of the book) may be anticipated, for the fourth significant figure will generally be liable to an error of 1 or 2 units, and angles, given in degrees and minutes, will usually be subject to a discrepancy of 1 or 2 minutes of arc.

The examples in the text are based on the *Nautical Almanac* for 1940; in this (or similar) annual publication the student will find detailed instructions as to its use. In the almanacs, British and foreign, there is considerable diversity as regards the symbol for the standard Greenwich time for which the elements of the heavenly bodies are tabulated; G.M.T. (Greenwich Mean Time), G.C.T. (Greenwich Civil Time) and U.T. (Universal Time) are all used; in this book, as in my *Spherical Astronomy*, I use G.C.T.

While the book was going through the press, the Astronomer Royal (Dr. H. Spencer Jones) announced the final result for the value of the solar parallax—the fruit of ten years' work—and fortunately it was possible to bring the book up to date in this very important matter.

The Greek alphabet is printed, as an Appendix, for general reference.

In conclusion, I have to acknowledge the valuable help of my assistant, Dr. T. R. Tannahill, in preparing the manuscript for press and in checking the answers to the examples. I am also grateful to the officials and staff of the Glasgow University Press for their attention and care.

W. M. S.

UNIVERSITY OBSERVATORY, GLASGOW
November, 1941

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CHAPTER I

THE GEOMETRY OF THE SPHERE

1. *Introduction.*

A fundamental department of astronomy is concerned with the relative directions in which the heavenly bodies (the sun, moon, planets, stars) are seen by an observer. As one views the heavens on any clear night, the stars—and any other heavenly body visible—appear to be situated on the surface of a vast sphere the centre of which is the particular observer concerned. We can then describe the relative directions of the several stars as being defined, at a given instant, by the positions which they appear to occupy on the *celestial sphere*, as it is called. As we are not concerned in many problems with the distances of the heavenly bodies from us, but only with their directions, the radius of the celestial sphere may be chosen in any way we wish. Our first discussion is devoted to the properties of the sphere.

2. *The great circle.*

Geometrically, a sphere is a solid body bounded by a surface every point of which is equidistant from a certain point called the *centre*. A straight line joining the centre to any point on the spherical surface is called a *radius*.

Any plane passing through the centre of the sphere cuts the surface in a circle whose centre is the centre of the sphere; such a circle is called a *great circle*. In Fig. 1 a great circle $ABCD$ is shown. Let POQ be the diameter of the sphere drawn at right angles to the plane of the great circle; the extremities P and Q of this diameter are called the *poles* of the great circle $ABCD$. Let OA and OB be the radii of the sphere (or of the great circle) corresponding to two points A and B on the great circle. Then these points divide the great circle into two circular arcs, namely AB and $BCDA$; in the figure, the former is, by inspection, less than a semi-circular arc and the latter is greater. If θ denotes the angle subtended at O by the smaller arc AB and s the length of the arc AB, we have, denoting the radius of the sphere by R,

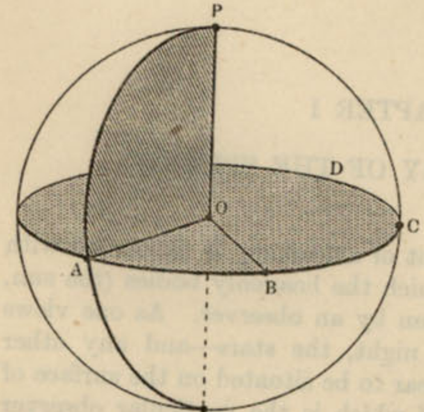

FIG. 1.

$$s = R\theta. \qquad \text{.........(1)}$$

In this formula θ is expressed in circular measure, that is, in radians. If the radius, R, of the sphere is taken to be unity, the length, s, of the arc AB is now given by

$$s = \theta, \qquad \text{..........(2)}$$

so that *the length of a great circle arc on a sphere of unit radius is equal to the angle (in circular measure) which this arc subtends at the centre of the sphere.* For example, if $A\widehat{O}B = \frac{\pi}{3}$ radians,* we see that the length of the great circle arc AB of the unit sphere is also $\frac{\pi}{3}$.

The length of a great circle arc such as AB is often referred to as the *angular distance* of A from B. It is important to remember that in this definition we are concerned with the great circle passing through A and B and with no other curve on the surface of the sphere.

In practice it is not always convenient to express the length of a great circle arc, such as AB, in circular measure; instead, we express AB in degrees, minutes and seconds so that if AB is $\frac{\pi}{3}$ radians we describe it simply as 60°.

Consider now a plane passing through the diameter POQ

* π radians $\equiv 180°$ ($\pi = 3{\cdot}1416$ or, approximately, $3\frac{1}{7}$); hence 1 radian $\equiv 57° \, 17' \, 45'' \equiv 206{,}265''$, or $3438'$ approximately.

(Fig. 1) and passing also through A; this plane cuts the surface of the sphere in a great circle of which only the semi-circle PAQ is shown. Since OP is perpendicular to the plane of the great circle $ABCD$, it is perpendicular to any straight line in this plane; hence $P\widehat{O}A$ is a right angle. We can express this otherwise by saying that the great circle arc AP subtends a right angle at the centre of the sphere. Thus we can write

$$AP = \frac{\pi}{2} \quad \text{or} \quad AP = 90°.$$

In other words, *the angular distance of the pole of a great circle from any point on that great circle is* 90°.

3. *Spherical angle.*

In Fig. 2 let PAQ and PBQ define two great circles having the common diameter POQ and let AB be the great circle* of which P and Q are the poles. Let PU be the tangent at P to the great circle AP and PV the tangent to the great circle BP. The angle UPV is defined to be the *spherical angle* at P between the great circles AP and BP. Now PU, being a tangent, is perpendicular to the radius OP of the great circle AP; also PU lies in the plane of the great circle. Again, since P is the pole of the great circle AB, OP is perpendicular to OA. It follows that PU is parallel to OA. Similarly, PV is parallel to OB. Hence the angle UPV is equal to the angle AOB. Accordingly, the

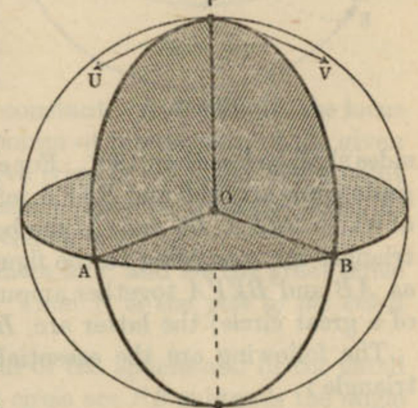

FIG. 2.

* We shall usually refer to a great circle in terms of an arc; thus, for example, we designate the great circle of which AB is an arc as the great circle AB. There is no ambiguity in this procedure, since a circle is uniquely specified if two points on its circumference, not at the extremities of a diameter, and its centre are given.

spherical angle at P—which we denote by $\widehat{APB}$—is equivalent to the arc of the great circle, of which P is the pole, intercepted between the great circles defining the spherical angle.

4. *Spherical triangle.*

A spherical triangle is bounded by three arcs of great circles. In Fig. 3, $ADEC$, $DBCF$ and $ABEF$ are three great circles. The figure ABC is called a spherical triangle. The elements of a spherical triangle consist of three *spherical angles* and three *sides*. Thus, in the spherical triangle ABC we have the three spherical angles BAC, ABC and ACB and the three sides are defined by the three great circle arcs AB, BC and CA.

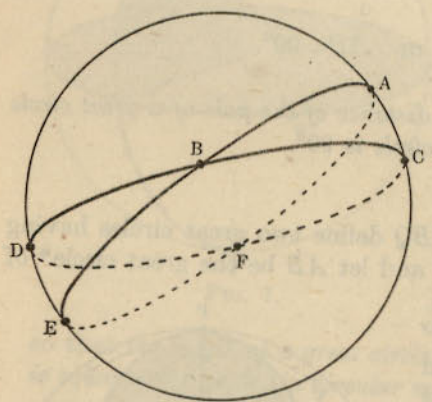

Fig. 3.

The definition of a spherical triangle includes the stipulation that each of the sides shall be less than 180°. For example, in Fig. 3 we have two great circle arcs AB and $BEFA$ joining A and B; but the figure of which $BEFA$, BC and CA are bounding arcs is not a spherical triangle for, according to the figure, AB is less than 180° and as AB and $BEFA$ together amount to 360° (the circumference of a great circle) the latter arc, $BEFA$, is greater than 180°.

The following are the essential properties of any spherical triangle:

(1) Any two sides are together greater than the third side.

(2) The sum of the three angles of a spherical triangle is greater than two right angles.

(3) Any angle is less than two right angles.

5. *Small circle.*

A plane which cuts a sphere but which does not pass through the centre of the sphere intersects the spherical surface in a circle called a *small circle*.

Consider a plane cutting the sphere in a curve BCD, as drawn in Fig. 4. We shall first prove that this curve is a circle. Let O be the centre of the sphere and let OA be the perpendicular from O to the given plane. Join A to any point B on the curve of intersection; join OB. Now, since OA is perpendicular to the given plane, it is perpendicular to any straight line in the plane; in particular, OA is perpendicular to AB. Hence, by Pythagoras' theorem,

$$OB^2 = OA^2 + AB^2. \quad \ldots(3)$$

But OB is the radius of the given sphere and is therefore a constant; again, OA is the perpendicular from O to the given plane and is also a constant. Hence, by (3), AB is a constant. Accordingly, the locus of B, that is, the locus of the points of intersection of the given plane with the sphere, is a curve lying in the plane, every point of the curve being at a constant distance from the point A. Thus BCD is a circle and its centre is A.

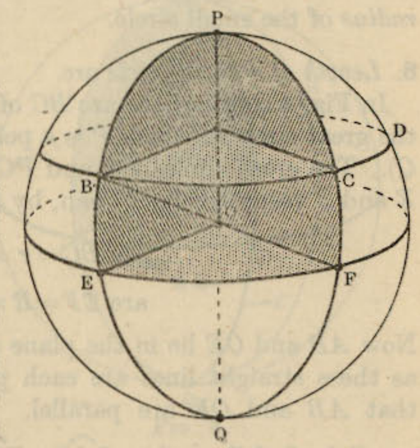

Fig. 4.

Produce OA to meet the sphere in P and let the great circle be drawn through P and B. Then P is the *pole of the small circle BCD*.

Let R and r denote the radii of the sphere and of the small circle respectively. The great circle arc PB subtends the angle BOP at the centre of the sphere and, as $\widehat{BAO}$ is 90°,

$$\sin \widehat{BOA} = \frac{AB}{OB}. \quad \ldots(4)$$

But $\widehat{BOA}$ is the same as $\widehat{BOP}$ and this angle defines the spherical arc PB; also, $OB = R$ and $AB = r$. Hence

$$\sin PB = \frac{r}{R}. \quad \ldots(5)$$

This formula shows that PB is constant for all positions of B on the small circle.

The great circle arc, such as PB, between the pole of the small circle and any point on the latter, is called the *angular radius* of the small circle.

6. *Length of a small circle arc.*

In Fig. 4 consider the arc BC of the small circle. Let EF be the great circle of which P is a pole (the other pole is shown at Q). The great circles PB and PC meet the great circle EF in E and F respectively. Then, by (1),

$$\text{arc } BC = r \times B\widehat{A}C, \quad \ldots\ldots (6)$$

$$\text{arc } EF = R \times E\widehat{O}F. \quad \ldots\ldots (7)$$

Now AB and OE lie in the plane of the great circle PBE and, as these straight lines are each perpendicular to OA, we see that AB and OE are parallel. Similarly, AC and OF are parallel. It follows that $B\widehat{A}C = E\widehat{O}F$. Hence, from (6) and (7), by division,

$$\frac{\text{arc } BC}{\text{arc } EF} = \frac{r}{R}. \quad \ldots\ldots (8)$$

Using (5), we then have

$$\frac{\text{arc } BC}{\text{arc } EF} = \sin PB. \quad \ldots\ldots (9)$$

This last formula gives the length of the small circle arc BC in terms of the corresponding great circle arc EF and of the angular radius PB of the small circle.

We can write (9) in a slightly different form; since $PB + BE = 90°$, we obtain

$$\frac{BC}{EF} = \cos BE. \quad \ldots\ldots (10)$$

7. *Application to the earth.*

The earth is one of the sun's family of planets; its form is so nearly spherical that, in this book, we shall make the simple assumption that it is actually a sphere. It has been found by what are essentially survey methods that its radius is 3960 miles.

We can apply the principles described in the previous sections to the specification of positions on the earth's surface. It is known that the earth spins about a diameter—called the *polar diameter* or *polar axis*—which is represented in Fig. 5 by POQ, the earth's centre being at O. We call P the *north pole* and Q the *south pole*. The great circle CDK of which P and Q are poles is the terrestrial *equator*.

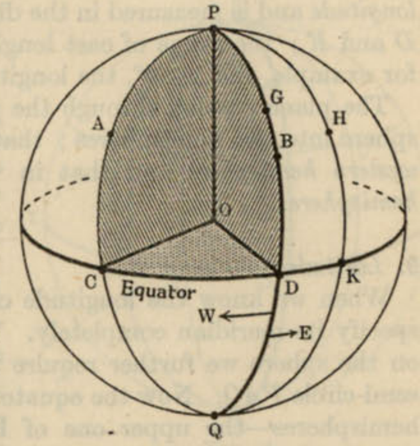

FIG. 5.

Consider a point A on the earth's surface. The *semicircle* PAQ passing through A and terminated by the poles P and Q is called a *meridian*. By international agreement, the meridian which passes through a certain telescope (the Airy transit instrument) at the Royal Observatory, Greenwich, is regarded as the *prime meridian*; it is shown as PGQ in Fig. 5.

8. *Longitude.*

With the prime meridian as basis, we can now specify the position of any other meridian such as PAQ. Let the meridians through A and G meet the equator in C and D respectively. Then the position of the meridian PAQ with reference to the prime meridian PGQ is defined by the equatorial arc DC, which is the same as the spherical angle BPA at P between the two meridians. The equatorial arc DC or the spherical angle BPA is called the *longitude* of A; a meridian such as that through A is generally designated a *meridian of longitude*. In Fig. 5 another meridian $PHKQ$ is shown on the other side of the prime meridian and its longitude is represented by the equatorial arc DK. To distinguish between the two cases, the longitude of such a point A is designated *west longitude* and is measured in the direction of the arrow between D and C; the range of west longitudes is from 0° to 180°. If, for example, DC is 60°, the longitude of A is designated 60° W.

Also, the longitude of such a point as H is designated *east longitude* and is measured in the direction of the arrow between D and K; the range of east longitudes is from 0° to 180°. If, for example, DK is 50°, the longitude of H is written 50° E.

The plane passing through the polar axis and G divides the sphere into two hemispheres; that in which A lies is called the *western hemisphere* and that in which H lies is the *eastern hemisphere*.

9. *Latitude.*

When we know the longitude of such a point as A we can specify its meridian completely. To specify the position of A on the sphere we further require to define its position on the semi-circle PAQ. Now the equator divides the sphere into two hemispheres—the upper one of Fig. 5 containing the north pole P is called the *northern hemisphere*, and the lower, containing the south pole Q, is called the *southern hemisphere*. In the figure, A is in the northern hemisphere and its position on the meridian PAQ will be specified if we know the arc CA of the meridian intercepted between the equator and the given point A; the arc CA, or the angle COA, is called the *latitude* of A. If the place concerned, such as A, is in the northern hemisphere, it is said to have a *north latitude*; if the place is in the southern hemisphere, it is said to have a *south latitude*. The latitude is measured in degrees.

In this way the position of a place on the earth's surface can be specified completely by means of longitude and latitude. It is to be remembered that this method of defining the position of the place depends on the choice of a particular semi-circle (the prime meridian) and a particular great circle (the equator) for the purpose of reference.

10. *Parallel of latitude.*

A small circle such as ABC (Fig. 6) of which P is the pole is called a *parallel of latitude*; the equator is shown as DEF and the Greenwich meridian as $PGHQ$. If ϕ denotes the latitude of A, all points on the parallel of latitude, ABC, will have the

same latitude ϕ. Since the great circle arc PAD is 90°, the angular radius, PA, of the parallel of latitude is $(90° - \phi)$.

In particular, the arc PA is called the *colatitude* of A, so that we have the relation

colatitude = 90° − latitude.

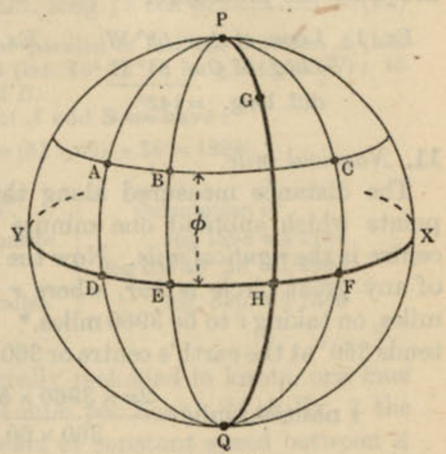

Fig. 6.

Now consider two positions, A and B, on the same parallel of latitude and both west of Greenwich. The longitude of A is measured by the equatorial arc HD and the longitude of B by the arc HE. The equatorial arc ED is then the *difference of longitude* of A and B, which we denote by *diff. long*. By (10), the arc AB of the parallel of latitude is given by

$$\frac{AB}{DE} = \cos BE = \cos \phi,$$

or $$AB = (\text{diff. long.}) \cos \phi. \qquad \text{.................(11)}$$

If two places (for example, A and B in Fig. 6) are in the western hemisphere, the corresponding diff. long. is obtained by simple subtraction of the two longitudes concerned. Thus if the longitudes of A and B are respectively 78° W and 46° W, the diff. long. is 32°. A similar procedure holds if the two places are in the eastern hemisphere.

To find the diff. long. between two places, one in west longitude and the other in east longitude, as A and C in Fig. 6, we proceed as follows. The longitude of A is measured by the arc HD and the longitude of C by the arc HF. The length of the arc FD which contains the point of intersection, H, of the Greenwich meridian with the equator is thus the *sum* of the longitudes of A and C. If this arc FHD is less than 180°, then FHD is the diff. long. between A and C. If, however, FHD

exceeds 180°, the diff. long. between A and C is measured by the arc $FXYD$, which is $360° - FHD$. The following examples illustrate the procedure.

Ex. 1. Long. of A =	58° W	Ex. 2. Long. of A =	68° W
Long. of C =	84° E	Long. of C =	166° E
diff. long. =	142°	Sum =	234°
		diff. long. =	126°

11. *Nautical mile.*

The distance measured along the great circle joining two points which subtend one minute of arc (1′) at the earth's centre is the *nautical mile*. Now the length of the circumference of any great circle is $2\pi r$, where r is the radius, or $2\pi . 3960$ miles, on taking r to be 3960 miles.* But the circumference subtends 360° at the earth's centre or 360×60 minutes of arc. Hence

$$1 \text{ nautical mile} = \frac{2\pi \times 3960 \times 5280}{360 \times 60} \text{ feet} = 6082 \text{ feet}.$$

The above definition of the nautical mile and the preceding calculation are based on the assumption that the earth is a sphere. However, the earth is not quite spherical and the strict definition of a nautical mile shows that it varies according to latitude, being 6046 feet at the equator and 6108 feet at the poles. But, in practice, this variation is nearly always ignored and the nautical mile is taken, in round figures, to be 6080 feet. We then have

$$1 \text{ nautical mile} = 6080 \text{ feet}.$$

In the sequel, it will be sufficient to assume that a nautical mile is the distance, along *any* great circle, between two points which subtend 1′ at the earth's centre and that a nautical mile is equivalent to 6080 feet.

12. *Departure.*

In Fig. 6, consider the points E and D on the equator; then the number of nautical miles between E and D is simply the number of minutes of arc in the equatorial arc ED. It follows from (11) that the distance between A and B, in nautical miles, measured along the parallel of latitude is (diff. long.) $\times \cos\phi$, in which diff. long. is expressed in minutes of arc and ϕ is the

* This refers to the statute mile which is equal to 5280 feet.

latitude of A or B. This distance is called the *departure* and is measured in nautical miles. Thus we have

$$\text{departure} = (\text{diff. long.}) \times \cos\phi. \quad \ldots\ldots\ldots\ldots\ldots(12)$$

Ex. 3. A ship steams along the parallel of latitude between A (lat. 40° 35′ N, long. 56° 28′ W) and B (lat. 40° 35′ N, long. 24° 50′ W); to find the departure between A and B.

By subtracting the longitudes of A and B we have:

$$\text{diff. long.} = 31° \; 38' = (31 \times 60)' + 38' = 1898'.$$

Hence, by (12),
departure = 1898 cos 40° 35′.
The calculation is shown opposite.
The result is:
departure = 1442 nautical miles.

Calculation:
log 1898 = 3·2784
log cos 40° 35′ = $\bar{1}$·8805
log. dep. = 3·1589

13. *The knot.*

The speed of a ship is generally reckoned in knots, one *knot* being defined as one nautical mile per hour. If in Ex. 3 the time required by a ship to steam at constant speed between A and B along the parallel of latitude is 72·1 hours, the speed is $\frac{1442}{72 \cdot 1}$ knots, that is, 20 knots; here the departure (= 1442 nautical miles) is the distance travelled along the parallel.

14.* *Formulae of spherical trigonometry.*

We first prove a formula connecting the three sides of a spherical triangle with any one of the angles. Let ABC (Fig. 7) be a spherical triangle in which the sides BC, CA and AB are denoted by a, b and c respectively. Taking A to be the angle to be associated with the three sides, we show that

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

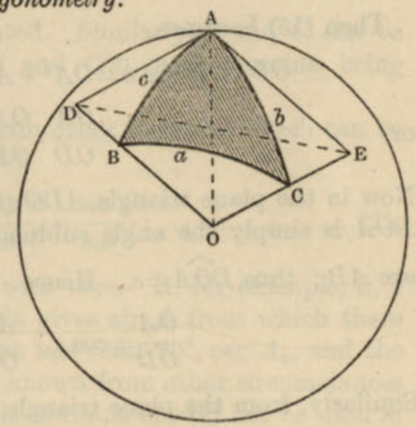

Fig. 7.

The spherical angle A is

* Sections and examples throughout the book marked with an asterisk may be omitted on a first reading.

defined as the angle between the tangents AD and AE drawn at A to the great circles AB and AC. Let the radii OB and OC of the sphere meet the tangents at D and E. Join DE. Then ADE is a plane triangle in which the angle DAE is A. Hence, by a well-known formula,

$$DE^2 = AD^2 + AE^2 - 2AD \cdot AE \cos A. \quad \text{..........(13)}$$

Similarly, ODE is a plane triangle in which the angle DOE is the angle subtended at the centre, O, by the great circle arc BC; by definition, this angle is a. Hence we have, for the plane triangle ODE,

$$DE^2 = OD^2 + OE^2 - 2OD \cdot OE \cos a. \quad \text{............(14)}$$

On subtracting (13) from (14) we have

$$2OD \cdot OE \cos a = (OD^2 - AD^2) + (OE^2 - AE^2) + 2AD \cdot AE \cos A. \quad \text{......(15)}$$

In the plane triangle AOD, the angle DAO is 90° since DA, being a tangent to the great circle AB at A, is perpendicular to the radius OA. Hence

$$OD^2 - AD^2 = OA^2.$$

Similarly, $$OE^2 - AE^2 = OA^2.$$

Then (15) becomes

$$OD \cdot OE \cos a = OA^2 + AD \cdot AE \cos A,$$

or

$$\cos a = \frac{OA}{OD} \cdot \frac{OA}{OE} + \frac{AD}{OD} \cdot \frac{AE}{OE} \cos A. \quad \text{.......(16)}$$

Now in the plane triangle ADO (right-angled at A), the angle DOA is simply the angle subtended at O by the great circle arc AB; thus $D\widehat{O}A = c$. Hence

$$\frac{OA}{OD} = \cos c, \quad \frac{AD}{OD} = \sin c.$$

Similarly, from the plane triangle AOE (right-angled at A), in which $A\widehat{O}E = b$,

$$\frac{OA}{OE} = \cos b, \quad \frac{AE}{OE} = \sin b.$$

Inserting these ratios in (16), we obtain

$$\cos a = \cos b \cos c + \sin b \sin c \cos A. \quad \text{............(17)}$$

This is the fundamental formula of spherical trigonometry, usually known as the *cosine-formula.*

The formula (17) is a relation involving all three sides (that is a, b and c) and one angle—in this instance A—of the spherical triangle.

There are clearly two other formulae of a similar nature involving B and C; they are:

$$\cos b = \cos c \cos a + \sin c \sin a \cos B. \quad \text{............(18)}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \quad \text{............(19)}$$

Problem 1. If, in a spherical triangle, two sides and the included angle are known, the third side can be calculated. For example, if we are given b, c and A, the third side, a, can be calculated by means of (17).

Problem 2. If, in a spherical triangle, all three sides are known, each angle can be calculated in turn. For example, from (17), we have—on rearranging—

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \quad \text{..................(20)}$$

from which A can be calculated. Similarly B and C can be calculated by means of (18) and (19), each formula being modified as in (20).

We give for reference * certain other formulae which can be derived from (17), (18) and (19).

Sine formula. $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}. \quad \text{.....................(21)}$$

This formula has to be used with care. If, for example, a, b and B are known, the formula gives $\sin A$ from which there are found two values of A, one less than 90°, say A_0, and the other $180° - A_0$. Unless it is known from other circumstances that A is, say, less than 90°, the formula should not be used in calculations.

* Proofs will be found in the author's *Spherical Astronomy* (Cambridge University Press).

Analogues of the cosine-formula.

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A, \quad\dots\dots\dots(22)$$

$$\sin a \cos C = \cos c \sin b - \sin c \cos b \cos A. \quad\dots\dots\dots(23)$$

There are clearly two others of a similar nature involving B on the right-hand side and two others involving C.

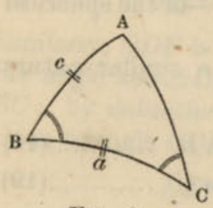

FIG. 8.

Four-parts formula. This is a formula involving two sides and two angles, taken in order, for example, c, B, a and C in Fig. 8. Of these one side, a, is between the two angles B and C and is called the *inner side*; the remaining side, c, is called the *other side*. Similarly, B is the *inner angle* and C the *other angle*. The formula is :

$$\cos a \cos B = \sin a \cot c - \sin B \cot C, \quad\dots\dots\dots(24)$$

which can be expressed as :

cos (inner side) cos (inner angle)
= sin (inner side) cot (other side)
− sin (inner angle) cot (other angle). …………(25)

Formula (20) can be expressed in alternative forms, one of which is as follows. Let

$$2s = a + b + c;$$

then

$$\sin \frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}. \quad\dots\dots\dots(26)$$

Thus, if all three sides of a spherical triangle are given, any one of the angles can be calculated.

Applications of some of the formulae in this section will be made later to certain astronomical problems.

EXAMPLES

1. Find the difference of longitude between two places A and B, given that their longitudes are :

(i) A, 36° W ; B, 48° W.
(ii) A, 20° E ; B, 175° E.
(iii) A, 47° 38′ W ; B, 28° 25′ E.
(iv) A, 52° 30′ E ; B, 140° 45′ W.
(v) A, 34° 28′ W ; B, 165° 52′ E.

2. Find the difference of latitude between two places A and B, given that their latitudes are :

(i) A, 20° N ; B, 38° N.
(ii) A, 37° 55′ S ; B, 23° 27′ S.
(iii) A, 52° 30′ N ; B, 29° 47′ S.

3. If, in Example 2, A and B are on the same meridian, find the distance between A and B (a) in nautical miles, (b) in statute miles. [1 statute mile = 5280 feet.]

4. A ship steams along the parallel of latitude between A and B; find the distance steamed in nautical miles given that the positions of A and B are :

(i) A (38° N, 42° W) ; B (38° N, 15° W).
(ii) A (42° S, 63° W) ; B (42° S, 10° E).
(iii) A (20° N, 170° E) ; B (20° N, 140° W).
(iv) A (32° 15′ N, 130° 20′ W) ; B (32° 15′ N, 146° 35′ E).

5. A ship steams eastwards along the parallel of latitude from A to B at 15 knots. Find the longitude of B, given that the position of A and the duration of the run are as follows :

(i) A (42° N, 50° W) ; 3 days 10 hours.
(ii) A (45° 30′ S, 59° W) ; 4 days 4 hours.
(iii) A (35° 40′ N, 162° E) ; 2 days 20 hours.

6.* In the spherical triangle ABC,
(i) given $b = 32°\ 12'$, $c = 56°\ 49'$, $A = 40°\ 33'$, find a, B and C.
(ii) given $b = 103°\ 17'$, $c = 27°\ 19'$, $A = 36°\ 15'$, find a, B and C.
(iii) given $a = 82°\ 11'$, $b = 59°\ 34'$, $C = 111°\ 47'$, find c, A and B.
(iv) given $a = 15°\ 23'$, $c = 33°\ 53'$, $B = 27°\ 58'$, find b, C and A.

7.* Find the great circle distances between A and B from the following data :

	A		B	
	Latitude	Longitude	Latitude	Longitude
(i)	37° N,	59° W ;	11° N,	122° W.
(ii)	52° N,	64° E ;	16° S,	15° W.
(iii)	12° 37′ N,	171° 43′ E ;	25° 13′ S,	109° 57′ E.
(iv)	62° 41′ S,	13° 33′ E ;	12° 17′ N,	45° 44′ W.

CHAPTER II

THE CELESTIAL SPHERE

15. *Zenith and horizon.*

We consider in this chapter some principles of fundamental character in relation to the specification of the positions of heavenly bodies on the celestial sphere.

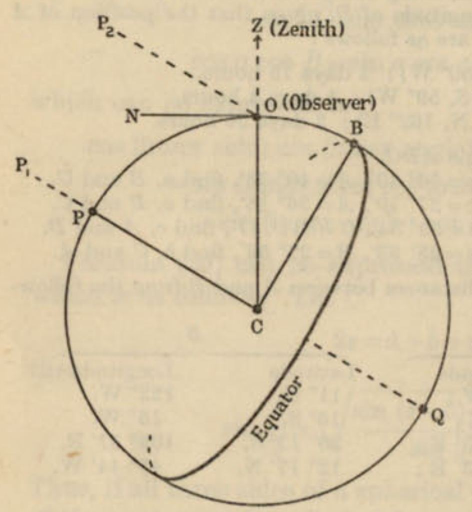

FIG. 9.

In Fig. 9 the earth is represented by the sphere, its centre being C; the axis about which it rotates is PCQ. We suppose that an observer is situated at O, in north latitude. The prolongation of the radius CO defines the direction of the observer's *zenith*; this direction is indicated by OZ and it may be regarded as determined by a plumb-line held by the observer at O.

The plane through O and perpendicular to OZ is the plane of the *celestial horizon* for the observer at O.

16. *The north celestial pole.*

We are all familiar with the daily and nightly movements of the heavenly bodies, from east to west, across the celestial sphere, and we ascribe these movements to the effect of the earth's rotation about its polar diameter. In the northern hemisphere there is one bright star visible (*Polaris* or the *Pole-*

star) which appears to maintain its position from hour to hour and to be unaffected by the *diurnal motion*, as the visible effect of the earth's rotation is called. We conclude that the Pole-star must be in the direction PP_1 given by the prolongation of the earth's axis (Fig. 9). As we can assume that any star is at an infinite distance (as compared with the dimensions of the earth) *Polaris* will be seen by the observer at O in the direction OP_2, where OP_2 is parallel to CP_1. The direction OP_2 defines, for the observer, the direction of the *north celestial pole.* We have found it convenient to use the Pole-star to illustrate the definition of the north celestial pole; actually, the Pole-star is a little over a degree away from the latter and the strict definition of the north celestial pole, for an observer at O, is that position on the celestial sphere, of which O is the centre, whose direction from O is parallel to the earth's axis of rotation.

In Fig. 9, $POBQ$ is the meridian of longitude for the observer at O, B being its intersection with the terrestrial equator. Accordingly, the arc BO (or the angle OCB) measures the latitude, ϕ, of the observer. Since PC is perpendicular to CB, the angle PCO is $90° - \phi$, that is, the colatitude of O. Also $P_2\widehat{O}Z = P\widehat{C}O$, since OP_2 and CP are parallel; hence $P_2\widehat{O}Z$ is also the colatitude $(90° - \phi)$, that is, *the angle between the direction of the zenith and the direction of the north celestial pole is* $90° - \phi$.

Let ON be drawn in the plane of the observer's meridian of longitude, $POBQ$, and perpendicular to OZ. Then the angle P_2ON is called the *altitude* of the north celestial pole. Since

$$P_2\widehat{O}N = 90° - P_2\widehat{O}Z = 90° - (90° - \phi) = \phi,$$

we have the result:

altitude of the pole = latitude of observer. ....(1)

17. *The observer's celestial sphere (north latitude).*

We now consider the celestial sphere for an observer at O (Fig. 10); we assume as before that the observer's latitude is north.

We first draw the direction, OZ, of the zenith Z upwards in the diagram to correspond with the fact that it is vertically upwards for the observer; P is the north celestial pole, OP being parallel to the earth's axis of rotation.

The great circle, $NWYSE$, of which Z is the pole, is the *celestial horizon*, or simply the horizon. The upper hemisphere is the visible hemisphere; in general, we are not concerned with the lower hemisphere.

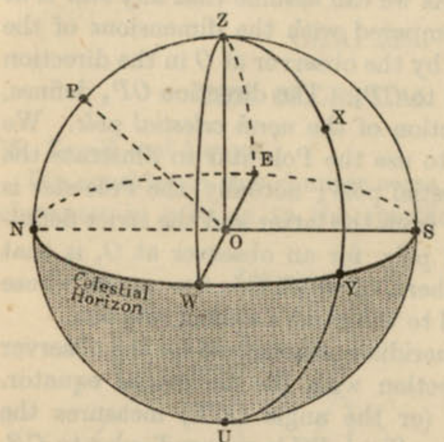

FIG. 10.

Any semi-great circle through Z, such as ZYU, is called a *vertical circle*, or simply a *vertical*. In particular the vertical passing through the pole P cuts the horizon in the *north point*, denoted by N. The position of any vertical * such as ZXY can then be specified with regard to the principal vertical, ZPN, either by means of the spherical angle NZY (or PZX) or by the arc NWY measured along the horizon; this angle or arc is called the *azimuth* (*west*) and its value ranges from 0° for the vertical ZN to 180° for the vertical ZS.

We have seen that the direction $\overrightarrow{ON}$ on the horizon is defined as the north direction; the opposite direction $\overrightarrow{OS}$ is the *south direction* (Fig. 10). Let EOW be a diameter of the horizon at right angles to NOS. The points E and W are defined to be the *east* and *west* points respectively; these points are placed in the diagram according to the convention that if we imagine ourselves facing the north point (that is, along $\overrightarrow{ON}$) the east point is towards our right hand and the west point towards our left hand. The points N, E, S and W are generally known as the *cardinal* points.

The verticals passing through the east and west points are called

* It is usually sufficient to consider only the part of the vertical in the visible hemisphere.

the *prime verticals*; in particular, we distinguish them as *prime vertical east* (that is ZE) and *prime vertical west* (that is ZW).

18. *Horizontal system of coordinates.*

Consider the position of a heavenly body at a given instant at X on the celestial sphere for an observer in north latitude. We distinguish two cases for the purpose of diagrammatic representation.

Case 1: *heavenly body in western hemisphere* (*N. lat.*). In this case the vertical circle through X meets the horizon in a point Y on the semi-circle NWS (Fig. 11). As we have seen, the position of the vertical ZXY is specified with reference to the vertical ZPN by the azimuth (west), which we denote by A. Thus in Fig. 11 the azimuth (west) is the arc NY measured along the horizon westwards from N or, alternatively, the angle PZX in the spherical triangle PZX. Also A can have any value between 0° and 180°.

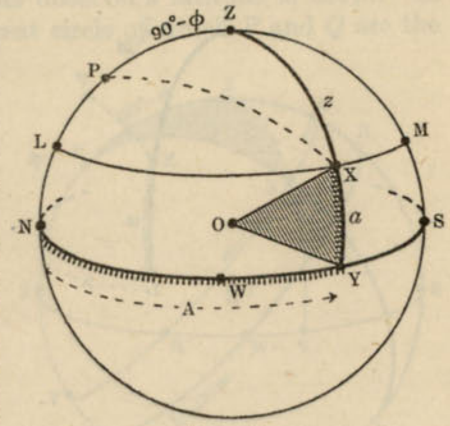

FIG. 11.

The position of X on the vertical ZXY is specified if we know the arc YX (equivalent to the angle YOX). We define YX to be the *altitude* of X and denote it by a.

Since the arc ZY is 90°, Z being the pole of the horizon, $ZX = 90° - XY = 90° - a$. The arc ZX is called the *zenith distance* * of the heavenly body X and is denoted by z. Thus

$$z = 90° - a. \qquad \ldots\ldots(2)$$

Accordingly, we can specify the position of a heavenly body at a given instant by the spherical coordinates (A, a) or (A, z).

The small circle LXM of which Z is the pole is a *parallel of altitude*.

* Abbreviated to $Z.D.$

Also, NP is the altitude of the north celestial pole and, accordingly, if ϕ is the observer's north latitude, $NP=\phi$. Hence

$$PZ=90^\circ-\phi. \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Case 2: *heavenly body in eastern hemisphere* (*N. lat.*). We represent Z and the horizon as before and place E in the middle of that part of the horizon shown by the heavy line (Fig. 12).

With the convention as to the cardinal points the north point, N, must be situated as indicated ($\overrightarrow{OE}$ to the right of $\overrightarrow{ON}$).

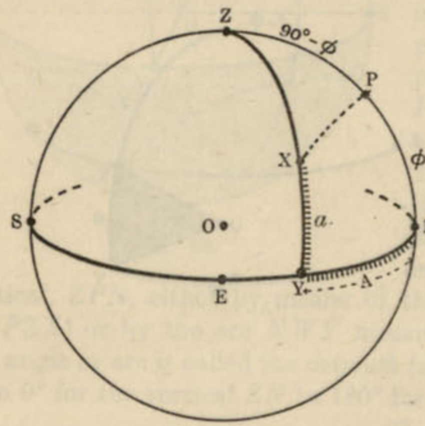

Fig. 12.

As the observer is in north latitude the north celestial pole, P, must be situated on the principal vertical ZN. The observer's latitude ϕ being known we can now insert P in the diagram so that $NP=\phi$. Also, the south point S can now be indicated.

The position of the vertical ZXY with respect to the principal vertical ZN is then specified by means of the arc NY on the horizon or by the spherical angle PZX; this is the *azimuth* (east) which we denote by A; the value of A lies between 0° and 180°. As before, the altitude, a, is defined by the arc YX of the vertical through X; also the zenith distance, z, of X is the arc ZX.

19. *Rules for drawing diagrams* (*observer in north latitude*).

1. Insert Z at top of diagram and draw the horizon.

2. (*a*) If the heavenly body is in the western hemisphere, insert W as in Fig. 11; N and S are then inserted according to the convention relating to the cardinal points.

(*b*) If the heavenly body is in the eastern hemisphere, insert E as in Fig. 12; N and S are then inserted according to the convention relating to the cardinal points.

3. In *both* cases, the north celestial pole, P, must be on the vertical ZN; insert P on this vertical so that $NP=\phi$.

4. Draw the vertical through the position, X, of the heavenly body.

5. In *both* cases, the azimuth is the angle PZX in the spherical triangle PZX and is called azimuth (west) or azimuth (east) according as the heavenly body is in the western or eastern hemisphere. In each case, the value of A lies between 0° and 180°.

20. *Observer's meridian.*

We still suppose that the observer's latitude is north. In Fig. 13 let RWT be the great circle of which P and Q are the poles; this great circle is the *celestial equator*, or, simply, the *equator*.

It is clear that we can specify the position of a heavenly body on the celestial sphere, at a given instant, with the equator as reference plane. For convenience of explanation we shall take the heavenly body to be a star.

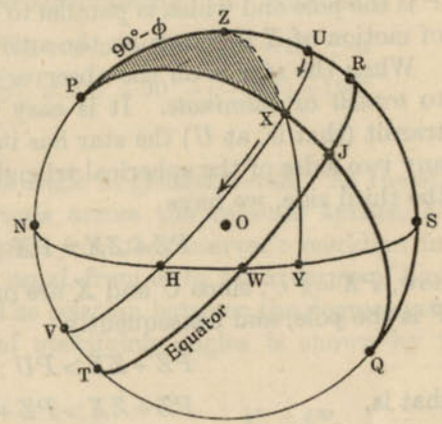

Fig. 13.

It is easily seen that the equator meets the horizon in the east and west points. If, as in Fig. 13, W denotes, for the moment, one of the points of intersection of the equator and horizon, the angular distance of W (a point on the horizon) from Z is 90°; also, the angular distance of W (a point on the equator) from P is 90°; thus W is the pole of the great circle through Z and P and hence WN is 90°; it follows that W is the west point. We can establish a similar result for the east point.

Any semi-great circle whose extremities are the poles P and Q is called a *celestial meridian* or, simply, a *meridian*. Thus the semi-great circle PXQ is the meridian of the star, X.

The meridian passing through the observer's zenith (*PZRSQ* in Fig. 13) is the *observer's meridian*; as this is of fundamental importance, it is shown in the figure by a heavy line.*

21. *The diurnal motion.*

We know from simple observation that the stars appear to travel across the sky from east to west; this is the *diurnal motion*, resulting from the earth's rotation. Moreover, this diurnal motion has no effect in increasing or decreasing the angular distance between any two given stars, as is obvious in a general way from simple observation. If we think of the pole, P, as being indicated by a star, it follows that the great circle arc PX remains constant as X moves across the celestial sphere. Accordingly, X describes a small circle UXV of which P is the pole and which is parallel to the equator; the direction of motion of X is shown by the arrow near X.

When the star is on the observer's meridian at U, it is said to *transit* or *culminate*. It is easy to show that at meridian transit (that is, at U) the star has its maximum altitude; for, any two sides of the spherical triangle PZX being greater than the third side, we have

$$PZ+ZX>PX;$$

now, $PX=PU$, since U and X are on the small circle of which P is the pole, and consequently

$$PZ+ZX>PU;$$

that is, $$PZ+ZX>PZ+ZU.$$

Hence $ZX>ZU$, so that the meridian zenith distance, ZU, is less than the zenith distance ZX; accordingly, the meridian altitude, SU, is greater than the altitude YX.

22. *Equatorial system of coordinates (hour angle and declination).*

In this system of coordinates, the equator is the principal circle of reference and the observer's meridian is the principal meridian. The position of a star at X is then specified by (*a*) the angle ZPX between the observer's meridian and the meridian through X and (*b*) the arc JX measured from the

* The student is strongly recommended to mark distinctively the observer's meridian in every diagram in which it appears.

equator to X along the star's meridian (Fig. 13). The angle ZPX—or its equivalent, the equatorial arc RJ—is called the *hour angle* of the star; we denote it by H.

The arc JX is called the *declination* (denoted by δ); the declination is *positive* or *north* if X is between the north celestial pole and the equator, and *negative* or *south* if the star is between the south celestial pole, Q, and the equator. The small circle $UXHV$ along which the star appears to travel owing to the diurnal motion is a *parallel of declination*. The angular distance PX of the star from the north celestial pole is called the *north polar distance* (N.P.D.), which is clearly related to the declination by the formula

$$\text{N.P.D.}=90°-\delta. \quad\quad (4)$$

In this formula, δ is used algebraically; for example if

$$\delta=+30° \text{ (or } 30° \text{ N), N.P.D.}=90°-30°=60°; \text{ if}$$

$$\delta=-40° \text{ (or } 40° \text{ S), N.P.D.}=90°-(-40°)=130°.$$

23. *Hour angle.*

We now consider hour angle in greater detail. As the stars appear to move westwards across the celestial sphere, *hour angle is measured westwards from the* observer's meridian from 0° to 360° or, as is more usual, from 0^h to 24^h in terms of hours, minutes and seconds. The relation between the degree-system and the hour-system of specifying angles is shown by the following:

$1^h=15°.$	$1°=4^m.$
$1^m=15'.$	$1'=4^s.$
$1^s=15''.$	$1''=\frac{1^s}{15}.$

When the star is on the observer's meridian, its hour angle is 0^h. Immediately after transit at U (Fig. 13) the diurnal motion carries the star *westwards* and as the earth's rotation is uniform, the star will appear to move along the parallel, UXV, of declination at a uniform rate; consequently, its hour angle increases uniformly. It is this fact which forms the basis of the measurement of time.

When the star reaches the horizon at H it is said to *set*.

Thereafter between H and V it is below the horizon (and therefore invisible). At V its angular distance below the horizon—in this case, NV—is a maximum and at V its hour angle is 180° or 12^h. We thus have the rule:

When a star is in the western hemisphere of the celestial sphere—that is, when its azimuth is west—its hour angle is between 0^h and 12^h.

Conversely, *if a star's hour angle is between 0^h and 12^h, it lies in the western hemisphere of the celestial sphere, that is to say, its azimuth is west.*

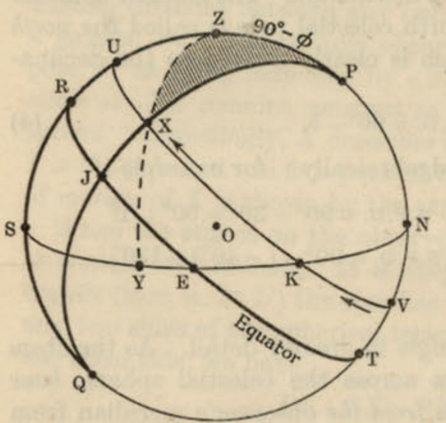

Fig. 14.

After reaching V, the star passes into the eastern hemisphere and we follow its progress in Fig. 14, in which the observer's meridian is shown by a heavy line. At V the star's hour angle is 12^h, and thereafter the star moves uniformly along the small circle $VKXU$ in the direction of the arrow. At K the star is said to *rise*. At X the star's hour angle, H, is $12^h + \text{arc } TJ$. Now $TJ = TR - RJ$ and, since TR is 12^h (a semicircle of the equator), we have

$$H = 12^h + (12^h - RJ) = 24^h - RJ.$$

As RJ is the same as the angle ZPX in the spherical triangle PZX, we obtain

$$H = 24^h - Z\widehat{P}X,$$

or

$$Z\widehat{P}X = 24^h - H. \qquad \text{.............................(5)}$$

Following the star's motion beyond X, we see that it reaches the observer's meridian at U, its hour angle there being 24^h or 0^h. We have the rule:

If a star is in the eastern hemisphere—that is, if its azimuth is east—its hour angle is between 12^h and 24^h; conversely, *if a star's hour angle is between 12^h and 24^h, it lies in the eastern hemisphere and its azimuth is east.*

24. *Meridian zenith distance.*

In Figs. 13 and 14 the star's zenith distance, when it is on the observer's meridian, is ZU. Now PU is the star's N.P.D., so that by (4)

$$PU = 90° - \delta.$$

Also, $PZ = 90° - \phi$. As $ZU = PU - PZ$, we obtain

$$ZU = \phi - \delta. \qquad \text{.................................(6)}$$

In this formula, δ is to be used in its algebraic sense.

Ex. 1. Given $\phi = 50°$ N, $\delta = 20°$ N $\equiv +20°$; then
Mer. Z.D. $= 50° - 20° = 30°$.

Ex. 2. Given $\phi = 50°$ N, $\delta = 20°$ S $\equiv -20°$; then
Mer. Z.D. $= 50° - (-20°) = 70°$.

25. *Circumpolar stars.*

Stars which are above the horizon, for a given observer, for all values of the hour angle are called *circumpolar stars*. In Fig. 15, NF is the parallel of declination passing through the north point, N, of the horizon and the corresponding declination is FR or TN. The condition that a star of declination δ should be circumpolar is evidently

$$\delta > NT.$$

Now

$$NT = PT - PN = 90° - \phi.$$

Hence the condition is

$$\delta > 90° - \phi. \text{ ......(7)}$$

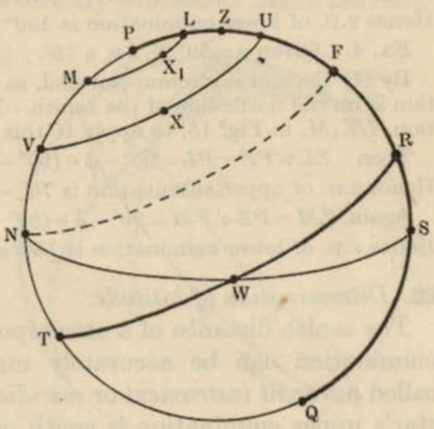

Fig. 15.

Consider two typical circumpolar stars, X and X_1, their parallels of declination being UXV and LX_1M respectively. The points at which their hour angles are zero (the stars are then on the observer's meridian) are U and L; the transit or culmination is then said to be *upper transit* or *upper culmination*.

When the hour angle is 12^h the star X is at V and the star

X_1 at M; they are then said to transit *below pole* or to be at *lower culmination*.

It will be noticed from Fig. 15 that the upper culmination of a circumpolar star can take place either on the south side of the zenith (as at U for star X) or on the north side of the zenith (as at L for star X_1). The additional condition for culmination on the south side of the zenith is evidently $PU > PZ$, that is, $90° - \delta > 90° - \phi$ or $\phi > \delta$; and the additional condition for culmination on the north side of the zenith is $PL < PZ$, that is, $90° - \delta < 90° - \phi$ or $\delta > \phi$.

The relation between latitude, declination and the zenith distance of culmination in any particular case is best worked out from the appropriate diagram.

Ex. 3. Given $\phi = 50°$ N; $\delta = +45°$; to find the star's Z.D. at upper and lower culmination. By (7) the star is circumpolar and, as $\phi > \delta$, the star's upper culmination is on the south side of the zenith. We take the parallel of declination, UXV, in Fig. 15 to apply to this star.

Then $ZU = PU - PZ = 90° - \delta - (90° - \phi) = \phi - \delta$. ....(8)

Hence Z.D. of upper culmination is $50° - 45°$ or $5°$.

Again, $ZV = PZ + PV = 90° - \phi + 90° - \delta = 180° - \phi - \delta$. ....(9)

Hence Z.D. of lower culmination is $180° - 50° - 45°$ or $85°$.

Ex. 4. Given $\phi = 50°$ N, $\delta = +75°$.

By (7) the star is circumpolar and, as $\delta > \phi$, the star's upper culmination is on the north side of the zenith. We take the parallel of declination, LX_1M, in Fig. 15, to apply to this star.

Then $ZL = PZ - PL = 90° - \phi - (90° - \delta) = \delta - \phi$.

Hence Z.D. of upper culmination is $75° - 50°$ or $25°$.

Again, $ZM = PZ + PM = 90° - \phi + (90° - \delta) = 180° - \phi - \delta$.

Hence Z.D. of lower culmination is $180° - 50° - 75°$ or $55°$.

26. *Determination of latitude.*

The zenith distance of a circumpolar star at upper and lower culmination can be accurately measured by an instrument called a *transit instrument* or *meridian circle*. Suppose that the star's upper culmination is south of the zenith as in Ex. 3 of the previous section.

From (8) and (9) we have

$$ZU = \phi - \delta; \quad ZV = 180° - \phi - \delta. \quad \text{..............(10)}$$

By subtraction we obtain

$$2\phi = 180° + ZU - ZV, \quad \text{.....................(11)}$$

which determines ϕ when ZU and ZV are derived from observations (we leave out here the consideration of the necessary corrections to be applied to the actual observations).

27. *Determination of declination (circumpolar star).*

By adding the equations (10) we obtain

$$2\delta = 180° - ZU - ZV,$$

which determines δ for a circumpolar star culminating south of the zenith, ZU and ZV being obtained from observations.

The results for stars culminating north of the zenith are obtained in a similar way.

If the latitude is known, we obtain δ easily by simply measuring the star's zenith distance ZU at upper culmination so that, by (10),

$$\delta = \phi - ZU. \quad \text{..........................(12)}$$

28. *Diagrams for an observer in south latitude.*

(*a*) *Star west.* Wherever an observer, O, may be on the surface of the earth his zenith is directly overhead. As in previous diagrams we draw OZ upwards (Fig. 16) and insert the horizon in the celestial sphere; as the star is west we place the west point, W, as shown; the remaining cardinal points N, S, and E are then inserted according to the usual convention. Just as in northern latitudes the north celestial pole is in the visible or upper hemisphere, so for an observer in south latitude the south celestial pole will be above the horizon. We shall denote the south celestial pole by P_1 (instead of Q, as we have hitherto done). If ϕ_1 denotes numerically the observer's south latitude, $P_1Z = 90° - \phi_1$; this enables us to place P_1 in the diagram, between Z and S, for the south celestial pole

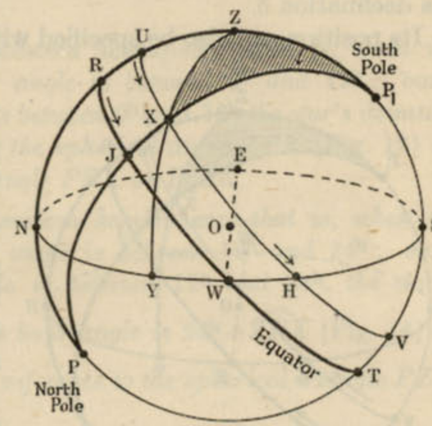

FIG. 16.

is situated on the vertical through S; we then draw the equator.

Meridians are defined, as before, to be semi-great circles terminated by P_1 and P; the meridian passing through the zenith is the *observer's meridian* (shown by a heavy line as P_1ZRNP).

Consider a star, X, in the western hemisphere of the celestial sphere; P_1XP is the star's meridian and ZXY its vertical. Now, wherever the observer may be, the diurnal motion carries a star from the observer's meridian *westwards*. The angle ZP_1X in the spherical triangle P_1ZX, or the equatorial arc RJ, measured *westwards from the observer's meridian* to meet the star's meridian (in the direction shown by the arrow), is defined as before to be the hour angle, H, of the star. For the star west, H lies between 0^h and 12^h.

In Fig. 16, JX is the *declination* and as X, in the figure, is between the equator and the south pole, P_1, its declination is south or negative. The declination is north or positive if the star is on the same side of the equator as the north pole P.

The position of X is thus specified by its hour angle H and its declination δ.

Its position can also be specified with the horizon as reference circle. The spherical angle P_1ZX in the triangle P_1ZX, or the arc SY measured on the horizon from the south point, is the *azimuth* and in this case it is *west*. Also, YX is the altitude and ZX the zenith distance. The star sets at H.

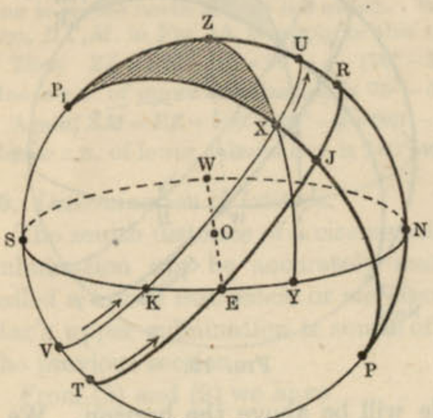

FIG. 17.

(*b*) *Star east.* In this case, we place E as shown in Fig. 17; then N, S and W are inserted according to the usual convention. P_1ZNP is the observer's meridian. The diurnal motion carries the star westwards from the observer's meridian, along the half, $VKXU$, of the parallel

of declination; at V the star's hour angle is 12^h and its angular distance below the horizon is then a maximum. Thereafter it moves along the parallel VKU, rising at K and reaching the observer's meridian at U. At X its hour angle is 12^h + spherical angle VP_1X or 12^h + equatorial arc TJ. Now $TJ = TR - JR$. Hence

$$H = 24^h - JR,$$

or

$$H = 24^h - \text{spherical angle } ZP_1X.$$

Thus for a star in the eastern hemisphere the hour angle lies between 12^h and 24^h. The star's declination is measured by JX, as before.

The azimuth is the spherical angle P_1ZX in the spherical triangle P_1ZX or the arc SY measured eastwards from the south point; in this case the azimuth is east. As before, YX is the altitude and ZX the zenith distance.

29. *General rules.*

1. *In* ALL *cases, hour angle is measured westwards from the observer's meridian.*

2. *If a star is in the western hemisphere—that is, when its azimuth is west—its hour angle is between* 0^h *and* 12^h; conversely, *if the hour angle is between* 0^h *and* 12^h *the star's azimuth is west; the hour angle is the spherical angle* ZPX (Fig. 13) *or* ZP_1X (Fig. 16) *in the triangle* PZX *or* P_1ZX.

3. *If a star is in the eastern hemisphere—that is, when its azimuth is east—its hour angle is between* 12^h *and* 24^h; conversely, *if the hour angle is between* 12^h *and* 24^h, *the star's azimuth is east; the star's hour angle is* $24^h - Z\widehat{P}X$ (Fig. 14) *or* $24^h - Z\widehat{P_1}X$ (Fig. 17) *with reference to the spherical triangle* PZX *or* P_1ZX.

4. *For an observer in north latitude the azimuth is measured from the north point westwards or eastwards, and the azimuth is then the spherical angle* PZX.

5. *For an observer in south latitude the azimuth is measured from the south point westwards or eastwards, and the azimuth is then the spherical angle* P_1ZX.

30. *The standard, or geocentric, celestial sphere.*

The dimensions of the earth are so small compared with the distance of even the nearest star that the directions of a given star as seen by two observers at widely different positions on the earth's surface can be regarded as parallel (the actual deviation from parallelism is far beyond the power of detection of even the most accurate instruments). Now the north polar distance of a star is the angle between the direction of the star and the direction defined by the earth's axis of rotation, and it follows in consequence that the north polar distance of the star as measured by an observer O is the same as the north polar distance as measured by another observer O_1. Since N.P.D. $= 90° - \delta$, the star's declination as we have defined it is independent of the observer's position on the earth's surface.

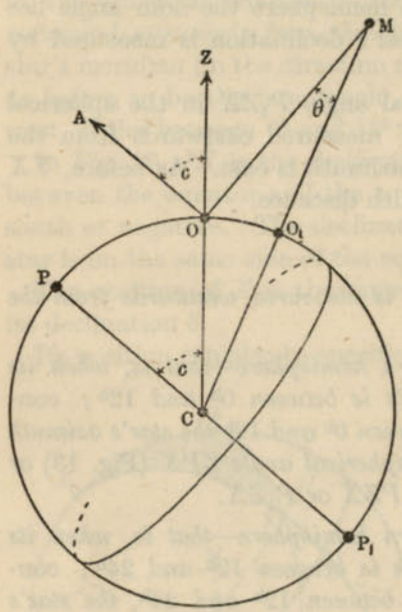

Fig. 18.

But the situation is different if we are dealing with a near object such as the moon, or sun, or a planet. Consider such a near body, M, and for simplicity suppose that at a given instant it is on the meridian of an observer, O, on the earth's surface (Fig. 18). The sphere represents the earth, centre C, and PCP_1 is the axis of rotation. Let OA be parallel to P_1CP. Since M is on the observer's meridian, the line joining C and M cuts the earth's surface at O_1 lying on the terrestrial meridian POP_1. Thus OA, OZ, OM and CM are coplanar; hence

$$\widehat{AOM} = \widehat{AOZ} + \widehat{ZOM}. \quad \text{.....................(13)}$$

Now $\widehat{AOM}$ is the N.P.D. of M, as previously defined for an observer at O; we denote $\widehat{AOM}$ by p_0. Also, since OA is parallel to CP, $\widehat{AOZ} = \widehat{PCO}$, that is to say, $\widehat{AOZ}$ is the colatitude, c, of the observer. Hence (13) becomes

$$p_0 = c + \widehat{ZOM}. \quad \text{..............................(14)}$$

Also, $\widehat{ZOM} = \widehat{OCM} + \widehat{OMC}$,

so that (14) becomes

$$p_0 = c + \widehat{OCM} + \theta, \quad \text{......................(15)}$$

in which we have written θ for $\widehat{OMC}$. But

$$c + \widehat{OCM} = \widehat{PCM},$$

hence $$p_0 = \widehat{PCM} + \theta. \quad \text{......................(16)}$$

Now $\widehat{PCM}$, being the angle between the earth's polar axis and the direction of M from the earth's centre, C, is independent of the observer; on the other hand, θ clearly depends on the observer's position for it is the angle between the direction of M as viewed from O and the direction of M from C. Thus the north polar distance p_0 (and, consequently, the declination) as previously defined varies according to the observer. This fact would create intractable difficulties as regards the tabulation in the *Nautical Almanac* (where information relating to the heavenly bodies is given) of the declination of such bodies as the sun, moon and planets.

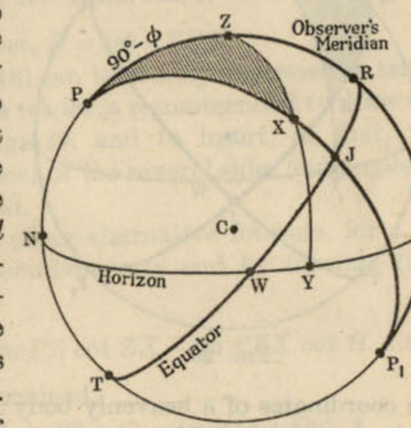

Fig. 19.

We define the north polar distance of M to be the angle PCM; the declination of M is then $90° - \widehat{PCM}$. This procedure is equivalent to regarding the earth's centre, C, as the centre of the celestial sphere, which can now be drawn as shown in Fig. 19. The direction of the observer's zenith is

CZ, and the celestial horizon (or, simply, horizon) is the great circle of which Z is the pole; the plane of the celestial equator is identical with the plane of the terrestrial equator; as before, the *observer's meridian* is the meridian PZP_1, *passing through the observer's zenith*. The declination of a heavenly body X—whatever its distance from the earth—is the arc JX; altitude, zenith distance, azimuth and hour angle are all defined as before.

The sphere as drawn in Fig. 19 is the *standard*, or *geocentric*, celestial sphere, and in subsequent pages when we refer to the celestial sphere it is the standard celestial sphere that must be understood.

31. *Relation between the horizontal and equatorial systems of coordinates.*

Fig. 20 shows the celestial sphere for an observer in north latitude ϕ, which we shall assume to be known. We specify

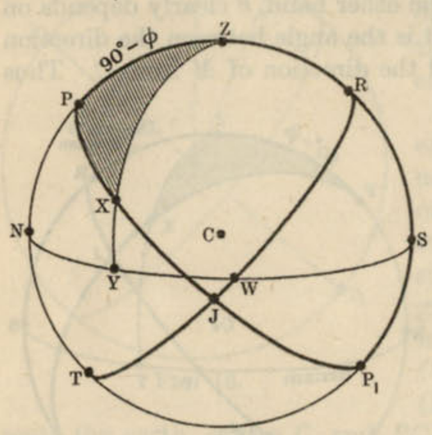

FIG. 20.

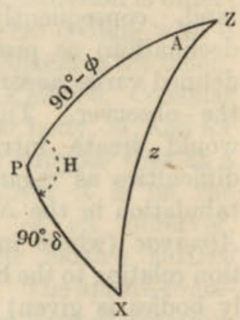

FIG. 21.

the coordinates of a heavenly body X in the horizontal system by its azimuth A and zenith distance z (or its altitude a) and in the equatorial system by its hour angle H and declination δ.

There are two principal problems. In the first it is assumed that A and z are obtained from observations and it is required to calculate H and δ; in the second problem it is assumed that H and δ are known and it is required to calculate A and z.

In these and similar problems we are concerned with the spherical triangle PZX (Fig. 21); if the latitude is south, we are concerned with the triangle P_1ZX.

Problem 1. Given ϕ, A and z; to calculate H and δ.

We require to find PX and $\widehat{ZPX}$ in the triangle PZX. We are given two sides (PZ and ZX) and the contained angle PZX; the third side PX is given by the fundamental formula (17), p. 13:

$$\cos PX = \cos PZ \cos ZX + \sin PZ \sin ZX \cos \widehat{PZX}, \quad \text{...(17)}$$

from which $\delta\,(\equiv 90° - PX)$ is obtained. For a northern latitude, as in Fig. 20, δ is positive or north if $PX < 90°$, and negative or south if $PX > 90°$; for a southern latitude δ is positive or north if $P_1X > 90°$ and negative or south if $P_1X < 90°$.

To calculate H we have now all three sides known; in particular, the two sides PZ and PX contain the hour angle H. Hence we can write

$$\cos ZX = \cos PZ \cos PX + \sin PZ \sin PX \cos \widehat{ZPX}. \quad \text{...(18)}$$

This formula determines $\widehat{ZPX}$. If the star is west, as in Fig. 20, $H = \widehat{ZPX}$; if the star is east, $H = 24^h - \widehat{ZPX}$.

The formulae (17) and (18) can be readily expressed in terms of ϕ, A, z, H and δ, but the reader is recommended to draw the celestial sphere, as in Fig. 20, and to insert, at first, the numerical values (in degrees) of the several sides and angles as they are given or calculated.

H can be found by use of an alternative formula, for z, A, PZ and H are four consecutive parts and by formula (25), p. 14,

$$\cos PZ \cos \widehat{PZX} = \sin PZ \cot ZX - \sin \widehat{PZX} \cot H \quad \text{....(19)}$$

from which H can be determined.

Problem 2. Given ϕ, H and δ; to calculate z and A.

We are thus given two sides PZ, PX and the contained angle ZPX (which is H if the star is west and $24^h - H$ if the star is east). The third side ZX is given by

$$\cos ZX = \cos PZ \cos PX + \sin PZ \sin PX \cos \widehat{ZPX}, \quad \text{......(20)}$$

and $\widehat{PZX}$ is then found from

$$\cos PX = \cos PZ \cos ZX + \sin PZ \sin ZX \cos \widehat{PZX} \quad \text{........(21)}$$

or by

$$\cos PZ \cos \widehat{ZPX} = \sin PZ \cot PX - \sin \widehat{ZPX} \cot \widehat{PZX}. \quad \text{....(22)}$$

Ex. 5. Given $\phi = 50^\circ$ N, $A = 48^\circ$ (west) and $z = 70^\circ$; to calculate δ and H.

We have $PZ = 40^\circ$, $ZX = 70^\circ$ and $\widehat{PZX} = 48^\circ$. Hence by (17)

$$\cos PX = \cos 40^\circ \cos 70^\circ + \sin 40^\circ \sin 70^\circ \cos 48^\circ,$$

which we write as

$$\cos PX = U + V,$$

where $U = \cos 40^\circ \cos 70^\circ$ and $V = \sin 40^\circ \sin 70^\circ \cos 48^\circ$. The calculations of U and V are shown below:

$\log \cos 40^\circ = \bar{1}{\cdot}8843$	$\log \sin 40^\circ = \bar{1}{\cdot}8081$
$\log \cos 70^\circ = \bar{1}{\cdot}5341$	$\log \sin 70^\circ = \bar{1}{\cdot}9730$
	$\log \cos 48^\circ = \bar{1}{\cdot}8255$
$\therefore \log U = \bar{1}{\cdot}4184$	$\therefore \log V = \bar{1}{\cdot}6066$
$\therefore U = 0{\cdot}2620$	$\therefore V = 0{\cdot}4042$

Hence

$$\cos PX = 0{\cdot}2620 + 0{\cdot}4042 = 0{\cdot}6662,$$

from which

$$PX = 48^\circ\ 14'.$$

The declination, δ, is then $90^\circ - 48^\circ\ 14'$, that is, $+41^\circ\ 46'$ or $41^\circ\ 46'$ N.

We calculate H by means of (18), leaving as an exercise to the student the calculation of H by the alternative formula (19).

Inserting the values of ZX, PZ and PX in (18) we have

$$\cos 70^\circ = \cos 40^\circ \cos 48^\circ\ 14' + \sin 40^\circ \sin 48^\circ\ 14' \cos H$$

which we write as

$$\cos 70^\circ = L + M \cos H. \quad \text{.........................(23)}$$

The calculation proceeds as follows:

$\log \cos 40^\circ = \bar{1}{\cdot}8843$	$\log \sin 40^\circ = \bar{1}{\cdot}8081$
$\log \cos 48^\circ\ 14' = \bar{1}{\cdot}8235$	$\log \sin 48^\circ\ 14' = \bar{1}{\cdot}8726$
$\therefore \log L = \bar{1}{\cdot}7078$	$\therefore \log M = \bar{1}{\cdot}6807$
$\therefore L = 0{\cdot}5102$	$\therefore M = 0{\cdot}4794$

Hence, from (23),

$$\cos H = \frac{\cos 70^\circ - L}{M} = \frac{0{\cdot}3420 - 0{\cdot}5102}{0{\cdot}4794};$$

$$\therefore \cos H = -\frac{0{\cdot}1682}{0{\cdot}4794} = -0{\cdot}3508$$

$\log 0{\cdot}1682 = \bar{1}{\cdot}2258$	
$\log 0{\cdot}4794 = \bar{1}{\cdot}6807$	
$\bar{1}{\cdot}5451$	

As the star is in the western hemisphere, H must lie between 0^h and 12^h or, in degrees, between 0° and 180°. Also, since $\cos H$ is negative, H must lie between 90° and 180°. Now

$$\cos(180^\circ - H) = -\cos H.$$

Hence

$$\cos(180^\circ - H) = +0{\cdot}3508,$$

from which we obtain

$$180^\circ - H = 69^\circ\ 28',$$

so that

$$H = 110^\circ\ 32'$$
$$\equiv 105^\circ + 5^\circ\ 30' + 2',$$

that is,

$$H = 7^h\ 22^m\ 8^s.$$

32. *Hour angle at sunset.*

In Fig. 22, drawn for an observer in north latitude, let UXV be the parallel of declination of the sun supposed to be of south declination. The sun reaches the celestial horizon at X when it sets. In the spherical triangle PZX we have $ZX = 90^\circ$, a circumstance which simplifies the formulae.

To find H we have by the fundamental formula

$$\cos ZX = \cos PZ \cos PX + \sin PZ \sin PX \cos \widehat{ZPX},$$

and since $\cos ZX = 0$, and $\widehat{ZPX} = H$, we obtain

$$\cos H = -\cot PZ \cot PX. \quad \text{..................(24)}$$

It is easily deduced from the diagram that for north latitude and south declination the hour angle at sunset is less than 6^h, and from a similar diagram representing the eastern hemisphere the sun's hour angle is greater than 18^h at sunrise. Hence the sun is above the horizon for less than 12^h.

Similarly it can be deduced that for north latitude and north declination the sun is above the horizon for more than 12^h.

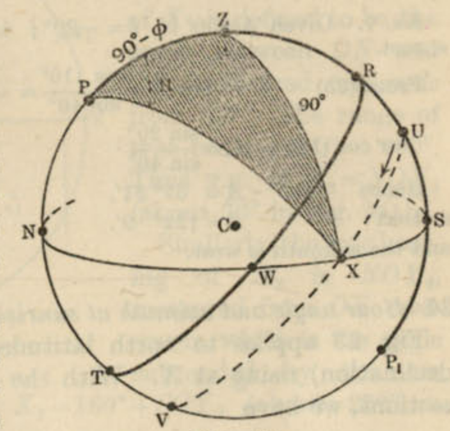

FIG. 22.

Corresponding results can be readily obtained for south latitude.

Ex. 6. Given $\phi = 50°$ N, $\delta = -20°$; to find the sun's hour angle at sunset. The data apply qualitatively to Fig. 22.

We have $PZ = 40°$, $PX = 110°$.

H is calculated by (24), in which

$$\cot PZ = \cot\ 40° = \tan 50°,$$
$$\cot PX = \cot 110° = -\tan 20°.$$

Thus $\cos H = \tan 50° \tan 20°$.

Hence $H = 64°\ 17'$
$= 60° + 4°\ 15' + 2'$,
or $H = 4^h\ 17^m\ 8^s$.

$\log\tan 50° = 0{\cdot}0762$
$\log\tan 20° = \bar{1}{\cdot}5611$
$\therefore \log\cos H = \bar{1}{\cdot}6373$

33. *Azimuth at sunset.*

To find the azimuth, A, or $P\widehat{Z}X$, we have from Fig. 22 by the fundamental formula

$$\cos PX = \cos PZ \cos ZX + \sin PZ \sin ZX \cos P\widehat{Z}X.$$

Now $\cos ZX = \cos 90° = 0$ and $\sin ZX = \sin 90° = 1$; hence

$$\cos PX = \sin PZ \cos A,$$

or

$$\cos A = \frac{\cos PX}{\sin PZ}. \quad \text{.............................(25)}$$

Ex. 7. Given $\phi = 50°$ N, $\delta = -20°$; to find the sun's azimuth at sunset.

From (25) $\cos A = \frac{\cos 110°}{\sin 40°} = -\frac{\sin 20°}{\sin 40°}$,

or $\cos(180° - A) = \frac{\sin 20°}{\sin 40°}$.

Hence $180° - A = 57°\ 51'$,
so that $A = 122°\ \ 9'$.
and the azimuth is west.

$\log\sin 20° = \bar{1}{\cdot}5341$
$\log\sin 40° = \bar{1}{\cdot}8081$
$\therefore \log\cos(180° - A) = \bar{1}{\cdot}7260$

34. *Hour angle and azimuth at sunrise.*

Fig. 23 applies to north latitude with the sun (of south declination) rising at X. With the data of the previous two sections, we have

$$PZ = 40°,\ \ ZX = 90° \text{ and } PX = 110°,$$

so that the spherical triangle PZX is the same as in the examples of these sections; accordingly

$$Z\widehat{P}X = 4^h\ 17^m\ 8^s,$$
$$P\widehat{Z}X = 122°\ 9'.$$

But if H is now the hour angle at sunrise,

$$Z\widehat{P}X = 24^h - H.$$

Hence

$$H = 19^h\ 42^m\ 52^s.$$

Also, the azimuth, A, is $122°\ 9'$ and it is *east*.

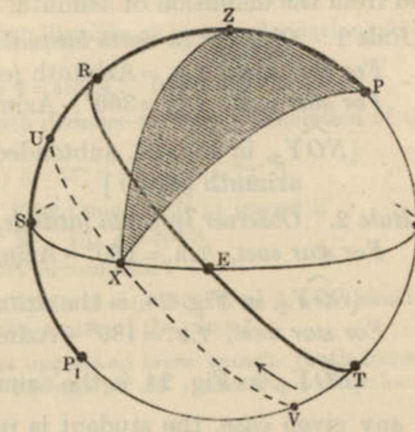

Fig. 23.

35. *True bearing.*

In Fig. 24 we represent the celestial horizon with the cardinal points N, E, S and W.

Let Y_1 and Y_2 be the points of intersection on the horizon of the verticals through two heavenly bodies X_1 and X_2.

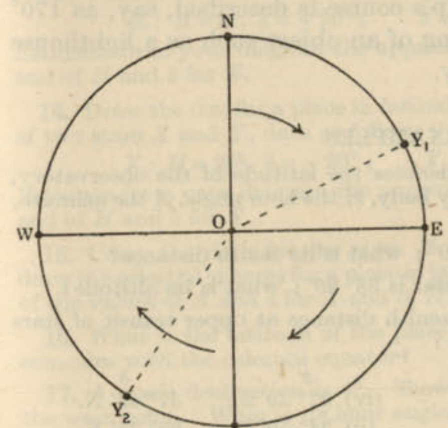

Fig. 24.

The *true bearing* (T.B.) of X_1 is defined to be the angle between ON and OY_1 measured *eastwards* from ON. The range of true bearing is 0° to 360°. Thus T.B. of $X_1 = NOY_1$ (about 70° in Fig. 24).

Similarly, the true bearing of X_2 is NOY_2, measured from ON eastwards, which is the angle subtended by the arc $NESY_2$. Thus T.B. of $X_2 = 180° + SOY_2$ (about 220° in Fig. 24).

The relation between true bearing and azimuth is summarised in the following rules, which follow from the principles stated and from the definition of azimuth.

Rule 1. *Observer in north latitude.*

For star east, T.B. = Azimuth (east).

For star west, T.B. = 360° − Azimuth (west).

[$\widehat{NOY_2}$, in Fig. 24, subtended by the arc NWY_2 is the azimuth (west)]

Rule 2. *Observer in south latitude.*

For star east, T.B. = 180° − Azimuth (east).

[$\widehat{SOY_1}$, in Fig. 24, is the azimuth (east).]

For star west, T.B. = 180° + Azimuth (west).

[$\widehat{SOY_2}$, in Fig. 24, is the azimuth (west).]

In any given case, the student is recommended to derive the relation between the true bearing and azimuth from a diagram rather than to memorise the rules given.

The method of specifying directions, with reference to the horizon, in terms of true bearing is used in graduating the dials of gyro-compasses with which most ships nowadays are provided. For example, a ship's course is described, say, as 170° or 325°, and the true bearing of an object such as a lighthouse is specified in a similar way.

EXAMPLES

(In the examples below, ϕ denotes the latitude of the observatory, δ the declination of the heavenly body, H the hour angle, A the azimuth, and z the zenith distance.)

1. The altitude of a star is 30°; what is its zenith distance?

2. The zenith distance of a star is 38° 20′; what is its altitude?

3. Find, for latitude ϕ, the zenith distance at upper transit of stars of given declination, as follows:

	ϕ	δ		ϕ	δ
(i)	30° N;	20° N.	(iv)	22° 39′ S;	37° 42′ N.
(ii)	50° N;	20° S.	(v)	34° 50′ N;	62° 37′ N.
(iii)	38° 25′ S;	35° 14′ S.	(vi)	45° 28′ S;	21° 14′ N.

4. A and B are two places on the same meridian. At A the zenith distance at upper transit of a star is 35° 40′ and at B the zenith distance similarly is 25° 20′. How far apart are A and B (i) in nautical miles, (ii) in statute miles? Assume that the star's transits are on the same side of the zenith.

5. If ϕ is 38° N, which of the following stars are circumpolar?
(i) δ = 22° N; (ii) δ = 22° S; (iii) δ = 55° N; (iv) δ = 35° N.

6. If ϕ is 42° N, find the zenith distance at lower culmination of the following stars:
(i) δ = 52° N; (ii) δ = 48° N; (iii) δ = 65° N.

7. If ϕ is 52° S, find the zenith distance at lower culmination of the following stars:
(i) δ = 42° S; (ii) δ = 55° S; (iii) δ = 72° S.

8. For *Sirius*, δ = 16° 38′ S. Find the altitude at transit if
(i) ϕ = 40° N; (ii) ϕ = 40° S.

At what latitude is *Sirius* just circumpolar?

9. What is the altitude of *Capella* (δ = 45° 56′ N) at lower transit if at upper transit the star is in the zenith of the place?

10. The altitudes of *Dubhe* at upper and lower transits (both transits are *north* of the zenith) are 79° 25′ and 23° 35′; find the declination of *Dubhe* and the latitude.

11. Convert the following into degrees, etc.:
(i) $13^h\ 16^m$; (ii) $5^h\ 37^m$; (iii) $14^h\ 10^m\ 48^s$; (iv) $7^h\ 35^m\ 20^s$.

12. Convert the following into hours, etc.:
(i) 73°; (ii) 143° 30′; (iii) 65° 38′; (iv) 139° 24′.

13. Draw the C.S. for a place in latitude 45° N and show the positions of two stars X and Y, data as follows:

X: $H = 3^h$, $\delta = +20°$; Y: $A = 50°$ W, $z = 60°$.

Estimate from your diagram the approximate values of A and z for X and of H and δ for Y.

14. Draw the C.S. for a place in latitude 45° N and show the positions of two stars X and Y, data as follows:

X: $H = 22^h$, $\delta = -20°$; Y: $A = 120°$ E, $z = 70°$.

Estimate from your diagram the approximate values of A and z for X and of H and δ for Y.

15. Using the data for the stars X and Y in examples 13 and 14 draw the celestial spheres for a place in latitude 45° S and make estimates of the values of A and z for X and of H and δ for Y.

16. What is the latitude of the place for which the celestial horizon coincides with the celestial equator?

17. A star's declination is 0°. Show that in all latitudes it sets at the west point. What is its hour angle then?

18.* *Calculate* the quantities which you are asked to estimate in examples 13, 14 and 15.

19.* Calculate the sun's hour angle and azimuth at sunset for a place in latitude 50° N when its declination is (i) 20° N, (ii) 15° S.

20.* Calculate the sun's hour angle and azimuth at sunrise for a place in latitude 42° S when its declination is (i) 15° N, (ii) 20° S.

CHAPTER III

RIGHT ASCENSION

36. *Introduction.*

In the previous chapter we have seen that the position of a star at any instant can be specified by its hour angle and declination. Disregarding, for the present, small changes in a star's declination due to causes which will be dealt with later, we take the declination of a star to be constant from day to day. The hour angle, however, increases at a uniform rate, which is simply the rate at which the earth rotates on its axis; expressed differently, the star's meridian moves away from the observer's meridian *westwards* at a uniform rate. Moreover, this rate is the same for all stars.

If we consider two stars with different hour angles we see that the angle between the two meridians on which the stars lie remains unaltered during the diurnal motion. If we regard one of these meridians (say, that through the first star) as a standard meridian the position of the second star can be uniquely specified, in relation to all the stars in the sky, with reference to the equator and this standard meridian in terms of declination and the angle between the two meridians. This latter angle—which we denote by α—may be regarded as analogous to the terrestrial longitude of a place with reference to the Greenwich

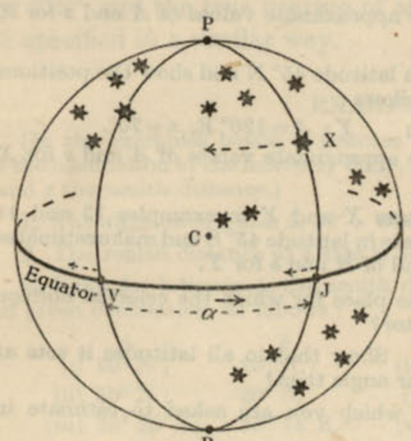

FIG. 25.

meridian as standard meridian, except in this respect, that α will be measured *eastwards* from the standard meridian from 0° to 360° or, as is more usual, from 0^h to 24^h.

In Fig. 25 we represent the celestial sphere of stars with the earth's centre, C, as centre of the sphere. Let PVP_1 be the selected standard meridian meeting the equator in V. We can regard V, if we please, as a particular star of declination zero. The meridian of a star X is $PXJP_1$. The diurnal motion carries X westwards (in the direction of the arrow near X) and it also carries V in the same direction. But the angle between the meridians through V and X remains constant, that is, VJ remains constant.

VJ, *measured eastwards* from V, is called the ***right ascension*** of the star X and, as previously stated, is denoted by α.

It remains to specify more particularly the equatorial star, or point, V from which right ascension is measured; this is done by means of the sun.

37. *The sun.*

We shall suppose, for simplicity, that an observer in north latitude ϕ measures the meridian zenith distance of the sun day by day during the year. When the sun is on the meridian, that is, when its hour angle is 0^h, it is then *apparent noon*; when the sun's hour angle is 12^h, it is *apparent midnight*.

By the method of section 27 as summarised in formula 12 we can find by observation the sun's declination, δ, on any day of the year. It is found that δ alters from day to day during the year; it is 0° on or about March 21, thereafter increasing to a maximum value of 23° 27′ N on or about June 21, then decreasing to 0° on or about September 21 and reaching its greatest southerly value—23° 27′ S—on or about December 21, from which date it decreases in numerical value until it is 0° again on or about March 21. The interval during which these cyclical changes occur is the *year*.

38. *The ecliptic.*

It is found that the sun appears to move eastwards against the background of the stars at a rate (which is not quite constant) of about 1° per day. A similar phenomenon is easily observable, without instrumental aid, in the case of the moon, which moves eastwards among the stars at the average rate of about 13° per day; we interpret this sequence of changes by saying that the moon revolves about the earth in a path, or orbit, which is deduced, by further considerations, to be nearly circular. In the same way the sun *appears* to move round the earth in an orbit which is also nearly circular.* Moreover, this *apparent orbit* of the sun, relative to the earth, lies in a plane called the *plane of the ecliptic*. Hence the sun *appears* to describe a great circle on the celestial sphere, with the earth as centre, in the course of a year; this great circle is the *ecliptic*. The changes in the sun's declination which we have mentioned in the preceding section show that the ecliptic must be inclined at an angle to the equator.

In Fig. 26 we have the equator with its pole, P. The ecliptic is shown with its pole at K. The equator and the ecliptic intersect in the two points designated ♈ and ≏. The yearly apparent motion of the sun, *against the background of the stars*, being easterly, is in the direction B♈A≏B. It is easily seen that when the sun is on the semi-circle ♈A≏ its declination is north; on the semi-circle ≏B♈, its declination is south.

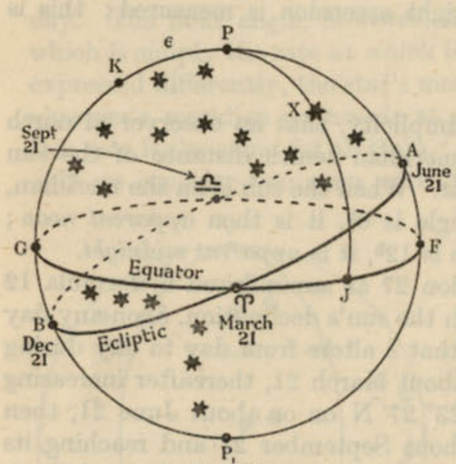

Fig. 26.

The point ♈ at which the sun's declination changes from south to north is called the *Vernal Equinox*, or the *First Point of Aries*, so called because, when this nomenclature was first adopted about 2,000 years ago, this point of intersection of the ecliptic with the equator lay in the constellation of *Aries* (the Ram). The sun is at ♈ on or about March 21. At A, on the ecliptic

* For a more precise statement, see section 45. Also, the motions of the moon and sun, relative to the earth, are elaborated in greater detail in section 68.

and 90° from ♈, the sun's declination is evidently a maximum; thus A corresponds to June 21. Similarly, ≏ and B correspond to September 21 and December 21 respectively.

≏ is called the *First Point of Libra*, or the *Autumnal Equinox*; A and B are respectively the *summer solstice* and the *winter solstice*.

39. *Right ascension.*

In defining the Right Ascension of a star such as X (Fig. 26) *we use as reference point the First Point of Aries* (♈)—this is the precise definition of V in Fig. 25. Thus for the star X, the right ascension α is the arc ♈J *measured eastwards from* ♈—that is, in the direction ♈JF.

For a given star the right ascension, α, and the declination, δ, are constant (we ignore at present certain small changes in these coordinates). Thus a star is definitely specified if α and δ are known. These coordinates are given, for the brightest stars, in the *Nautical Almanac* and for other stars in star-catalogues.

When the sun is at ♈, its right ascension is zero and it will be easily seen from Fig. 26 that as the sun moves along the ecliptic (for example, from ♈ to A) its right ascension increases. This is the reason for measuring right ascension in the eastward direction.

The inclination of the ecliptic to the equator, that is, the spherical angle A♈F is called the *obliquity of the ecliptic*, which we denote by ϵ. Since ♈ is the pole of the great circle PAF, the obliquity is also given by the arc FA and also by KP. But FA is the sun's maximum declination during the year; hence the value of the obliquity is 23° 27′.

40. *Sidereal time.*

If we think of ♈ as being identified with a star, we see that ♈ shares in the diurnal motion like any other star. Consider Fig. 27 for an observer in north latitude. The First Point of Aries is a point (or a star) lying in the equator and at a given instant we suppose its position to be as shown in the figure. The dotted great circle is the ecliptic at this instant. The hour angle of ♈ is the arc RW♈ measured westwards from the observer's meridian. The hour angle of ♈ at the given instant and for the given observer is called the *sidereal time* for the observer's meridian, or the *local sidereal time*.

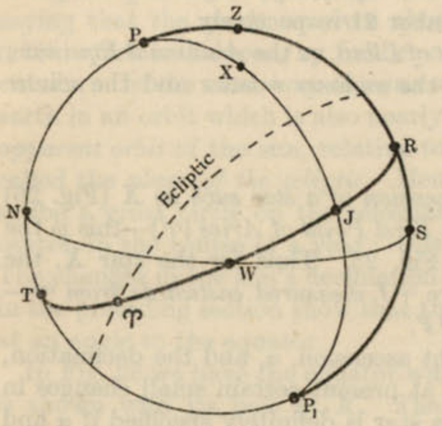

FIG. 27.

When ♈ is at R (on the observer's meridian), the hour angle of ♈ is 0^h and consequently the local sidereal time at this instant is 0^h. The interval between two consecutive passages of ♈ across the observer's meridian—due to the diurnal motion—is called a *sidereal day*, which is subdivided into sidereal hours, minutes and seconds as follows:

1 sidereal day = 24 sidereal hours.
1 sidereal hour = 60 sidereal minutes.
1 sidereal minute = 60 sidereal seconds.

A sidereal day measures the period of rotation of the earth about its axis.

A *sidereal clock* is a clock keeping the sidereal time for the meridian of the observatory; if the clock is correctly set, its dial—which is graduated from 0^h to 24^h—shows 0^h when ♈ is on the meridian. We shall later show how the clock is "set" in accordance with this principle.

41. *An important formula.*

In Fig. 27 let PXP_1 be the meridian through a star X, at a given instant, meeting the equator in J. We have:

$$\text{Hour angle of star} \equiv \text{H.A.}X. = \overrightarrow{RJ}.$$

$$\text{Right ascension of star} \equiv \text{R.A.}X. = \overrightarrow{♈J}.$$

$$\text{Local sidereal time} \equiv \text{H.A. of } ♈ = \overrightarrow{R♈}. \quad \text{............(1)}$$

But $RJ + ♈J = R♈$. Hence

$$\text{H.A.}X. + \text{R.A.}X. = \text{local sidereal time.} \quad \text{........(2)}$$

This is a formula of fundamental importance.

42. *Setting of an equatorial telescope.*

This instrument can be set on a given star provided its hour angle and declination at a given instant are known. Assuming that the observatory's sidereal clock keeps the correct sidereal time appropriate to the observatory's meridian, we can easily calculate the hour angle of the star by means of (2), the right ascension of the star being taken from the *Nautical Almanac* or a star-catalogue. The declination is also taken from the same source. The telescope can rotate about the polar axis of the instrument which is set parallel to the earth's axis of rotation; a scale enables the correct hour angle to be set and a clock-work moves the telescope in the same direction and at the same rate as the diurnal motion. The telescope can also rotate about its "declination axis" and a scale enables the correct declination to be set. The star ought now to be in the field of view and, with the clock-work in action, it remains in the field of view for as long as the star is above the horizon. For a more detailed account of the equatorial telescope, see Chapter XV.

Ex. 1. $\alpha = 3^h\ 45^m\ 38^s$ and local sidereal time $= 14^h\ 32^m\ 41^s$; to find the star's hour angle.

Local sidereal time	14^h	32^m	41^s
α	3	45	38
Hour angle	10	47	3

by subtraction, using (2).

Ex. 2. $\alpha = 21^h\ 16^m\ 58^s$ and local sidereal time $= 8^h\ 53^m\ 48^s$.

Local sidereal time	8^h	53^m	48^s
α	21	16	58
Hour angle	11	36	50

In this example α is greater than the local sidereal time; we therefore add 24^h to the latter, making it $32^h\ 53^m\ 48^s$, from which we now subtract α in accordance with (2).

43. *Setting of the sidereal clock.*

Assume that the right ascension, α, of a certain star is known. By means of the transit instrument we can determine the instant, by the clock, when the star is on the observer's meridian. At this instant the star's hour angle is 0^h; hence, by (2),

$$\text{Local sidereal time of transit} = \alpha. \quad \text{..............(3)}$$

Thus at the moment of transit the sidereal clock should show the star's right ascension. As it is mechanically impossible to construct a clock to keep time with perfect accuracy, the clock-time of transit will generally differ from the true sidereal time; the transit observation will then provide the means of determining the *clock-error*, which will be *fast* or *slow* according as the clock-time is in advance of or behind the true sidereal time.

From frequent observations, the rate at which the clock-error increases or decreases—these rates are *gaining* or *losing* rates respectively—can be easily determined.

Ex. 3. The observed times of consecutive transits of a star whose right ascension is $13^h\,51^m\,14^s{\cdot}3$ are $13^h\,50^m\,36^s{\cdot}5$ and $13^h\,50^m\,37^s{\cdot}1$; to find the rate of the clock.

At the first transit, the clock-error is

$$13^h\,50^m\,36^s{\cdot}5 - 13^h\,51^m\,14^s{\cdot}3 \text{ or } 37^s{\cdot}8 \textit{ slow}.$$

Similarly at the next transit, the error is $37^s{\cdot}2$ *slow*. In one sidereal day the clock evidently gains $0^s{\cdot}6$; its gaining rate is thus $0^s{\cdot}6$ per sidereal day.

EXAMPLES

(In the examples below, α denotes right ascension; the obliquity of the ecliptic is 23° 27′.)

1. What are the sun's right ascension and declination on (i) March 21, (ii) June 21, (iii) September 21, (iv) December 21?

2. What is the sun's hour angle at (i) sunset, (ii) sunrise on March 21? Do these hour angles depend on the latitude?

3. The sun is in the zenith of a place at meridian transit, its declination being 15° N. Find the latitude of the place.

4. What is the sun's declination if it is in the zenith of a place in latitude 12° S at meridian transit?

5. Draw the celestial sphere for a place in latitude 45° N at local sidereal time 6^h. Sketch the position of the ecliptic at this instant and show the sun's position if its R.A. is 3^h.

6. Draw the celestial sphere for a place in latitude 45° S at local sidereal time 18^h. Sketch the position of the ecliptic at this instant and show the sun's position if its R.A. is 22^h.

7. Find the latitudes at which the ecliptic is perpendicular to the horizon at some time during the day; what is then the local sidereal time?

8. Calculate the hour angles of certain stars from the following data:

	α	Local sidereal time
(i)	$3^h\,21^m\,35^s$	$5^h\,44^m\,55^s$
(ii)	15 55 13	7 59 50
(iii)	21 50 20	3 4 5
(iv)	8 10 34	12 19 22

9. Find the values of α for certain stars given the following:

	Hour angle	Local sidereal time
(i)	$2^h\,18^m\,59^s$	$3^h\,45^m\,17^s$
(ii)	19 41 27	17 48 39
(iii)	15 22 44	19 25 16
(iv)	7 11 12	5 22 55

10. The observed times (by a sidereal clock) of consecutive transits of a star for which $\alpha = 5^h\,12^m\,15^s{\cdot}2$ are $5^h\,11^m\,48^s{\cdot}7$ and $5^h\,11^m\,49^s{\cdot}2$. Find the error of the clock at each transit and also its rate.

CHAPTER IV

MEAN TIME

44. *Apparent solar time.*

So far as the heavenly bodies are concerned the sun is the one which has by far the greatest influence in regulating human activities; for example, our waking hours are mostly those when the sun is above the horizon and our sleeping hours when it is below the horizon. Before the invention of clocks, the passage of time was indicated by the diurnal motion of the sun as shown, for example, by the sun-dial. The year, too, is a solar unit of time based on the sun's apparent motion on the celestial sphere, relative to the earth, and measured as the interval required by the sun to make a complete circuit of the ecliptic.

The rotating earth is our natural clock, and the measurement of time is based on the diurnal motion of either a star (or the vernal equinox) or the sun. In the first case we are concerned with sidereal time—of the greatest importance in the observatory, but of little relevance in our ordinary workaday activities; in the second, we are concerned with *apparent solar time.* We say that an interval of one hour of apparent solar time corresponds to the increase of one hour in the sun's hour angle, and that one *apparent solar day* is the interval between two consecutive transits of the sun over a given meridian. Suppose we have a sidereal clock keeping sidereal time accurately; it is found from transit observations of the sun that the length of the apparent solar day, as measured by the sidereal clock, varies unmistakably from day to day during the year. The sun is consequently an irregular time-keeper and unsuitable for the *uniform* measurement of time. This unsuitability is due to two causes. First, the sun *appears* to move at a non-uniform rate in the ecliptic (see Fig. 26), that is to say, the line joining

the earth's centre to the sun's centre does not sweep out equal angles in equal times. Second, the sun's apparent orbit is in the plane of the ecliptic and not in the plane of the equator; the measurement of time is the measurement of hour angle which is fundamentally related to the equator and not to the ecliptic. To avoid the irregularities of apparent solar time, a fictitious body called *the mean sun* is introduced which moves uniformly along the equator, its rate of motion being the average, throughout the year, of the sun's angular motion in the ecliptic. In defining the mean sun more specifically, we require to consider the sun's apparent orbit in some detail.

45. *The sun's apparent orbit.*

Relative to the earth, the sun appears to move around the earth in a path, called an *ellipse*, shown in Fig. 28; this is a

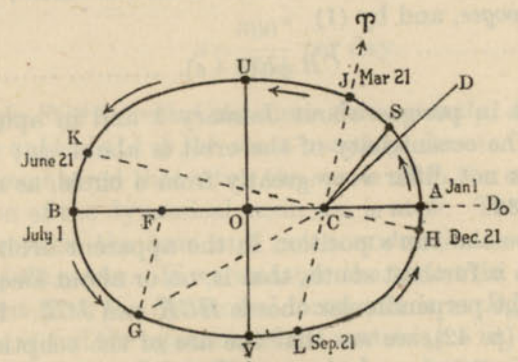

Fig. 28.

more precise statement than that required, at the time, in section 38. The curve is symmetrical about two perpendicular diameters *AOB* and *UOV*, called the *major axis* and *minor axis* respectively, *O* being the *centre* of the ellipse. *OA* or *OB* is called the *semi-major axis* (denoted by *a*) and *OU* or *OV* the *semi-minor axis*. Associated with the given ellipse are two points, *C* and *F*, on the major axis and equidistant from *O*, called the *foci*. If *G* is any point on the ellipse, the sum of the distances *CG* and *FG* is constant (equal to 2*a*) for all positions of *G* on the ellipse; this may be regarded as the fundamental property of the ellipse. The ratio of OC to OA is called the *eccentricity* of the ellipse, denoted by e. Then

$$OC = ae. \qquad \text{(1)}$$

The sun's *apparent orbit* is described with reference to the earth which is situated at one of the foci (we have assumed the earth to be at C in Fig. 28). The sun is shown at S and the direction in which it moves in its orbit (as viewed from a point on the same side of the ecliptic as the north celestial pole) is shown by the arrows. The distance CS of the sun from the earth is called the *geocentric distance* of the sun and the line CS is called a *radius vector*. When the sun is at A, the geocentric distance is a minimum and A is called *perigee*; by (1),

$$CA = a(1-e). \qquad \text{(2)}$$

When the sun is at B, it is furthest from the earth; this point is called *apogee*, and by (1)

$$CB = a(1+e). \qquad \text{(3)}$$

The sun is in perigee about January 1 and in apogee about July 1. The eccentricity of the orbit is about $\frac{1}{60}$; hence the ellipse does not differ very greatly from a circle, as mentioned in section 38.

Let H be the sun's position in the apparent orbit when its declination is furthest south, that is, on or about December 21. We draw the perpendicular chords HCK and JCL. If we refer to Fig. 26 (p. 42), we see that the arc of the ecliptic between the winter solstice and the vernal equinox is 90°; hence J corresponds to the sun's position on March 21. Similarly, K and L correspond to the sun's position on June 21 and September 21 respectively.

The sun's non-uniform angular motion in its apparent orbit may be illustrated by means of Fig. 28. Suppose that the angle SCJ is equal to the angle BCK; then the time required by the radius vector CS to move through the angle SCJ is *not* equal to the time required by the radius vector CK to move through the angle KCB; this is an example of the general rule for an elliptic orbit that equal angles are, in general, *not* described in equal times.

46. *Dynamical mean sun.*

To avoid the complications produced by the non-uniform angular motion of the sun in its apparent orbit, we introduce a fictitious body, called the *dynamical mean sun*, D, in Fig. 28, the corresponding position of the sun being at S. Although we are only concerned with the direction as given by CD, we may suppose for simplicity that the dynamical mean sun moves around C in a circle of radius a. We assume, first, that when the sun is in perigee (that is, at A), the dynamical mean sun is at D_0, on the prolongation of CA; and, second, that D moves uniformly along its circular path, completing the circuit in the same time as that required by the sun to move around the apparent orbit. Thus the radius vector, CD, moves through 360° in a year (that is, $365\frac{1}{4}$ days) at a constant rate called the *mean angular motion*; this is denoted by n. Thus, if n is measured in degrees, the unit of time being the day,

$$n = \frac{360°}{365\frac{1}{4}} \text{ per day}. \qquad \text{(4)}$$

The angle SCA is called the sun's *true anomaly*, denoted by v, and the angle DCA is the *mean anomaly*, denoted by M; it is to be remembered that when the sun is at S, the corresponding position of the dynamical mean sun is at D. The difference $(v-M)$ between the true and mean anomalies is called the *equation of the centre*. If we measure the time t (in days) from the moment that the dynamical mean sun is at D_0, the mean anomaly is nt and the equation of the centre is $v-nt$. From the known eccentricity of the ellipse, v can be calculated for time t, and so the equation of the centre can be found at any instant.

Since ♈ is the point on the ecliptic giving the direction of the sun at the vernal equinox, its direction is represented in the plane of the apparent orbit by the direction of CJ, as shown in Fig. 28. The angle ♈CS, measured from C♈ in the direction of the arrows, is the sun's *true longitude* (denoted by L) and the angle ♈CD is the sun's *mean longitude* (denoted by l). Also, the angle ♈CA is called the *longitude of perigee* (denoted by ϖ_1). In each case the longitude concerned is measured, from 0° to 360°, from C♈ in the direction of the sun's motion, as shown by the arrows.

The semi-major axis, a, the eccentricity, e, and the longitude of perigee, ϖ_1, specify the size, shape and position of the orbit, the latter with reference to the direction of the vernal equinox; they are *elements* of the apparent orbit.

47. *The mean sun.*

At a particular time on a given day let the sun's direction from the earth's centre, C, be represented by S on the celestial sphere; let D be the corresponding position of the dynamical mean sun; the direction of perigee is represented by A. We have seen that D, by definition, moves round C with uniform angular motion in the ecliptic. We define a second fictitious body, M, moving round the equator with the same angular motion as D and we suppose that, when D is at ♈, M is also at ♈. Hence, in the figure, ♈M = ♈D. This fictitious body, M, is called the *mean sun*. It describes the complete circuit of the equator, against the background of the stars, in the same time required by the dynamical mean sun, D, to describe a complete circuit of the ecliptic, that is, a year; the angular motion of M is thus n, as given in (4).

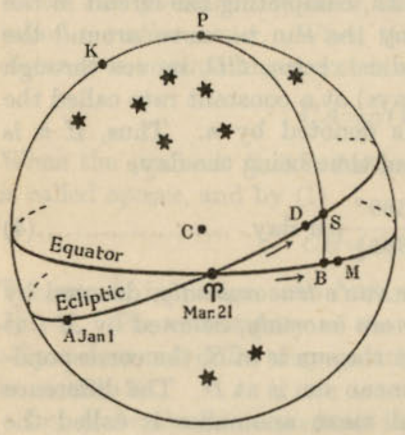

FIG. 29.

Let the meridian through S (Fig. 29) meet the equator in B. Then ♈B is the sun's right ascension, which we denote by R.A.$\odot$; also ♈M is the right ascension of the mean sun, denoted by R.A.M.S.; the difference, namely BM, is called the *equation of time* (E.T.)*, and is given algebraically by

$$\text{E.T.} = \text{R.A.M.S.} - \text{R.A.}\odot. \quad \text{......................(5)}$$

It is found that during the year the equation of time varies

* In older textbooks, E.T. is defined by

$$\text{E.T.} = \text{R.A.}\odot - \text{R.A.M.S.}$$

between $-14\frac{1}{4}^{\text{m}}$ and $+16\frac{1}{4}^{\text{m}}$ approximately. We shall later discuss the equation of time more fully.

48. *Mean time.*

Consider now the standard celestial sphere for an observer in north latitude. At a given instant we shall suppose the vernal equinox and the mean sun, M, to be situated as shown in Fig. 30. At this instant the hour angle of ♈ is R♈ (measured westwards from the observer's meridian), and this is the observer's, or the local, sidereal time. Owing to the diurnal motion, ♈ moves with uniform angular motion in the direction of increasing hour angle, that is, in the direction RWT; also, M moves, relative to ♈ and in the opposite direction, with uniform angular motion, n. Since the diurnal angular motion is very much greater than n, the meridian of M will move westwards from the observer's meridian with a *uniform* angular motion somewhat less than the diurnal motion. The mean sun, M, thus fulfils the necessary condition for the uniformity of time-measurement.

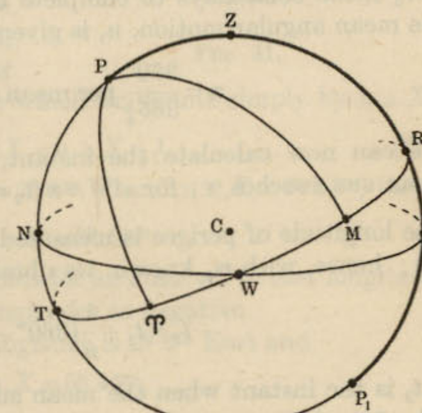

FIG. 30.

The interval between two consecutive transits of the mean sun, M, over a given meridian is called a *mean solar day*, divided into 24 hours of mean solar time with further subdivisions into minutes and seconds.

The hour angle of the mean sun, H.A.M.S., measures *mean solar time* (M.S.T.) or, simply, *mean time*. When the H.A.M.S. is 0^{h}, it is *mean noon*, and when the H.A.M.S. is 12^{h}, it is *mean midnight*.

From Fig. 30, RM + ♈M = R♈. Hence

$$\text{H.A.M.S.} + \text{R.A.M.S.} = \text{Local sidereal time.} \quad \text{..........(6)}$$

Also, if X denotes a star, we have the formula (2), p. 44 :

$$\text{H.A.}X+\text{R.A.}X=\text{Local sidereal time.} \quad\text{.............(7)}$$

Hence, from (6) and (7)

$$\text{H.A.}X+\text{R.A.}X=\text{H.A.M.S.}+\text{R.A.M.S.} \quad\text{.............(8)}$$

49. *Calculation of R.A.M.S.*

We shall measure intervals of time in terms of the mean solar day as unit. It is assumed that we know the instant, t_1, when the sun is in perigee (A in Fig. 29), and also the value of the longitude, ϖ_1, of perigee. We assume that the sun requires $365\frac{1}{4}$ mean solar days to complete the circuit of the ecliptic ; the mean angular motion, n, is given by

$$n=\frac{360°}{365\frac{1}{4}} \text{ per mean solar day.} \quad\text{.................(9)}$$

We can now calculate the instant, t_2, when the dynamical mean sun reaches ♈, for $A♈=n(t_2-t_1)$; also $A♈=360°-\varpi_1$ (the longitude of perigee is measured in the sense $♈\overrightarrow{DS}$ in Fig. 29) ; hence, with ϖ_1 known, we obtain t_2 from

$$t_2=t_1+\frac{1}{n}(360°-\varpi_1).$$

If t_3 is the instant when the mean sun reaches the position M (Fig. 29), then

$$♈M=n(t_3-t_2),$$

that is,

$$\text{R.A.M.S.}=n(t_3-t_2).$$

Hence, for a given t_3, we can calculate the corresponding value of R.A.M.S.

50. *The relation between the hour angle of a heavenly body for the Greenwich meridian and any other meridian.*

In Fig. 31 we represent the earth with centre C and polar axis pCp_1. Let g denote the position of Greenwich and h the position of an observer in *west* longitude, λ. We draw also the standard celestial sphere with C as centre. The plane of the terrestrial meridian pgp_1 cuts the celestial sphere in the celestial meridian PGU and the plane of the terrestrial meridian php_1

cuts the celestial sphere in the celestial meridian PHV. It is evident that the spherical angles GPH and gph are equal. But $\widehat{gph}$ is the west longitude, λ; hence $\widehat{GPH}=\lambda$.

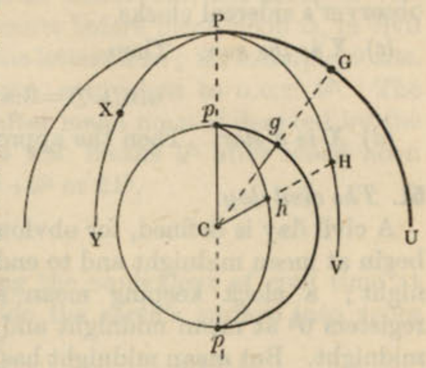

FIG. 31.

Now consider a heavenly body X with PXY as its celestial meridian. The spherical angle GPX is the hour angle of X for the Greenwich meridian ; we denote it by G.H.A.X. Also, the angle HPX is the hour angle of X for the observer which we denote simply by H.A.X. We then have

$$\text{G.H.A.}X=\text{H.A.}X+\lambda. \quad\text{.......................(10)}$$

For example, if $\lambda=32°\text{ W}\equiv 2^h\,8^m\text{ W}$ and H.A.$X=6^h\,45^m$, then, by (10),

$$\text{G.H.A.}X=8^h\,53^m.$$

It is easily seen that (10) holds for an observer in east longitude provided we regard east longitudes as negative.

Thus, if the observer's longitude is $2^h\,8^m$ East and

$$\text{H.A.}X=6^h\,45^m,$$

we obtain

$$\text{G.H.A.}X=6^h\,45^m-2^h\,8^m=4^h\,37^m.$$

We regard (10) as a general formula, λ *to be taken positive if the longitude is west, and negative if the longitude is east.*

Formula (10) is true whatever the heavenly body may be. We consider the following cases.

(*a*) *X is the mean sun.* Let G.H.A.M.S. denote the hour angle of the mean sun for the Greenwich meridian. Then

$$\text{G.H.A.M.S.}=\text{H.A.M.S.}+\lambda. \quad\text{....................(11)}$$

(*b*) *X is the vernal equinox.* Then

$$\text{G.H.A.}♈=\text{H.A.}♈+\lambda.$$

But G.H.A.♈ is the sidereal time at Greenwich and H.A.♈ is the local sidereal time ; hence

$$\text{Greenwich sidereal time}=\text{Local sidereal time}+\lambda. \quad\text{...(12)}$$

This formula gives the relation between the Greenwich and observer's sidereal clocks.

(*c*) *X is the sun.* Then

$$\text{G.H.A.}\odot = \text{H.A.}\odot + \lambda. \quad \text{......................(13)}$$

(*d*) *X is a star.* Then the appropriate formula is (10).

51. *The civil day.*

A civil day is defined, for obvious reasons of convenience, to begin at mean midnight and to end at the following mean midnight; a clock keeping mean solar time accurately thus registers 0^h at mean midnight and 24^h or 0^h at the next mean midnight. But mean midnight has been defined as the moment when H.A.M.S. is 12^h, and this can only mean the H.A.M.S. for a given observer. It would obviously be impracticable for every owner of a watch to carry about with him the mean time appropriate to the particular longitude in which he found himself, and so the mean time of a country such as Great Britain is legally defined to be the mean time of a standard meridian —in this case, the meridian of Greenwich. Thus, before the introduction of Summer Time (see p. 62) any mean time clock in Great Britain keeps, or tries to keep, the mean time appropriate to the meridian of Greenwich. This mean time is sometimes called *Greenwich Civil Time* (G.C.T.), sometimes *Greenwich Mean Time* * (G.M.T.), and sometimes *Universal Time* (U.T.) —the last from the fact that the Greenwich meridian is internationally regarded as the standard meridian for the earth.

Since the G.H.A.M.S. is 12^h at mean midnight and the Greenwich mean time clock then registers 0^h, we obtain the following relation between G.H.A.M.S. and G.C.T. (or its equivalents):

$$\text{G.H.A.M.S.} = \text{G.C.T.} \pm 12^h. \quad \text{....................(14)}$$

In this formula we take whichever sign is more convenient.

Ex. 1. Given G.H.A.M.S. $= 14^h$.
Then G.C.T. $= 14^h - 12^h = 2^h$.

Ex. 2. Given G.H.A.M.S. $= 5^h$.
Then G.C.T. $= 5^h + 12^h = 17^h$.

* Only since 1925 in this significance.

In particular, at mean noon at Greenwich, the G.C.T. is 12^h. The time of an event that occurs before mean noon is, in civil usage, generally denoted by the letters A.M.; for example, 9 A.M. is, for the Greenwich meridian, equivalent to G.C.T. 9^h. The time of an event occurring after mean noon is denoted by the letters P.M.; for example, 9 P.M. means 9^h after mean noon and is the same as G.C.T. $12^h + 9^h$ or 21^h.

52. *Time zones.*

For the purpose of keeping the equivalent of civil time at sea, it is convenient to divide the earth's surface into zones bounded by meridians of longitude at intervals of 15° (or 1^h) apart; within a zone the mean time appropriate to its central meridian is kept. For example, the zone between the meridians of $7\frac{1}{2}$° E and $7\frac{1}{2}$° W (or, in time-measure, $0^h\ 30^m$ E and $0^h\ 30^m$ W) is the Greenwich Zone (designated Zone 0) of which the Greenwich meridian is the central meridian. This zone is shown in Fig. 32, the boundaries being the meridians PAP_1 and PBP_1. The next zone to the westward of the Greenwich Zone lies between the meridians of $0^h\ 30^m$ W and $1^h\ 30^m$ W; its central meridian is 1^h W and the zone is designated Zone +1. Similarly, if PO_1P_1 in Fig. 32 is the meridian of 5^h W, the zone between the meridians PCP_1 and PDP_1 ($4^h\ 30^m$ W and $5^h\ 30^m$ W respectively) is designated Zone +5. In the same way zones to the eastward of the Greenwich Zone are centred at 1^h E, 2^h E, etc., and these zones are designated Zone −1, Zone −2, etc. The only exception to this hourly division into zones concerns the meridian of 12^h E. Now, this meridian is the same as that of 12^h W and, according to our definition so far, Zone −12 is identical with Zone +12. To prevent confusion, the *half* of

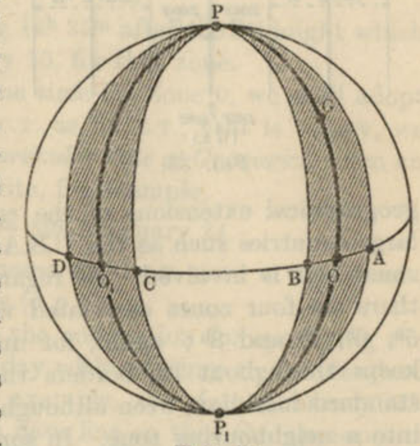

Fig. 32.

the hourly zone lying in the eastern hemisphere—that is, between the meridians $11^h\ 30^m$ E and 12^h E—is called Zone -12 and the half in the western hemisphere—between the meridians of $11^h\ 30^m$ W and 12^h W—is called Zone $+12$. This is illustrated in Fig. 33.

The meridian of 12^h E is called the *date line* * (Fig. 33); it is the meridian where a given day, say February 14, first begins.

A similar procedure, in principle, applies to the keeping of time within the boundaries of a country. For example, Germany, Austria, Italy, etc., keep Mid-European Time which is associated with the meridian of 1^h E; the boundaries of the zone, however, need not be the meridians of $\frac{1}{2}^h$ E and $1\frac{1}{2}^h$ E, but are governed by the geographical extensions of the country concerned. In very large countries such as the U.S.A. and Siberia more than one zonal belt is involved. As regards the U.S.A., for example, there are four zones associated with the meridians (west) of 5^h, 6^h, 7^h and 8^h; again, for uniformity, a particular state keeps throughout its borders the time associated with the standard meridian, even although part of its territory extends into a neighbouring zone. In some countries and islands the longitude of the standard meridian is not chosen to be an exact number of hours east or west; for example, the civil time of the Federated Malay States is based on the meridian of $7^h\ 20^m$ E; the civil time for India (except Calcutta) corresponds to the meridian of $5^h\ 30^m$ E, while Calcutta time is associated with the meridian $5^h\ 53^m\ 20^s{\cdot}8$ E.

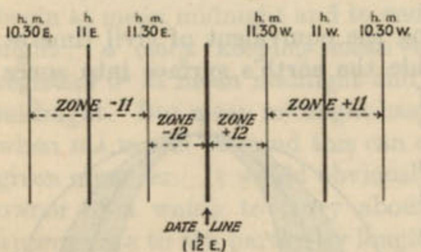

FIG. 33.

The mean time of a country associated with a particular meridian, as in the examples mentioned above, is generally designated *Standard Time*; however, in this book, we shall refer to any standard time, at sea or on land, as simply *Zone*

* The actual date-line is slightly different from the meridian of 12^h E where the latter passes through the eastern end of Siberia and certain groups of islands forming geographical or political units.

Time and the Zone will be designated by the longitude of its central or standard meridian.

53. *Zone time.*

The civil day in any zone or country begins at the mean midnight associated with the central or standard meridian and ends at the following mean midnight; thus, the H.A.M.S. for the central or standard meridian at mean midnight is 12^h.

We use the term *Zone Time* (Z.T.) to denote the civil mean time, appropriate to the particular zone concerned, together with the day of the year. Thus we write

Z.T. $14^h\ 35^m$ January 15

to denote an event occurring $14^h\ 35^m$ after the midnight which begins the civil day, January 15, for that zone.

As G.C.T. is simply the zone time for Zone 0, we shall adopt the same convention for G.C.T. as for Z.T., that is to say, we shall associate it with the *particular date at Greenwich* when an event occurred; thus we write, for example,

G.C.T. $19^h\ 28^m\ 52^s$ February 24.

It is convenient to have a name for G.C.T. used in this sense; we call it the *Greenwich Date* (G.D.).

The question arises as to the convention for specifying the beginning of any particular day with reference to the standard time zones. Consider, for example, the date February 22. This day first begins at the date line so that the appropriate Z.T. (Zone -12) is 0^h February 22; at this instant the H.A.M.S. for the meridian 12^h E is 12^h, and hence, by (11), the corresponding G.H.A.M.S. is 0^h; the civil day, February 22, at Greenwich begins 12 hours later—that is, when the G.H.A.M.S. has increased to 12^h. This is a particular example of a general principle which we investigate in detail in the next section.

54. *Relation between zone time* (*with date*) *and G.C.T.*

Let PAP_1 be the date line (longitude -12^h E) and PGP_1 the Greenwich meridian. Suppose that a civil day—say, February 14—is beginning at the date line. At this instant, the H.A.M.S. for the date line is 12^h and, as the mean sun moves in the celestial equator, the line joining the earth's centre to the mean sun cuts the earth's equator at M_1, where $AM_1 = 12^h$; M_1 is thus on the Greenwich meridian. For convenience we refer to M_1 as the position of the mean sun at the beginning of February 14 for the date line.

Let PBP_1 be a meridian with east longitude. The civil day, February 14, will not begin for the meridian PBP_1 until the mean sun has moved from M_1 to M_2, where $M_1M_2 = AB$ and the arc BM_1M_2 is 12^h. Thus, with the mean sun at M_2 the Zone Time (with date) for the meridian PBP_1 is 0^h, February 14.

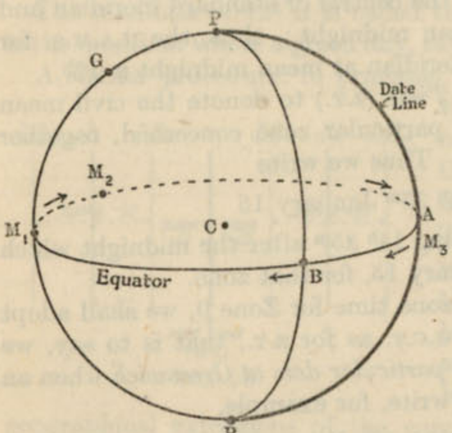

FIG. 34.

In the same way, the civil day, February 14, at Greenwich will not begin until the mean sun has moved from M_1 to M_3, M_3 being coincident with A. Thus, with the mean sun at M_3, the G.C.T. (with date) is 0^h, February 14. The interval between Zone Time 0^h, February 14, for the meridian PBP_1 and G.C.T. 0^h, February 14, is the interval required by the mean sun to move from M_2 to M_3. But the arc M_2M_3 is the same as the arc M_1B, that is, the easterly longitude of the meridian PBP_1 which we shall suppose to be 9^h E; thus the interval concerned is 9^h. Hence at G.C.T. 0^h, February 14, the Zone Time (with date) for the meridian PBP_1 is 9^h, February 14.

After a further interval of, say, 4^h the G.C.T. becomes 4^h, February 14, and the Zone Time for PBP_1 becomes 13^h, February 14.

It is evident that the relation between G.C.T. and the Zone Time for PBP_1 (each with the appropriate date attached) is given by

$$\text{G.C.T.} = \text{Z.T.} + \lambda, \quad \text{......................(15)}$$

where λ is the easterly longitude of PBP_1, an *east* longitude being taken as *negative* as in the formulae (10) to (13).

In the same way, the day February 14 begins in a place of longitude 2^h W, say, 2 hours after the beginning of the civil day February 14 at Greenwich. Thus, for Zone +2, Z.T. 0^h, February 14 corresponds to G.C.T. 2^h, February 14, and Z.T. 5^h, February 14, will therefore correspond to G.C.T. 7^h, February 14. Again, the relation between Zone Time and G.C.T. (each with date) is evidently given by (15), in which λ is taken *positive* for zones in *west* longitude.

55. *The Greenwich date.*

The official British astronomical publication is the *Nautical Almanac*, in which certain information concerning the sun, moon, planets and a selection of stars is given for G.C.T. 0^h on each day throughout the year; in some instances, the information is tabulated at intervals of one hour, or of two hours, for each civil day at Greenwich.* For example, if the moon has been observed by a navigator in the Indian Ocean at an instant corresponding to G.C.T. $10^h\ 45^m$, January 24, and if the moon's declination is required for the reduction of the observation, the appropriate value of the declination can then be obtained by interpolation between the entries in the almanac for G.C.T. 10^h and G.C.T. 11^h for the day January 24. It is thus of the utmost importance that the observer should know the Greenwich Date at which an observation was made; it is assumed that he knows the corresponding Zone Time with date. In the following examples, 3 to 5, it is required to find the Greenwich Date, given the Zone Time (with date) and the Zone.

Ex. 3. Zone −4; Z.T. $8^h\ 44^m$ May 22.

We arrange the work as follows, applying formula (15).

	Z.T.	$8^h\ 44^m$ May 22
	Zone	−4
	G.D. ≡ G.C.T.	$4^h\ 44^m$ May 22

Ex. 4. Zone +11; Z.T. $22^h\ 35^m$ July 10.

	Z.T.	$22^h\ 35^m$ July 10
	Zone	+11
	G.D.	$33^h\ 35^m$ July 10
or	G.D. ≡ G.C.T.	$9^h\ 35^m$ July 11

* The *Nautical Almanac* uses "G.M.T." instead of "G.C.T."; the *American Nautical Almanac* uses "U.T. or G.C.T."

Ex. 5. Zone − 8 ; Z.T. $5^h\ 23^m$ June 28.

Z.T. $5^h\ 23^m$ June 28 ≡	$29^h\ 23^m$ June 27
Zone - - -	− 8
G.D. ≡ G.C.T.	$21^h\ 23^m$ June 27

Ex. 6. A ship in Zone + 12 is on the point of crossing the date line at Z.T. 4^h, February 20 (according to the time-reckoning for Zone + 12). What is the Zone Time (with date) after the ship has crossed the date line?

We use the Greenwich Date as an intermediary.

Before crossing :	Z.T.	4^h February 20
	Zone	+ 12
	G.D.	16^h February 20, using (15).
After crossing :	G.D.	16^h February 20
	Zone	− 12
For Zone − 12,	Z.T.	28^h February 20, using (15)
or	Z.T.	4^h February 21.

By crossing the date line from west to east the ship has thus passed instantaneously from February 20 to February 21.

In the same way it is seen that for a ship crossing the date line from east to west, the new date will be, say, April 10 if the date before crossing is April 11.

56. *Summer time.*

In many countries mean time clocks are advanced, in normal years, by one hour on a date in April fixed by statute, and restored to the normal system of time-keeping on some date in October usually (this applies to countries in the northern hemisphere ; an analogous principle applies to countries in the southern hemisphere). For example, British Summer Time (B.S.T.) is one hour ahead of G.C.T. so that

$$\text{G.C.T.} = \text{B.S.T.} - 1^h.$$

If we compare this with (15) we see that B.S.T. is simply the zone time for the Zone − 1 ; in other words, B.S.T. is the mean time appropriate to the meridian of 1^h East.

As a war measure in Great Britain, B.S.T. was, in 1940, extended to apply to the whole of the year. Also, in 1941 the clocks were advanced an additional hour between May 3 and August 9, the clocks thus keeping the time of Zone − 2.

57. *Greenwich mean time chronometers.*

These are accurate instruments keeping G.C.T. and used for noting the exact time of an observation made, for example, on board ship or in field-survey work. In practice no chronometer can be expected to keep mean time perfectly, and the error between G.C.T. and the time shown by the chronometer is obtained by means of the daily Wireless Time-Signals. The errors are designated " fast on G.C.T." or " slow on G.C.T. ". When the chronometer error has been applied, it is then necessary to attach the correct civil day at Greenwich to the corrected G.C.T. so that the particular elements required, such as the sun's declination, may be obtained from the *Nautical Almanac*.

The correct Greenwich Date is found easily if the *approximate* Z.T. of the observation is noted.

Ex. 7. The chronometer time of an observation made by a navigator in Zone − 10 was $22^h\ 13^m\ 4^s$; the approximate Z.T. was $8^h\ 10^m$ April 20 ; chronometer error, $1^m\ 23^s$ fast on G.C.T. To find the correct G.D.

We first find the approximate G.D. as follows :

	Approximate Z.T.	$8^h\ 10^m$ April 20	
	Zone - - -	− 10	
	Approximate G.D.	$22^h\ 10^m$ April 19.	...............(*a*)
Then,	chronometer time -	$22^h\ 13^m\ \ 4^s$	
	Error (fast) - -	− 1 23	
	Correct G.D. ≡ G.C.T.	$22^h\ 11^m\ 41^s$ April 19.	............(*b*)

The approximate G.D. in line (*a*) contains the correct *date* ; *this date* (*April* 19) *must be attached to the* G.C.T. *in line* (*b*).

If the sun's declination for the time of the observation is subsequently required, it is obtained from the *Nautical Almanac* for the correct G.D. in line (*b*).

In this example it is assumed that the chronometer has a 24-hour dial. If, however, the chronometer dial is graduated from 0^h to 12^h only, as in ordinary watches, a reading such as $3^h\ 10^m$ may mean either G.C.T. $3^h\ 10^m$ *or* G.C.T. $15^h\ 10^m$ (omitting the chronometer error) ; in other words, we are uncertain as to whether we have to add 12^h to the chronometer reading or not. This difficulty is resolved by remembering that in line (*a*) —we refer to the example worked out above—we have the approximate G.C.T. without ambiguity and hence line (*b*) must be made to correspond. Thus if, in the previous example, the chronometer time given by a 12^h chronometer is $10^h\,13^m\,4^s$, we have to add 12^h to this reading to make line (*b*) agree approximately with line (*a*).

58. *H.A.M.S. for an observer in a given longitude.*

We suppose that an observer, say at sea, notes the time of an bservation by means of a Greenwich Mean Time chronometer and that he also notes the approximate Z.T. By the procedure illustrated in the preceding section, he obtains the correct G.D. at which his observation was made. Now G.C.T. and G.H.A.M.S. differ by 12^h, so that, by (14),

$$\text{G.C.T.} = \text{G.H.A.M.S.} \pm 12^h. \qquad (16)$$

This enables the G.H.A.M.S., corresponding to the time of the observation, to be found.

If λ is the observer's longitude, supposed known, we obtain the H.A.M.S. for the observer by means of (11), namely,

$$\text{G.H.A.M.S.} = \text{H.A.M.S.} + \lambda, \qquad (17)$$

λ being positive for west longitudes and negative for east longitudes.

Ex. 8. An observer in longitude 56° 30′ W. keeping the mean time of Zone +4, made an observation at approximate Z.T. $21^h\,30^m$ May 8; chronometer time, $1^h\,28^m\,56^s$; error of chronometer $0^m\,15^s$ slow on G.C.T. To find H.A.M.S. for the observer.

	Approx. Z.T. -	$21^h\,30^m$ May 8
	Zone - - -	+ 4
	Approx. G.D. -	$25^h\,30^m$ May 8
	that is,	$1^h\,30^m$ May 9
	Chronometer time	$1^h\,28^m\,56^s$
	Error (slow) -	+ 15
	G.D. ≡ G.C.T. -	$1^h\,29^m\,11^s$ May 9
Hence		
	G.H.A.M.S. is	$13^h\,29^m\,11^s$ (see Note 1)
	Long. (W)	$3^h\,46^m\,0^s$ (see Note 2)
	H.A.M.S. -	$9^h\,43^m\,11^s$ (see Note 3)

Note 1. From (16), by adding 12^h to G.C.T.
Note 2. In time measure, $56°\,30' \equiv 45° + 11\tfrac{1}{2}° = 3^h\,46^m$.
Note 3. From (17); λ is $+3^h\,46^m$, since the longitude is west.

59. *Equation of time.*

In section 47, formula (5), we defined the equation of time as follows:

$$\text{E.T.} = \text{R.A.M.S.} - \text{R.A.}\odot.$$

Since $\quad \text{H.A.M.S.} + \text{R.A.M.S.} = \text{Local sidereal time}$

and $\quad \text{H.A.}\odot + \text{R.A.}\odot = \text{Local sidereal time},$

we have, by subtraction,

$$\text{R.A.M.S.} - \text{R.A.}\odot = \text{H.A.}\odot - \text{H.A.M.S.},$$

whence
$$\text{E.T.} = \text{H.A.}\odot - \text{H.A.M.S.}. \qquad (18)$$

Fig. 35 shows the observer's celestial sphere with the sun, $\odot$, and mean sun, M, at a given instant. The meridian through the sun meets the equator at Y. Now

$$RY = \text{H.A.}\odot,$$

and
$$RM = \text{H.A.M.S.}$$

Hence, by (18), MY is the equation of time. Also, by (18), E.T. is *positive* if H.A.$\odot$ is greater than H.A.M.S. and *negative* if H.A.$\odot$ is less than H.A.M.S.

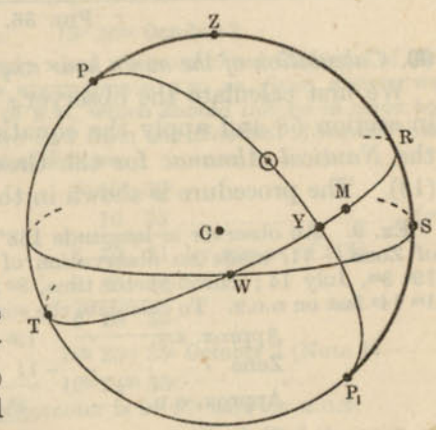

FIG. 35.

The values of E.T. throughout the year are tabulated in the *Nautical Almanac*, and the particular value to be used is that which corresponds to the Greenwich Date of the observation.

Fig. 36 shows how the equation of time varies throughout the year. It will be noticed that the equation of time vanishes four times during the year—on or about April 16, June 14, September 1 and December 25.

The values tabulated in the *Nautical Almanac* are obtained from (18) by calculation.

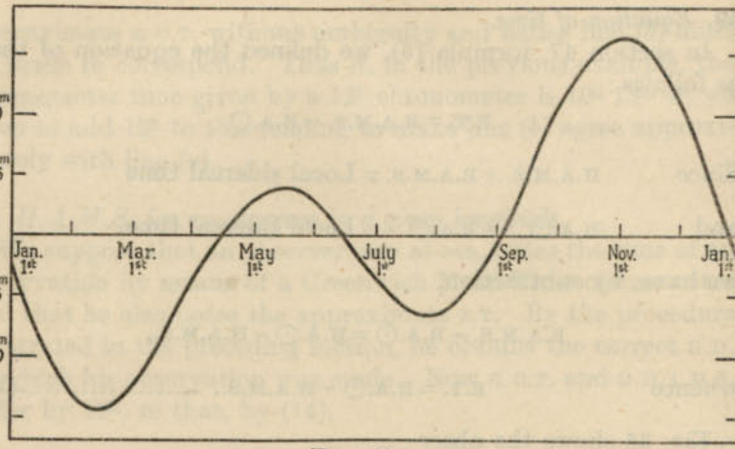

FIG. 36.

60. *Calculation of the sun's hour angle.*

We first calculate the observer's H.A.M.S. as in the example in section 58 and apply the equation of time, extracted from the *Nautical Almanac* for the Greenwich Date, by means of (18). The procedure is shown in the following example.

Ex. 9. An observer in longitude 168° 15′ E, keeping the mean time of Zone −11, made an observation of the sun at approximate Z.T. 19^h 3^m, July 14; chronometer time, 8^h 2^m 49^s; error of chronometer, 1^m 14^s fast on G.C.T. To calculate the sun's hour angle for the observer.

	Approx. Z.T. -	19^h 3^m July 14	
	Zone - - -	−11	
	Approx. G.D. -	8^h 3^m July 14	
	Chronometer time	8^h 2^m 49^s	
	Error (fast) -	−1 14	
	G.D. ≡ G.C.T. -	8^h 1^m 35^s July 14 (see Note 1)	
Hence,	G.H.A.M.S. is	20^h 1^m 35^s	
	Long. (E) - -	11 13 0	
	H.A.M.S. - -	7 14 35 (see Note 2)	
	E.T. - - -	−4 47	
	H.A.⊙ - - -	7^h 9^m 48^s	

Note 1. With this G.D. we find from the *Nautical Almanac* that E.T. is -4^m 47^s.

Note 2. By (11); the addition of G.H.A.M.S. and Long. (E) gives 31^h 14^m 35^s; as hour angle is reckoned from 0^h to 24^h we subtract 24^h and obtain the result given.

Note 3. In the *Nautical Almanac, abridged for the use of seamen*, a quantity E, equal to 12^h + E.T., is tabulated at two-hourly intervals. The H.A.⊙ is then found by (i) adding E to the G.C.T. (this gives G.H.A.⊙) and (ii) applying the longitude.

61. *Calculation of the chronometer error from the sun's hour angle.*

We assume that the sun's hour angle can be deduced from observation. The procedure is readily seen from the following example.

Ex. 10. An observer in long. 58° 48′ 30″ W, keeping the mean time of Zone +4, made an observation of the sun at approximate Z.T. 14^h 25^m, October 2; chronometer time 18^h 24^m 38^s. From the observation the sun's hour angle was deduced to be 2^h 40^m 59^s. To find the error of the chronometer.

Approx. Z.T.	14^h 25^m October 2
Zone - -	+4
Approx. G.D.	18^h 25^m October 2

We can find the correct G.D. only when we have found the error of the chronometer; but, using the approximate G.D. as given above, we obtain an approximate value of E.T., which should not be in error by more than a second. Thus, we find from the *Nautical Almanac* that E.T. is $+10^m$ 38^s. We proceed as follows.

	H.A.⊙ - -	2^h 40^m 59^s
	E.T. (+) - -	10 38 ....(*a*)
	H.A.M.S. - -	2 30 21, by means of (18)
	Long. (W) - -	3 55 14
	G.H.A.M.S. - -	6 25 35
Hence,	G.C.T. is - -	18^h 25^m 35^s October 2 (Note 1)
But chronometer time is - -		18^h 24^m 38^s.

Consequently, the error of chronometer is 0^m 57^s slow on G.C.T.

Note 1. The date, October 2, must agree with that of the approximate G.D.

If we require extreme accuracy, we repeat the above calculation with the value of E.T., taken from the *Nautical Almanac*, for the G.C.T. immediately above (in the same line as Note 1).

62. *Relation between mean time and sidereal time intervals.*

We start from formula (6), p. 53, which gives for a given instant and for a given meridian the sidereal time in terms of H.A.M.S. and R.A.M.S., namely,

$$\text{H.A.M.S.} + \text{R.A.M.S.} = \text{Local sidereal time,}$$

which we write in the form

$$H_1 + R_1 = T_1, \quad \text{.........................(19)}$$

If we know H_1 and R_1 we obtain the sidereal time T_1 as indicated by the observer's sidereal clock at the given instant (we suppose for simplicity that the clock's error and rate are both zero).

After an interval of one mean solar day we have the formula, similar to (19),

$$H_2 + R_2 = T_2, \qquad \text{(20)}$$

where T_2 is now the reading shown by the sidereal clock. From (19) and (20), by subtraction, we obtain

$$H_2 - H_1 + R_2 - R_1 = T_2 - T_1. \qquad \text{(21)}$$

Now in the interval of one mean solar day the H.A.M.S. has increased by 24^h, so that

$$H_2 - H_1 = 24^h. \qquad \text{(22)}$$

Also, the R.A.M.S. has increased by n (the mean daily angular motion) which is given from (9) by

$$n = \frac{360°}{365\frac{1}{4}},$$

or, writing 24^h for 360°, by

$$n = \frac{24^h}{365\frac{1}{4}};$$

hence

$$R_2 - R_1 = \frac{24^h}{365\frac{1}{4}}. \qquad \text{(23)}$$

From (21), (22) and (23) we obtain

$$T_2 - T_1 = 24^h + \frac{24^h}{365\frac{1}{4}}.$$

Now $T_2 - T_1$ is the interval of sidereal time equivalent to 24^h of mean solar time; hence

$$24^h \text{ M.S.T.} \equiv 24^h \left(1 + \frac{1}{365\frac{1}{4}}\right) \text{ sidereal time} \qquad \text{(24)}$$

$$\equiv 24^h \left(\frac{366\frac{1}{4}}{365\frac{1}{4}}\right) \text{ sidereal time.} \qquad \text{(25)}$$

Formula (24) or (25) enables us to convert an interval of mean solar time into its equivalent value of sidereal time. From (24) we can readily construct the following table.*

TABLE I

Conversion of mean solar time into sidereal time

24^h M.S.T.	$\equiv (24^h + 3^m\ 56^s{\cdot}556)$	sidereal time.
1^h ,,	$\equiv (1^h + 9^s{\cdot}8565)$	,, ,,
1^m ,,	$\equiv (1^m + 0^s{\cdot}1643)$	,, ,,
1^s ,,	$\equiv (1^s + 0^s{\cdot}0027)$	,, ,,

From (25) we can clearly write

$$24^h \text{ sidereal time} = 24^h \left(\frac{365\frac{1}{4}}{366\frac{1}{4}}\right) \text{ M.S.T.}$$

$$= 24^h \left(1 - \frac{1}{366\frac{1}{4}}\right) \text{ M.S.T.}$$

The following table can then be constructed.

TABLE II

Conversion of sidereal time into mean solar time

24^h sidereal time	$\equiv (24^h - 3^m\ 55^s{\cdot}910)$	M.S.T.
1^h ,, ,,	$\equiv (1^h - 9^s{\cdot}8296)$	,,
1^m ,, ,,	$\equiv (1^m - 0^s{\cdot}1638)$	,,
1^s ,, ,,	$\equiv (1^s - 0^s{\cdot}0027)$	,,

The *Nautical Almanac* contains more detailed tables for converting mean solar time intervals into sidereal time intervals, and vice versa.

63. *Greenwich sidereal time.*

The sidereal time at Greenwich is tabulated in the *Nautical Almanac* for G.C.T. 0^h for each day of the year.

Suppose that an observation is made at G.C.T. $10^h\ 25^m\ 30^s$ April 4 by an observer in longitude λ and that the corresponding local sidereal time is required. We proceed in two steps.

The first step is to find the Greenwich sidereal time at the given G.C.T. We find from the *Nautical Almanac* that

$$\text{Gr. sid. time at G.C.T. } 0^h \text{ April } 4 = 12^h\ 48^m\ 38^s.$$

* Instead of 365¼ the value 365·2422 is used for this purpose.

The interval between G.C.T. 0^h April 4 and the G.C.T. of the observation is $10^h\ 25^m\ 30^s$ M.S.T.

We convert this mean time interval into the corresponding sidereal time interval by means of Table I. Thus

$$10^h \text{ M.S.T.} = 10^h\ 1^m\ 38^s{\cdot}565 \text{ sidereal time.}$$
$$25^m \quad ,, \quad = \quad 25 \quad 4{\cdot}107 \quad ,, \quad ,,$$
$$30^s \quad ,, \quad = \quad 30{\cdot}081 \quad ,, \quad ,,$$

Hence the mean time interval is equivalent to a sidereal time interval of $10^h\ 27^m\ 12^s{\cdot}753$, or $10^h\ 27^m\ 13^s$, to the nearest second.

The Greenwich sidereal time * at the time of the observation is then

$$12^h\ 48^m\ 38^s + 10^h\ 27^m\ 13^s,$$

or

$$23^h\ 15^m\ 51^s.$$

The second step is to apply formula (12), p. 55, namely,

$$\text{Gr. sid. time} = \text{Local sid. time} + \lambda,$$

from which the observer's sidereal time can be obtained. For example, if the observer's longitude is 96° 30′ W ($6^h\ 26^m$) we have

Gr. sid. time for observation	-	$23^h\ 15^m\ 51^s$
Long. W - - - -	-	6 26 0
Local sid. time for observation	-	$16^h\ 49^m\ 51^s$

64. *Calculation of the hour angle of a star.*

We assume that a star is observed at a known G.C.T. (the method is applicable to the moon or a planet) by an observer in a given longitude. The method of the previous section enables us to calculate the local sidereal time of the observation. The right ascension of the star (or moon or planet) is obtained from the *Nautical Almanac* or other book of reference. We then apply formula (7), p. 54, which enables us to derive the hour angle required.

* In the *Nautical Almanac, abridged for the use of seamen*, a quantity *R*, equal to R.A.M.S. ± 12^h, is tabulated at two-hourly intervals. The Greenwich sidereal time is then found by adding *R* to the G.C.T.

Ex. 11. To find the hour angle of *Betelgeuse* (α Orionis) for an observer in longitude 85° 15′ E (Zone − 6); given, approximate Z.T. of observation, $18^h\ 25^m$ January 2; chronometer time, $12^h\ 24^m\ 38^s$; chronometer error, $1^m\ 33^s$ slow on G.C.T.

Approx. Z.T.	-	$18^h\ 25^m$	January 2
Zone - - -		− 6	
Approx. G.D.	-	12 25	January 2
Chronometer time		12 24 38	
Slow - - -		1 33	
G.D. ≡ G.C.T.	-	12 26 11	January 2

It is found from the *Nautical Almanac* that the Greenwich sidereal time at G.C.T. 0^h January 2 is $6^h\ 41^m\ 58^s$. Also, $12^h\ 26^m\ 11^s$ of mean time is equivalent, by using Table I, p. 69, to $12^h\ 28^m\ 14^s$ sidereal time.

Gr. sid. time at G.C.T. 0^h Jan. 2		$6^h\ 41^m\ 58^s$	
Add sid. time interval -	-	12 28 14	
Gr. sid. time for observation	-	19 10 12	
Long. (E) - - -	-	5 41 0	
Local sid. time - -	-	24 51 12,	using (12)
R.A. *Betelgeuse* - -	-	5 51 57,	from N.A.
H.A. *Betelgeuse* - -	-	18 59 15,	using (7)

The procedure of finding the hour angle of a star, the moon or a planet by the method of this section, or of the sun by the method of section 60, is fundamental in many problems. For example, the zenith distance and azimuth of any heavenly body can be calculated for any observer whose latitude and longitude are known and who is equipped with a chronometer; for, with the H.A. determined as above, two sides (*PZ* and *PX*) of the spherical triangle *PZX* and the included angle *ZPX* are known so that, as in Problem 2, p. 33, the zenith distance and azimuth can be found.

65. *The seasons.*

The year is divided astronomically into four seasons called spring, summer, autumn and winter. In the northern hemisphere, *spring* begins when the sun, in its circuit of the ecliptic, reaches the First Point of Aries, ♈, and ends when it reaches the summer solstice. During spring the sun's right ascension increases from 0^h to 6^h and its declination from 0° to 23° 27′ N. *Summer* is the interval between the passage of the sun through the summer solstice and through the autumnal equinox (or First Point of Libra); the sun's right ascension increases from 6^h to 12^h and its declination changes from 23° 27′ N to 0°. *Autumn* begins when the sun is at ♎ and ends at the winter solstice; the sun's right ascension increases from 12^h to 18^h and its declination changes from 0° to 23° 27′ S. *Winter* begins with the sun at the winter solstice and ends with the return of the sun to ♈; the right ascension increases from 18^h to 24^h (or 0^h) and the declination changes from 23° 27′ S to 0°. In the southern hemisphere spring begins when the sun is at ♎ and ends when its declination is 23° 27′ S. The other seasons follow in succession.

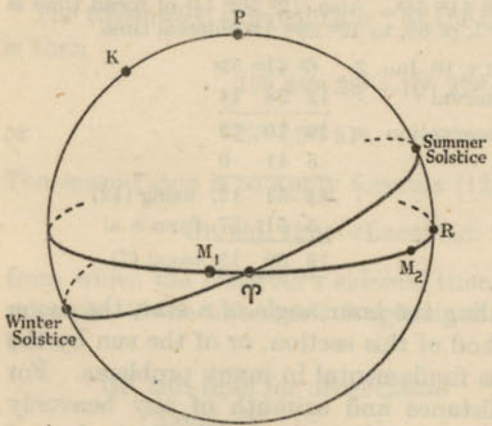

FIG. 37.

The length of the northern spring can be found as follows. In Fig. 37 suppose that when the sun is at ♈ the mean sun is at M_1, where the arc M_1♈ is the numerical value of the equation of time. From the *Nautical Almanac* it is found that, when the sun's declination is 0°, on or about March 21, the equation of time is $-7^m\,28^s$. Now

$$\text{E.T.} = \text{R.A.M.S.} - \text{R.A.}\odot, \quad\quad (26)$$

so that if the sun is at ♈ (R.A.⊙ being then zero) we have

$$\text{R.A.M.S.} = -7^m\,28^s \text{ or } 24^h - 7^m\,28^s.$$

Thus M_1 is to the left of ♈ in the figure and M_1♈ $= 7^m\,28^s$. Again, when the sun is at the summer solstice its right ascension is 6^h and the equation of time (from the *Nautical Almanac*) is $-1^m\,34^s$, so that if M_2 denotes the corresponding position of the mean sun, we have from (26):

$$♈M_2 \equiv \text{R.A.M.S.} = 6^h - 1^m\,34^s.$$

We then obtain easily that

$$M_1M_2 = 6^h\,5^m\,54^s;$$

or, expressed in degrees,

$$M_1M_2 = 91°\,28\tfrac{1}{2}'.$$

Spring may now be described as the interval required by the mean sun to describe 91° 28½′ of the equator. But the mean sun describes 360° in one year; consequently spring is the fraction $\frac{91°\,28\frac{1}{2}'}{360°}$ of one year, which is rather more than one quarter of a year. The lengths of the seasons (for the northern hemisphere) are found in this way to be as follows.

Spring	- -	92 days 20·2 hours
Summer	- -	93 days 14·4 hours
Autumn	- -	89 days 18·7 hours
Winter	- -	89 days 0·5 hours

During the northern spring and summer the sun's declination is north and hence, by section 32, the sun is above the horizon for more than 12 hours in each day; similarly during the northern autumn and winter the sun is above the horizon for less than 12 hours in each day.

Moreover, for a northern observer the meridian zenith distance of the sun is, in general, smaller when the sun's declination is north than when it is south (compare formula (6), p. 25, namely, $ZU = \phi - \delta$).

We conclude that (i) during the northern spring and summer the hours of daylight are longer than during autumn and winter, and (ii) the sun has greater noon-time altitudes during spring and summer than in the other two seasons. These two facts determine the relative amounts of the solar heat falling at a particular place on the earth's surface and explain, to a large extent, why the terrestrial temperatures are higher in spring and summer than in autumn and winter.

66. *The terrestrial zones.*

The parallel of latitude 23° 27′ N is called the *Tropic of Cancer* and the parallel of latitude 23° 27′ S is called the *Tropic of Capricorn*; the zone between these two parallels is the *Torrid*

Zone. At any place within the Torrid Zone the sun is in the zenith (or very nearly so) on two days of the year; for if the latitude is 20° N, say, the meridian zenith ($=\phi-\delta$) is zero or very nearly zero when the sun's declination is exactly equal, or nearly equal, to 20° N, that is, about May 21 and July 24.

The parallels of latitude 66° 33′ N and 66° 33′ S are called the *Arctic Circle* and the *Antarctic Circle* respectively; the zone between the Tropic of Cancer and the Arctic Circle is called the *North Temperate Zone* and the zone between the Tropic of Capricorn and the Antarctic Circle is the *South Temperate Zone.*

For a place on the Arctic Circle the meridian zenith distance of the sun on December 21 is given by $\phi-\delta$ where $\phi=66° \ 33'$ and $\delta=-23° \ 27'$; thus the sun's meridian zenith distance is 90°, that is to say, the sun is below the horizon throughout the whole of this day. It is easily seen in a similar way that within the temperate zones the sun is above the horizon for some part of every day of the year.

If we consider a place north of the Arctic Circle, say in latitude 80° N, the meridian zenith distance of the sun will be less than 90° provided that $\phi-\delta$ is less than 90°, that is if $\delta>-10°$. The dates corresponding to $\delta=-10°$ are February 21 and October 20; it follows that between October 20 and February 21 the sun never appears above the horizon in this latitude.

Again, by (7) of section 25, the sun will be circumpolar, that is to say, above the horizon for all hour angles, if $\delta>90°-\phi$. For latitude 80° N this condition becomes $\delta>10°$. The dates corresponding to $\delta=10°$ are April 17 and August 28; hence between these dates the sun never sets and we have the phenomenon of the *midnight sun.*

It is also easily seen that at the north pole the sun is continuously above the horizon between March 21 and September 21 and continuously below the horizon during the remainder of the year.

67. *Twilight.*

For some time after the sun has set, we receive a diminishing amount of diffused sunlight, reflected and scattered by the

earth's atmosphere. This is called *twilight* which steadily diminishes as the sun moves further below the horizon. Astronomical twilight is said to end when the sun's centre is 18° below the horizon.

In Fig. 38, $\widehat{ZPX}$ is the hour angle of sunset and $\widehat{ZPY}$ is the hour angle when the sun is 18° below the horizon, that is, when its zenith distance (ZY in the figure) is 108°. The interval between sunset and the end of twilight is called the duration of evening twilight; it is measured as the difference of the two hour angles ZPY and ZPX and is strictly an interval of apparent solar time; but for all practical purposes it may be taken to be equivalent to an interval of mean solar time. The calculation proceeds in two steps; we first calculate the hour angle ZPX from the spherical triangle PZX in which

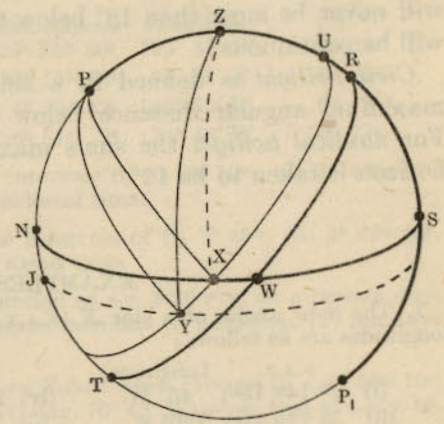

FIG. 38.

$$PZ=90°-\phi, \ PX=90°-\delta \text{ and } ZX=90°,$$

using the formula (24) of section 32. The second step is to calculate the hour angle ZPY from the spherical triangle PZY in which we know all three sides, namely

$$PZ=90°-\phi, \ PY=90°-\delta \text{ and } ZY=108°.$$

The difference between the two hour angles so found is the duration of evening twilight.

Morning twilight is defined and found in a similar way.

Twilight lasts continuously—that is, the beginning of morning twilight coincides with the end of evening twilight—if at apparent midnight the sun is not more than 18° below the horizon. The limiting case is when the sun is at J (Fig. 38) at apparent midnight and then twilight is just continuous. Now in this case $ZJ = 108°$ and $PJ = 90° - \delta$, and, since $ZJ = PZ + PJ$, the condition in the limiting case becomes

$$108° = 90° - \phi + 90° - \delta,$$

or

$$\delta = 72° - \phi.$$

Thus if $\phi = 62°$, twilight is just continuous when $\delta = 10°$, that is, on April 17 and August 28. Between these dates the sun will never be more than 18° below the horizon and so twilight will be continuous.

Civil twilight is defined in a similar way, with the sun's maximum angular distance below the horizon taken as 6°. For *nautical twilight* the sun's maximum distance below the horizon is taken to be 12°.

EXAMPLES

1. The hour angles of a star X (H.A.X.) for the places in the given longitudes are as follows:

	H.A.X.	Longitude		H.A.X.	Longitude
(i)	$3^h\,14^m\,15^s$;	45° W.	(iv)	$23^h\,5^m\,32^s$;	35° 45′ E.
(ii)	$7^h\,27^m\,37^s$;	120° E.	(v)	$2^h\,39^m\,42^s$;	163° 22′ E.
(iii)	$14^h\,25^m\,56^s$;	156° W.			

Find in each case the corresponding hour angle at Greenwich.

2. The hour angles of a star X at Greenwich (G.H.A.X.) are as follows. Find the corresponding hour angles for the given longitudes.

	G.H.A.X.	Longitude		G.H.A.X.	Longitude
(i)	$17^h\,53^m\,14^s$;	175° W.	(iii)	$22^h\,15^m\,37^s$;	42° 15′ E.
(ii)	$3^h\,47^m\,9^s$;	108° W.	(iv)	$5^h\,11^m\,48^s$;	68° 30′ E.

3. Find the H.A.M.S. for the given longitudes, corresponding to the G.C.T. shown.

	G.C.T.	Longitude		G.C.T.	Longitude
(i)	$17^h\,11^m\,22^s$;	48° E.	(iii)	$23^h\,19^m\,42^s$;	124° 15′ E.
(ii)	$3^h\,35^m\,55^s$;	155° W.	(iv)	$22^h\,37^m\,55^s$;	73° 30′ W.

4. Find the Greenwich Date, given the zone and the zone time as follows:

	Zone	Zone Time		Zone	Zone Time
(i)	+ 5;	6.30 P.M. July 4.	(iii)	+ 8;	$19^h\,36^m$ August 10.
(ii)	− 11;	5.44 A.M. July 8.	(iv)	− 7;	$8^h\,27^m$ August 20.

5. Find the correct Greenwich Date, given the zone, the approximate zone time, the chronometer time and the chronometer error as follows:

	Zone	Approx. Z.T.	Chronometer	Error
(i)	− 3	$8^h\,14^m$ May 1	$5^h\,15^m\,36^s$	$1^m\,10^s$ slow
(ii)	+ 9	$22^h\,10^m$ May 2	$7^h\,12^m\,15^s$	$2^m\,15^s$ fast
(iii)	− 12	$5^h\,10^m$ May 4	$17^h\,8^m\,40^s$	$2^m\,4^s$ fast
(iv)	+ 4	$17^h\,25^m$ May 5	$21^h\,25^m\,30^s$	$0^m\,10^s$ slow

6. Find the hour angle of the sun for places in the given longitudes, from the following data (take the chronometer error to be $1^m\,10^s$ slow in all cases; the values of the equation of time (E.T.) are taken from the *Nautical Almanac*):

	Zone	Approx. Z.T.	Chronometer	Longitude	E.T.
(i)	+ 7	1630 Jun. 1	$23^h\,31^m\,20^s$	103° 40′ W	+ $2^m\,18^s$
(ii)	− 6	0425 Nov. 3	$22^h\,23^m\,50^s$	91° 12′ E	+ $16^m\,22^s$
(iii)	+ 10	1830 Aug. 2	$4^h\,29^m\,42^s$	149° 52′ W	− $6^m\,9^s$
(iv)	+ 2	0450 Feb. 12	$6^h\,51^m\,3^s$	34° 49′ W	− $14^m\,21^s$

7. Express the mean time intervals of (i) $9^h\,10^m$, (ii) $3^h\,40^m\,10^s$, (iii) $19^h\,45^m\,36^s$ as intervals of sidereal time.

8. Express the sidereal time intervals of (i) $7^h\,45^m$, (ii) $9^h\,22^m\,48^s$, (iii) $21^h\,35^m\,16^s$ as intervals of mean time.

9. *Sirius* crosses a given meridian at Z.T. 9.30 P.M. on a certain day. At what Z.T. will it cross the same meridian (i) next day, (ii) ten days later?

10. Zone + 5. Find the Z.T. on Feb. 3 when *Procyon* (R.A. $7^h\,36^m\,10^s$) crosses the meridian of Ottawa (Long. 75° 43′ W) given that at G.C.T. 0^h, Feb. 3, the Greenwich sidereal time is $8^h\,48^m\,8^s$.

11. Find the local sidereal time, for the given longitudes, and the corresponding hour angles of *Regulus* (R.A. $10^h\,5^m\,11^s$) from the following data; take the chronometer error in each case to be $2^m\,5^s$ slow. From the *Nautical Almanac* the Greenwich sidereal time at G.C.T. 0^h, January 4, is $6^h\,49^m\,52^s$.

	Zone	Approx. Z.T.	Chronometer	Longitude
(i)	+ 4	$3^h\,14^m$ Jan. 4	$7^h\,12^m\,56^s$	58° 20′ W
(ii)	− 7	$17^h\,26^m$ Jan. 4	$10^h\,27^m\,2^s$	103° 48′ E
(iii)	+ 9	$18^h\,50^m$ Jan. 3	$3^h\,50^m\,0^s$	134° 10′ W
(iv)	− 10	$9^h\,18^m$ Jan. 4	$23^h\,17^m\,50^s$	148° 26′ E
(v)	0	$16^h\,44^m$ Jan. 4	$16^h\,43^m\,48^s$	6° 14′ E
(vi)	+ 2	$23^h\,7^m$ Jan. 3	$1^h\,8^m\,14^s$	29° 39′ W

12. The values of the equation of time at the equinoxes or solstices on or near March 21, June 21, September 21 and December 21 are respectively:

$$-\,7^m\,28^s;\quad -\,1^m\,34^s;\quad +\,7^m\,32^s \text{ and } +\,1^m\,34^s.$$

Find the lengths of the seasons correct to one-tenth of a day.

13. Find the most northerly latitude at which the phenomenon of the midnight sun is just possible between the two dates in summer for which the sun's declination is 20° N.

14. Find the latitude of a place at which twilight (astronomical) just lasts all night, the sun's declination being (i) 14° N, (ii) 22° S.

15. Can twilight last all night at the equator or at any place within the Torrid zone?

16. What are the limits of declination if twilight lasts all night in latitude 58° S?

17. Calculate the duration of evening twilight (astronomical) for latitude 50° N when (i) $\delta = +5°$, (ii) $\delta = -5°$.

18. Explain how you would find the zone time of apparent noon for an observer in a given longitude on a given date.

19. On a certain day at a place on the Greenwich meridian the sun rose at G.C.T. $5^h\ 4^m\ 10^s$ and set at G.C.T. $19^h\ 8^m\ 50^s$. If the equation of time is assumed constant between sunrise and sunset, find its value.

20. Zone +2. At a place in longitude 31° 15′ W, the sun rose at Zone Time $7^h\ 11^m\ 40^s$ and set at $16^h\ 47^m\ 36^s$. Assuming that the equation of time is constant, find its value and also the Zone Time of apparent noon at the place.

21. At Zone Time 1800 (Zone −2) on February 4, 1940, a ship's position was 38° S, 34° 30′ E. The ship steams eastwards along the parallel of latitude for 2850 nautical miles at 15 knots. Find (i) the ship's longitude and (ii) the Zone Time with date, at the end of the run.

22. At Zone Time 0600 (Zone +11) October 14, a ship's position was A (42° 30′ S, 162° 40′ W). At Zone Time 1400 (Zone +5), October 27, she reached the position B (42° 30′ S, 79° 35′ W), her course being along the parallel of latitude of 42° 30′ S. Find her speed in knots.

CHAPTER V

THE SOLAR SYSTEM AND THE LAW OF GRAVITATION

68. *Motions relative to the earth.*

In this chapter we consider the sun and its family of planets, the satellites (or moons) of the planets, comets and meteors, the whole forming what is known as the *Solar System*. Starting with the Moon—the only satellite of the Earth—we can easily infer by simple observation the principal features of its motion in the sky. Suppose that we observe—say, at 10 P.M., on a particular day—the moon's position, M_1, with reference to the principal stars in its neighbourhood. Next day, at 10 P.M., it will be seen that its position, M_2, relative to the same stars, is now about 13° to the eastward of its first position; 24 hours later its position is about 13° still further to the eastwards. These simple observations are represented in Fig. 39.

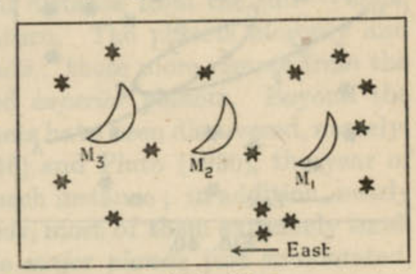

FIG. 39.

Continued observation will show that after about $27\frac{1}{3}$ days the moon will return approximately to the position, relative to the stars, denoted by M_1 in the figure. We infer that the moon describes an orbit round the earth in a period, with reference to the background of the stars, of about $27\frac{1}{3}$ days; this period is called the moon's *sidereal period* of revolution or *orbital period*.

In a somewhat similar way we infer that the sun appears to move around the earth in a period of $365\frac{1}{4}$ days with reference to the background of the stars. The stars, of course, are not visible to the naked eye in day-time, but we can proceed as follows. At apparent noon we can measure the sun's declination. At the apparent midnight following, when the sun's hour angle is 12^h, the point in the heavens exactly opposite to the sun (the declination of this point will be opposite in sign to that of the sun) will be on the meridian and its position with reference to the stars can be noted—allowing for the change in the sun's declination in the interval. Observations on successive nights will show that the point diametrically opposite to the sun will appear to move in 24 hours about 1° eastwards with respect to the stars. In this way we infer that the sun appears to move eastwards against the stellar background in a period of about $365\frac{1}{4}$ days. We emphasise that the sun *appears* to revolve around the earth and its orbit is the *apparent orbit* with which we were concerned in section 45. On the celestial sphere the *ecliptic* is the path of the sun with reference to the stars and is a great circle.

If we make a similar set of observations of a planet such as Mars, for example, it is seen that the apparent path of the planet with reference to the background of the stars is very much more complicated than in the case of the moon or sun. This is illustrated in Fig. 40 where the curve *ABCDEFG* represents some of the typical features of a planet's path in the sky. Along the section *ABC* the planet appears to move eastwards with reference to the stars, its angular motion decreasing as *C* is approached. Along the section *CDE* the planet appears to move westwards relative to the stars until *E* is reached, after which its motion again becomes eastward along the section *EFG*. The eastward motion (sections *ABC* and *EFG*), corresponding to increasing right ascension, is said to be *direct*, and the westward motion (section *CDE*), corresponding to decreasing right ascension, is said to be *retrograde*.

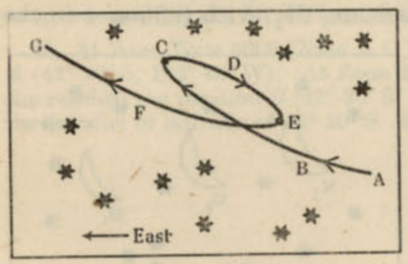

FIG. 40.

69. *The Ptolemaic and Copernican theories.*

The first rational attempt to account for the phenomena described in the previous section is embodied in the *Ptolemaic theory*. The earth was supposed to be the fixed and immovable centre of the universe (the Ptolemaic theory is thus a *geocentric* theory) and round it circulated the sun and moon. The stars were imagined to be situated on the surface of a transparent sphere which rotated westwards about an axis in the period of what we now call a sidereal day, thus accounting for the diurnal motion, that is, the westward motion of the heavenly bodies across the sky. The peculiar apparent motions of the planets as illustrated in Fig. 40 necessitated ingenious geometrical devices which became more and more complicated as further divergences between observation and "theory" were noticed. The Ptolemaic system survived until it was challenged by the *heliocentric* theory of Copernicus (1473-1543). According to Copernicus, the sun is the centre of the system of planets; the Earth rotates about an axis, thus giving rise to the diurnal motion of the heavens, and is itself a planet describing a circular orbit, like the other planets, about the sun. The planet nearest the sun is Mercury, and the others known at the time of Copernicus are—in order of distance from the sun—Venus, Earth, Mars, Jupiter and Saturn. The planets Mercury and Venus are called *inferior planets*; those more remote from the sun than the earth are called *superior* planets. Beyond the orbit of Saturn three new planets have been discovered, namely, Uranus (1781), Neptune (1846) and Pluto (1930), the year of discovery being indicated in each instance; in addition, nearly 2,000 *minor planets* or *asteroids*, most of them extremely small in size as compared with the *major planets* just enumerated, have been discovered—mostly within recent years—moving in orbits between those of Mars and Jupiter. The Copernican theory gave a simple explanation of the principal phenomena of planetary motions, including direct and retrograde motions (which we shall further consider in section 76), but its complete expression had to await the researches of Kepler and Newton.

70. *Synodic period (superior planet).*

We assume that the planetary orbits are circles and coplanar. The assumption that the planetary orbits are coplanar is equivalent to saying that the planets move in the plane of the ecliptic, since this is the plane defined by the earth's orbit—or the sun's *apparent* orbit round the earth. Actually, the orbital planes of the inferior and superior planets are inclined at small angles to the plane of the ecliptic—the inclinations are given on p. 259.

A planet is said to be in *opposition* when the direction of the planet as viewed from the earth is opposite to the direction of the sun. In Fig. 41 the orbits of the earth and a superior planet are shown, and opposition corresponds to the configuration shown by S, E_1 and P_1. Thus, at apparent midnight the planet will be on the observer's meridian and the planet will be above the horizon between sunset and the next sunrise.

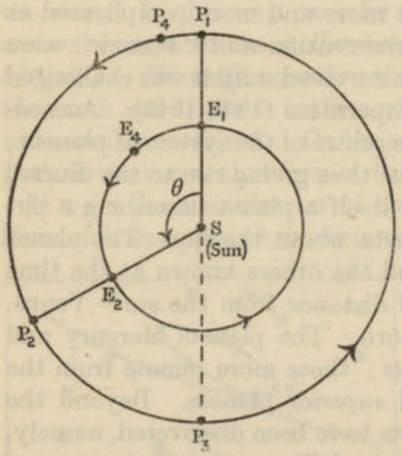

Fig. 41.

If the directions of the sun and planet coincide, as in the configuration E_1, S and P_3, the planet is said to be in *conjunction*.

The interval between two successive oppositions, or between two successive conjunctions, is called the planet's *synodic period*. Let SE_1P_1 define a particular opposition and SE_2P_2 the next opposition. Now, as we shall see later, the orbital periods of revolution of the planets increase from Mercury outwards. Hence in Fig. 41 the orbital period of the planet P is greater than that of the earth. In other words, the angular motion of E is greater than the angular motion of P. Thus, if the planet moves from P_1 to P_4 in the same time as that required by the earth to move from E_1 to E_4, the angle subtended by the arc E_1E_4 at S is greater than the angle subtended by the arc P_1P_4 at S. It follows that, since the planet has described the angle P_1SP_2, or θ, in the direction of the arrow when the next opposition takes place, the earth has described $360° + E_1\widehat{S}E_2$, that is, $360° + \theta$.

Let n_1, n_2 denote in degrees the daily angular motion of the earth and the planet and T_1, T_2 the corresponding orbital periods in days. We have

$$n_1 = \frac{360°}{T_1}, \quad n_2 = \frac{360°}{T_2}. \quad \text{.................(1)}$$

Let S denote the synodic period in days; it is the interval required by the earth to move through $360° + \theta$ and also the interval required by the planet to move through θ. Thus

$$360° + \theta = n_1 S, \quad \text{and} \quad \theta = n_2 S.$$

Eliminating θ by subtraction, and using (1), we obtain

$$360° = (n_1 - n_2)S = 360° \left(\frac{1}{T_1} - \frac{1}{T_2}\right) S,$$

whence

$$\frac{1}{S} = \frac{1}{T_1} - \frac{1}{T_2}. \quad \text{..............................(2)}$$

The synodic period, S, can be readily obtained by observing the interval between two successive oppositions and, taking T_1 to be $365\frac{1}{4}$ days, we can then determine the orbital period of the planet. For example, the synodic period of Saturn is 378·1 days, and from (2) we deduce that the orbital period is 29·5 years approximately.

71. *Synodic period (inferior planet).*

We define the synodic period for an inferior planet as the interval between two successive conjunctions. It is easily seen from Fig. 42 that there is never any possibility of opposition, since the earth E can never lie directly between the inferior planet V and the sun. When the planet is directly between S and E, as at V, it is said to be in *inferior conjunction*; at V_2 it is in *superior conjunction*.

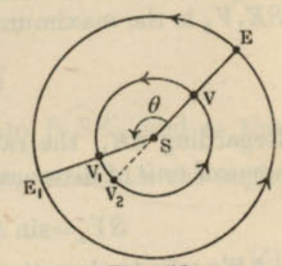

Fig. 42.

Let SVE and SV_1E_1 denote the configurations for two successive inferior conjunctions. Since the angular motion of V is greater than the angular motion of the earth, E, the earth will move through the angle ESE_1, or θ, in the synodic period S while V moves through the angle $360°+\theta$. Let n_1, T_1 refer as before to the earth and n_2, T_2 to the planet. Then

$$\theta=n_1S \quad \text{and} \quad 360°+\theta=n_2S,$$

whence
$$360°=(n_2-n_1)S,$$

and as $n_1=360°/T_1$ and $n_2=360°/T_2$, we obtain

$$\frac{1}{S}=\frac{1}{T_2}-\frac{1}{T_1} \quad \text{.............................(3)}$$

The orbital and synodic periods of the planets will be found on p. 259; it will be seen that the orbital periods increase from Mercury outwards from the Sun.

72. *Elongation.*

The angle between the directions of a planet and of the sun as seen from the earth at any instant is called the *elongation*.*

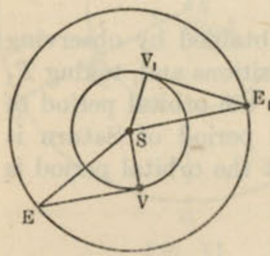

FIG. 43.

In Fig. 43, which is drawn for an inferior planet, the elongation is $\widehat{SEV}$. The elongation of an inferior planet cannot exceed a certain angle which depends on the radii of the orbits of the earth and planet. It is clear from the figure that the maximum elongation occurs when the straight line joining the earth and planet is tangential to the planet's orbit. Thus, if E_1V_1 is a tangent to the circular orbit of the inferior planet, the angle SE_1V_1 is the maximum elongation. In this instance,

$$\sin SE_1V_1=\frac{SV_1}{SE_1}. \quad \text{........................(4)}$$

Regarding SE_1, the radius of the earth's orbit, as the *astronomical unit* of distance † we obtain from (4)

$$SV_1=\sin \widehat{SE_1V_1} \text{ astronomical units.}$$

* We refer to elongation as *east* or *west* elongation according as the planet is east or west of the sun, the limits of elongation being 0° and, at the most, 180°.

† The astronomical unit is more particularly defined as the semi-major axis of the earth's orbit; its value is 93·0 million miles (see section 73).

Hence, if $\widehat{SE_1V_1}$ is obtained from observations, we can determine the radius of the orbit of an inferior planet in terms of the astronomical unit. This method was used by Copernicus.

For a superior planet the elongation, east or west, can take any value between 0° and 180°. At conjunction the sun and planet are in the same direction and consequently the elongation is 0°; at opposition the elongation is 180°. When the elongation is 90°, the planet is said to be *in quadrature*.

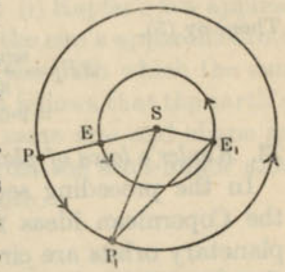

FIG. 44.

The approximate distance of a superior planet from the sun can be determined as follows. Let PES be the configuration at opposition (Fig. 44). Some time later, say t days after opposition, the earth is at E_1 and the planet at P_1. The angle SE_1P_1 is the elongation which we suppose to be found from observation. Let n_1, n_2 denote as before the daily angular motions of the earth and planet respectively; from Fig. 44 we have

$$\widehat{ESE_1}=n_1t \quad \text{and} \quad \widehat{PSP_1}=n_2t,$$

so that
$$P_1\widehat{SE_1}=(n_1-n_2)t=360°\left(\frac{1}{T_1}-\frac{1}{T_2}\right)t.$$

Hence, if S is the synodic period for the planet, we obtain from (2)

$$P_1\widehat{SE_1}=360°\frac{t}{S}.$$

With t and S known we can calculate $P_1\widehat{SE_1}$, and as the elongation $\widehat{SE_1P_1}$ is also supposed known we obtain the angle SP_1E_1. Now

$$\frac{SP_1}{SE_1}=\frac{\sin SE_1P_1}{\sin SP_1E_1}. \quad \text{........................(5)}$$

As SE_1 is the astronomical unit this formula enables us to calculate SP_1 (the radius of the planet's orbit) in astronomical units.

Ex. 1. Calculate the radius of Saturn's orbit given that, 40 days after opposition, the elongation is 137° 53′.

From section 70, the synodic period for Saturn is 378·1 days. Hence

$$P_1\widehat{SE}_1=\frac{360°\times 40}{378·1}=38°\ 5'.$$

Also $P_1\widehat{SE}_1+S\widehat{E_1}P_1=38°\ 5'+137°\ 53'=175°\ 58'$; hence

$$S\widehat{P_1}E_1=4°\ 2'.$$

Then, by (5),

$$SP_1=\frac{\sin 137°\ 53'}{\sin 4°\ 2'}\text{ astronomical units}$$
$$=9·54\text{ astronomical units.}$$

73. *Kepler's laws of planetary motions.*

In the preceding sections we have assumed—according to the Copernican ideas relating to the solar system—that the planetary orbits are circles. It was soon found, however, that the hypothesis of circular orbits led to small discrepancies between the predicted and observed positions of the planets, and this was notably so in the case of Mars for which accurate observations had been made by the Danish astronomer Tycho Brahe (1546-1601). His pupil and successor, Kepler (1571-1630), investigated the subject afresh, and was finally led to enunciate the three great laws of planetary motions which bear his name. These laws are as follows.

Kepler's first law.

The orbit of a planet is an ellipse with the sun situated at a focus.

Fig. 45 shows an ellipse with the sun situated at a focus, S; C is the centre of the ellipse and AB is the major axis. The semi-major axis CA (or CB) is denoted by a. The ratio of CS to CA is the eccentricity e; hence $CS=ae$. The point on the orbit nearest S is A and this is called *perihelion*; the perihelion distance, SA, is equal to $CA-CS$, that is, $a(1-e)$. The point on the orbit most remote from S is B, called *aphelion*; the aphelion distance SB is $SC+CB$ or $a(1+e)$. If the planet is at L, the straight line SL joining the planet to the sun is called a *radius vector*. These definitions are analogous to those already referred to in Chapter IV, section 45, in connection with the sun's *apparent* orbit with the earth at a focus.

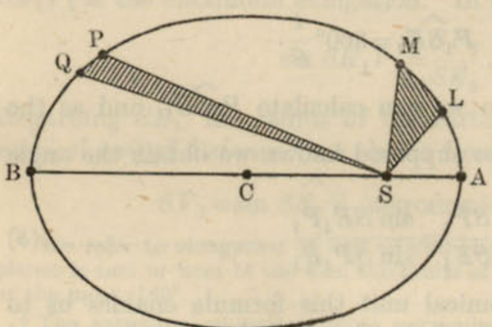

FIG. 45.

The following points should be noted: (i) Kepler's law applies to all planets, including the earth; (ii) the sun's apparent orbit described in Chapter IV, section 45, is the path which the sun *appears to describe relative to the earth*; it follows that the earth's elliptic orbit around the sun is of the same size and shape as the sun's apparent orbit; in other words, the semi-major axis and the eccentricity are the same for both.

Kepler's second law.

The radius vector, joining the sun to a planet, sweeps out equal areas in equal times.

The law is illustrated in Fig. 45, where L, M and P, Q are two pairs of positions in the orbit occupied by the planet, the interval required by the planet to travel from L to M being equal to the interval between P and Q; according to the second law the area LSM is equal to the area PSQ.

If we suppose the interval to be short, M will be near to L and Q to P. The area LSM is then approximately $\frac{1}{2}r^2\sin L\widehat{SM}$—where LS is denoted by r—or $\frac{1}{2}r^2L\widehat{SM}$, the angle LSM being expressed in circular measure. If $L\widehat{SM}$ is equivalent to θ degrees, the area is then $\pi r^2\theta/360$. Similarly if r_1 and θ_1 refer to SP and $P\widehat{SQ}$, the area PSQ is approximately $\pi r_1{}^2\theta_1/360$. Hence by the second law

$$r^2\theta=r_1{}^2\theta_1=\text{a constant.} \quad\text{.....................(6)}$$

If the interval is t minutes of time and ω denotes the angular velocity, expressed in degrees per minute, with which the radius vector SL moves to SM, we have $\omega t=\theta$. Similarly, if ω_1 is the angular velocity between P and Q, $\omega_1 t=\theta_1$. We then derive from (6)

$$r^2\omega=r_1{}^2\omega_1=h, \quad\text{...........................(7)}$$

where h is a constant. The formula shows that the smaller the radius vector in a given orbit the greater is the angular motion of the planet. Accordingly, the angular motion is greatest at perihelion and least at aphelion.

Kepler's third law.

The first two laws refer to the orbit of any given planet; the third law states a relation between certain of the elements of two or more planets. Let $a_1, a_2, a_3, \ldots$ be the semi-major axes of the orbits of the planets and $T_1, T_2, T_3, \ldots$ the corresponding periods of revolution in their orbits. The third law is expressed algebraically as:

$$\frac{a_1^3}{T_1^2}=\frac{a_2^3}{T_2^2}=\frac{a_3^3}{T_3^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Suppose a_1 and T_1 refer to the earth. Then a_1 is the *astronomical unit of distance*; this definition replaces the earlier one when we were concerned with circular orbits which were regarded as approximations to the true orbits. If we know the orbital period (expressed in years) of a planet such as Saturn, we can calculate by means of (8) the semi-major axis of its orbit in terms of the astronomical unit. Suppose that a_2 and T_2 refer to Saturn. Then from (8), putting $a_1=1$ astronomical unit and $T_1=1$ year, we have

$$a_2^3=T_2^2,$$

T_2 being expressed in years. Hence $a_2=(T_2)^{\frac{2}{3}}$.

Ex. 2. For Saturn, $T_2=29{\cdot}46$ years. Hence the semi-major axis of Saturn's orbit is $(29{\cdot}46)^{\frac{2}{3}}$ or 9·54 astronomical units.

In this way we determine the lengths of the semi-major axes of all the planetary orbits in terms of the astronomical unit. With further knowledge of the directions in which the several major axes point and the inclinations of the orbital planes to the ecliptic, we can construct a model of the solar system. If we want to know the distance of a planet from the sun in miles, we have to make one further step, namely, to express the astronomical unit in miles. This latter problem will be discussed in Chapter VII. Meanwhile we may state the result:

1 astronomical unit = 93,003,000 miles
= 149,674,000 kilometres.

74. *Newton's law of gravitation.*

The three laws of planetary motion as stated by Kepler are independent of one another. It was Newton (1643-1727) who showed that all three could be deduced from one single law—the law of gravitation. Newton's law is as follows.

Every particle of matter in the universe attracts every other particle with a force varying directly as the product of their masses and inversely as the square of the distance between them.

Newton also proved that if M is the mass of a spherical body such as the sun, its attraction on a particle outside the sphere is the same as if the solar mass were concentrated at the centre of the sphere. Except for certain refinements which we need not consider here, we can thus replace for theoretical purposes the sun and planets by point-masses. The law of gravitation then asserts that if M and m are the masses of the sun and a planet and r is the distance between their centres, the sun attracts the planet with a force F equal to GMm/r^2 and that the planet attracts the sun with an equal force. Here G is a universal constant, called the *constant of gravitation*; its value in C.G.S. units is $6{\cdot}670\times10^{-8}$.

75. *Deductions from the law of gravitation.*

It is impracticable within the scope of this book to show how all three of Kepler's laws for elliptic orbits can be deduced from the law of gravitation; the reader is referred to the author's *Spherical Astronomy* for a more detailed discussion. We state, however, some results.

(*a*) *Accurate statement of Kepler's third law.* Let M, m denote the masses of the sun and a planet, a the semi-major axis of the planet's orbit and T the period of revolution in the orbit. Then, as Newton showed,

$$\frac{4\pi^2}{T^2}\cdot a^3=G(M+m). \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

For another planet with mass m_1 and elements a_1, T_1, we have similarly

$$\frac{4\pi^2}{T_1^2}a_1^3=G(M+m_1). \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

From (9) and (10) we have, by division,

$$\frac{a^3}{T^2}\cdot\frac{1}{\left(1+\frac{m}{M}\right)}=\frac{a_1^3}{T_1^2}\cdot\frac{1}{\left(1+\frac{m_1}{M}\right)}, \quad\text{.............(11)}$$

which is the accurate statement of Kepler's third law so far as these two planets are concerned. Formula (8) is thus an approximation to (11) if m/M and m_1/M are very small quantities. Actually, the sun is 330,000 times more massive than the earth and 1047 times more massive than Jupiter, which is the most massive planet of all. It is thus seen that Kepler's third law is an exceedingly good approximation to the accurate formula (11).

(*b*) *Determination of the mass of a planet.* As before, let m, a and T refer to a planet which is accompanied by one or more satellites. Then for the planet we have formula (9) connecting M, m, a and T. Let m_1 be the mass of a satellite, a_1 the semi-major axis of its orbit round the planet and T_1 the orbital period of revolution. Considering only the planet and its satellite, we notice that they are bound together by the universal law of gravitation, and it follows that the orbital motion of the satellite about the planet implies a formula similar to (9), namely,

$$\frac{4\pi^2 a_1^3}{T_1^2}=G(m+m_1)=Gm\left(1+\frac{m_1}{m}\right). \quad\text{............(12)}$$

In most cases we may assume that the mass of the satellite is negligible in comparison with the planet's mass, so that we can omit m_1/m in (12). From (9) and (12), by division, we have

$$\frac{m}{M+m}=\left(\frac{a_1}{a}\right)^3\left(\frac{T}{T_1}\right)^2, \quad\text{......................(13)}$$

or, with sufficient accuracy in most cases,

$$\frac{m}{M}=\left(\frac{a_1}{a}\right)^3\left(\frac{T}{T_1}\right)^2. \quad\text{........................(14)}$$

We assume, first, that a and T are known for the planet, the former in astronomical units (or miles) and the latter in years, and, second, that a_1 and T_1 are known similarly for the satellite.

The right-hand side of (14) is then easily calculated, and so we determine the ratio of the planet's mass to the sun's mass.

Ex. 3. To determine the ratio of the mass of Uranus to the sun's mass from the following data :

Uranus : $a=19{\cdot}19$ astronomical units, $T=84{\cdot}02$ years.

Titania (the third satellite of Uranus) :

$$a_1=0{\cdot}00293 \text{ astronomical units, } T_1=8{\cdot}706 \text{ days}=\frac{8{\cdot}706}{365\frac{1}{4}} \text{ years.}$$

Then, by (14),

$$\frac{m}{M}=\left(\frac{0{\cdot}00293}{19{\cdot}19}\right)^3\cdot\left(\frac{84{\cdot}02\times 365\frac{1}{4}}{8{\cdot}706}\right)^2$$

$$=\frac{1}{22610}$$

which is close to the value obtained from more accurate data.

The masses of planets which have no satellites (such as Mercury and Venus) are determined by indirect methods. Consider the earth ; its orbit round the sun is an ellipse when only the mutual gravitation of the sun and earth are taken into account. But Mercury and Venus and all the planets attract the earth, with the result that the earth's path deviates from an ellipse by small but calculable amounts depending on the masses of the "disturbing" planets. It is possible from observations to determine the deviation due to, say, Venus and so to derive the mass of the planet in terms of the sun's mass as unit. The deviations which we have been considering are called *perturbations*.

(*c*) *The linear velocity of a planet.* Let V denote the linear velocity of a planet at a point in its orbit (elliptic) at a distance r from the sun. Then it is deduced from the law of gravitation that

$$V^2=\mu\left(\frac{2}{r}-\frac{1}{a}\right), \quad\text{.........................(15)}$$

where $\mu=G(M+m)$. This formula shows that V will be greatest when r is least and least when r is greatest, that is, at perihelion and aphelion respectively. At perihelion, $r=a(1-e)$; and if V_1 denotes the linear velocity at perihelion,

$$V_1^2=\frac{\mu(1+e)}{a(1-e)}.$$

Similarly, if V_2 denotes the velocity at aphelion,

$$V_2^2 = \frac{\mu(1-e)}{a(1+e)}.$$

From these two formulae we derive

$$\frac{V_1}{V_2} = \frac{1+e}{1-e}.$$

For example, the eccentricity of Mercury's orbit is about $\frac{1}{5}$; hence, in this case, $V_1/V_2 = 3/2$, so that Mercury's velocity at perihelion is half as great again as its velocity at aphelion.

If the eccentricity of a planetary orbit is neglected so that we may regard the orbit as circular, (15) becomes, on putting $r=a$, where a is now the radius of the orbit,

$$V^2 = \frac{\mu}{a}. \qquad \text{.................................(16)}$$

But V is given in this case by the length of the circular path divided by the period of revolution; that is

$$V = 2\pi a/T. \qquad \text{.................................(17)}$$

Hence we obtain, from (16) and (17),

$$\frac{4\pi^2}{T^2} a^3 = \mu,$$

which is the form assumed by (9) for a circular orbit of radius a.

From (17) we can readily find the velocity of the earth in its orbit, assumed circular, for $a = 93{\cdot}0 \times 10^6$ miles and T (the number of seconds in a year) $= 31{\cdot}56 \times 10^6$. Hence

$$V = \frac{2\pi \times 93{\cdot}0}{31{\cdot}56} = 18{\cdot}5 \text{ miles per second.}$$

To find the velocity of any other planet in its orbit (assumed circular) we proceed as follows. Let V and a refer to the earth and V_1 and a_1 to the planet. Then, neglecting the masses of the earth and planet in comparison with the sun's mass, we have by (16)

$$V^2 = \frac{GM}{a}, \quad V_1^2 = \frac{GM}{a_1},$$

from which

$$\frac{V_1}{V} = \left(\frac{a}{a_1}\right)^{\frac{1}{2}}. \qquad \text{.................................(18)}$$

It follows from this formula that a planet nearer to the sun moves more rapidly than a planet more remote.

Ex. 4. For Venus, $a_1 = 0{\cdot}723$ astronomical units; hence

$$V_1/V = (1/0{\cdot}723)^{\frac{1}{2}} = 1{\cdot}176,$$

and as $V = 18{\cdot}5$ miles per second, $V_1 = 21{\cdot}8$ miles per second.

76. *Direct and retrograde motions.*

We have already remarked (p. 81) that the Copernican theory accounted in a simple manner for the direct and retrograde motions of the planets as illustrated in Fig. 40. We now give the explanation, assuming for simplicity that the planetary orbits are circular and coplanar.

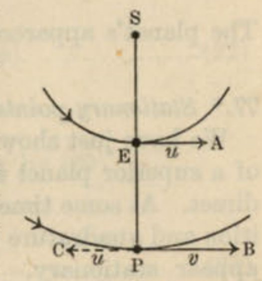

FIG. 46.

Consider a superior planet P at opposition (Fig. 46), and let u and v be the linear velocities of the earth, E, and the planet in their orbits, the radii of the orbits being a and b respectively. Then, by (18),

$$\frac{u}{v} = \left(\frac{b}{a}\right)^{\frac{1}{2}}; \qquad \text{.................................(19)}$$

hence, since $b>a$, we have $u>v$. Now, as we observe P from E, its linear motion *relative to* E will be compounded of its orbital velocity v along PB and the earth's orbital velocity reversed, that is to say, u along PC. Since $u>v$ the planet will appear to have a linear velocity $u-v$, as viewed from the earth, in the direction PC, and this direction is *opposite* to that of the orbital motion; accordingly, the planet will appear to move in the *retrograde* direction at opposition.

Consider now the figure corresponding to quadrature (Fig. 47), that is, when the elongation, $\widehat{SEP}$, is 90°. Let β denote $\widehat{SPE}$ and let PD be perpendicular to EP, so that $\widehat{BPD}$ is β. The earth's orbital velocity is u along EA and, consequently, relative to the earth, the planet's motion consists of u reversed (that is, u along PC) and v along PB. Now v along PB is equivalent to $v \cos \beta$ along PD and $v \sin \beta$ along PE. Hence, relative to E the planet's motion consists of (i) $u - v \sin \beta$ along PC, and (ii) $v \cos \beta$ along PD. Now (i) has no effect in changing the direction of P as viewed from the earth; hence the apparent motion of P is due to (ii) alone. It follows that at quadrature the planet appears to move, relative to E, in the anti-clockwise direction, that is, in the same direction as the orbital motion. The planet's apparent motion at quadrature is thus *direct*.

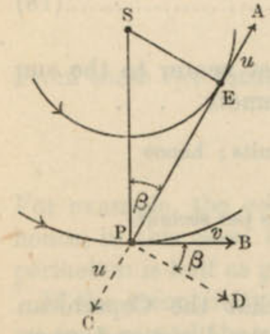

FIG. 47.

77.* *Stationary points.*

We have just shown that at opposition the apparent motion of a superior planet is retrograde and that at quadrature it is direct. At some time between opposition and quadrature the planet must appear stationary. Let Fig. 48 be the corresponding diagram. Then, relative to E, the motion of P is compounded of (i) the orbital velocity v along PB, and (ii) the earth's reversed velocity u along PD (DPG is drawn parallel to EA); the condition for a stationary point is that the resultant of (i) and (ii) should lie along EP produced. We have thus the parallelogram of velocities, $PBCD$, in which $PB = DC = v$ and $BC = PD = u$.

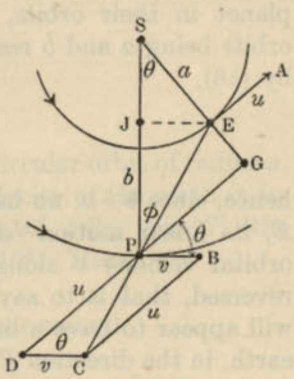

FIG. 48.

Let θ denote $\widehat{PSE}$ and let ϕ denote $\widehat{SPE}$. Since EA and PG are perpendicular to ES and PB is perpendicular to PS, then $\widehat{GPB} = \theta$. Hence $\widehat{EPG} = \widehat{DPC} = \widehat{PCB} = 90° - \theta - \phi$. Also

$$\widehat{CPB} = 90° + \phi.$$

From the triangle PCB we obtain

$$\frac{\sin(90° + \phi)}{u} = \frac{\sin(90° - \theta - \phi)}{v},$$

whence

$$\cos \phi = \frac{u}{v} \cos(\theta + \phi). \quad \text{.....................(20)}$$

If we draw a perpendicular EJ from E to SP we have

$$SP - SJ = PJ,$$

so that

$$b - a \cos \theta = PE \cos \phi. \quad \text{.....................(21)}$$

Let SE produced meet DPG in G; since $SG - SE = EG$, $\widehat{PEG} = \theta + \phi$ and $\widehat{SGP} = 90°$, we have

$$b \cos \theta - a = PE \cos(\theta + \phi). \quad \text{.....................(22)}$$

Dividing (21) by (22), we obtain

$$\frac{b - a \cos \theta}{b \cos \theta - a} = \frac{\cos \phi}{\cos(\theta + \phi)} = \frac{u}{v}, \text{ by (20)},$$

from which

$$\cos \theta = \frac{au + bv}{av + bu}. \quad \text{.....................(23)}$$

But, by (16), $u = \sqrt{\mu/a}$ and $v = \sqrt{\mu/b}$; hence (23) becomes

$$\cos \theta = \frac{a^{\frac{1}{2}} b^{\frac{1}{2}} (a^{\frac{1}{2}} + b^{\frac{1}{2}})}{a^{\frac{3}{2}} + b^{\frac{3}{2}}}. \quad \text{.....................(24)}$$

This formula gives the value of θ when the planet is stationary. If α, in degrees, is the value of θ between 0° and 90° given by (24), the planet's motion, relative to the earth, is retrograde for θ between $360° - \alpha$ and α, since it is retrograde at opposition when $\theta = 0$. α corresponds to the stationary point represented in Fig. 48; $360° - \alpha$ corresponds to the second stationary point, not shown in the figure, when the radius vector SE is α degrees behind the radius vector SP. Now α is the angle through which the radius vector SE has moved ahead of the radius vector SP between opposition and the moment at which the planet is stationary. If S is the synodic period of the planet in days, the motion is consequently retrograde for $\frac{2\alpha}{360} \cdot S$ or $\frac{\alpha}{180} S$ days. For the remainder of the synodic period, that is, $\frac{180-\alpha}{180} \cdot S$ days, the motion is direct.

78.* *The elongation at a stationary point.*

In Fig. 48 let $\widehat{SEP}$, the elongation, be denoted by E. Then $E = 180° - \theta - \phi$. From triangle SEP we have

$$\sin\phi = \frac{a}{b} \cdot \sin E$$

and (20) can be written, since $\cos E = -\cos(\theta+\phi)$,

$$\cos\phi = -\frac{u}{v} \cdot \cos E.$$

For the stationary point shown in Fig. 48, ϕ lies between 0° and 90°; the above equations show that E lies between 90° and 180°. Similarly for the other stationary point. The value of the elongation, east or west, is thus between 90° and 180°.

Squaring and adding the previous equations, we obtain

$$\frac{u^2}{v^2} \cdot \cos^2 E + \frac{a^2}{b^2} \cdot \sin^2 E = 1,$$

from which we derive, using (19),

$$\frac{b}{a} \cos^2 E - \frac{a^2}{b^2} \cos^2 E = 1 - \frac{a^2}{b^2}.$$

We obtain

$$\cos^2 E = \frac{a(a+b)}{a^2+ab+b^2}, \quad \text{.....................(25)}$$

and hence

$$\sin^2 E = \frac{b^2}{a^2+ab+b^2}. \quad \text{.....................(26)}$$

From (25) and (26), we derive

$$\tan^2 E = \frac{b^2}{a(a+b)}. \quad \text{.........................(27)}$$

This gives the value of E (between 90° and 180°) corresponding to the two stationary points.

79. *Phases of the planets.*

The planets are non-luminous bodies but they appear bright owing to the reflection of sunlight from their surfaces or atmospheres. Let the sphere in Fig. 49 represent a planet (we assume it to be spherical) of radius r, with its centre at P. If S is the sun—we have to imagine that, in the figure, S is at a great distance from P—the hemisphere towards the sun will be illuminated and the hemisphere away from S will be in darkness. Let E be the earth and APB the diameter of the sphere, in the plane SEP, perpendicular to EP. Similarly, let GPH be the diameter in this plane perpendicular to SP. From H draw HC perpendicular to PB. Then the fraction of the diameter of the planet's disc seen illuminated from E is AC/AB—we must imagine that, in the figure, E is at a great distance from P. This fraction measures the *phase*. When the phase is unity, the planet is seen full; when the phase is zero, the planet is invisible.

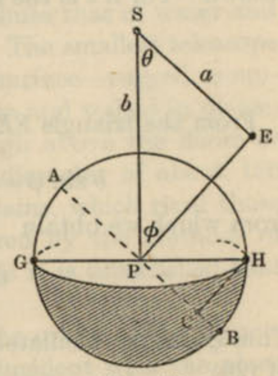

FIG. 49.

Let ϕ denote $\widehat{SPE}$ and θ denote $\widehat{PSE}$. It is easily seen that $\widehat{HPC} = \phi$, so that $PC = PH \cos\phi = r\cos\phi$. We then have

$$\text{Phase} = \frac{AC}{AB} = \frac{r+PC}{2r} = \frac{1+\cos\phi}{2}. \quad \text{..............(28)}$$

For a superior planet the phase is unity when $\phi = 0$; this corresponds to opposition. Also, ϕ is always less than 90°, whatever the configuration of S, E and P may be, so that the phase is always more than $\frac{1}{2}$ and consequently more than half the disc is visible. The appearance of the superior planet is then said to be *gibbous*.

For an inferior planet it can be seen from (28) that the phase is zero at inferior conjunction—for, then, $\phi = 180°$—and unity at superior conjunction—for, then, $\phi = 0°$. Between these two limits the planet is seen to pass through all the phases from a thin crescent to "half-full", followed by the gibbous phase until the "full" stage is reached.

We can calculate the phase given by (28) as follows. From the known interval between opposition (superior planet) or inferior conjunction (inferior planet) and the time of observation we can determine θ, assuming the synodic period, S, known. For if t is the interval, θ is given in degrees by

$$\theta = \frac{360^\circ t}{S}.$$

From the triangle SEP, we have

$$b \sin \phi = a \sin \widehat{SEP} = a \sin (\theta + \phi),$$

from which we obtain

$$\tan \phi = \frac{a \sin \theta}{b - a \cos \theta}. \quad \text{.....................(29)}$$

Thus ϕ can be calculated and then the phase is found by means of (28).

80. *Description of the sun, moon and planets.*

In this section we give a brief description of these bodies, the numerical details being found in the Table on p. 259.

The sun. The sun is a luminous, gaseous globe of radius 432,000 miles, or 695,500 kms., and is a star of very ordinary dimensions. At its radiating surface the temperature is about 6000° on the centigrade scale; it is inferred that the central temperature is of the order of 40 million degrees. The brightness of the planets is due to the reflection by their surfaces of the sunlight falling on them. By the gravitational methods described earlier in this chapter it is deduced that the sun's mass is 330,000 times the mass of the earth. By experiments such as the Cavendish Experiment it is found that the earth's mass is about 5000 million million million tons, from which the sun's mass can be deduced. The average density of the sun is about 1·4 times that of water, the average density of the earth being about 5·5.

Usually sunspots are seen on the sun's disc and, from their apparent motion across the disc, it is inferred that the sun rotates about an axis in about 25 days; this applies to the solar equatorial layers—in higher solar latitudes the rotational period increases to about 28 days. Sunspots appear dark against the brilliantly luminous disc, but it is found that their temperatures are about 4000°. The solar atmosphere will be described in some detail in section 170.

The moon. The moon is the earth's only satellite. Its radius is 1080 miles, its average density is $3\frac{1}{3}$ times that of water and its mass is about $\frac{1}{81}$ that of the earth. The smallest telescope reveals the chief features of the lunar surface—rugged mountain ranges and a large number of craters and walled-in plains, many with central mountains rising high above the floors of the craters. The largest crater has a diameter of about 150 miles. The heights of the lunar mountains, which rival those on the earth in height, can be measured by the methods of section 138. By a variety of arguments it is established that the moon has no atmosphere.

The planets. We first remark that the orbits of the planets are ellipses whose planes are nearly coincident with the plane of the ecliptic. The angle which the plane of a planetary orbit makes with the plane of the ecliptic is called the *inclination*; it will be seen from the Table on p. 259 that with the exception of Mercury and Pluto the inclinations are all remarkably small. Except for Pluto, the planets are all within 8° on either side of the ecliptic, and this zone is called the *Zodiac*. The zodiac is divided into twelve *signs*, or constellations, each occupying 30° of the ecliptic; the signs are:

Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricornus, Aquarius, Pisces;

or Ram, Bull, Heavenly Twins, Crab, Lion, Virgin, Scales, Scorpion, Archer, He-goat, Water-carrier, Fishes.

We remark in the second place that the planets rotate about axes just as the earth rotates about an axis and in the same direction.

Mercury. This planet is a difficult object to observe, as its maximum elongation is only 28°. Mercury has no atmosphere, and as it normally shows no surface markings its axial period of rotation is not known with any certainty. However, it is believed to be 88 days, which is the period of orbital revolution; if this is so, Mercury presents nearly the same hemispherical surface to the sun—it would be continuously the same hemisphere if its orbit were exactly circular, in which event one half of Mercury's surface would be in perpetual darkness and the other half would be perpetually illuminated by the sun. Mercury has no satellite.

Venus. The orbit of Venus is the closest approach to a circle of all the planetary orbits, the eccentricity being about $\frac{1}{150}$ only. Venus resembles the earth very closely in size and mass. It is surrounded by a dense atmosphere which obscures its surface features; spectroscopic observations have not revealed the presence of oxygen and water-vapour in its atmosphere. Its axial period of rotation is not known with any certainty, but it is possible that it is 225 days, which is the orbital period; if so, Venus presents almost continuously the same hemisphere towards the sun. The planet has no satellite.

Mars. This planet, which has a diameter about half that of the earth, has a rather rare atmosphere through which one can see reddish and bluish-green tracts, the former being presumed to be deserts and the latter areas where some form of vegetation still survives. Mars has two white polar caps—similar in appearance to the Arctic and Antarctic snowfields of the earth —from which it is concluded that water is present, perhaps only in small quantities, on the Martian surface. The "Canals" of Mars, which are faint, narrow markings crossing the surface in all directions, have given rise to much controversy as to their interpretation and even to their objective reality. Mars has two satellites, Phobos and Deimos, which are small bodies, probably only about a dozen miles in diameter.

Jupiter. Jupiter is the largest and most massive of the planets and possesses eleven satellites, two of which were discovered only as recently as 1938. The planet is surrounded by a dense atmosphere consisting mainly of methane (or marsh-gas) and ammonia-gas. Jupiter departs notably from the spherical form, its polar diameter being 82,800 miles and its equatorial diameter 88,700 miles. In the telescope Jupiter shows several dark bands parallel to the planet's equator and also many smaller markings by means of which the axial period of rotation can be found with great accuracy.

Saturn. This planet has nine satellites and is similar to Jupiter as regards the chemical constitution of its atmosphere. The distinguishing feature of Saturn is its unique and magnificent system of rings which make Saturn one of the most beautiful objects in the heavens. The rings actually consist of innumerable tiny satellites, so close together as to give the impression of a continuous surface except where "Encke's" and "Cassini's divisions" occur.

Uranus. This planet was discovered in 1781 by Sir William Herschel. It is surrounded by an atmosphere resembling that of Jupiter and Saturn. The planet has four satellites which revolve in planes almost perpendicular to the plane in which Uranus revolves about the sun, the orbital planes of all other satellite systems being in, or nearly in, the orbital plane of revolution of the planet concerned.

Neptune. After Uranus had been observed for several years it became increasingly evident that there were small unexplained discrepancies between the observed positions and the theoretical positions as calculated according to the law of gravitation; these theoretical positions included the small disturbances of its path due to the gravitational action of all the other planets. Acting on the supposition that these discrepancies were due to the influence of an unknown planet, presumably more remote than Uranus, J. C. Adams and U. J. J. Le Verrier—independently and unknown to each other—determined the orbit of the hypothetical planet by mathematical analysis which enabled its position in the sky to be specified at any given date. It was found at once, in 1846, when the telescope was directed to the predicted position. Neptune has one satellite and its atmosphere contains methane and ammonia-gas.

Pluto. Pluto was discovered photographically in 1930. It is a faint object, visible only in the largest telescopes. It is almost certain that its mass is less than that of the earth and, probably, approximately that of Mars. Owing to the short time during which it has been observed, its orbit is not known with the accuracy associated with the orbits of the other planets.

81. *Bode's "law".*

Previous to the discovery of Uranus it was noticed that the semi-major axes of the planetary orbits could be approximately represented by a numerical relationship, known now as Bode's "law". Write down the following series of numbers in which any number after the second is double the preceding one :

0, 1, 2, 4, 8, 16, 32, 64, 128, 256.

Then multiply by 3 throughout and add 4 to the products. We thus get line (1) :

(1)	4	7	10	16	28	52	100	196	388	772
(2)	Mer.	Ven.	Earth	Mars	Minor planets	Jup.	Sat.	Ur.	Nep.	Pluto
(3)	3·9	7·2	10	15	—	52	95	192	301	396

Underneath the series 4, 7, 10, ... we write down the name of the planet to which each number applies ; lines (1) and (2) constitute Bode's "law". Underneath the names of the planets we insert in line (3) the actual values of the semi-major axes, taking the value of the earth's major axis to be 10. It will be seen that the actual values, in line (3), of the semi-major axes for all the major planets up to Saturn agree fairly closely with the values, in line (1), as given by Bode's "law". When Uranus was discovered it was seen to fit also into the scheme satisfactorily. However, with Neptune and Pluto the numerical relationship breaks down and it is to be concluded that Bode's "law" is very probably no more than an accidental relationship which is useful, however, for giving the approximate values of the semi-major axes for the major planets up to Uranus.

82. *The minor planets.*

The first of the minor planets to be discovered was Ceres (January 1, 1801) ; the total number now known is close to 2000 and each year sees a substantial addition to the number. The orbits of the minor planets lie between the orbits of Mars and Jupiter. The diameter of Ceres—the largest—is about 500 miles, but the great majority have diameters of only a score or so of miles. It is surmised that the minor planets are the débris of a disrupted planet which occupied the position in Bode's table indicated by the number 28.

83. *Comets.*

The naked eye comets are distinguished by extensive tails, which sometimes stretch over 30 or 40 degrees of the sky. The faint comets, only visible in the telescope, are generally nebulous objects without any indication of the tail that characterises the bright comets. Cometary orbits have large eccentricities and sometimes they are hardly distinguishable from parabolas

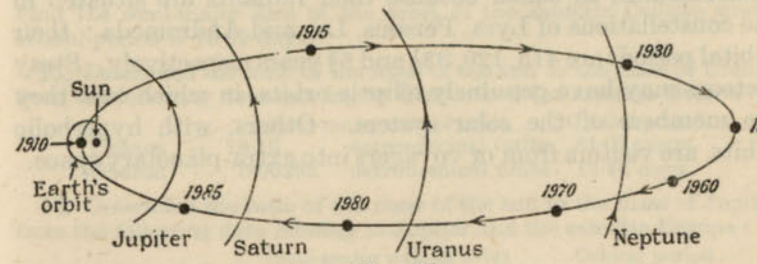

FIG. 50.

or hyperbolas—forms of conics describable under the law of gravitation, as Newton proved. Comets with elliptic orbits are periodic—the shortest period is that of Encke's Comet, namely 3·3 years ; the period of Halley's Comet, perhaps the best known of all the comets, is 75½ years ; it was last visible in 1910. Its orbit is shown in Fig. 50 and it is typical of the periodic comets.

84. *Meteors.*

Meteors are tiny objects, in most instances perhaps no bigger than a grain of sand, which are rendered luminous owing to air-resistance when they crash into the earth's atmosphere with speeds of twenty to forty miles per second ; they are then described as "shooting-stars". Sometimes a large number of shooting-stars are visible during a short interval of two or three hours and on such occasions they all appear to radiate from a particular point in a constellation, called the *radiant*. This is an effect of perspective and we conclude that the meteors are travelling in a parallel stream to which an orbit around the sun can be ascribed. For example, the Leonids, which are visible about November 14, are a swarm of meteors moving round the sun in an elliptic orbit which intersects the ecliptic in a point near which the earth is situated about November 14. As a display of meteors occurs annually about this date, it is concluded that the meteors are strung out all along the orbit. There is a similarity between the orbits of meteor swarms and certain periodic comets and it is inferred that the meteors are fragments of comets, partially or wholly disrupted. The principal meteor showers are the Lyrids, Perseids, Leonids and Andromedids, so called because their radiants are situated in the constellations of Lyra, Perseus, Leo and Andromeda; their orbital periods are 415, 120, 33$\frac{1}{3}$ and 6$\frac{2}{3}$ years respectively. Stray meteors may have genuinely elliptic orbits, in which case they are members of the solar system. Others, with hyperbolic orbits, are visitors from or voyagers into extra-planetary space.

EXAMPLES

(Take a year to be 365$\frac{1}{4}$ days and the astronomical unit to be 93,000,000 miles.)

1. The synodic period of Venus is 583·9 days; find the planet's sidereal period in days.

2. The sidereal period of Saturn is 29·46 years; find the synodic period.

3. The synodic period of Jupiter is 398·9 days; find its sidereal period in years.

4. Find the maximum elongation of Venus, given that the radius of its orbit (assumed circular) is 0·723 astronomical units.

5. Find the interval between opposition and the next quadrature for Jupiter, given that the radius of Jupiter's orbit (assumed circular) is 5·20 astronomical units and that the synodic period is 398·9 days.

6. From Kepler's third law, calculate in days the orbital periods of Mercury and Venus, given that the semi-major axes of their orbits are 0·3871 and 0·7233 astronomical units respectively.

7. A minor planet has a semi-major axis of 3·55 astronomical units; find its orbital period in years.

8. A minor planet has an orbital period of 9·86 years; find the semi-major axis of its orbit in miles.

9. The orbital periods of the Martian satellites, Phobos and Deimos, are 0·3189 and 1·262 days respectively. Find the semi-major axes of their orbits in (i) astronomical units, (ii) miles, given that the sun's mass is $3{\cdot}093 \times 10^6$ times the mass of Mars.

10. The earth's orbital speed is 18·5 miles per second; find the orbital speed of Venus (assume its orbit to be a circle of radius 0·723 astronomical units).

11. With the data for Venus given in exs. 1 and 4, find how long a transit of the planet centrally across the sun's disc would last. Assume that Venus moves in the ecliptic; take the sun's S.D. to be 16′ and neglect the S.D. of the planet and the earth's rotation.

12. The orbit of Jupiter's satellite Ganymede has a semi-major axis of 0·007156 astronomical units and the orbital period is 7·155 days. Find the semi-major axis of the orbit of the satellite Callisto whose orbital period is 16·69 days.

13. Determine the ratio of the mass of the sun to the mass of Uranus from the following data relating to Uranus and the satellite Oberon:

	Semi-major axis of orbit		Orbital period
Uranus -	19·19	astronomical units	84·02 years.
Oberon -	0·00392	astronomical units	13·46 days.

14. Determine the ratio of the mass of the sun to the mass of Jupiter from the following data relating to Jupiter and the satellite Europa:

	Semi-major axis of orbit		Orbital period
Jupiter -	5·203	astronomical units	11·86 years.
Europa -	0·004486	astronomical units	3·551 days.

15.* Find the elongation of Jupiter at a stationary point; take the radius of the orbit (assumed circular) to be 5·20 astronomical units.

16. Given that the radius of the orbit of Venus is 0·723 astronomical units, calculate (i) the maximum elongation and the corresponding phase, (ii) the two values of the phase when the elongation is 20°.

17. The phase of Venus is $\frac{1}{16}$; calculate its elongation.

18. The synodic period of Mars is 780 days and the radius of its orbit (assumed circular) is 1·52 astronomical units. Calculate its phase (i) at quadrature, (ii) 150 days before opposition.

CHAPTER VI

ATMOSPHERICAL REFRACTION

85. *Introduction.*

In empty space a ray of light is rectilinear, but as soon as the ray enters a transparent medium its direction is altered; this phenomenon is known as *refraction*. Now the earth is surrounded by an atmosphere which is transparent to the light of heavenly bodies; accordingly, a ray of light from a star will suffer a change of direction on entering the atmosphere and further changes as it penetrates through atmospheric layers of increasing density until it reaches the observer at the earth's surface. The star will be seen by the observer in a direction somewhat different from that in which it would be seen if the earth's atmosphere were non-existent. This difference in direction depends on the star's zenith distance, as will be shown later, and so it is necessary to correct any observation for the effects of refraction produced by the atmosphere.

86. *The laws of refraction.*

Consider a uniform slab of a transparent medium, M_1, such as glass, bounded by two parallel planes intersecting the plane of the paper in EF and GH (Fig. 51). Let AB be a ray *in vacuo* falling on the upper surface at B, and let BN be the perpendicular, or the *normal*, to the surface at B; the angle ABN, denoted by i, is called the *angle of incidence*. On entering the medium M_1 at B the direction of the ray is changed to BC which makes an angle r_1 with the normal at B.

The first law of refraction states that the refracted ray BC lies in the plane defined by the incident ray AB and the normal BN.

The second law of refraction states that

$$\frac{\sin i}{\sin r_1} = \mu_1, \quad \text{.................................(1)}$$

where $\mu_1 (>1)$ is a constant depending only on the optical properties of the medium M_1; μ_1 is called the *refractive index* of the medium. Thus for a given value of i we can calculate r_1, if the index of refraction is known. Since $\mu_1 > 1$, it follows from (1) that $r_1 < i$; in other words, the effect of refraction is to change the direction of the ray to a direction nearer that of the normal.

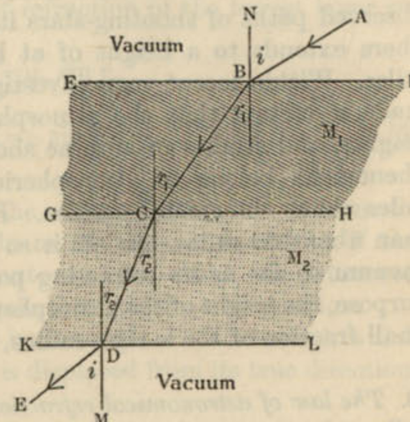

Fig. 51.

Suppose now that a uniform slab of a medium M_2, bounded by GH and KL, is placed below the first medium. The ray BC is now incident on the upper surface of M_2 at C and is refracted into the medium M_2 along some such direction as CD, making an angle r_2 with the normal at C. If the ray then passes into a vacuum along the path DE, it is known from experiments that DE is parallel to AB so that, if DM is the normal to KL at D, the angle EDM is i. Now a ray passing in the reversed direction along ED will be refracted along DC so that, if μ_2 is the index of refraction for the medium M_2,

$$\frac{\sin i}{\sin r_2} = \mu_2. \quad \text{.................................(2)}$$

Hence, from (1) and (2), we obtain

$$\sin i = \mu_1 \sin r_1 = \mu_2 \sin r_2. \quad \text{.........................(3)}$$

This formula evidently holds for any number of slabs; for three slabs it is

$$\sin i = \mu_1 \sin r_1 = \mu_2 \sin r_2 = \mu_3 \sin r_3,$$

and for n slabs

$$\sin i = \mu_1 \sin r_1 = \mu_2 \sin r_2 = \ldots = \mu_n \sin r_n. \quad \text{..........(4)}$$

87. *The earth's atmosphere.*

It is well known that the density of the air diminishes with distance above the earth's surface. From studies of the observed paths of shooting-stars it is inferred that the atmosphere extends to a height of at least three or four score of miles. Within recent years investigations on radio-waves have made it certain that the atmosphere reaches to still greater heights, and studies of aurorae show unmistakably that these phenomena belong to atmospheric regions up to about 500 miles above the earth's surface. However, at heights greater than about 50 miles, the air is so rare as to be practically a vacuum so far as its refracting power is concerned; for our purpose, the height of the atmosphere may then be regarded as a small fraction of the earth's radius, the latter being 3960 miles.

88. *The law of astronomical refraction.*

Consider the path of a ray from a star which, after its passage through the atmosphere, eventually reaches an observer O on the earth's surface (Fig. 52). Owing to the small height of the atmosphere as compared with the earth's radius we can assume, for moderate values of the zenith distance, that the atmosphere consists of a large number of horizontal layers in which the density—and, consequently, the index of refraction—increases from the uppermost layer towards the surface layer.

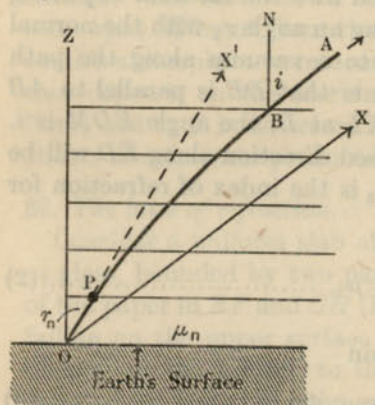

FIG. 52.

AB is the ray incident at the highest layer at B, the angle of incidence being i. If we draw OX parallel to BA, $\widehat{ZOX}$ is the *true zenith distance* of the star; we denote this angle by z. Thus $\widehat{ZOX} = z = i$. In the figure PO is the final element of the path of the ray before it reaches the observer, so that OP, or OX', gives the direction in which the observer sees the star.

We refer to $\widehat{ZOX'}$ as the *observed zenith distance* and denote it by ζ, or by r_n according to the nomenclature of the previous section. If μ_n is the index of refraction of the lowest layer we have from (4)

$$\sin i = \mu_n \sin r_n$$

so that

$$\sin z = \mu \sin \zeta, \quad \text{.........................(5)}$$

in which we write μ for μ_n.

It is to be noted (i) that the effect of refraction is to make the star appear *nearer to the zenith* than it would be if the atmosphere were non-existent, and (ii) that the observed position is on the vertical circle through the true position.

The *angle of refraction*, R, is the angle XOX' through which the star's observed direction is displaced from its true direction.

By definition,

$$R = z - \zeta. \quad \text{.........................(6)}$$

Hence, from (5) and (6),

$$\sin (R + \zeta) = \mu \sin \zeta,$$

or

$$\sin R \cos \zeta + \cos R \sin \zeta = \mu \sin \zeta. \quad \text{...............(7)}$$

Actually, R is a small angle and, if R is expressed in circular measure, we can write with sufficient accuracy

$$\sin R = R \text{ and } \cos R = 1,$$

so that (7) becomes

$$R = (\mu - 1) \tan \zeta. \quad \text{.........................(8)}$$

If R is now expressed in seconds of arc we have, since 1 radian = 206265″,

$$R = 206265 (\mu - 1) \tan \zeta,$$

or

$$R = k \tan \zeta, \quad \text{.................................(9)}$$

where k depends on the value of μ at the earth's surface; k is called the *constant of refraction*. Corresponding to the standard barometric height of 30 inches and the temperature of 50° F., the value of k is found from observations to be 58″·2; we can thus write (9) in the alternative form:

$$R = 58''\!\cdot\!2 \tan \zeta. \quad \text{.........................(10)}$$

In this case R is called the *mean refraction*.

In actual observations it is the angle ζ that is involved directly in the measures; by means of (10) we calculate R and then the true zenith distance, z, is given, from (6), by $\zeta+R$. In this way we eliminate the effect of astronomical refraction from our observations.

The formula (10) shows that the angle of refraction is proportional to the tangent of the observed zenith distance, and by reason of the approximations introduced into its derivation it may be regarded as valid for zenith distances up to about 50° and as a fairly good approximation up to about 70°, if exceptional accuracy is not required. For large zenith distances the appropriate accurate formula is a much more complicated one and for zenith distances close to 90° the refraction cannot be represented by a practicable formula at all.

89. *Refraction for non-standard conditions.*

The constant of refraction, k, has been defined with reference to standard atmospherical conditions. If R_1 denotes the refraction when the barometric height is B inches and the temperature is T degrees on the Fahrenheit scale, the relationship between the actual refraction R_1 and the mean refraction R is given by

$$\frac{R_1}{R}=\frac{17B}{460+T}, \qquad \text{.............................(11)}$$

so that

$$R_1=\frac{17B}{460+T}\,k\tan\zeta. \qquad \text{.....................(12)}$$

90. *Measurement of the constant of refraction.*

One method depends on measuring the observed zenith distances of a circumpolar star at upper and lower culmination by means of the meridian circle. We suppose for simplicity that the observations are made at standard atmospheric pressure and temperature; if these conditions do not apply, we can make use of (12) at each culmination.

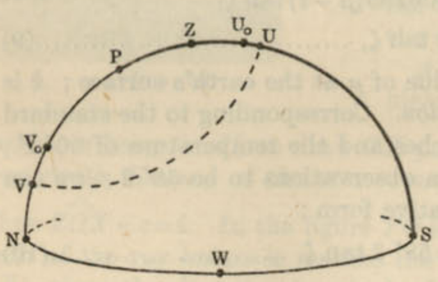

FIG. 53.

Let ϕ be the latitude which will be assumed known—we take ϕ to be north latitude—and let δ be the true declination of the star. Then, in Fig. 53, $PZ=90°-\phi$, and $PU=90°-\delta=PV$. We consider the case where the star transits south of the zenith. The star is observed at upper culmination at U_0 (on the vertical circle ZU) and at lower culmination at V_0 (on the vertical circle ZV). We denote by ζ and ζ_1 the observed zenith distances ZU_0 and ZV_0.

Now $$ZU=ZU_0+U_0U=\zeta+k\tan\zeta,$$

and since $PU=PZ+ZU$, we obtain

$$90°-\delta=90°-\phi+\zeta+k\tan\zeta,$$

or

$$\delta=\phi-\zeta-k\tan\zeta. \qquad \text{........................(13)}$$

At lower culmination, $VV_0=k\tan\zeta_1$, and since

$$ZV=PZ+PV=180°-\phi-\delta,$$

we obtain

$$180°-\phi-\delta=\zeta_1+k\tan\zeta_1. \qquad \text{...............(14)}$$

Adding (13) and (14) so as to eliminate δ, which may not be known with the requisite accuracy, we have

$$180°-2\phi=\zeta_1-\zeta+k(\tan\zeta_1-\tan\zeta). \qquad \text{............(15)}$$

As ϕ is supposed known and ζ_1 and ζ are observed angles, we can calculate k from this formula.

With the value of k accurately determined the declination of any star can be derived by means of (13) or (14) or by a combination of these two equations found by eliminating ϕ (so as to dispose of any uncertainty in the value of ϕ); this relation is

$$2\delta=180°-\zeta-\zeta_1-k(\tan\zeta+\tan\zeta_1). \qquad \text{............(16)}$$

In Fig. 53 we have considered a star culminating at U, south of the zenith; the condition for this is that $\delta<\phi$. If $\delta>\phi$ the star's upper culmination occurs at a point between P and Z; the appropriate formulae, corresponding to (13), (14), (15) and (16) can be readily obtained from a figure.

Ex. 1. In latitude 60° N the observed zenith distances of a circumpolar star ($\delta<\phi$) are 3° 19′ 56″·6 and 63° 18′ 04″·4 at upper and lower transits. To calculate k (we assume standard atmospherical conditions for each observation).

From (15) we have

$$60° = 63° \, 18' \, 04''{\cdot}4 - 3° \, 19' \, 56''{\cdot}6 + k(\tan \zeta_1 - \tan \zeta).$$

Inserting the values of $\tan \zeta_1$ and $\tan \zeta$ we obtain

$$1' \, 52''{\cdot}2 = k(1{\cdot}9883 - 0{\cdot}0582),$$

so that $$112''{\cdot}2 = 1{\cdot}9301k,$$

whence $$k = 58''{\cdot}1.$$

91. *Horizontal refraction.*

The refraction for a heavenly body observed on the horizon, that is when its observed zenith distance is 90°, is 35′; this is called for convenience the *horizontal refraction.*

One effect of refraction is to increase the interval during which the sun is seen above the horizon. Near sunset, for example, when the true zenith distance of the sun's centre is 90°, the sun is observed to be still above the horizon, the altitude of its centre being about 35′. Sunset will not occur till some time later when the observed zenith distance of the sun's centre has increased to 90°, the true zenith distance of the sun's centre being then 90° 35′. Similarly, sunrise will occur somewhat earlier owing to refraction. It is found, for example, that at the equator on March 21 or September 21 the interval during which the sun is observed above the horizon is increased, owing to refraction, by about $4\frac{1}{2}$ minutes.

Another effect of refraction is to make the sun at or near sunset appear slightly oval in shape. If A and B in Fig. 54 are the ends of a vertical diameter of the sun's disc in its true position near the horizon, the point A will be raised by refraction towards the zenith to A_1 and B to B_1. But, inasmuch as the zenith distance of A_1 is greater than the zenith distance of B_1, the displacement of A to A_1 will be greater than the displacement of B to B_1—in other words, the vertical angular diameter A_1B_1 as observed will be somewhat less than the true angular diameter AB, AB being equivalent to the horizontal diameter, the length of which (about 32′) will remain unaffected by refraction. The sun will accordingly appear slightly oval in shape, as represented in the upper portion of Fig. 54.

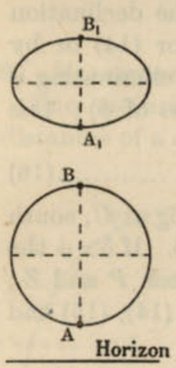

FIG. 54.

92. *Dip of the sea horizon.*

Let O be an observer at a height h ($\equiv AO$) above sea-level (Fig. 55). In a given vertical plane the sea horizon will be in the direction OT which is drawn tangential to the circle in which the given plane cuts the spherical earth; we omit for the present the effects of refraction. OH is perpendicular to OZ, the direction of the zenith, and gives the direction of the theoretical horizon. The angle HOT, which we denote by θ, is called the *dip of the horizon.* It is readily seen that $\theta = \widehat{TCA}$, where C is the earth's centre. Let a denote the earth's radius. Then $OC = a + h$ and

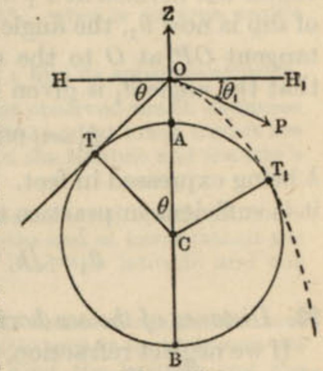

FIG. 55.

$$\cos \theta = \frac{TC}{OC} = \frac{a}{a+h}, \qquad \text{(17)}$$

whence

$$1 - 2\sin^2\frac{\theta}{2} = \frac{a}{a+h},$$

so that

$$\sin\frac{\theta}{2} = \sqrt{\frac{h}{2(a+h)}}.$$

Now θ is a small angle, for h is small in comparison with a, and if we express θ in circular measure, we have, with sufficient accuracy,

$$\theta = \sqrt{\frac{2h}{a}} \text{ radians.}$$

Express θ now in minutes of arc. Now 1 radian = 3438′; hence

$$\theta = 3438\sqrt{\frac{2h}{a}} \text{ minutes of arc.}$$

If h is expressed in feet, $a = 3960 \times 5280$ feet, and it is found that

$$\theta = 1{\cdot}06\sqrt{h} \text{ minutes of arc.} \qquad \text{(18)}$$

The effect of refraction is shown in Fig. 55. The air being less dense at O than at sea-level, the path of a ray from the visible sea horizon at T_1 will be curved as indicated. The angle of dip is now θ_1, the angle between the horizontal OH_1 and the tangent OP at O to the curved path OT_1. It can be shown that the angle θ_1 is given by

$$\theta_1 = 0{\cdot}98\sqrt{h} \text{ minutes of arc,}$$

h being expressed in feet. For values of h up to about 100 feet, it is sufficient in practice to write

$$\theta_1 = \sqrt{h} \text{ minutes of arc.........................(19)}$$

93. *Distance of the sea horizon.*

If we neglect refraction, the distance of the sea horizon is OT. Now, by a theorem in geometry, AB being the diameter through A,

$$OT^2 = OA \,.\, OB = h(2a+h),$$

so that, with sufficient accuracy,

$$OT = \sqrt{2ah}.$$

If OT is to be expressed in statute miles we then have (h being expressed in feet)

$$OT = \sqrt{\frac{2 \times 3960h}{5280}} = \sqrt{\frac{3}{2}h} = 1{\cdot}225\sqrt{h} \text{ statute miles.}$$

It is usual to express OT in nautical miles. Now one nautical mile is 6080 feet. We then find that

$$OT = 1{\cdot}06\sqrt{h} \text{ nautical miles. ....(20)}$$

If we take refraction into account the distance of the sea horizon is the distance between O and T_1 in Fig. 55. It can be shown that

$$OT_1 = 1{\cdot}15\sqrt{h} \text{ nautical miles. ....(21)}$$

The effect of refraction is thus to diminish the dip and to increase the distance of the sea horizon.

EXAMPLES

(Take the constant of refraction to be 58″·2.)

1. The apparent altitude of a star is 45°; what is its true zenith distance?

2. The true zenith distance of a star is 60°; find its apparent altitude.

3. At an observatory in north latitude the observed zenith distances of a star at upper transit (south of the zenith) and at lower transit are 7° 14′ 28″ and 68° 29′ 30″ respectively. Find the latitude and the star's declination.

4. At an observatory in north latitude the observed zenith distances of a star at upper transit (north of the zenith) and at lower transit are 19° 28′ 49″ and 61° 47′ 42″ respectively. Find the latitude and the star's declination.

5. At an observatory in south latitude the observed zenith distances of a star at upper transit (south of the zenith) and at lower transit are 23° 14′ 52″ and 65° 58′ 22″ respectively. Find the latitude and the star's declination.

6. Find the dip of the sea horizon when the height of eye is (i) 16 feet, (ii) 28 feet, (iii) 92 feet.

7. Find the true zenith distance of the star concerned from the following altitudes above the sea horizon: (i) 60° 24′, H.E., 15 feet; (ii) 28° 40′, H.E., 25 feet; (iii) 39° 16′, H.E., 52 feet.

8. Find the distance of the sea horizon for observers whose heights of eye are: (i) 15 feet, (ii) 25 feet, (iii) 45 feet.

9. The height of a lighthouse is 150 feet above sea-level. What is the greatest distance at which it can be seen by the captain of a ship on his bridge, his height of eye being 49 feet above sea-level?

10. At what distance will the top of a mountain 3000 feet high be seen by a look-out in a ship, his height of eye being 80 feet above sea-level?

CHAPTER VII

PARALLAX

94. *Angle of parallax.*

The principle involved in the determination of the distance of a heavenly body is fundamentally the same as the principle underlying the surveyor's method of measuring the distance of an inaccessible object from a given point. Suppose O is an object whose distance from A the surveyor requires (Fig. 56). He measures out carefully a suitable *base-line* AB, and by means of a theodolite he measures the angles OAB and OBA. If p denotes the angle $AOB (\equiv 180° - O\widehat{A}B - O\widehat{B}A)$ and d the distance AO, we have

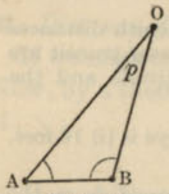

FIG. 56.

$$d = AB \cdot \frac{\sin O\widehat{B}A}{\sin p}, \quad \text{.........................(1)}$$

from which d can be calculated. The angle p is called the *angle of parallax* with reference to the base-line AB; it is the angle subtended at the object by the base-line.

Consider now a near heavenly body such as the moon or a planet which we denote by M (Fig. 57). C is the earth's centre, O the observer and OZ the direction of his zenith. The angle ZOM, or z_0, is the zenith distance of M as defined for the observer—we assume that the effect of refraction has been removed from the observed zenith distance—and this angle is the *apparent zenith*

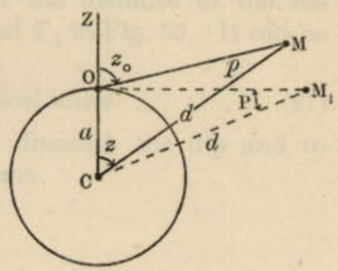

FIG. 57.

distance. We refer to the angle ZCM, z, as the *true zenith distance.* From the figure, denoting $O\widehat{M}C$ by p, we have

$$p = z_0 - z, \quad \text{..............................(2)}$$

and in this case p is the *angle of parallax*; it is the angle subtended at M by that radius of the earth terminated by the observer.

Since $z_0 > z$ the *effect of parallax is to displace the heavenly body in a vertical circle away from the zenith,* the direction CM being independent of the observer.

Let a be the earth's radius and d the geocentric distance CM of M. From the triangle COM we have

$$\sin p = \frac{a}{d} \sin z_0. \quad \text{..........................(3)}$$

Suppose for the moment that we know both a and d and that z_0 has been obtained from observation; then by (3) we can calculate p and by means of (2) we then can obtain z. As we explained in section 30 the declinations of the heavenly bodies such as the sun, moon and planets are defined with reference to the standard or geocentric celestial sphere, that is, the sphere centred at C. Consequently, if we have to make use of an observed zenith distance—as in the problem of determination of position at sea—in conjunction with the data of the *Nautical Almanac*, it is first necessary to correct z_0 for the effect of parallax; in other words, we require to find the value of z, the true zenith distance.

95. *Horizontal parallax.*

In Fig. 57 let M_1 be the position of the moon or planet when it is on the observer's horizon, the angle ZOM_1 being 90°. The angle OM_1C is called the *horizontal parallax*, which is denoted by P. It is evident that

$$\sin P = \frac{a}{d} \cdot \quad \text{..............................(4)}$$

Using (3), we obtain

$$\sin p = \sin P \sin z_0. \quad \text{..........................(5)}$$

For the sun, moon and planets d is very much larger than a so that in (5) p and P are small angles (for the moon, P is about

1° and, for the sun, 8″·8). We accordingly write $\sin p = p$ and $\sin P = P$, where p and P are in circular measure; (5) then becomes

$$p = P \sin z_0. \qquad (6)$$

It is evident that this formula holds if p and P are both expressed in seconds of arc. The values of P are given for the moon in the *Nautical Almanac* for every day of the year.

It is to be remarked that as the moon, for example, moves in its elliptic orbit around the earth, the geocentric distance d varies throughout the orbit. The horizontal parallax corresponding to the mean geocentric distance is called the *mean horizontal parallax*, or, in the case of the sun, simply the *solar parallax*; for the moon its value is 3423″ and for the sun 8″·790.

96. *Measurement of the moon's distance.*

The base-line utilised in measuring the moon's geocentric distance is effectually defined by two selected observatories in or near the same meridian of longitude, one in the northern hemisphere and the other in the southern. In Fig. 58 let O' and O'' be the two observatories which we shall suppose to be on the same meridian $PO'AO''$, A being the intersection of this meridian with the equator. Then if ϕ', ϕ'' denote the latitudes of O' and O'', $O'\widehat{C}A = \phi'$ and $O''\widehat{C}A = \phi''$. Hence

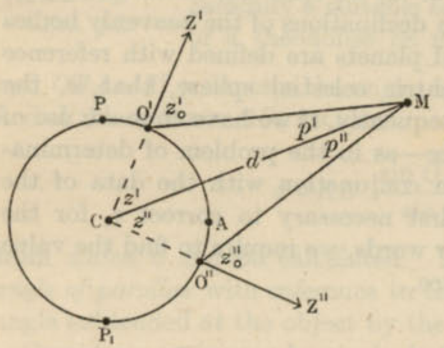

Fig. 58.

$$O'CO'' = \phi' + \phi''. \qquad (7)$$

Simultaneous observations with the meridian circle at O' and O'' give the apparent zenith distances z_0' and z_0'' corresponding to the true zenith distances z' and z'' as shown in the figure. But $z' + z'' = O'\widehat{C}O''$. Hence by (7)

$$z' + z'' = \phi' + \phi''. \qquad (8)$$

Let p' and p'' be the angles of parallax for O' and O''. Then by (6),

$$p' = P \sin z_0', \quad p'' = P \sin z_0'', \qquad (9)$$

whence $\quad p' + p'' = P(\sin z_0' + \sin z_0''). \qquad (10)$

But $\quad p' + p'' = z_0' - z' + z_0'' - z''$

$$= z_0' + z_0'' - \phi' - \phi'', \text{ using (8)}$$

$$= \theta, \text{ say.}$$

With the values of z_0' and z_0'' determined from observations and ϕ', ϕ'' known, the value of θ is readily obtained. Hence by (10) we have

$$P(\sin z_0' + \sin z_0'') = \theta, \qquad (11)$$

from which P is obtained. The geocentric distance d is then found by means of (4).

The observatories at Greenwich and the Cape of Good Hope satisfy the conditions approximately—the difference of longitude is about $1^h\ 14^m$; allowance has to be made in the calculations for the difference in declination of the moon between its meridian passage at the Cape and at Greenwich.

Successive observations will give successive values of d, from which the semi-major axis and the eccentricity of the lunar orbit can be deduced. It is found that the semi-major axis is about 240,000 miles and that the eccentricity is 0·0549. Thus the geocentric distance varies between about 222,000 and 253,000 miles, corresponding to perigee and apogee respectively.

It is to be noted that in the explanation given it has been assumed that the zenith distances of the moon's centre have been observed. In practice a well-defined lunar crater or mountain is observed and the value of d deduced is then the distance of the crater or mountain from the earth's centre. With a knowledge of the moon's radius in miles—we consider this in the next section—the geocentric distance of the moon's centre is finally obtained.

97. *Semi-diameter.*

The angle subtended at the earth's centre by the radius of a near heavenly body such as the moon or sun or a planet is called the angular semi-diameter or, simply, *semi-diameter*. In

Fig. 59, if CL is tangential to the disc of the moon the angle MCL is the semi-diameter, denoted by S. Then, r being the moon's radius in miles, and a the earth's radius in miles,

$$\sin S = \frac{r}{d} = \frac{r}{a} \cdot \frac{a}{d} = \frac{r}{a} \sin P\ ; \quad \ldots.(12)$$

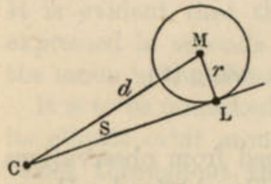

FIG. 59.

or, since S and P are comparatively small angles, we can write

$$S = P \cdot \frac{r}{a}. \quad \ldots\ldots\ldots\ldots.(13)$$

If we deduce the horizontal parallax P as in the previous section and measure S we obtain r from (13) in miles, a being of course known. In this way it is found that the radii of the moon and sun are 1080 and 432,000 miles respectively.

To obtain S from measures it is to be remembered that the distance of the observer from the moon's centre is slightly different from the geocentric distance d. Referring to Fig. 58 and denoting $O'M$ by ρ, the semi-diameter measured at O'—we denote it by S_0—will be given by

$$\sin S_0 = \frac{r}{\rho} = \frac{r}{d} \cdot \frac{d}{\rho},$$

from which, by (12) and by making the usual approximation,

$$S_0 = S \cdot \frac{d}{\rho}.$$

From the triangle $O'CM$ we obtain the ratio of d to ρ, and we have

$$S_0 = S \cdot \frac{\sin z_0'}{\sin z'} = S \cdot \frac{\sin z_0'}{\sin (z_0' - p')}. \quad \ldots\ldots\ldots\ldots.(14)$$

With S_0, z_0' and p' known we obtain S.

The sun's semi-diameter varies between $16' \, 17''\cdot5$ and $15' \, 45''\cdot4$ and the moon's between $16'\cdot8$ and $14'\cdot7$, these variations being due to the variations in the geocentric distances resulting from motions in elliptic orbits.

From these data we can deduce the eccentricity of the sun's apparent orbit. If d_1, d_2 denote the minimum and maximum geocentric distances of the sun and S_1, S_2 the corresponding semi-diameters, we have from (12)

$$\sin S_1 = \frac{r}{d_1}, \quad \sin S_2 = \frac{r}{d_2},$$

so that, with the usual approximation,

$$S_1/S_2 = d_2/d_1.$$

But, if a denotes the semi-major axis of the sun's apparent orbit and e the eccentricity, then $d_1 = a(1-e)$ and $d_2 = a(1+e)$.

Hence $$S_1/S_2 = (1+e)/(1-e),$$

whence $$e = \frac{S_1 - S_2}{S_1 + S_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots.(15)$$

Inserting the values of S_1 and S_2 already given we find that $e = 0\cdot0167$, agreeing approximately with the value obtained by more accurate methods. The approximate eccentricity of the moon's orbit can be deduced in the same way.

98. *Measurement of a planet's distance.*

First method. This is essentially the same method as described in section 96. The zenith distances of the planet at the two observatories can be measured by means of the meridian circle.

Second method. The principal disadvantage of the previous method (or any variant of it) is due to the necessity for co-operation between two widely separated observatories, employing different instruments under generally different conditions, and to the possibility of unfavourable weather conditions at one or other of the observatories. In the method to be described the observations can be undertaken by a single observatory, the essential base-line being that defined by the change in the position of the observatory due partly to the diurnal motion and partly to the earth's orbital motion round the sun. The observations are made within a few weeks of opposition—the method is only applied to a superior planet. At opposition the planet is on the observer's meridian at apparent midnight, and so it is generally practicable to observe the planet when it is three or four hours east of the meridian (evening observation) and when it is three or four hours west of the meridian (morning observation). Now the effect of parallax, as mentioned in section 94, is to displace the planet's position along a vertical circle and away from the observer's zenith, and this is equivalent to a calculable displacement in hour angle—or in right ascension—and in declination. Consider Fig. 60 drawn for an observer in north latitude ϕ. Let A be the true, or geocentric, position of the planet on the celestial sphere. Then the direction of the planet as seen by the observer will be at B, say, which lies on the vertical circle through A, with $ZB > ZA$, and such that AB is the angle, p, of parallax (see, for example, Fig. 57). The meridians through A and B cut the equator in C and D. Thus it is clear from the figure that, with the planet west, the R.A. of the apparent position B is less by CD than the R.A. of the geocentric position A. In the same way the effect of parallax is to decrease the declination by an amount $(AC - BD)$.

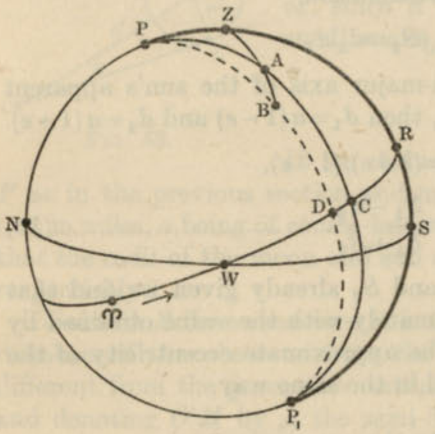

FIG. 60.

Let α_1, δ_1, H_1 denote the R.A., declination and hour angle of the position A. Let α' denote the R.A. of B. Then it is found that

$$\alpha' - \alpha_1 = -\frac{a}{d_1}\cos\phi\sec\delta_1\sin H_1, \quad \ldots\ldots\ldots\ldots(16)$$

in which d_1 is the geocentric distance of the planet and a is the earth's radius.

At the previous evening observation (with the planet east) we denote the corresponding R.A., declination and hour angle of the geocentric position by α_2, δ_2 and H_2, and we denote the geocentric distance by d_2; also α'' denotes the R.A. of the apparent position. Then

$$\alpha'' - \alpha_2 = -\frac{a}{d_2}\cos\phi\sec\delta_2\sin H_2. \quad \ldots\ldots\ldots\ldots(17)$$

Let α_0 be the R.A. of a star very close to the planet in the sky. At the morning observation the difference between the R.A.'s of the planet (apparent position B in Fig. 60) and of the star is measured. This can be done by means of the meridian circle (if the objects concerned are not too faint), or by an accurate instrument called the filar micrometer, or most easily and accurately by means of photography. Let m_1 denote $\alpha' - \alpha_0$, so that $\alpha' = m_1 + \alpha_0$. The left-hand side of (16) can now be written $(m_1 - \alpha_1 + \alpha_0)$. In the same way the left-hand side of (17) can be written $(m_2 - \alpha_2 + \alpha_0)$ where m_2 denotes the difference $\alpha'' - \alpha_0$, the star's R.A. being constant. By subtraction of the two equations we obtain

$$m_1 - m_2 = \alpha_1 - \alpha_2 + a\cos\phi\left\{\frac{\sec\delta_2\sin H_2}{d_2} - \frac{\sec\delta_1\sin H_1}{d_1}\right\}. \ldots(18)$$

In this formula m_1 and m_2 are derived directly by observation; by processes which we need not specify, the difference $\alpha_1 - \alpha_2$ can be estimated with sufficient accuracy; also, at or near opposition, the value of d_1 will differ very little from d_2 so that in (18) we may put $d_1 = d_2 = d$; further, for our purpose, it will be sufficient to suppose that $\delta_1 = \delta_2 = \delta$, the value of δ being taken to be that at transit. Then (18) becomes

$$m_1 - m_2 = \alpha_1 - \alpha_2 + \frac{a}{d}\cos\phi\sec\delta(\sin H_2 - \sin H_1). \ldots(19)$$

This equation enables us to calculate d in miles, all the other quantities being known, including a which is 3960 miles; we can suppose that d refers to the planet's geocentric distance midway between the evening and morning observations.

If the displacements in declination are utilised the general procedure is analogous to that just described.

In these observations the star is used as an intermediary to provide a direction of reference to which the apparent positions of the planet can be specified accurately. In observations of a minor planet such as Eros, which is a very faint object, a star of comparable faintness is chosen (in actual practice several stars are employed) and, as its right ascension α_0 is hardly likely to be known with the accuracy required, the procedure represented in the derivation of (18) amounts to the elimination of the uncertain, or unknown, quantity α_0.

99. *The solar parallax.*

The determination of the distance of a planet at a given instant enables us to derive the earth's distance from the sun. It has been remarked in section 73 that a knowledge of the orbital periods of the planets enables us to construct a chart or model of the solar system in which distances are expressed in terms of the *astronomical unit* or, alternatively, to calculate the geocentric distance of any planet at a given instant in terms of the astronomical unit. If we determine the planet's geocentric distance in miles at this instant by one of the methods outlined in the previous section we obtain the value of the astronomical unit in miles. The result is

$$1 \text{ astron. unit} = 93{,}003{,}000 \text{ miles} = 149{,}674{,}000 \text{ km.} \quad \ldots(20)$$

and the solar parallax is $8''\cdot 790$.

At the opposition of the minor planet Eros in 1900-1, the planet was observed at nearly a score of observatories, and after many years spent in reducing the observations the value of the solar parallax was obtained. A favourable opposition occurred again in 1930-1, and the results of the observations (as given above) have just (1941) been announced.

The selection of Eros for these long investigations into the value of the solar parallax is due, firstly, to the fact that the planet can come, owing to the large eccentricity of its orbit, comparatively close to the earth so that the value of its geocentric distance at a suitable opposition can be determined with a minimum percentage error and, secondly, to the fact that its disc—unlike that, say, of Mars or Jupiter—is extremely minute so that little error is possible in estimating where the centre of the disc is situated.

The method of deriving the solar parallax from observations of transits of Venus across the sun's disc is only now of historical interest and will not be further referred to.

100. *Stellar parallax.*

The dimensions of the earth are so small in comparison with the immense distances of the stars that it is impossible to use base-lines, as defined in the previous sections, for the determination of stellar parallax. Accordingly, a very much larger

base-line must be found, and this is provided as a consequence of the earth's annual motion round the sun. For this reason stellar parallaxes are sometimes called *annual parallaxes*.

Let X be a near star at a distance d from the sun, S (Fig. 61). Let a be the radius of the earth's orbit which we assume, for simplicity, to be circular. We shall further assume, for simplicity, that the direction of X from S, with respect to the background of the very faint, and, presumably, very distant stars, remains constant. We shall refer to the direction SX as the *heliocentric* direction of the star. On a given date the earth will be at some such position E in its orbit so that the star will be seen from the earth in the direction EX which makes an angle p with the heliocentric direction SX. As the earth moves in its orbit around the sun, the direction EX of the star as viewed from the earth will change with reference to the constant direction SX and also with reference to the background of the faint stars. It is by measuring the changes in the geocentric directions that we are enabled to find the star's distance.

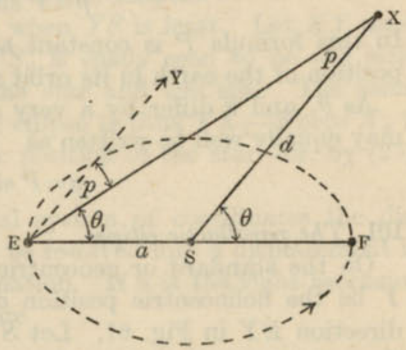

FIG. 61.

Let EY be drawn parallel to SX. Then EY gives the heliocentric direction of the star when the earth is at E. Denote the angles XES and XSF in Fig. 61 by θ_1 and θ. Then

$$\frac{\sin p}{a} = \frac{\sin \theta_1}{d},$$

or

$$\sin p = \frac{a}{d} \sin \theta_1, \quad \ldots\ldots(21)$$

in which we suppose that p is expressed in seconds of arc.

We define the parallax, P, of the star by

$$\sin P = \frac{a}{d}. \quad \ldots\ldots(22)$$

P is the angle, in seconds of arc, subtended at the star by a radius of the earth's orbit perpendicular to the heliocentric direction of the star. From (21) and (22) we have

$$\sin p = \sin P \sin \theta_1 ;$$

or, since p and P are very small angles,

$$p = P \sin \theta_1 . \quad \text{.........................(23)}$$

In this formula P is constant and p varies according to the position of the earth in its orbit round the sun.

As θ_1 and θ differ by a very small angle, the formula (23) may equally well be written as

$$p = P \sin \theta . \quad \text{.........................(24)}$$

101. *The parallactic ellipse.*

On the standard or geocentric celestial sphere (Fig. 62) let Y be the heliocentric position of the star as defined by the direction EY in Fig. 61. Let S be the position of the sun in the ecliptic on the date concerned. Now, it is seen from Fig. 61 that the directions EY, EX and ES are coplanar; hence the geocentric position of the star will be at a point X (Fig. 62) on the great circle joining Y and S. The arc $YS = \theta$, since this arc is the same as the angle between EY and ES in Fig. 61; similarly $YX = p$. Hence, by (24),

$$YX = P \sin YS . \quad \text{...(25)}$$

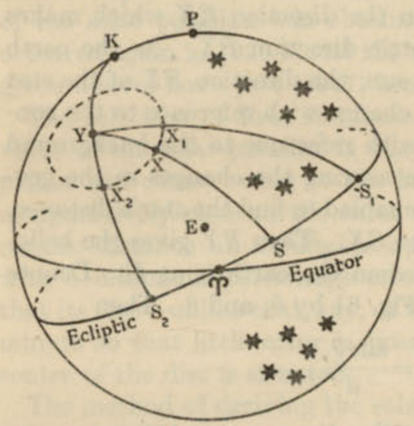

FIG. 62.

As S describes its apparent path around the earth, X will describe a path on the celestial sphere around Y; this path is found to be an ellipse, called the *parallactic ellipse* (shown on a magnified scale by a dotted line in the figure). Now, from (25), YX will be a maximum when the angular distance of the sun from Y is 90°, corresponding to the sun's position S_1, say, on the ecliptic, YS_1 being 90°. As S_1 is 90° from Y and 90° from K (since K is the

pole of the ecliptic), S_1 must be the pole of the great circle KY; accordingly, $\widehat{KYS_1}$ is 90°. If X_1 is the corresponding geocentric position of the star, YX_1 is the semi-major axis of the parallactic ellipse and it is, by (25), equal to P; moreover, since $\widehat{KYX_1}$ is 90°, YX_1 is parallel to the ecliptic.

By (25), YX will be least when YS is least. Let KY meet the ecliptic in S_2. Then YS_2 is easily seen to be the least angular distance between the star and the sun. The semi-minor axis of the parallactic ellipse is thus YX_2—where X_2 is the corresponding geocentric position of the star—or, by (25), $P \sin YS_2$ or $P \cos KY$.

Referred to the equatorial system of coordinates the displacement from Y to X can be resolved into a displacement in right ascension and in declination. If α is the right ascension of Y and α_1 of X, we can write

$$\alpha_1 - \alpha = F_1 . P, \quad \text{.........................(26)}$$

where F_1 is called the *parallax factor* in right ascension, corresponding to the particular date to which the position of S refers. F_1 depends on the star's equatorial coordinates, the sun's longitude (♈S) and the obliquity of the ecliptic. The dates of observation are chosen so that the numerical value of the parallax factor is as large as possible. The formula for declination, corresponding to (26), is seldom employed in practice.

To take a simple example, we consider a star at ♈ (heliocentric position). By (25), the maximum displacement of the geocentric position will occur when S is 90° from ♈, that is, on June 21 or December 21; the displacement is then P along the ecliptic, and since P is small this gives a displacement $P \cos \epsilon$ in right ascension. Thus the maximum parallax factor for this star is numerically $\cos \epsilon$, corresponding to observations made at the summer and winter solstices.

102. *The measurement of stellar parallax.*

The method adopted in practice is the differential method—analogous to the second method in section 98. It involves, in principle, the measurement of the difference in right ascension between the parallax star (the star whose parallax is to be measured) and a faint star close to it in the sky, the latter being presumed to be practically at an infinite distance from the earth. We have, then, the measure m_1 given by $m_1 = \alpha_1 - \alpha_0$, where α_0 is the right ascension of the faint star. But, from (26),

$$\alpha_1 - \alpha_0 - (\alpha - \alpha_0) = F_1 \,.\, P,$$

or

$$m_1 - (\alpha - \alpha_0) = F_1 \,.\, P. \quad \text{.....................(27)}$$

About six months later the observation is repeated and we have in this case a similar formula :

$$m_2 - (\alpha - \alpha_0) = F_2 \,.\, P. \quad \text{.....................(28)}$$

Subtracting (27) and (28) we obtain

$$m_1 - m_2 = P \,.\, (F_1 - F_2), \quad \text{.....................(29)}$$

from which P is calculated. In (29) the sign of F_2 is opposite to that of F_1.

The observations nowadays are all made photographically ; also three or four stars are used, each giving an equation of the type (29).

In 1838 Bessel announced the first successful measurement of stellar parallax ; he found that the parallax of 61 Cygni was 0″·314, the best modern result being 0″·300. This achievement was quickly followed by two more—Henderson's measurement of the parallax of α Centauri and Struve's result for Vega. The nearest star is α Centauri with a parallax of 0″·758, and there are, so far as is known at present, a score of stars with parallaxes of 0″·25 or greater.

103. *The parsec.*

Just as it is convenient to use the astronomical unit instead of miles for distances in the planetary system, so it is convenient to use a particular unit for the immense stellar distances. This unit is called the *parsec*, and is defined as the distance corresponding to a parallax of one second of arc. Now, by (22), $\sin P = a/d$; as P is expressed in seconds of arc, its value in circular measure is $P/206265$ radians (1 radian = 206265″). Also, since P is small, we can write (22) in the form

$$\frac{P}{206265} = \frac{a}{d}. \quad \text{.............................(30)}$$

Hence, if $P = 1''$, the corresponding distance d is 206265 a. Now a is one astronomical unit. Hence

1 parsec = 206,265 astronomical units. ....(31)

Since 1 astronomical unit = $93{\cdot}003 \times 10^6$ miles,

1 parsec = $19{\cdot}183 \times 10^{12}$ miles, or $30{\cdot}872 \times 10^{12}$ kilometres.

The distance of a star whose parallax is p seconds of arc is $1/p$ parsecs.

Ex. 1. The distance of 61 Cygni (parallax 0″·3) is 1/0·3 or $3\frac{1}{3}$ parsecs, or 64×10^{12} miles.

Ex. 2. The distance of a star whose parallax is 0″·025 is 1/0·025 or 40 parsecs, or 767×10^{12} miles.

104. *The light-year.*

This is a unit of distance much used in popular writings in place of the more technical unit, the parsec. One light-year is defined as the distance travelled by light in one year. Now the velocity of light is 186,271 miles or 299,774 kilometres per second, and as one year is $31{\cdot}56 \times 10^6$ seconds, one light-year is equal to $5{\cdot}88 \times 10^{12}$ miles or $9{\cdot}46 \times 10^{12}$ kilometres. The relation between the parsec and the light-year is easily found ; it is

1 parsec = 3·26 light-years.

Thus, the distance of 61 Cygni is $3\frac{1}{3}$ parsecs, which is equivalent to $10\frac{5}{6}$ light-years. Interpreted in another way, this means that the light which enters our eye when we observe 61 Cygni on a given date has been travelling through interstellar space for nearly $10\frac{5}{6}$ years previously.

EXAMPLES

(In these examples, unless it is otherwise stated, the correction for parallax only is to be applied.)

1. The sun's horizontal parallax is 8″·8 ; find its true zenith distance if the observed zenith distance is (i) 30°, (ii) 70°.

2. Find the sun's distance in miles, given that the horizontal parallax is 8″·8, and the earth's radius is 3960 miles.

3. If the moon's horizontal parallax is 60′·8, find the true zenith distance if the observed altitude above the celestial horizon is 38° 45′.

4. On a given day the moon's horizontal parallax is 54′·9 and its semi-diameter is 15′·0. What would be (i) its horizontal parallax when its semi-diameter is 16′·6, and (ii) its semi-diameter when its horizontal parallax is 54′·0?

5. If the earth's radius is 3960 miles, find (i) the moon's geocentric distance when the horizontal parallax is 57′·2, and (ii) the horizontal parallax when its geocentric distance is 230,000 miles.

6. The moon's maximum and minimum geocentric distances are 252,120 miles and 225,880 miles respectively. Find the corresponding horizontal parallaxes; the earth's radius is 3960 miles.

7. The semi-diameter and horizontal parallax of the moon on a certain day are 16′·1 and 59′·2. If the earth's radius is 3960 miles, find the moon's radius in miles.

8. At meridian transit in latitude 51° 30′ N, the observed zenith distance of the moon's upper limb is 35° 20′ 45″; the moon's semi-diameter is 16′ 9″, and the horizontal parallax is 59′ 16″. Taking refraction into account, find the moon's declination.

9. The moon transits in the zenith of a certain place, and its measured semi-diameter is 15′ 46″. What semi-diameter would you expect to obtain from the *Nautical Almanac*, given that the horizontal parallax is 57′ 52″?

10. Find the eccentricity of the moon's orbit given that the maximum and minimum values of the semi-diameter are 16′·5 and 14′·8.

11. The parallaxes of two stars are 0″·300 and 0″·023. Find (i) their distances in parsecs, (ii) in miles, (iii) in light-years.

12. A star is 72 light-years away; what is its parallax?

13. A star is 35 parsecs away; what is its parallax?

14. A star A of parallax 0″·050 is 3·5 times further away than a star B. What is the parallax of B?

15. What must be the parallax of a star, the light from which reaching us in 1941 has been travelling through space since the Battle of Bannockburn (1314)?

CHAPTER VIII

ABERRATION, PRECESSION AND NUTATION

105. *Introduction.*

It has not infrequently happened in the history of astronomy that a new phenomenon has been brought to light from a series of observations designed to solve a wholly different problem. Aberration was discovered by Bradley by means of observations with the meridian circle begun in 1725 for the purpose of measuring the parallax of γ Draconis. Bradley selected this bright star because its point of meridian transit almost coincided with the zenith of his observatory; thus, owing to its extremely small zenith distance at culmination, its declination could be deduced with considerable accuracy even if the value of k in the formula for refraction were somewhat uncertain. The right ascension of the star happens to be not much different from 18^h, and so, for purposes of illustration, we shall suppose its right ascension to be precisely 18^h.

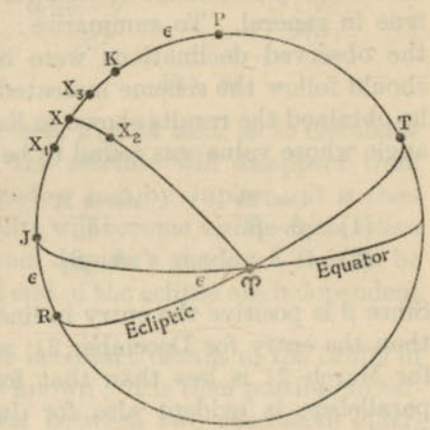

FIG. 63.

In Fig. 63 we show the equator and ecliptic with their poles P and K. Since the star's right ascension is 18^h, it will lie on the meridian PKR, say, at X; this position we will take to be the heliocentric position as described in the previous chapter. Now, on December 21 when the sun is at R, the star will be displaced, owing to parallax, to X_1, where [by (25), p. 126]

$XX_1 = P \sin XR$, P being the star's parallax. If δ denotes the declination of the heliocentric position X and δ_1 the declination of the observed position X_1, we have from the figure that $\delta_1 = \delta - P \sin XR$. Also it is seen from the figure that $XR = XJ + JR = \delta + \epsilon$, where ϵ is the obliquity of the ecliptic. Now, for γ Draconis δ is about 51° and, as ϵ is about $23\frac{1}{2}°$, XR is about $74\frac{1}{2}°$, so that $P \sin XR$ is positive; the latter quantity we write, for convenience, as β, so that the star's observed declination on December 21 would be $\delta - \beta$, in which β is positive.

On March 21 the sun is at ♈ and, due to parallax, the star's observed position would be at X_2 on the great circle X♈. But X♈ is perpendicular to PX, so that for a small displacement XX_2, the position of X_2 would practically lie on the parallel of declination through X; in other words, the declination of X_2 would be simply δ for all practical purposes. On June 21 the sun is at T and, due to parallax, the star will be observed at X_3; its declination is then $\delta + P \sin XT$, or $\delta + P \sin XR$ (since $XR + XT = 180°$) or, in our notation, $\delta + \beta$. Similarly, on September 21 the star's observed declination would be δ. It will be noticed that of the observations considered δ is the mean of any two made at an interval of six months and this is true in general. To summarise: Bradley anticipated that if the observed declinations were influenced by parallax they should follow the scheme indicated in line (1) below. Instead, he obtained the results shown in line (2), β_1 denoting a positive angle whose value was found to be close to 20″.

	Dec. 21	March 21	June 21	Sept. 21
(1)	$\delta - \beta$	δ	$\delta + \beta$	δ
(2)	δ	$\delta - \beta_1$	δ	$\delta + \beta_1$

Since β is positive the entry in line (1) for March 21 is greater than the entry for December 21, whereas in line (2) the entry for March 21 is less than that for December 21; this non-parallelism is evident also for June 21 and September 21. Further, we may summarise the contents of the two lines by saying that line (1) is out of step, with reference to the observations in line (2), by three months. It was concluded by Bradley that his observations could not be explained in terms of

parallax, and so he was forced to seek a different explanation, particularly as he observed a similar sequence of changes for other stars.

106. *Velocity of light.*

Half a century before (in 1675) Roemer had deduced from observations of the eclipses of Jupiter's satellites that light is propagated with a finite velocity. On the Copernican theory the earth moves round the sun with a velocity which we now know to be $18\frac{1}{2}$ miles per second. Bradley found in the combination of these two velocities an explanation of his observations of γ Draconis and, incidentally, a further confirmation of the Copernican system.

Roemer's discovery may be illustrated by means of Fig. 64. Let S, E and J denote the sun, earth and Jupiter and consider one of Jupiter's satellites Io. For simplicity we assume that the orbits of E and J around the sun and the orbit of Io around Jupiter are circular and coplanar. Now planets and satellites are visible only because of the sunlight which illuminates

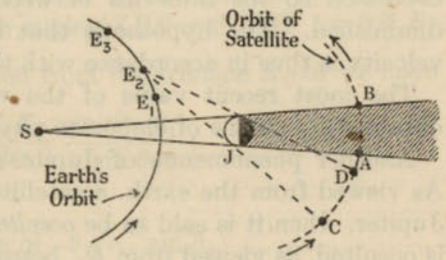

Fig. 64.

their surfaces, and it is clear from Fig. 64 that, as Io circulates around Jupiter in its orbit, the satellite will disappear from view at A as it enters the shadow cast by Jupiter; it is then said to be *eclipsed*. The satellite will become visible again when it reaches B as it emerges from Jupiter's shadow. It is to be noted that the beginning and end of the eclipse are independent of the earth's position.

We can assume that all the necessary details of the orbits of the earth, Jupiter and Io are known; it is then possible to calculate accurately the interval between two successive emergences of Io from Jupiter's shadow; this interval is, in fact, the satellite's synodic period with reference to Jupiter. Suppose an emergence occurs when the earth is at E_1. We can then calculate the time of the second, third, ... emergence, and we shall suppose that the relative configurations of the bodies concerned are indicated in Fig. 64 with the earth at E_2, E_3, ... Roemer found, in effect, that the *observed* times of emergence occurred progressively later than the predicted times as the earth's distance from Jupiter continued to increase. Now Io's emergence at B out of Jupiter's shadow is in the nature of a light signal, and the character of the discrepancies between the observed and predicted times of emergence suggested that these discrepancies represented the intervals required by light to traverse the additional distances between B and the positions E_1, E_2, E_3 ... . As the radius of Io's orbit is only 0·0028 astronomical unit (which is negligible compared with the earth's distance from Jupiter), these additional distances are practically the increases in the distances of E_1, E_2, E_3 ... from Jupiter. In the same way, as the earth's distance from Jupiter decreased so the intervals between the *observed* emergences diminished. The hypothesis that light travels with a finite velocity is thus in accordance with the observed phenomena.

The most recent value of the velocity of light has been obtained by means of elaborate physical experiments.

Another phenomenon of Jupiter's satellites may be noted. As viewed from the earth, a satellite such as Io passes behind Jupiter, when it is said to be *occulted*. In Fig. 64 the satellite is occulted, as viewed from E_2, between C and D.

107. *Aberration.*

Let c denote the velocity of light and v the earth's orbital velocity in the direction EF (Fig. 65). Let XE denote the path of a ray of light from a star which is seen by an observer when his eye is at E, which represents the eye-end of a telescope. If the observer were stationary at E, EX would be the true direction of the star at the time of observation (we neglect the effects of refraction and parallax); also, EX is parallel to the heliocentric direction of the star, since we may assume for our purpose here that the star is at an infinite distance. But when the ray of light enters the object-glass, O, of the telescope the observer is not actually at E, but at some such point as E_1 given by

$$\frac{E_1E}{EO}=\frac{v}{c}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This is simply the condition that the time required by the observer and the telescope to move from E_1 to E is the same as the time required by the ray to pass from O to E through the telescope (only the initial and final positions of the telescope are shown in Fig. 65).

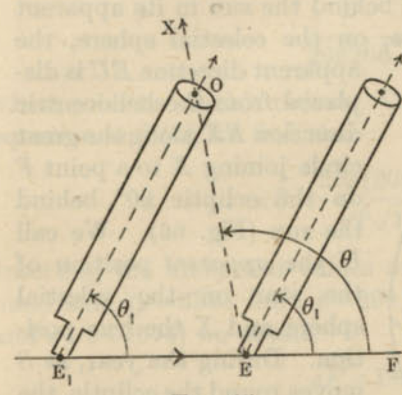

Fig. 65.

It follows that the direction in which the observer requires to point his telescope is E_1O or EU, where EU is parallel to E_1O. We designate EU as the *apparent direction* of the star at the moment of observation.

Let θ and θ_1 denote the angles XEF and UEF (or OE_1E), so that $E_1\widehat{O}E=\theta-\theta_1$. Then from the triangle E_1OE we have

$$\frac{E_1E}{EO}=\frac{\sin(\theta-\theta_1)}{\sin\theta_1},$$

or, using (1),

$$\sin(\theta-\theta_1)=\frac{v}{c}\sin\theta_1. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

As v/c is a small quantity ($v=18\frac{1}{2}$ miles per second and $c=186{,}271$ miles per second), we can write θ for θ_1 on the right of (2) without appreciable error. Also, expressing $\theta-\theta_1$ in seconds of arc, (2) becomes

$$\theta-\theta_1=206{,}265\frac{v}{c}\sin\theta; \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

or, putting

$$\kappa=206{,}265\frac{v}{c}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$\theta-\theta_1=\kappa\sin\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

In this last formula we refer to κ (expressed in seconds of arc) as the *constant of aberration*; $\theta-\theta_1$ is the angle of aberration. The direction of F is called the *apex of the earth's way*.

If we regard the earth's orbit as circular, the direction of the earth's motion is perpendicular to the radius joining the sun to the earth; with reference to the earth, the direction EF in Fig. 65 is easily seen to be 90° behind the sun in its apparent orbit round the earth. Hence, on the celestial sphere, the apparent direction EU is displaced from the heliocentric direction EX along the great circle joining X to a point F on the ecliptic 90° behind the sun (Fig. 66). We call U the *apparent* position of the star on the celestial sphere and X the *true* position. During the year, as S moves round the ecliptic, the apparent position U will describe a curve—it is an ellipse called the *aberrational ellipse*, shown as a dotted curve—around the true position X. As in the case of parallax, it can be similarly deduced that the major axis is parallel to the ecliptic; the semi-major axis corresponds to $\theta = 90°$ and its magnitude is κ. The semi-minor axis is $\kappa \sin XJ$; it is zero for a star on the ecliptic, and κ for a star at the pole of the ecliptic.

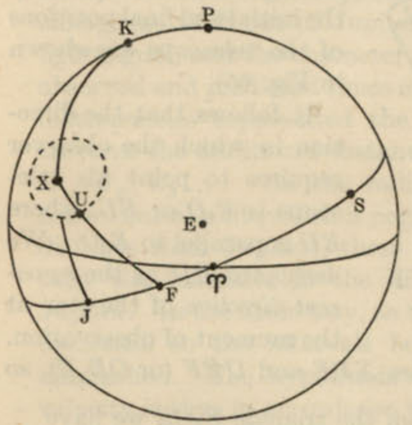

FIG. 66.

108. *The constant of aberration.*

If we regard the earth's orbit as circular, of radius a, the orbital velocity v is given in kilometres per second by $2\pi a/T$, where T is the number of seconds in a year, a being expressed in kilometres. Hence, from (4),

$$\kappa = 206{,}265 \frac{2\pi a}{cT}.$$

More accurately, the constant of aberration is defined by

$$\kappa = 206{,}265 \cdot \frac{2\pi a}{cT\sqrt{1-e^2}}, \quad \text{.....................}(6)$$

which takes account of the fact that the earth's orbit is an ellipse of eccentricity e, a now denoting the semi-major axis.

We can express (6) in a form involving the solar parallax P. If P is expressed in seconds of arc,

$$\frac{P}{206265} = \frac{\rho}{a},$$

where ρ is the earth's radius. On eliminating a between this equation and (6), we find that

$$\kappa P = \frac{2\pi\rho\,(206265)^2}{cT\sqrt{1-e^2}}.$$

Inserting the numerical values of the constants on the right-hand side ($\rho = 6368$ km., $c = 299{,}774$ km./sec., $T = 31{\cdot}56 \times 10^6$ and $e = 0{\cdot}01674$) we obtain

$$\kappa P = 180{\cdot}2. \quad \text{..............................}(7)$$

It follows that, if we can determine κ from observations of the stars, we can deduce the value of the solar parallax from (7) and hence the sun's distance from the earth. We refer to this again in the next section.

109. *Measurement of the constant of aberration.*

One method of determining κ is by means of meridian observations in which the observed declinations of suitable stars are obtained. Let δ_1 be the measured declination of the apparent position U in Fig. 66, and δ the declination of the true position X. The difference $\delta_1 - \delta$ will depend directly on κ, so that we can write

$$\delta_1 - \delta = \kappa Q_1,$$

where Q_1 is a quantity depending on the star's coordinates and on the position of the sun (or of F) in the ecliptic. Some months later we shall have a similar equation

$$\delta_2 - \delta = \kappa Q_2.$$

By subtraction we obtain

$$\delta_2 - \delta_1 = \kappa (Q_2 - Q_1).$$

As δ_1 and δ_2 are found from meridian observations and Q_1 and Q_2 can be calculated from the appropriate formulae, we can consequently derive the value of κ.

If we take the observational value of κ, derived in the way just described, to be $20''\cdot47$, the solar parallax is found, by means of (7), to be $8''\cdot804$ from which the sun's mean distance from the earth is $92\cdot9 \times 10^6$ miles or $149\cdot5 \times 10^6$ kilometres. At first sight this method of determining the solar parallax (or solar distance) seems the simplest and most accurate but, actually, the measurement of κ is bound up with a phenomenon known as the *variation of latitude*. This is due to a minute alteration in the direction of the earth's axis of rotation with reference to the earth's crust; accordingly, the latitude and longitude of a place on the earth's surface are not quite constant, although the variation is never more than about $0''\cdot5$. The difficulty of disentangling the variation of latitude from observations of the apparent declinations of the stars leads to some uncertainty in the deduced value of κ and, if (7) is used, in the calculated value of the solar parallax.

It is to be remarked that the *Nautical Almanac* contains data for the easy calculation of the effects of aberration on the right ascensions and declinations of the stars. If (α, δ) and (α_1, δ_1) are the equatorial coordinates of the true and apparent positions of a star (X and U in Fig. 66), then the appropriate formulae can be expressed in the forms:

$$\left.\begin{aligned} \alpha_1 - \alpha &= Cc + Dd \\ \delta_1 - \delta &= Cc' + Dd' \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

in which C and D depend on the sun's position in the ecliptic; their values are given for every day of the year in the *Nautical Almanac*. The quantities c, d, c' and d' depend on the star's coordinates (α, δ) and when they are once calculated they can be used on all relevant occasions.

110.* *Diurnal aberration.*

Diurnal aberration results from the observer's velocity due to the rotation of the earth. If ρ is the earth's radius, the observer (in latitude ϕ) is carried round a circle of circumference $2\pi\rho \cos\phi$ in one sidereal day. If κ_1 is the constant of diurnal aberration, we have as in (4)

$$\kappa_1 = 206265 \frac{2\pi\rho\cos\phi}{cT_1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

where T_1 is the number of seconds in one sidereal day. Inserting the usual numerical values we find that

$$\kappa_1 = 0''\cdot32 \cos\phi. \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

It is only in meridian observations that diurnal aberration is of importance. In particular, as the observer is being whirled eastwards by the earth's rotation, the observed position of the star (declination δ) will be to the east of its geocentric position; hence, the effect of diurnal aberration is to delay the transit by an amount which is found to be $\frac{1}{15} \times 0''\cdot32 \cos\phi \sec\delta$, or $0^s\cdot021 \cos\phi \sec\delta$. As the observer's velocity is perpendicular to his meridian, the meridian zenith distances of the stars are unaffected by diurnal aberration.

111. *Precession of the equinoxes.*

Up to the present we have assumed that the celestial equator and the ecliptic are constant in position relative to the background of the stars or, in other words, that the direction of the earth's axis of rotation and the plane in which the earth moves around the sun are fixed. As any two points, not exactly 180° apart, define a great circle on the celestial sphere, a fixed plane or great circle may be regarded as being defined by two faint stars which may be presumed to be at an infinite distance. It is in this sense that we consider the fixity, or otherwise, of the ecliptic and equator.

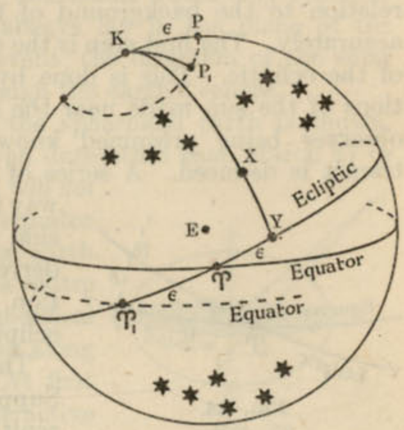

Fig. 67.

Observations of the positions of the bright stars over a number of years led Hipparchus about 125 B.C. to two conclusions which we illustrate in Fig. 67. (i) The latitude of each star—that is, its angular distance from the ecliptic—remained substantially constant (for the star X, the *latitude* is YX, KXY being a great circle arc through the pole, K, of the ecliptic); (ii) the longitude of each star increased by about 50″ per annum (in the figure ♈Y is the *longitude* of X). Hipparchus deduced from (i) that the ecliptic is, or is very nearly, a fixed great circle on the celestial sphere with reference to the background of the stars, and from (ii) that the equator alters in such a way that ♈, one of its points of intersection with the ecliptic, moves backwards along the ecliptic, say to ♈$_1$, at the rate of 50″·2 per annum. Further, as no change in the obliquity of the ecliptic was detected it followed that the pole P of the equator describes, at a uniform rate, a small circle of which K is the pole and the angular radius is ϵ. (It is easily seen from Fig. 67 that ♈♈$_1 = P\widehat{K}P_1$, P_1 being the pole of that equator which intersects the ecliptic in ♈$_1$.) The backward motion of ♈—and also, of course, of ♎—is called the *precession of the equinoxes*, or, simply, *precession*.

112. *Determination of* ♈.

The importance of ♈ in fundamental astronomy naturally leads to the question as to how this point in the heavens, in relation to the background of the stars, can be determined accurately. The first step is the determination of the obliquity of the ecliptic. This is done by means of meridian observations of the sun made near the solstices; the latitude of the observer being presumed known, the sun's declination at transit is deduced. A series of declinations obtained in this way for two or three days on either side of a solstice enables us to derive the sun's maximum declination, which is the obliquity of the ecliptic.

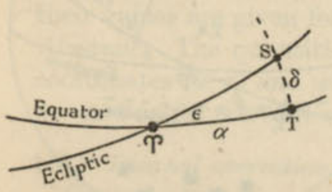

FIG. 68.

The second step is as follows. Suppose that the sun's meridian zenith distance is observed near an equinox; in the usual way we measure the sun's declination δ (TS, in Fig. 68). Let α be the sun's right ascension, ♈T, at the time of transit. Then, if we regard the triangle ♈ST as a plane triangle, we have $\alpha = \delta \cot \epsilon$. A more accurate formula is

$$\sin \alpha = \tan \delta \,.\, \cot \epsilon,$$

which would be used for observations made at a longer interval from an equinox. In either case, an error in ϵ will prove innocuous so far as the calculation of α is concerned, since α and δ are both small quantities. We thus can calculate the sun's right ascension, α, at transit, and this is the observer's sidereal time; the sidereal clock can now be set accurately or, as is more convenient in practice, its error can be accurately obtained. By observing the transits of stars we deduce their right ascensions. With their declinations also measured, we can then unambiguously define the position of ♈ and the equator with respect to the stars.

113. *Explanation of precession.*

Newton was the first to explain the phenomenon of precession. Consider the gravitational attraction of the sun on the earth. If the earth were a sphere, this attraction would act along the straight line joining the earth's centre to the sun's centre and it would have no effect in altering the direction of the earth's rotational axis. But, the earth being a spheroid, symmetrical about the equator, this result would only continue to hold if the sun's position always lay in the plane of the earth's equator; for, in this event, the direction of the solar attraction would still pass through the earth's centre.

Consider Fig. 69, in which the spheroidal earth is shown, with the sun at S. At any given date other than March 21 or September 21, the position of S will *not* lie in the plane of the earth's equator and the attraction of S on the earth will not pass through the earth's centre E—since the earth is not symmetrical about the line ES—but will act along some such direction as FS. At first sight it would seem that the attractive force in the direction FS would tend to make the axis PFP_1 take up a position perpendicular to ES so that, eventually, the earth's axis would be perpendicular to the plane of the ecliptic (the plane swept out by ES in the course of a year) or, in other words, so that the earth's equatorial plane would become coincident with the

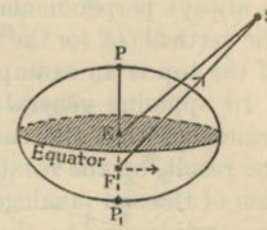

FIG. 69.

plane of the ecliptic. The actual result, however, is quite different.

We illustrate the dynamical principles involved by first considering the familiar behaviour of a spinning top. Let OGA be the axis about which it is rapidly spinning, the point O being on a rough horizontal floor. The weight W of the top acts vertically downwards through its centre of gravity, G. At first sight it would seem that the effect of the weight would tend to make the top lie on the floor; instead, the top behaves in such a way that its rotational axis moves round the vertical OK, tracing out a conical surface in the process. If KDC is a horizontal plane intersecting the cone in the circle CDE, the motion of the axis OAC may be described in terms of the motion of C towards D, along the circumference of the circle CDE, in the direction of the arrow. Now, the motion of C is, instantaneously, perpendicular to the radius KC; consequently, the direction of motion of the top's axis is always perpendicular to the plane containing the axis and the vertical OK (or the vertical through G). This conical motion of the top is an example of precession.

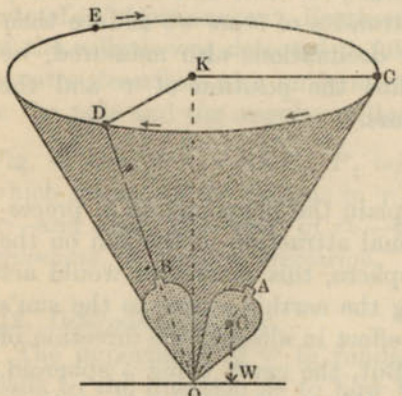

FIG. 70.

In applying general dynamical principles to the earth, we remember that the earth is spinning rapidly about its axis; the result, on the rotating earth, of the unsymmetrical attraction of the sun (analogous to the weight of the top) is to cause the earth's axis to change its direction continuously at right angles to the plane defined by the instantaneous position of the axis and the perpendicular to the ecliptic. In Fig. 67 the plane just mentioned is that defined by the great circle KP when the earth's axis points to P; consequently, the instantaneous motion of P is perpendicular to KP on the celestial sphere. When the pole is at P_1, the instantaneous motion is perpendicular to KP_1, and so on. The result is that the celestial pole P moves, owing to the sun's action, along a small circle of which K is the pole and whose angular radius is ϵ, the obliquity of the ecliptic.

The effect of the moon's attraction is mostly of a similar nature (we omit, for the present, certain additional effects), and the uniform motion of the celestial pole P along the small circle PP_1 due to the combined action of the sun and moon is called *luni-solar precession*. This leads to the continuous motion of the equator against the background of the stars, and, consequently, to the continuous and uniform motion of ♈ in the direction ♈♈$_1$ (Fig. 67). The rate at which ♈ moves backwards along the ecliptic is 50″·2 per annum. Thus, in about 26,000 years, P makes a complete circuit of the small circle PP_1 and ♈ makes a complete circuit of the ecliptic; this period is called the *precessional period*.

114. *Changes in a star's coordinates due to precession.*

To illustrate, in some simple cases, the changes due to precession in the right ascensions and declinations of the stars, we shall consider, in Fig. 71, (*a*) the pole P, vernal equinox ♈ and the equator, all for the beginning of 1940; (*b*) the pole P_1, equinox ♈$_1$ and equator for one quarter of the precessional period later, i.e. in (1940 + 6500) or 8440 A.D. (*c*) the pole P_2 one half of the precessional period from now, i.e. in 14,940 A.D.

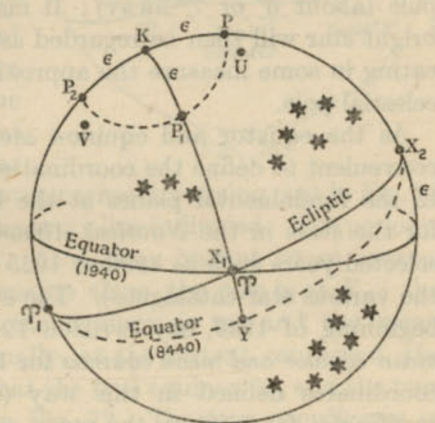

FIG. 71.

As regards (*b*), the arc PP_1 is one quarter of the circumference of the small circle PP_1P_2; it follows that the angle PKP_1 is 90°; consequently, the great circle KP_1 passes through ♈ and consequently ♈$_1$ lies on the great circle KP_2 so that ♈$_1$♈ = 90°. Suppose there is a star X_1 with R.A. = 0^h and declination = 0° in 1940;

its position is thus at ♈. The star's R.A. in 8440 A.D. is given by $♈_1 Y$ and, as $♈_1$ is the pole of the great circle $KP_1♈$, the R.A. is 90° or 6^h. Also, $P_1X_1 = P_1Y - X_1Y = 90° - X_1Y$ and $P_1X_1 = KX_1 - KP_1 = 90° - \epsilon$; hence $YX_1 = \epsilon$ or $23\frac{1}{2}°$. Thus, in 8440 A.D. the star's coordinates are (6^h, $23\frac{1}{2}°$).

With regard to (c), consider for example a star at X_2. Its present coordinates are (6^h, $+23\frac{1}{2}°$). In 14,940 A.D. the vernal equinox (not shown in Fig. 71) will coincide with the present position of ♎ (the first point of Libra), so that in 14,940 A.D. the star's R.A. will be 18^h. Also, the present north polar distance of the star is PX_2 or $90° - \epsilon$; its north polar distance in 14,940 A.D. will be P_2PX_2 or $2\epsilon + PX_2$ or $90° + \epsilon$. Hence its declination will then be $-\epsilon$ or $-23\frac{1}{2}°$.

In 1940 Polaris or the "pole-star" (U in Fig. 71) indicates approximately the north celestial pole P, but after one or two thousands of years, this star will no longer be close to the celestial pole. The present position of Vega, one of the brightest stars in the sky, is ($18^h\ 35^m$, $+38°\ 44'$); it is shown at V in Fig. 71. We deduce from the figure that, about 10 to 12 thousand years hence, V will then be fairly close to the north celestial pole (about 6° or 7° away); it may be anticipated that this bright star will then be regarded as the "pole-star"—as indicating in some measure the approximate position of the north celestial pole.

As the equator and equinox are continually altering, it is convenient to define the coordinates of the stars with reference to the fundamental planes at the beginning of each year (as for the stars in the *Nautical Almanac*) or at the beginning of selected years such as 1900 or 1925 or 1950 (as for the stars in the various star-catalogues). The equator and equinox for the beginning of 1940, say (written 1940·0), are described as the *mean equator and mean equinox* for 1940·0. If (α, δ) denote the coordinates defined in this way (they are called the *mean coordinates* for 1940·0), the mean coordinates (α_1, δ_1) for 1941 are given by the formulae

$$\left.\begin{aligned}\alpha_1 - \alpha &= 3^s{\cdot}073 + 1^s{\cdot}336 \sin\alpha \tan\delta \\ \delta_1 - \delta &= 20''{\cdot}04 \cos\alpha\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(11)$$

As the annual rates of change of the R.A. and declination are given by the right-hand sides of (11), it is easy to find accurately the mean coordinates for the beginning of any previous or subsequent year provided the intervals concerned are not more than a score or two of years; otherwise, if adequate accuracy is required, the procedure is more complicated.

115. *Nutation.*

Owing to the fact that the moon's orbital plane is not coincident with the ecliptic and, further, that this orbital plane is not constant with respect to the background of the stars, the effect of the moon's attraction on the spheroidal earth consists of two parts: (*a*) its contribution to the uniform motion of ♈ backwards along the ecliptic, that is, to the luni-solar precession already discussed; and (*b*) a periodic displacement of the true pole about the small circle PP_1 of Fig. 67. This latter phenomenon was discovered by Bradley shortly after his discovery of aberration. In addition there are other periodic effects due to the elliptic character of the lunar and apparent solar orbits. The result may be summarised by saying that in Fig. 72 the *true pole* travels along the wavy line indicated. The periodic effects are called *nutation*. At a given instant let us assume that, due to luni-solar precession alone, the pole is at P_1; the corresponding position of the equinox is $♈_1$ and the mean equator is as shown. Actually, at the instant concerned, the true pole is at P_2, say, so that the true equinox $♈_2$ and the true equator are as indicated in Fig. 72. The true coordinates of the star at the given instant are defined with reference to the true equinox and the true equator.

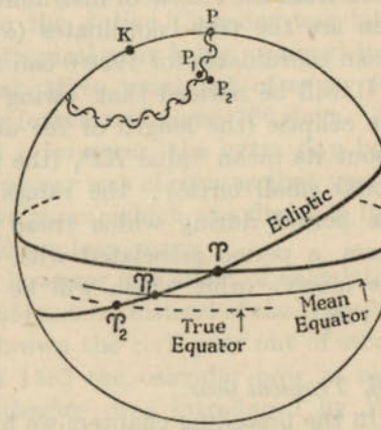

FIG. 72.

If (α, δ) are the mean coordinates of a star, say for 1940·0, and (α_1, δ_1) the true coordinates for G.C.T. 10^h, May 28, 1940, the differences in the coordinates due to precession and nutation for the interval concerned are given by formulae of the type

$$\left.\begin{aligned}\alpha_1-\alpha&=Aa+Bb+E\\ \delta_1-\delta&=Aa'+Bb'\end{aligned}\right\}. \qquad \ldots\ldots\ldots\ldots(12)$$

In these formulae A, B and E are quantities whose values are tabulated in the *Nautical Almanac* for every day of the year, and a, b, a' and b' depend on the star's coordinates and on the numerical quantities appearing in (11).

If the observed coordinates of a star are obtained by means of the meridian circle on the date concerned, their values when freed from the effects of instrumental errors, refraction, aberration are the true coordinates (α_1, δ_1); by means of (12) the mean coordinates for 1940·0 can then be computed.

It will be noticed that, owing to nutation, the obliquity of the ecliptic (the length of the arc KP_2 in Fig. 72) fluctuates about its mean value KP_1 (the angular radius of the precessional small circle); the range on either side is 9″·2 and the period during which these changes occur is about $18\frac{2}{3}$ years, a period associated with a phenomenon pertaining to the moon's orbit which will be discussed later (section 139, p. 176).

116. *Tropical year.*

In the preceding chapters we have used the term "year" to denote the interval required by the sun to make a complete circuit of the ecliptic. This interval is more properly particularised as the *sidereal year*. For civil purposes the most convenient interval is that between two successive passages of the sun through the vernal equinox, for such a passage marks the beginning of spring and is thus related to the seasons as a whole. Now, the sun's motion in the ecliptic is "direct", whereas the motion of the equinox is "retrograde" or backwards along the ecliptic and is also uniform if we ignore nutation. It follows that the interval defined with reference to the moving equinox is *less* than the sidereal year; this interval is called the *tropical year*, the length of which is found from observations over a long period (thus eliminating the effects of nutation) to be 365·2422 mean solar days (M.S.D.). The relation

between the tropical year and the sidereal year is easily obtained for, evidently,

$$\frac{1 \text{ tropical year}}{360^\circ - 50''\cdot 2} = \frac{1 \text{ sidereal year}}{360^\circ},$$

from which one sidereal year = 365·2564 M.S.D.

117. *The calendar.*

The *civil year* contains an integral number of mean solar days but, owing to the fact that the tropical year is not equal to an integral number of mean solar days, the civil year would soon get out of step with the seasons unless an adjustment were made. For this reason the *Julian Calendar* was introduced by Julius Caesar, the tropical year being assumed to be $365\frac{1}{4}$ M.S.D. In this calendar, three years out of every four were given 365 days and the fourth was given 366 days. The year of 366 days was called a *leap-year*, the extra day being called February 29; the leap-year was chosen as that year, in any group of four consecutive years, which was divisible by 4. Thus, 136, 140, ... 1936, 1940 are leap-years.

By the sixteenth century, however, the effect of calculations based on a tropical year of $365\frac{1}{4}$ M.S.D., instead of the true value of 365·2422, had seriously thrown the civil year out of step in relation to the seasons; in 1582 the calendar now in use—known as the *Gregorian Calendar*—was introduced by Pope Gregory. In this calendar a leap-year is defined as before, with the exception that any given century, such as 1600, is a leap-year only if the number of hundreds (in this case, 16) is divisible by 4. Thus 1700 is an ordinary year of 365 days, while 1600 contains 366 days. In a period of 400 civil years the Julian calendar would contain 100 leap-years but in the Gregorian calendar the number is 3 less, that is, 97. Hence, as regards the Gregorian calendar,

$$400 \text{ civil years} = (400 \times 365 + 97) \text{ M.S.D.},$$

from which the average civil year is 365·2425 M.S.D. As this last value is almost exactly the length of the tropical year no change in the Gregorian calendar will be necessary during the next thousand years.

EXAMPLES

(In examples 1-3 consider only aberration effects and take the constant of aberration to be 20″·47.)

1. The longitude of a star lying in the ecliptic is 90°. What is the star's apparent longitude (i) on March 21, (ii) on June 21, (iii) on December 21, (iv) when the sun's longitude is 30°?

2. What is the effect of aberration during the year on the apparent position of a star whose heliocentric position is at the pole of the ecliptic?

3. The longitude of a star lying in the ecliptic is 42°. What is its apparent longitude (i) on June 21, (ii) on September 21, (iii) when the sun's longitude is 135°?

4. In 1940, a star's coordinates are: R.A. $= 18^h$, Dec. $= -50°$. Find its coordinates after an interval equal to half the precessional period.

5. In 1940, a star's coordinates are: R.A. $= 18^h$, Dec. $= -23° 27'$. Find its coordinates one quarter of the precessional period later.

6. In 1940, a star's coordinates are $(0^h, 0°)$: find its coordinates three-quarters of the precessional period later.

7.* On March 20 at G.C.T. $11^h 52^m 27^s$ the sun's declination was found to be $-0° 6' 21''$, and on March 21 at G.C.T. $11^h 52^m 15^s$ the declination was $+0° 17' 22''$. Taking the obliquity of the ecliptic to be 23° 27′, calculate the sun's R.A. at the two G.C.T.'s given and find the G.C.T. and the date of the spring equinox.

8.* The mean coordinates of a star for 1900·0 are $(4^h, +32°)$. Calculate the mean coordinates for 1910·0.

CHAPTER IX

DETERMINATION OF POSITION ON THE EARTH

118. *The sextant.*

We begin by describing the main features of the sextant, a small hand-instrument used at sea for measuring the altitude of a heavenly body above the sea-horizon. It is also used by navigators of aircraft in a somewhat different form, called the "bubble sextant". The chief details of the sextant are illustrated schematically in Fig. 73, which represents a plan of the instrument. Two mirrors I (the index mirror) and H (the horizon glass) are mounted on a framework and I is capable of rotation in the plane of the framework. The lower half of the horizon glass is silvered and so this part acts as a mirror. A telescope T is fixed to the framework, pointing towards the fixed horizon glass H; its axis is parallel to the plane of the framework and is in such a position that a distant object beyond the horizon glass can be seen in the telescope by means of the rays passing through the clear part of H and entering the upper half of the telescope's object glass. Rays reflected from the mirror section of H enter the lower half of the object glass. The remaining features of the instrument are (i) a graduated arc OQ whose plane we may define as the plane of the framework and whose centre is the point about which I is pivoted; (ii) a rod IP on which I is mounted (we call IP simply a pointer) with an arrow at P, by means of which we determine the reading on the graduated arc corresponding to a particular position of P.

To measure the altitude of a star above the sea-horizon the procedure is as follows. The sextant is held in a vertical plane containing the star, with the telescope pointing to the horizon. The index mirror I is rotated until the image of the star appears in the field of view. The arm IP is then clamped to the arc and a slow-motion screw enables us to move IP gradually until the star's image appears to rest on the line of the horizon. The reading of the graduated arc, given by P, is the reading required.

The optical path of the rays from the star which enable the latter to be seen in the telescope is as follows. An incident ray

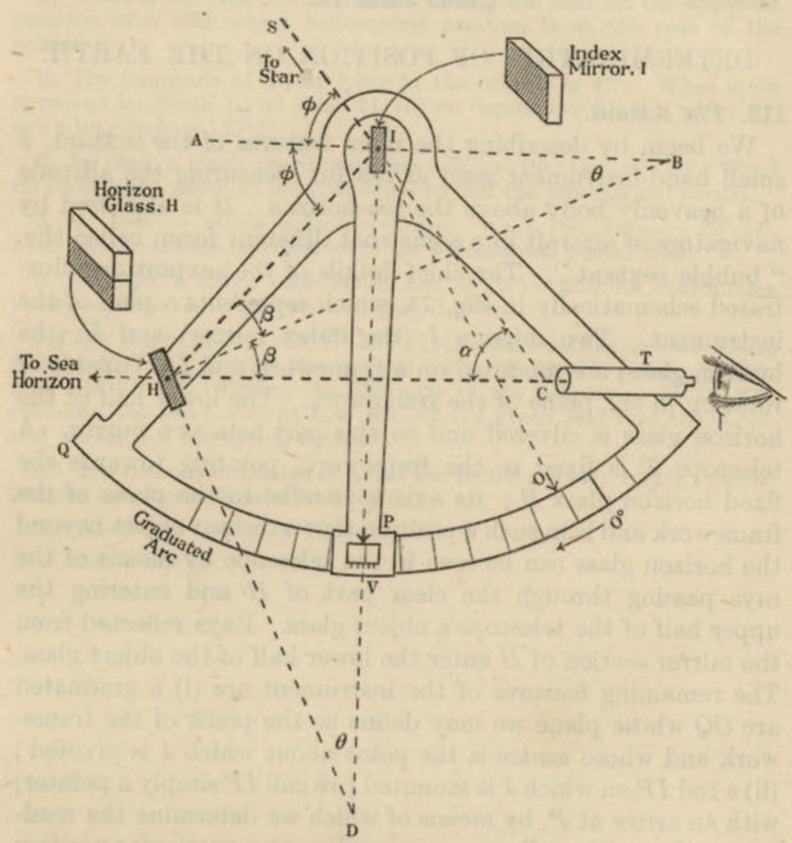

Fig. 73.

SI is reflected by the mirror I into the direction IH. This ray IH is reflected by the mirror section of H along HC and so enters the lower half of the telescope's object glass. The altitude, a, to be measured is the angle SCH, and this is related simply to the angle between the mirrors—that is, the angle

HDI. Let θ denote $\widehat{HDI}$. This is also the angle between the normals, AIB and HB, to the mirrors. Since, by the laws of reflection, AI bisects $\widehat{SIH}$ and HB bisects $\widehat{IHC}$, we have—denoting $\widehat{AIH}$ by ϕ and $\widehat{IHB}$ by β—from the triangle IHC, of which $\widehat{SIH}$ is an external angle,

$$2\phi = a + 2\beta.$$

In a similar way, the triangle IBH gives

$$\phi = \theta + \beta.$$

These two formulae give at once

$$a = 2\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In other words, the altitude is twice the angle between the mirrors.

Let IO denote the direction of the moveable arm IP when the mirror I is parallel to the fixed mirror H. Then $\widehat{OIP}$, in the figure, is equal to θ so that the altitude is, by (1), twice the angle subtended by the arc OP at I. To save multiplication by the factor 2, the arc is graduated in half-degrees but the divisions are marked as degrees. The arc is further subdivided, each division representing 10′. A vernier, V, attached to the moving arm IP enables altitudes to be measured with an accuracy of 6″ or 10″, depending on the class of sextant concerned.

It is clear from the formula (1) that zero altitude corresponds to $\theta = 0$, that is, to the arrow at O (since OI is parallel to the fixed mirror H). O is thus the true instrumental zero. The actual zero ($O°$) of the scale may not coincide with O; the difference is called the *index error* (I.E.), which may be positive or negative. Thus in Fig. 73 the correct reading is to be measured from O, whereas the scale-reading as measured from $O°$ is too large; in this case the index error is negative, so that it has to be subtracted numerically from the scale-reading. In the same way the index error may be positive and, to facilitate its measurement, the arc is graduated for a degree or two on the right-hand side of $O°$ in the figure.

119. *Errors of the sextant.*

(i) *Index error.* This can be measured by observing a distant object such as the sun or the horizon. Set the pointer P near $O°$ and point the telescope towards the horizon, for example. The latter will be seen directly through the clear field of the mirror H and an image produced by reflections at I and H will also be seen in the field of view. By actuating the slow-motion screw, the two images can be made to coincide. The position of P then gives the true instrumental zero and the index error is then read off on the scale.

If the sun is observed, the direct image of the solar disc and the doubly reflected image can be made to touch in two different ways; the mean of the corresponding positions of the pointer P gives the position of O. As a check, the interval on the arc between these two positions is four times the sun's semi-diameter, the value of which can be found from the *Nautical Almanac.*

(ii) *Error of perpendicularity.* This refers to the index mirror I, which ought to be perpendicular to the plane of the graduated arc.

(iii) *Side error.* This refers to the horizon glass H, which ought to be perpendicular to the plane of the arc.

(iv) *Collimation error.* This refers to the telescope, whose optical axis ought to be parallel to the plane of the arc.

(v) *Centering error.* This refers to errors in the readings on the graduated arc resulting from the fact that the point about which I pivots may not be exactly the centre of the arc. The errors vary according to the readings on the arc.

The errors (ii), (iii) and (iv) can be removed by suitable adjustments.*

The errors (v) are determined at some official establishment, such as, in England, the National Physical Laboratory and are recorded on the sextant-box at intervals of the scale.

120. *Correction of the sextant altitude.*

We consider two of the principal problems in which the sextant is used: (*a*) determination of position at sea, (*b*) determination of position on land. In both problems we require, for our subsequent calculations, to obtain from the sextant reading for altitude the *geocentric zenith distance* of the heavenly body observed—we call this the G.Z.D. The corrections to be applied to the sextant reading are as follows.

* For details the reader is referred to more exhaustive treatises, e.g. *Admiralty Manual of Navigation.*

(i) We first correct the sextant reading for index error.

(ii) We then correct for *dip* (see section 92). Since the angle measured is the altitude above the sea-horizon, the correction is *minus* to the sextant reading; the result is the altitude above the celestial horizon.

(iii) The next correction is that due to *refraction*; it is given (see section 88) by $58''\cdot2 \tan$ Z.D. or $58''\cdot2 \cot$ (altitude). As refraction makes the heavenly body appear displaced towards the zenith, the correction is *minus* to the sextant reading. Owing to the uncertainty of refraction for low altitudes, it is usual to restrict observations to heavenly bodies with a minimum altitude of 15°.

(iv) If the heavenly body is the sun, moon or a planet, we observe in practice the altitude of the lower limb (or edge)—or the upper limb—above the sea-horizon; the altitude of the heavenly body's centre is obtained by *adding* the semi-diameter (S.D.), taken from the *Nautical Almanac*, to the sextant reading if the lower limb is observed, or subtracting the S.D. if the upper limb is observed.

(v) If the heavenly body is as in (iv) we require to remove the effect of *parallax* (see section 95) so as to obtain the geocentric position of the body. Now, the effect of parallax is to increase the observer's zenith distance over the geocentric zenith distance by $P \sin$ (Z.D.)—where P is the horizontal parallax of the body concerned—or by $P \cos$ (altitude), the altitude used in this formula being that obtained after the application of corrections (i) to (iv). Alternatively stated, the geocentric altitude is greater than the altitude for the observer by $P \cos$ (altitude); parallax, as given by the preceding formula, is thus additive to the sextant reading.

The corrections (i) to (iii) only are applied to the observations of stars and planets (in practice, the corrections (iv) and (v) for the planets are generally ignored, as they are usually too small to make any substantial alteration in the results).

The corrections (i) to (v) are applied to observations of the sun and moon.

In each case the corrected sextant reading is subtracted from 90° ; the result is the O.Z.D.

Ex. 1. The sextant altitude of the moon's lower limb (above the sea-horizon) at G.C.T. $18^h\ 30^m$, November 8, 1940, was 48° 10′·3 ; I.E. − 2′·3 ; height of eye, 25 feet. To find the O.Z.D.

Sextant alt.	48° 10′·3	
I.E.	− 2 ·3	
Dip	− 5 ·0	[Formula (19) p. 114]
Refraction	− 0 ·9	[≡58″·2 cot 48°·0]
S.D.	+ 14 ·9	[from *Nautical Almanac*]
	48° 17′·0	
Parallax	+ 36′·4	[*P*, from *N.A.*, = 54′·7 and parallax = 54′·7 cos 48° 17′]
	48° 53′·4	
∴ O.Z.D. =	41° 6′·6	[Subtract from 90°]

121. *Artificial horizon.*

As regards problem (*b*) the sea-horizon is not available, and the observations, if made with the sextant, depend on the use of the "artificial horizon", which consists simply of a shallow trough containing mercury and shielded by a glass cover from the wind.

Let *AB* denote the mercury surface (Fig. 74). The sextant is held in a vertical plane with the telescope pointing in the direction *TC* so that the ray *SC* from a star, say, passes, after reflection at the mercury surface, into the telescope along the direction *CT*. The star is thus seen as if it were viewed directly in the direction *CD*. The image of the star formed by reflections at the index and horizon mirrors of the sextant is brought into the field of view by moving the pointer arm in the usual way, and when the two images coincide we obtain a measure of the angle *SCD*. Now, by the laws of reflection, $\widehat{SCA} = \widehat{TCB}$, the latter being equal to the

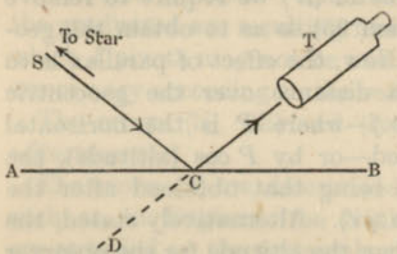

Fig. 74.

vertically opposite angle $\widehat{ACD}$. Thus $\widehat{ACS}$, which is the altitude above the celestial horizon, is one-half of $\widehat{SCD}$, the angle measured by the sextant.

The procedure in correcting an observation is as follows. First apply index error to the sextant reading and then divide the result by two ; this gives the altitude above the celestial horizon. Then apply, as previously indicated, the correction for refraction and, in the case of the sun or moon, the corrections for semi-diameter and parallax. Finally, subtract the result from 90° to obtain the O.Z.D.

Ex. 2. The sextant reading for a star observed in the artificial horizon was 86° 28′·5 ; I.E. + 3′·7.

Sextant reading	86° 28′·5	
I.E.	+ 3 ·7	
2	86° 32′·2	[Divide by 2]
	43° 16′·1	
Refraction	− 1 ·0	[58″·2 cot 43°·3]
	43° 15′·1	
∴ O.Z.D. =	46° 44′·9	[Subtract from 90°]

The land-surveyor's observations with a theodolite and the air-navigator's with a bubble sextant are reduced as in the example of section 120 above, except that there is no correction for dip ; if the heavenly body is a star or a planet there is, of course, no correction for S.D. and for parallax, these being assumed negligible for the planet.

122. *The geographical position of a heavenly body.*

We now consider a further step in determining position on the earth's surface, and for convenience we restrict our discussion to the problem as it concerns the sea-navigator ; the fundamental principles are the same for the air-navigator and for the surveyor. The navigator's actual observation of a heavenly body consists of two parts :

(i) the measurement of altitude above the sea-horizon ;

(ii) the noting of the G.C.T. at which the observation was made.

In this section we are concerned with (ii).

In all ships one or more chronometers, keeping G.C.T., are carried. Actually, no chronometer is always accurate, but its error can be determined once a day—or more frequently—by means of radio time-signals. Applying the known error determined in this way, the navigator obtains the G.C.T. Also, knowing the approximate Zone Time of his observation together with the date for the zone concerned he obtains, as in section 57, the G.C.T. with date. Thus he can extract from the *Nautical Almanac* such quantities, referring to the heavenly body observed, as will be required in his later calculations. If the body concerned is, for example, the sun, the necessary quantities are the sun's declination and the equation of time. The latter enables him to calculate the sun's hour angle at Greenwich at the moment of his observation, for *

$$\text{G.H.A.}\odot = \text{G.C.T.} \pm 12^{h} + \text{E.T.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Consider Fig. 75, showing the earth and the celestial sphere, the earth being the smaller sphere; *pga* denotes the Greenwich meridian on the earth; the radii *Cp* and *Cg* are produced to meet the celestial sphere in *P* and *G*, so that *PGA* is the celestial meridian for Greenwich. Let *X* be the sun's position on the celestial sphere at the moment of observation. The radius *CX* cuts the earth's surface at *U*. Thus, *U* is the point on the earth's surface at which the sun is vertically overhead at the moment of the observation. *U* is called the sub-solar point or the sun's *geographical position*, this latter term being used generally for any heavenly body.

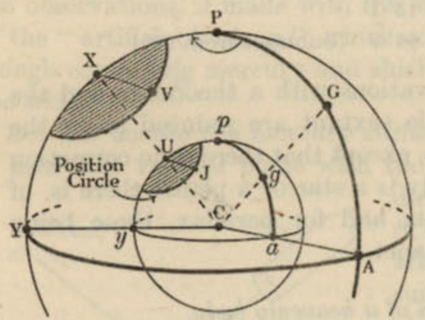

FIG. 75.

The position of *U*—its latitude and longitude—can be easily obtained. The celestial meridian through *X* is *PXY* (*Y* being

* In the *Nautical Almanac, abridged for the use of seamen*, a quantity *E*, equal to 12^{h} + E.T., is tabulated at two-hourly intervals throughout the year.

on the celestial equator) so that $\widehat{YCX}$ is the sun's declination. The radius *CY* cuts the terrestrial equator in *y* and $\widehat{yCU}$ is the latitude of *U*. But $\widehat{YCX} = \widehat{yCU}$. Hence, the *latitude of the sun's geographical position is the sun's declination.*

Again, *GPX* is the G.H.A.☉ at the moment of observation, and it can be found by means of (2). But $\widehat{GPX} = \widehat{gpU}$, which is, in Fig. 75, the *west* longitude of *U*. Hence, if G.H.A.☉ is less than 12^{h}, the west longitude of the sun's geographical position is the G.H.A.☉.

Evidently, if G.H.A.☉ is greater than 12^{h}, the east longitude of the geographical position is (24^{h} − G.H.A.☉).

If the heavenly body is the moon or a planet or a star, the latitude of the geographical position is the body's declination and the west longitude is the Greenwich hour angle of the body (obtained by the method of section 64), both found for the instant at which the observation was made; if the G.H.A. is greater than 12^{h}, the longitude of the geographical position is (24^{h} − G.H.A.) east.

Thus, from a knowledge of the G.C.T. (with date) at which an observation of a heavenly body is made, we can specify uniquely a point on the earth's surface—the geographical position, *U*—at which the body is vertically overhead at the moment of observation.

123. *The position circle.*

We consider now the information which we can deduce from our knowledge of the O.Z.D. (derived from the sextant observation) and of the geographical position. Now, the O.Z.D. is the true angular distance of the heavenly body from the observer's zenith, all corrections having been applied, or—put differently—the O.Z.D. is the true angular distance of the observer's zenith from the heavenly body, the position of the latter on the Greenwich celestial sphere being definitely known. Suppose, for example, that the O.Z.D. is 30°. It follows that the observer's zenith must be *situated on a small circle of which X* (Fig. 75) *is the pole and whose angular radius is* 30°. If the observer's zenith happens to be at *V*, the observer's position on the earth's surface must be at J where the radius CV cuts the earth's surface; clearly, J is at an angular distance of 30° from the geographical position U. Hence, with the O.Z.D. known, we conclude generally that *the observer's position on the earth's surface lies on the small circle of which the geographical position is the pole and whose angular radius is the* O.Z.D.

This small circle on the earth's surface is called a *position circle.*

If another heavenly body is observed at the same G.C.T., this observation gives a second position circle which will cut the first in two points. The observer must therefore be at one or other of these points of intersection; there is no difficulty, in practice, in selecting the actual position.

124. *The intercept.*

To make practical application of the knowledge to which the observation of a heavenly body leads the navigator—namely, that at the instant of observation his actual position must be at some point of the derived position circle—he proceeds as follows. After leaving harbour the navigator knows the ship's course and speed from hour to hour, and by plotting on his chart he keeps a continuous record of the ship's position derived in this way. This process is called *dead reckoning,** and the ship's position deduced in this way at any instant is called the *dead reckoning position* (D.R.) It is to be noted that the D.R. position cannot be expected to be an accurate position, for the course and speed are not actually known with complete accuracy and, in addition, there are the incalculable effects of winds and currents.

At the moment of the observation of the heavenly body, the navigator can find from his chart the D.R. position; in Fig. 76 this is represented by D, which must be regarded only as the approximate position of the ship at the time of observation—the best that can be deduced from the ship's course and estimated speed. The true position of the ship, however, lies on the position circle, and as, in modern navigation, the errors in estimating the position of D are usually not considerable—say, 5 or 10 or 20 nautical miles—the position of D must be in the vicinity of the position circle.

* "Dead" is a corruption of "deduced".

Let the great circle joining D to the geographical position U cut the position circle in J. Then, having regard to the nature of the possible errors in estimating D, we conclude that the true position of the ship must be on the position circle in the neighbourhood of J; this part of the position circle is shown in Fig. 76 by a heavy line, and it is the object of the navigator to represent on his chart this portion of the position circle. The angular distance of J from D is called the *intercept.*

125. *The position line.*

Let Z be the zenith of D (Fig. 76). The great circle arc XZ cuts the small circle AVB in V, the angular radius of this small circle being the O.Z.D. derived from the sextant observation. It is evident that the angular distance of Z from V is the same as the intercept DJ. This fact enables us to calculate the intercept by considering the spherical triangle PZX for, as we shall show, ZX can be calculated and, as XV is the O.Z.D., we obtain by subtraction the intercept ZV (or DJ).

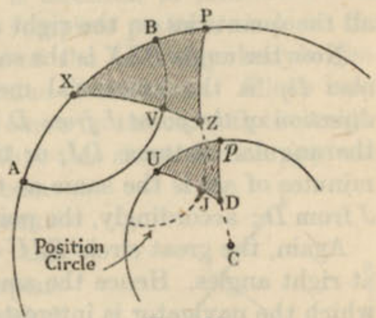

Fig. 76.

Now, the latitude, ϕ, and longitude, L, of D are known, these being taken from the chart. Hence $PZ = 90° - \phi$. Also, $Z\widehat{P}X$ is the hour angle of the heavenly body at the time of observation *for the position D*; now, the Greenwich hour angle (G.H.A.) is known, as already described; hence

$$Z\widehat{P}X = \text{G.H.A.} - L, \quad \text{.........................}(3)$$

L being regarded as positive if the longitude of D is west and negative if the longitude of D is east. If δ is the declination of the body at the time of observation, $PX = 90° - \delta$. We then have—by formula (18), p. 13—

$$\cos ZX = \cos PZ \cos PX + \sin PZ \sin PX \cos Z\widehat{P}X. \quad \text{...}(4)$$

As all the quantities on the right-hand side of (4) are known, we can calculate the value of ZX.

The value of ZX, which is the zenith distance, at the time of observation, of the heavenly body calculated *for the position D*, is called the *calculated zenith distance* (C.Z.D.). We thus have

$$\text{Intercept} = \text{C.Z.D.} - \text{O.Z.D.} \quad \text{.....................(5)}$$

We can also calculate * the azimuth, $P\widehat{Z}X$, of the heavenly body with respect to the position D for, by the same procedure as in (4), we have

$$\cos PX = \cos PZ \cos ZX + \sin PZ \sin ZX \cos P\widehat{Z}X,$$

from which

$$\cos P\widehat{Z}X = \frac{\cos PX - \cos PZ \cos ZX}{\sin PZ \sin ZX}; \quad \text{............(6)}$$

all the quantities on the right of this equation are now known.

Now the angle PZX is the same as the angle UDp in Fig. 76; also Dp is the terrestrial meridian through D. Hence *the direction of the point J from D can be drawn on the chart.* Also, the angular distance DJ, or the intercept, when expressed in minutes of arc is the same as the distance in nautical miles of J from D; accordingly, the point J can be plotted on the chart.

Again, the great circle DJU cuts the position circle (Fig. 76) at right angles. Hence the small part of the position circle in which the navigator is interested (the heavy line on either side of J) is represented on the chart by a straight line passing through the position J on the chart and drawn perpendicular to DJ.

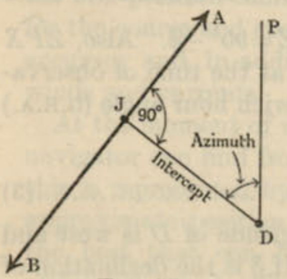

FIG. 77.

The procedure is illustrated in Fig. 77, where DP is the meridian, on the chart, through D, the angle PDJ is the azimuth found by means of (6) and DJ is the intercept. The line AB on the chart, drawn perpendicular to DJ, is called the *position line.* The ship's actual position must lie, on the chart, somewhere on AB.

In Fig. 76, the D.R. position D has been represented as lying

* This calculation is unnecessary if "Azimuth Tables" are available. These give the values of the azimuth for every degree of latitude, declination and hour angle. Simple interpolation for the actual latitude, declination and hour angle gives the azimuth required.

outside the position circle so that the O.Z.D. is less than the C.Z.D. In this case the intercept is drawn from D in the direction given by the azimuth PZX—that is, *towards* the geographical position U.

If, however, D lies within the position circle—the O.Z.D. is then greater than the C.Z.D.—the intercept must then be drawn from D in the direction *opposite* to that given by the azimuth.

126. *Remarks on the derivation of the position line.*

The procedure of the previous section has enabled us to represent on the chart the small part of the position circle of real practical importance. Although we have utilised the D.R. position in our calculations, it is essential to remember that the actual representation of the complete position circle on the chart depends on the position of the heavenly body's geographical position and on the value of the O.Z.D. All we have done is to select that part of the curve on the chart in the neighbourhood of D and to assume that this short section approximates to a straight line. As the complete position circle is independent of the D.R. position so, practically, the position line on the chart is independent of the D.R. position, provided that any other position, say D_1 (Fig. 78), used in the calculations is within a short distance of D—say, within 10 or 20 nautical miles. Suppose we used D_1 in the calculations; we should then derive an intercept D_1J_1 and an azimuth A_1. But, since D_1 and D are close together, the values of A_1 and A (the azimuth calculated for D) will be very nearly equal—differing not more than by about $\frac{1}{4}°$, say—so that *practically* D_1J_1 will be parallel to DJ and the corresponding position line will be perpendicular to D_1J_1. But this second position line represents on the chart a small portion of the complete position circle in the neighbourhood of D_1, and so the two position lines (derived by using D and D_1 in the calculations) will be in practice identical; in other words, J_1 will lie on the position line AB derived by using D.

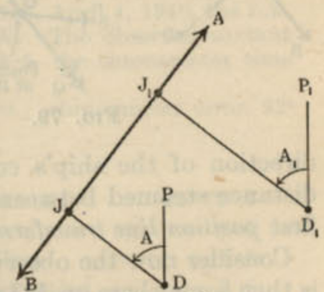

FIG. 78.

127. *Determination of position.*

The actual position of the ship at a given instant can only be determined if we can derive two position lines. Consider, first, simultaneous observations of two heavenly bodies X and Y. From each observation we can derive the corresponding position line, and the chart-position of the ship must then be the point of intersection of these position lines ; this is called the *observed position.*

If the observations are not made simultaneously we proceed as follows. Suppose that X is observed at Zone Time 1800, the D.R. position then being denoted by D (Fig. 79) ; the resulting position line is AB. Hence the ship's position must lie somewhere on AB at 1800. If the observation of Y is made at 1900, say, the ship's position must lie at 1900 (as the result of the first observation) on PQ, which is obtained by drawing JD_1 in the direction of the ship's course and making JD_1 equal to the distance steamed between the observations. PQ is called the first *position line transferred.*

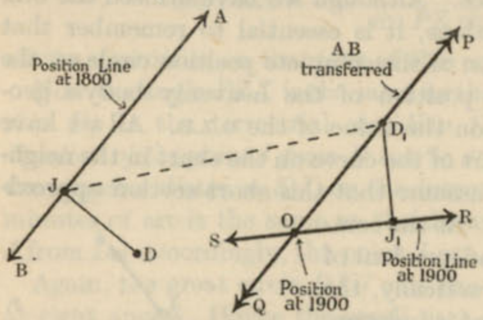

Fig. 79.

Consider now the observation of Y made at 1900. The ship is then somewhere on PQ and, by the principles of the previous section, we can take D_1 as the approximate position of the ship and use D_1 for calculating the intercept and azimuth for the observation of Y. Let RS be the resulting position line. Then the ship's position at 1900 is at O, the intersection of PQ and RS ; this is the *observed position.*

To minimise the effects of the unavoidable errors in observation, the acute angle between the position lines PQ and RS should be as near 90° as practicable or not less than about 30° or 40°. But in Fig. 79, $\widehat{POR}$ is equal to the angle between DJ and D_1J_1, so that the rule in practice is that the difference between

the azimuth of X and the azimuth of Y (or its opposite) should be between 30° or 40° and 90°.

Instead of observing two different heavenly bodies we can determine the ship's position by observing a single body, such as the sun, at two different times of the day, the position lines obtained from the earlier observation being transferred as in Fig. 79 to the time of the second observation ; the only practical stipulation is that the change in the sun's azimuth between the observations should be at least 30° and not more than 150°.

128.* *Example of a position line.*

We work out, in this section, an example in detail. Normally, 5-figure logarithmic tables are used in such calculations, as these give an accuracy of 0′·1. However, here we shall use 4-figure tables. (We retain decimals of 1′, but with 4-figure tables the calculation of the C.Z.D. has a spurious accuracy.)

Ex. 3. Zone +2 ; at approximate Z.T. 0755, April 4, 1940, the D.R. position of a ship was 34° 25′ N, 28° 42′ W. The observed sextant altitude of the sun's lower limb was 25° 52′·5, the chronometer time being $9^h\ 49^m\ 58^s$.

Index error, +1′·5 ; height of eye, 25 feet ; chronometer error, 22^s fast on G.C.T.

To find the position line.

Appr. Z.T.		0755 Apr. 4
Zone		+ 2
Appr. G.D.		0955 Apr. 4
Chron. Time	=	$9^h\ 49^m\ 58^s$
Error	=	− 22
G.D. ≡ G.C.T.	=	$9^h\ 49^m\ 36^s$ Apr. 4
From *N.A.*,		
⊙'s Dec.	=	+ 5° 42′·5
E.T.	=	$-3^m\ 3^s$
S.D.	=	16′·0
G.H.A.M.S.	=	$21^h\ 49^m\ 36^s$
E.T.	=	− 3 3
G.H.A. ⊙	=	$21^h\ 46^m\ 33^s$
Long. W	=	1 54 48
H.A. ⊙	=	$19^h\ 51^m\ 45^s$
	≡	297° 56′

Sext. alt.	=	25° 52′·5
I.E.	=	+ 1·5
Dip	=	− 5·0
Refraction	=	− 2·0
S.D.	=	+ 16·0
		26° 03′·0
Parallax	=	+ 0·1
		26° 03′·1
∴ O.Z.D.	=	63° 56′·9

We now use the formula

$$\cos ZX = \cos PZ \cos PX + \sin PZ \sin PX \cos \widehat{ZPX},$$

in which $\widehat{ZPX}$ is $360° - \text{H.A.}\odot$. As $PZ = 55° \, 35'$, $PX = 84° \, 17'{\cdot}5$, and $\cos 297° \, 56' = \cos 62° \, 4'$, we have

$$\cos ZX = \cos 55° \, 35' \cos 84° \, 17'{\cdot}5 + \sin 55° \, 35' \sin 84° \, 17'{\cdot}5 \cos 62° \, 4'.$$

The azimuth is given by

$$\cos \widehat{PZX} = (\cos PX - \cos PZ \cos ZX)/(\sin PZ \sin ZX),$$

in which ZX is the zenith distance already calculated.

Calculation of ZX

	log.	number
cos PZ	$\bar{1}{\cdot}7522$	
cos PX	$\bar{2}{\cdot}9976$	
	$\bar{2}{\cdot}7498$	$+0{\cdot}0562$
sin PZ	$\bar{1}{\cdot}9164$	
sin PX	$\bar{1}{\cdot}9978$	
cos $\widehat{ZPX}$	$\bar{1}{\cdot}6706$	
	$\bar{1}{\cdot}5848$	$+0{\cdot}3844$
cos ZX		$+0{\cdot}4406$

ZX = 63° 51′·5

This is the C.Z.D.

Calculation of azimuth

	log.	number
cos PX		$+0{\cdot}0095$
cos PZ	$\bar{1}{\cdot}7522$	
cos ZX	$\bar{1}{\cdot}6441$	
	$\bar{1}{\cdot}3963$	$+0{\cdot}2491$
Numerator *	$\bar{1}{\cdot}1750n$	$-0{\cdot}1496$
sin PZ	$\bar{1}{\cdot}9164$	
sin ZX	$\bar{1}{\cdot}9532$	
Denominator	$\bar{1}{\cdot}8696$	
cos $\widehat{PZX}$	$\bar{1}{\cdot}3054n$	

$\widehat{PZX} = 101°{\cdot}6$

Hence azimuth is 101°·6 E.

Since the O.Z.D. is 63° 56′·9 and the C.Z.D. (ZX) is 63° 51′·5 the intercept is 5′·4. It is readily seen that the D.R. position is within the position circle; hence the intercept must be drawn in the direction *opposite* to the calculated azimuth. The position line is then as shown in Fig. 79*a*.

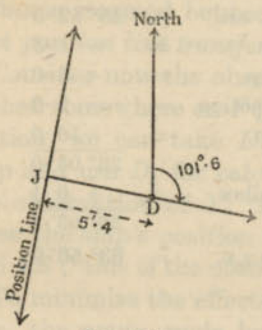

FIG. 79*a*.

129. *Position lines derived from meridian observations.*

The example given in the previous section gives some idea of the amount of calculation necessary in deriving a position line if no "short-cuts" are available. The calculations become extremely simple, however,

* The letter *n* placed after a logarithm means that the number is negative.

when the heavenly body is observed on the observer's meridian. The sextant altitude is treated in the usual way and the C.Z.D. is simply $\phi - \delta$ (ϕ being the latitude of the D.R. position). The intercept is then found as in the general case. As the direction of the heavenly body on the meridian is either north or south, the position line runs east and west; in other words, the meridian observation gives the latitude of the ship. Because of this simplicity in calculation, a navigator loses no opportunity of observing the sun, for example, at meridian passage—or within two or three minutes of meridian passage, for then the C.Z.D. is substantially $\phi - \delta$ and the direction of the heavenly body is practically north or south. Another observation, made two or three hours later, is combined with the meridian observation in the way illustrated in Fig. 79.

130. *The plotting-chart.*

Although the navigator plots all his positions and draws all his courses and position lines on the mercator chart, the problems with which we are concerned here can be dealt with by means of ordinary squared paper which, when used in this connection, is called a *plotting-chart*. If we refer to Fig. 76 we notice that the part of the sphere with which we are concerned as regards the actual plotting forms a small area of the earth's surface in the neighbourhood of J or D. As this small part of the surface may be assumed plane, it is legitimate to use squared paper on which to represent the geometrical processes involved in drawing and transferring position lines. Suppose that we choose, as scale, one inch to represent one nautical mile. If the vertical scale corresponds to latitude, the horizontal scale corresponds to *departure*. Thus, if we require the latitude and longitude of O in Fig. 79, which we may take to be a plotting-chart, the position of the D.R. position D being supposed known, the vertical difference between O and D gives the d. lat. between these points and so we obtain at once the latitude of O. Again, the horizontal distance between O and D gives the departure between these points and the difference of longitude between them is given by the formula (12), p. 11, which we write in the form

$$\text{d.long.} = \text{dep. sec } \phi,$$

where ϕ is taken to be the mean of the latitudes of O and D.

EXAMPLES

1. Find the true zenith distance of the sun's centre, corrected for index error, dip, refraction, semi-diameter and parallax from the following observations of the sun's lower limb. Take the sun's horizontal parallax to be 8″·8.

	Sextant altitude	I.E.	H.E. (in feet)	S.D.
(i)	43° 15′·0	− 1′·5	49	16′·1
(ii)	75° 44′·8	+ 2′·3	25	15′·8
(iii)	35° 28′·9	− 3′·7	32	16′·0

2. Find the zenith distance of the moon's centre, corrected for I.E., dip, refraction, S.D. and parallax from the following observations of the moon's limb, upper or lower (U. or L.):

	Sextant altitude	I.E.	H.E. (in feet)	S.D.	H.P.
(i)	27° 45′·2 (L.)	− 2′·3	16	15′·4	56′·5
(ii)	33° 14′·9 (U.)	+ 4′·2	36	16′·5	60′·6
(iii)	37° 28′·2 (U.)	− 3′·5	22	15′·1	55′·4

3. Find the sun's *geographical position* at the following G.C.T.'s (the requisite data are given):

	G.C.T.	Dec.	E.T.
(i)	$13^h\ 14^m\ 45^s$	− 3° 13′	+ $10^m\ 19^s$
(ii)	$6^h\ 43^m\ 29^s$	+ 15° 20′	+ $3^m\ 3^s$

4. Find the geographical positions of the stars, whose R.A. and Dec. are given, at the following G.C.T.'s (the Greenwich sidereal time at G.C.T. 0^h on February 2 is $8^h\ 40^m\ 15^s$):

	G.C.T.	Date	R.A.	Dec.
(i)	$6^h\ 20^m\ 35^s$	Feb. 2	$5^h\ 12^m\ 15^s$	+ 45° 56′
(ii)	$15^h\ 17^m\ 55^s$	Feb. 1	$1^h\ 35^m\ 29^s$	− 57° 32′
(iii)	$3^h\ 55^m\ 22^s$	Feb. 2	$10^h\ 5^m\ 11^s$	+ 12° 16′

5. The instantaneous observations of two stars *X* and *Y*, when worked out for the position 48° 20′ N, 36° 50′ W, gave the following results:

Star	O.Z.D.	C.Z.D.	True bearing
X	38° 25′	38° 20′	048°
Y	52° 37′	52° 41′	135°

Find by means of a plotting-chart the observed position.

6. A ship was steaming 030° at 10 knots. At 0600 the ship's D.R. was 33° 57′ S, 18° 4′ E. The observations of two stars *X* and *Y* observed at 0600 and 0700 respectively gave (both observations being worked out with the D.R. position at 0600) the following:

	Z.T.	O.Z.D.	C.Z.D.	True bearing
X	0600	52° 28′	52° 24′	310°
Y	0700	38° 35′	38° 40′	050°

Find by means of a plotting-chart the observed position at 0700.

***7.** Calculate the intercept and sketch the position line from the following sextant altitudes of the sun's lower limb (assume, in all examples: I.E. − 3′·4; H.E. 36 feet; chronometer $2^m\ 15^s$ fast on G.C.T.), the latitude and the longitude being given.

	Zone	Lat.	Long.	Approx. Z.T.	Chronometer
(i)	+ 1	51° 5′ N	9° 32′ W	1220 Mar. 14	$13^h\ 20^m\ 58^s$
	Sextant altitude 35° 50′·0		Sun's Dec. − 2° 27′	E.T. − $9^m\ 18^s$	S.D. 16′·1
(ii)	− 12	43° 42′ S	173° 15′ E	0825 Dec. 22	$20^h\ 27^m\ 15^s$
	Sextant altitude 37° 16′·0		Sun's Dec. − 23° 26′	E.T. + $2^m\ 0^s$	S.D. 16′·3
(iii)	− 8	31° 38′ N	121° 58′ E	0940 Dec. 2	$1^h\ 48^m\ 52^s$
	Sextant altitude 29° 38′·0		Sun's Dec. − 21° 55′	E.T. + $11^m\ 2^s$	S.D. 16′·2

CHAPTER X

THE MOON

131. *The phases of the moon.*

In a previous chapter we have discussed the orbits of the planets and we have referred to the fact, easily deduced from simple observations, that the moon moves in an elliptic orbit about the earth. In the earlier parts of this chapter we shall assume for simplicity that the moon's orbit is circular, with its plane coinciding with that of the ecliptic.

We consider first the moon's phases on the same lines as in section 79 dealing with the phases of the planets. Fig. 80

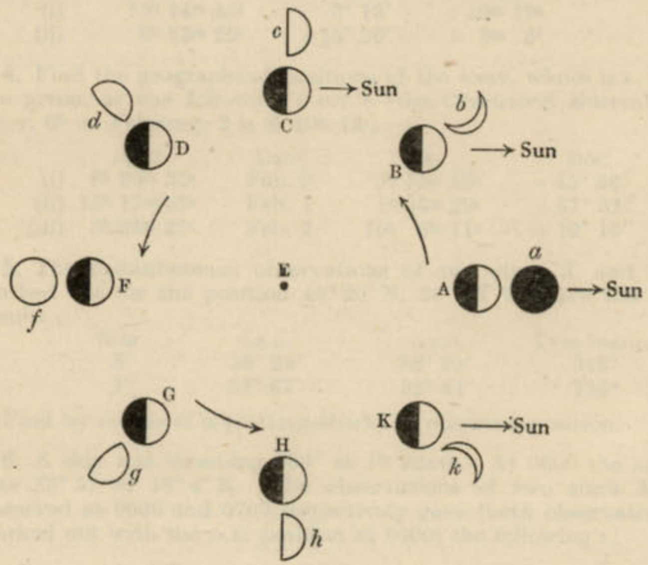

FIG. 80.

shows the moon in different parts of its orbit, the arrow indicating the direction of orbital motion. As the moon's geocentric

distance is only about 240,000 miles on the average, as compared with the sun's geocentric distance of nearly 93 millions of miles, we can assume that the direction of the sun from any one of the moon's positions in the orbit is parallel to the sun's direction, S, from the earth, E. Thus the surface of the moon illuminated by the sun's light will consist of the hemisphere bounded by a plane perpendicular to the direction ES. We show in Fig. 80, corresponding to the orbital positions $A, B, \dots$, the moon's appearance at $a, b, \dots$ as seen from the earth.

At A the moon is directly between the earth and the sun, this position corresponding to *conjunction* or *new moon*. At F the moon appears *full*; this position corresponds to *opposition* or *full moon*. At C half the moon's surface is visible; this is called *first quarter*; at H half the moon's disc is visible, and this position is called *third quarter*. Between A and C and between H and A, the area seen illuminated is less than half the moon's disc. Between first and third quarters more than half the moon's disc is visible; the moon's appearance is then said to be *gibbous*.

The visible part of the moon's disc, as shown for example at k, is bounded by a semi-circle and a semi-ellipse; the points of intersection of these curves are called *cusps*, and the elliptical boundary separating the illuminated part of the disc from the dark part is called the *terminator*.

132. *The moon's synodic period.*

The interval between two successive new moons is called the *synodic period* or *lunation* or *lunar month*. Taking into account the complexities of the moon's motion, discussion of which we omit at this stage, the *average* value of the synodic period is found accurately from ancient and modern solar eclipses, which occur at or near new moon. It is found in this way that the average synodic period is 29·53059 days.

133. *The moon's sidereal period.*

This is the interval required by the moon to make a complete circuit of its orbit with respect to the stars. Let S denote the synodic period, M the sidereal period and T the earth's sidereal orbital period, all in days. The relation between S and M is easily found as follows by the method of section 71. Let E, L and S in Fig. 81 denote the positions of the earth, moon and sun at the time of a new moon; let F and R be the positions of the earth and moon at the next new moon. During this interval—the synodic period, S—the earth has described the angle θ (in circular measure) of its orbit; therefore the time required by the earth to travel from E to F is $\frac{\theta}{2\pi}T$; but this is S; hence

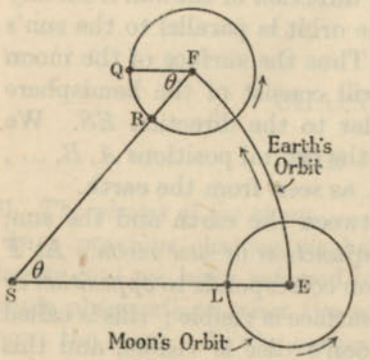

FIG. 81.

$$\theta = \frac{2\pi S}{T}. \quad \text{..........(1)}$$

Let FQ be parallel to EL. Then the sidereal period M is the interval required by the moon to describe an angle of 2π, for we can assume that, if the direction ES coincides with the direction of a certain star, the direction of this star as viewed from F will be parallel to ES, since the star's distance is practically infinite as compared with the sun's geocentric distance. It is then clear from the figure that at the time of the second new moon (configuration FRS) the moon has moved through the orbital angle $2\pi+\theta$. But as it describes 2π in the sidereal period M, it will require the time $\frac{2\pi+\theta}{2\pi}M$ to reach R from the previous new moon; this is again the synodic period S; hence

$$\left(\frac{2\pi+\theta}{2\pi}\right)M = S,$$

or

$$2\pi+\theta = \frac{2\pi S}{M}. \quad \text{......................(2)}$$

Subtracting (1) from (2) we obtain

$$2\pi = 2\pi\left(\frac{S}{M} - \frac{S}{T}\right),$$

whence

$$\frac{1}{M} = \frac{1}{S} + \frac{1}{T}. \quad \text{..............................(3)}$$

As we have seen in the previous section, $S = 29{\cdot}53059$ days; also $T = 365{\cdot}2564$ days; we then find that the sidereal period M is $27{\cdot}3217$ days.

The *age of the moon* at a given moment is the number of days that have elapsed since the previous new moon; it is given in the *Nautical Almanac* for G.C.T. 0^h of every day throughout the year.

134. *Formula for the phase.*

Let S, M, and E denote the positions of the sun, moon and earth at a given instant, and let d, θ and β denote the angles at M, E and S (Fig. 82); in particular, θ is the elongation. As was shown in section 79, the phase—the ratio of the illuminated area to the area of the disc—is defined by the expression $\frac{1}{2}(1+\cos d)$. Now β is a small angle, since EM is small compared with ES; the maximum value of β occurs when EM is perpendicular to ES. Now the ratio of EM to ES is about 1/360; hence the maximum value of β is about 10′. Thus we can regard MS as approximately parallel to ES so that

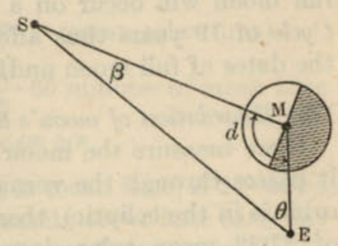

FIG. 82.

$$d = 180^\circ - \theta.$$

Hence the expression for the phase is $\frac{1}{2}(1-\cos\theta)$.

If A is the age of the moon, corresponding to the elongation θ in Fig. 82, we have θ (in degrees) given by

$$\theta = 360\cdot\frac{A}{S}, \quad \text{..............................(4)}$$

S being the synodic period. Thus, when the moon's age is known, we can calculate θ by (4) and then obtain the phase by means of the expression previously stated.

For example, if we assume that the moon's position B in Fig. 80 is half-way between new moon and first quarter, the elongation θ is 45° and the phase represented at b in Fig. 80 is $\frac{1}{2}(1-\cos 45^\circ) = 0{\cdot}15$ approximately; thus, at B a little more than one-seventh of the moon's disc is seen illuminated.

135. *The Metonic cycle.*

It was discovered by Meton in 433 B.C. that the number of days in 19 years is almost an exact multiple of the number of days in a lunation. Taking the year to be $365\frac{1}{4}$ days, we find that

$$19 \text{ years} = 6939{\cdot}75 \text{ days}.$$

Also,

$$235 \text{ lunations} = 235 \times 29{\cdot}53059 \text{ days} = 6939{\cdot}689 \text{ days}.$$

The difference between the two cycles is thus about $1^{\mathrm{h}}\,38^{\mathrm{m}}$ only. Consequently, if a full moon occurs on a certain date, full moon will occur on a date 19 years later. The *Metonic Cycle* of 19 years thus affords a simple method of predicting the dates of full moon and, also, of new moon.

136. *Retardation of moon's transit.*

If we measure the moon's sidereal period from the moment it passes through the vernal equinox (we still assume that its orbit is in the ecliptic), then, during the sidereal orbital period of 27·32 mean solar days, the moon's right ascension has increased by 360° or 24^{h} (we neglect the precessional motion of ♈).

It is convenient in this problem to make use of the sidereal day rather than the mean solar day. Now

$$1 \text{ sidereal day} = \frac{365\frac{1}{4}}{366\frac{1}{4}} \text{ mean solar day,}$$

so that the sidereal orbital period is $\dfrac{27{\cdot}32 \times 1465}{1461}$ sidereal days.

Hence the moon's R.A. increases on the average by

$$\frac{24 \times 1461}{27{\cdot}32 \times 1465} \text{ hours per } \textit{sidereal day},$$

that is, by $0^{\mathrm{h}}{\cdot}8750$ per sidereal day, or by 7/192 hour per *sidereal hour*, on the average.

Let S_1 be the sidereal time of a transit over a given meridian and S_2 the sidereal time at the next transit. If α_1 and α_2 are the right ascensions of the moon at the corresponding transits, we have

$$S_1 = \alpha_1 \text{ and } S_2 = 24^{\mathrm{h}} + \alpha_2.$$

Let x, in sidereal hours, denote the interval between transits. Then

$$x = S_2 - S_1 = 24^{\mathrm{h}} + (\alpha_2 - \alpha_1). \quad \text{.....................(5)}$$

But $\alpha_2 - \alpha_1$ is the increase in R.A. during x sidereal hours; hence the average value of $\alpha_2 - \alpha_1$ is $\dfrac{7x}{192}$ hours. Formula (5) becomes

$$x = 24^{\mathrm{h}} + \frac{7x}{192}, \quad \therefore\ x = \frac{24 \times 192}{185},$$

or

$$x = \frac{24}{185}(185 + 7) = 24^{\mathrm{h}} + \frac{168^{\mathrm{h}}}{185} \text{ (sidereal time).}$$

Now 24^{h} (sidereal time) $= 23^{\mathrm{h}}\,56^{\mathrm{m}}\,4^{\mathrm{s}}$ (mean solar time)

and

$$\frac{168^{\mathrm{h}}}{185} \text{(sidereal time)} = \frac{168}{185} \cdot \frac{1461}{1465} \cdot 60 \text{ minutes of mean time} = 54^{\mathrm{m}}{\cdot}34 = 54^{\mathrm{m}}\,20^{\mathrm{s}}.$$

Hence the interval between transits, in mean time, is $24^{\mathrm{h}}\,50^{\mathrm{m}}{\cdot}4$ on the average. Thus, the moon transits $50^{\mathrm{m}}{\cdot}4$ *later* each day on the average; this is usually designated the *retardation of transit*.

137. *Harvest moon.*

If we suppose, for a moment, that the moon's declination is constant, the moon will rise at a constant interval before transit. As the retardation of transit is $50^{\mathrm{m}}{\cdot}4$ on the average, it follows that, under the given circumstances, the retardation of moon-rise will also be $50^{\mathrm{m}}{\cdot}4$ on the average; in other words, the moon will rise $50^{\mathrm{m}}{\cdot}4$ minutes *later* each day on the average.

Consider now the changes in the time of moon-rise due to changes in the declination and suppose that the moon's declination is

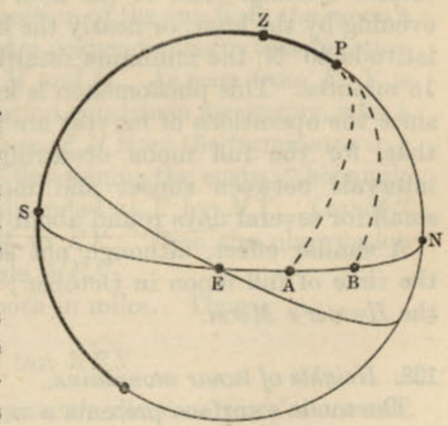

Fig. 83.

increasing between two consecutive risings. In Fig. 83, drawn for a northern latitude, we suppose the moon to rise at A and B on successive days, the declination at B being greater than the declination at A.

The first moon-rise will occur before transit by the interval required by the moon to describe the angle ZPA; the second moon-rise will occur before transit by the interval required by the moon to describe the angle ZPB. Thus, when the declination is increasing, the time of moon-rise will occur *earlier*, with respect to the time of transit, by the interval required to describe the angle APB. If this interval is denoted by I and T_1, T_2 are the mean times of successive transits, the moon will rise *later* in the day on the second occasion by $(T_2-T_1)-I$. This expression is the *retardation of moon-rise*. As T_2-T_1 is the retardation of transit and is $50^{m}\cdot 4$ on the average, the retardation of moon-rise will be a minimum when T_2-T_1 is as small as possible and when I is as large as possible; the latter condition requires, as can be easily seen from the figure, that the rate of change of declination should be as large as possible. Now, the rate of increase of declination is greatest when the moon is at ♈. If the moon is then full, the sun must be at the autumnal equinox, so that the date is September 21. Also, it can be shown that, on this date, T_2-T_1 is a minimum. Hence, about this date the full, or nearly full, moon will rise on successive evenings close to the hour of sunset, being later each evening by the least, or nearly the least, possible amount. For latitude 50° N, the minimum retardation of moon-rise is about 18 minutes. This phenomenon is known as the *Harvest Moon*, since the operations of harvest are greatly assisted by the fact that, for the full moon occurring near September 21, the intervals between sunset and moon-rise are comparatively small for several days round about the time of full moon.

A similar effect, although not so pronounced, occurs near the time of full moon in October; this phenomenon is called the *Hunter's Moon*.

138. *Heights of lunar mountains.*

The moon's surface presents a very rough appearance in the telescope; its principal features are mountains, mountain-

ranges, craters and plains. It is known, also, from a variety of considerations, that the moon has no atmosphere.

One method of measuring the height h of a lunar mountain is as follows.

In Fig. 84 let E be the position of the earth relative to the moon and sun, and let M denote the summit of a mountain,

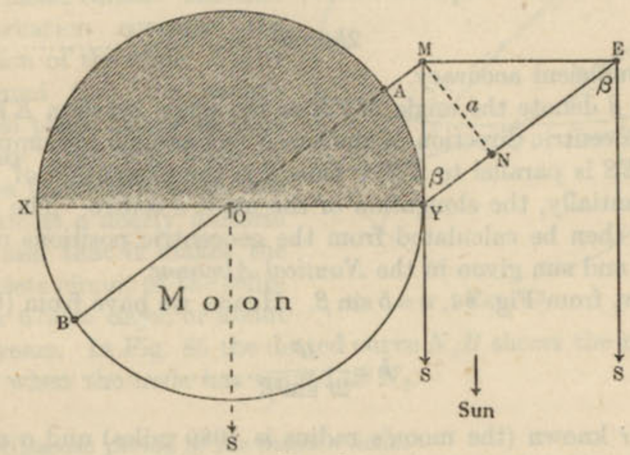

FIG. 84.

situated in the invisible part of the moon's surface, at the moment when the summit is illuminated by the sun's rays. Then MY is parallel to the direction of the sun from the moon's centre O, XY being the diameter perpendicular to this direction and lying in the plane of S, M and O. As seen from E, Y is one of the cusps. The observation consists in measuring, with a micrometer, the angular distance of M from the terminator in a direction perpendicular to the line joining the cusps; this angle is equivalent to the angle subtended at E by MY. Let MN be the perpendicular from M to EY. Then the observation consists in measuring the angle MEN.

Let $MN=a$ and $MY=b$, both in miles. Then

$$a = EN \tan \widehat{MEN},$$

and as EN is effectively the moon's distance from the earth, we can calculate a.

Let r be the moon's radius in miles; we suppose, also, that the height $h (\equiv AM)$ of the mountain is in miles. Now, by geometry, since MY is a tangent to the lunar disc at Y,

$$MA \,.\, MB = b^2,$$

AB being the diameter through A. Hence $h(2r+h)=b^2$ or, since h is small compared with r,

$$2hr = b^2 \quad \text{.................................(6)}$$

with sufficient accuracy.

Let β denote the angle MYE or the angle between EY and the geocentric direction of the sun S (we assume for simplicity that ES is parallel to YS); thus β is the elongation of Y or, substantially, the elongation of the moon's centre. The angle β can then be calculated from the geocentric positions of the moon and sun given in the *Nautical Almanac*.

Now, from Fig. 84, $a = b \sin \beta$. Hence, we have from (6),

$$h = \frac{a^2}{2r \sin^2 \beta} \cdot \quad \text{..............................(7)}$$

With r known (the moon's radius is 1080 miles) and a and β found as already indicated, we obtain the height in miles.

Some of the lunar mountains rise to a height of 20,000 feet, thus rivalling the terrestrial mountains; it is found, too, that some crater-walls rise to heights approaching 15,000 feet.

Another method consists of measuring the lengths of the shadows cast by the mountains, but usually difficulties arise owing to the irregular nature of the lunar surface over which the shadows fall.

139. *The moon's orbit.*

The moon's orbital plane is inclined at an average angle, i, of 5° 9′ to the plane of the ecliptic; projected on the celestial sphere (Fig. 85), NA represents part of the moon's path in the sky. The point N at which the moon crosses the ecliptic from south to north is called the *ascending node*, the other point of intersection being the *descending node* (not shown in the figure). The angle $AN♈$ is the inclination i.

The eccentricity e of the elliptic orbit is 0·0549 and the semi-

major axis is 238,000 miles. Owing mainly to the attraction of the sun on the earth-moon system, the elements of the orbit undergo changes called *perturbations*; as regards i, e and a, these effects are seen in small fluctuations about their mean values. Another perturbation concerns the position of the node, N. It is found that the moon's orbital plane moves in such a way that the node N moves backwards along the ecliptic at a nearly uniform rate and that it makes the complete circuit of the ecliptic in 6793·5 days, or about $18\frac{2}{3}$ years. In Fig. 85 the dotted curve N_1B shows the moon's path when the node has moved to N_1.

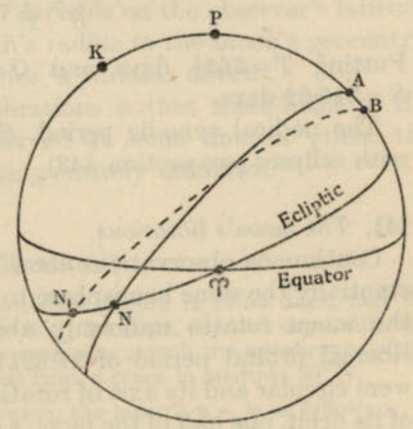

FIG. 85.

140. *Synodic period of the moon's node.*

In Fig. 85 let N denote the position of the node at the moment when the sun, in its apparent path along the ecliptic, coincides with it. Let N_1 denote the position of the node when the sun next coincides with it. The interval between successive coincidences with the node is the *synodic period of the node*, which we denote by S. Let Q denote the period of revolution of the node in the ecliptic and T the sidereal period of the sun's apparent motion in the ecliptic.

Then, if $NN_1 = \theta$ (in circular measure), we have that the synodic periodic S is the time required by the node to move through θ. Hence

$$S = \frac{\theta}{2\pi} \cdot Q.$$

Also, in the synodic period S, the sun has described $2\pi - \theta$; hence

$$S = \frac{2\pi - \theta}{2\pi} T.$$

Eliminating θ, we obtain

$$\frac{1}{S}=\frac{1}{T}+\frac{1}{Q}.$$

Putting $T=365\frac{1}{4}$ days and $Q=6793\frac{1}{2}$ days, we find that $S=346{\cdot}62$ days.

The nodical synodic period, S, is important in connection with eclipses (see section 143).

141. *The moon's librations.*

Continuous observations show that the moon presents substantially the same hemisphere to the earth. It is inferred that the moon rotates uniformly about an axis in precisely its sidereal orbital period of 27·3217 days. If the moon's orbit were circular and its axis of rotation perpendicular to the plane of its orbit, one half of the moon's surface would remain entirely unknown to us owing to the equality of the orbital and rotational periods (we omit, for the moment, a slightly modifying effect depending on parallax). The consequences due to the actual departures from these simple but hypothetical conditions are known as *librations*. We consider them in turn.

(a) The moon's rotational axis is inclined at an angle of about $6\frac{1}{2}°$ to the perpendicular to the orbital plane. The result is that, at certain times, we can see up to $6\frac{1}{2}°$ beyond the north pole of the moon's axis, with a corresponding effect for the south pole. This effect, known as *libration in latitude*, enables us to map out rather more than half the lunar surface.

(b) Owing to the elliptic nature of the moon's orbit, the moon is sometimes ahead of and sometimes behind the positions it would occupy if its angular motion in its orbit were uniform. The consequence is that we sometimes see an additional part of the moon's surface to the east and sometimes an additional part to the west. This is known as *libration in longitude*.

(c) The third libration, called *diurnal libration*, results from the fact that the observer is continually changing his position, due to the rotation of the earth, relative to the line joining the centres of the earth and moon. Now, leaving out (a) and (b), the moon turns the same hemisphere towards the earth's centre; hence, during the interval between two successive transits of the moon over his meridian, the observer sees a little more of the moon's surface to the east and to the west. The amount of this libration depends on the observer's latitude and on the ratio of the earth's radius to the moon's geocentric distance; it is, in consequence, a parallax effect.

The result of all three librations is that three-fifths of the moon's surface can be observed at some time or other, the remaining two-fifths remaining entirely unknown.

EXAMPLES

1. Given that the moon's synodical period is 29·53 days, find the phase of the moon when its age is (i) 5·4 days, (ii) 12·7 days, (iii) 20·5 days, and (iv) 23·4 days, and in each case sketch the appearance of the moon at meridian transit as seen from a place in latitude 50° N.

2. At upper transit at Greenwich, the moon's R.A. is as follows:

Feb. 15: $\alpha=2^h\ 58^m\ 57^s$,

16: $\alpha=3^h\ 51^m\ 54^s$.

If the Greenwich sidereal times at G.C.T. 0^h on Feb. 15 and Feb. 16 are $9^h\ 35^m\ 27^s$ and $9^h\ 39^m\ 23^s$ respectively, find

(i) the G.C.T. of transit at Greenwich on Feb. 15;
(ii) the retardation of transit for Feb. 15–Feb. 16.

3. Full moon occurred at G.C.T. 11^h on Dec. 25, 1920. Find the date of full moon in December, 1939.

4. Find the eccentricity of the moon's orbit given that the values of the S.D. at perigee and apogee are $16'\ 28''{\cdot}3$ and $14'\ 45''{\cdot}4$ respectively.

5. Find the eccentricity of the moon's orbit given that the values of the horizontal parallax at perigee and apogee are $60'\ 19''{\cdot}5$ and $54'\ 9''{\cdot}7$ respectively.

6. From the data in example 4 find the geocentric distances of the moon at perigee and apogee and the semi-major axis of the orbit given that the moon's radius is 1080 miles.

7. From the data in example 5, find the geocentric distances of the moon at perigee and apogee and the semi-major axis of the orbit given that the earth's radius is 3960 miles.

***8.** The angular distance of the illuminated summit of a lunar mountain from the terminator is 30″; if the moon's elongation is 30°, its radius 1080 miles and its distance 240,000 miles, find the height of the lunar mountain.

CHAPTER XI

ECLIPSES OF THE MOON AND SUN

142. *Introduction.*

An eclipse of the moon occurs when the moon passes into the shadow cast by the earth; this can take place only at opposition, that is, at full moon. The circumstances are illustrated in Fig. 86, where the sun and earth are shown at S and

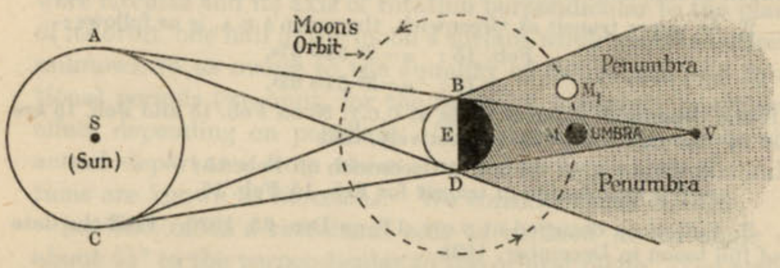

FIG. 86.

E, with the moon in opposition at M. The shadow cast by the earth is a cone whose vertex is V; it is called the *umbra* and is shown in dark shading. The space outside the umbra, between the direct and transverse tangents with respect to the spheres representing the sun and earth, is called the *penumbra*; it is shown in light shading. If the moon is at M_1 in the penumbra, part only of the sun's disc is hidden by the earth; the light from the remaining part of the solar disc illuminates the moon (which is still near full), but the brightness is diminished.

Even when the moon is within the umbra, as at M, a certain amount of light still reaches it owing to the refractive and scattering effects of the earth's atmosphere; these effects are more pronounced for red light, and, consequently, when the moon is technically eclipsed as at M it is still visible as a red or copper-coloured disc.

A solar eclipse occurs when the earth passes into the shadow cast by the moon; this can only occur at new moon.

If the moon's orbital plane coincided with the ecliptic, a lunar and a solar eclipse would occur in every lunar month or lunation. Actually, eclipses can only take place when the moon is full or new and its position is at or near one of its nodes.

Solar eclipses can be of three types; total, partial or annular; lunar eclipses are either total or partial. If, for example, a solar eclipse is total, the moon's disc must then cover the sun's disc completely; the necessary condition is that the moon's semi-diameter must exceed the sun's semi-diameter. If this condition does not hold, the moon's disc is unable to cover up the whole of the solar disc, thus leaving, at mid-eclipse, a ring still shining; the eclipse is said to be *annular*.

Eclipses are *partial* when the body eclipsed passes only partly into the shadow concerned.

The sun's semi-diameter varies between $15'{\cdot}8$ and $16'{\cdot}3$, while the moon's semi-diameter varies between $14'{\cdot}7$ and $16'{\cdot}8$; thus, if the latter is less than $15'{\cdot}8$ a solar eclipse cannot be total and is either annular or partial; if the moon's semi-diameter is greater than $16'{\cdot}3$, a solar eclipse cannot be annular and is either total or partial.

143. *The Saros.*

In section 140 we found that the synodic period of the moon's nodes is 346·62 days; also, a lunation is 29·53059 days. Now,

$$19 \text{ synodic periods} = 6585{\cdot}78 \text{ days},$$

$$\text{and} \quad 223 \text{ lunations} = 6585{\cdot}32 \text{ days}.$$

These two intervals are very nearly equal. The interval of 6585 days is known as the *Saros*; its importance in connection with eclipse phenomena is as follows. If, at new moon, say, a solar eclipse occurs, the moon must be at or near one of its nodes with the sun at or near the same node; 6585 days later the sun must be at or near the same node, since practically 19 synodic periods have elapsed; also, it must be again new moon practically, since 223 lunations have elapsed. Thus, the conditions are approximately the same as for the eclipse at the beginning of the Saros, and so a solar eclipse—not necessarily of the same kind—again occurs.

A similar argument applies to lunar eclipses. Hence, after intervals of 6585 days or 18 years 11 days (or 18 years 10 days, if the Saros contains 5 leap-years), eclipses will be repeated. This fact was discovered by the Chaldean astronomers three or four thousand years ago; they were thus able to predict the occurrences of lunar and solar eclipses. As an example of the use of the Saros, we take the famous total eclipse of the sun on 1919 May 29 at which the first successful verification of Einstein's relativity theory was made. If we add 18 years 10 days to this date (there are 5 leap-years during this interval) we obtain the date 1937 June 8, on which a solar eclipse ought to have occurred; the records show that it duly took place.

144. *The angular radius of the umbral cone at the moon's geocentric distance.*

For lunar and solar eclipses we use the following notation, the earth being regarded as a sphere: P, P_1 are the horizontal parallax of the sun and moon respectively; S, S_1 are the semi-diameters of the sun and moon; r, r_1 are the geocentric distances of the sun and moon; s is the angular radius (with respect to the earth's centre) of the umbral cone at the moon's distance from the earth.

In Fig. 87 let AX be a direct common tangent to the sections of the sun and moon in the moon's orbital plane. Let v denote

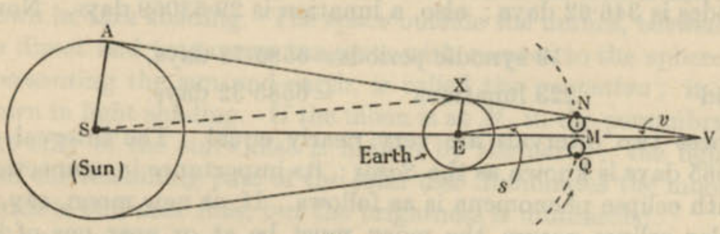

FIG. 87.

the semi-vertical angle XVE of the umbral cone. Let the moon's orbit cut SEV in M and AXV in N. Then MN, being practically perpendicular to the axis EV of the umbral cone, is the radius of the circular section of the cone at the moon's

distance from the earth. The angular radius s, which we require, is the angle subtended at E by MN—that is, the angle MEN.

Now the external angle XNE of the triangle ENV is equal to $s+v$. Also, XN is perpendicular to EX and $\widehat{XNE}$ is the angle subtended by the earth's radius at the moon, at N; accordingly, $\widehat{XNE}$ is the moon's horizontal parallax, P_1. Hence

$$P_1 = s + v. \quad \text{.....(1)}$$

In the same way, the angle XSE is the sun's horizontal parallax P (the angle v is so small that we may take EX to be perpendicular to ES). Now, $\widehat{AXS}$ is the angle subtended at X by the sun's radius; this angle is substantially the same as $\widehat{AES}$, that is, the sun's semi-diameter S. Hence $\widehat{AXS} = S$. Now

$$\widehat{AXS} = \widehat{XVS} + \widehat{XSE};$$

hence

$$S = v + P. \quad \text{.....(2)}$$

From (1) and (2) we obtain the result required, namely,

$$s = P + P_1 - S. \quad \text{.....(3)}$$

In the same way, if s_1 denotes the corresponding radius of the penumbral cone, we find that

$$s_1 = P + P_1 + S. \quad \text{.....(4)}$$

145. *Duration of total lunar eclipses.*

We consider the simplest case when the moon is exactly at a node at mid-eclipse. The circumstances are then represented by Fig. 87. If we take $P = 9''$, $P_1 = 57'$ and $S = 16'$, we find from (3) that $s = 41'$ approximately. The angular diameter QN is thus $82'$. It is easily seen that the moon will be totally eclipsed during the time required by the moon's centre to move over $82'$ less twice its angular semi-diameter. If $S_1 = 16'$, the time required corresponds to the moon's synodic motion through $82' - 32'$ or $50'$. Now, the moon's synodic period is 29·53 days, so that its average synodic angular motion is 360°/29·53 per day, or 30′·3 per hour. Thus the duration of *totality*, as it is called, is 50/30·3 hours, or, approximately, $1^h\,40^m$. Fig. 86 shows that the total phase is, at any instant, visible from one hemisphere of the earth. But, owing to the diurnal motion, the additional area of the earth's surface from which the total phase may be visible for at least an instant is $\frac{5}{3}\cdot\frac{1}{24}$ or $\frac{5}{72}$ of the total terrestrial surface.

146. *Length of the earth's shadow.*

We require the distance EV in Fig. 87. Now

$$XE = EV \sin v. \quad\ldots\ldots(5)$$

But, by (1), $v = P_1 - s$. Taking $P_1 = 57'$ and $s = 41'$ as obtained in the previous section, we obtain $v = 16'$. Hence, by (5), since $EX = 3960$ miles,

$$EV = \frac{3960}{\sin 16'} \text{ miles.}$$

But $\sin 16' = \frac{16}{3438}$ approximately; hence

$$EV = 851{,}000 \text{ miles.}$$

As this distance is very much greater than the moon's geocentric distance, the vertex V of the umbral cone lies well outside the moon's orbit; consequently, a lunar eclipse must always take place if the moon is in opposition at one of its nodes.

147. *Conditions for a lunar eclipse.*

We consider in this section the effect of the moon's orbital inclination i on the circumstances of lunar eclipses. A total eclipse of the moon will *just* take place if the whole of the moon is, for a moment, completely within the umbral cone. Fig. 88 represents a part of the celestial sphere with the earth as centre and we suppose that a total eclipse just occurs. The point C is the centre of the shadow as seen from E—it corresponds

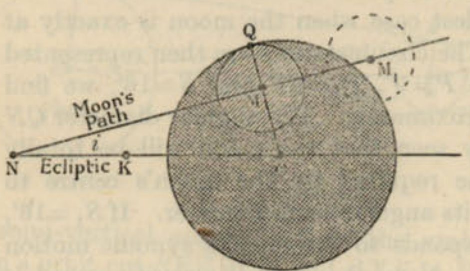

FIG. 88.

to the direction of the axis, EV, of the cone in Fig. 87—and, consequently, the coordinates of C are the geocentric coordinates of the point on the sphere diametrically opposite to the direction of the sun; the position of C can thus be calculated at any time. A part of the moon's apparent path in the sky is shown by NM, where N is a node (here taken to be the ascending node). If we draw a great circle arc CQ perpendicular to the great circle NM, a total eclipse will just take place if the moon's disc touches the circle, representing the shadow, internally at Q. Now the angular radius CQ is the same as the angle between EV and EN in Fig. 87; thus $CQ = s$. Hence the angular distance CM between the moon's centre and the centre of the shadow is $s - S_1$; accordingly, we have from (3),

$$CM = P + P_1 - S - S_1. \quad\ldots\ldots(6)$$

We can thus evaluate CM at the time of full moon.

Let K be the position of the centre of the shadow when the moon passes through the node N. Denote NK by l. It is found, under the circumstances represented in Fig. 88, that

$$l = 10{\cdot}3\,CM,$$

or, from (6),

$$l = 10{\cdot}3\,(P + P_1 - S - S_1). \quad\ldots\ldots(7)$$

The angle l is called the *ecliptic limit*. If we take $S = 16'$, $S_1 = 15'{\cdot}6$, $P = 9''$, $P_1 = 57'$, we obtain $l = 4^\circ{\cdot}6$. The interpretation of this result is that, if the centre of the shadow cone is not more than $4^\circ{\cdot}6$ from N when the moon passes through the node (this time can be obtained easily from the data of the *Nautical Almanac*), a total eclipse will take place.

In the same way, the condition that a partial eclipse will just occur is that

$$l = 10{\cdot}3\,(P + P_1 - S + S_1), \quad\ldots\ldots(8)$$

and with the values already used for P, P_1, S and S_1, we obtain $l = 9^\circ{\cdot}9$.

For any given eclipse the value of l depends on the actual values of P, P_1, S and S_1, and also on the inclination i, which varies by 11′ approximately about its mean value of 5° 9′. The greatest possible value of l in (8) is $12^\circ{\cdot}1$, and the least possible value is $9^\circ{\cdot}5$; these are called the *superior* and *inferior ecliptic limits*. A partial eclipse must occur if l is less than

$9°{\cdot}5$, and is impossible if l is greater than $12°{\cdot}1$; if l lies between these limits, an eclipse may or may not take place.

148. *Solar eclipses.*

When an eclipse of the sun occurs the observer must be situated in the shadow cast by the moon; as in Fig. 86, this shadow is partly *umbra* and partly *penumbra*. A total eclipse occurs if the observer is within the umbra, and a partial eclipse occurs if the observer is within the penumbra. Fig. 89

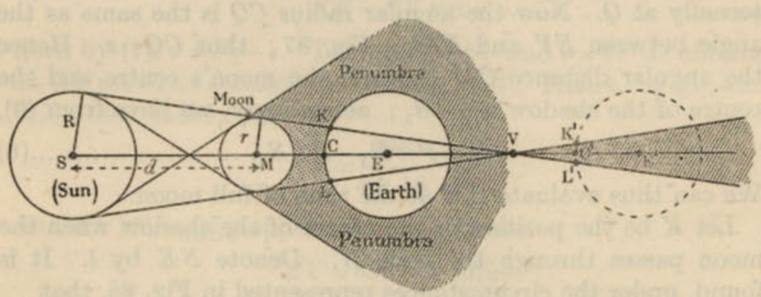

FIG. 89.

shows the configuration of the sun, moon and earth when the centres of these bodies are exactly collinear; this occurs at new moon with the moon at a node. It is clear that the eclipse will be total for the part of the earth's surface between K and L, the point C in the figure being between M and the vertex, V, of the umbral cone.

The distance of V from M is obtained as follows. We have, by similar triangles,

$$\frac{MV}{SV}=\frac{r}{R},$$

where r and R are the linear radii of the moon and sun. Let $SM=d$ and $VM=x$. Then

$$\frac{x}{d+x}=\frac{r}{R},$$

from which

$$x=\frac{r}{R-r}d. \qquad \text{(9)}$$

Now $R=432{,}000$ miles and $r=1080$ miles. Also, $d=ES-EM$, and the mean value of d is then $92{\cdot}76\times10^6$ miles, using $93{\cdot}0\times10^6$ miles and 239,000 miles as the mean values of ES and EM. We then obtain from (9) that

$$VM=232{,}500 \text{ miles}.$$

We may regard this as the mean value of the distance of the vertex of the umbral cone from the moon's centre.

To find the variations in the value of VM we notice first that $ES=d+EM$ and, as EM is small compared with ES, the variations in d are substantially equal to the variations in ES. But, as the eccentricity of the earth's orbit is $\frac{1}{60}$, ES may be greater or less than its mean value by $\frac{1}{60}$ of this value. Hence, the values of d may be greater or less than its mean value by $\frac{1}{60}$ of this value. Accordingly, from (9), the values of x may be greater or less than the mean value by $\frac{1}{60}$ of 232,500 miles (which is the mean value of x) or by 3900 miles approximately. We conclude that VM lies between 228,600 miles and 236,400 miles.

Now, from Fig. 89, it is seen that a total eclipse will be observable if C lies between M and V. As the earth's radius is 4000 miles approximately, EM must not exceed $MV+4000$. Thus, under the conditions represented in Fig. 89, a total eclipse occurs if EM is less than 232,600 miles, and may occur if EM does not exceed 240,500 miles. The limits of the moon's geocentric distance EM are 225,880 miles and 252,120 miles. If the moon is in perigee, a total eclipse is possible; but if the moon is in apogee, a total eclipse cannot take place.

If EM is greater than $MV+4000$, the corresponding position of the earth is shown in Fig. 89 by the dotted circle, centre E'. The eclipse is then *annular* in the region of the earth's surface indicated by $K'L'$.

149. *The geocentric angular distance between the centres of the sun and moon at the beginning or end of a total solar eclipse.*

Fig. 90 shows the configuration when the point C on the earth's surface is on the boundary of the umbra. We require to find the angle MES, which we denote by D. Now

$$D=M\widehat{E}B-B\widehat{E}S.$$

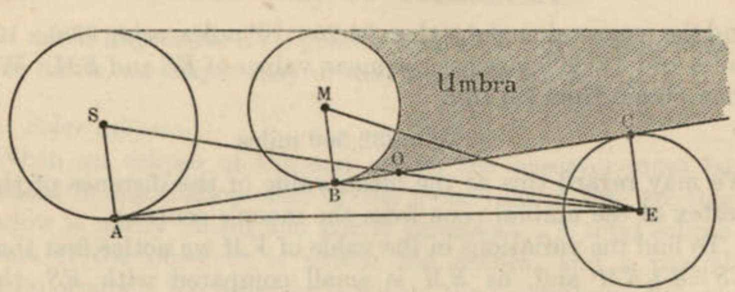

FIG. 90.

Since MB is nearly perpendicular to EB or to EM, $\widehat{MEB}$ is substantially the moon's semi-diameter S_1. Hence

$$D = S_1 - \widehat{BES}.$$

Also, $$\widehat{BES} = \widehat{COE} - \widehat{CBE}.$$

Now $\widehat{CBE}$ is the parallax of B and, therefore, very nearly the parallax of M, that is, P_1. Hence $\widehat{BES} = \widehat{COE} - P_1$, so that

$$D = S_1 + P_1 - \widehat{COE}.$$

Now, $$\widehat{COE} = \widehat{CAE} + \widehat{OEA} = \widehat{CAE} + \widehat{SEA};$$

$\widehat{CAE}$ is the parallax of A or, very nearly, the sun's parallax P; $\widehat{SEA}$ is the sun's semi-diameter S. Hence

$$D = S_1 + P_1 - S - P. \quad \text{......................(10)}$$

With the appropriate values of S_1, P_1, S and P at the time of new moon, the value of D can be found corresponding to the beginning or end of a total eclipse.

It is readily found in a similar way that the corresponding value of D for the beginning or end of a partial eclipse—we denote this value by D'—is given by

$$D' = S_1 + P_1 + S - P. \quad \text{......................(11)}$$

150. *The ecliptic limits.*

The ecliptic limit, L, for a partial solar eclipse is defined to be the angular distance of the sun from the appropriate node at the moment of new moon. The value of L depends on

the value of D' given by (11) and on the inclination, i, of the moon's orbit to the ecliptic.

The *superior ecliptic limit* is the maximum value of L if a partial eclipse is just possible; it is found to be 18°·5. If the sun's distance from a node at new moon exceeds 18°·5, an eclipse is impossible.

The *inferior ecliptic limit* is the minimum value of L; it is 15°·4. If the sun's distance from a node is less than 15°·4 a partial eclipse must take place.

151. *Calculation of eclipses.*

The actual calculation of all the circumstances of an eclipse, lunar or solar, is a matter of considerable complexity, especially for the latter. For a total solar eclipse, it is necessary to calculate the belt on the earth's surface within which the eclipse will be total, the exact beginning and end of totality for any desirable location within the belt, and several other circumstances that may require to be known.

The approximate circumstances of all solar and lunar eclipses (8000 of the former and 5200 of the latter) between 1207 B.C. and A.D. 2162 are to be found in the *Canon of Eclipses* by Oppolzer, with appropriate maps for the most important eclipses.

Each volume of the *Nautical Almanac* contains the results of accurate calculation and other necessary details for the eclipses occurring during the particular year concerned.

The least possible number of eclipses that can occur during any year is two, and these are both solar eclipses.

The greatest possible number of eclipses during a year is seven; either four are solar and three lunar, or five are solar and two lunar.

CHAPTER XII

THE STARS

152. *The constellations.*

At some early stage in astronomical history the bright stars were associated in groups called constellations. Most of the names which the constellations bear to-day have come down from the time of the great Greek astronomers who flourished in the centuries immediately preceding the beginning of the Christian era; the familiar constellations of Orion, Ursa Major, Cassiopeia are typical examples. The delimitation of the constellations in the extreme southern heavens was the work of astronomers in comparatively recent times.

153. *Naming the stars.*

The brightest stars bear individual names—mainly of Greek, Latin or Arabian origin—such as Capella, Sirius, Altair, to mention only a few. In his atlas, published in 1603, Bayer introduced the system of using the Greek letter α to designate the brightest star of a constellation, β to designate the second brightest, and so on; on the exhaustion of the Greek alphabet the Roman alphabet was pressed into service; for example, α Aurigae (Capella), β Aurigae, ... a Aurigae, b Aurigae ...; it is to be noted that the Latin genitive of the constellation's name is used in this connection. After the invention of the telescope, when myriads of stars were revealed for the first time, a numerical system was introduced by Flamsteed (the first Astronomer Royal); for example, 1 Aurigae, 2 Aurigae, ... 13 Aurigae (Capella) ..., the numerical order being the order of right ascension.

The formation of star-catalogues, from meridian circle observations or from photographs, provides another method of nomenclature, for the number of a star in such a catalogue (the

star's R.A., declination and magnitude being specified there) serves to identify it; for example, Lalande 21185 is a star with this number in the catalogue compiled by the French astronomer Lalande in the eighteenth century.

Other catalogues of a more specialised nature may be used to designate certain classes of stars; for example, A.D.S. 700 specifies a double star in the *New General Catalogue of Double Stars within* 120° *of the North Pole*, compiled by Dr. R. G. Aitken and published in 1932.

154. *Magnitude.*

The grading of the naked-eye stars according to their brightness appears to have been first undertaken by Hipparchus, as recorded in Ptolemy's *Almagest*. The twenty brightest stars in the sky, known to the Greeks, were designated stars of the *first magnitude*, the next fifty in order of brightness were stars of the *second magnitude*, and so on up to the *sixth magnitude* which referred to the faintest stars visible to the naked eye. This system is now extended to the faint stars visible only in the telescope, the fainter the star the larger being the magnitude number.

The definition of magnitude has now been placed on a precise mathematical basis as follows. Let B_1, B_2 denote the brightness of two stars X_1, X_2 and m_1, m_2 their magnitude-numbers (or, simply, magnitudes); then the *difference* of their magnitudes, namely $m_1 - m_2$, is defined by

$$\frac{B_1}{B_2} = 10^{-0\cdot4(m_1-m_2)}. \quad\quad (1)$$

The ratio of brightness ($B_1 : B_2$) of the two stars can be measured by suitable means; hence the difference of magnitude can be calculated from this formula. It is easily seen from (1) that, if the ratio $B_1 : B_2$ is exactly 100 : 1, the difference of magnitude is exactly 5. Thus if the magnitude of X_1 is 1·2 and X_1 is 100 times brighter than X_2, the magnitude of X_2 is 1·2 + 5 or 6·2.

To complete the specification of the magnitude-scale, we require to assign a magnitude-number to a given star—call it X_0. On the scale of visual magnitudes now in use the magnitude of Aldebaran is designated 1·06. By comparing the brightness of Aldebaran with, say, δ Persei, we obtain the difference of magnitude of the two stars by (1); hence, knowing the magnitude of Aldebaran, we deduce the magnitude of δ Persei.

Ex. 1. It is found that Aldebaran is 6·55 times brighter than δ Persei; to find the magnitude m_2 of the latter star, given that the magnitude m_1 of Aldebaran is 1·06.

From (1) we have, taking logarithms to base 10,

$$\log_{10}\frac{B_1}{B_2} = -0{\cdot}4(m_1 - m_2).$$

But $\frac{B_1}{B_2} = 6{\cdot}55$, and $\log 6{\cdot}55 = 0{\cdot}8162$. Hence

$$m_2 - m_1 = \tfrac{5}{2} \times 0{\cdot}8162 = 2{\cdot}04,$$

from which $m_2 = 1{\cdot}06 + 2{\cdot}04 = 3{\cdot}10.$

If we think of X_0 as a standard star whose brightness is denoted by B_0 and magnitude-number by m_0, the brightness B_1 and magnitude m_1 of any other star are related, by (1), by the formula

$$\frac{B_1}{B_0} = 10^{-0\cdot4(m_1 - m_0)},$$

which can be written

$$B_1 = B_0 \,.\, 10^{0\cdot4m_0} \,.\, 10^{-0\cdot4m_1};$$

or, writing

$$C = B_0 \,.\, 10^{0\cdot4m_0},$$

we obtain

$$B_1 = C \,.\, 10^{-0\cdot4m_1}. \quad \text{..........................(2)}$$

In this formula C is a constant which depends on the standard star X_0.

In problems involving magnitudes we may find it more convenient to use (1), as in the previous example, or (2) in such a problem as the following.

The star α Crucis is easily seen in the telescope to be a double star, the magnitudes of the components being 1·58 and 2·09; what is the magnitude of the star seen as a single star by the naked eye?

If B_1, m_1 and B_2, m_2 refer to the two components, we have from (2)

$$B_1 = C \,.\, 10^{-0\cdot4m_1}, \qquad B_2 = C \,.\, 10^{-0\cdot4m_2},$$

from which

$$B_1 + B_2 = C(10^{-0\cdot4m_1} + 10^{-0\cdot4m_2}).$$

If m denotes the magnitude corresponding to the naked-eye observation of what appears to be a single star, we have, by (2),

$$B_1 + B_2 = C \,.\, 10^{-0\cdot4m}.$$

Hence $$10^{-0\cdot4m} = 10^{-0\cdot4m_1} + 10^{-0\cdot4m_2}$$
$$= 10^{-0\cdot632} + 10^{-0\cdot836}.$$

But $10^{-0\cdot632}$ is the number whose logarithm is $-0{\cdot}632$ or $\bar{1}{\cdot}368$, in the usual notation. Hence from the antilogarithm tables we find that the number is 0·2333. Similarly, $10^{-0\cdot836} = 0{\cdot}1459$. Hence

$$10^{-0\cdot4m} = 0{\cdot}2333 + 0{\cdot}1459 = 0{\cdot}3792.$$

But $$0{\cdot}3792 = 10^{\bar{1}\cdot5788} = 10^{-0\cdot4212}.$$

Hence $$-0{\cdot}4m = -0{\cdot}4212,$$

whence, to two decimal places,

$$m = 1{\cdot}05.$$

It is to be remarked that a few of the brightest stars in the sky have negative magnitude-numbers; for example, the magnitude of Sirius is $-1{\cdot}58$. In (1) and (2) we regard m_1 and m_2 as algebraical quantities which may, on occasion, have negative values.

155. *The different magnitude systems.*

The instrument used in measuring the brightness of the stars is called a *photometer*. If the comparison is made with respect to the light which ordinarily enters the eye from the stars the magnitudes are said to be *visual* magnitudes.

When a part of the sky is photographed, the brightest stars generally produce the largest images on the photographic plate. The ordinary plate is most sensitive to blue light and, as starlight is a blend of the rainbow colours in varying proportions, the images on the plate give a measure of the intensity of the blue light emitted by the different stars. As the photographic brightness of the stars can be correlated with the diameters of the images on the plate, the measurement of the diameters enables the stars to be graded on a definite magnitude system; the magnitudes derived by means of (1) are then called *photographic* magnitudes.

If a yellow piece of glass (called a filter) is placed in front of a special kind of photographic plate which is sensitive to yellow light, only the yellow light radiated by the stars is transmitted by the filter to form the stellar images on the plate. As the eye is most sensitive to this particular kind of light, the magnitudes derived are essentially on the visual scale; owing to the fact that a photographic procedure is employed, such magnitudes are called *photovisual* magnitudes.

When light falls on such metals as sodium, potassium, caesium or rubidium, electrons are ejected from the atoms of these substances and the rate of ejection is proportional to the intensity of the light falling on the particular element concerned. This rate can be measured by a sensitive instrument called a photoelectric photometer, and so the ratio of photoelectric brightness of two stars can be derived; the magnitudes derived in this way are called *photoelectric* magnitudes and they have reference to the quality of light to which the particular element used is most sensitive.

All these instruments are concerned with measuring only a part (which varies according to the particular photometer used) of the radiation from a star. The sum-total of the radiation from a star can be measured by an instrument called a *bolometer*, and the derived magnitudes are called *bolometric* magnitudes.

In all cases the magnitudes obtained directly from observation are designated *apparent magnitudes* and, if we refer to one of the magnitude systems described, we may specify the magnitude of a star more particularly as apparent visual or apparent photographic, and so on.

156. *Absolute magnitude.*

The fact that a star A *appears* brighter than a star B—in other words, that the apparent magnitude of A is less than the apparent magnitude of B—does not necessarily mean that A is *intrinsically* brighter than B. For example, a pocket electric torch held a few feet away would appear brighter than a

powerful motor-car headlight a mile away; and of two motor-car headlights of equal candle-power, the nearer would appear brighter than the one more remote. In these examples the factor of distance evidently enters into the relation between intrinsic brightness and apparent brightness. In the same way, for example, if the physical characteristics of two stars A and B happened to be identical in every respect and if B were further off than A, it is evident that A would appear brighter than B—in other words, the apparent magnitude of A would be less than the apparent magnitude of B. In order, then, to compare the intrinsic brightnesses—or *luminosities*—of the stars, we require to compare their brightnesses if they were all at the same distance from us. The standard distance chosen for this purpose is 10 parsecs, and the magnitudes which the stars would then have, corresponding to this standard distance, are called *absolute magnitudes*.

157. *The relation between apparent and absolute magnitudes.*

Let m be the apparent magnitude of a star at a distance of d parsecs, and let b be the corresponding apparent brightness. Let M be the absolute magnitude corresponding to the standard distance of D parsecs, the numerical value of D being 10, and let B be the corresponding brightness.

By (1) the ratio of brightness, $b:B$, is given by

$$\frac{b}{B}=10^{-0\cdot4(m-M)}.$$

Now the brightness of a star varies inversely as the square of its distance from us; hence

$$\frac{b}{B}=\frac{D^2}{d^2},$$

so that $$10^{0\cdot4(M-m)}=\frac{D^2}{d^2}.$$

On taking logarithms of both sides (to base 10) we have

$$\tfrac{2}{5}(M-m)=2\log D-2\log d,$$

from which

$$M=m+5\log D-5\log d. \quad \text{.............(3)}$$

But $D=10$ parsecs and $d=1/P$ parsecs, where P is the star's parallax in seconds of arc. Hence (3) becomes

$$M=m+5+5\log P. \qquad\qquad (4)$$

This is an important formula connecting apparent magnitude, absolute magnitude and parallax; it is to be remembered that the logarithm in the formula is with respect to the base 10.

Ex. 2. The parallax of 11 Ursae Minoris is $0''\cdot024$ and its apparent visual magnitude m is $5\cdot10$; to calculate its absolute visual magnitude M.

We have by (4),

$$\begin{aligned} M &= 5\cdot10+5+5\log 0\cdot024 \\ &= 10\cdot10+5[\bar{2}\cdot3802] \\ &= 10\cdot10+5[-2+0\cdot3802] \\ &= 2\cdot00. \end{aligned}$$

158. *Giant and dwarf stars.*

It is found by the method of the previous section that the absolute magnitudes of the stars range from about -5 to about $+15$. A star A of absolute magnitude -5 is thus 20 magnitudes more luminous than a star B of absolute magnitude $+15$. As a difference of 5 magnitudes corresponds to a ratio of 100 : 1 in brightness, the star A is thus $(100)^4$ times or 100 million times more luminous intrinsically than B. There is thus an enormous disparity in the luminosities of the stars. The very luminous stars (absolute magnitudes in the range -5 to about $+2$) are called *giants*, and the less luminous stars (absolute magnitudes from about $+2$ to $+15$) are called *dwarfs*. In particular, the absolute magnitude of the sun is found to be $+4\cdot85$, so that the sun belongs to the category of dwarf stars.

The luminosity of a star depends mainly on two factors: (i) the surface temperature (or *effective temperature* as it is usually described), and (ii) the extent of the surface area. According to physical principles the luminosity varies as the fourth power of the absolute temperature T (on the centigrade scale T is measured from the absolute zero, $-273°$). It will be shown later how effective temperatures are estimated; it is found, for example, that the value of T for the sun is about 6000°.

Again, the surface area of a star is proportional to the square

of its radius r. It follows from (i) and (ii) that, if L_1 and L_2 denote the intrinsic luminosities of two stars, T_1 and T_2 their effective temperatures and r_1 and r_2 their radii, then

$$\frac{L_1}{L_2}=\left(\frac{T_1}{T_2}\right)^4\left(\frac{r_1}{r_2}\right)^2. \qquad\qquad (5)$$

It is to be remarked that, in the application of the physical law (Stefan's law) relating to the fourth power of the effective temperature, it is assumed that the star concerned radiates its heat and light as if it were what the physicist calls a "perfect radiator", and that the luminosity concerned refers to the total output of radiation of all wave-lengths. The absolute magnitude associated with the luminosity is then the absolute bolometric magnitude.

If M_1 and M_2 denote the absolute bolometric magnitudes of the two stars, we have

$$\frac{L_1}{L_2}=10^{-0\cdot4(M_1-M_2)},$$

so that, by (5),

$$10^{-0\cdot4(M_1-M_2)}=\left(\frac{T_1}{T_2}\right)^4\left(\frac{r_1}{r_2}\right)^2. \qquad\qquad (6)$$

This formula enables us to calculate the radius of a star, for which M_1 and T_1 are known, in terms of the sun's radius r_2 as unit, M_2 and T_2 being known for the sun.

Actually, for reasons into which we cannot enter here, formula (6) does not give very satisfactory results in practice if T_1 or T_2 is greater than about 7000°. In such cases, a more detailed formula is used.

Ex. 3. To calculate the radius of a star for which $T_1=5400°$ and $M_1=-0\cdot35$; for the sun, we assume: $T_2=6000°$, $M_2=+4\cdot85$ and $r_2=695{,}500$ kilometres or 432,000 miles.

From (6),
$$10^{-0\cdot4(-0\cdot35-4\cdot85)}=\left(\frac{5400}{6000}\right)^4\left(\frac{r_1}{r_2}\right)^2.$$
$$\therefore \left(\frac{r_1}{r_2}\right)^2=\left(\frac{10}{9}\right)^4\times10^{2\cdot08};$$
$$\therefore \frac{r_1}{r_2}=\frac{100}{81}\times10^{1\cdot04}.$$

Hence $r_1=13\cdot54r_2$, so that $r_1=9\cdot4\times10^6$ kms. $=5\cdot8\times10^6$ miles.

It is found by the preceding method and by others of a different character that the range in stellar radii is exceedingly great. At the lower end of the range we have stars such as the Companion of Sirius (or Sirius B) whose radius is 12,000 miles (about three times the earth's radius and somewhat less than the radius of Uranus). At the other end of the range we have stars such as Betelgeuse and Antares with radii of about 125 million and 200 million miles respectively. As the biggest stars are also stars of very great luminosity and the smallest stars are of very small luminosity, the terms "giants" and "dwarfs" are associated with size as well as luminosity.

It is of interest to note that the calculations of the diameters of about a dozen of the largest stars have been confirmed by direct observations made with the *interferometer* attached to the 100-inch telescope of Mount Wilson Observatory.

159. *Binary stars.*

A double star, as it name implies, consists of two stars seen very close together in the telescope. This visual proximity of the stars is evidently due to one of two causes; first, the two stars (A and B) and the sun may be in nearly the same straight line, so that B will appear close to A in the sky; if, in addition, A and B are at widely different distances from the sun, their close appearance is then the result of their fortuitous geometrical relation with respect to the sun and to nothing more. Such stars are known as *optical doubles* and are of little further interest. Second, the double star may actually be a physical system, analogous to the physical system of the earth and moon, in which the moon describes an orbit relative to the earth as a result of the mutual gravitation of the two bodies. In this case, the double star is known as a *binary*.

In 1782 Sir William Herschel published his first catalogue of double stars as a preliminary to an attempt to measure parallaxes. Some years later, re-observation of the catalogue stars showed that several were in relative motion. This, in itself, might indicate nothing more than that each star had its own distinctive proper motion (see § 181). But, in a few cases, still later observations gave indisputable evidence of orbital motion.

The brighter component of a binary is called the *primary*,

usually denoted by A; the fainter is the *secondary*, denoted by B. Thus, Sirius A and Sirius B refer to the binary system of Sirius which, to the naked eye, appears to be a single star; Sirius B is sometimes designated the *Companion of Sirius*.

The observation of a binary consists of the measurement—with a filar micrometer (see section 211) or by means of photography—of the angular separation, ρ, of the components and the position angle, θ, of B relative to A (Fig. 91). In this way observations spread over many years suffice, in many cases when the period of observation is long enough, to determine what is known as the *apparent orbit*; this is the path traced out by B, relative to A, on the celestial sphere, illustrated by the dotted curve in Fig. 91. The dimensions of the apparent orbit are expressed in seconds of arc. The plane of the actual orbit in space (the *true orbit*) will not be, in general, perpendicular to the line of sight; consequently, the apparent orbit is the projection, on the celestial sphere, of the true orbit.

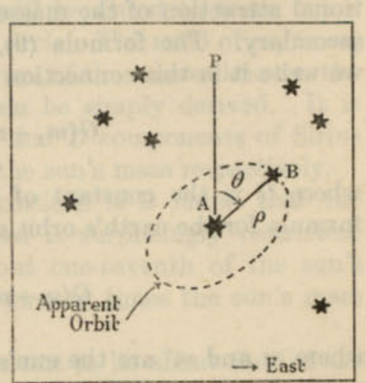

Fig. 91.

160. *Stellar masses.*

From the apparent orbit the following elements—among others—of the true orbit can be deduced: the orbital period T (in years), the semi-major axis, α, of the elliptic orbit (in seconds of arc) and the eccentricity e. Most of the eccentricities are found to be large, varying between 0·3 and 0·7, approximately.

If we know the distance d (in kilometres) of the binary from the sun, we can obtain the linear semi-diameter, A, in kilometres, for evidently

$$\sin \alpha = \frac{A}{d}.$$

Also, the parallax P is given by

$$\sin P = \frac{a}{d},$$

where a is the earth's orbital radius in kilometres. Hence, α and P being small angles, we obtain by division,

$$\frac{\alpha}{P}=\frac{A}{a}. \qquad \text{.............................(7)}$$

Now, the true orbit is described owing to the mutual gravitational attraction of the masses m_1 and m_2 of the primary and secondary. The formula (9), section 75, is then applicable; we write it in this connection

$$G(m_1+m_2)=\frac{4\pi^2A^3}{T^2}, \qquad \text{.....................(8)}$$

where G is the constant of gravitation. The corresponding formula for the earth's orbit about the sun is

$$G(m+m')=\frac{4\pi^2a^3}{T_0^{\,2}}, \qquad \text{........................(9)}$$

where m and m' are the sun's and earth's mass and T_0 is one year. In (9) we can neglect m' in comparison with m so that dividing (8) by (9) we obtain

$$\frac{m_1+m_2}{m}=\left(\frac{A}{a}\right)^3\cdot\frac{1}{T^2},$$

which becomes, by means of (7), taking the solar mass to be unity,

$$m_1+m_2=\left(\frac{\alpha}{P}\right)^3\cdot\frac{1}{T^2}. \qquad \text{....................(10)}$$

This formula gives the sum of the masses of the components of the binary in terms of the solar mass as unit. If α and T are obtained from the study of the apparent orbit and P is measured in the usual way, we can calculate the mass of the binary as a whole.

Ex. 4. For Sirius, it is found that $\alpha=7''{\cdot}57$, $T=50{\cdot}0$ years and $P=0''{\cdot}371$. Hence

$$m_1+m_2=\left(\frac{7{\cdot}57}{0{\cdot}371}\right)^3\cdot\frac{1}{2500}=3{\cdot}40.$$

Thus, the binary system of Sirius has a mass equal to 3·4 times the sun's mass.

It is possible in the case of several binaries to use some other observational factor to obtain the ratio of m_1 to m_2. For example, if a star is single, proper motion will carry the star along a great circle on the celestial sphere. If the star is a binary, it is the centre of gravity of the system that moves in this way, and if the position (R.A. and Dec.) of, say, the primary A is measured year by year it will be found to trace out a curved path which is not a great circle. The ratio of m_1 to m_2 can then be inferred. With the sum of the masses known, the mass of each component can then be simply derived. It is deduced in this way that the A and B components of Sirius have masses 2·44 and 0·96 times the sun's mass respectively.

From these and other considerations it is found that the range of individual stellar masses is surprisingly restricted. The smallest stellar mass is about one-seventh of the sun's mass, while masses greater than twenty times the sun's mass are extremely rare.

The average stellar mass comes out to be about that of the sun. This fact is used in deriving *dynamical parallaxes*; thus, if P is not known, we can obtain a fairly reliable value of the parallax by means of (10), by putting $m_1+m_2=2$; then,

$$P=\alpha\left(\frac{1}{2T^2}\right)^{\frac{1}{3}}. \qquad \text{........................(11)}$$

Parallaxes, obtained in this way, are useful in supplementing the parallax values of other stars obtained by direct measurement.

161. *Eclipsing binaries.*

It was discovered in 1783 by Goodricke that the brightness of Algol (β Persei) fluctuated in a regular manner, the changes occurring, as now more accurately known, in a period of 2·8673 days. Since then hundreds of stars have been found of a similar nature and their study has led to important additions to our knowledge of the dimensions and densities of the stars. Briefly, a star such as Algol which varies in brightness is called a *variable*, and since, as we shall see later, Algol is a binary whose components mutually eclipse one another, it is more particularly known as an *eclipsing binary*.

The changes in brightness are conveniently represented by a *light-curve* which gives the observed apparent magnitude of the variable plotted against the time. The observations are made by comparing the brightness of the variable with a neighbouring star of constant brightness; the observations

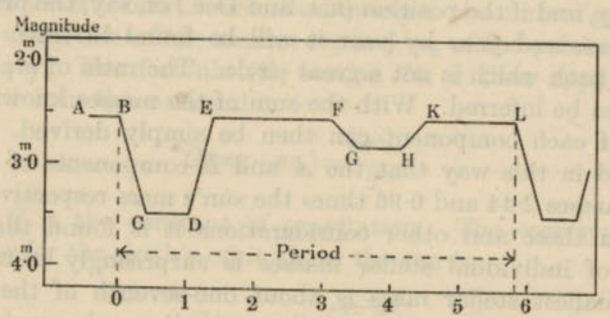

FIG. 92.

may be either visual, photographic or photoelectric. A typical light-curve of an eclipsing binary is illustrated in Fig. 92. The horizontal scale gives the time in days and the vertical scale the magnitude at any time.

Along AB the variable is constant in brightness, the magnitude being 2·5. At B its brightness begins to diminish until C is reached at magnitude 3·5. Between C and D it remains constant in brightness, and at D it begins to increase in brightness, reaching its previous maximum at E. At F the brightness decreases to the level of G, about magnitude 2·7, remaining constant along GH. The brightness increases from H to K at the original maximum brightness. At L the above sequence of changes is repeated. The interval between B and L is the *period* of the light-curve (in Fig. 92 we arbitrarily begin the period at B).

The minimum brightness represented by CD is the *principal minimum*; the minimum represented by GH is the *secondary minimum.*

162. *Explanation of the light-curve.*

The features of such a light-curve as is shown in Fig. 92 can be explained by supposing that the variable star is a binary

and that the minima are the results of eclipses. This explanation is supported by spectroscopic observations. Fig. 93 illustrates the orbit of the smaller star about the primary P. As eclipses are assumed to occur, the line of sight must lie in, or nearly in, the orbital plane. For convenience in explanation we assume that the primary is more luminous per unit of surface area than the companion. When the latter is at A, we obtain the maximum possible light; this remains constant

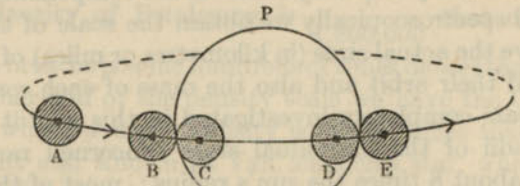

FIG. 93.

until the companion reaches B when it is just on the point of beginning its eclipse of the primary; between B and C the companion covers up a progressively larger area of the primary until at C the eclipsing effect is a maximum; this corresponds to the downward part BC of the light-curve (Fig. 92). Between C and D the companion cuts off a constant amount of light from the primary and this corresponds to the part CD of the light-curve. Between D and E the effect of the eclipse diminishes until at E we receive again the maximum amount of light from the system; this stage corresponds to the part DE of the light-curve. Later on, the companion passes behind the primary and we have the sequence of light-changes represented by the secondary minimum of the light-curve.

In the system just described the companion is completely eclipsed at the secondary minimum, and at the constant part CD of the principal minimum it eclipses the primary to the greatest possible extent. But not all eclipsing systems involve total eclipses of one of the components. As regards Algol, for example, the eclipse is only partial (the line of sight is inclined to the orbital plane at an angle greater than that required for a total eclipse to occur) and the principal minimum, for example, is V-shaped; as soon as minimum brightness is reached the light immediately begins to increase, and this is easily seen to correspond to the geometrical circumstances of a partial eclipse.

163. *Analysis of light-curves.*

The analysis of a light-curve such as is shown in Fig. 92 leads to information with which we can construct a model of the particular system concerned—that is to say, we can express the radius of each component and the radius of their relative orbit in terms of any arbitrary unit. If the components can also be observed spectroscopically we obtain the scale of the model and so derive the actual sizes (in kilometres or miles) of the components and their orbit and also the mass of each component.

Of the stars completely investigated in this way it is found that the radii of the individual stars concerned range from about $\frac{3}{5}$ to about 8 times the sun's radius; most of the masses are between one-half and eight times the solar mass, with a few approaching twenty times the sun's mass. The semi-major axes of the relative orbits—or the average distance between components—range from $1\frac{1}{2}$ to nearly 20 million kilometres.

Another feature that emerges is that many of the stars are not spherical but ellipsoidal in shape, due to tidal action. The eclipsing system of β Lyrae is noteworthy in this respect; it consists of two elongated stars almost in contact.

164. *Stellar densities.*

If r_0 and m_0 denote the sun's radius and mass, the average density ρ_0 of the sun is given by

$$\rho_0 = m_0 / \tfrac{4}{3}\pi r_0^3.$$

Similarly, if r, m and ρ refer to a star,

$$\rho = m / \tfrac{4}{3}\pi r^3.$$

These formulae give, by division,

$$\frac{\rho}{\rho_0} = \frac{m}{m_0} \Big/ \left(\frac{r}{r_0}\right)^3.$$

It is convenient to take each of r_0, m_0 and ρ_0 as unit; hence, if r, m and ρ are now supposed expressed in these units,

$$\rho = \frac{m}{r^3}. \quad \text{.................................(12)}$$

The sun's density is 1·4 times that of water, and this enables us to transform ρ in (12) into more familiar units.

The densities of eclipsing binaries range from about $\frac{1}{30}$ to about $2\frac{1}{2}$ times the sun's density, but this range is far from giving an adequate idea of the enormous variation of stellar densities. We have seen that the radius of Betelgeuse is 125 million miles or 300 times the sun's radius. The mass is estimated to be 15 times that of the sun. Hence, by (12), the average density of Betelgeuse is $\frac{1}{1{,}800{,}000}$ that of the sun. The air we breathe is some hundreds of times denser than this star. At the other end of the density scale we have the Companion of Sirius, whose average density is 50,000 times that of water, and a faint star known as Van Maanen's star, whose density is estimated to be about half a million times that of water.

165. *Star colours.*

To the naked eye the stars vary in colour, from the blue of Rigel to the red of Betelgeuse; in the telescope the stellar colours are more readily discernible. Colour is an index of the effective temperature T of a star—the red stars are the comparatively cool stars with T about 3000°, and the blue stars are the hottest with T of the order of 20,000° or more. As we shall see later, the light of a star consists of light of all colours; in the case of a blue star, the blue light predominates greatly over the other colours, while in the case of a red star the red light predominates. In this sense the sun is a yellow star, and for this reason the eye has become, through the long ages of human evolution, most sensitive to yellow light. On the other hand, the ordinary photographic plate is most sensitive to blue light. Thus, if two stars, one blue and one red, appear to the eye to be of equal brightness, the image of the blue on a photographic plate will appear larger than the image of the red star, the exposures being equal; in other words, the blue star will be brighter photographically than the red star. If m_p denotes the photographic magnitude of a star and m_v its visual (or photovisual) magnitude, the difference of these is called the star's *colour index* (C.I.); we have

$$\text{C.I.} = m_p - m_v. \quad \text{..........................(13)}$$

The visual and photographic magnitude scales are arbitrarily reconciled by taking the photographic magnitude to be the same as the visual magnitude for a star whose spectral type is $A0$ (we refer to this spectral classification later).

166. *The spectrum.*

Sunlight is a combination of light of different colours—red, orange, yellow, green, blue, indigo and violet; these are the colours of the rainbow, which is produced under certain conditions as the result of the breaking-up of sunlight by raindrops. The rainbow colours are produced easily by means of a glass prism (Fig. 94). Suppose a narrow parallel beam of sunlight falls on one side of the prism. Red rays are refracted, by the glass, less than blue rays, and so the emergent rays are spread out into a diverging beam. If a sheet of paper is held in this

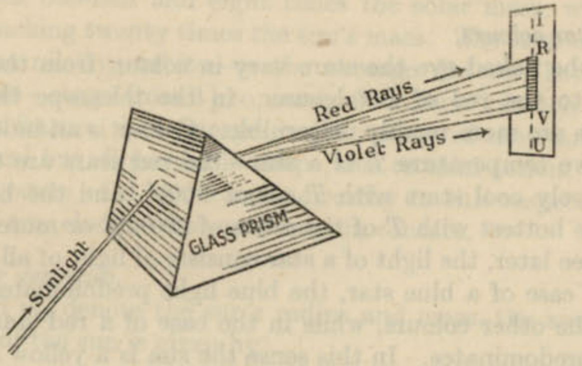

FIG. 94.

beam a narrow strip RV will be seen with the succession of the rainbow colours from red at R to violet at V. This strip is called a *spectrum*. But the visible spectrum RV does not represent the sum-total of sunlight. Beyond R, towards I in Fig. 94, is the *infra-red* region, and between V and U is the *ultra-violet* region. The radiations in the ultra-violet are unable to penetrate the earth's atmosphere. The radiations in the infra-red merge into heat radiations, which can be detected by suitable thermometric means.

The spectroscope is an optical instrument, including one or more prisms, used to produce the spectrum; it is attached to

the eye-end of the telescope. Used photographically, its function is to record the spectrum. At the focus of the telescope is placed a narrow slit through which the light from the heavenly body concerned is allowed to pass; the rays of this beam are rendered parallel by a lens. This parallel beam now falls on one side of the prism, and the spectrum rays which emerge are brought to a focus by a suitable lens; they can then be photographed.

167. *The sun's spectrum.*

When the sun's spectrum is viewed in a spectroscope the rainbow spectrum is seen to be crossed by a large number of dark lines, some faint, some prominent, others of intermediate "strength". If the spectrum is photographed and a "positive" made, the same dark lines are shown. The complete rainbow spectrum is called a *continuous spectrum* and the dark lines are called *absorption lines*. In a relatively few stars, several bright lines may appear, and these are called *emission lines*. It was Wollaston who, in 1802, first observed a few of the strongest absorption lines in the solar spectrum, but it was left to Fraunhofer some years later to improve the observing technique whereby he detected, and later mapped out, several hundreds of lines, to the most prominent of which he assigned letters, e.g. A, ... D, ... H, ... K, ..., etc.; some of these are in use to-day. Fig. 95 is intended to illustrate some features

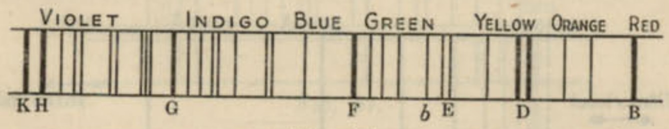

FIG. 95.

only of the solar spectrum; it shows several of the most prominent Fraunhofer lines, but not the continuous spectrum; the colours of the latter are, however, indicated in their approximate positions.

168. *Kirchhoff's laws.*

About the middle of last century Kirchhoff formulated three laws which enable us to interpret the spectra of the sun and stars; these laws are based on laboratory experiments.

First Law. If a gas under great pressure is made luminous by means of an electric spark, say, or if a solid is rendered incandescent, the spectrum obtained in each case is a continuous spectrum.

Second Law. If the pressure of the gas, or of a volatilised metal such as sodium, is low, the spectrum obtained consists of one or more bright lines depending on the substance experimented with. Thus, hydrogen commonly gives a series of bright lines, one line being red, another green, and so on until the end of the series is reached as a close bundle of violet lines. This is known as the Balmer series of hydrogen. The second law may be more fully stated : a luminous gas or vapour under low pressure gives a spectrum of bright lines whose number and position in the spectrum are characteristic of the gas or vapour.

Third Law. If the light from a source producing a continuous spectrum passes through a low-pressure gas or vapour at a lower temperature, the spectrum consists of the continuous spectrum crossed by dark lines whose positions in the spectrum are exactly those of the bright lines which are characteristic of the gas or vapour.

It should be remarked that these dark or absorption lines are not completely devoid of light, but they appear dark in contrast with the brilliant continuous spectrum adjoining. The gas or vapour has the property of *absorbing* most of the

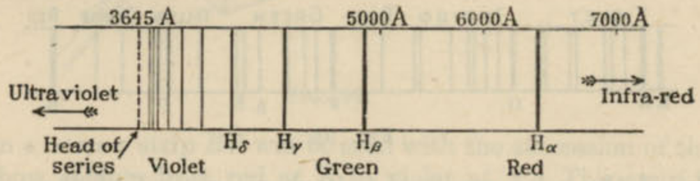

FIG. 96.

light from the continuous spectrum precisely in those places where the characteristic bright lines of the low-pressure gas or vapour occur. Fig. 96 shows the Balmer absorption spectrum of hydrogen. The lines are designated H_α, H_β, H_γ, ... , etc.

169. *Wave-length.*

Light is a wave-motion, and to each distinctive colour, or element of the continuous spectrum, can be assigned a definite

wave-length. The unit of wave-length is the *angstrom unit*, denoted by A, which is 10^{-8} cm., that is to say, 100 million angstrom units are equal to 1 cm. The wave-lengths of the visible spectrum range from about 3000 A in the violet to nearly 7000 A in the red. The wave-lengths of the lines in a spectrum, such as the Balmer Series of hydrogen, can be measured in the laboratory ; for example, the wave-lengths of the red (H_α) and green (H_β) lines are 6562·793 A and 4861·327 A respectively.

170. *Interpretation of the sun's spectrum.*

To facilitate the measurement of the absorption lines in the solar spectrum (or in a stellar spectrum), a "*comparison spectrum*" is photographed above and below the solar (or stellar) spectrum. This spectrum is produced generally by means of glowing iron vapour rendered incandescent by electrical means. The spectrum of iron consists of hundreds of lines in the visible spectrum, and as their wave-lengths are known accurately from laboratory measurements, the wave-lengths of all the solar or stellar lines can be deduced. This is illustrated schematically in Fig. 97, the spectra denoted by (*a*)

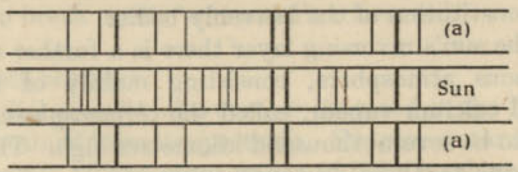

FIG. 97.

being the comparison spectra. It is to be remarked that if a "positive" is made from the actual photograph, the positive shows the solar absorption lines as dark lines, while the emission spectrum of iron is shown as a series of bright lines on a dark background. In Fig. 97, instead of representing the comparison spectrum in this way, the emission lines are represented, for convenience, as dark lines.

The first observational fact that may be noted is that the series of iron lines can be matched exactly in the solar spectrum. The interpretation is as follows. The sun's continuous spectrum is produced by gases under high pressure ; the boundary surface from which the light producing the continuous spectrum emanates is called the *photosphere*, the temperature of which is the "effective temperature" of the sun. Above the photosphere lies an atmosphere of diminishing pressure and temperature consisting of gases and vapours of such elements as iron; this atmosphere is called the *reversing layer*, the height of which above the photosphere is estimated to be 200 or 300 kilometres. As the reversing layer contains the vapour of iron which is at a lower temperature than the photosphere, this vapour absorbs the light from the photosphere precisely in those wave-lengths characteristic of the iron spectrum (Kirchhoff's third law). It is in this way that the presence of iron in the sun is inferred. The total number of chemical elements known on the earth is 92—this includes two believed to exist, but hitherto undiscovered. By spectrum analysis it is deduced that about 50 of the terrestrial elements are present in the sun; there are reasons, which we cannot discuss here, why it is not possible to detect the remainder, assuming that they exist in the sun. Thus the chemical constitution of the earth and of the sun's atmosphere are very much alike. The spectroscope is evidently a very powerful instrument for the analysis of the chemical constitution of the heavenly bodies.

Above the sun's reversing layer there is a further and much more tenuous atmosphere, consisting mainly of hydrogen, helium and calcium vapour, called the *chromosphere*, which is estimated to be several thousand kilometres high. The absorption lines due to these elements appear in the normal solar spectrum. The existence of the chromosphere (and of the reversing layer) can be demonstrated during a total eclipse of the sun at the moment when the moon has just covered the photosphere, still leaving the atmosphere exposed. The spectrum obtained at this instant is called the *flash-spectrum*; it consists of bright lines, thus showing that the light producing the spectrum emanates from a low-pressure atmosphere of the elements concerned; moreover, the heights to which the several elements extend above the photosphere can be measured. Beyond the chromosphere and extending to two or three solar diameters is the *corona*, which is such a notable feature of total eclipses of the sun.

171. *Stellar spectra.*

The spectra given by the stars are, except in a few instances, absorption spectra. Stellar spectra vary in an apparently inexplicable manner. However, spectroscopists have evolved order out of seeming chaos and the spectra are classified according to definite general principles. The principal spectral classes are denoted by the letters

$$O,\ B,\ A,\ F,\ G,\ K,\ M;$$

further, stars belonging to a particular type or class, say A, are sub-divided as $A0$, $A1$, $A2$, ... $A9$. The spectral series from O to M is one of diminishing temperature and also of variation of colour from the blue of the O type to the red of the M type. Without going into details of the methods of classification we may mention some of the salient features. The spectra of the red stars show the absorption lines or bands of chemical compounds such as titanium oxide. Evidently the temperature of the stellar atmospheres is not too low to cause the dissociation of such compounds into their constituent elements. But in type F, for example, the evidence of compounds disappears, from which we conclude that the temperature is high enough to break up the compounds concerned.

172. *Radial velocities.*

The radial velocity of a star, or its speed in the line of sight, can be obtained from its spectrum. Suppose a star is approaching the earth. Then more waves of light reach us per second than if the star had no radial velocity; the effect in this case

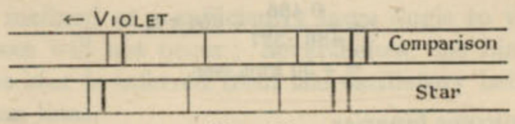

FIG. 98.

is the apparent decrease of a normal wave-length, λ, from λ to λ', say. Thus each absorption line is displaced towards the violet end of the spectrum. This is shown schematically in Fig. 98, in which, for convenience, only the lines in the stellar spectrum appearing also in the comparison spectrum are shown.

If u is the star's velocity of approach in kilometres per second, it is found that, for a given line of wave-length λ,

$$u=\frac{\lambda-\lambda'}{\lambda}\cdot c, \quad\text{.........................(14)}$$

where c is the velocity of light ($=299{,}774$ kms. per sec.).

The measured displacement $(\lambda-\lambda')$ of each line, inserted in (14), gives a result for u, λ and c being known; the mean of the results for various lines gives the definitive value of u.

If the star is receding with velocity v, a line of wave-length λ is displaced towards the red end of the spectrum. If λ_1 is its measured wave-length,

$$v=\frac{\lambda_1-\lambda}{\lambda}\cdot c. \quad\text{.........................(15)}$$

The results obtained by means of (14) and (15) include the effects of the orbital motion of the earth around the sun. These effects can be calculated and when they are removed from the observed results we obtain the radial velocity of the stars relative to the sun. This is the sense in which the term radial velocity is to be generally understood.

Velocities of recession are designated + and velocities of approach −. It is important to note that the spectroscopic observations give directly the radial velocities in kms. per sec.

Ex. 5. The measured wave-length of H_β in a stellar spectrum is 4861·813 A; given that λ for H_β is 4861·327 A, find the star's radial velocity.

Here the measured wave-length λ_1 is greater than λ, so that the star is receding. Also $\lambda_1-\lambda=0{\cdot}486$ A. Hence, by (15),

$$v=\frac{0{\cdot}486}{4861{\cdot}327}\times 299{,}774$$
$$=+30 \text{ kms./sec.}$$

173. *Spectroscopic binaries.*

These are systems which are detected to be binary by the spectroscope. Consider, for example, an eclipsing binary. As a system it will have some constant velocity in space and therefore a constant radial velocity, V. But each component describes an orbit around the centre of mass so that, at any instant, the radial velocity of one star will consist of V together with the component of the orbital motion in the line of sight. This component may be + or −, so that during the orbital period a particular spectrum line will appear to oscillate about a mean position given by the displacement corresponding to V. In the same way, a spectrum line due to the second star of the binary will appear to oscillate about the mean position just referred to. At any instant the two spectrum lines produced by the system will be on opposite sides of the mean position. At some definite instant the two components will be moving at right angles to the line of sight, so that at this instant the components of radial velocity relative to the centre of mass will both be zero; the lines thus coincide and the spectrum will then consist of a series of single lines.

Curves showing the radial velocities of the components throughout their orbital period can be constructed from the observations, and when the curves are analysed the ratio of the masses of the two stars can be obtained; also the minimum values of the mass of each star and the minimum values of the semi-major axis of each orbit (a_1 and a_2) with respect to the centre of mass. Thus,* for example, $m_1 : m_2$ for the spectroscopic binary α Virginis is 9·6 : 5·8, and the minimum values of m_1 and m_2 are 9·6 and 5·8 times the solar mass; also the minimum values of a_1 and a_2 are 6·4 and 4·9 millions of kilometres.

Not all spectroscopic binaries show the doubling of their spectral lines, for one component may be comparatively faint so that its spectrum is not recorded on the photograph.

In the preceding explanation we have considered an eclipsing binary, which can also be described as a spectroscopic binary by reason of the method of observation. But if the line of sight is inclined at a sufficiently large angle to the orbital plane eclipses will not occur; nevertheless, the binary character of the star is inferred from the oscillatory behaviour of the spectrum lines.

174. *Stellar temperatures.*

If we know from laboratory experiments the temperature, under given conditions of pressure, at which a compound such as titanium oxide just ceases to exist as a compound, we have

* Here, m_1 and m_2 denote the masses of the stars.

evidently a means of estimating the temperature of a star in a particular sub-division of the spectral sequence for which the characteristic lines disappear. This is one example of a general method applied extensively.

Again, the character of the continuous spectrum varies throughout the spectral sequence; in red stars, for example, the continuous spectrum is most intense in the red part of the spectrum, fading away in brilliance towards the violet. With suitable instruments an energy-curve can be drawn which gives the intensity of light throughout the range of wave-length measurable. Physical theory enables us to construct similar curves for the light-energy radiated by bodies (called perfect radiators) at different temperatures. Comparison of the observed energy-curve of a star of a particular spectral type with the theoretical curves then indicates the "effective temperature" of the type concerned.

The following table * gives the effective temperature T of the principal spectral classes; the temperatures of giants and dwarfs of the same spectral class are found to be slightly different; these are indicated by the letters g and d from class $G0$ where the separation into giants and dwarfs most clearly begins.

TABLE:

Effective Temperatures of the Stars

Class	T	Class	T (g)	T (d)
$O5$	35,000°	$G0$	5,600°	6,000°
$B0$	23,000	$G5$	4,700	5,600
$B5$	15,000	$K0$	4,200	5,100
$A0$	11,000	$K5$	3,400	4,400
$A5$	8,500	$M5$	2,700	3,400
$F0$	7,400			
$F5$	6,500			

175. *The Hertzsprung-Russell diagram.*

About thirty years ago E. Hertzsprung and H. N. Russell independently discovered a general relation between absolute

* Mainly from *Astronomy* by Russell, Dugan and Stewart.

magnitude and spectral type. When the absolute magnitudes are plotted against spectral type the great bulk of the stars are represented approximately in the diagram (Fig. 99) by the

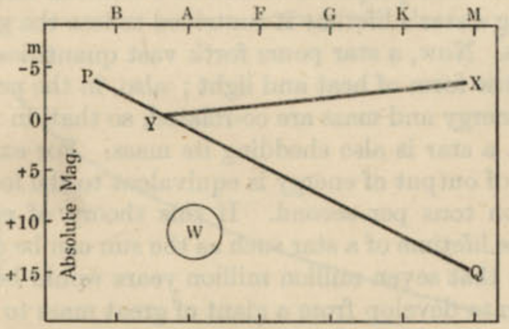

FIG. 99.

lines XY and PQ. Stars which give points on XY are the giants, and the stars which give the locus PQ are said to belong to the *main sequence.* The few white dwarfs—the Companion of Sirius is one—for which reliable data are known are situated in the region denoted by W. If we consider the stars of class M, for example, it is found that they are sharply divided into two classes, the giants clustering near X in the diagram and the very feeble dwarfs in the neighbourhood of Q; there is a distinct gap between absolute magnitudes $+2$ and $+12$, approximately, within which M type stars are almost entirely absent. The gaps between the giants and dwarfs of types K and G are also evident from the statistics. As we later approach type B in the diagram, the gaps are less apparent.

Until 1924, the diagram was supposed to furnish evidence of the evolution of the stars. It was then believed that a luminous star began its evolutionary career as a diffuse giant of type M; as it contracted under gravitation it became hotter and denser, passing through the spectral classes in the order M to A or B. After this stage contraction still continued but temperature began to diminish, so that the star progressed along the main sequence, eventually reaching class M as a comparatively dense star of low luminosity. The sequence of changes just indicated is in accordance with our knowledge of stellar densities. Moreover, the sequence of changes is also a sequence of diminishing mass, for the giants of class M are 10 to 20 times the solar mass, whereas the dwarf M-stars have but a fraction of the sun's mass. The proposed course of evolution thus involved that during a star's lifetime it contrived to lose the greater part of its mass. Now, a star pours forth vast quantities of radiant energy in the form of heat and light; also, in the new physical concepts energy and mass are co-related, so that, in the process of shining, a star is also shedding its mass. For example, the sun's rate of output of energy is equivalent to the loss of about four million tons per second. If this theory of evolution is correct, the lifetime of a star such as the sun can be calculated; it is found that seven million million years would be necessary for the sun to develop from a giant of great mass to its present condition. The long time-scale involved in this evolutionary theory, besides introducing a hypothetical physical process whereby mass is transformed into radiation, is in direct opposition to the total time-scale of about 5,000 million years required by what are believed, at the present time, to be more reliable considerations. If the short time-scale is finally accepted, the evolutionary course suggested by the Hertzsprung-Russell diagram must be discarded and the diagram itself regarded as an unexplained statistical relation between absolute magnitude and spectral type.

176. *The mass-luminosity relation.*

The relation between mass and luminosity (or absolute magnitude) as applied to actual stars, was discovered by Sir Arthur Eddington in 1924; it is represented by the curve in Fig. 100; for convenience the *logarithm* of the mass (sun's mass as unit) is plotted against absolute magnitude. The curve itself is simply a representation of a formula, involving mass and luminosity, obtained from the mathematical and physical theory of the constitution of gaseous stars, such as the giant diffuse stars undoubtedly are. These stars agreed with the curve as might have been expected, unless, of course, the theory were unsound in any way; in addition dense stars, such as the sun and the K and M dwarfs with densities greater than that of water, satisfied the curve. The inference is that

such dense stars must be in the state of a " perfect gas ", with which the theory was concerned. This might, at first, seem an absurd conclusion, but it is to be remembered that, at the

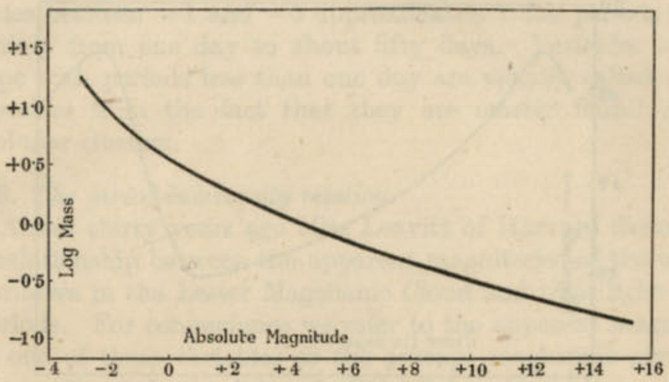

Fig. 100.

enormous temperatures existing within the stars, the comparatively bulky atoms of the chemical elements are broken up into the much more minute protons and electrons which have then as much freedom of movement as the atoms and molecules of a terrestrial gas.

The curve in Fig. 100 has a practical use in those investigations for which the values of stellar masses are required when these are not available by any of the usual observational methods. It is assumed that the absolute magnitudes of such stars are known; the curve then gives the corresponding masses.

177. *Cepheid Variables.*

As we have seen earlier, an eclipsing variable is a binary system whose changes in observed brightness are due entirely to a particular geometrical relation between the plane of the binary and the line of sight. A Cepheid variable is a single star whose *physical* state changes in a well-defined period, and one way in which this change of state is shown is in the rhythmical alteration of the star's brightness. These variables take their name from the well-known variable δ Cephei; the light-curve of this star is shown in Fig. 101 (from photoelectric observations by the author), the period being 5·36629 days. It is impossible to account for this light-curve on the hypothesis that the star is an eclipsing system, and some other explanation

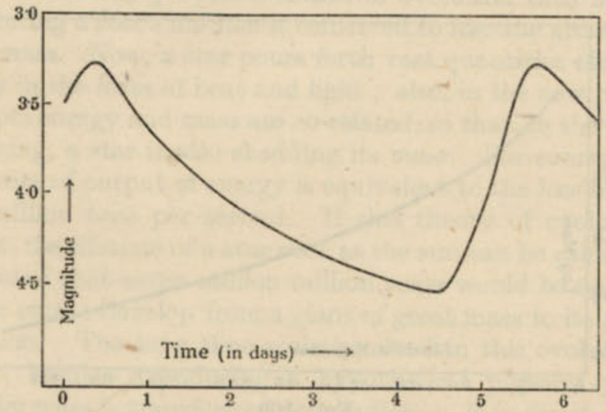

FIG. 101.

must be looked for. It is also found that the observed radial velocity and the spectral type fluctuate in the same period as that given by the light-curve.

The theory that best fits the known facts is the *pulsation theory*. This postulates that the star is rhythmically expanding and contracting. The theory accounts for the changes in radial velocity since, during expansion, the star's atmosphere is moving outwards, that is, towards us, and during contraction it is moving away from us, these changes being superimposed on the presumably constant radial velocity of the star as a whole.

The changes in spectral type are also explained, for the character of the spectrum at any time depends on the physical characteristics, pertaining to pressure and temperature, of the star's atmosphere. During expansion, both pressure and temperature decrease and the spectral type changes in the direction indicated, for example, by $F \rightarrow G$; during contraction the spectral type changes in the reverse direction, that is, $G \rightarrow F$.

Further, a decrease in temperature during expansion is followed by a decrease in intrinsic luminosity which is modified, however, by the increase in the radiating surface of the star; changes in the apparent brightness are thus to be expected,

with changes in the opposite direction following from the contraction of the star. These considerations account for the light-curve.

The Cepheids, as a class, are giant stars with absolute magnitudes between -1 and -5 approximately; the periods range mainly from one day to about fifty days. Variables of this type with periods less than one day are usually called *cluster variables* from the fact that they are mostly found in the globular clusters.

178. *The period-luminosity relation.*

About thirty years ago Miss Leavitt of Harvard discovered a relationship between the apparent magnitudes of the cluster variables in the Lesser Magellanic Cloud and their light-curve periods. For convenience we refer to the apparent magnitude of one of these variables as the average magnitude given by the maximum and minimum of the light-curve. It was found

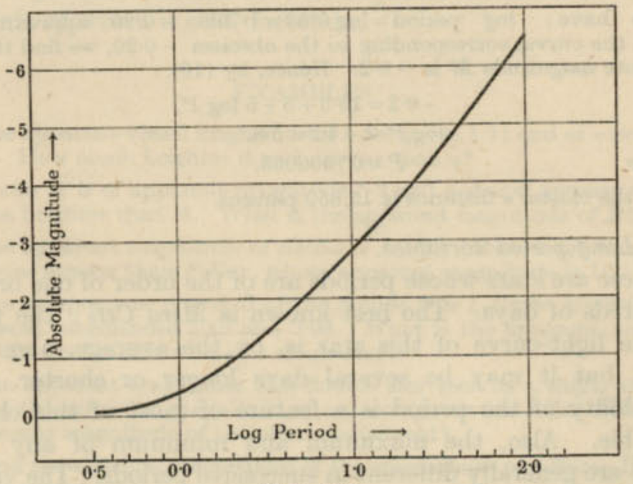

FIG. 102.

that the brighter the star the greater was its period, and that apparent brightness and period increased together in a definite way. Now, all the stars of the Cloud are at practically the same distance from us, so that the relation actually found can be expressed as a relation between absolute magnitude and period. Studies of Cepheids showed that this relation also held for them. Fig. 102 gives the period-luminosity relation in a graphical form; for convenience, absolute magnitude is plotted against the *logarithm* of the period.

The curve enables us to find the distance of any remote object in which Cepheids are found. The distance of a star in a globular cluster, for example, cannot be measured in the usual way, but, if it is a variable, it is comparatively easy to obtain its light-curve photographically; we thus have (*a*) the period, and (*b*) the average magnitude, m. Now, from the curve in Fig. 102 we obtain, using the known period, the corresponding absolute magnitude M. Hence, the parallax P (in seconds of arc) of the star—and, effectively, of the cluster—is found by means of the formula (p. 196)

$$M = m + 5 + 5 \log P. \qquad \text{.......................(16)}$$

Ex. 6. For a cluster variable $m = 15{\cdot}8$ and the period is 0·63 days; to find the distance of the cluster.

We have: log period = log 0·63 = $\bar{1}{\cdot}799 = -0{\cdot}20$ approximately. From the curve, corresponding to the abscissa $-0{\cdot}20$, we find that the absolute magnitude M is $-0{\cdot}2$. Hence, by (16),

$$-0{\cdot}2 = 15{\cdot}8 + 5 + 5 \log P,$$

or $$\log P = -4{\cdot}2 = \bar{5}{\cdot}8.$$

Hence $$P = 0''{\cdot}000063.$$

Thus the cluster's distance is 15,850 parsecs.

179. *Long-period variables.*

These are stars whose periods are of the order of one or more hundreds of days. The best known is *Mira Ceti*; the period of the light-curve of this star is, on the average, about 330 days, but it may be several days longer or shorter. This variability of the period is a feature of most of this class of variable. Also, the maximum and minimum of any light-curve are generally different in successive periods. The various maxima of the light-curve of Mira, for example, vary between the first and fifth magnitudes; the minima are similarly erratic, varying between the eighth and tenth magnitudes. The cause of the light changes in these stars is not known with certainty, but possibly they are irregularly pulsating stars, thus belonging to the Cepheid type.

180. *Novae.*

A *nova* (or *new star*) is a star which suddenly flares up from the insignificant state of a star, say, of the 15th magnitude to the prominence associated with the naked-eye stars. This rapid rise in brightness generally occurs within a day or so. Shortly after maximum has been reached, the star begins to fade slowly in brightness and after several weeks or months it returns to the obscurity of its earlier state. The most notable nova in astronomical records is that now known as Tycho Brahe's Nova (called after the famous Danish astronomer); this star suddenly appeared on November 7, 1572, and within a day or two surpassed in brilliance every star in the sky, so much so that it was visible in broad daylight.

If we take as an example of a nova a star which rises from the 15th magnitude to magnitude 0, this change means that the maximum luminosity is one million times the original luminosity. It is evident that cataclysmic forces are at work, the origin of which is still uncertain.

EXAMPLES

1. The apparent visual magnitude of λ *Scorpii* is 1·71 and of η *Scorpii* is 3·44. How much brighter does λ appear than η?

2. A star A is of apparent magnitude 5·2 and a star B appears to be 7·4 times brighter than A. What is the apparent magnitude of B?

3. The apparent magnitude of *Sirius* is $-1{\cdot}58$. How much brighter does *Sirius* appear than *Pollux*, whose apparent magnitude is 1·21?

4. The components A and B of the double star ζ *Ursae Majoris* are of apparent magnitudes 2·40 and 3·96. What is the apparent magnitude of the double star seen as a single star?

5. The apparent magnitude of a double star seen as a single star is 3·54 and the apparent magnitude of one component is 4·82. What is the apparent magnitude of the other component?

6. Find the absolute magnitudes of the stars whose parallaxes P and apparent magnitudes m are given:

	P	m
(i)	$0''{\cdot}060$;	3·54.
(ii)	$0''{\cdot}377$;	$-1{\cdot}58$.
(iii)	$0''{\cdot}003$;	7·32.

7. Find the ratio of the luminosities of the following pairs of stars (A and B) whose parallaxes and apparent magnitudes are given:

(i) A: $0''{\cdot}042$, $3^m{\cdot}56$; B: $0''{\cdot}005$, $7^m{\cdot}48$.
(ii) A: $0''{\cdot}291$, $0^m{\cdot}48$; B: $0''{\cdot}377$, $-1^m{\cdot}58$.

8. Taking the sun's luminosity to be unity and its absolute magnitude to be 4·85, find the luminosities of the stars whose parallaxes and apparent magnitudes are given :

(i) 0″·050, 4^m·52. (iii) 0″·012, 5^m·95.
(ii) 0″·101, 3^m·80. (iv) 0″·175, 10^m·48.

9. The stars whose absolute magnitudes are given have the same effective temperature as the sun (radius 432,000 miles, absolute magnitude 4·85) ; calculate the radii of the stars.

(i) 3·56. (ii) 5·72. (iii) − 0·52.

10. Calculate the radii of the following stars whose effective temperatures (T) and absolute magnitudes (M) are given (take T and M for the sun to be 6000° and 4·85).

	T	M
(i)	4800° ;	3·85.
(ii)	6600° ;	1·23.
(iii)	5400° ;	− 1·56.

11. The angular semi-major axes of the true orbit (α), the orbital period (T) and the parallax (P) of the following binaries are given below ; find the mass of the system (take sun's mass as unity).

	α	T (years)	P
(i)	1″·83 ;	42·2 ;	0″·169.
(ii)	4″·50 ;	87·7 ;	0″·192.
(iii)	4″·83 ;	152·8 ;	0″·168.
(iv)	0″·054 ;	0·285 ;	0″·063.

12. Calculate the dynamical parallaxes of the following stars for which α and T (see example 11) are given :

(i) 1″·62, 153·0 years ; (ii) 3″·56, 230 years.

13. If λ denotes the wave-length of a line in the comparison spectrum and λ' the measured wave-length in a stellar spectrum, calculate the radial velocities of the stars from the following data :

	λ	λ'
(i)	4861·327 ;	4861·102.
(ii)	5167·497 ;	5168·254.
(iii)	6027·059 ;	6026·372.

14. Use Fig. 102 to find the distance in parsecs of the Cepheid variables whose average apparent magnitudes and periods are as follows :

(i) 11^m·5, 10 days ; (ii) 14^m·5, 0·5 day ; (iii) 6^m·8, 41 days.

CHAPTER XIII

STELLAR MOTIONS

181. *Proper motion.*

Until a little over two centuries ago it was believed—for lack of evidence to the contrary—that the stars were fixed in space ; in favour of this attitude was the fact that the visible stars of the various constellations maintained, or appeared to maintain, their well-known configurations from year to year and from century to century. In 1718, however, Halley announced that the positions of the three bright stars Sirius, Aldebaran and Arcturus had changed unmistakably, with reference to their neighbours, since the time of Hipparchus about nineteen centuries before. This may be illustrated simply by means of Fig. 103, in which we have represented a number of faint stars in a small area of the sky (say, in the field of view of a telescope) together with a star A ; Halley's discovery is equivalent to noticing that, when an observation is made some years later, the star A now occupies a different position B with respect to the apparently unchanged configuration of the faint stars. Consequently, it must be concluded that Halley's three stars—and star A in Fig. 103—must be moving in space relative to the sun. It is to be noted that the actual observations of any given star, made from the earth, include (amongst others) the annual effects of aberration and parallax ;

Fig. 103.

when these are removed, we obtain the heliocentric direction of the star concerned, and it is the changes * in such heliocentric directions with which we are concerned here. The annual angular change in heliocentric direction on the celestial sphere due to the star's spatial motion is called *proper motion*. Proper motion is a feature of the stars in general, although, for the faintest and most remote stars, it is practically negligible.

In Fig. 104 let S be the sun and X a near star which has a velocity V, relative to the sun, in the direction XY which makes an angle β with the line of sight SXQ. Let the magnitude of V be represented by the side XY of the triangle XQY in which YQ is drawn at right angles to the line of sight. This triangle is the triangle of velocities by means of which we resolve V into two components: (i) v, along the line of sight and expressed in magnitude by XQ, and (ii) u, perpendicular to the line of sight and expressed in magnitude by QY. The component v simply alters the heliocentric distance of the star without altering its direction as viewed against the fixed background of the extremely remote, faint stars. The component, u, however, carries the star at right angles to the line of sight and changes the direction of X as viewed against the background of the faint stars. We have, from the triangle of velocities shown in Fig. 104,

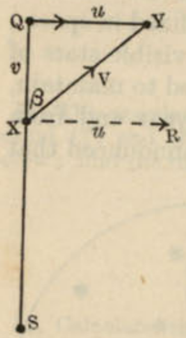

FIG. 104.

$$u = V \sin \beta, \quad v = V \cos \beta. \qquad \text{.....................(1)}$$

The component, v, in the line of sight is the *radial velocity* of the star, defined in section 172. If v is such that X recedes from S (as in the figure) the radial velocity is defined to be positive; if the radial velocity is one of approach, it is negative. Radial velocities are almost invariably expressed in kilometres per second.

* We eliminate the effects of precession and nutation by referring the positions of the stars, corresponding to observations made at various times, to a definite mean equator and equinox, say, for 1900·0.

182. *Components of proper motion.*

Let X in Fig. 105 be the position of a star on the celestial sphere at a given date and let Y be its position a year later. With reference to the very faint stars the change in the star's direction is given by the great circle arc XY. We denote XY by μ; then μ is called the *total annual proper motion*. Let XY make an angle θ with the northward direction of the meridian through X; θ is called the *position angle* of the proper motion and is measured from 0° to 360° in the sense indicated by the arrow.

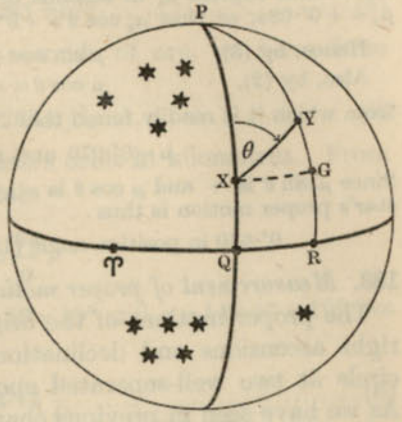

FIG. 105.

Let PXQ and PYR be the meridians through X and Y, and XG the parallel of declination through X. The total proper motion, μ, results in an annual change of right ascension, which we denote by μ_α, and an annual change in declination denoted by μ_δ. Thus

$$QR = \mu_\alpha \quad \text{and} \quad GY = \mu_\delta.$$

But, if δ is the star's declination, by section 6 we have $XG = QR \cos \delta$; hence

$$XG = \mu_\alpha \cos \delta. \qquad \text{............................(2)}$$

Now, at the most, the total proper motion μ is a small quantity (the largest proper motion known is about 10″ per annum), so that XYG may be regarded as a small plane triangle in which $Y\widehat{G}X$ is 90° and $Y\widehat{X}G$ is 90° − θ. Hence

$$XG = \mu \sin \theta, \quad GY = \mu \cos \theta;$$

and finally, using (2),

$$\mu_\alpha \cos \delta = \mu \sin \theta, \quad \mu_\delta = \mu \cos \theta. \qquad \text{................(3)}$$

It is to be remembered that μ_α is generally expressed in seconds of time per annum and μ_δ in seconds of arc per annum.

Ex. 1. The components of proper motion of a star of declination + 60° are :

$$\mu_\alpha = +0^s{\cdot}0056, \quad \mu_\delta = -0''{\cdot}056.$$

To find the values of μ and θ.

We first express μ_α in seconds of arc. Since $1^s = 15''$, we find that $\mu_\alpha = +0''{\cdot}084$, so that $\mu_\alpha \cos\delta = +0''{\cdot}084 \cos 60° = +0{\cdot}042$.

Hence, by (3), $\qquad \mu \sin\theta = +0''{\cdot}042.$

Also, by (2), $\qquad \mu \cos\theta = -0''{\cdot}056,$

from which it is readily found that

$$\mu = 0''{\cdot}070 \text{ and } \tan\theta = -0{\cdot}75.$$

Since $\mu \sin\theta$ is + and $\mu\cos\theta$ is −, θ is in the second quadrant. The star's proper motion is thus

$$0''{\cdot}070 \text{ in position angle } (180° - 36°\ 52') \text{ or } 143°\ 8'.$$

183. *Measurement of proper motion.*

The proper motions of the bright stars are derived from the right ascensions and declinations measured by the meridian circle at two well-separated epochs (say, in 1850 and 1930). As we have seen in previous chapters the observed coordinates of a star are affected by parallax, aberration, precession and nutation *; the changes due to these are all calculable, and the residual changes in the coordinates, divided by the interval between the observations (say, 80 years) give the values of μ_α and μ_δ.

The components of proper motion of the fainter stars are found by photographic methods.

184. *The transverse speed of a star.*

When the parallax of a star is known, we can find its linear speed, u, at right angles to the line of sight, expressed as so many kilometres per second ; this is called the star's *transverse linear speed.* If n is the number of seconds in a year the star will be displaced, in a year, through a distance of nu kilometres (from X to Z in Fig. 106, where ZX is perpendicular to SX) ; as μ is the corresponding angular displacement in one year, we have

$$\tan\mu = nu/d,$$

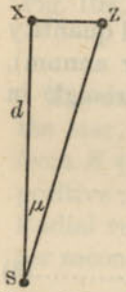

Fig. 106.

in which d is the star's distance from the sun in

* See the footnote on p. 224.

kilometres. In this formula μ is a small angle, and we can write $\tan\mu = \mu$ where μ is in circular measure, or, expressing μ in seconds of arc, we have $\tan\mu = \mu/206265$. The formula then becomes

$$\mu = 206265\ nu/d. \quad \text{.........................(4)}$$

If P is the star's parallax in seconds of arc, we have from formula (30), p. 128,

$$P = 206265\ a/d, \quad \text{.........................(5)}$$

where a is the radius of the earth's orbit in kilometres. From (4) and (5), by division, we obtain

$$\frac{\mu}{P} = \frac{nu}{a},$$

from which, on putting $n = 31{\cdot}56 \times 10^6$ and $a = 149{\cdot}67 \times 10^6$, we obtain

$$u = 4{\cdot}74\,\frac{\mu}{P}. \quad \text{.........................(6)}$$

Ex. 2. The parallax and components of proper motion of Arcturus ($\delta = +19°\ 30'$) are as follows :

$$P = 0''{\cdot}085, \quad \mu_\alpha = -0^s{\cdot}0786, \quad \mu_\delta = -1''{\cdot}996.$$

To find the transverse linear speed.

$$\mu_\alpha \cos\delta = -15'' \times 0{\cdot}0786 \cos 19°\ 30' = -1''{\cdot}111.$$

From (3), $\qquad \tan\theta = \dfrac{-1{\cdot}111}{-1{\cdot}996}$, from which

$$\theta = 180° + 29°\ 6' = 209°\ 6'.$$

Also, from (3), $\qquad \mu = \mu_\delta \sec\theta = -1''{\cdot}996 \sec 209°\ 6' = 2''{\cdot}284.$

Then, from (6), $\qquad u = \dfrac{4{\cdot}74 \times 2{\cdot}284}{0{\cdot}085}$

$$= 127{\cdot}4 \text{ kilometres per second.}$$

185. *The total linear speed of a star relative to the sun.*

If u can be found as in the previous section and the radial velocity is known from spectroscopic observations, we can determine the star's linear speed V, and also β, from the formulae (1) of section 181.

Ex. 3. For *Arcturus* we have found $u = 127{\cdot}4$ kms. per sec. and v is $-5{\cdot}1$ kms. per sec., as given by spectroscopic observations. Thus, from (1),

$$V \sin\beta = 127{\cdot}4, \qquad V\cos\beta = -5{\cdot}1.$$

From these we obtain, by division,

$$\tan\beta = -\frac{127{\cdot}4}{5{\cdot}1},$$

from which

$$\beta = 92^\circ\ 18'.$$

Also $V \sin 92^\circ\ 18' = 127{\cdot}4$, and we find that

$$V = 127{\cdot}5 \text{ kms. per sec.}$$

The linear velocity of *Arcturus* is very much higher than the average velocity of the stars as a whole; the great majority of stellar motions do not exceed 30 kms. per sec.

186. *Moving clusters.*

In the sky about two hundred "open clusters" are known; these are groupings of stars occupying a limited volume of space and being generally concentrated within a comparatively small area of the sky. As such clusters appear to be permanent formations, it is presumed that all the stars of a given cluster are moving with the same velocity relative to the sun and in the same direction in space; hence the alternative name *moving cluster*. It is only in the case of a few of the nearest clusters, however, that the motions of the cluster stars have been measured; the best known of these are the *Pleiades*, the *Hyades* (alternatively known as the *Taurus Cluster*) and the cluster in *Ursa Major*. The *Pleiades* is a cluster * of about 150 stars, all situated within about a degree of *Alcyone* (the brightest member of the group) and with practically identical proper motions and position angles. The *Hyades*, on the other hand, consists of rather less than a hundred members scattered over a much larger area of the sky (the area is roughly bounded by meridians about 20° apart and parallels of declination also about 20° apart). In this case the total proper motions and the position angles of the cluster stars vary appreciably, and it is this circumstance that leads to information of considerable value.

187. *Convergent point of a moving cluster.*

We assume that all the cluster stars have a common velocity, V, parallel to some such direction as SC with respect to the sun S (Fig. 107). Let X be a cluster star. We resolve V into

* Six or seven of the Pleiades are easily visible to the naked eye on a clear night.

two components u and v in the plane defined by SX and the direction of V (or, of SC); u is the transverse linear component and v is the radial component. We have, as in section 181,

$$u = V\sin\beta, \quad v = V\cos\beta, \quad \text{.........(7)}$$

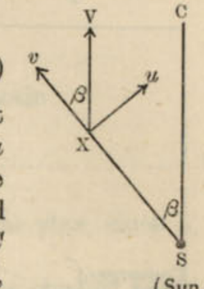

Fig. 107.

where β is the angle between the line of sight and the direction of V. The component u gives rise to a total proper motion μ, the direction of which lies in the plane defined by SX and SC. Hence, if in Fig. 108, C defines on the celestial sphere, centred at S, the direction of the common velocity V, the total proper motion of X will be directed along the great circle XC. Similarly, the total proper motion of a cluster star at Y will be directed along the great circle YC. Simple considerations show how the position of C can be found. Since the position angle of the proper motion of X is known (that is, the angle PXC), this information, together with the known position of X, defines the great circle XC; similarly, the position angle PYC at Y, together with the known position of Y, defines the great circle YC. The intersection of the two great circles thus gives us the position of C on the celestial sphere; C is called the *convergent point* of the moving cluster.

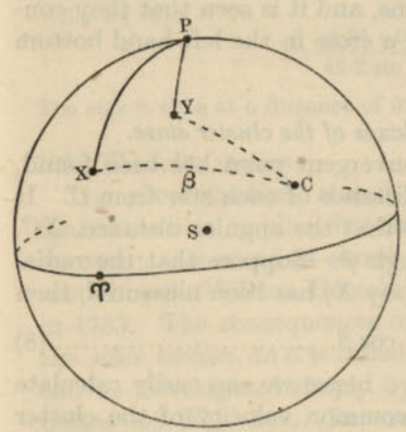

Fig. 108.

In practice the position angles of the proper motions of all the cluster stars are used to determine the position of C on the celestial sphere. If the stars are very close together in the sky, as in the Pleiades cluster, the unavoidable errors in the proper motions make the determination of the position of the convergent point wholly impracticable. The stars of the Taurus cluster, however, do not suffer from this disability, and so the convergent point is known with high accuracy. This is illustrated by Fig. 109; the arrows point in the directions given by the position angles

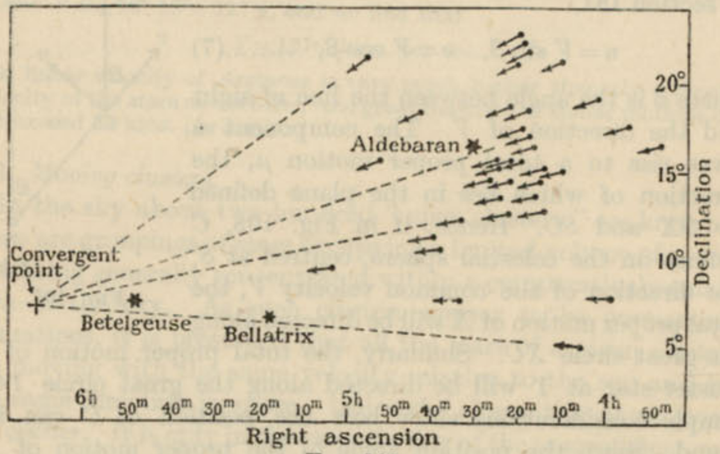

Fig. 109.

of the individual proper motions, and it is seen that they converge to the point marked with a cross in the left-hand bottom corner of the figure.

188. *Determination of the parallaxes of the cluster stars.*

When the position of the convergent point has been found, we can calculate the angular distance of each star from C. It is seen from Figs. 107 and 108 that the angular distance, XC, of the star X from C is the angle β. Suppose that the radial velocity, v, of one cluster star (say X) has been measured, then by (7),

$$v = V \cos \beta. \quad \text{.............................(8)}$$

In (8), v and β are now known; hence we can easily calculate the value of V, which is the common velocity of the cluster stars. In practice we utilise all the available radial velocity measures of the cluster stars to determine V with as great an accuracy as possible.

The transverse velocity, u, of any other star can now be found in kilometres per second, for it is given by

$$u = V \sin \beta,$$

in which u and β refer to the particular star concerned. The total proper motion, μ, corresponding to u, is known from the observations; hence, by (6), we have

$$u \equiv V \sin \beta = 4{\cdot}74 \frac{\mu}{P},$$

where P is the star's parallax. We thus obtain

$$P = \frac{4{\cdot}74\mu}{V \sin \beta}, \quad \text{..............................(9)}$$

from which we calculate the parallax of the star, since μ, V and β are now all known.

The parallaxes of the stars in the Taurus cluster, found in this way, have a very high degree of accuracy.

Ex. 4. The angular distance of a member of the Taurus cluster from the convergent point is $21°{\cdot}3$, its radial velocity is $39{\cdot}3$ kms. per sec. and its total proper motion is $0''{\cdot}101$. Find (i) the velocity of the cluster, and (ii) the parallax of the star.

(i) Here, $v = 39{\cdot}3$, $\beta = 21°{\cdot}3$. Hence, by (8),
$V = 39{\cdot}3 \sec 21°{\cdot}3 = 42{\cdot}2$ kms. per sec.

(ii) By (9),

$$P = \frac{4{\cdot}74 \times 0''{\cdot}101}{42{\cdot}2 \sin 21°{\cdot}3} = 0''{\cdot}031.$$

The star is thus at a distance of 32 parsecs or 104 light-years.

189. *The solar motion.*

The generalisation, stated earlier, that all the stars have spatial motions implies that the sun has an individual motion of its own; this was first corroborated by Sir William Herschel in 1783. The consequences of the *solar motion*, as it is called, can be investigated simply by means of Fig. 110. Let us suppose that it is possible to choose a system of rectangular axes (OB, OC, OD) fixed in the space occupied by the visible stars.

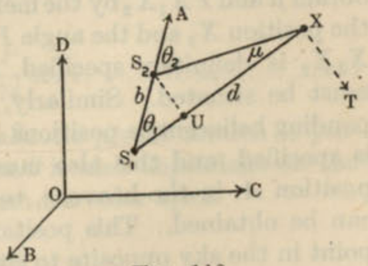

Fig. 110.

With respect to these axes, the sun will move in the course of a year from S_1 to S_2, say; we denote the distance S_1S_2 in kilometres by b. If the solar velocity is V kms. per sec. and n is the number of seconds in a year, we have

$$b = nV. \quad \text{......................(10)}$$

Let X be a star at a distance d (in kilometres) from S_1. We shall assume at first that X is stationary with respect to the axes. The effect of the solar motion is to change the heliocentric direction of the star from S_1X to S_2X in the course of a year. Now, the direction of the solar motion will be indicated by some such point as A (whose position we shall later find) on the celestial sphere of which the sun, S, is the centre (Fig. 111). If the heliocentric position of the star corresponding to the heliocentric direction S_1X in Fig. 110 is denoted by X_1, the heliocentric position of the star at the end of the year will be at X_2 lying on the great circle joining A and X_1 and such that $X_1X_2 = \theta_2 - \theta_1 = \mu$. Thus the solar motion gives rise to a total proper motion μ in position angle PX_1X_2. If the components of proper motion are obtained from observation we obtain μ and $P\widehat{X_1}X_2$ by the method of section 182. Hence, since the position X_1 and the angle PX_1X_2 are known, the great circle X_1X_2 is definitely specified, and on this circle the point A must be situated. Similarly, if another star has the corresponding heliocentric positions Y_1 and Y_2, the great circle Y_1Y_2 is specified, and this also must pass through A. Hence, the position A, in the heavens, towards which the sun is moving, can be obtained. This position is called the *solar apex*; the point in the sky opposite to the solar apex is the *solar antapex*. It will be noticed that the procedure in finding A is similar to that in finding the convergent point of a moving cluster for, in the former instance, each star has, relative to the sun, a

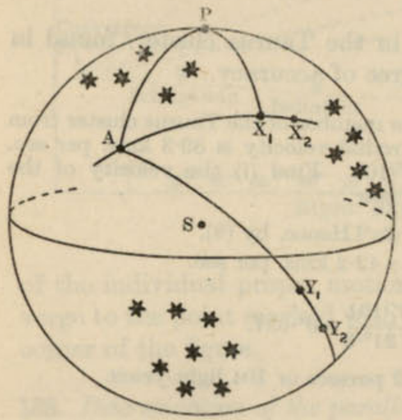

FIG. 111.

velocity equal in magnitude, but opposite in direction, to the solar motion.

The proper motion μ which arises from the solar motion *alone* is called the *parallactic proper motion*.

From Fig. 110 we have

$$\sin \mu = \frac{b}{d} \sin \theta_2. \quad \text{......................(11)}$$

Since μ is a very small angle this formula remains practically unaltered if we write θ_1 for θ_2 on the right-hand side; we shall write, for simplicity, θ for θ_1 or θ_2, so that (11) becomes

$$\sin \mu = \frac{b}{d} \sin \theta. \quad \text{......................(12)}$$

Also, if a denotes the earth's orbital radius in kilometres and P the star's parallax, we have

$$\sin P = \frac{a}{d};$$

hence, using (11), we obtain, μ and P being small angles and both measured in seconds of arc,

$$\frac{\mu}{P} = \frac{b}{a} \sin \theta, \quad \text{......................(13)}$$

or, by (10),

$$\frac{\mu}{P} = \frac{nV}{a} \sin \theta. \quad \text{......................(14)}$$

Putting $a = 149{\cdot}67 \times 10^6$ kms. and $n = 31{\cdot}56 \times 10^6$ seconds, we obtain

$$\mu = \frac{VP}{4{\cdot}74} \sin \theta. \quad \text{......................(15)}$$

This formula shows that the parallactic proper motion is proportional to the star's parallax and is also dependent on the angular distance, θ, of the star from the solar apex.

190. *The random motion of the stars.*

In the previous section we based our arguments for finding the solar apex on (*a*) the practicability of choosing a set of rectangular axes as indicated in Fig. 110 and (*b*) the zero-velocity of the star X (and others) with respect to these axes. As we know, the stars have their own individual velocities, so that the second assumption is no longer true. We modify (*b*) as follows. Instead of a single star at X, let us suppose there is a large number in the vicinity of X, each with its own individual linear velocity. These velocities will have components along XT in Fig. 110, or in the opposite direction, XT being perpendicular to the star's heliocentric direction and being also in the plane S_1S_2X. We replace (*b*) by the assumption that the number of stars with components of velocity along XT (which give rise to the parallactic proper motion) is equal to the number with components in the opposite direction, and that the average value of the components along XT is equal, numerically, to the average value of the components in the opposite direction. Such a distribution of velocities—if it holds for each of any three perpendicular directions—is called a *random distribution*. Further, we apply this hypothesis of random motions to every other region of space occupied by the stars with which we are dealing.

This general hypothesis is equivalent to saying that it is possible to choose rectangular axes, such as are shown in Fig. 110, in such a way that the components of velocity of all the stars with respect to each of these axes form a random distribution.

191. *Determination of the solar apex.*

If we consider N stars in the neighbourhood of X in Fig. 110, we have for each star the *observed* proper motion components μ_α and μ_δ. But μ_α consists of two parts: (i) the effect produced by the star's individual velocity, and (ii) the effect produced by the solar motion. If we take the algebraic mean of the N values of μ_α, the effects of the random velocities will tend to cancel out and, consequently, the average will be effectively the parallactic motion component in right ascension. This procedure is evidently equivalent to the assumption that all the stars are at rest with respect to the axes. A similar argument applies to the proper motion components μ_δ. As before, we have now sufficient information with which we can specify the great circle X_1X_2 (Fig. 111), and by considering a different region of the sky, say at Y_1, we are enabled to specify the great circle Y_1Y_2. The intersection of these two great circles gives the position, A, of the solar apex. In practice the method is applied to all the stars for which the essential data are known. The equatorial coordinates of the solar apex are found to be: R.A. 18^h, Dec. $+34°$, a position which is close to the bright star Vega.

192. *The solar velocity.*

The magnitude of the solar motion is derived from radial velocity measures. Consider the N stars in the neighbourhood of X (Fig. 110). Due to the solar motion the distance of a star at X has decreased, in the course of a year, by $S_1X - S_2X$ or by S_1U where S_2U is perpendicular to S_1X (as μ is a very small angle, S_2X and UX are practically equal). Hence the radial velocity due to the solar motion is $-S_1U/n$ kms. per sec., n being the number of seconds in a year. Also,

$$S_1U = b\cos\theta = nV\cos\theta,$$

by (10). Hence the radial velocity is $-V\cos\theta$. This is called the *parallactic radial velocity*.

The *observed* radial velocity of a star at X will consist of two parts: (i) the radial velocity due to its individual velocity, and (ii) the parallactic radial velocity. As before, we assume that the individual velocity components along the line of sight are random in character, so that if we take the algebraic mean of the observed radial velocities of the N stars at X, we obtain simply the parallactic radial velocity. If this mean is found to be R kms. per sec., we then have

$$R = -V\cos\theta.$$

As θ can be readily found, we obtain the solar velocity V.

Actually, the usual method of deriving V from radial velocities of stars all over the sky determines the position of the solar apex as well; this position is generally found to be within one or two degrees of the position found from the proper motions; the results of the two methods are thus in satisfactory agreement as regards the solar apex.

The value of V is found to be 19·5 kms. per sec. Thus, the complete specification of the solar motion with respect to the stars for which proper motions and radial velocities are available may be taken to be 19·5 kms. per sec. towards the apex (18^h, +34°).

193. *Secular parallaxes.*

The secular parallax of a star is defined as the angle subtended at the star by a base-line equal in length to the distance through which the sun moves in a year. If ω denotes the secular parallax,

$$\sin \omega = \frac{b}{d}. \quad \text{.............................(16)}$$

Also, the parallactic proper motion μ is given by (12), viz.,

$$\sin \mu = \frac{b}{d} \sin \theta,$$

so that, by dividing this equation by (16) we obtain, remembering that μ and ω are small angles,

$$\mu = \omega \sin \theta. \quad \text{............................(17)}$$

The relation between the annual parallax P and the secular parallax ω is easily found from (15) and (17); these give

$$\frac{VP}{4{\cdot}74} = \omega.$$

Inserting $V = 19{\cdot}5$ kms. per sec. in this last formula, we obtain

$$P = 0{\cdot}243\omega. \quad \text{...........................(18)}$$

The formula $\mu = \omega \sin \theta$, (17), is used extensively for determining the secular parallaxes of faint stars within small magnitude ranges, for example, from magnitude 10·5 to 11·0. Now, it is comparatively easy to determine the observed proper motions of these stars and, if we consider a group in a small region of the sky, the mean of the observed proper motions will be substantially the parallactic proper motion μ. Thus, θ being known, we easily determine ω from (17), and then P from (18). In this procedure we are making the hypothesis that all the stars of magnitude 10·5 to 11·0—if this is the magnitude range—are virtually at the same distance from the sun or, in other words, that they are all of very nearly the same

luminosity. As we have seen, the stars vary very greatly in luminosity, and so secular parallaxes are statistical in character, indicating only a kind of average value pertaining to the stars of the magnitude class dealt with. Secular parallaxes of the faint stars are used in a variety of practical and theoretical problems.

EXAMPLES

(In these examples, P denotes the parallax of a star.)

1. Find the total proper motion, the corresponding position angle and the transverse linear velocity (in kilometres per second) from the following data relating to certain stars:

	δ	μ_α	μ_δ	P
(i)	+38° 10′	$+0^s{\cdot}0122$	$+0''{\cdot}037$	$0''{\cdot}037$
(ii)	+45° 56′	$+0^s{\cdot}0076$	$-0''{\cdot}421$	$0''{\cdot}071$
(iii)	− 9° 58′	$-0^s{\cdot}0064$	$+0''{\cdot}747$	$0''{\cdot}112$
(iv)	−16° 38′	$-0^s{\cdot}0375$	$-1''{\cdot}208$	$0''{\cdot}377$

2. The linear transverse velocity (u), the radial velocity (R) and the parallax for certain stars are given below. Find the star's speed relative to the sun and its minimum distance from the sun.

	u (kms./sec.)	R (kms./sec.)	P
(i)	23·5	+14·6	$0''{\cdot}050$
(ii)	48·5	− 9·4	$0''{\cdot}123$
(iii)	19·8	−27·5	$0''{\cdot}028$
(iv)	28·7	+49·8	$0''{\cdot}010$

3. The angular distance of a member of the *Ursa Major* cluster from the convergent point is 140°·2, its radial velocity is −14·6 kilometres per second, and its total proper motion is $0''{\cdot}106$. Calculate the velocity of the cluster and the parallax of the star.

4. Assuming that the velocity of the *Taurus* cluster is 42·5 kilometres per second, find the parallax of a star at an angular distance of 34° 34′ from the convergent point and with a total proper motion of $0''{\cdot}151$. What would you expect its radial velocity to be?

5. A star is at an angular distance of 30° from the solar apex and its random velocity is zero. If the solar speed is 19·5 kilometres per second, find the star's total proper motion and radial velocity, its parallax being $0''{\cdot}075$.

CHAPTER XIV

CLUSTERS AND NEBULAE

194. *The galactic system.*

The vast assemblage of stars visible in the telescope form what is known as the galactic system. This stellar system is not unique, for it has now been established that the thousands and millions of *spiral nebulae* are independent stellar systems comparable in magnitude and stellar population with the galactic system, of which the sun, our nearest star, is but a very ordinary member. Although the dimensions of any given stellar system are large, the distance between any two neighbouring systems is, in general, very much larger.

A noticeable feature of the galactic system is the Milky Way—to the naked eye, a band of misty light stretching right round the sky. In the telescope the Milky Way is resolved into myriads of faint stars, and counts of stars show unmistakably that the numbers of stars diminish with increasing angular distance from the Milky Way. From such and other considerations it is deduced that the galactic system is spheroidal in form, the plane of the Milky Way representing the directions in which the stars extend to the greatest distances. In Fig. 112 the plane of the Milky Way defines the *galactic equator*, the centre of the system being at C; the poles are called the *north galactic* and *south galactic poles* respectively, the former being on the same side of the galactic equator as the north pole of the earth's equator. The angular distance of a star from the galactic equator is called the *galactic latitude*.

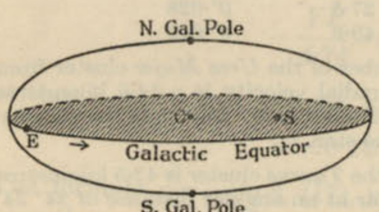

Fig. 112.

If we imagine a sphere drawn with C as centre, the directions of the north and south galactic poles and of the star define a semi-great circle on the sphere, called a *galactic meridian*. If E defines a reference direction in the galactic equator, the angle between the meridians through the star and through E is called the *galactic longitude* (which is measured in the direction of the arrow in Fig. 112 from 0° to 360°). The direction of E is chosen as that point of intersection of the celestial and galactic equators on the celestial sphere where declinations change from negative to positive values as the galactic longitude increases; in other words, E is the ascending node of the galactic equator on the celestial equator.

It is found that the north galactic pole has the equatorial coordinates: R.A. 190°, Dec. +28°. When the equatorial coordinates of a star are known, its galactic coordinates can be readily calculated. From the fact that the Milky Way defines effectively a great circle in the sky, we deduce that the sun must be situated in, or very nearly in, the galactic equator. Also, the sun is situated at a considerable distance from the centre C of the galactic system (in Fig. 112 the sun is situated at S). This fact can be deduced qualitatively from the following considerations. Suppose that S is at some distance from C. Then if we look at the sky in the direction SC, the boundary of the galactic system will be at a much greater distance in this direction than in the opposite direction; accordingly, if we assume that the distribution of stars throughout the spheroid is roughly symmetrical, we shall expect to see a much greater number of stars in the direction SC than in the opposite direction. Further, the number of stars visible in the field of a telescope ought actually to be a maximum in the direction SC, for the distance of the boundary from S is greatest in this direction. These expectations are realised, for it is found that the stars are most numerous at galactic longitude 325° (corresponding to a direction in the constellation of *Sagittarius*). We thus infer that S must be at some considerable distance from the centre of the galactic system and that the direction of the galactic centre C is in galactic longitude 325°. Further, it has been found in various ways that the sun is about 10,000 parsecs from the galactic centre, that the equatorial diameter of the system is about 30,000 parsecs, and that its polar diameter is about 5000 parsecs.

195. *Open clusters.*

These are aggregations of stars, numbering usually a few score, forming compact systems of which the best examples are the Hyades or Taurus Cluster (discussed in section 186) and the Pleiades. It is found that the Hyades cluster consists of about 70 stars, half of which lie within three parsecs of the centre of the cluster, the remainder extending up to a distance of ten parsecs. The cluster is roughly spherical in form, and the distance of its centre from the sun is 35 parsecs, a result of considerable accuracy. Certain clusters, such as the Pleiades, which occupy a small area of the sky, are inferred to be cluster formations from the fact that the proper motions of the individual stars are substantially of the same magnitude and in the same position angle; these are the so-called moving clusters. As regards the distant open clusters, the proper motions of the individual stars are so small that the criterion of common proper motion becomes unreliable, and the cluster-property is inferred mainly from the close grouping of the stars in the sky.

Altogether about 200 open clusters are known, and they are all situated close to the galactic equator.

196. *The distances of open clusters.*

Except in the case of a very few clusters the distances of these systems are obtained by indirect methods. We suppose that the spectral types of a number of cluster stars have been determined. Associated with each spectral class, we have a known absolute magnitude M; as the apparent magnitude m is easily measured, we obtain the parallax of a particular star from the formula (4) of section 157, namely,

$$M = m + 5 + 5 \log P. \qquad \text{.......................(1)}$$

Let r be the distance of the star in parsecs. Then as P is supposed expressed in seconds of arc, $r = 1/P$, and we obtain

$$M = m + 5 - 5 \log r,$$

or

$$5 \log r = m - M + 5. \qquad \text{...........................(2)}$$

In applying (2) we obtain the value of $m - M$ for the individual stars from the necessary observational measures, and as the cluster-stars may be regarded as substantially at the same distance from us, the mean value of $m - M$ for all the stars concerned is used to determine the distance of the cluster.

For example, 30 stars of the open cluster known as Messier 36 (or M36)—the number 36 refers to the number of the object in Messier's catalogue—of spectral types between $B4$ and $A2$ gave the mean value of $m - M$ to be 11·12. Hence r is given by

$$5 \log r = 11{\cdot}12 + 5, \text{ or } \log r = 3{\cdot}224,$$

so that

$$r = 1675 \text{ parsecs.}$$

In the derivation of (1) it was assumed in section 157 that the apparent brightness of a star varies inversely as the square of its distance from us, and this involves the implicit assumption that interstellar space is empty. It has been found within recent years that the latter assumption can no longer be maintained, for the existence of an interstellar cloud of very diffuse matter has been definitely established. This cloud absorbs a proportion of the light from a star, the absorption being in general larger for the more distant stars. Thus, a star of given absolute magnitude at a given distance will appear rather fainter than it would be if absorption were entirely absent; in other words, if we could remove the effects of absorption our values of the apparent magnitude m to be used in (2) should be smaller than the observed values, with the consequence that the distances of the clusters must be reduced.

If we consider magnitudes on the photographic scale it has been found that the values of m are increased on the average by about $\frac{3}{4}$ per 1000 parsecs as the result of absorption. In the case of the open cluster M36 considered earlier, the distance of 1675 parsecs has been obtained from the *observed* apparent photographic magnitude m, that is, on the assumption that space is empty. Taking into consideration the average effect of absorption we can calculate that the actual distance of the cluster is reduced to 1170 parsecs.

In this way the distances of the open clusters have been obtained; the most remote cluster is at a distance of about 5000 parsecs from the sun. As the angular diameters of the clusters are readily obtained, the linear diameters can then be easily calculated; these vary from about two to twenty parsecs.

Further investigation shows that the absorbing cloud is mainly concentrated towards the galactic equatorial plane (in or near which the open clusters lie); various estimates of its effective thickness have been made—these are of the order of two or three hundred parsecs.

197. *Globular clusters.*

These are very dense aggregations of stars, of which about a hundred are known. They are roughly spherical in form and in many the number of stars amounts to a figure of the order of 100,000. The distribution of the clusters in the sky is noteworthy. First, none are found in or near the Milky Way; as the distances of these clusters are now known to be very great, the apparent absence of clusters from the Milky Way regions is interpreted as due to the effects of the absorption of the galactic layer discussed in the previous section. Second, the great majority of the clusters are found in one hemisphere of the sky only, with roughly equal numbers on each side of the Milky Way. We deal with this point later.

The distances of many of the clusters have been inferred as the result of the discovery of Cepheid variables and the application of the period-luminosity relation discussed in section 178. These distances vary from 5000 to 50,000 parsecs. It is then found that the globular clusters form a roughly spherical system, the members of which are mainly outside the galactic stellar system. If it be assumed—as is most probable—that the system of clusters is symmetrically disposed with relation to the galactic system, the centres of each being the same, we can obtain the direction and distance of the galactic centre by means of the globular clusters. This was the method adopted by Dr. Shapley some years ago; the galactic centre was placed in galactic longitude 325°, as already stated, and its distance from the sun was first estimated at about 15,000 parsecs, subsequent investigations, however, showing that this distance should be reduced to about 10,000 parsecs. These results are confirmed by methods of a wholly different character.

As we have seen, the globular clusters are observed to be mainly in one hemisphere of the sky. This feature of their distribution is a result simply of the great distance of the sun from the galactic centre and from the centre of the system of clusters; effectively, the sun is not far from the outer boundary of this system, so that the great majority of the clusters will be situated on one side of a plane drawn through the sun and perpendicular to the galactic equator and to the direction of the galactic centre, only a very small number lying on the other side of this plane.

198. *Galactic rotation.*

The spheroidal form of the galactic system suggests that its stability is maintained by a process similar to that found in the planetary system. As we have seen, the planets revolve in orbits around the sun, which acts as the centre of attraction, and the stability of the planetary system as a unit depends on this orbital motion. If every planet were suddenly stopped in its orbit, the gravitational attraction of the sun would simply cause all the planets to fall into the sun. In the same way, the galactic system of stars would collapse unless the stars moved individually in orbits around the centre where undoubtedly there exists a dense concentration of galactic matter.

It is only in recent years that this phenomenon of galactic rotation, as it is called, has been verified observationally.

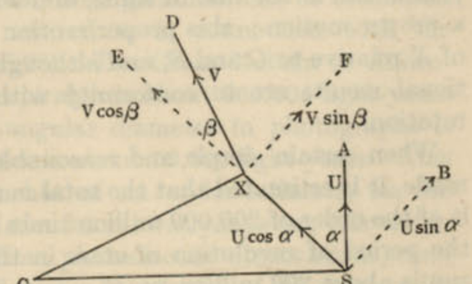

FIG. 113.

To simplify matters, let us suppose that the stars revolve in circular orbits about the centre, and let us consider only stars lying in the galactic equatorial plane. We assume that the sun, S, lies in this plane. In Fig. 113, C is the galactic centre and X a star, the plane CXS being the plane of the galactic equator. Let U denote the orbital velocity of the sun in the direction SA, which is perpendicular to SC; similarly, let V denote the orbital velocity of X in the direction XD which is perpendicular to XC. We resolve both U and V along the direction SX and at right angles to this direction. Then U is equivalent to $U \cos \alpha$ along SX and $U \sin \alpha$ along SB; similarly, V is equivalent to $V \cos \beta$ along XE (or SX) and $V \sin \beta$ along XF (which is parallel to SB). Hence, *relative to S*, the star X will have the components

(i) $V \cos \beta - U \cos \alpha$, along SX;

(ii) $V \sin \beta - U \sin \alpha$, along XF.

The first component means that X has a radial velocity relative to S, and this velocity, it can be shown, depends on the position of X in the galactic equatorial plane relative to the line CS and on the distance of X from S, being greater for the remote stars; for the latter reason the faint and distant O and B type stars are mainly observed spectroscopically for the measurement of the radial velocities used in this connection. It is found that the hypothesis of galactic rotation is verified; moreover, the direction of the galactic centre from the sun is found from the observations, and this direction is substantially in agreement with that found (galactic longitude 325°) from the study of the globular clusters.

The second component of velocity (that along XF) is perpendicular to the line of sight, and so the star at X will have a proper motion; this proper motion depends on the position of X relative to C and S, and although it is small, the observational results are in conformity with the theory of galactic rotation.

When certain simple and reasonable hypotheses are further made, it is estimated that the total mass of the galactic system is of the order of 200,000 million times the sun's mass, and that the period of revolution of stars in the neighbourhood of the sun is about 200 million years.

199. *The diffuse nebulae.*

These are of two kinds, luminous and dark, and they are found only in or near the Milky Way. A dark nebula is an extensive cloud of diffuse matter which is sufficiently absorbing as to blot out the light of the stars beyond it; its presence

is revealed by the almost complete absence of stars in a region of the otherwise densely populated tracts of the Milky Way. The luminous diffuse nebulae are beautiful objects—the Great Nebula in Orion is perhaps the best known. Physically they are of the same character as the dark nebulae, but they owe their luminosity to the action of neighbouring stars of high temperature, which stimulate the atoms of the cloud into luminescence. The characteristic green colour of the diffuse nebulae is due to the light emitted by oxygen and nitrogen atoms in the cloud under the stimulation of these hot stars.

200. *Planetary nebulae.*

About a hundred planetary nebulae are known. In the telescope these objects are seen as small, fairly regular and luminous discs with, in most instances, a central star of type O —the hottest of the stellar types. A planetary nebula is believed to represent a stage in the life-history of a nova at some later time after its outburst.

201. *Spiral nebulae.*

These objects, which have a definite spiral structure, are not found close to the Milky Way, presumably because of the absorbing galactic cloud. As Cepheids have been discovered in some of the nearer spirals, their distances can be estimated by the method already referred to in connection with the globular clusters (section 178). The Great Nebula in Andromeda—one of the nearest spirals—is about 900,000 light-years distant. As its greatest angular diameter in photographs is about 3°, its linear diameter is about 50,000 light-years. This is almost certainly somewhat of an under-estimate, as the photographs fail to reveal the fainter outlying stars (the existence of these is inferred from photoelectric observations), and it is very likely that its diameter is actually of the same order of magnitude as that of the galactic system. As parts of the cloudy regions of this nebula have been resolved into stars by the telescope—in the same way as the Milky Way clouds are resolved—it is certain that such objects are independent stellar systems comparable in size and constitution with the galactic system. As they are far outside the galactic system they are known also as *extra-galactic nebulae*, a designation that also includes all the various types of nebulae in which a spiral structure has not been recognised.

The distances of the more remote extra-galactic nebulae are estimated mainly from the measures of their angular diameters on the hypothesis that, as regards their linear dimensions, they are all much of the same size. Thus a spiral with an angular diameter of $0^{\circ}\!\cdot\!3$ would be nearly 10 million light-years away, taking the known features of the Andromeda nebula as basis.

It is found from spectroscopic observations that the extra-galactic nebulae are all receding from us with enormous velocities and that these velocities are greatest for the most remote objects, there being a definite relation of a simple nature between radial velocity and distance. For example, the radial velocity and distance of a nebula in *Ursa Major* are 9600 miles per second and 85 million light-years; the corresponding measures for a nebula in *Gemini* are 14,300 miles per second and 135 million light-years. The greatest velocity recorded is 24,400 miles per second for a nebula in *Boötes*; on the basis of the velocity-distance relation its distance must be about 250 million light-years.

The enormous recessional velocities of the extra-galactic nebulae, which form the main units of the universe, have suggested the theory that it is space that is expanding. Here we are brought into the most recondite regions of astronomical and mathematical theory, where the best minds and the most powerful observational equipment are at present engaged in attempting to penetrate the innermost secrets of the universe.

CHAPTER XV

TELESCOPES

202. *The refracting telescope.*

In this chapter we shall describe the main features of several astronomical instruments, beginning with the refracting telescope, or refractor. The chief parts (Fig. 114) are (*a*) the

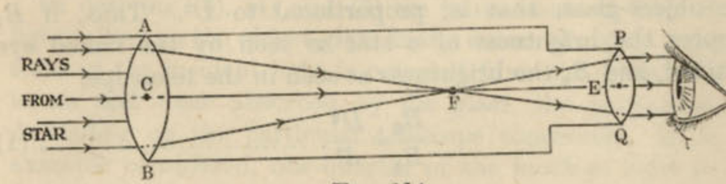

FIG. 114.

object-glass (or *objective*) AB and the *eye-piece* PQ; for simplicity these are represented in the figure by single lenses the surfaces of which are spherical (in the modern instruments the single lens is replaced in each case by a combination of lenses). Consider a parallel beam of rays from a star falling on the object-glass; we assume here that a ray of light is propagated in a straight line in any given medium. Each ray on passing through the lens AB suffers a deviation in its direction, according to the laws of refraction, in such a way that all the emergent rays converge to a point F called the *focus*. The distance CF is called the *focal length* of the object-glass, C being the centre of the lens; it depends on the radii of the spherical surfaces of the lens and on the optical properties of the glass used.

If the eye-piece lens PQ (centre E) is placed in such a position that EF measures the focal length of PQ, the rays diverging from F will, after passing through PQ, be transformed into a parallel beam which then enters the eye. The eye itself is an optical " instrument ", the lens of which focusses the image of the star on the retina. The star is then seen sharply.

203. *Limiting magnitude.*

One of the main properties of a telescope is its capacity to collect light, and this depends on the size of the object-glass. As Fig. 114 indicates, the large beam of light falling on the object-glass is concentrated into the small beam which enters the eye. Let D be the cross-sectional diameter of the beam entering the object-glass (D is called the *aperture*), and d the cross-sectional diameter of the beam entering the eye. If the star is viewed by the eye unaided, the quantity of light entering the eye at a given instant is proportional to the cross-sectional area of the beam, that is, proportional to d^2. With the telescope, the quantity of light now entering the eye is proportional to the cross-sectional area of the beam entering the object-glass, that is, proportional to D^2. Thus, if B_1 denotes the brightness of a star as seen by the naked eye unaided, and B_2 the brightness as seen in the telescope,

$$\frac{B_2}{B_1}=\frac{D^2}{d^2}. \quad \text{.................................(1)}$$

Suppose B_1 refers to a faint star of magnitude m_1; then, compared with a standard star, we can write—by formula (2) of section 154—

$$B_1=C\,.\,10^{-0\cdot 4m_1}.$$

Similarly, if m_2 is the magnitude, as compared with the standard star, corresponding to the brightness B_2, we have

$$B_2=C\,.\,10^{-0\cdot 4m_2}.$$

Hence, we obtain, using (1),

$$\frac{B_2}{B_1}=\frac{D^2}{d^2}=10^{-0\cdot 4(m_2-m_1)}, \quad \text{.......................(2)}$$

from which, by taking logarithms to base 10,

$$\log\left(\frac{D^2}{d^2}\right)=-\tfrac{2}{5}(m_2-m_1), \quad \text{.......................(3)}$$

whence

$$m_1=m_2+5\log\left(\frac{D}{d}\right). \quad \text{.......................(4)}$$

We interpret this result by saying that the faint star of magnitude m_1 is seen in the telescope as of the same brightness as a star of magnitude m_2 would be seen by the naked eye.

The limit m_1, of faintness, of stars visible in the telescope is given by (4), where m_2 is taken to be the limiting magnitude seen by the naked eye.

If we take the diameter of the pupil of the eye to be one-third of an inch and apply (4) to the Yerkes telescope, which has an aperture D of 40 inches, we obtain

$$m_1=m_2+10\cdot 4. \quad \text{............................(5)}$$

Taking magnitude 6·0 to be the faintest magnitude visible to the naked eye, the magnitude of the faintest star visible in the Yerkes telescope is 16·4, by (5).

In deriving (3), (4) and (5) we have assumed that *all* the light falling on the object-glass finally enters the eye. Actually, some of the incident light is reflected from the surfaces of the lenses and some absorbed by the glass, the proportion lost depending on the particular telescope concerned. If, in the example considered, one-quarter of the incident light is lost, formula (3) becomes

$$\log\left(\frac{3}{4}\,\frac{D^2}{d^2}\right)=-\frac{2}{5}(m_2-m_1),$$

so that, for the Yerkes telescope,

$$m_1=m_2+10\cdot 1$$

and the limiting magnitude is 16·1.

204. *Magnifying power.*

The second property of a telescope is its capacity to "separate" two close objects in the sky, such as the components of a close double star which, to the naked eye, would be seen as a single star. We illustrate this property first by reference to the *photographic telescope* (Fig. 115), the object-glass of which, for simplicity, we suppose to be a single lens. A photographic plate is placed in the focal plane of the object-glass, that is, the plane through the focus F of the object-glass and perpendicular to the optical axis CF of the lens AB (there is, of course, no eye-piece in this form of telescope when the plate is in position).

The image of a star X on the axis CF is formed at F; if CY is the direction of a second star its image will be formed at G on the plate in the direction YC produced. If F denotes the focal length of the object-glass and α the angle of separation, $X\widehat{C}Y$, of the two stars, the distance FG (denoted by b) between

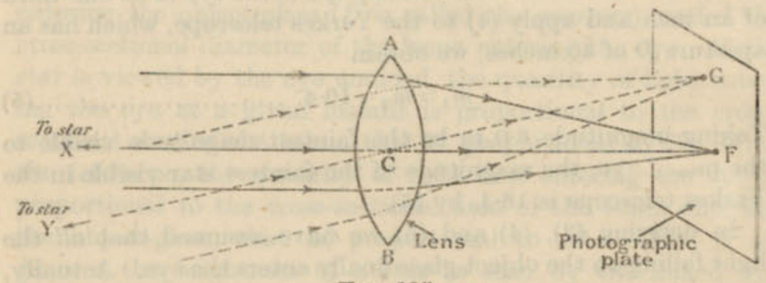

FIG. 115.

the images on the plate is given, evidently, by $b = F \tan \alpha$. Expressing the small angle α in seconds of arc, we have with sufficient accuracy

$$b = \frac{F\alpha}{206265}. \quad \text{.........................(6)}$$

For example, if $F = 40$ feet and $\alpha = 10''$, the value of b from (6) is 0·024 inch.

In this case, the *scale* of the photographic plate is described by saying that one second of arc is represented by 0·0024 inch on the plate.

From (6) it follows that the larger the focal length, the larger is the linear separation, on the plate, of the images of two given stars; consequently, the larger the focal length, the more accurately can the angular separation α be measured.

Suppose now that the plate is removed and the eye-piece restored. If E (not shown in Fig. 115) is the centre of the eye-piece, the angular separation, β, of the two stars now seen by the eye is the angle subtended at E by the distance FG. If f is the focal length, EF, of the eye-piece,

$$\tan \beta = \frac{FG}{f} = \frac{b}{f}. \quad \text{.........................(7)}$$

But $\tan \alpha = b/F$; hence, regarding β and α as small angles and using (7), we obtain

$$\frac{\beta}{\alpha} = \frac{F}{f}. \quad \text{..............................(8)}$$

The ratio β/α or F/f is called the *magnifying power* or the *magnification.*

Thus, if $F = 40$ feet and $f = 1$ inch, the magnifying power is 480; consequently, a double star, for which the angular separation, α, of the two components is $10''$, will be seen with an angular separation, β, equal to $80'$.

205. *Aberrations and resolving power.*

The principal defects of a convergent lens such as is shown in Fig. 114 are known as *spherical aberration* and *chromatic aberration.* We have supposed in section 202 that all the rays of a parallel beam pass through a single point F, the focus. Actually, the rays passing through the central portions of the object-glass have a focus slightly further away than the rays passing through the outer portions. This is known as *spherical aberration.*

Again, the deviation of the incident rays as they pass through the object-glass is governed by the refractive properties of the glass, and the index of refraction for any given glass depends on the colour of the light. For example, the focus for red light is further from the object-glass than the focus for blue light. This is known as *chromatic aberration.*

The result of these aberrations is that the star is not seen with perfect sharpness. The art of the optician is devoted to the minimising, as far as possible, of these defects by designing compound object-glasses, the components of which are made of different kinds of glass, with specially calculated radii of curvature of the surfaces.

In section 202 we assumed that a ray of light is propagated in a straight line. Actually, light is a wave-motion, and the result is that the image of a point of light (a star) is not a geometrical point but a disc. Two close stars will then be separated in the eye-piece only if the discs are completely separated. The *resolving power* of a telescope is its capacity to separate effectively two close objects in this way. The resolving power depends on the aperture, D, of the object-glass and on the effective wave-length of the light of the star. Taking yellow light, it is found that if a double star, of angular separation α, is just resolved,

$$\alpha = \frac{5''\cdot 4}{D}. \quad\text{..............................(9)}$$

For example, with a visual telescope of aperture 40 inches, a double star with a separation of $\frac{1}{7}$ of a second of arc will just be resolved; this is regarded as the resolving capacity of the telescope. If the angle of separation is $0''\cdot 5$, the minimum aperture required is about 11 inches.

206. *The reflecting telescope.*

The reflecting telescope, or reflector, depends for its light-gathering power on a concave mirror MN (Fig. 116) which brings a beam of rays, parallel to PQ, from a star to a focus at F.

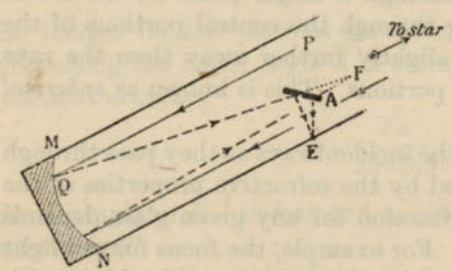

Fig. 116.

In the *Newtonian* form a small plane mirror is placed at A so that the beam of rays is brought to a focus at E, at the side of the tube where the star can be conveniently seen with the help of an eye-piece, or photographed on a plate placed at E.

In the *Cassegrain* form the mirror at A is replaced by a small convex mirror, whose axis is parallel to that of MN, so that the converging beam of rays is reflected back towards the principal mirror MN. A small hole in MN enables the beam to pass through to a focus on the under-side of MN where the star can be viewed or photographed in the usual way.

207. *Advantages of refractors and reflectors.*

The refractor with the largest aperture is that at Yerkes Observatory, Williams Bay, Wis., U.S.A., the aperture being 40 inches. The practical difficulties of constructing object-glasses greater than this seem to be almost insuperable. On the other hand, there are many reflectors in use with mirrors greatly exceeding in diameter the Yerkes instrument. The largest in actual use in 1941 is the 100-inch reflector of the Mount Wilson Observatory in California, but it is expected that within a short time the giant 200-inch reflector at Mount Palomar, also in California, will be ready for use. So far as the practicable size of aperture, or of light-gathering power, is concerned, the reflector is distinctly superior to the refractor. The mirror of a reflector, however, has to be frequently silvered —otherwise, it loses its reflecting power—or aluminised, at rarer intervals, as is the case with many instruments to-day; the object-glass of a refractor, on the other hand, requires comparatively little attention.

As we have seen, the refractor suffers from the defects of spherical and chromatic aberration to which the reflector is not liable. For work of precision over the field represented by the photographic plate used, the refractor has definite advantages over the reflector, and, for this reason, the measurement of stellar parallaxes, for example, is undertaken mostly by the refractor.

208. *The equatorial mounting.*

Fig. 117 shows the principal schematic details of one of the commonest forms of the equatorial mounting of a telescope (in this instance, a refractor). The instrument is supported by a massive metal pillar L, which is firmly secured to a stone pier extending several feet below ground-level. The *polar axis* of the telescope is indicated by PQ, and this is adjusted for the latitude of the observatory so as to be parallel to the earth's rotational axis. AB is the *declination axis*, perpendicular to PQ; at one end the telescope is fixed and at the other a heavy counterpoise C is adjusted to ensure the balance of the telescope. The telescope is capable of rotation about the axis AB; the declination of the heavenly body to be observed is set by means of the graduated circle D; when the telescope-axis EO is parallel to QP, the reading on D should be the declination of the pole, that is, 90°. A clamp fixes the telescope to the axis AB when the correct declination setting has been made The telescope, together with the metal framework surrounding AB, can then rotate as a single unit about the polar axis PQ. The graduated circle H is the hour-angle circle, which should read 0^h when the telescope is pointed to an object on the observer's meridian. A subsidiary mechanism—known as the "clock"

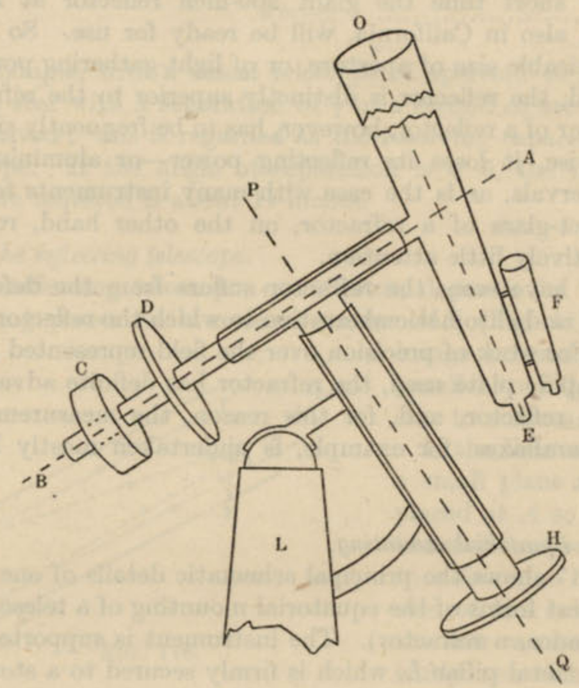

FIG. 117.

(not shown in Fig. 117)—drives the telescope about the axis PQ at the same angular rate as that of the diurnal motion; consequently, if a heavenly body is in the field of view when the telescope is connected up to the clock, it will remain in view so long as the clock is operating normally. A small telescope F—called the "finder"—with its axis parallel to OE is used for a rough setting. Slow-motion screws, which alter the hour-angle and declination readings by small amounts, enable the object to be set in the middle of the field of view of F; it ought then to be in or near the centre of the field of view of the main telescope.

In the photographic telescope a plate-holder is mounted in the focal plane of the object-glass (there is, of course, no eyepiece as shown at E in Fig. 117); there is provision for the correct adjustment of the distance of the plate-holder from O.

Such are, in broad outline, the principal features of the equatorial mounting.

209. *The alt-azimuth.*

The simplest instrument of this type is the *theodolite* used by surveyors. The main part of the instrument can rotate about a vertical axis, with the telescope itself capable of rotation about a horizontal axis. Suitable graduated circles enable the altitude and the azimuth of a heavenly body to be measured.

210. *The meridian circle.*

The meridian circle, or transit circle, consists of a telescope capable of rotation about a fixed horizontal axis, lying east and west. In the focal plane of the telescope several spider webs are fixed, as indicated in Fig. 118 (we consider a simple type of instrument). The web H is called the horizontal wire and M the middle wire. The line joining C to the centre O of the object-glass is the *collimation-axis*, and if all the instrumental adjustments are correct, this line OC sweeps out the plane of the meridian as the telescope is made to rotate about the horizontal axis. When a star enters the field of view the telescope is altered in altitude by a slow-motion screw, so that the star appears to travel along the horizontal wire H; a graduated circle attached to the horizontal axis enables the altitude of the star to be measured. The latitude of the telescope being known, the star's declination can then be found, corrections such as that due to refraction being, of course, applied.

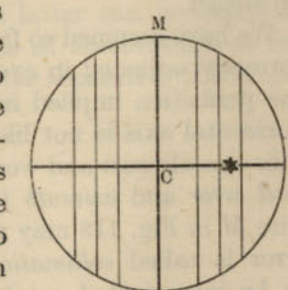

FIG. 118.

When the star passes through C it is at this instant on the observer's meridian; if the sidereal time is noted by means of a sidereal clock keeping the sidereal time appropriate to the longitude of the telescope, this sidereal time is equal to the star's right ascension. In practice, the times of passage of the star over each of the wires on either side of the middle wire M are also noted; the mean of all these times gives a much more accurate value of the time of meridian transit—and, consequently, of the star's right ascension—than if the single wire M were alone used. With modern instruments of the larger kind the times of the star's passage over the several wires are recorded automatically by means of a *chronograph.*

In actual practice, the sidereal clock is almost certain to have an error on the true sidereal time; on any night this error is found by observing the times of transit of certain selected stars whose right ascensions are accurately known; such stars are known as *clock stars.*

At any instant, then, the true sidereal time is known and, the longitude of the telescope being also presumed known, the G.C.T. (or any other standard time) can be calculated for the instant concerned. Comparison with a mean time clock gives the error of this clock. Accordingly, radio time-signals can be accurately broadcast.

We have assumed so far that the transit instrument has been correctly adjusted in every way, but no instrument possesses the perfection implied in this assumption. For example, the horizontal axis is not likely to be exactly horizontal, nor will it lie exactly east and west—the corresponding errors are called *level error* and *azimuth* (or *deviation*) *error.* Also the central wire M in Fig. 118 may not be correctly placed; the resulting error is called *collimation error.* These various errors have to be investigated specially and their effects calculated and applied to the observations.

211. *The filar micrometer.*

This instrument is used for observing visually the relative position of two close objects in the sky, in particular, the separation and position angle of a double star. It consists of a circular plate, graduated from 0° to 360°, attached to the eye-end of the telescope, an eye-piece being mounted at the centre. In one form, Fig. 119, two fixed perpendicular spider-webs, XY and CD, lie in the focal plane, together with a movable web EF

parallel to CD and actuated by a micrometer drum. Consider the observation of a double star. The plate is rotated and the telescope is adjusted by the slow-motion screws in right ascension and declination so that the primary A lies at the intersection of XY and CD and XY passes through the component B. If P is a pointer fixed on the tube of the telescope, readings of the scale give the angle PAY. We may suppose that P is adjusted so that the direction AP corresponds to the meridian or to position angle 0°. The scale-reading of Y then gives the position angle of B relative to A (in the figure, we suppose that the direction PCY gives the direction of increasing position angle). The web EF is moved by the micrometer drum so as to pass through B. The micrometer reading gives the separation AB in terms of the micrometer scale; as one unit of the latter can be expressed in terms of seconds of arc by means of observations of two objects of known angular separation, the separation AB is then found in seconds of arc.

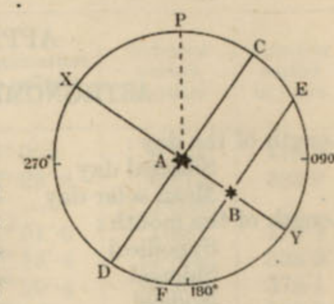

FIG. 119.

APPENDIX (i)

ASTRONOMICAL CONSTANTS

Length of the day:	
Sidereal day	$=23^h56^m4^s.091$ mean solar time.
Mean solar day	$=24^h3^m56^s.555$ sidereal time.
Length of the month:	
Synodical	= 29·5306 mean solar days.
Sidereal	= 27·3217 " " "
Nodical	= 27·2122 " " "
Length of the year:	
Tropical	=365·2422 mean solar days.
Sidereal	=365·2564 " " "
Solar parallax	=8″·7900 (1941).
1 astronomical unit	=93,003,000 miles = 149,674,000 kms.
Moon's equatorial horizontal parallax	=3422″·7.
Mean geocentric distance of moon	=238,900 miles = 384,400 kms.
Constant of aberration	=20″·47.
Constant of general precession	=50″·2655 (1941).
Constant of nutation	=9″·21.
Obliquity of ecliptic	=23° 26′ 49″ (1941).
Velocity of light	=186,271 miles per sec.
	=299,774 kms. per sec.
1 parsec	$=19{\cdot}18\times10^{12}$ miles $=30{\cdot}87\times10^{12}$ kms.
	=206,265 astronomical units
	=3·263 light-years.
1 light-year	$=5{\cdot}878\times10^{12}$ miles $=9{\cdot}460\times10^{12}$ kms,
	=63,233 astronomical units
	=0·3064 parsec.

Light travels 1 astronomical unit in 499·3 seconds.

Constant of gravitation	$=6{\cdot}670\times10^{-8}$ C.G.S. units.
Mass of earth	$=5{\cdot}98\times10^{27}$ grams.
Mass of sun	$=2{\cdot}00\times10^{33}$ grams.
Solar apex	R.A. = 270°, Dec. = +34°.
Galactic pole	R.A. = 190°, Dec. = +28°.

APPENDIX (ii)

ELEMENTS OF THE PLANETARY ORBITS

Planet	Semi-major axis (in astron. units)	Eccentricity	Inclination to ecliptic	Sidereal period in years	Synodic period in days
Mercury	0·387	0·2056	7° 0′·2	0·241	115·9
Venus	0·723	0·0068	3° 23′·6	0·615	583·9
Earth	1·000	0·0167	—	1·000	—
Mars	1·524	0·0934	1° 51′·0	1·881	779·9
Jupiter	5·203	0·0484	1° 18′·4	11·862	398·9
Saturn	9·539	0·0557	2° 29′·4	29·458	378·1
Uranus	19·182	0·0472	0° 46′·4	84·015	369·7
Neptune	30·057	0·0086	1° 46′·5	164·788	367·5
Pluto	39·457	0·2485	17° 8′·6	247·697	366·7

APPENDIX (iii)

DIMENSIONS OF SUN, MOON AND PLANETS

Planet	Radius		Mass (fraction of Sun's mass) 1÷	Density (in terms of density of water as unity)
	Miles	Kilometres		
Sun	432,000	695,500	—	1·41
Moon	1,080	1,738	$27{\cdot}2\times10^6$	3·34
Mercury	1,504	2,420	$9{\cdot}0\times10^6$	3·73
Venus	3,788	6,096	403,500	5·21
Earth	3,963 (Eql.) 3,950 (Polar)	6,378 6,357	329,300	5·52
Mars	2,108	3,392	$3{\cdot}09\times10^6$	3·94
Jupiter	44,350 (Eql.) 41,390 (Polar)	71,370 66,620	1047·3	1·34
Saturn	37,530 (Eql.) 33,580 (Polar)	60,400 54,050	3501·6	0·69
Uranus	15,440	24,850	22,870	1·36
Neptune	16,470	26,500	19,310	1·32
Pluto	Not known with any accuracy.			

APPENDIX (iv)

ELEMENTS OF THE SATELLITES

Planet	Satellite	Mean distance from Planet (in astron. units)	Sidereal period (in days)	Eccentricity of orbit	Diameter (in miles)
Earth -	Moon	0·002 571	27·322	0·0549	2160
Mars -	I Phobos	0·000 063	0·3189	0·0170	—
	II Deimos	0·000 157	1·2624	0·0031	—
Jupiter -	I Io	0·002 820	1·7691	Small	2109
	II Europa	0·004 486	3·5512	and	1865
	III Ganymede	0·007 156	7·1546	vari-	3273
	IV Callisto	0·012 586	16·6890	able	3142
	V Unnamed	0·001 207	0·4982	—	—
	VI ,,	0·076 605	250·62	0·1550	—
	VII ,,	0·078 516	260·07	0·2073	—
	VIII ,,	0·157 20	738·9	0·38	—
	IX ,,	0·158	745	0·248	14
	X ,,	0·077 334	254·2	0·1405	—
	XI ,,	0·150 834	692·5	0·2068	—
Saturn -	I Mimas	0·001 240	0·9424	0·0190	370
	II Enceladus	0·001 591	1·3702	0·0046	460
	III Tethys	0·001 969	1·8878	0·0000	750
	IV Dione	0·002 522	2·7369	0·0020	900
	V Rhea	0·003 523	4·5175	0·0009	1150
	VI Titan	0·008 166	15·9455	0·0289	3550
	VII Hyperion	0·009 893	21·2767	0·119	—
	VIII Iapetus	0·023 798	79·3308	0·029	—
	IX Phoebe	0·086 593	550·45	0·166	—
Uranus -	I Ariel	0·001 282	2·5204	—	—
	II Umbriel	0·001 786	4·1442	—	—
	III Titania	0·002 930	8·7059	—	—
	IV Oberon	0·003 919	13·4633	—	—
Neptune -	Triton	0·002 363	5·8768	—	—

APPENDIX (v)

GREEK ALPHABET

α	Alpha	η	Eta	ν	Nu	τ	Tau
β	Beta	θ	Theta	ξ	Xi	υ	Upsilon
γ	Gamma	ι	Iota	o	Omicron	ϕ	Phi
δ	Delta	κ	Kappa	π, ϖ	Pi	χ	Chi
ϵ	Epsilon	λ	Lambda	ρ	Rho	ψ	Psi
ζ	Zeta	μ	Mu	σ	Sigma	ω	Omega

ANSWERS *

CHAPTER I

1. (i) 12°. (ii) 155°. (iii) 76° 3′. (iv) 166° 45′. (v) 159° 40′.

2. (i) 18°. (ii) 14° 28′. (iii) 82° 17′.

3. (*a*) (i) 1080. (ii) 868. (iii) 4937.
(*b*) (i) 1244. (ii) 999½. (iii) 5684.

4. (i) 1276. (ii) 3256. (iii) 2819. (iv) 4216.

5. (i) 22° 25′ W. (ii) 23° 20′ W. (iii) 177° 4′ W.

6. (i) *a*, 36° 41′ ; *B*, 35° 27′ ; *C*, 114° 22′.
(ii) *a*, 81° 1′ ; *B*, 144° 22′ ; *C*, 15° 57′.
(iii) *c*, 104° 22′ ; *A*, 71° 45′ ; *B*, 55° 45′.
(iv) *b*, 21° 24′ ; *A*, 19° 56′ ; *C*, 134° 14′.

7. (i) 61° 55′. (ii) 95° 59′. (iii) 71° 4′. (iv) 87° 42′.

CHAPTER II

1. 60°. **2.** 51° 40′.

3. (i) 10°. (ii) 70°. (iii) 3° 11′. (iv) 60° 21′. (v) 27° 47′. (vi) 66° 42′.

4. (i) 620. (ii) 714. **5.** Star (iii).

6. (i) 86°. (ii) 90°. (iii) 73°. **7.** (i) 86°. (ii) 73°. (iii) 56°.

8. (i) 33° 22′. (ii) 66° 38′. Latitude, 73° 22′ S. **9.** 1° 52′.

10. $\delta = 62° 5' N$, $\phi = 51° 30' N$.

11. (i) 199°. (ii) 84° 15′. (iii) 212° 42′. (iv) 113° 50′.

12. (i) $4^h 52^m$. (ii) $9^h 34^m$. (iii) $4^h 22^m 32^s$. (iv) $9^h 17^m 36^s$.

13. By calculation (see Ex. 18). $X : z = 44° 38'$; $A = 108° 56'$ W.
$Y : \delta = 48° 22'$ N ; $H = 6^h 13^m 50^s$.

14. By calculation (see Ex. 18). $X : z = 70° 31'$; $A = 150° 6'$ E.
$Y : \delta = 5° 11'$ S ; $H = 20^h 20^m 48^s$.

* See Preface.

15. By calculation (see Ex. 18). X (Ex. 13) : $z=76° 49'$; $A=136° 58'$ W.
Y (Ex. 13) : $\delta=48° 22'$ S ; $H=6^h 13^m 50^s$.
X (Ex. 14) : $z=35° 11'$; $A=125° 23'$ E.
Y (Ex. 14) : $\delta=5° 11'$ N ; $H=20^h 20^m 48^s$.

16. 90° N or 90° S. **17.** 6^h. **18.** See Exs. 13, 14, 15.

19. (i) $H=7^h 42^m 52^s$; $A=57° 51'$ W. (ii) $H=4^h 45^m 32^s$; $A=113° 45'$ W.

20. (i) $H=18^h 55^m 52^s$; $A=110° 23'$ E. (ii) $H=16^h 43^m 28^s$; $A=62° 36'$ E.

CHAPTER III

1. (i) 0^h, 0°. (ii) 6^h, 23° 27′ N. (iii) 12^h, 0°. (iv) 18^h, 23° 27′ S.

2. (i) 6^h. (ii) 18^h. No. **3.** 15° N. **4.** 12° S.

7. 23° 27′ N, 6^h; 23° 27′ S, 18^h.

8. (i) $2^h 23^m 20^s$. (ii) $16^h 4^m 37^s$. (iii) $5^h 13^m 45^s$. (iv) $4^h 8^m 48^s$.

9. (i) $1^h 26^m 18^s$. (ii) $22^h 7^m 12^s$. (iii) $4^h 2^m 32^s$. (iv) $22^h 11^m 43^s$.

10. $26^s{\cdot}5$ slow, $26^s{\cdot}0$ slow ; $0^s{\cdot}5$ gaining.

CHAPTER IV

1. (i) $6^h 14^m 15^s$. (ii) $23^h 27^m 37^s$. (iii) $0^h 49^m 56^s$. (iv) $20^h 42^m 32^s$.
(v) $15^h 46^m 14^s$.

2. (i) $6^h 13^m 14^s$. (ii) $20^h 35^m 9^s$. (iii) $1^h 4^m 37^s$. (iv) $9^h 45^m 48^s$.

3. (i) $8^h 23^m 22^s$. (ii) $5^h 15^m 55^s$. (iii) $19^h 36^m 42^s$. (iv) $5^h 43^m 55^s$.

4. (i) $23^h 30^m$ July 4. (ii) $18^h 44^m$ July 7. (iii) $3^h 36^m$ Aug. 11.
(iv) $1^h 27^m$ Aug. 20.

5. (i) $5^h 16^m 46^s$ May 1. (ii) $7^h 10^m 0^s$ May 3. (iii) $17^h 6^m 36^s$ May 3.
(iv) $21^h 25^m 40^s$ May 5.

6. (i) $4^h 40^m 8^s$. (ii) $16^h 46^m 10^s$. (iii) $6^h 25^m 15^s$. (iv) $16^h 18^m 36^s$.

7. (i) $9^h 11^m 30^s{\cdot}4$. (ii) $3^h 40^m 46^s{\cdot}2$. (iii) $19^h 48^m 50^s{\cdot}8$.

8. (i) $7^h 43^m 43^s{\cdot}8$. (ii) $9^h 21^m 15^s{\cdot}8$. (iii) $21^h 31^m 43^s{\cdot}8$.

9. (i) $9^h 26^m 4^s{\cdot}1$ P.M. (ii) $8^h 50^m 40^s{\cdot}9$ P.M. **10.** $22^h 46^m 20^s$.

11. (i) $10^h 12^m 44^s$, $0^h 7^m 33^s$. (ii) $0^h 15^m 54^s$, $14^h 10^m 43^s$.
(iii) $1^h 45^m 55^s$, $15^h 40^m 44^s$. (iv) $16^h 3^m 24^s$, $5^h 58^m 13^s$.
(v) $0^h 3^m 26^s$, $13^h 58^m 15^s$. (vi) $6^h 1^m 47^s$, $19^h 56^m 36^s$.

12. Spring, $92^d{\cdot}8$; summer, $93^d{\cdot}6$; autumn, $89^d{\cdot}8$; winter, $89^d{\cdot}0$.

13. 70° N. **14.** (i) 58° N. (ii) 50° S. **15.** No ; No.

16. 14 S to 23° 27′ S. **17.** (i) $1^h 59^m 48^s$. (ii) $1^h 52^m 52^s$.

19. $-6^m 30^s$. **20.** $+5^m 22^s$; $11^h 59^m 38^s$.

21. (i) 94° 46′ E. (ii) 2000 Feb. 12 (zone −6). **22.** 11·70 knots.

CHAPTER V

1. $224^d{\cdot}7$. **2.** $378^d{\cdot}0$. **3.** $11^y{\cdot}85$. **4.** 46° 18′.

5. $87^d{\cdot}46$. **6.** $87^d{\cdot}96$; $224^d{\cdot}7$.

7. $6^y{\cdot}69$. **8.** $427{\cdot}7 \times 10^6$ miles.

9. (i) 0·0000627, (ii) 5830 miles. (i) 0·0001569, (ii) 14590 miles.

10. 21·76 miles/sec. **11.** $7^h 57^m$. **12.** 0·01259.

13. 22,550. **14.** 1047. **15.** 115° 35′.

16. (i) 46° 18′ ; $\frac{1}{2}$. (ii) 0·940, 0·060. **17.** 20° 29′. **18.** (i) 0·876. (ii) 0·890.

CHAPTER VI

1. 45° 0′ 58″·2. **2.** 30° 1′ 40″·8.

3. $\phi=59° 21' 19''$ N, $\delta=52° 6' 43''{\cdot}5$ N.

4. $\phi=49° 20' 40''$ N, $\delta=68° 49' 49''{\cdot}5$ N.

5. $\phi=45° 22' 5''$ S, $\delta=68° 37' 22''$ S.

6. (i) 4′·0. (ii) 5′·3. (iii) 9′·6.

7. (i) 29° 40′·5. (ii) 61° 26′·8. (iii) 50° 52′·4.

8. (i) 4·5. (ii) 5·7. (iii) 7·7 nautical miles (n.m.).

9. 22·1 n.m. **10.** 73·3 n.m.

CHAPTER VII

1. (i) 29° 59′ 55″·6. (ii) 69° 59′ 51″·7. **2.** $92{\cdot}8 \times 10^6$.

3. 50° 27′·6. **4.** (i) 60′·7. (ii) 14′·7.

5. (i) 238,000. (ii) 59′·2. **6.** 54′·0 ; 60′·3.

7. 1078. **8.** 16° 26′ 56″ N.

9. 15′ 30″. **10.** 0·0543.

11. (i) $3\frac{1}{4}$, (ii) $63{\cdot}9 \times 10^{12}$, (iii) 10·87 ; (i) 43·48, (ii) 834×10^{12}, (iii) 141·8.

12. 0″·045. **13.** 0″·029. **14.** 0″·175. **15.** 0″·0052.

CHAPTER VIII

1. (i) 90°. (ii) 89° 59′ 39″·53. (iii) 90° 0′ 20″·47. (iv) 89° 59′ 49″·77.

2. " Aberrational ellipse " becomes a small circle of angular radius κ. On June 21, the star's apparent longitude and latitude are 0° and $90°-\kappa$ respectively.

3. (i) 41° 59′ 46″·30. (ii) 42° 0′ 15″·22. (iii) 42° 0′ 1″·07.

4. 6^h, −3° 6′. **5.** 0^h, 0°. **6.** 18^h, −23° 27′.

7. *$23^h 59^m 1^s{\cdot}4$, $0^h 2^m 40^s{\cdot}2$; G.C.T., $18^h 17^m 56^s$ March 20.

8. *$4^h 0^m 37^s{\cdot}96$, $+32° 1' 40''{\cdot}2$.

CHAPTER IX

1. (i) 46° 38′·3. (ii) 14° 2′·4. (iii) 54° 25′·8.
2. (i) 61° 17′·6. (ii) 56° 14′·1. (iii) 52° 12′·2.
3. (i) 3° 13′ S, 21° 16′ W. (ii) 15° 20′ N, 78° 22′ E.
4. (i) 45° 56′ N, 147° 24′ W. (ii) 57° 32′ S, 24° 41′ E. (iii) 12° 16′ N, 37° 46′ W.
5. 48° 13′·5 N, 36° 51′·2 W.
6. 33° 55′·0 S, 18° 10′·0 E.
7. (i) 8′ in direction 010°. (ii) 2′ in direction 088°. (iii) 7′ in direction 329°.

CHAPTER X

1. (i) 0·295. (ii) 0·952. (iii) 0·671. (iv) 0·368.
2. (i) 17h 20m 39s. (ii) 48m 53s.
3. Dec. 26.
4. 0·0549.
5. 0·0538.
6. 225,400, 251,600, 238,500 miles.
7. 225,600, 251,300, 238,450 miles.
8. 11910 ft.

CHAPTER XII

1. 4·92. 2. 3·03. 3. 13·06. 4. 2·17. 5. 3·94.
6. (i) +2·43. (ii) +1·30. (iii) −0·29.
7. (i) 1 : 1·91. (ii) 1 : 3·98.
8. (i) 5·40. (ii) 2·58. (iii) 25·1. (iv) 0·00182.
9. (i) 782,500. (ii) 289,400. (iii) 5,123,000 miles.
10. (i) 1,070,000. (ii) 1,892,000. (iii) 10,210,000 miles.
11. (i) 0·71. (ii) 1·67. (iii) 1·02. (iv) 7·76.
12. (i) 0″·045. (ii) 0″·075.
13. (i) −13·9. (ii) +43·9. (iii) −34·2 kms./sec.
14. (i) 7400 parsecs. (ii) 8300. (iii) 2300.

CHAPTER XIII

1. (i) 0″·149, 75° 35′, 19·0 kms./sec. (ii) 0·″428, 169° 20′, 28·6. (iii) 0″·753, 352° 47′, 31·9. (iv) 1″·323, 204° 3′, 16·6.
2. (i) 27·7 kms./sec., 17·0 parsecs. (ii) 49·4, 8·0. (iii) 33·9, 20·9. (iv) 57·5, 49·9.
3. 19·0 kms./sec., 0″·041.
4. 0″·0297 ; +35·0 kms./sec.
5. 0″·154, −16·9 kms./sec.

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