FIELD ASTRONOMY FOR SURVEYORS

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PREFACE

This textbook is the outcome of a long period of close collaboration between the authors in the teaching of field astronomy at the University of New South Wales, Australia. The scope of this book is confined to those aspects of astronomical theory and practice, which are appropriate for observations made with a modern single second theodolite. Included in the book are many examples of observations taken in both hemispheres. The calculation of these observations is given in greater detail than that normally required, in order to help the student reduce his own observations. Astronomical methods of high accuracy required for the geodetic control of continental areas have not been included.

In such a well-established subject of study in surveying education as field astronomy, it may be presumed that there is very little new that can be written. However, over their years of teaching, the authors found that there was no textbook for student reference which used conventions, which were not biased to one hemisphere and which also covered a systematic treatment of predicting observing programmes and analysing the results of observations made with a theodolite. One of the overriding considerations in the writing of this book has been that everything should be generalised so that strict mathematical rules could be used without the need for a host of auxiliary rules governing a change of terrestrial or celestial hemisphere.

The need for the practical application of field astronomy in land surveying and exploration will decline as greater use is made of earth satellite methods of position fixing and as horizontal control surveys are extended into unsurveyed areas. However, a practical need is not the only criterion by which a course of study at a tertiary institution should be judged. It is the opinion of the authors that a study of field astronomy has many desirable features, which make it attractive as a discipline and as a subject of interest to both students and experienced surveyors. Besides gaining an understanding of celestial phenomena, a study of field astronomy exercises the student in spherical trigonometry, convergence of meridians, error theory and least squares methods as well as theodolite construction and adjustment, all of which complement the instruction in other surveying subjects. The student also gains the satisfaction of being able to find his geographical position and determine the azimuth of a terrestrial line with a high degree of accuracy with little more equipment than is required for normal surveying operations.

In field astronomy, the work of surveyors has been greatly simplified by improvements in theodolite construction and by the wide availability of simple cheap and accurate time keeping and time recording devices, short wave radios and powerful continuous radio time signal services. But perhaps the greatest single influence, in recent years, has been the widespread use of small electronic calculators. The labour of repetitive and complex calculations has been removed. Individual observations, in preference to mean values, may be reduced quickly and the results of all observations analysed, even under field conditions, using simple calculator programmes. Furthermore, the necessity for making independent check calculations, preferably by a different person, with alternative formulae, has been eliminated provided the calculator programmes are thoroughly tested beforehand and the input data and output results carefully checked. No longer is it necessary to restrict observations to circumstances and time limitations to suit special simple mathematical relationships, which are mainly in the form of rapidly converging series. However, to provide some continuity with past practice and also to maintain the flow of explanation in the text, the proofs of these and some other relationships have been included in an appendix at the end of the book.

It is with a great deal of pleasure that the authors record their gratitude to Mrs. Susan Kiriazis, who has, with cheerful patience and efficiency, typed the whole of the manuscript.

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PREFACE TO REVISED EDITION

In this revised edition of the book some minor changes have been made to the original manuscript. In Chapter 1 the examples on pages 4, 5 and 6 have been arranged differently and the distinction drawn between meridian and grid convergence. Also calculator methods of time conversion have been included in the appendix.

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The Uses of Field Astronomy

INTRODUCTION

THIS is a question asked of the surveyor with such frequency and such incredulity that an answer to it must be provided. The answer will no doubt be disappointing to the uninformed layman, who, in many cases, hardly realises that there is any difference between astronomy and astrology.

The surveyor's interest in astronomy is very much a practical one as he, unlike the astronomer with a scientific interest in the stars, is chiefly interested in how he can make use of the stars for the purposes of his survey requirements.

The Uses of Field Astronomy

THE surveyor uses Field Astronomy for two main purposes. These are

(a) Determination of the Position of Points on the Earth

and (b) Determination of Orientation.

The accuracy required for these determinations varies naturally with the purpose of each task. One can appreciate that no hairsplitting accuracy is needed in laying out, for the devout Mohammedan, a line pointing to Mecca so that he may make his obeisances in the correct direction. On the other hand, the highest accuracy in an astronomical determination is needed to define the relationship between the geoid and the mathematical surface to which a geodetic survey is referred.

Position determination is used to correlate the Fundamental Station of a continental survey network with the geoid. This also requires that the survey network be orientated with respect to the meridian.

Position Determination

1.11 Several examples of position determination from astronomical sights to a lesser accuracy come readily to mind. A geophysical expedition was mounted to traverse the Kalahari Desert for making measurements for mineral prospecting. This expedition was accompanied by a surveyor who determined, from star sights, the position of each night's camp. If the geophysical observations obtained were later found to be of sufficient interest to be followed up, the surveyor's work could be used to lead them back to within about 200 metres of the point, at which these observations were obtained.

1.12 Another example is one, in which observations were used to determine the positions of points identifiable on air photographs, so that the set of overlapping air photographs could be set up in the form of a mosaic, with these fixed points providing control for both position and scale. This controlled mosaic provided its information at the fairly small scale of 1/100 000.

During the Second World War, astronomical methods were used in North Africa by the Long Range Desert Groups, who made long journeys deep into the featureless inhospitable desert. This desert, like the sea, can be traversed with little restriction and the methods of navigation used were similar to those of the sailor and fixes to within a kilometre were, in many cases, quite satisfactory. Sun compasses were mounted on the vehicles to overcome the difficulties of magnetic compasses close to steel.

1.13 Another example is that of placing marks, which are to define a property, such as mining lease in unmapped country, and whose positions are specified in terms of the geographical coordinates, latitude ϕ and longitude λ. The surveyor navigates himself by some rough means into the vicinity of the required position $\phi_o \lambda_o$. He determines the position where he sets up his theodolite, from sun or star sights as $\phi_1 \lambda_1$. The distance and direction from $\phi_1 \lambda_1$ to $\phi_o \lambda_o$ is taken out. If the distance is short $\phi_o \lambda_o$ is set out by placing a mark on the calculated direction at the calculated distance. If, however, the distance between $\phi_1 \lambda_1$ and $\phi_o \lambda_o$ is long, the direction is set out roughly (say by compass) and the distance run down and measured by speedometer. At this point, the position $\phi_2 \lambda_2$ is determined astronomically and the short distance between $\phi_2 \lambda_2$ and $\phi_o \lambda_o$ is set out in the required direction as indicated above.

Azimuth Determination

1.21 The determination of orientation is probably of greater importance to the ordinary surveyor especially if he is working in an area, in which there is no national survey network, or one, in which only the first stage of such a network has been carried out and the geodetic stations are therefore too far apart for his convenient use.

Determination of orientation consists in determining the azimuth of a line, say PQ, in a survey. This is the horizontal angle round towards the east from the northern branch of the local meridian through P to the line PQ. This observation serves to orientate the survey with respect to True North at the point P. This kind of determination provides a very valuable means for checking the quality of a survey because the azimuth carried forward from a line, whose azimuth has been previously determined, can be checked by determining an azimuth from star sights along any successive line of the survey (see sections 1.23 and 1.41).

When a survey has been oriented by astronomical methods, its orientation can be re-established at a later date with ease and certainty. This is not the case in orientation by magnetic methods, whose accuracy is low in any case and whose re-establishment is uncertain. In some cases, the datum for azimuth is merely an assumed one and its re-establishment is therefore impossible.

1.22 A requirement for a good azimuth is that for monitoring the performance of the gyro- theodolite, which itself determines azimuth.

Since it is not possible to adjust the various axes of the gyro- theodolite into the precisely correct relationships, one to the other, the gyro - theodolite's azimuth will be subject to a zero error, which can only be determined, if the azimuth of the line of reference of the gyro - theodolite is known from an astronomic sight (see section 1.42).

1.23 A surveyor is to carry out the survey of a very long traverse for a pipe line for natural gas. He is also required to fix its position with respect to the boundaries or properties traversed by this pipe line. Unfortunately, no national survey has been carried across this portion of the country. The surveyor starts by determining the azimuth of the first leg of his traverse from astronomical observations. He then runs his survey traverse and he carries forward a direction based on the azimuth of the first leg of his traverse and the angles between the successive lines of the traverse. After the traverse has been carried forward some distance, a check on the correctness of his angular observations is obtained by determining the azimuth of a line of the traverse and comparing this with the value obtained from the previous azimuth determined and the angles measured between the successive traverse lines. In this case, allowance for Meridian Convergence must be made. (see section 1.41)

1.24 The third example is one, which sometimes must be applied to all surveys carried out under a specific Survey Act. This often occurs when it is proposed to cover a state or country with a national survey. Up to such time as the stations of this survey are established in sufficient density on the ground, so that the cadastral surveyor may be left with only a small amount of additional work to link his surveys to the national survey, regulations may have been promulgated requiring any surveyor to arrange the orientation of his survey, from an initial astronomical azimuth determination, to agree with that of the State Survey when it is carried into his area. This requires the application of meridian convergence to his determination, so that the surveyor sets out the grid north of the national survey at his own local survey to anticipate the orientation of the state survey before its actual arrival.

Corrections to Azimuths for Meridian Convergence

1.31 On the earth, considered to be a sphere, the azimuth along a line PQ varies from a value A_P at P to a value A_Q at Q, because the meridians at the two ends of a line are not parallel but converge, except when the line lies along the equator. (see Fig 1.1)

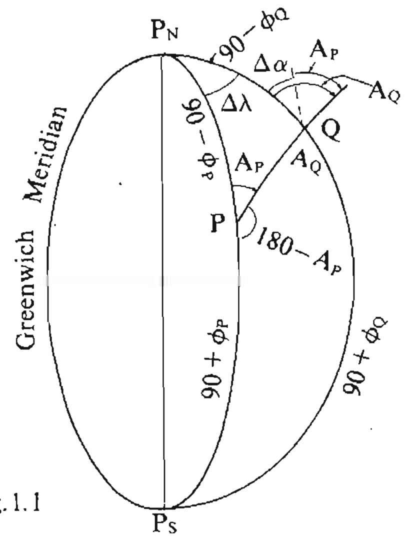


Fig. 1.1

This meridian convergence $\Delta\alpha$ is given as the difference between the azimuths along the line at its end points

$$\therefore \qquad \Delta\alpha = A_Q - A_p$$

From Napier's Analogies in the spherical triangle WXY (vide section 2.62)

$$\tan\{\tfrac{1}{2}(X+Y)\} = \cos\{\tfrac{1}{2}(x-y)\}\sec\{\tfrac{1}{2}(x+y)\}\cot\{\tfrac{1}{2}W\}$$

Substitution in the southern of the two possible spherical triangles gives

$$\tan\{\tfrac{1}{2}(A_Q+180-A_p)\} = \cos\{\tfrac{1}{2}(90+\phi_p-90-\phi_Q)\}\sec\{\tfrac{1}{2}(90+\phi_p+90+\phi_Q)\}\cot\{\tfrac{1}{2}(\lambda_Q-\lambda_p)\}$$

This on reduction gives the following result

$$\tan\{\tfrac{1}{2}\Delta\alpha\} = \sec\{\tfrac{1}{2}\Delta\phi\}\sin\bar{\phi}\tan\{\tfrac{1}{2}\Delta\lambda\}$$

in which ϕ is the latitude measured positive northwards from the equator,
λ is the longitude measured positive eastwards from the Greenwich meridian,
$\Delta\phi$ is the latitude difference between the ends of the line PQ,
$\Delta\lambda$ is the longitude difference between the ends of the line PQ,
and $\bar{\phi}$ is the latitude of the mid point of the line PQ

Substitution in the northern of the two possible spherical triangles gives exactly the same result as above. The sign of the correction follows from keeping track of the signs of the defined quantities latitude and longitude, and also the signs of the trigonometrical functions involved. Its sign can also be easily determined from a simple sketch.

Since in survey practice, lines are comparatively short, $\Delta\phi$ and $\Delta\lambda$, and as a result $\Delta\alpha$, are small angles. The above relationship is therefore given, to the first order of correctness, as

$$\Delta\alpha = \Delta\lambda \sin \bar{\phi}$$

It is very often convenient to substitute, for the difference $\Delta\lambda$ in longitude, the corresponding distance or the difference in easting in coordinates ΔE. The difference in longitude $\Delta\lambda$ corresponds to an east west distance of ΔE along the small circle of latitude $\bar{\phi}$. Since this small circle has a radius of $R \cos \phi$

$$\Delta\lambda = \frac{\Delta E}{R \cos \bar{\phi}} \rho$$

But

$$\Delta\alpha = \Delta\lambda \sin \bar{\phi} = \frac{\Delta E}{R \cos \bar{\phi}} \rho \sin \bar{\phi} = \frac{\Delta E}{R} \rho \tan \bar{\phi}$$

in which relationship ΔE and R are in the same units, and the angular units and the value of ρ are in accord.

1.41 Example. Given the following data

(1)

Station	Latitude	Local System Coordinates Easting (m)	Northing	Line	Observed Azimuth
A	33°34'10" N	+60 850.5	+33 008.7	AB	169°27'30"
G	33 30 40 N	+75 906.2	+26 445.8	GF	283 44 40

in which the latitudes are given to the nearest 10".

(2) The sum of the clockwise angles of the traverse at the stations B, C, D, E, and F amounted to 834°11'20".

(3) The radius of the earth is 6 380 kilometres.

Determine the angular closure of the traverse between the stations A and G.

To deduce the azimuth GF from that of AB. (see Fig 1.2)

Azimuth of AB	169°27'30"
Sum of Angles	834 11 20
	-720 00 00
Meridian Convergence between A and G, $\Delta\alpha$	00 05 23
Deduced Azimuth GF	283 44 13
Observed Azimuth GF	283 44 40
Angular Misclosure	27

$$\Delta\alpha = \frac{(E_G - E_A)}{R} \rho \tan \bar{\phi}$$

$$= \frac{15055.7}{6\ 380\ 000}\ 206\ 265'' \tan 33°32'30''$$

$$= 323''$$

$$= 0°05'23''$$

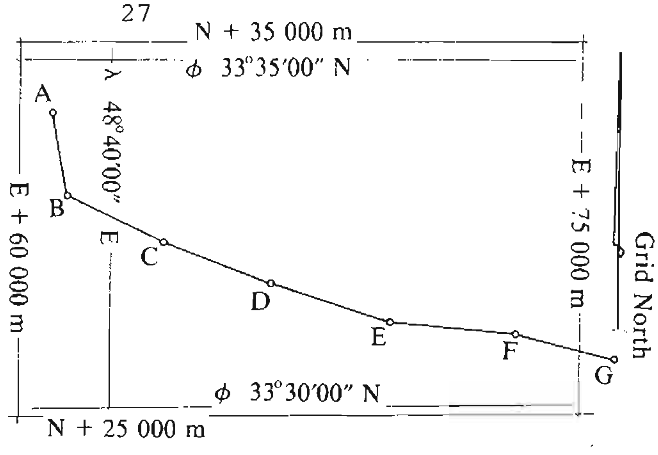


Fig. 1.2

The sign of term $\Delta\alpha$ is easily determined from Figure 1.3 where AN_A & GN_G are the local meridians at A and G respectively and $\Delta\alpha$ shows the angle of convergence between these two meridians.

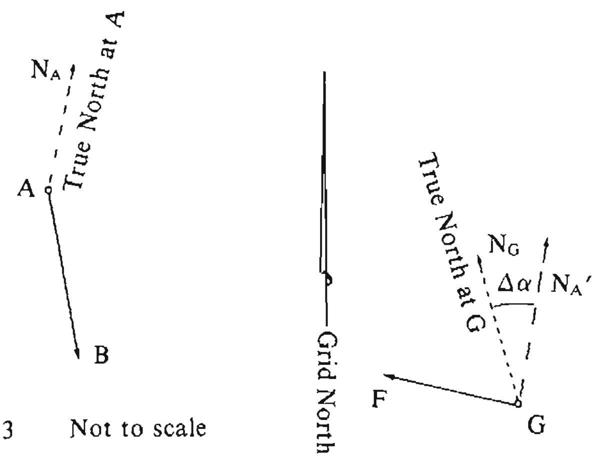


Fig. 1.3   Not to scale

1.42 Example. In a country in the northern hemisphere, a gyro-theodolite, set up at Trig. Station T and sighted towards Trig. Station S, was used to determine a direction value for this line. Determine from the following information, the zero correction to be applied to the gyro-theodolite direction to obtain an azimuth. (see Fig 1.4)

Station	Universal Transverse Mercator System Zone 39 Coordinates		Line	Bearing from these Coordinates
	Easting (m)	Northing		
T	227 929.4	3 794 910.5	TS	192°56'42"
S	226 805.4	3 790 020.5		

The gyro-theodolite direction value as obtained from observing from T towards S = 191°19'20"

The following facts must be known about this projection

(i) The meridians in the northern hemisphere converge towards the central meridian and towards the north. In the southern hemisphere, they converge towards the central meridian and the south.

(ii) The ray sighted on the earth between two stations shows on the projected plane as a curved line between these two points with a bulge away from the central meridian. The angle τ between this arc and the chord between the two stations is easily computed.

(iii) The bearing from coordinates gives the clockwise angle round from Grid North to the straight line chord between the two points.

From the tables published for the Universal Tranverse Mercator System, the geographical coordinates of Station T were computed as

$$\phi_T \quad 34°15'35" \text{ North}$$

$$\lambda_T \quad 48°02'43" \text{ East}$$

Likewise from these tables the grid convergence for Station T was computed as

$$\gamma = 1°39'52"$$

Grid convergence is a special case of meridian convergence, in which one of the two points is taken to lie on the central meridian. The longitude λ_o of the central meridian of this Zone is 51°E.

$$\therefore \quad \Delta\lambda = \lambda_T - \lambda_o = 51°00'00" - 48°02'43" = 2°57'17"$$

A good approximation for grid convergence on this projection is given by

$$\gamma = \Delta\lambda \sin\phi = 1^\circ 39' 48''$$

This compares very well with the accurate value taken out above from the tables.

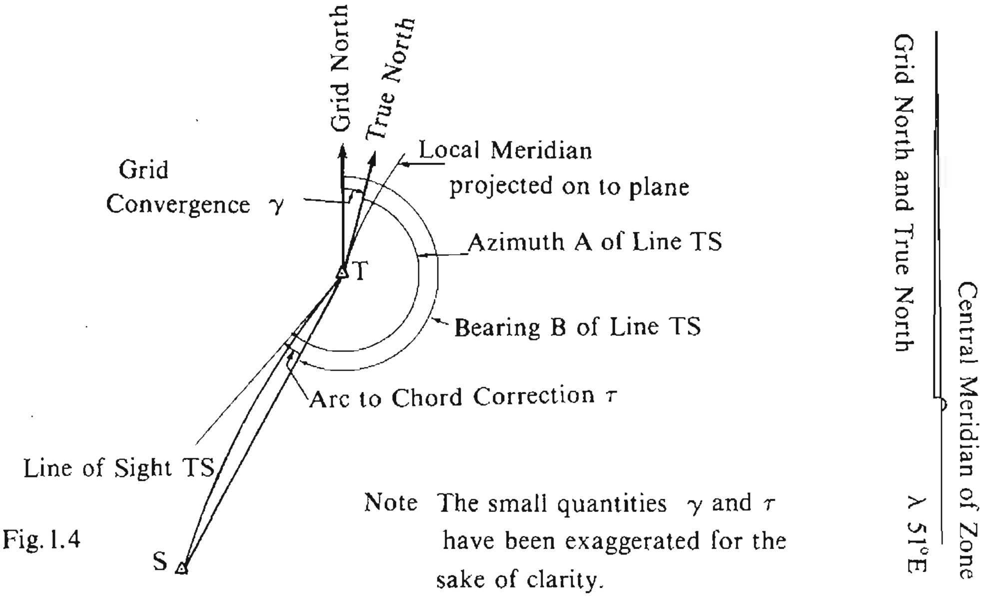


Fig. 1.4

The Arc to Chord Correction

$$\tau'' = \frac{2E_T + E_S - 1\ 500\ 000^*}{6R^2\ k_o^{\ 2}}\ \rho''(N_S - N_T)$$

$$= \frac{2\text{x}227\ 929 + 226\ 805 - 1\ 500\ 000}{6 \times 6.378^2 \times 10^{12} \times 0.9996^2}\ 206\ 265\text{x}4\ 890$$

$$= 3''$$

* False Origin -500 000

∴ Azimuth TS = A $= B + \tau - \gamma$ (see Fig 1.4)

$= 192^\circ 56' 42'' + 3'' - 1^\circ 39' 52''$

$= 191^\circ 16' 53''$

Observed Gyro Azimuth TS $= 191^\circ 19' 20''$

Zero Correction $= -2' 27''$

The Use of Laplace Stations

1.51 As a preliminary to the discussion to follow, brief and somewhat generalized descriptions must be given of the surfaces, to which various portions of a continental survey are referred.

The first surface is the topographical one, namely that of the actual earth, because the survey is carried out in order to provide information about this surface. The next one is a smooth mathematical surface, to which the observations, calculations and results of the survey can be referred. This is usually an ellipsoid of revolution or spheroid, selected to approximate very closely to the earth's shape and dimensions at sea level.

The third is an equipotential surface, which closely fits the surface corresponding to mean sea level and its hypothetical continuation under the land surfaces and which is called the geoid. The normal to the geoid at a point will intersect the topographical surface at a second point immediately above the first. This normal will coincide very closely indeed with the direction in which gravity is acting at the topographic surface and in which, therefore, the vertical axis of a theodolite is set when this instrument is levelled.

The topographical surface of the earth can be seen to be an irregular one. The ellipsoid of revolution selected is very slightly elliptical. It has a smooth surface, on which the normal at any given point can be specified mathematically. The geoidal surface is *not* a smooth one; but, because it is

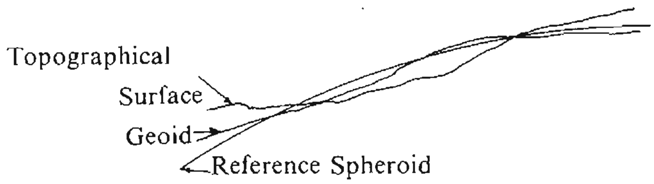


Fig. 1.5

not a visible surface its unevenness can be inferred only from indirect observations. These irregularities are due to the material, of which the earth is composed, being homogeneous neither in density nor in distribution, and they therefore affect the position of the vertical axis of the observer's theodolite, when this is levelled. Since this axis defines the observer's zenith at this point, and since the irregularities are largely random, the observer's zenith cannot be referred exactly to the other two surfaces.

1.52 A high precision survey, carried over a continental area is known as a geodetic survey. At the beginning of such a survey, one of its stations is selected as the Fundamental Station. At this point, an astronomical determination of its position is made with the very highest accuracy. The astronomical azimuth of the line from this station to one of the adjacent stations of the survey is likewise determined. The estimated standard deviation of the internal accuracy of these determinations is of the order of ± 0.3 seconds of arc.

Then, at the Fundamental Station, the astronomical position $\phi_A \lambda_A$ is taken to be the same as that of its geodetic position $\phi_G \lambda_G$, which are position values referred to the spheroid of reference selected for the survey. Similarly, the survey is then orientated by means of the astronomical azimuth A_A of the observed line; in other words, the astronomical azimuth A_A is equated to the geodetic azimuth A_G of the survey.

As the survey progresses, the geodetic position of successive stations can be derived in terms of the accepted position of the Fundamental Station and the quantities observed at each station of the survey. If, at a station, which is not the Fundamental Station, an astronomical determination of position and azimuth is made with the highest accuracy, this station is known as a Laplace Station. At such a station, there will be two sets of data available for position and azimuth and they will not necessarily coincide with each other, even if the observations, from which they are derived, are absolutely free of the random errors of human observation. This discrepancy may come about as a result of several factors.

The technique, described above, of equating astronomical and geodetic values at the Fundamental Station sets the tangent of the reference spheroid parallel to that of the geoid, i.e. perpendicular to the direction of the plumb line, at this point. The geodetic position of the Laplace Station is obtained from the computation on the surface of the reference spheroid. In this computation, those measurements made on the earth's surface in the course of the geodetic survey are used and the position of the new Laplace Station, relative to the Fundamental Station, will therefore be known within a few metres.

1.53 The Laplace Station's astronomical position, however, is determined by the position of the observer's zenith at this station. The theodolite's vertical axis, after the theodolite has been levelled at this station, defines this zenith and, since the levelling is done by means of a bubble or other device under the influence of the force of gravity, this zenith is determined by the normal to the equipotential surface. The relationship between the plumbline and the normal to the reference spheroid at this station is not known and therefore the normals to the two surfaces at this point do not necessarily coincide, even though they were set to coincide at the Fundamental Station. The difference in position $(\phi_A - \phi_G, \lambda_A - \lambda_G)$ provides a very good criterion of the relative position of the reference spheroid with respect to the geoid. When there is evidence of a divergence between the two surfaces, as will be indicated by a systematic difference between the astronomical and the geodetic values of position, it becomes necessary to re- appraise the assumptions made at the Fundamental Station and also to consider possible changes in the dimensions of the reference spheroid adopted.

As the geodetic survey proceeds, its orientation will become increasingly uncertain, as the distance from the Fundamental Station increases, because of the accumulation of errors of observation. However, this can be rectified by making use of the value of the astronomical azimuth, which has also been obtained at the Laplace Station, in a relationship known as the Laplace Equation, which states

$$A_A - A_G = (\lambda_A - \lambda_G) \sin \phi$$

This has been derived in the Appendix in section A.91.

The most significant feature of this equation is that it makes it possible to orient the geodetic survey on the adopted spheroid, irrespective of which one has been adopted and also irrespective of how it was oriented at the Fundamental Station.

With a spheroid of well chosen dimensions and orientation to the geoid, the difference between the astronomical and geodetic values of position and azimuth will be small, except possibly in those regions of disturbance, where the local geoid is unusually uneven.

2

The Solar System, The Celestial Sphere and The Astronomical Triangle

INTRODUCTION

2.11 AN observer, looking out at the sky at night sees the black firmament dotted with a host of points of light, which wheel across the sky from east to west. On further observation all, with a few exceptions, appear to maintain their relative positions unaltered, and it is known that they have done so over the period of recorded history. These points of light of differing brilliances are the stars. For convenience in identification, they are grouped in sets as constellations, which are named from their appearance. In some cases, there is some justification for the name, e.g. the Scorpion, the Lion, the Southern Cross, but in others the name has only a fanciful relationship to the constellation's shape.

Constellation boundaries have been adopted and agreed upon. These are shown in star atlases. Originally the brightest star in any constellation was designated α, the next brightest β and so down the Greek alphabet in diminishing brightness. However, since this system was laid down, the actual order of brightness, due to some natural cause, may have changed but the alphabetical order has, for convenience, not been altered. In the star catalogues, the brightness of each star is given by a number, called its magnitude.

The ancient astronomers ranked the stars according to their brightness on an arbitrary whole number scale of "magnitudes" varying between one and six. Stars having a magnitude of six were just visible to the naked eye under very favourable observing conditions, and the brightest stars were considered to be of the first-magnitude. In the 19th century, it was discovered that a first magnitude star was about 100 times as bright as a star of the sixth magnitude, and this fact is now used as a basis for the present scale of magnitudes. Furthermore, this scale is divided in a logarithmic manner, in order to be able to represent the magnitudes of very bright and very dim bodies by small numbers. The scale also takes into account fractional magnitudes and extends beyond the original limits of one and six, with the brightest of the celestial bodies having negative magnitudes.

Let a star of magnitude m_1 have a brightness of b_1 and a star of magnitude m_2 have a brightness of b_2. These quantities may then be related on a logarithmic scale by

$$m_1 = k_1 + k_2 \log b_1$$

and

$$m_2 = k_1 + k_2 \log b_2$$

where k_1 and k_2 are constants.

The difference between these two equations gives

$$m_1 = m_2 + k_2 \log \frac{b_1}{b_2}$$

where $\frac{b_1}{b_2}$ is the ratio of brightness of the two stars.
However from before when $m_1 = 1$ and $m_2 = 6$, $\frac{b_1}{b_2} = 100$ and thus $k_2 = -2.5$

$$\therefore \quad m_1 = m_2 - 2.5 \log \frac{b_1}{b_2}$$

which relationship is the basis of the modern scale of magnitudes.

The following table of magnitudes for some selected celestial bodies will indicate the characteristics of this scale of magnitudes.

Table 2.1

Body	Magnitude	Remarks
Sun	-27	Approximate value
Moon	-12	Approximate value at full moon
Venus	- 3.4	Average value
Sirius(α Canis Majoris)	- 1.6	Brightest star
Betelgeuse(α Orionis)	0 - 1	Varies between 0 & 1
Polaris(α Ursae Minoris)	2.1	
σ Octantis	5.5	

The reader will find it instructive to calculate the brightness ratio for some of the bodies cited in the table.

Since the stars are almost infinitely distant, so that the image is a point source of light, a magnifying telescope will not show any enlargement of the star image. The images of the sun, the planets and their satellites are not point sources of light and magnification will enlarge their images and therefore show them up as discs.

If the sky is kept under observation, it is quickly seen that, during the year, the constellations shift across the sky so that certain of them can be seen in the eastern sky immediately after sunset at one time of the year; about six months later, these constellations will appear in the western sky after sunset.

It will also soon appear that a few of the points of light in the sky behave in a different way from the majority, because they appear to wander across the background of the fixed star pattern. They take part in the overall movement of the whole sky from east to west, but in addition this small band of wanderers also appears to move relatively with respect to the unchanging star background in an irregular manner. They appear to move at differing rates, sometimes appearing to stop and sometimes even to move retrogressively. These stars are called planets from the Greek word for a wanderer. Five planets are visible to the naked eye. Venus and Mercury are always in the vicinity of the sun. The other three visible planets are Mars, Jupiter and Saturn. All lie in a fairly narrow belt in the sky.

2.12 A prominent object in the sky is the moon, which waxes and wanes over a period of approximately a month. At the start of a cycle, it can be seen as a thin crescent in the western sky after sunset. Gradually this crescent grows until it appears as a semi-circle and then as a full circle at Full Moon. After this it wanes to the semi-circle and finally disappears. The moon can be seen to take part in the motion of the sky from east to west. If it is watched at night, it can be seen to move across the fixed background at a considerable rate towards the east. Between successive nights, the moon appears to traverse an arc of about 13^{o} across the sky. The moon also remains, over the years, within a fairly narrow band in the sky.

2.13 At dawn, the sun rises over the eastern horizon and its light then hides the stars and planets from the observer's view. It moves across the sky from east to west, reaches its maximum altitude at noon and then sets in the evening over the western horizon.

Because the sun's light obscures the stars, it is not so easy to see that the sun also moves across the star background. If at sunrise or at sunset, the stars in the sun's vicinity are noted over a period, it will quickly be seen that they appear to be catching up with and overtaking the sun. From this, it is clear that the sun, as well as the moon, is moving eastwards across the star background. The sun, however, moves at the much slower rate of about one degree per day and, after a year, appears to reach the same point with respect to the star background.

The sun's noon altitude at a particular place varies from day to day over a yearly cycle. The range of this variation in altitude in temperate latitudes is 47°. For instance, at Sydney in Australia, at a latitude of 34° South, the sun's maximum midday altitude is $79\frac{1}{2}^{\circ}$ in midsummer, which occurs late in December, and its minimum noonday altitude is $32\frac{1}{2}^{\circ}$ in midwinter six months later.

2.14 The earth, which is the observer's platform, is a planet. Like other planets, it travels round the sun in an orbit, which is slightly elliptical with the sun at one of the focal points. The earth's path round the sun defines the orbital plane and the elliptical path, which is very nearly circular, has an average radius of approximately 150 000 000 kilometres.

The earth itself approximates closely to a sphere of radius 6 380 kilometres. It spins round its own axis once in a day, which motion produces for all persons, outside the Arctic Circle or the Antarctic Circle, alternate periods of daylight and darkness. In addition, the earth travels round its orbit once in a year.

The moon is a satellite of the earth, around which it travels at an average distance of approximately 390 000 kilometres. If the planets are observed by means of a telescope, it will be seen that Jupiter has four main satellites, which are easily visible, and that Saturn has a flat ring encircling it.

2.15 The stars, with the exception of one, which is the sun, are incredibly distant from the earth. Even though they are, or may be, moving with high individual velocities in individual directions with respect to each other, these movements have, over the period of recorded history, produced no obvious changes in the constellation patterns.

Light from the sun takes about 8 minutes to travel from sun to earth. Light from the next nearest star, Proxima Centauri, takes $4\frac{1}{2}$ years to travel from this star to the earth. This gives the information that the 150 000 000 kilometre radius of the earth's orbit subtends, at this star, the minute angle of three quarters of a second of arc. From this information, one may deduce that the earth's orbit can be considered to have a point dimension at the centre of a sphere of infinite radius and that the stars may, for all practical purposes, be considered to be situated on this sphere, which is known as the Celestial Sphere. On this account, therefore, the earth and the sun may be considered to lie at the centre of the celestial sphere.

It is interesting to note here that, while most people subscribe to the heliocentric view of Copernicus, the pre-Copernican idea of a geocentric system is used as the model for much explanation in astronomy. The success of this is due to the fact that the observer has all the time a geocentric view of the sky and the celestial bodies.

Reference Circles on the Surface of the Earth

2.21 Any plane, intersecting a sphere and containing its centre, cuts its surface along a circle of radius equal to that of the sphere. Such a circle is a Great Circle. Any plane, which intersects the sphere, but does not contain its centre, cuts its surface along a circle of radius less than that of the sphere. (see Fig 2.1(a)) Such a circle is a Small Circle. *The relationships of spherical Trigonometry apply only to Great Circles.*

Two intersecting planes A and B, containing the centre of a sphere, produce, at their intersection, a straight line, which also is a diameter of the sphere. If a third plane C intersects planes A and B and has their line of intersection

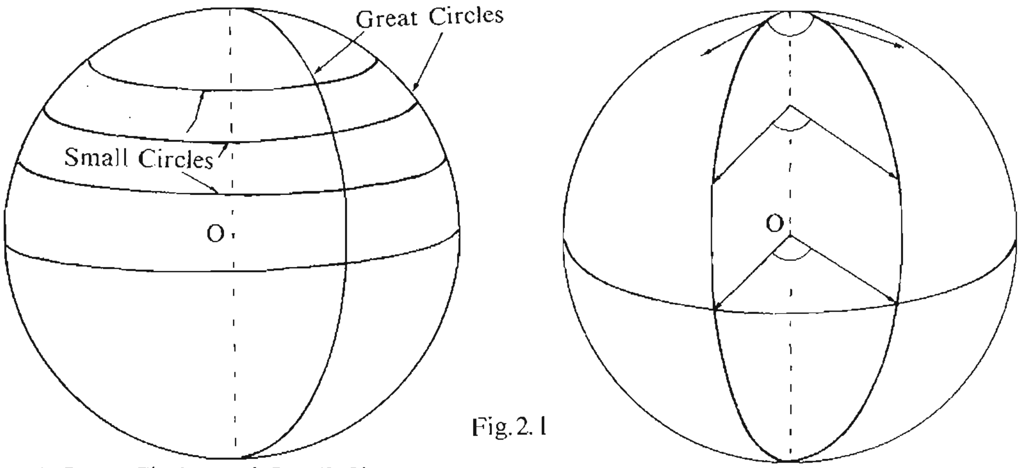


Fig.2.1

(a) Great Circles and Small Circles (b) The Dihedral Angle

as a normal, two additional lines of intersection are produced. The angle between these two lines is the Dihedral Angle between the planes A and B. This dihedral angle is produced, wherever plane C may lie, so long as it has the original line of intersection as its normal. When C is tangential to the sphere, the two lines defining the dihedral angle are tangents to the two great circles formed by the intersection of planes A and B with the sphere. Since these two circles are also tangential to the plane C, the dihedral angle is equal to the spherical angle on the sphere between these two great circles. (see Fig 2.1(b))

Any plane, which contains the earth's rotational axis, cuts its surface along a great circle called a Meridian. All meridians (see Fig 2.1(c)) therefore pass through the two terrestrial poles and the angle at each pole between any two meridians is equal to the dihedral angle between them. The meridian, which passes through Greenwich, has been selected as the Prime Meridian. Any other meridian is referred to the prime meridian by quoting its dihedral angle from the prime meridian. This quantity is the Longitude λ , which will be assumed to be positive eastwards round from the Greenwich Meridian and negative westwards round. In practice, longitudes are usually quoted from Greenwich east or west from 0° to 180°. Longitude east will be denoted by a positive sign or by means of the letter E, and a west longitude by a negative sign or by the letter W. (see Fig 2.1(d))

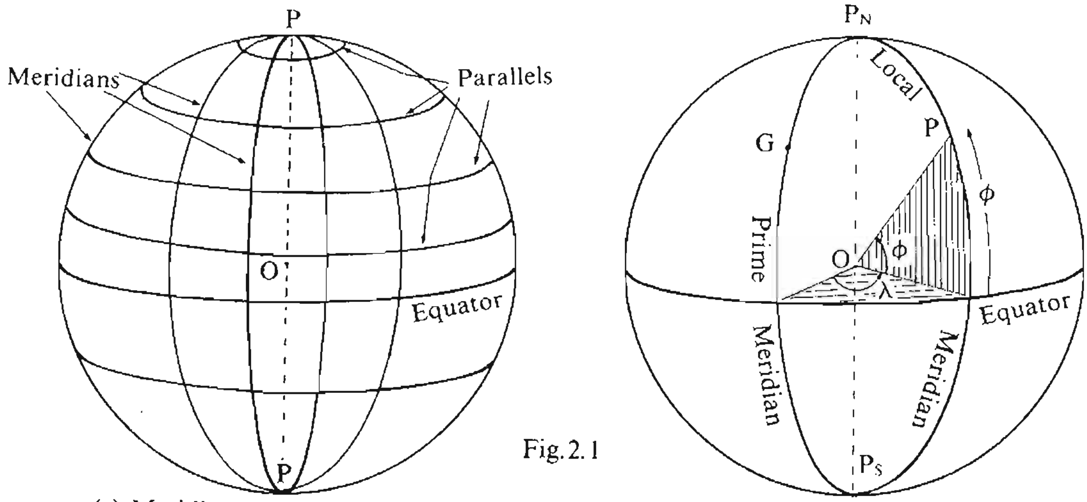


Fig.2.1

(c) Meridians and Parallels (d) Latitude and Longitude

2.22 Any plane, which intersects the earth and has its rotational axis as a normal, cuts the earth's surface in a circle called a Parallel of Latitude. One of these is a great circle known as the Equator, which is the prime parallel of latitude. The others are all small circles.(see Fig 2.1(c)).

The position of a place on the earth can be defined by specifying the meridian and the parallel, on which it lies. This is equivalent to giving the angle λ between the Greenwich and the local meridians, which are both great circles. This has been defined above as the longitude of the place. Since parallels of latitude are small circles, produced by a set of parallel planes, their positions, relative to the equator, cannot be defined by the angle between them, but must be defined by means of an arc length on the sphere along the local meridian from the equator to the local parallel. This arc length is known as the latitude ϕ. Since the length between two points on a sphere is defined as the angle, which the arc of a great circle between these two points subtends at the centre of the sphere, the latitude ϕ is therefore the angle, which its arc length, defined immediately above, subtends at the earth's centre (see Fig 2.1(d)). Latitude starts from zero at the equator and is considered to be positive northwards, negative southwards. Latitude north may, therefore, be denoted by a positive sign or by means of the letter N beside the value, and a south latitude by a negative sign or the letter S.

Reference Circles on the Celestial Sphere

2.31 The earth rotates about its own axis. If this terrestrial axis is produced outwards in both directions into the sky, it will intersect the celestial sphere at two points known respectively as the North Celestial Pole and the South Celestial Pole.

The stars appear to rotate about these poles. In the constellation of the Lesser Bear, there is a bright star (α Ursae Minoris) of magnitude 2.1 called Polaris, because it lies very close to the north celestial pole. As a result of this, it has held a very special place in man's reckoning as it shows continuously where north is and, for an observer at a particular station, it maintains its altitude practically unaltered unlike other stars. This is not so, in the southern hemisphere, where there is a dearth of visible stars in the vicinity of the south celestial pole. However, there is a faint star, σ Octantis, of magnitude 5.5, within one degree of the south pole. It cannot, like the northern Pole Star, be seen easily by the naked eye but requires a telescope, with which it may be viewed at night.

The earth's axis of rotation very nearly maintains its direction in space. In other words the earth's axis points to a certain spot in the sky and only departs from this spot at a very slow rate indeed, except for minor periodic effects. In other words, the two pole stars cited above will remain close to the celestial pole for many years to come.

If the earth's Equatorial Plane is extended out to cut the celestial sphere, it will intersect this along a great circle called the Celestial Equator, which lies mid-way between the Celestial Poles. All planes parallel to the equatorial one will cut the celestial sphere in small circles, called Parallels of Declination. The celestial equator is, of these, the only one which is a great circle and it therefore is taken as the prime declination circle, to which the others are referred.

2.32 Fig 2.2 shows a plan view of the earth's orbit with the sun at one focus point of this elliptical path. Its eccentricity is much exaggerated in this diagram, as the actual orbit differs only slightly from a circle. The north pole of the earth is shown to project upwards out of the plane of the paper at an oblique angle to the orbital plane. The upper side of the earth's equator is also shown. The earth's axis maintains a constant spatial direction. The earth moves round its orbit from A to B to C to D and back to A in one year. The earth-sun radius vector is the line from earth to sun and therefore, for an observer on the earth, the sun appears to lie always on the orbital plane, projected out on to the celestial sphere. This great circle is known as the Ecliptic. On account of the earth's movement round its orbit, the sun therefore appears to traverse the ecliptic once a year and therefore to shift across the star background in this time.

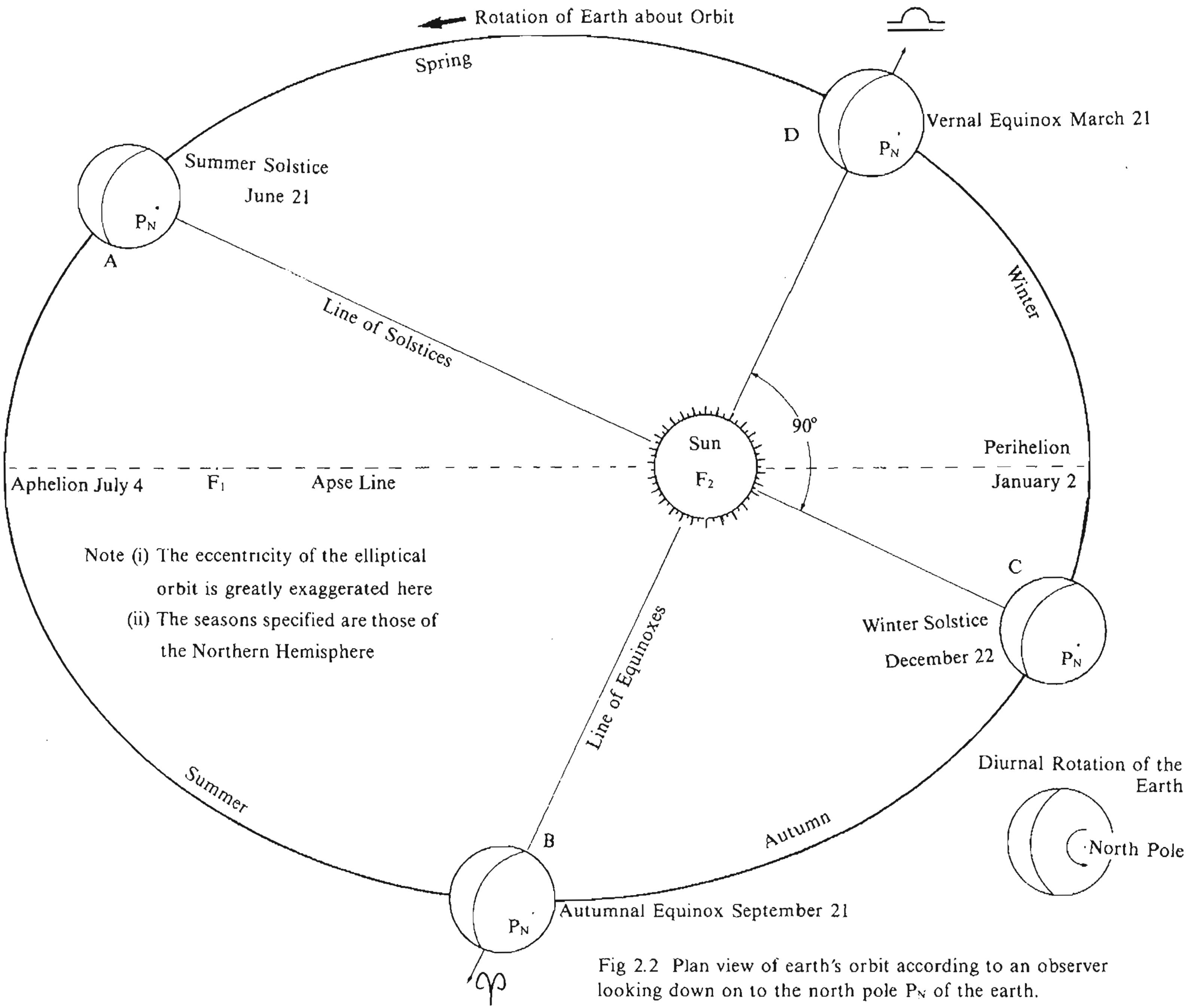


Fig 2.2 Plan view of earth's orbit according to an observer looking down on to the north pole P_N of the earth.

The seasons are produced not because the sun is sometimes closer to and sometimes further from the earth, but because the earth's axis of rotation is inclined and not perpendicular to the plane of its orbit. When the earth is on the side towards A of the Equinoctial Line BD, the sun is north of the equator and it shines with increasing intensity on the northern hemisphere of the earth until it reaches its maximum distance north of the equator at the summer solstice in June. The sun then comes back to the equator when the earth moves towards the equinoctial point B and summer passes through autumn, with the sun moving to a position south of the equator. Winter now ensues with the sun shining less intensely on the northern hemisphere. (see Fig 2.2) *It should be noticed that all these seasonal terms refer to the Northern Hemisphere.*

2.33 The information in section 2.15 indicates that the radius of the earth's orbit, large as it may seem to the earth dweller, may be considered to be infinitesimally small as a length by astronomical standards. The whole orbit may therefore be considered to be so small that any point in it may be taken as lying at the centre of the celestial sphere. (see Fig 2.3)

The ecliptic is the great circle produced, on the celestial sphere, by extending the orbital plane to intersect this sphere. This has poles as shown lettered EP_N and EP_S. The earth's rotational axis produced out intersects the celestial sphere as shown at P_N and P_S. This axis is not normal to the

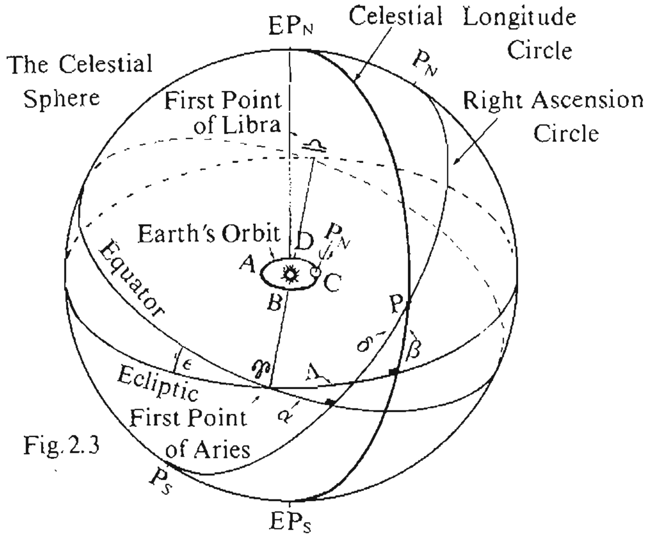


Fig. 2.3

orbital plane, but deviates from this normal by about $23\frac{1}{2}^{\circ}$ between it and the ecliptic and there is a line of intersection between the two planes. This line of intersection defines the two points of intersection which, on the celestial sphere are shown as the two equinoctial points. The point at B in Fig. 2.2 is projected out onto the ecliptic. This point (see Fig 2.3) is the one, where the sun appears to be at the Vernal Equinox when it is crossing the equator from south to north. This point is known as the First Point of Aries, and is indicated by the zodiacal symbol of the ram's horns. Opposite the vernal equinoctial point is the First Point of Libra which is indicated by the symbol of the scales.

2.34 If a series of planes, each containing the earth's axis is extended out to intersect the celestial sphere, each will produce a great circle of Right Ascension, and each will also intersect the two celestial poles. That one, which also passes through the First Point of Aries, can be considered as the prime right ascension circle. This circle has a value of zero right ascension. The right ascension value of any other such circle is given by the dihedral angle between it and the prime right ascension circle. The direction of this numbering system is such that the right ascension circles cross a given meridian in the order of their numbering. A useful method of remembering is to consider that right ascension increases towards the east.

The declination of a point P is equal to the angle subtended at the centre of the celestial sphere by the arc length along the right ascension circle through P from the equator to the point P. If north, it is lettered N or North and, if south, S or South (see Fig 2.4). The position of a point on the celestial sphere can now be specified by defining the parallel of declination and the right ascension circle on which it lies. Such coordinates are known as right ascension RA_h or α and declination δ. The former is given in time units from 0^h to 24^h and the latter in degrees from 0° to 90°.

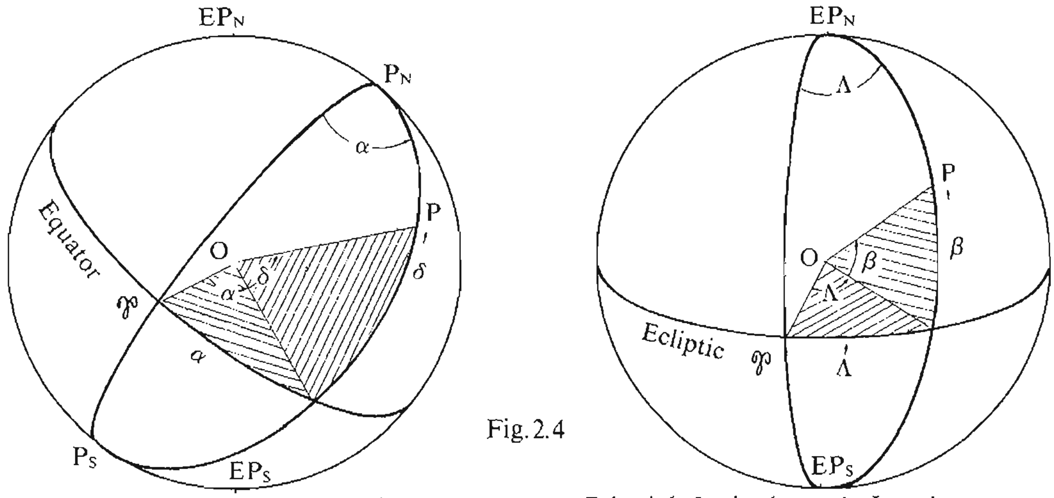


Fig. 2.4

Right Ascension and Declination

Celestial Latitude and Longitude

It is obvious from these explanations that terrestrial latitude and longitude are not the *same* as, but are exactly *analogous* to, right ascension and declination. The first is the system for defining a *terrestrial* position and the second that for defining a *celestial* position. Each of these systems is independent of time because, for each, the reference circles are carried around together with the surface, to which the system refers.

2.35 There is, on the celestial sphere, a secondary system (see Fig 2.4) for defining position on it. In this system any plane parallel to the ecliptic cuts the celestial sphere, in a series of parallels of celestial latitude having the ecliptic as the reference parallel of zero celestial latitude. Northern celestial latitudes are labelled N or North and southern ones S or South. The circles on the celestial sphere produced by planes containing the poles of the ecliptic provide a set of celestial longitude circles. That one passing through the First Point of Aries is the zero celestial longitude circle and the longitude values increase also towards the east. (see Fig 2.4)

The position of a point on the celestial sphere is specified by defining the parallel of celestial latitude and the circle of celestial longitude, on which it lies. The symbols used for celestial longitude will be Λ and for celestial latitude β.

Observation Circles linking the Terrestrial and the Celestial Spheres

2.41 The two main systems, one of latitude and longitude on the earth and the other of right ascension and declination on the celestial sphere have been set out. They have the earth's axis of rotation common to them. Due to the earth's rotation, there appears to be a relative motion of one system with respect to the other. This can be expressed either way, because the motion is relative, in the statement that the sky rotates about the earth from east to west or that the earth rotates from west to east with respect to the sky. The relationship between these two systems is a straightforward time rotation one. The time system relationships will be dealt with in Chapter 3.

2.42 In addition, there is the surveyor's or the observer's system for defining the position of a point on the celestial sphere. This is a gravity dependent system, which uses the local vertical as the reference line and the local meridian as the reference direction. It has long been used because the simple levelling bubble made it possible to define the vertical and the horizontal so easily.

The surveyor's theodolite is the instrument constructed to measure in this system. The theodolite has a vertical axis, which can easily be set very closely, but never exactly, except by occasional chance, into the vertical at any station. Attached to the vertical axis is a graduated circle, so constructed as to be horizontal when the vertical axis is set vertical, and a vertical circle is also attached so that it then occupies a vertical plane.

When the theodolite is set up, i.e. when its vertical axis is set vertical, horizontal directions and vertical angles can be observed. If now the horizontal circle is set to read zero when the telescope is pointed northwards along the meridian through the theodolite, the observer has a reference system for setting out, by means of a horizontal circle reading and a vertical circle reading, any point he wishes to define. Most modern theodolites have vertical circles so graduated that on one face the zero coincides with the zenith, 90° & 270° with the horizontal and 180° with the nadir.

2.43 The zenith is defined as the point on the celestial sphere where the vertical axis of the theodolite, projected upwards, intersects it. Similarly, the nadir is the point on the celestial sphere where this axis, projected downwards, intersects the celestial sphere. A plane, passing through the zenith and the nadir of a particular station, cuts the celestial sphere in a great circle called a Vertical or an Azimuth Circle. That circle, which coincides with the northern branch of the local meridian, is the zero azimuth circle. The azimuth numbering then increases from this zero northwards to 90° towards the east and so right round to 360°. That azimuth circle, perpendicular to the meridian, is called the Prime Vertical and its azimuth eastwards is 90° and westwards 270°.

If a theodolite is set up and levelled, the telescope clamped at a specific angle of altitude and the whole alidade rotated about the vertical axis, the line of sight in the telescope will then describe a small circle of specific altitude on the celestial sphere. Such a family of circles produces a set of

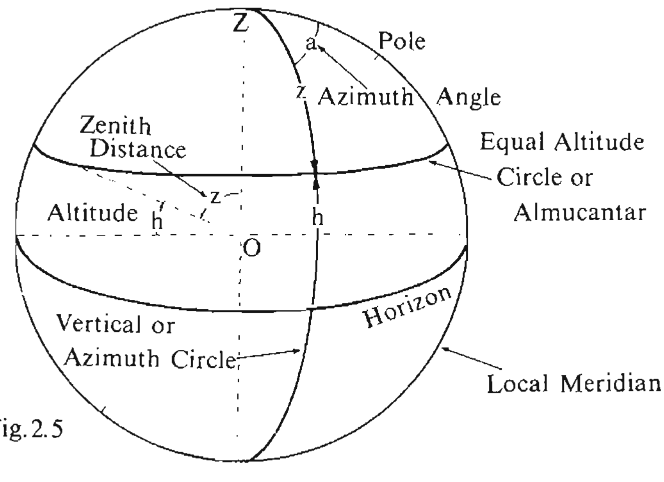


Fig. 2.5

parallels of altitude, with the main circle the great circle perpendicular to the vertical axis and therefore having zero altitude. This circle is the sensible horizon, which, as far as referring it to the celestial sphere is concerned, is the same as the plane parallel to the horizon and passing through the earth's centre. This distinction must not be forgotten when the comparatively close sun, moon and planets are observed, when a correction (parallax) must be applied to observations of altitude made from the earth's surface instead of, as they should be, from the earth's centre (see section 4.54).

The parallels of altitude are often more conveniently dealt with by referring them to the zenith and providing the zenith distance instead of the altitude. From this, it is clear that altitude h and zenith distance z

are complementary quantities, i.e. z = 90-h. A specific parallel of altitude is often called an equal altitude circle or an almucantar. The angle a at the zenith between the meridian towards the elevated pole and the vertical circle through the star is the azimuth angle (see Fig 2.5). The observer can now use this system for defining a point by specifying its azimuth or its azimuth angle and its altitude or zenith distance.

The Link between the Systems

2.51 The crux in the understanding of field astronomy lies in the understanding of the linking up of the celestial systems of right ascension and declination and of celestial latitude and longitude with those of azimuth and altitude.

2.52 The local meridian of a place or station P is that one passing through P and the two terrestrial poles. The plane of this circle, extended outwards to the celestial sphere, will therefore pass through the two celestial poles and also through the zenith and the nadir of the station P. These two points are those, at which the line containing the vertical axis of a theodolite set up at P will intersect the celestial sphere.

The upper branch of this local meridian goes from one pole through the zenith of P to the other pole, while its lower branch likewise goes from one pole through the nadir of P to the other one. Half the local meridian circle is visible from the station P. This semi-circle goes from one side of the horizon through the elevated celestial pole and the zenith of P to the other side of the horizon. (see Fig 2.6)

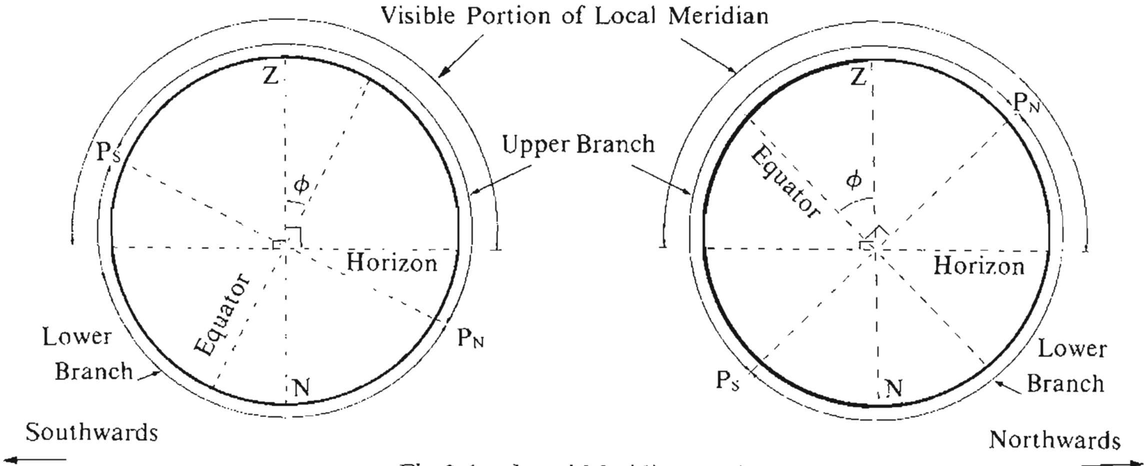


Fig. 2.6 Local Meridian Section

Of the upper branch of the local meridian, that portion from the elevated celestial pole over the zenith and down to the horizon is visible from P. In the opposite direction from this pole, that section of the local meridian visible from the elevated celestial pole down to the horizon is part of the lower branch of this meridian. A star, crossing over the upper branch of the meridian is said to be making its Upper Transit, whereas one crossing the lower branch is said to be making its Lower Transit. At both these times, the star is moving horizontally, but for the former it moves from east to west and for the latter in the opposite direction.

The local meridian of P sweeps continuously across the celestial sphere. For clarity and easier understanding, the meridian may be considered stationary and the stars to be moving across it.

2.53 A plane, passing through the two celestial poles and a star, cuts the celestial sphere in a great circle called an Hour Circle. Such a circle is similar to a right ascension circle but it is unnumbered. The dihedral angle between an hour circle through a particular celestial body and a local meridian at a particular moment is the local hour angle of this body at this instant

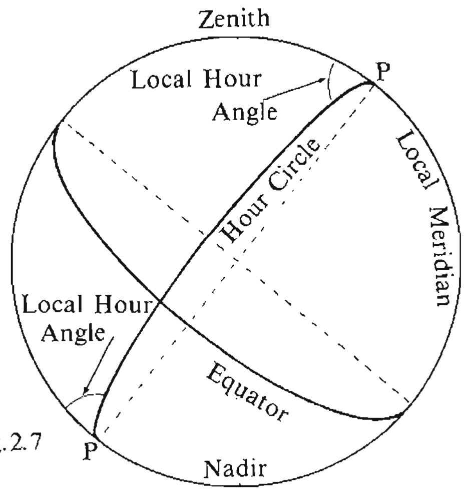


Fig. 2.7

(see Fig 2.7). This angle is measured from the meridian as zero and it varies directly with time, because the celestial body is continuously rotating with respect to the local meridian.

2.54 It is now necessary to bring the various systems together in Fig 2.8 to be able to consider their relationships with respect to each other. Fig 2.6 shows a particular meridian, on which lie the celestial pole P and the zenith Z. The two great circles equator and horizon, of which these two

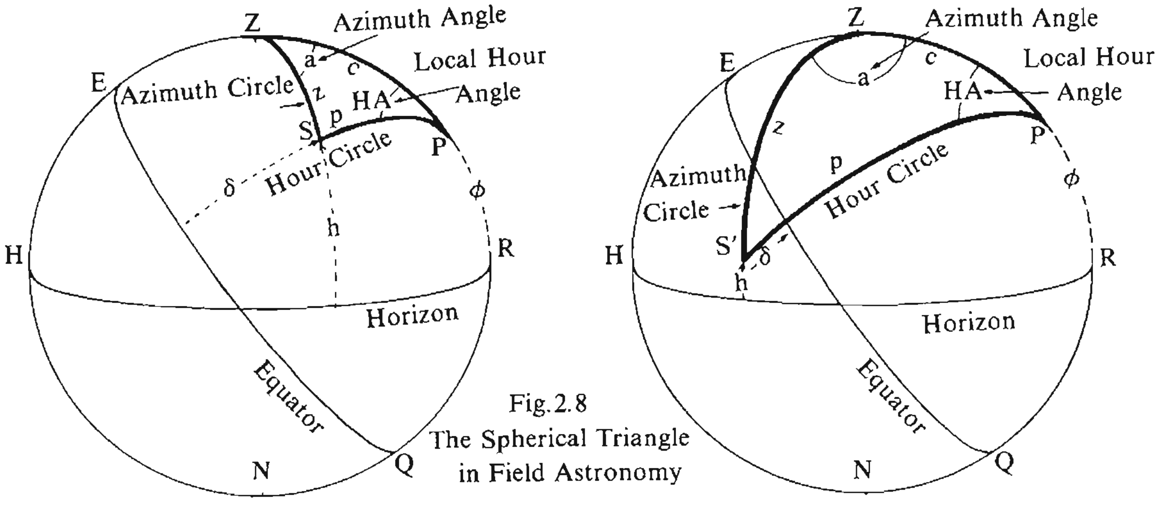


Fig. 2.8
The Spherical Triangle
in Field Astronomy

points are respectively the poles, are also shown. Fig 2.7 shows the hour circle and the local hour angle with respect to the local meridian. The length along the meridian from R to P in Fig 2.8 is equal to the altitude of the elevated pole. A glance at Fig 2.6 enables one to demonstrate that "the altitude of the elevated pole at a specific station is equal to the latitude at this station". In this statement, the sign of the latitude value is dispensed with.

THE SPHERICAL TRIANGLE OF FIELD ASTRONOMY

2.61 AS Fig 2.8 shows, this triangle on the sphere is the spherical triangle PZS bounded by portions of a local meridian, an hour circle and an azimuth circle. The apex points of this triangle are the elevated pole, the observer's zenith and a star. In it are linked the three systems of altitude and azimuth, right ascension and declination as well as latitude and longitude. It should be noted that this spherical triangle could be one in which the star S did not lie on the same side of the equator as the observer and his elevated pole. This is shown by placing a second star at S' in Fig 2.8. The spherical triangle would then be the one with apex points at P, Z and S' and, in this case, the zenith distance and polar distance would cross over the equator. Normally, in this system, the observer always uses the elevated pole. He therefore changes his elevated pole on crossing the equator. Furthermore, the restriction is imposed that no element of this spherical triangle shall exceed 180°.

The spherical triangle of field astronomy therefore is defined by the three apex points, *elevated* pole, zenith and star. The lengths of its sides are then:-

ZS or ZS'	=	zenith distance	z	=	90 - h
PZ	=	colatitude	c	=	90 - ϕ
PS or PS'	=	polar distance	p	=	90 $\pm$ δ

in which ϕ, δ and h are considered to be unsigned quantities and the positive sign in the expression for p is used only when δ and ϕ are of contrary name, i.e. they are on opposite sides of the equator.

The elements in this spherical triangle must be deduced from the astronomical elements, e.g. if the azimuth is given as 330°, the azimuth angle a in the corresponding spherical triangle is either 30° or 150°. Similarly elements obtained from the solution of the spherical triangle must be translated into the corresponding astronomical elements.

The above method can be used, but the solutions from the spherical triangle are encumbered with sets of special rules for the various situations encountered. The restrictions of the spherical triangle and the constant manipulation of the information for, and the answers of, each solution from this triangle make the method laborious and liable to error. The cumbersome navigation tables and the need for differentiating between the case of "Latitude and Declination of the Same Name" and that of "Latitude and Declination of Opposite Name" are being rendered obsolete now because computations are being carried out by means of the electronic computer. For this reason, it is better to develop a system, which copes automatically with all the possible variations of the spherical triangle, encountered in astronomy. For these and other reasons, which will be seen as the following sections are read, a generalized system for solving the Astronomical Triangle has been worked out and expanded further ahead in this chapter.

The Relationships of Spherical Trigonometry

2.62 For the computation, the equations of spherical trigonometry are required. Because they will be constantly referred to, they are given here. In the spherical triangle WXY the following relationships hold:

Cosine Formula $$\cos w = \cos x \cos y + \sin x \sin y \cos W$$

Five Parts Formula $$\sin w \cos X = \cos x \sin y - \sin x \cos y \cos W$$

Sine Formula $$\frac{\sin W}{\sin w} = \frac{\sin X}{\sin x} = \frac{\sin Y}{\sin y}$$

Four Parts Formula $$\cot y \sin x = \cot Y \sin W + \cos x \text{ cox } W$$

Polar Cosine Formula $$-\cos W = \cos X \cos Y - \sin X \sin Y \cos w$$

Polar Five Parts Formula $$\sin W \cos x = \cos X \sin Y + \sin X \cos Y \cos w$$

Differential Relationships

$$\sin w \, dY = -\cos X \sin y \, dW - \cos w \sin Y \, dx + \sin X \, dy$$

$$dw = \sin x \sin Y \, dW + \cos Y \, dx + \cos X \, dy$$

$$-dW = -\sin X \sin y \, dw + \cos y \, dX + \cos x \, dY$$

$$\cos w \sin X \, dw = -\sin w \cos X \, dX + \cos x \sin W \, dx + \sin x \cos W \, dW$$

Additional Formulae

Half Angle Formulae

$$\sin^2 \tfrac{1}{2}W = \sin(s-x) \sin(s-y) \operatorname{cosec} x \operatorname{cosec} y$$

$$\cos^2 \tfrac{1}{2}W = \sin s \quad \sin(s-w) \operatorname{cosec} x \operatorname{cosec} y$$

$$\tan^2 \tfrac{1}{2}W = \sin(s-x) \sin(s-y) \operatorname{cosec} s \operatorname{cosec}(s-w)$$

in which $2s = w + x + y$

Napier's Analogies

$$\tan \tfrac{1}{2}(X+Y) = \cos \tfrac{1}{2}(x-y) \sec \tfrac{1}{2}(x+y) \cot \tfrac{1}{2}W$$

$$\tan \tfrac{1}{2}(X-Y) = \sin \tfrac{1}{2}(x-y) \operatorname{cosec} \tfrac{1}{2}(x+y) \cot \tfrac{1}{2}W$$

$$\tan \tfrac{1}{2}(x+y) = \cos \tfrac{1}{2}(X-Y) \sec \tfrac{1}{2}(X+Y) \tan \tfrac{1}{2}w$$

$$\tan \tfrac{1}{2}(x-y) = \sin \tfrac{1}{2}(X-Y) \operatorname{cosec}\tfrac{1}{2}(X+Y) \tan \tfrac{1}{2}w$$

The polar form of any one of the formulae of spherical trigonometry can be derived by substituting 180-W for W, 180-x for X etc. This has been done below in the Cosine Formula as an illustration. It should be noticed that the Four Parts and the Sine Formulae transform back into themselves when this process of changing an angle for a side, and vice versa, is carried out.

The Polar Cosine Formula, for instance, is obtained by making the above substitution in the Cosine Formula

$$\cos w = \cos x \cos y + \sin x \sin y \cos W$$

as follows:

$$\cos(180-W) = \cos(180-X) \cos(180-Y) + \sin(180-X) \sin(180-Y) \cos(180-w)$$

$$-\cos W = \cos X \cos Y - \sin X \sin Y \cos w$$

For the solution of a spherical triangle, four elements, out of the total of six, must be linked in one of the spherical trigonometry relationships. These four elements may be made up of the following sets:-

Case I
- (a) Three sides and one angle
- (b) Three angles and one side.

Case II

Two angles and two sides with
- (a) four cyclically consecutive elements
- (b) two angular elements lying opposite two side elements.

Table 2.2 below sets out the twelve various ways of solving for the fourth unknown, when three of the elements in the spherical triangle WXY are given.

Table 2.2

No.	Given	Sought	Solution
1	Two sides x,y and included angle W	Angle opposite one of the two sides x,y	Four Parts Formula
2	" "	Third side w	Cosine Formula
3	Two angles X,Y and included side w	Side opposite one of the two angles X,Y	Four Parts Formula
4	" "	Third angle W	Polar Cosine Formula
5	Three sides w,x,y	Angle W	Cosine Formula
6	Three angles W,X,Y	Side w	Polar Cosine Formula
7	Two sides w,x and angle W opposite w	Angle X opposite other side x	*Sine Formula
8	" "	Angle Y contained between two sides w,x	**Four Parts Formula in implicit form for Y
9	" "	Third side y	**Cosine Formula in implicit form for y
10	Two angles W,X and a side x opposite X	Side w opposite W	*Sine Formula
11	" "	Side y not opposite either W or X	**Four Parts Formula in implicit form for y
12	" "	Third angle Y	**Polar Cosine Formula in implicit form for Y

* In these two cases, the ambiguity may be resolved from the rule that, according as the sum(w+x) is greater or less than 180°, so the sum(W+X) is greater or less than 180°.

** These formulae are all in the implicit form for the unknown sought. They can be solved by dropping perpendiculars from apex Y to base y and solving the two right angled triangles from which the unknown can be obtained (see section A.41 in the appendix for one such example). Ambiguities must then be resolved by means of some additional piece of information.

Therefore, when a solution is being sought, it is necessary to note what information has been given and what is required, to fit this into the twelve possible cases given above and then to name the elements by lettering the apex points of the spherical triangle with W, X and Y at the appropriate points.

2.63 The differential relationships of section 2.62 give the result of small variations in the elements of the spherical triangle. The first, the second and the fourth relationships are obtained by differentiation of the Four Parts, the Cosine and the Sine Formula respectively (see section A.31 in the appendix for this detail). The third relationship is the polar form of the second one. If this method is applied to the first and the fourth relationships, they come back to themselves, just as do the functions, from which these differential relations are derived.

THE ASTRONOMICAL TRIANGLE OF FIELD ASTRONOMY AND THE GENERALIZED CONVENTIONS IN THIS TRIANGLE

2.71 THE elements of the spherical triangle of field astronomy of Fig 2.8 and of section 2.61 have been generalized and conventionalized in the astronomical triangle of field astronomy to develop a more efficient system of calculation and mathematical manipulation of these elements and to take advantage of the great power of the electronic computer.

Some of these elements have already been conventionalized, e.g. latitude and declination have long been considered positive north and negative south of the equator and right ascension likewise has been taken to increase eastwards. The implications of these conventions are, however, not so obvious. They are that the first two are quantities, which exist only in the first and the fourth quadrants, whereas right ascension exists in all four.

The further conventions postulated below give rise to the generalized spherical triangle, which, it is proposed, should be called the Astronomical Triangle. In it, these generalized conventions liberate this triangle from the restrictions and anomalies of the spherical triangle of field astronomy to produce a much more flexible and a much more efficient system.

The Astronomical Triangle

2.72 This is the spherical triangle with the apex points at either of the celestial poles (which may or may not be the observer's elevated pole) the observer's zenith and the star (see Fig 2.9). In this triangle, the viewer is considered to be looking down on to the observer's zenith from outside the celestial sphere. Both the northern and the southern astronomical triangles are shown and also the situations of star west and of star east of the meridian.

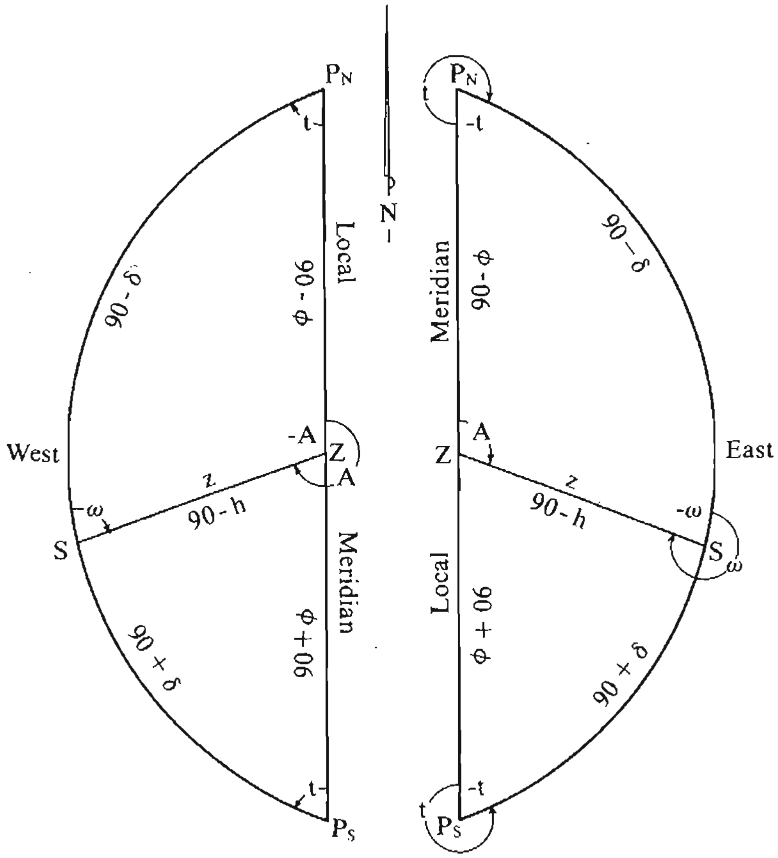


Fig. 2.9 The Astronomical Triangle

In this triangle, neither sides nor angles need be restricted in size. The generalized conventions enable the astronomical triangle to be solved without ambiguity, even if the computation or the manipulation is complex, provided these conventions are used and the resulting signs of the trigonometrical functions are carefully followed through the computation (see section 2.73). The generalized system postulated above succeeds in releasing the self-imposed bonds, inherent in working with the spherical triangle of field astronomy, but the ambiguous solutions, indicated in Table 2.2, still remain present in both systems, except that this has been overcome in the example for the general latitude solution as shown in section 5.21.

2.73 The definitions and conventions for the generalized system are given below.

Name	Symbol	Definition and Sign Convention	Possible Quadrants
Longitude	λ	Angle at terrestrial poles or dihedral angle between local meridian and Greenwich Meridian; positive eastwards from the Greenwich Meridian as zero	1 2 3 4
Latitude	ϕ	Angle subtended at centre of the terrestrial sphere by arc of meridian from equator to point specified; positive northwards and negative southwards from equator as zero	1 - - 4
Right Ascension	RA or α	Angle at celestial poles or dihedral angle between the hour circle of the First Point of Aries and that of a point specified; positive eastwards round from zero at the First Point of Aries	1 2 3 4
Declination	δ	Angle subtended at centre of the celestial sphere by arc of hour circle from equator to point specified, positive northwards and negative southwards from equator as zero	1 - - 4
Local Hour Angle	t	Angle at poles or dihedral angle between the local meridian and the hour circle through the point specified; positive westwards from this meridian as zero	1 2 3 4
Azimuth	A	Angle at zenith or dihedral angle between the northern branch of the local meridian and the azimuth circle to the point specified; positive eastwards round from north as zero	1 2 3 4
Altitude	h	Vertical angle from horizon upwards to the zenith as positive and downwards to the nadir as negative with the horizon as zero	1 - - 4
Zenith Distance	z	Vertical angle downwards from zenith to nadir with zenith as zero	1 2 - -
Parallactic Angle	ω	Angle at star between its hour circle northwards as zero eastwards round to the azimuth circle through the star	1 2 3 4

2.74 The spherical trigonometry relationships of section 2.62 can now be generalized for use in the astronomical triangle by substituting in them the elements of this triangle as shown in Fig 2.9. Any one of the four possible variants may be used as starting point and, if the relevant elements are substituted, *the same end result will be obtained*. This is demonstrated for the four cases with respect to one of the spherical trigonometry relationships (see appendix section A.21 for manipulation of complex angles and their trigonometrical functions).

In the northern astronomical triangle, the Sine Formulae become

(i)	to the west	$\frac{\sin t}{\sin(90-h)} = \frac{\sin(360-A)}{\sin(90-\delta)} = \frac{\sin \omega}{\sin(90-\phi)}$ $\frac{\sin t}{\cos h} = -\frac{\sin A}{\cos \delta} = \frac{\sin \omega}{\cos \phi}$
(ii)	to the east	$\frac{\sin(360-t)}{\sin(90-h)} = \frac{\sin A}{\sin(90-\delta)} = \frac{\sin(360-\omega)}{\sin(90-\phi)}$ $\frac{\sin t}{\cos h} = -\frac{\sin A}{\cos \delta} = \frac{\sin \omega}{\cos \phi}$

In the southern astronomical triangle, these become

(iii)	to the west	$\frac{\sin t}{\sin(90-h)} = \frac{\sin(A-180)}{\sin(90+\delta)} = \frac{\sin(180-\omega)}{\sin(90+\phi)}$ $\frac{\sin t}{\cos h} = -\frac{\sin A}{\cos \delta} = \frac{\sin \omega}{\cos \phi}$
(iv)	to the east	$\frac{\sin(360-t)}{\sin(90-h)} = \frac{\sin(180-A)}{\sin(90+\delta)} = \frac{\sin(\omega-180)}{\sin(90+\phi)}$ $\frac{\sin t}{\cos h} = -\frac{\sin A}{\cos \delta} = \frac{\sin \omega}{\cos \phi}$

By means of a similar approach, the same can be shown to hold for any of the other relationships of section 2.62 to obtain generalized relationships for the astronomical triangle.

Because so much calculation or computation is required in a course of field astronomy, it is considered to be an advantage to have these generalized relationships readily available. The substitutions have therefore been made for the frequently needed cases and these are given below in section 2.75. The derivations of first and second differential relationships are given in sections A.31 to A.33.

2.75 Generalized Spherical Trigonometry Formulae for Use in the Astronomical Triangle

Cosine Formula	$\sin\delta = \sin h \sin\phi + \cos h \cos\phi \cos A$ $\sin\phi = \sin\delta \sin h + \cos\delta \cos h \cos\omega$ $\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t$
Polar Cosine Formula	$-\cos A = \cos t \cos\omega - \sin t \sin\omega \sin\delta$ $-\cos\omega = \cos A \cos t + \sin A \sin t \sin\phi$ $-\cos t = \cos\omega \cos A + \sin\omega \sin A \sin h$
Five Parts Formula	$\cos\delta \cos t = \sin h \cos\phi - \cos h \sin\phi \cos A$ $\cos\delta \cos\omega = \sin\phi \cos h - \cos\phi \sin h \cos A$ $\cos\phi \cos A = \sin\delta \cos h - \cos\delta \sin h \cos\omega$ $\cos\phi \cos t = \sin h \cos\delta - \cos h \sin\delta \cos\omega$ $\cos h \cos\omega = \sin\phi \cos\delta - \cos\phi \sin\delta \cos t$ $\cos h \cos A = \sin\delta \cos\phi - \cos\delta \sin\phi \cos t$
Polar Five Parts Formula	$-\sin A \sin h = \cos t \sin\omega + \sin t \cos\omega \sin\delta$ $-\sin A \sin\phi = \cos\omega \sin t + \sin\omega \cos t \sin\delta$ $-\sin\omega \sin\delta = -\cos A \sin t + \sin A \cos t \sin\phi$ $-\sin\omega \sin h = \cos t \sin A - \sin t \cos A \sin\phi$ $-\sin t \sin\phi = \cos\omega \sin A - \sin\omega \cos A \sin h$ $-\sin t \sin\delta = -\cos A \sin\omega + \sin A \cos\omega \sin h$
Four Parts Formula	$\tan\delta \cos\phi = \sin\phi \cos t - \sin t \cot A$ $\tan\delta \cos h = \sin h \cos\omega - \sin\omega \cot A$ $\tan\phi \cos h = \sin h \cos A - \sin A \cot\omega$ $\tan\phi \cos\delta = \sin\delta \cos t + \sin t \cot\omega$ $\tan h \cos\delta = \sin\delta \cos\omega + \sin\omega \cot t$ $\tan h \cos\phi = \sin\phi \cos A - \sin A \cot t$
Sine Formula	$\sin t/\cos h = -\sin A/\cos\delta = \sin\omega/\cos\phi$
Differential Relationships	$dh = \cos\phi \sin A\, dt + \cos A\, d\phi + \cos\omega\, d\delta$ $d\phi = -\cos\delta \sin t\, d\omega + \cos t\, d\delta + \cos A\, dh$ $d\delta = \cos\phi \sin t\, dA + \cos\omega\, dh + \cos t\, d\phi$ $dA = \sec h \cos\omega \cos\delta\, dt + \tan h \sin A\, d\phi + \sec h \sin\omega\, d\delta$ $d\omega = -\sec h \cos A \cos\phi\, dt + \sec h \sin A\, d\phi + \tan h \sin\omega\, d\delta$ $dt = -\sec\phi \cos A \cos h\, d\omega + \tan\phi \sin t\, d\delta + \sec\phi \sin A\, dh$ $dA = \sec\phi \cos t \cos\delta\, d\omega + \sec\phi \sin t\, d\delta + \tan\phi \sin A\, dh$ $d\omega = \sec\delta \cos t \cos\phi\, dA - \sec\delta \sin t\, d\phi + \tan\delta \sin\omega\, dh$ $dt = \sec\delta \cos\omega \cos h\, dA + \tan\delta \sin t\, d\phi - \sec\delta \sin\omega\, dh$

Second differential coefficients with ϕ and δ being held constant are

$$\frac{d_2h}{dt^2} = \frac{dh}{dt}\{ \cot t + \tan h \frac{dh}{dt} \}$$

$$= - \cos \phi \cos A(\tan h \cos \phi \cos A - \sin \phi)$$

$$\frac{d_2t}{dh^2} = - \frac{dt}{dh} (\cot t \frac{dt}{dh} + \tan h)$$

$$= - \sec \phi \cot A \operatorname{cosec} A(\tan\phi \operatorname{cosec} A - \tan h \cot A)$$

$$\frac{d_2A}{dt^2} = \cos \phi \sec^2 h \sin A(\sin \delta \cos h \ -2 \cos \phi \cos A)$$

$$\frac{d_2A}{dh^2} = - \sec^2 h \cot \omega(\sin h + 2 \cot A \operatorname{cosec} 2 \omega)$$

It must be realized that, in this generalized system, Napier's Rule of Circular Parts for a right-angled spherical triangle should not be used, because this rule does not always differentiate between an angle of 90° and one of 270°. As a result, the sign convention is destroyed. Therefore, if an angle of 90° or 270° occurs, its value must be inserted into the generalized relationship and the rule of signs observed. This does not give trouble, when the cosine of this angle is used as each gives a zero for this function.

Calculation Example

2.76 Determine the remaining elements of the astronomical triangle in which the latitude is 26° North, the declination 50° South and the hour angle 3 hours east.

i.e. $\phi = + 26°$ $\delta = - 50°$ $t = 21^h = 315°$

If only the relationships of section 2.62 are available, then W is equated with the north pole, X with the zenith and Y with the star. This gives the hour angle at the north pole as the angle included between the latitude and the declination sides.

The Cosine Formula of section 2.62 gives

$$\cos w = \cos x \cos y + \sin x \sin y \cos W$$

$$\cos(90 - h) = \cos(90-\delta) \cos(90-\phi) + \sin(90-\delta) \sin(90-\phi) \cos t$$

$$\sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos t$$

which is exactly what the relationships of section 2.75 give.

$$\therefore \sin h = \sin \delta \sin \phi + \cos \delta \cos \phi \cos t$$

$$= \sin(-50) \sin(+26) + \cos(-50) \cos(+26) \cos(315)$$

$$= 0.07271$$

$$\therefore h = + 4°10'10'' \quad \text{or} \quad +175°49'50''$$

But by definition and convention, the altitude lies only in the first or fourth quadrants and therefore

$h = + 4°10'10''$ or $4°10'10''$ *above* the horizon

To solve for the azimuth, it is necessary to link the four consecutive elements A, ϕ, t and δ. This requires the Four Parts Formula. For an illustration, and only for demonstration purposes, W will be placed at the south pole, Y at the zenith and X at the star.

$$\cot y \sin x = \cot Y \sin W + \cos x \cos W$$

$$\cot(90+\delta)\ \sin(90+\phi) = \cot(180-A)\ \sin(360-t) + \cos(90+\phi)\ \cos(360-t)$$

$$-\tan\delta \cos\phi = (-\cot A)\ (-\sin t) + (-\sin\phi)\cos t$$

$$-\tan\delta \cos\phi = \cot A \sin t - \sin\phi \cos t$$

$$\therefore \quad \cot A \sin t = \sin\phi \cos t - \tan\delta \cos\phi$$

$$\cot A = \sin\phi \cot t - \tan\delta \cos\phi \operatorname{cosec} t$$

$$= \sin(+26) \cot 315 - \tan(-50) \cos(+26) \operatorname{cosec} 315$$

$$= -1.95\ 319$$

$$\therefore \quad A = 332°53'20'' \quad \text{or} \quad 152°53'20'' \quad \text{to the nearest } 10''$$

Before A can be unambiguously determined, a second piece of information is required. The cotangent of the azimuth is negative as shown above. If the sign of either its sine or cosine is known or can be determined, then A is known uniquely.

Now from the Sine Formula

$$\sin A = -\sin t \cos\delta \sec h$$

in which $\cos\delta$ and $\sec h$ are both positive since δ and h exist only in first or fourth quadrants. Therefore, $\sin A$ has sign opposite to that of $\sin t$. But $\sin t = \sin 315$ = negative and so $\sin A$ is positive. Thus, with its sine positive and its cotangent negative, the azimuth lies in the second quadrant (see section 2.77 for a superior alternative).

$$\therefore \quad A = 152°53'20''$$

Another way of doing the same is to remember that the hour angle of 315° indicates that the star was east of the meridian and therefore of the two possible values of the azimuth, the value 152°53'20" is the one on the eastern side of the meridian.

From a similar calculation and reasoning, ω is calculated by linking the four consecutive elements ω, δ, t and ϕ in the Four Parts Formula. W is put at the pole, Y at the star and X at the zenith and the required relationship for the astronomical triangle is then determined and used to calculate ω as 320°24'50" without any ambiguity.

If the relationships of section 2.75 are available, then these relationships can be used directly without the necessity of deriving them in the manner demonstrated above. The calculation is checked thoroughly by substitution of the elements computed in the Five Parts Formula and less thoroughly by substitution in the Sine Formula, which gives only a partial check and, in some cases, when any of the elements is near a right angle, the check lacks accuracy. It is to be noted that such lack of accuracy does not occur when the tangent or the cotangent is used as these functions are sensitive over the whole of their range, even when the functions have very large values indeed.

2.77 From the above, it comes out that at least two independent facts must be known before an unambiguous solution can be obtained. One fact known leads to double answers and the second fact enables the unique answer to be selected. If one takes the determination of hour angle t from an observed altitude h, a known declination δ and a known latitude ϕ, the relationship connecting these elements is the Cosine Formula which, taken direct from the generalized relationships of section 2.75, is

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t$$

$$\therefore \quad \cos t = \sec\phi \sec\delta \sin h - \tan\phi \tan\delta$$

From this solution for t from the cosine, double values, either one in first quadrant and one in fourth or one in second quadrant and one in third, are obtained. It is not possible to get the correct value without knowing whether the observation was made towards the west or towards the east.

One function, however, can be used to give unique answers because it is derived from two pieces of information. This function is the tangent, which can be expressed in the form of a numerator and denominator as

$$\tan x = \frac{\sin x}{\cos x} = \frac{N}{D}$$

x is then uniquely determined by the pair of signs specified by N and D as the quadrant is determined from the signs shown in Table A.1 of section A.21. For example, from section 2.75,

$$\tan A = \frac{\sin A}{\cos A} = \frac{-\sin t \cos \delta \sec h}{(\sin \delta \cos \phi - \cos \delta \sin \phi \cos t)\sec h}$$

in which the numerator comes from the Sine Formula and the denominator comes from the Five Parts Formula.

$$\therefore \quad \tan A = \frac{-\sin t}{\tan \delta \cos \phi - \sin \phi \cos t}$$

because $\cos \delta$ and $\cos h$ are always positive.
Likewise

$$\tan \omega = \frac{\sin \omega}{\cos \omega} = \frac{\sin \omega \cos h}{\cos \omega \cos h}$$

$$= \frac{\sin t \cos \phi}{\sin \phi \cos \delta - \cos \phi \sin \delta \cos t}$$

$$= \frac{\sin t}{\tan \phi \cos \delta - \sin \delta \cos t}$$

because $\cos \phi$ and $\cos h$ are always positive.
When the values of the elements used in the example of section 2.76 are substituted in these relationships, then

$$\tan A = \frac{-\sin 315}{\tan(-50)\cos(+26)-\sin(+26)\cos(315)}$$

$$= \frac{+0.707\ 107}{-1.381\ 116}$$

This indicates a second quadrant angle for the azimuth and therefore

$$A = 152°53'20''$$

Likewise

$$\tan \omega = \frac{\sin 315}{\tan(+26)\cos(-50)-\sin(-50)\cos 315}$$

$$= \frac{-0.707\ 107}{+0.855\ 184}$$

This indicates a fourth quadrant angle for the parallactic angle and therefore

$$\omega = 320°24'50''$$

2.78 Example. To illustrate the power and certainty of computing in the astronomical triangle rather than in the spherical triangle, let it be required to find the azimuth to the above star at an instant 20 minutes of time earlier.

i.e. $\Delta t = -20^m = -5° = -18\ 000''$

and $t = 310°$ at this moment.

From a Taylor Series expansion,

$$A_{310} = A_{315} + \frac{dA}{dt_{315}} \Delta t + \tfrac{1}{2} \frac{d_2A}{dt^2_{315}} (\Delta t)^2 \quad \text{------}$$

From the spherical trigonometry relationships, the change in an angle dY resulting from changes in the included angle dW and the two included sides dx and dy is given in section 2.62 by

$$\sin w \, dY = - \cos X \sin y \, dW - \cos w \sin Y \, dx + \sin X \, dy$$

To obtain the generalized relationship from this and also to illustrate the manipulative process, the southern astronomical triangle of Fig 2.9 with the star east will be taken for the starting point.

The change in the azimuth is to be found in terms of a change in the hour angle, which is the included angle, a change in the declination and in the latitude sides, which are the including sides. W is therefore put at the South Pole, at which the hour angle lies. The change in azimuth dA is to be associated with the azimuth A at the zenith so that Y is put at this point. X then falls at the third apex, i.e. at the star.

Then the differential relationship above becomes

$$\sin(90-h)\, d(180-A) = -\cos(\omega-180)\sin(90+\delta)\, d(360-t)$$
$$-\cos(90-h)\sin(180-A)\, d(90+\phi)$$
$$+\sin(\omega-180)\, d(90+\delta)$$

$$-\cos h \, dA = -\cos\omega\cos\delta\, dt - \sin h \sin A \, d\phi - \sin\omega\, d\delta$$

$$\frac{dA}{dt} = \sec h \cos\omega\cos\delta + \tan h \sin A \frac{d\phi}{dt} + \sec h \sin\omega \frac{d\delta}{dt}$$

With ϕ and δ held constant $\frac{d\phi}{dt}$ and $\frac{d\delta}{dt}$ are each zero,

and then $\frac{dA}{dt} = \sec h \cos\omega\cos\delta$

$$\therefore \quad \frac{d_2 A}{dt^2} = \frac{d}{dt}\left(\frac{dA}{dt}\right) = -\sec h \sin\omega \frac{d\omega}{dt}\cos\delta + \sec h \tan h \frac{dh}{dt}\cos\omega\cos\delta$$

But $\frac{dh}{dt} = \cos\phi\sin A$ and $\frac{d\omega}{dt} = -\sec h \cos A \cos\phi$ when ϕ and δ are held constant.

$$\therefore \quad \frac{d_2 A}{dt^2} = \sec^2 h \cos\delta\cos\phi\,(\sin\omega\cos A + \cos\omega\sin h \sin A)$$

Substituting numerical values gives

$$\frac{dA}{dt} = \sec(+4°10')\cos(320°25')\cos(-50°)$$
$$= 0.49671$$

$$\frac{d_2 A}{dt^2} = \sec^2(+4°10')\cos(-50°)\cos(+26°)\,[\sin(320°25')\cos(152°53')$$
$$+ \cos(320°25')\sin(+4°10')\sin(152°53')]$$
$$= 0.3442$$

$$\therefore \quad A_{310} = 152°53'20'' + (0.49671)(-18000'')$$
$$+ \tfrac{1}{2}.(0.3442)(-18000'')^2 \sin 1''$$
$$= 152°53'20'' - 2°29'01'' + 0°04'30''$$
$$= 150°28'49''$$

But $A_{310} = 150°28'50''$ by direct solution

The use of a series, which embodies the differential coefficients, which, so often in field astronomy, are somewhat awkard, is avoided, because the direct solution with the altered data can nowadays so easily be repeated.

3
Time and Time Keeping

INTRODUCTION

THE rotation of the earth about its own axis at the centre of the celestial sphere provides the basic requirements of a time system.. These are regular recurrences of a phenomenon, which can be observed and which continues unfailingly. The basic unit of a time system should be of uniform length. It is most convenient if this interval is a reasonably short one, so that recurrences of the event are frequent and so that the interval between successive events can be successfully bridged, without too much difficulty, to obtain an accurate uniform subdivision of the unit of time.

The time units available are the year, the month and the day. The year served as the husbandman's indicator of the seasons and also as the historian's means of recording the sequential occurrence of events. The lunar unit of the month served to divide the rather long period of the year into smaller units. The day, as a unit of time, satisfied the civil need of timing the daily round. The relationships between the lengths of these units are not simple ones. In times gone past, these relationships were not accurately known. This, combined with time counting being a function of the priestly orders and not of the state, led to considerable confusion in the calendar. Julius Caesar, with the help of the astronomer Sosigenes, revised the calendar in 46 B.C. This served well up to the time of Pope Gregory XIII, who made a further revision and introduced it in the catholic countries of Europe in 1582. Other countries adopted the Gregorian calendar after this date.

TIME SYSTEMS

Sidereal Time

3.11 A less obvious system of time keeping, than that associated with the sun, is one connected with the stars. Such a system depends, as with a system based on the sun, upon the rotation of the earth around its polar axis; a rotation which, for the purpose of preliminary explanation, may be considered uniform.

For this time system a marker is selected on the celestial sphere and the basic time unit, *the sidereal day*, is defined as the interval between successive passages of this marker over a selected meridian. The time marker has been chosen as the First Point of Aries, the marker from which right ascensions are reckoned, see Fig 2.3.

The sidereal day starts at the instant at which the upper branch of the selected meridian, which for convenience will be taken as that of Greenwich, crosses the First Point of Aries.

There are imperfections in using such a system for civil time reckoning because the sidereal day, although very nearly constant in length, is not fixed in relation to the hours of light and darkness. However, the constancy in the length of the sidereal day can be used to advantage to explain the irregularities in a time system based on the motion of the sun.

Solar Time

3.12 The obvious time keeper for civil purposes is the sun, because of its division, for the overwhelming majority of people in the world, of the day into alternating periods of light and darkness. On this alternating cycle, the daily pattern of civil activity is based.

Apparent Solar Time

3.13 Due to the earth's rotation about its axis, each meridian will transit over the sun successively. For convenience of explanation, the Greenwich meridian will be used. The upper branch of this meridian will cross the sun at approximately the midpoint of the daylight period at Greenwich and the lower branch will likewise cross the sun at about the midpoint of the period of darkness. The interval between successive passages of the lower branch of the Greenwich meridian across the sun is a "day with respect to the actual sun" or an *apparent solar day*. Since the date in a civil timekeeping system is of great importance, it would be inconvenient to have a date change in the middle of the period of daylight, when social activity is at its height.

The Greenwich apparent solar day is considered to start from the moment, at which the sun is on the lower branch of the Greenwich meridian.

The actual sun, therefore, provides the Apparent Solar Time system, which is the system obtained from observations of time from a sundial. This system has many advantages but, as will be seen, it lacks the basic requirement of a uniform unit of time, which is possessed by the sidereal system.

The solar or tropic year, not the calendar year, begins when the sun, in its ecliptic passage, occupies the First Point of Aries in passing from the southern to the northern celestial hemisphere, (see section 2.32 and Fig 2.2). The year ends when the sun again occupies the same point, during which time the sun appears to make 365.2422... revolutions with respect to a fixed meridian.

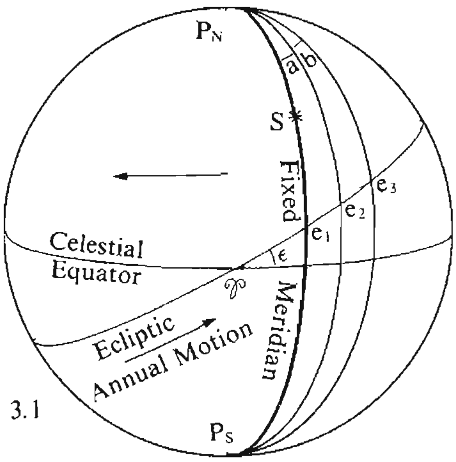


Fig. 3.1

3.14 Fig 3.1 shows the sun in three positions, e_1, e_2 and e_3 on the ecliptic, the spacing between each point being exactly 1/365.2422... of the circumference of the ecliptic i.e. the sun is assumed to move at constant speed in the path of the ecliptic. The duration of the first apparent solar day will be the time taken for the star marked S to occupy the fixed meridian after one revolution of the earth, i.e. one sidereal day, *plus* the time taken for the sun to move through the angle a back to the fixed meridian. Likewise the duration of a second solar day will be one sidereal day *plus* the time taken for the sun to move through the angle b. It is obvious that a and b are not equal, even though the distances e_1e_2 and e_2e_3 are equal. Thus the apparent solar day is not of constant length throughout the year and would only be so if the sun moved at constant speed in the path of the celestial equator, i.e. the obliquity of the ecliptic ε was zero.

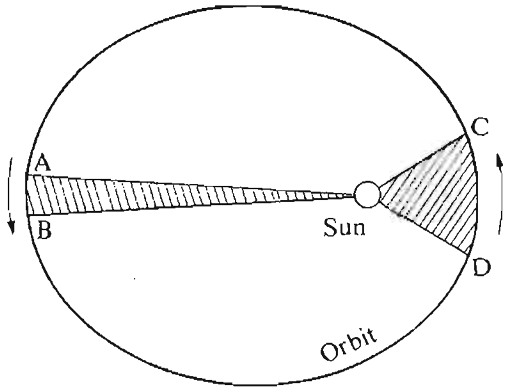


Fig.3.2 Plan view of Earth's Orbit

3.15 The earth is a planet of the sun and its motion is therefore subject to Kepler's laws of planetary motion, the first of which states that a planet's orbit around its parent body is an ellipse with the parent body at one of the focal points of this ellipse. The second law states that the variable length radius vector, planet to parent body, sweeps out equal areas in equal times. Therefore in Figure 3.2, the earth is seen to move faster in its orbit between C and D than between A and B where the two hatched sections have been made equal in area. Thus the previous assumption that the sun moves at constant speed in its ecliptic path is incorrect and would only be true if the earth's orbit around the sun was circular and not elliptical.

Mean Solar Time

3.16 The irregularity in the length of the apparent solar day caused by the obliquity of the ecliptic and the ellipticity of the earth's orbit was of no great concern to man until he was able to construct accurate timekeepers for scientific measurement. To overcome the inadequacies of the apparent solar time system, a fictitious mean sun moving at constant speed in the equator, was devised. Thus the intervals between successive transits of this sun across a fixed meridian were made equal, i.e. the day was of constant length.

This system, called the Mean Solar Time system, retains the convenience of the sun as an approximate time marker for civil purposes, and yet has a uniform unit of time for its base.

This unit, the *mean solar day*, is equal to the average length of all the apparent solar days in a year and starts when the mean sun is on the lower branch of the Greenwich meridian. It should be noted that the duration of the year is a fixed length of time and that the numbers of apparent and mean solar days in this period are identical. In the mean solar system, the unit of subdivision is uniform.

Standard or Zone Time

3.17 The Mean Time System is the basis of civil time keeping throughout the world. If one lived in a area, well removed from the meridian of Greenwich, it would be convenient to set ones watch so that it kept mean time for a nearby meridian. In order to do this, one would take into account the longitude of this selected meridian by converting the value of longitude from angular units to time units on the basis that $360° = 24^h$ etc. Such a time system is called Local Mean Time (LMT).

3.18 If individuals or individual communities were to adopt this practice independently of one another, there would be great confusion in the coordination of daily activity. To avoid this a meridian is selected near the centre of the country, and all clocks and watches are set to give the LMT of this meridian, which is called the standard meridian. The area selected on either side of the standard meridian is called a Time Zone. Zone Times or Standard Times are mean times, which usually differ from Greenwich Mean Time by a number of whole hours (15°).

Australia keeps three time zones; the eastern states of Australia keep a zone time, called Australian Eastern Standard Time, AEST, which is 10 hours east of Greenwich. To put this another way, the eastern states of Australia all keep mean time provided by the 150^{th} degree meridian east. South Africa keeps South African Standard Time, SAST, which is 2^h east of Greenwich. The United States of America, keeps four zone times, Eastern, Central, Mountain and Pacific Standard Times, which are respectively 5, 6, 7 and 8 hours west of Greenwich.

RELATIONSHIPS BETWEEN TIME SYSTEMS

IT is now necessary to know the exact relationships between the time systems to be able to convert from one to another.

The Relationship between Mean and Apparent Solar Time

3.21 Observations are often made on the real (apparent) sun and timing is made with watches keeping mean (solar) time. The difference between these time systems at any instant in the year is called the Equation of Time. This is the algebraic sum of the accumulation of the changes in the length of the apparent solar day described in sections 3.14 and 3.15. Fig 3.3 shows the two components of the Equation of Time, one due to the ellipticity of the orbit and the other to the obliquity of the ecliptic.

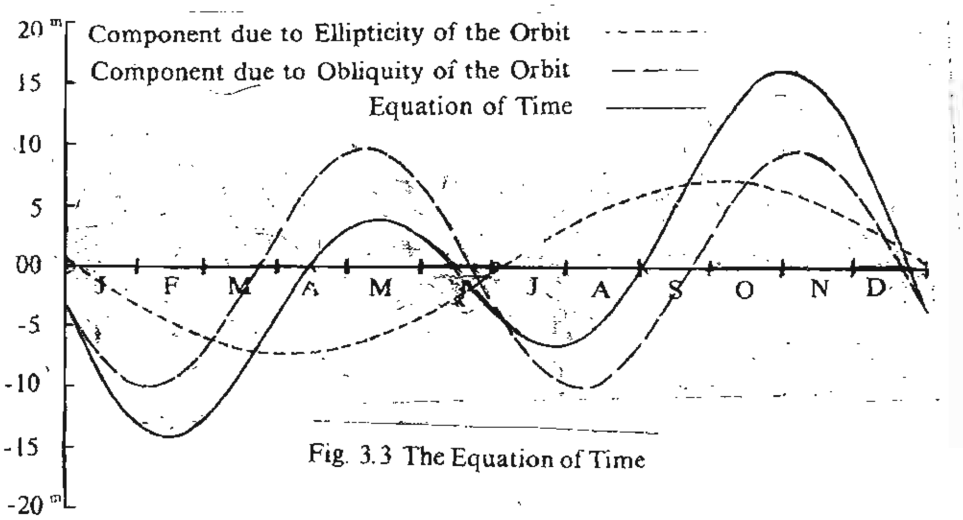


Fig. 3.3 The Equation of Time

It will be noted that the variation in sign of the Equation of Time indicates that the apparent sun leads or lags with respect to the mean sun. The Equation of Time is defined as,

Equation of Time = Greenwich Apparent Time - Greenwich Mean Time

In modern almanacs and ephemerides, a quantity E, which does not change its sign, is tabulated. This quantity is given by the relationship

$$E = 12^h + \text{Equation of Time}$$

and is tabulated for every 6^h of UT in the Star Almanac for Land Surveyors for the current year. A table is also provided for interpolating between these values.

The Relationship between Mean Solar and Sidereal Time

3.22 Greenwich Mean Time, GMT, or Universal Time, UT (the terms are used synonymously throughout the text) and Greenwich Sidereal Time are in phase when the First Point of Aries and the Mean Sun are diametrically opposite one another on the celestial sphere. (see sections 3.11 and 3.13) This situation occurs at the time of the Autumnal Equinox on or about the 21st of September. When these two points are together the time systems are 180° or 12^h out of phase and this occurs at the time of the Vernal Equinox on or about the 21st of March.

3.23 In addition to knowing the phase relationship at a particular time, it will be necessary to know the ratio between the lengths of the sub-divisions of the year in each time system. It was stated in section 3.13, in considering the variation in the length of the apparent solar day, that during the course of one year there were 365.2422... mean or apparent solar days. Furthermore it may be seen from section 3.14, that with respect to a star or the First Point of Aries, there are 366.2422... sidereal days in the year because of the retrograde motion of the sun through the background of stars. This motion accumulates to one complete revolution. The ratio between the sidereal and mean time units is therefore,

$$\frac{366.2422\ldots}{365.2422\ldots} = 1.0027379 \quad \ldots \quad = F$$

Thus, if a time interval is measured as M mean time units, the corresponding measure of this interval in sidereal time units is M x F and conversely if a time interval is expressed as S sidereal time units the corresponding measure of this interval in mean time units is $\frac{S}{F}$.

3.24 In order to facilitate conversion between these two time systems, astronomical almanacs or ephemerides publish a table giving the Greenwich Sidereal Time corresponding to the moment at which each Greenwich mean solar day starts. This table of GST at GMT 0^h or GST at UT 0^h is published in Table II in "The Apparent Places of Fundamental Stars" (FK4) for the current year. It is also published as the quantity R at UT $0^h(R_o)$ in the sun data section of "The Star Almanac for Land Surveyors" of the current year. In addition, tables giving conversion of intervals of time up to 24 hours from one system to the other are given in Tables III and IV of the FK4.

However, in order to avoid conversion tables which cover the whole 24 hour period of either mean or sidereal time, the Star Almanac for Land Surveyors provides, for each day, a quantity at UT 6^h, 12^h and 18^h, in addition to R_o. These will be referred to as R_6, R_{12} and R_{18}. These values of R are *not GST at their associated times of UT* but interpolated values of R_o at 6 hour intervals. To find intermediate values of R, an "Interpolation Table for R", which is a table of mutual conversion of intervals of mean and sidereal time, is provided.

E.g. At UT0^h

$$GST = R_o$$

At UT6^h

$$GST = 6^h + \Delta R \text{ for } 6^h \textit{ mean time} + R_o$$
$$= 6^h + R_6$$

where ΔR is obtained from the "Interpolation Table for R". Therefore, in general for any instant of UT,

$$GST = UT + R$$

where R is an interpolated value.

Further explanation of the inter-relationships between the time systems in the form of diagrams and examples is given in the next sections.

3.31 THESE are line diagrams which demonstrate in a simple way the relationships between quantities, which are associated with time and its measurement. The basic diagram consists of a circle representing the celestial equator with the earth in the centre of the circle. The reader imagines himself to be outside the celestial sphere looking down the terrestial axis onto the north pole. Terrestrial meridians are projected out to the celestial sphere thus appearing as radial lines with the prime or Greenwich meridian drawn vertically up the page and marked G. The earth is considered to be stationary so that the celestial bodies, scattered about in the circle, appear to rotate in a clockwise direction indicated by the arrow outside the diagram.

The diagram, explained above, forms the basis of further diagrams which will now be used to demonstrate a number of relationships.

3.32 An observer's meridian P and the mean sun MS, which lies on the celestial equator ($\delta=0$), are plotted on the diagram shown in Fig 3.4. By definition, when the mean sun is on the lower branch of the Greenwich meridian GMT = 0^h and when on the upper branch GMT = 12^h. Similarly for an observer at longitude λ, when the mean sun occupies the lower and upper branches of that meridian LMT is 0^h and 12^h respectively.

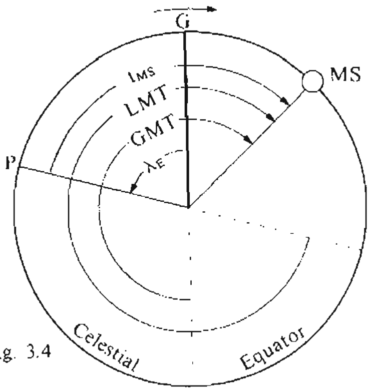


Fig. 3.4

At any other instant it will be seen that

$$LMT - GMT = \lambda = \lambda_E = -\lambda_W$$

and by extension, the time value with respect to one meridian at a particular instant can be converted into the time value with respect to another meridian, provided both times are in the same system, by the direct application of the longitude difference between the two meridians. It will be readily seen too that, if λ is the time zone longitude, then similar relationships are true for the difference between Zone Time and GMT or between two different time zones.

Also on this diagram the local hour angle of the mean sun t_{MS}, which is the amount by which the mean sun has advanced since crossing the upper branch of the observer's meridian, is given by

$$LMT = 12^h + t_{MS}$$

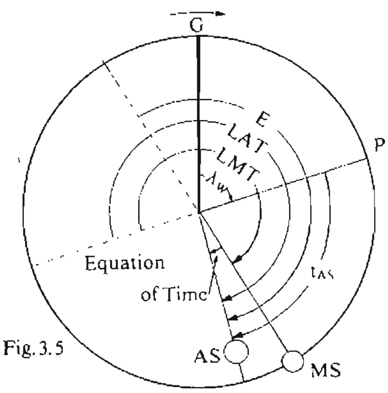


Fig. 3.5

3.33 Similarly, in Fig 3.5, the apparent sun AS is shown in advance of the mean sun by an angle equivalent to the Equation of Time.

$$LAT = 12^h + t_{AS}$$

where t_{AS} is the local hour angle of the apparent sun.

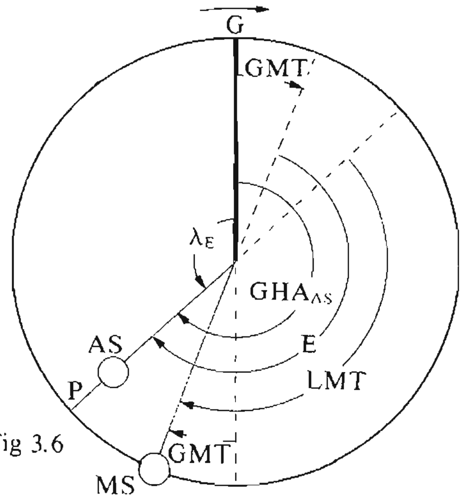


Fig 3.6

In section 3.431 it is required to find the Greenwich hour angle of the apparent sun, which can be deduced from Fig 3.6, as

$$\text{GHA of the apparent sun} = GMT + E$$

Also in section 3.432 it is required to find the LMT when the apparent sun is on the upper branch of the observer's meridian i.e. when LAT = 12^h. This circumstance, called Local Apparent Noon (LAN), is also shown in Fig 3.6, where it will be seen that

$$\text{LMT of LAN} = 24^h - E$$

3.34 Fig 3.7 shows the prime right ascension circle, which passes through the

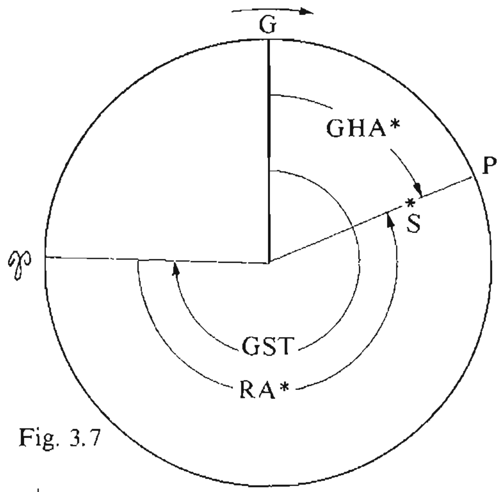


Fig. 3.7

First Point of Aries and a star, S, of right ascension, RA*, on its right ascension circle. By definition this latter quantity is measured in anti-clockwise direction from the First Point of Aries, see section 2.34. Also by definition GST is equal to the GHA of the First Point of Aries and the figure shows that

$$GST = RA^* + GHA^*$$

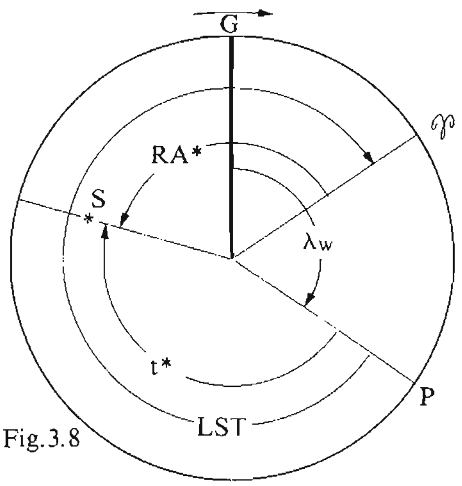


Fig.3.8

A similar situation exists with respect to the local meridian and from Fig 3.8, one may see that

$$LST = RA^* + t^*$$

3.35 Fig 3.9 shows the movement of the mean sun throughout a day, at the end of which time it is seen that the First Point of Aries has moved clockwise by about 1°.

Over the course of a year, this phase change accumulates until the phase relationship is as it was at the beginning. In addition, it will be seen that, when the mean sun and the First Point of Aries are diametrically opposite one another, the two time systems are in phase, i.e. GST = UT, which occurs at the Autumnal Equinox; when the two points are in coincidence, the time systems differ by 12^h, which occurs at the Vernal Equinox.

TECHNIQUES OF TIME CONVERSION

METHODS of converting an instant of time from one time system to the corresponding instant in another time system will now be illustrated by examples. The examples chosen are such that they cover various techniques and most of the situations which may arise in practice.

General methods for use with a calculator are given in section A.101.

Conversion between the Mean and Sidereal Time Systems

3.411 Example. Find the Local Sidereal Time corresponding to Atlantic Standard Time $1^h14^m27^s.3$ on September 12th 1977, at Fredericton, New Brunswick, Canada. The longitude of Fredericton is $4^h26^m34^s.1$ West, and the Time Zone is 4^h west of Greenwich.

Atlantic Standard Time of instant	12 September	$1^h14^m27^s.3$	
Zone longitude		4 W	(1)
Corresponding UT(GMT) of instant	12 September	5 14 27.3	
R at UT0^h on 12th September		23 23 32.5	(2)
ΔR for *mean time* interval of $5^h14^m27^s.3$		+51.7	(3)
Corresponding GST of instant		4 38 51.5	(4)
Local Longitude		4 26 34.1W	(5)
Local Sidereal Time of instant	12 September	12 17.4	

(1) The circumstances are referred to the Greenwich Meridian by the addition of a West or the subtraction of an East Time Zone, because the relationship between the sidereal and mean time systems is given for the Meridian of Greenwich with UT as argument. The mariner's mnemonic *Longitude East, Greenwich Time Least; Longitude West, Greenwich Time Best,* is particularly useful when performing this operation.

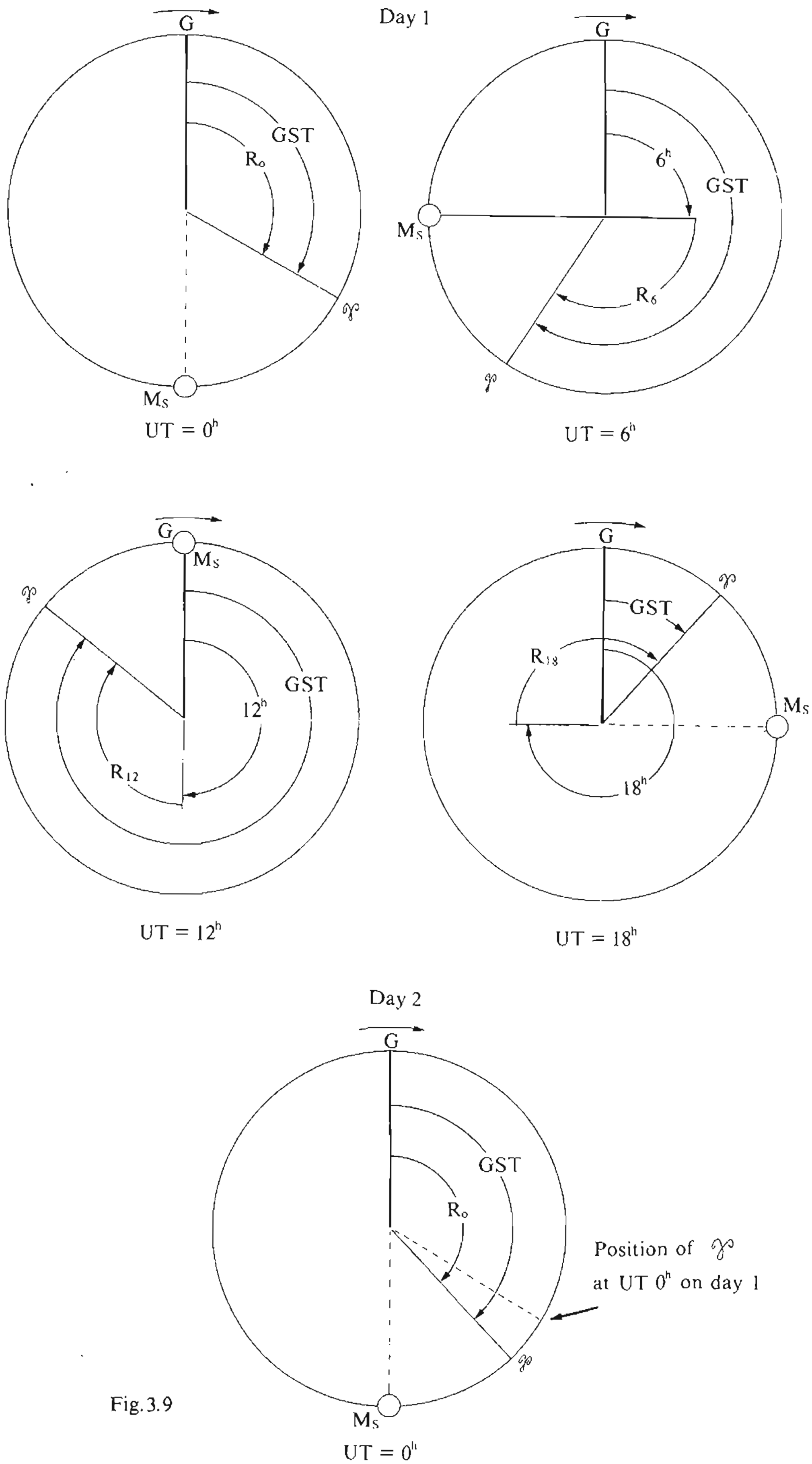

Day 1
G
GST
R₀
Ms
UT = 0h
G
6h
GST
Ms
R6
UT = 6h
G
Ms
GST
12h
R12
UT = 12h
G
GST
R18
Ms
18h
UT = 18h
Day 2
G
GST
R₀
Position of ♈ at UT 0h on day 1
Ms
UT = 0h

Fig.3.9

(2) This value is taken from the Star Almanac for Land Surveyors. R at $UT0^h$ or R_o = GST at $UT0^h$ is also to be found tabulated for every day in the Apparent Places of Fundamental Stars (FK4) for the current year.

(3) ΔR is taken from the table of the mutual conversion of intervals of solar and sidereal time given in the Star Almanac for Land Surveyors.

(4) The sum of the values in lines 3, 4 and 5 has been reduced by 24^h to give a value between 0^h and 24^h.

(5) The circumstances are referred back to the local meridian by the addition or subtraction of the longitude, reversing the signs given in note (1).

3.412 Example. The previous example will now be worked in reverse i.e. to find the Standard Time corresponding to $0^h12^m17^s.4$ Local Sidereal Time.

Local Sidereal Time of instant	12 September	$0^h12^m17^s.4$	
Local Longitude		4 26 34.1 W	
Corresponding GST of instant	12 September	4 38 51.5	
R at $UT0^h$ on 12th September		23 23 32.5	
Sidereal interval since $UT0^h$		5 15 19.0	(6)
ΔR for *sidereal time* interval of $5^h15^m19^s.0$		-51.7	(7)
Corresponding UT(GMT) of instant		5 14 27.3	
Zone Longitude		4 W	
Atlantic Standard Time of instant	12 September	1 14 27.3	

(6) To effect the subtraction of line 4 from line 3, 24^h is added to $4^h38^m51^s.5$.

(7) The table of mutual conversion of time in the Star Almanac for Land Surveyors is used to find this quantity ΔR.

3.413 Example. Find the Local Sidereal Time corresponding to Australian Eastern Standard Time (AEST) 8^h a.m.* on April 28th 1977 at Melbourne, Victoria. The longitude of Melbourne is $9^h39^m51^s.0$ East, and the Time Zone is 10^h east of Greenwich.

*Many watches have a 12 hour dial, in which case 12^h must be added to times in the afternoon and evening hours (p.m.), in order to express those times in a 24^h system.

AEST of instant	28 April	$8^h00^m00^s$	
Zone longitude		10 E	
Corresponding UT of instant	27 April	22 00 00	(8)
R at $UT0^h$ on 27th April		14 19 27.9	
ΔR for *mean time* interval of 22^h		+ 3 36.8	(9)
Corresponding GST of instant		12 23 04.7	
Local Longitude		9 39 51.0E	
Local Sidereal Time of instant	28 April	22 02 55.7	

(8) See Note (6). A change of date occurs here because the subtraction of the Time Zone brings the time value across the zero, or 24 hour time marker.

(9) The value of ΔR used here may be obtained partly from the 6 hour table of ΔR in the Star Almanac for Land Surveyors. Additional constants required are $59^s.1$, $1^m58^s.3$ and $2^m57^s.4$ for 6^h, 12^h and 18^h respectively. In this example ΔR for 4^h is $39^s.4$ and for 18^h is $2^m57^s.4$, therefore ΔR for 22^h is $39^s.4$ + $2^m57^s.4$ = $3^m36^s.8$. A more direct way of effecting this conversion is to use the 24 hour tables of conversion which are to be found in various publications such as the FK4.

A simpler and more accurate way of performing the conversion is to multiply the value of UT by 1.0027379, which may be done with a few key strokes on a calculator, thus rendering obsolete methods which require auxiliary tables. An additional advantage is that ΔR is calculated and added to UT simultaneously.

3.414 Example. The previous example will now be worked in reverse i.e. to find the Standard Time corresponding to $22^h02^m55^s.7$ Local Sidereal Time.

Local Sidereal Time of instant	28 April	$22^h02^m55^s.7$		
Local Longitude		9 39 51.0	E	
Corresponding GST of instant	27 April	12 23 04.7		(10)
R at $UT0^h$ on 27th April		14 19 27.9		
Sidereal interval since $UT0^h$		22 03 36.8		
ΔR for *sidereal time* interval of $22^h03^m36^s.8$		- 3 36.8		(11)
Corresponding UT of instant		22 00 00.0		
Zone Longitude		10	E	
AEST of instant	28 April	8 00 00.0		

(10) A change of date occurs here because the subtraction of the longitude brings the time value across the sidereal time marker corresponding to midnight i.e. R at $UT0^h$. R at $UT0^h$ on the 28th April = $14^h 23^m 24^s.5$.

An exception to this rule occurs when an observation has been made at an instant of Standard Time which lies within a range of $3^m55^s.9$ on either side of midnight. In this situation two *identical values* of LST on the *same date* can occur. However, these values are so far removed in time from one another that the choice of which of the two values is the correct one is obvious. Dates are *always* associated with the Mean Time and not with the Sidereal Time system.

(11) The value of ΔR used here may be obtained from the 6 hour table of ΔR in the Star Almanac for Land Surveyors. Additional constants required are $59^s.0$, $1^m58^s.0$ and $2^m56^s.9$ for 6^h, 12^h and 18^h respectively. This time conversion may be effected in a similar way to that shown in note (9). The sidereal time interval is divided by 1.0027379.

The Star Almanac for Land Surveyors also provides values of R corresponding to UT 6^h, 12^h and 18^h as well as at $UT0^h$. Using these values of R, one may solve problems of time conversion without using the constants for 6^h, 12^h and 18^h referred to in note (9) and note (11).

3.415 Example. Find the Local Sidereal Time corresponding to South African Standard Time $18^h32^m43^s.2$ on June 16th, 1977 at Cape Town. The longitude of Cape Town is $1^h13^m44^s.0$E, and the Time Zone is 2^h east of Greenwich.

South African Standard Time of instant	16 June	$18^h32^m43^s.2$		
Zone Longitude		2	E	
Corresponding UT of instant	16 June	16 32 43.2		
R at $UT12^h$ on 16th June		17 38 34.0		(12)
ΔR for *mean time* interval $4^h32^m43^s.2$		+44.8		(13)
Corresponding GST of instant		10 12 02.0		
Local Longitude		1 13 44.0	E	
Local Sidereal Time of instant	16 June	11 25 46.0		

(12) Select the tabulated value of R whose associated value of UT immediately precedes the given value of UT i.e. $UT12^h$ immediately precedes UT $16^h32^m43^s.2$, therefore choose R at $UT12^h$.

(13) The mean time difference $16^h32^m43^s.2 - 12^h = 4^h32^m43^s.2$ is used as the argument to find ΔR from the 6^h table.

3.416 Example. The previous example will now be worked in reverse i.e. to find the Standard Time corresponding to $11^h25^m46^s.0$ Local Sidereal Time.

Local Sidereal Time of instant	16 June	$11^h25^m46^s.0$		
Local Longitude		1 13 44.0	E	
Corresponding GST of instant	16 June	10 12 02.0		
R at $UT12^h$ on 16th June		17 38 34.0		(14)
Difference		16 33 28.0		
ΔR for *sidereal time* interval of $4^h33^m28^s.0$		-44.8		(15)
Corresponding UT of instant		16 32 43.2		
Zone Longitude		2	E	
South African Standard Time of instant	16 June	18 32 43.2		

(14) Select the tabulated value of R whose associated value of UT immediately precedes the UT of the instant. In this case UT is not as yet known, but a value of UT of sufficient accuracy for this purpose may be found from UT $\simeq$ GST - R i.e. $10^h12^m - 17^h40^m = 16^h32^m$, therefore choose R at $UT12^h$. This can be done mentally.

In situations when the approximate value of UT lies close to a multiple of 6^h the incorrect tabulated value of R may be chosen. However, the mistake will be seen immediately and rectified when it is found that the argument to find ΔR is not in the range 0 to 6^h.

(15) The sidereal time difference $16^h33^m28\overset{s}{.}0 - 12^h = 4^h33^m28\overset{s}{.}0$ is used as the argument to find ΔR in the 6^h table.

The aforegoing time conversion procedure is not an obvious one but may be explained by examining the working in detail.

$$\text{GST at UT } x^h = 10^h12^m02\overset{s}{.}0, \text{ where UT } x^h \text{ is not known}$$
$$\text{GST at UT}12^h = 12^h + \text{R at UT}12^h = 12^h + 17^h38^m34\overset{s}{.}0 = 5^h38^m34\overset{s}{.}0$$
$$\text{Difference in GST} = 10^h12^m02\overset{s}{.}0 - 5^h38^m34\overset{s}{.}0 = 4^h33^m28\overset{s}{.}0$$
$$\text{Difference in UT} = 4^h33^m28\overset{s}{.}0 - \Delta R \text{ for sidereal interval of } 4^h33^m28\overset{s}{.}0$$
$$\therefore \quad \text{UT} = 12^h + 4^h33^m28\overset{s}{.}0 - \Delta R$$
$$\text{UT} = 16^h33^m28\overset{s}{.}0 - 44\overset{s}{.}8 = 16^h32^m43\overset{s}{.}2$$

3.42 A calculation which frequently occurs in the reduction of observations in field astronomy is that required for determining the local hour angle of a star from an observed watch time. For this a relationship, which incorporates the time conversion and is particularly suitable for use with a calculator, is as follows,

$$t = 15\ [\ (WT + WC - Z)F + R_o - RA + \lambda\ ]$$

where

WT is the watch time of observation,
WC the watch correction to give Standard Time (+Slow, -Fast),
Z the longitude of the standard meridian (+East, -West),
R_o R at $UT0^h$ on the Greenwich date *equal* to the local date of observation,
RA the right ascension of the star and
λ the longitude of the station (+East, -West)
all the above quantities being expressed in hours and decimals.
F is the time conversion constant, 1.0027379 and
t the local hour angle of the star expressed in degrees and decimals.

With the exception of WT, all the other quantities on the RHS of the equation have constant values unless the watch or clock is gaining or losing rapidly over the period of observation on the star.

Conversion between the Mean and Apparent Solar Time Systems

3.431 Example. Find the local hour angle of the apparent sun at $8^h42^m14\overset{s}{.}0$ Australian Eastern Standard Time (AEST) at Sydney, N.S.W., on April 4th, 1977. The longitude of Sydney is $10^h04^m55\overset{s}{.}9$ East, and the Time Zone is 10^h east of Greenwich.

AEST of instant	4 April	$8^h42^m14\overset{s}{.}0$		
Zone Longitude		10	E	
Corresponding UT of instant	3 April	22 42 14.0		(8)
E at $UT18^h$ (ΔE_s for 6^h = $+4\overset{s}{.}4$)		11 56 46.6		(16)
ΔE for $4^h42^m14\overset{s}{.}0$		+ 3.4		(17)
Sum = Greenwich hour angle of sun = UT+E		10 39 04.0		(18)
Longitude		10 04 55.9	E	
Local hour angle of sun		20 43 59.9		

(16) Select the tabulated value of E whose associated value of UT immediately precedes the UT of the instant.

(17) ΔE is taken from the table for the interpolation of the sun given in the Star Almanac for Land Surveyors.

(18) In section 3.33 it was shown that the GHA of the apparent sun = UT + E

3.432 Example. Find the Eastern Standard Time at which the sun crosses the upper branch of the meridian of Washington, D.C., U.S.A., on November 24th, 1977. The longitude of Washington is $5^h 08^m 15^s.7$ W and the Time Zone is 5^h west of Greenwich.

LAT	24 November	$12^h 00^m 00^s$	(19)
Longitude		5 08 15.7 W	
GAT		17 08 15.7	(20)
$-(E \text{ at } UT12^h - 12^h) = -13^m\ 16^s.5$		−13 16.5	
Difference ≃ UT		16 54 59.2	(21)
$-(\Delta E \text{ for } 4^h\ 54^m \cdot 59^s.2) = +3^s.6\ (\Delta E \text{ for } 6^h = -4^s.4)$		+ 3.6	
Difference = UT		16 55 02.8	
Zone Longitude		5 W	
Eastern Standard of LAN	24 November	11 55 02.8	

(19) The apparent sun is on the upper branch of the local meridian (upper transit) when Local Apparent Solar Time (LAT) is 12^h. This is abbreviated to LAN (Local Apparent Noon).

(20) In section 3.33 it was shown that

$$\text{LMT of LAN} = 24^h - E$$

$$\therefore \quad \text{UT of LAN} = 24^h - E - \lambda$$
$$= 12^h - \lambda - (E - 12^h)$$
$$= 12^h - \lambda - (E_{n6} + \Delta E - 12^h)$$

where E_{n6} is the tabulated value of E whose associated value of UT immediately precedes the UT of the instant, where n is an integer. However, UT is not known but GAT = $12^h - \lambda$ is, and will be sufficiently accurate for selecting E_{n6}. In this case select E_{12}.

(21) A better approximation to UT will be

$$12^h - \lambda - (E_{n6} - 12^h)$$

which differs from the accurate value of UT by ΔE, a quantity which is seldom greater than a few seconds of time. The variation in ΔE by this amount is negligible for our purpose. Using this value of UT, ΔE is found from the tables referred to in note (17).

Then from before

$$\text{UT of LAN} = 12^h - \lambda - (E_{n6} - 12^h) - \Delta E$$

$$\therefore \quad \text{Standard Time of LAN} = 12^h - \lambda - (E_{n6} - 12^h) - \Delta E + \text{Time Zone}$$

The examples given have been worked and explained in great detail. For those with some experience, short cuts are obvious and the working can be reduced accordingly.

DETERMINATION OF TIME

3.51 THE unquestioned time keeper up to the end of the 19th century was the earth's period of rotation. Theoretical considerations, of which the main one was probably that of tidal friction, indicated that the period of rotation of the earth would be a lengthening one, i.e. that the earth's angular velocity was slowing with time. The clocks available up to this time were not of sufficient long period accuracy or constancy to be able to detect any slowing down or any irregularity in this angular velocity.

The introduction of the Riefler clock about 1890 and later in 1921 the

Shortt free-pendulum clock confirmed the theories of non-uniform earth rotation, and the subsequent development of the quartz crystal clock and atomic frequency standards have resulted in great increases in the accuracy of time keeping and of preserving constancy over long periods. Interesting variations in the earth's rotational period have been discovered. One effect is a seasonal one and another is due to the plastic deformation of the earth which causes it to rotate about an axis, which is not quite a stationary one, but one which has a slight wobble.

3.52 An international body, the Bureau Internationale de l'Heure, (B.I.H.) has been established and given as one of its tasks the monitoring of the earth's rotational period. Those observatories, making observations on this period, pass their information to the B.I.H., which then correlates and analyses the results and publishes definitive relations between the various time scales used. These relations are published a few months after the observations have been made.

In order to distinguish between the various time scales the following definitions are given:-

UTO is Universal Time (formerly Greenwich Mean Time) established from observation made at fixed observatories.

UT1 is UTO corrected for polar motion.

UT2 is UT1 corrected for seasonal variations in the earth's rotation.

UTC (Universal Coordinated Time) is related to the international atomic time scale (IAT). This atomic time scale is based on the frequency corresponding to a certain resonance of the caesium atom and differs from UTC by an integral number of seconds.

In 1972, by international agreement, UTC was adopted as a basis for broadcast time signals. In most radio time signal transmissions, a coded signal is included such that one may deduce a correction DUT1, enabling the user to establish UT1 to an accuracy of $0^{s}.1$ from the relationship

$$UT1 = UTC + DUT1$$

This accuracy is quite sufficient for a large proportion of the users of time signals, but for precise astronomical work the time scale UT2 should be used and the difference UT2 - UTC can be obtained from the publications of the BIH previously referred to.

The difference UT2 - UT1 is not greater than a few hundredths of a second.

Since there is a continuous phase shift between UTC and UT1, the DUT1 correction varies continuously. To keep it manageable, the time signal values are kept within a maximum of $0^{s}.9$ of UT1. When the DUT1 correction runs up towards the end of this range a whole second, called a leap second, is introduced in the counting of UTC in the broadcast time signal. Before this is done, eight week's warning about the proposed change is given. When the leap second occurs, the counting is shifted and the DUT1 correction changes, for instance, from -0.5^{s} to $+0.5^{s}$ in order to take up the omitted second.

Time Signals

3.53 The Star Almanac for Land Surveyors for the year has a list of Radio Time Signals on pages 60 and 61 and Notes on Radio Time Signals on page 61. This list of radio time signals is restricted to the principal signals, that are likely to be used by land surveyors and should be consulted for the signals most likely to be best received in the area. Details of the signals, frequencies used and identification data are given.

At present one has access to time signals of great accuracy at any place in the world, if one is provided with a suitable short wave receiver, because there are many continuous time signal transmitters of high power and their emissions are carefully controlled by atomic standards of high stability and accuracy. The chief disruptions in time signal receiption are those caused by ionosphere disturbance produced by sunspot activity. When this occurs one faces the problem, which existed when radio time signals were broadcast over only two five minute periods in the day. If these were received, the surveyor could obtain his clock readings corresponding to these time signals with considerable accuracy, but he than had to rely on the clock for bridging the long gap between successive time signals. It required a very good clock to subdivide this rather long interval in an accurately linear manner, as the observer had to assume that the clock kept a uniform rate over this period. The technique was to make one's star observations so that the clock comparison was made close to or even during the observing period, so that the extrapolation period on the clock was a short one. This, however, was not always possible as the time signal transmission might have occurred at times which were remote from the observing periods.

This difficulty is overcome nowadays as a result of two improvements. One, as stated before, is the introduction of continuous time signals and the other is the wide availability of the quartz clock in a portable form for use in the field by the surveyor. The first enables clock comparison with the time signal to be made during any observing period, provided the occasional "radio black-outs" do not occur at the same time. When this does occur with unusual frequency, as for instance in high latitudes in Canada or in Antarctica, the quartz clock can be used to bridge the gap in the reception of the time signals with great efficiency, because its stability of rate is far superior to that of the mechanical clock.

The Time Keeper

A good time keeper is one which has a stable rate, so that it can be relied on to subdivide a time interval accurately. The mechanical clock is now being replaced by the electronic clock, which uses a quartz crystal to provide a steady frequency source as the basis of its time keeping ability.

When a quartz crystal is cut in a certain manner and a steadily alternating voltage is applied to opposite faces of the crystal and the frequency of this voltage is close to the natural frequency of the crystal, the crystal itself will maintain its natural frequency of vibration to a highly stable degree. This stabilised oscillation, which is usually at a high frequency, can then be used by means of suitable dividing circuits to provide time units of great steadiness and stability. This property depends on the temperature of the crystal, which in good quality clocks is placed in a thermostatically controlled oven.

Determination of Clock Correction

3.61 In all but a few types of star observation, the clock correction required for obtaining the Greenwich time corresponding to an observed clock time must be determined. This quantity is defined as the amount to be added algebraically to a clock reading to obtain the Greenwich time.

Greenwich Time = Clock Time + Clock Correction on Greenwich Time

$$GT = CT + CC_{GT}$$

If the clock correction with respect to a specific meridian of longitude λ is required, each of the above must be increased by

$$GT + \lambda = CT + CC_{GT} + \lambda$$

or

$$LT = CT + CC_{LT}$$

i.e., Local Time = Clock Time + Clock Correction on Local Time

The above definition of clock correction implies that a positive value means that the clock is slow and a negative one that it is fast. Therefore, if a clock is losing with respect to a specific time, the clock correction will increase with time; if gaining, the clock correction will decrease with time. Thus a losing rate is a positive one and a gaining rate a negative one.

It is convenient to have a clock with a small rate with respect to a specified time system. Clocks, which purport to keep mean or sidereal time, are made to have such rates that they depart very slowly from the nominal time rate.

The following example illustrates a determination of rate:

Clock Reading	Corresponding Greenwich Mean Time	Clock Correction on Greenwich Mean Time	Rate
$08^h04^m02\overset{s}{.}1$	$08^h03^m17\overset{s}{.}4$	$-00^h00^m44\overset{s}{.}7$	$+12\overset{s}{.}4$ in $2^h30^m37^s$
10 34 39.1	10 34 06.8	-00 00 32.3	= + 4.94 seconds per clock hour, losing

No information is available here as to whether this rate is a linear one or not. Any departure from a linear rate is found from an examination of more then two clock comparisons.

Methods of Determining the Clock Correction

3.62 The general procedure for such determination is the observation of the clock time of an instant whose Greenwich Time is known. Each pair of such values gives a point on a clock correction graph, which should always be plotted in the first instance, whichever method of reduction may be used. Such a plot gives a good broad picture and the first estimate of the quality of the determination of the clock correction as a whole. Various methods of observation of varying accuracies can be used in practice to suit the accuracy of the astronomical quantity desired.

3.63 The Eye and Ear Method, a rough method, consists in the observer estimating the clock time to a fraction of the second as he hears the time signal pulse, whose Greenwich Time is known. A much better method is one, in which a stopwatch is used to obtain a much more accurate comparison. The observer starts the stopwatch on a known signal and then stops it on an observed clock time. If he prefers it, he may reverse the order of observing. A note of the order used should be made in the field book. Several such comparisons should be made over the observing period in order to enable blunders to be detected and to determine whether the clock's rate is stable and thus linear. This latter requirement is more important with a mechanical clock than with a quartz clock.

The seconds of the stopwatch being used may not be of the same length at those of the time signal or of the clock. The length of the stopwatch second can be compared directly with these. In addition, the effect of this sort of error is kept small by keeping the stopwatch intervals small or by arranging the stopwatch observations to eliminate their effect.

The stopwatches in the aforegoing method measure simple time intervals i.e., they bridge the short gap between the chronometer and time signal instants. If, however, a stopwatch with a split hand or a digital stopwatch, which has the same facility is used the chronometer may be dispensed with. This type of stopwatch combines the function of both chronometer and stopwatch because events on a continuous time scale are being recorded.

3.64 A very much more sophisticated device of very high accuracy is the printing chronograph. One of these is the Omega printing chronograph, which is portable and in which a temperature controlled quartz crystal oscillator governs the speed of a synchronous electric motor. Counting wheels, on which the numerical values of the time readings are marked are driven by this motor. When an observation is made, a circuit is closed and a printing pad is driven sharply up against the counting wheels and the reading of time is printed on paper tape held on top of the printing pad. This time record can be read accurately to nearly three decimal places of a second of time (see Fig 3.10). This device can be arranged to be operated by means of impersonal methods.

The Calculation of Clock Correction

3.65 Clock comparisons with the time signal require the clock time corresponding to a specific time signal pulse to be observed. This means that the identification of the time signal pulse is necessary. In the field, positive identification of signal time is made at beginning and at end of the

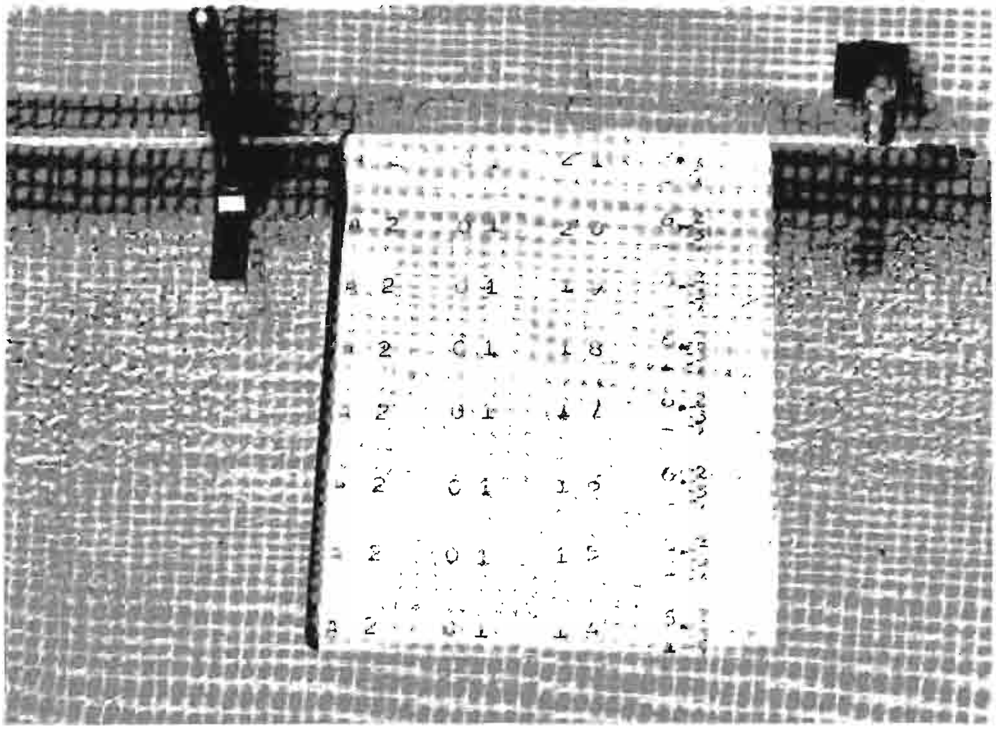


Fig.3.10 A Printing Chronograph Record

observing period and usually at suitable points in between these. It is also convenient if clock readings are made at whole minute points or at points identified by voice in the transmission. Further identification in detail can then be subsequently carried out in the office.

From these observations the clock corrections at the various times can be determined and the results plotted graphically and reduced numerically.

3.66 This numerical solution will be that of the method of Least Squares, by which a straight line of best fit will be determined so that the clock corrections at the times of observation may be determined for use in the subsequent reduction.

For simplicity here, the clock correction will be referred to Greenwich time as (GT) and the rate D will be taken as a linear one. From definition in section 3.61 therefore,

$$CC = GT - CT = CC_o + CT \times D$$

in which CC_o is the clock correction at zero clock time. The clock time CT is subject to a random error of observation and the above equation becomes

$$GT_i - (CT_i + v_i) = CC_o + (CT_i + v_i)D$$

but v_iD is a minute quantity.

$$\therefore \quad - v_i = CC_o + CT_i \times D + (CT_i - GT_i)$$

If $n > 2$, which should be so, normal equations are formed to give the unique solution of greatest probability from this set of observations. These equations take the form

$$n\,CC_o + [CT_i]\,D + [CT_i - GT_i] = 0$$

$$[CT_i]\,CC_o + [CT_i^2]\,D + [CT_i(CT_i - GT_i)] = 0$$

in which the square brackets indicate a summation of terms.

3.67 Example of a clock correction determination for a mean time clock. The data for the clock corrections of section 9.81 will be used to illustrate the Least Squares solution.

It should be noted that

(a) the clock used was a Heuer splithand stopwatch.

(b) each value of CT_i results from the mean of five observations.

(c) the comparison was recorded for a whole minute of signal time; in each case five determinations of the decimal of the second of time were made in the vicinity of the whole minute.

Observations

GMT of Signal	Observed Clock Time	Clock Corrn.	
GT_i	CT_i	$GT_i - CT_i$	v_i
$10^h17^m00^s.6$	$2^h36^m55^s.44$	$+7^h40^m05^s.16$	$+0^s.05$
10 35 00.6	2 54 54.93	05.67	-0.03
10 43 00.6	3 02 54.68	05.92	-0.04
10 45 00.6	3 04 54.60	06.00	-0.02
10 55 00.6	3 14 54.34	06.26	-0.09
11 12 00.6	3 31 53.65	06.95	+0.05
11 17 00.6	3 36 53.45	07.15	+0.08
11 20 00.6	3 39 53.40	07.20	+0.03
12 06 00.6	4 25 51.96	08.64	-0.03
12 15 00.6	4 34 51.65	08.95	-0.01

Normal equations

CC_o	D	Absolute Term = 0
10	+34.7328	-76.6855
	+124.2730	-266.3523

Solutions

$$CC_o = +7^h.6666624 = +7^h39^m59^s.985$$

$$D = 5.443 \times 10^{-4} \text{ hr/hr}$$

$$\text{or } 3600\,D = +1.959 \text{ sec/hr}$$

Back substitution in the individual correction equations gives the required v's from which the estimated standard deviation of an individual observed clock correction is obtained

$$\sqrt{\frac{\Sigma\, v_i v_i}{n-2}} \approx \sqrt{\frac{0.0243}{8}} = \pm\, 0^s.06$$

A number of points emerge from this,

(1) In this example, it was assumed that a linear relationship existed between the clock correction and the clock time of observations. In some cases, this may not be appropriate and the data would be better approximated by a second or higher degree curve.

(2) The calculation given is often referred to as fitting a regression line with paired data values - a calculation which occurs so frequently in science and technology that hard wired sub-routines for this calculation are often incorporated in small calculators.

(3) It should be noted that all the data must be converted to decimal form before the least squares solution is attempted and the inverse conversion made to obtain the solutions in sexagesimal form.

(4) The calculator solution of the line of best fit is made to a very high accuracy so that clock corrections are determined with no loss of accuracy even if the period is long or the rate of the clock large. For example, the identical method is quite appropriate for finding clock corrections and rates from comparisons between a sidereal chronometer and a time signal. In this case, there is a large rate of about ten seconds per hour.

4
Observations

INTRODUCTION

IN the astronomical triangle, the observer can measure one quantity directly with a theodolite and obtain two others from related observations and other quantities. The zenith distance can be measured directly. Directions can be measured on the horizontal circle of the theodolite and azimuth is associated with these. A clock or some such timing device can be used for determining an hour angle. Since an hour angle is associated with a specific meridian, it is usually necessary to link the clock time, from which the hour angle is determined, with the time associated with some specific meridian, which is usually the prime meridian, namely that of Greenwich. The techniques associated with the relation of clock times to Greenwich times are dealt with in section 3.61. Before any observations are made by means of a telescope, its focussing must be perfected. The observer achieves this by sighting the sky and focussing the crosshairs by rotating the eyepiece cell until the crosshairs stand out absolutely clearly. He then directs the telescope to a distant reference object. This mark is brought to sharp focus by means of the focussing screw for the main telescope. When this is completed, the observer tests for parallax by moving his eye relative to the eyepiece lens. If the image of the crosshair then moves relative to the image of the reference object, focussing is not perfect and this procedure is repeated until all parallax is eliminated.

Observing on Both Faces of the Theodolite

4.11 The practice of observing both face left and face right with a theodolite is adhered to in order to eliminate the effects, on the observed quantities, of any maladjustments present in the theodolite. Since this elimination is only exactly achieved when such maladjustments are relatively small quantities, it is good practice to keep the instrument in a state of good adjustment so that the horizontal collimation error, the horizontal or trunnion axis error and the vertical circle index error are always kept small.

Horizontal collimation error is the amount, by which the line of sight in the telescope departs from lying perpendicular to the horizontal axis. It can be reasonably easily adjusted by the user. The horizontal axis error is the amount, by which this axis departs from lying perpendicular to the vertical axis. In the modern optical theodolite, this error cannot be adjusted by the user, but the instrument should be sent to the servicing agent for this adjustment, because any cant imparted to the horizontal axis may disturb the focussing of the optical train, by which the circles of this type of instrument are read. The vertical circle index error is the difference between the reading obtained for a level sight and what should be obtained for such a sight. It is very easily adjusted by the user, who usually should determine this index error before any set of vertical circle observations is made. If this error is large, it should be reduced by adjustment mostly for convenience only, but an instance is known of the occurrence of a vertical index error of 10 arc minutes causing a problem in the calculation.

If, as is normally done, observations are made on both faces to a stationary object, the mean of the two observed values will be free of the effects of the three errors cited above. Observations to a star however are not made to a stationary, but to a moving object, which is therefore changing altitude. Since the effect of the collimation error and that of the horizontal or trunnion axis error depend on both the magnitude of the error and the altitude of the sight, these errors should be kept small and no time wasted between the observations made on the star on each face. The vertical circle index error effect is independent of altitude, and vertical observations on each face need not be made very quickly one after another.

4.12 Observing on both faces of a theodolite *does not* get rid of the effects on horizontal circle readings of residual error in the levelling up of the theodolite. This error leaves the vertical axis not quite vertical, but slightly tilted with respect to the vertical line by an unknown amount with the direction of tilt also unknown. *It cannot be sufficiently stressed that,* however many face left and face right observations are made on the horizontal circle, the means of corresponding pairs will not eliminate the effect of vertical axis error (i.e. non-verticality of the vertical axis) of a theodolite. This is particularly important in field astronomy, in which steep sights are observed, because the effects of the vertical axis error are proportional to the tangent of the altitude. The remedy is therefore to keep this error in verticality small by levelling very carefully with the most sensitive means available. The first of these is the vertical circle or alidade bubble with the split image viewing device, by which the two images of the ends of the bubble can be accurately brought into coincidence. The second is the automatic compensator (liquid or pendulum type) for indexing the vertical circle, when the theodolite, to which it is attached, is a single second one.

The vertical circle bubble, with its viewing device, is used as follows for levelling the theodolite accurately:-

i) After the theodolite has been levelled by means of the plate bubble, the alidade is rotated until the alidade bubble lies parallel to the line joining the two footscrews A and B. The bubble is then trimmed by means of the bubble adjusting screw, so that the bubble ends coincide with each other in the viewer.

ii) The alidade is now rotated through 180°. If the bubble ends are no longer coincident, they are brought halfway back towards coincidence by a rotation of footscrews A and B by equal amounts in opposite directions The ends are then brought the rest of the way back to coincidence by means of the bubble adjusting screw (Steps (i) and (ii) are repeated if necessary).

iii) The alidade is then rotated through 90°. If the bubble ends then do not remain coincident, they are brought all the way back into coincidence by means of the third footscrew C.

iv) The whole process should be repeated until the bubble ends stay coincident for any position of the alidade

The automatic compensator on a single second theodolite is used as follows for levelling the theodolite accurately:-

i) After the theodolite has been levelled by means of the plate bubble, the alidade is rotated until the plane of the vertical circle is parallel to the line joining two footscrews A and B. The telescope is clamped and left unaltered throughout the levelling procedure. The vertical circle reading is now observed.

ii) The alidade is rotated through 180° and the vertical circle reading is again observed.

iii) The two vertical circle readings are meaned and footscrews A and B are rotated equal amounts in opposite directions until the vertical circle reading is equal to the mean value computed.

iv) The alidade is rotated through 90° and, by means of the third footscrew C, the vertical circle reading is caused to be the same as the mean value computed above.

v) The vertical circle reading should now remain constant for any position of the alidade. If it does not, the whole process should be repeated (see *The Australian Surveyor*, December, 1976, vol.28, No. 4).

Some theodolites have the collars of the horizontal or trunnion axis left exposed so that a striding level may be mounted on this axis. When this has been done and the bubble has come to rest, its position in the bubble tube is noted. The striding level is then lifted, turned end for end and replaced on the axis. When the bubble is stationary, its position is again observed. From these readings, the inclination of the horizontal axis in this position is deduced and from this a correction to the horizontal circle reading, corresponding to this position, can be evaluated and applied to remove the error in the horizontal circle reading.

4.13 Observing on both faces constitutes good practice, but this does mean that the observer, after he has completed half his observations on one face, must transit and pick up the same star for the other half of his observations. The following is an effective method of achieving this but it is of course not the only one, which can be used.

Let it be assumed that a series of vertical circle readings with their corresponding times have been observed and noted in a field book by the recorder, who now informs the observer that he must change face by saying to him "Transit to your back-bearing". The observer immediately makes a quick observation of the horizontal reading to the nearest 5 or 10 minutes of arc, calling it out to the recorder, who notes it. The observer immediately swings the telescope round to a horizontal circle reading differing from the observed one by 180° and clamps the horizontal circle there. In the meantime, the recorder estimates say two minutes as the time, which will be spent in this procedure of transiting and, from the list of vertical circle observations already observed, he estimates the change in vertical circle reading in this period and works out the reading, which would have been obtained at this time. This reading is then converted to a corresponding value on the other face, for example on a Wild T2 theodolite, if the face left reading is predicted as $42^{\circ}25'$, the converted value would be $317^{\circ}35'$ if the index error of the vertical circle is small. By this time, the observer should have completed his part and he will be asking for the setting to be put up on the vertical circle. He is told the value and runs the telescope up to this value, at which the telescope is clamped. He now looks through the telescope and the star should be somewhere near the centre of the field of view of the telescope. It is of value for an observer to learn to estimate star magnitudes with reasonable accuracy, as this adds to certainty in relocating the correct star in the field of view. This estimation of the magnitude is not difficult to learn, but it must be remembered that the star sighted is infinitely distant and that therefore its image is not magnified by viewing it through the telescope.

Some instruments are fitted with a diopter sight on the telescope instead of gunsights. This enables pointings to be made with considerable accuracy because, if the one eye is put close up to the sight, the cross can be seen, even if there is only very little scattered light about. If at the same time, the other eye is kept open, the cross can be placed accurately on the desired star before transiting and the star identified in this way after transiting, provided that the sight is in correct alignment with the line of sight in the telescope.

The above methods of relocating the same star on transiting are used when the star is moving parallel to one of the two main cross hairs. When this is so, it is much easier to relocate the star on the other face, because only one of two settings is varying. When a meridian or circum-meridian observation is made, the star's altitude is hardly changing. If this is so, the azimuth change should be precomputed or estimated from the fruits of past experience. If on setting the value for this azimuth perhaps somewhat tardily, the star is not found in the field of view, it can be picked up by the technique of "hosepiping", i.e. by leaving the vertical circle as set, unclamping the horizontal circle and rotating the telescope slowly about the vertical axis, while the observer looks through the telescope. The star is usually found quite easily by this method.

When a circum-elongation azimuth observation is made, a similar hosepiping technique can be used with rotation of the telescope about the horizontal axis, because the star is then moving very slowly in the horizontal direction. When such an observation has been made on one face, the telescope is transited to the backbearing and then elevated with the observer looking through the telescope. If the star is not found by this searching in a vertical direction, the recorder will be able to supply the vertical circle setting at which the observer can clamp the telescope and, on looking through it, he will normally find the star in the field of view.

This whole section has been dealt with on the assumption that an infinitely distant star, which is seen as a point of light of no breadth is being observed. There is, however, one star, namely the sun, which is not infinitely distant and therefore subtends a broad disc of light on which observations must be made. Observations to the sun will be dealt with under that heading (see Chapter 8).

If a predicted programme is being observed, the above process of "transiting to your backsight" can be dispensed with, if the preliminary computations are such that the predicted values are computed at points, which are 10 to 15 minutes apart in time and a table of values at every second minute has been produced by a linear interpolation between the two computed points.

OBSERVING TECHNIQUES

Vertical Circle Observations for the Determination of Timed Altitudes

4.21 In this type of observation, the star or body sighted should be observed at the point of intersection of the vertical and the horizontal crosshairs. It is most important that this should be done if the horizontal crosshair is seen to be out of the horizontal. This seldom occurs but, with a newly acquired instrument, the horizontal crosshair should be tested for deviation from the horizontal and adjusted, if this is found to be necessary. If then, a vertical observation is made at a point on the horizontal crosshair adjacent to, but not exactly on, the intersection with the vertical hair, an error will be introduced. This will depend on the altitude itself as well as the distance from the intersection, but it is comforting to find out from investigation that this error is a surprisingly small one, and therefore, only when particular accuracy is required, need the intersection point itself be used to sight for the measurement of a vertical angle. The thoughtful observer should have got into the good habit of observing alternately on either side of the point of intersection of the crosshairs, so that there would be the tendency to eliminate any residual effect of the possible skewness of the horizontal crosshair.

That the error, produced by this method of observation, is small, will now be shown. Fig. 4.1 shows the horizontal or trunnion axis PQ of a theodolite with its line of sight OX. If a vertical circle observation is made not at the intersection point X of the vertical and horizontal crosshairs but at a point A as shown, the measured zenith distance z will be that corresponding to the point X and not to the point A. If the distance AX, equal to β, and the reading z are known, the correction Δz, which must be added to z, can be determined from the right-angled spherical triangle ZXA from the Cosine Formula. The vertical crosshair defines a vertical circle passing through the zenith Z, while the horizontal crosshair defines a great circle, which is perpendicular to that of the vertical crosshair and which contains the horizontal axis PQ

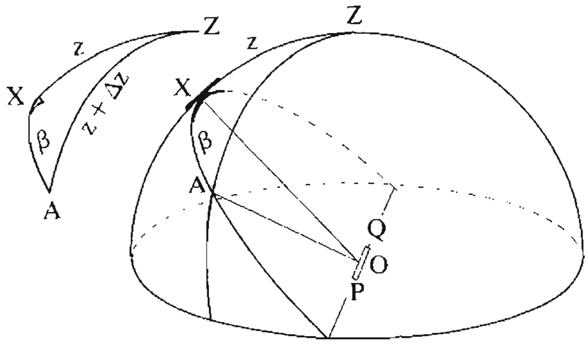


Fig 4.1

The Cosine Formula gives

$$\cos(z + \Delta z) = \cos z \cos \beta + \sin z \sin \beta \cos 90$$

$$\therefore \quad \cos(z + \Delta z) = \cos z \cos \beta$$

$$\therefore \quad \cos z - \Delta z \sin z \simeq \cos z\,(1 - 2\sin^2 \beta/2)$$

$$\Delta z \sin z \simeq 2 \sin^2 \beta/2$$

Since β is a small angle

$$\Delta z \simeq \frac{\beta^2}{2\rho \sin z}$$

Even if, for instance a high sight of zenith distance 30° is assumed and a rather large value of β of, for instance, one tenth of the distance from the centre hair to the lateral stadia hair is postulated, the value of the correction Δz still comes out very small indeed. In the above example, $\beta = 0.5 \times 0.01 \times 0.1 = 0.005$ radian = 103 arc seconds and from this

$$\Delta z = (103^2 \operatorname{cosec} 30^{\circ}) \div (2 \times 206265) = 0.05 \text{ arc seconds}$$

This correction is small and therefore, in all but the higher class work, this type of observation may be made on the horizontal hair just off from the intersection of the two crosshairs. But this does not mean that an observation may be made anywhere along the horizontal crosshair, with the expectation that a good vertical circle observation will then be obtained.

If the star to be observed is moving diagonally across the field of view, such as one from which a longitude from timed altitudes is obtained, then the vertical circle reading observation is made by setting the horizontal hair somewhat ahead of the star with the cross in such a position that the star, when it reaches the horizontal hair, will be close to the cross. The star is then allowed to make its own passage across the horizontal hair and the instant of its passage is timed. The alidade or vertical circle bubble is carefully trimmed to centre and the vertical circle then read. If, however, the star to be observed is moving with a small component in the vertical direction, the vertical circle reading observation is made by setting the horizontal hair exactly on to the star, with the vertical hair close to, but not necessarily on, the star. As the star is accurately bisected by the horizontal hair, the time is noted. The vertical circle bubble is set to centre and the vertical circle is then read. This is the type of observation made in determining latitude from meridian or circum-meridian observations.

Horizontal Circle Observations for Determination of Time Azimuths

4.22 In this type of observation, the star is observed on the vertical crosshair with the horizontal crosshair close to, but not necessarily exactly on, the star. If the star being observed is approaching the vertical crosshair fast, this crosshair is set ahead of the star in such a position that the star will cross the vertical hair near the point where the horizontal hair crosses it. The time of passage is then noted and the horizontal circle reading is observed. This type of observation occurs when a star or the sun, being observed for azimuth, is not sighted at the special positions, such as elongation, where its rate of change of azimuth with respect to time is small.

If the star being observed is approaching the vertical crosshair slowly, this hair at a point near the horizontal crosshair is placed on the star to bisect it accurately. The time of the instant of bisection is noted and finally the horizontal circle reading is observed. This is the type of observation when a star near to elongation or very close circumpolar star, such as Polaris or Sigma Octantis, is being sighted.

For an azimuth determination, horizontal circle readings to a mark must also be observed. The mark, used for reference, must be placed or selected so that it is sufficiently distant to require no change in the stellar focusing of the telescope. This requires the mark to be further from the theodolite than about two kilometres. The reference object for night work is a lamp or a light source and care must be taken that it is accurately centred over the gound mark, which indicates the station's position. The light should provide an image, which resembles a third magnitude star and to achieve this, the lamp should be provided with suitable stops for this purpose.

Altazimuth Observations

4.23 In this type of observation, the star is observed *exactly* at the intersection of the vertical and the horizontal crosshairs. If the star is moving in a vertical direction at a greater rate than in the horizontal direction, the horizontal hair is placed ahead of the star. When it gets close to this hair, the vertical hair is shifted by manipulation of the horizontal slow motion screw to bisect the star. This bisection of the star is maintained until it reaches the horizontal hair, at which instant tracking is stopped. The alidade bubble is trimmed and both circles are read and the readings noted. If the relative rates are reversed, the vertical hair is set ahead of the star and tracking is carried out by means of the vertical slow motion screw until the star is bisected by both crosshairs.

Sometimes the time of the instant, at which the star is bisected by both hairs, is required. This is observed as well as the readings on both circles This type of observation is required when an identification sight may be needed. This type of sight serves as a means of determining the right ascension and declination of the star sighted. (see section 10.91)

The Technique of Orienting the Horizontal Circle from Star Sights

4.31 Normally the observer has precomputed the azimuths to two stars, which are usually bright ones, which he knows so that he may orientate his horizontal circle on one and then check this on another about five minutes later. For most purposes, such orientation is sufficiently accurate if it produces a result, which is within one tenth of a degree. Further detail is available in section 10.11.

4.32 When the observer has the precomputed values for orientation to a known star, he sets the predicted altitude on the vertical circle and then reads this setting back to his recorder for checking. At this point, some theodolites require the horizontal circle to be set so that its reading is the same as the precomputed azimuth value. With others, however, the required reading can be set after the sight is made. At about three minutes before the time predicted for these settings, the telescope is directed towards the known star and the star caught in the field of view of the telescope. If the star is reasonably bright and it is not found, the vertical circle reading should be checked. If this is correct, there is always the possibility that the telescope is not focussed for infinity. If all is well, however, the star is tracked accurately by means of the appropriate horizontal slow motion screw so that the cross is on the star, as this reaches the horizontal hair; the observer calls out when this occurs and the recorder notes the clock time. Tracking is then stopped and, if the theodolite is one, which requires the horizontal circle reading to be set at this point, this setting is carried out. Immediately thereafter, the line of sight is directed to the reference object. This is bisected and the horizontal circle reading is observed and noted. This gives the azimuth to the reference object and it is recorded so that it can later be used if the orientation is, for some reason, disturbed. The next orienting star is then observed in exactly the same way. If all is well, the observer can then proceed with the rest of his programme. To be doubly sure, the inexperienced observer will probably have three orienting stars in his list, with values precomputed at five minute intervals.

The Observing of a Predicted Programme

4.41 THE following points are applicable to practically any type of predicted astronomical programme.

The observing party should arrive at the observing station with plenty of time for carrying out all the preliminary tasks well before the predicted programme is to be started. It is absolutely necessary that everything is ready, as any sense of hurry is most distracting. All must be calm and everything well under control.

This implies that the whole programme has been well thought out and proper preparations have been made in the time before the observing season, so that the equipment is in first class condition before the observing party leaves base. When the observing site is reached, the equipment must be set up and given a final testing. The lighting apparatus is tested, the radio set up and tuned to receive the time signals. A referring object, which is usually a light, must be set up over a distant mark. Sometimes, as in the observing of geodetic azimuths, the light is set over an adjacent primary station of the geodetic survey and this may be up to fifty kilometres distant. The chronometer, chronograph or timing apparatus being used should likewise be set up and tested.

The theodolite is set up and levelled and it is most important that it should be properly focussed on a distant object and tested for any parallax. Determination of vertical circle index correction should be carried out. The clock should be set so that it is close to local sidereal Time or Zone Time. If the latter is set on the clock, a wrist watch should be set to read LST so that it can be used in conjunction with the Working List, on which the predictions are set out with respect to LST, and not Zone Time, if stars are being observed.

At about 10 minutes before the start of the predicted programme, the first clock comparison with respect to the radio time signals is made. In this set, there should be included a time signal, which is absolutely certainly identified. The working list now starts with the orienting sequence previously described and then the programme goes on to the observing sequences predicted. As the time for observation on the first star predicted approaches, any preliminary observation required before the actual star is sighted is carried out, such as for instance the horizontal circle readings to the RO, if azimuth is being observed. Then the theodolite is swung round in azimuth until the predicted reading is obtained and this is set on the horizontal circle. The telescope is elevated until the required vertical circle setting is reached. This is set and both circles read out aloud to the recorder, who checks these values against the working list. About two minutes before the predicted time the star should appear in the field of view and at the predicted time it should be close to the intersection of the crosshairs. Under the direction of the recorder, whose task it is to assume overall responsibility for controlling the times of making the observations to achieve the balancing conditions, included in the prediction of the programme, the observer is led through the whole observing programme.

CORRECTIONS TO OBSERVED QUANTITIES

Theorectical considerations indicate that observed quantities are subject to discrepancies, for which allowance must be made. In some cases, the theory also enables the correction to be evaluated and it is then applied. In other cases, the correction cannot be evaluated but often a certain procedure of observation, by means of which the effects of the discrepancies can be eliminated, can be worked out.

Index Corrections to Vertical Circle Observations

4.51 Observed altitudes, derived from observation on one face of a theodolite, normally have the index error of the vertical circle applied to them so that the quantities calculated from them can be in fair agreement with the values derived from the observations made on the other face.

Astronomical Refraction

4.52 Light, coming from a celestial body towards an observer on the earth, travels in a straight line through the vacuum of outer space until it enters the earth's atmosphere. Since it then continues through a medium constantly increasing in density, the light ray, if not normal to the outer boundary of the atmosphere, will be bent towards the normal. This bending will take place in a vertical plane; occasionally there is a minute deviation laterally from the vertical plane. The total amount of bending through the atmosphere is the angle of refraction and it depends on the composition and state of the atmosphere, being traversed, and also on the size of the incident angle.

A simple proof of a formula for refraction, according to Newcomb, will be given. It is not a rigorous proof, but one that gives results close to those of more complex theories. Because the distance to the outer limit of the earth's atmosphere is small compared with its radius, it will be

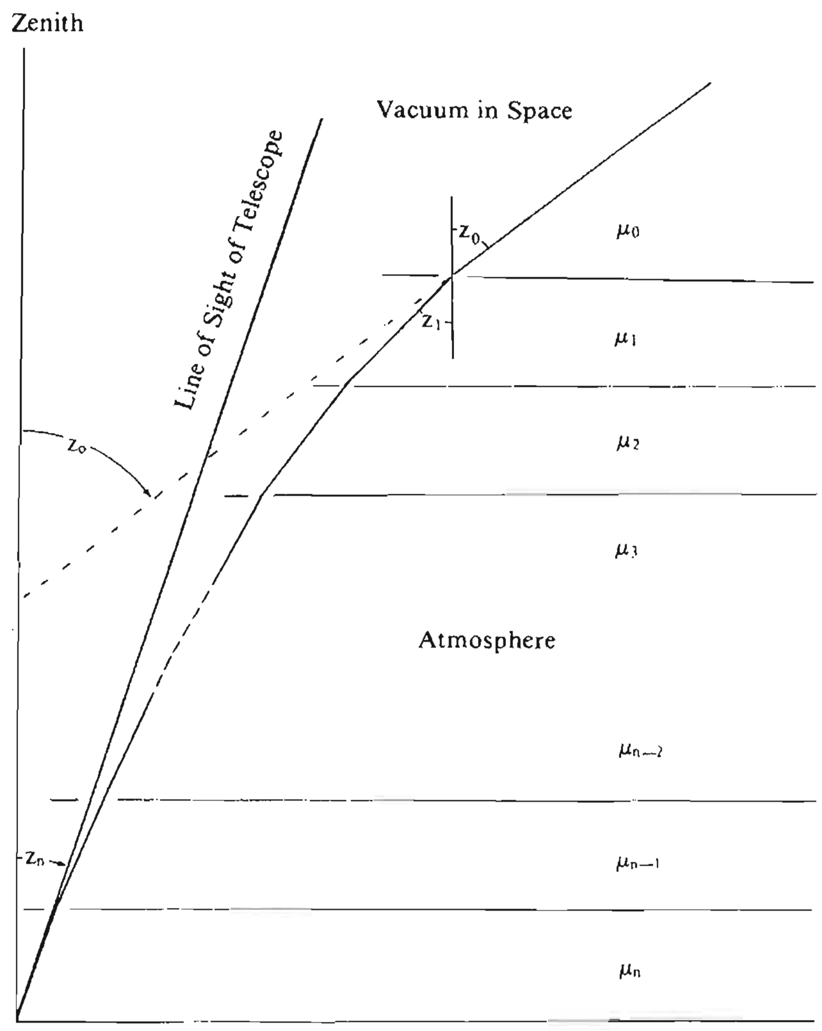


Fig. 4.2

assumed that the atmosphere in the vicinity of the observer's station consists of thin plane layers, each with a constant absolute refraction index μ and that the light path through each layer is straight.

Figure 4.2 shows a section through the earth at an observer's station P. Snell's law of refraction at the interface between adjacent layers gives

$$\mu_0 \sin z_o = \mu_1 \sin z_1$$

$$\mu_1 \sin z_1 = \mu_2 \sin z_2$$

$$\vdots$$

$$\mu_{n-2} \sin z_{n-2} = \mu_{n-1} \sin z_{n-1}$$

$$\mu_{n-1} \sin z_{n-1} = \mu_n \sin z_n$$

Thus it can be seen that

$$\mu_o \sin z_o = \mu_n \sin z_n$$

and that the refraction is given by

$$r = z_o - z_n$$

Substitution therefore gives

$$\mu_o \sin(z_n + r) = \mu_n \sin z_n$$

Substituting $\mu_o = 1$ and expanding in a Taylor's Series, to first order terms only, gives

$$\sin z_n + r \cos z_n = \mu_n \sin z_n$$

$$\therefore \qquad r = \rho\,(\mu_n - 1) \tan z_n$$

in which r and ρ are in the same units and ρ is the number of such units in one radian.

Putting an average value of the refractive index of air into the above gives a value of mean refraction of

$$r''_o = 60.1 \tan z$$

in which z is the observed zenith distance.

A comparison of refraction given by this formula and that given by more sophisticated formulae is shown in the following table:-

Zenith Distance	0°	30°	60°	75°	90°
Refraction from tables	0"	34"	1'40"	3'34"	1°06'29"
$r''_o = 60.1 \tan z$	0	35	1 44	3 44	∞

The simple refraction formula is seen to give results, which, in the light of the great simplifications made, are, up to zenith distances of 60°, in very close agreement with the values obtained from the more sophisticated formulae. If the simple formula is modified to take into account the spherical shape of the atmosphere's layers as well as the variations of pressure and temperature from those of the standard atmosphere, the following relationship is produced

$$r'' = \frac{P}{1013.25} \; \frac{273.2}{273.2 + T} \,(60.1 \tan z - 0.07 \tan z \sec^2 z)$$

in which P is in millibars and T is in degrees Celsius.

This relationship gives refraction values, which are adequate up to zenith distances of about 75° but beyond this, the values become inaccurate. If vertical angle observations with any pretensions to accuracy are to be made, the sights should be made at zenith distances not exceeding 75°; then in this way uncertainties in the refraction itself are avoided.

The refraction tables of "The Star Almanac for Surveyors" will suffice where very high accuracy is not required. These are based on the Harzer formulae and may be relied on to 1" up to 60° and 2" from 60° to 80° zenith distance. For a detailed discussion and derivation of astronomical refraction, articles by J. Saastamoinen in the Bulletin Geodesique 1972-1973 Nos. 105, 106, 107 should be consulted.

All formulae for refraction assume that refraction is independent of azimuth. In practice this may not be so, especially if the country in the vicinity of the observing station has considerable variation in relief or in vegetation. The temperature measurements made during an observing period should be made with the thermometer held well above the ground and, if made in sunshine, the thermometer must be shaded.

Differential Refraction

4.53 For observations, in which zenith distances are observed, predicted programmes are arranged on the assumption that there may be a difference between the calculated and the actual values of refraction. This difference is assumed to remain constant for a short period of time for a given zenith distance. Thus, if all observations are made in quick succession at about the same zenith distance, then this difference will take on the characteristics of a constant error and arrangements in the observing programme can be made to eliminate the effects of such a type of error.

If however, the observing period for a set of observations is of any length of time, observations of pressure and temperature should be made through the observing period, so that all observed zenith distances can be corrected to values at a common pressure and temperature. To obtain these changes in refraction for changes in pressure, temperature and zenith distance, the refraction formula is differentiated as follows:-

$$r'' = \frac{P}{1013.25} \frac{273.2}{273.2 + T} 60.1 \tan z$$

with P in millibars, T in degrees Celsius and z in sexagesimal units.

$$dr'' = 16.20461 \left[\frac{dP \tan z}{(273.2 + T)} - \frac{P\, dT \tan z}{(273.2 + T)^2} + \frac{P \sec^2 z\, dz}{(273.2 + T)}\right]$$

$$dr'' = 16.20461 \frac{P \tan z}{(273.2 + T)} \left[\frac{dP}{P} - \frac{dT}{(273.2 + T)} + \frac{dz}{\rho \sin z \cos z}\right]$$

$$= r''_{PTz} \left[\frac{dP}{P} - \frac{dT}{(273.2 + T)} + \frac{dz}{\rho \sin z \cos z}\right]$$

4.531 Example. Find the refraction r for P = 1023.25 mb, T = 10°C and z = 60°, and then find the change in refraction brought about in this value by a decrease of 10 mb, an increase of 2.8°C and a 1° decrease in zenith distance. The value of r_{PTz} may be calculated with sufficient accuracy for this purpose from the following:-

$$r''_{PTz} = 16.20461 \frac{1023.25}{273.2 + 10} \tan 60° = 101.4''$$

$$dr'' = 101.4 \left[\frac{(-10)}{1023.25} - \frac{(+2.8)}{283.2} + \frac{(-1)}{57.296 \sin 60 \cos 60}\right]$$

$$= -0.99 \qquad -1.00 \qquad -4.09$$

$$= -6.08''$$

Find by how much this quantity is incorrect if dP is incorrect by 4 mb, and dT by 0.5°C and dz by 10 minutes of arc.

Each quantity calculated above must change proportionally to the changes computed immediately above.

$\therefore$ Error in dr $= (\frac{4}{10})\ 0.99'' = 0.40''$ for the error in the pressure value

$= \frac{0.5}{2.8}\ 1.00 = 0.18$ for the error in the temperature value

$= \frac{10}{60}\ 4.09 = 0.68$ for the error in the zenith distance value

The above example is instructive in showing up the magnitudes of the errors, which arise from discrepancies, which have purposely been taken as rather large ones. If good work is to be done, altitudes less than 30° should not be used and also a method of observing should be used which minimises the effects of any systematic errors due to refraction.

4.532 It is not necessary to have instruments, which give the exactly correct readings of the ambient temperature and pressure of the atmosphere, provided the observation programme has been designed to minimize the effects of systematic errors in refraction. Such instruments should, however, be able to monitor precisely any changes in the temperature and pressure of the air throughout the observing programme. A simple example to demonstrate the principles implied will be given.

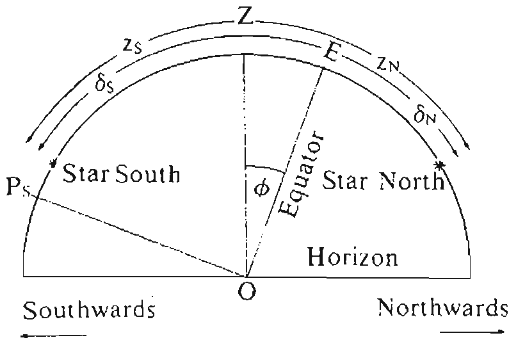


Fig.4.3

4.533 Fig 4.3 shows a meridian section at a station O. The two stars shown were observed at meridian transit in order to find the station's latitude.

Observations made:-

Atmospheric Readings	Temperature	=	18°C
	Pressure	=	930 millibars
Declination of star north		=	39°10'23" north of equator
" " " south		=	77°03'48" south " "
Observed zenith distance to star north		=	59°09'58" north of zenith
" " " " " south		=	57°01'25" south of zenith

Relationships used

Refraction (see section 4.52)

$$r'' = \frac{P}{1013.25} \frac{273.2}{273.2 + T} (60.1 \tan z - 0.07 \tan z \sec^2 z)$$

Differential Refraction (see section 4.53)

$$dr'' = r''\left[\frac{dP}{P} - \frac{dT}{(273.2 + T)} + \frac{dz}{\rho \sin z \cos z}\right]$$

By inspection, the latitude is about 20° South

Direct calculation of the latitude

		Star north		Star south
Observed Meridian ZD	z_N	59°09'58"	z_S	57°01'25"
Refraction	r_N	1 26.3	r_S	1 19.4
	z_N+r_N	59 11 24.3	z_S+r_S	57 02 44.4
Declination	δ_N	39 10 23	δ_S	77 03 48
Latitude		20 01 01.3	Latitude	20 01 03.6
Mean Latitude		20°01'02.5		

Alternative calculation of the latitude from the Mean Differences

From inspection, latitude $= (z_N + r_N) - \delta_N$ for star north

latitude $= \delta_S - (z_S + r_S)$ for star south

$\therefore$ Mean latitude $= \frac{1}{2}(\delta_S - \delta_N) + \frac{1}{2}(z_N - z_S) + \frac{1}{2}(r_n - r_S)$

$= \frac{1}{2}(\delta_S - \delta_N) + \frac{1}{2}(z_N - z_S) + \frac{1}{2}\, dr$

in which dr has the same sign as $dz = (z_N - z_S)$

$$\tfrac{1}{2}dr'' = \bar{r}''\left(\frac{\frac{1}{2}\, dz}{\rho \sin \bar{z} \cos \bar{z}}\right)$$

in which $\bar{r}''$ is found for the mean value z in the Star Almanac for Land Surveyors. For this calculation δ_S and δ_N are taken as unsigned quantities.

δ_S	77°03'48"		
δ_N	39 10 23		
$(\delta_S - \delta_N)$	37 53 25	$\frac{1}{2}(\delta_S - \delta_N) =$	18°56'42.5"
z_N	59 09 58		
z_S	57 01 25		
$(z_N - z_S)$	2 08 33	$\frac{1}{2}(z_N - z_S) =$	1°04'16.5" = 3856.5"

$$\tfrac{1}{2}dr'' = 83''\left(\frac{+\ 3856.5''}{206265''\sin 58°06'\ \cos 58°06'}\right) = 3.5''$$

Sum = Mean Latitude 20°01'02.5" South

Demonstration of the Effect of Systematic Error

If the true values of the atmospheric readings were not 18°C and 930 millibars but were 23°C and 910 millibars, it is required to determine the error produced in the latitude sought.

Observed Meridian ZD	z_N	59°09'58"	z_S	57°01'25"
Refraction	r_N	1 23.0	r_S	1 16.4
	z_N+r_N	59 11 21.0	z_S+r_S	57 02 41.4
Declination	δ_N	39 10 23	δ_S	77 03 48
Latitude		20 00 58.0		20 01 06.6
	Mean Latitude	20°01'02.3"		

This example demonstrates two points. The large discrepancies in the atmospheric readings produce quite small changes in the refraction. Also the changes so produced affect the two latitude values computed in opposite ways so that the mean value is almost unaffected. The elimination of the effects of this kind of systematic error is a feature common to all observations on balanced sets of stars.

Parallax

4.54 An observer makes his observations to celestial bodies from a position on the earth's surface. Since data for these bodies is geocentric data, the surface observations must also be reduced to the centre of the earth. This correction must be made only when the earth's radius has an effect on the quantity measured. Horizontal circle readings are not affected but vertical circle readings are.

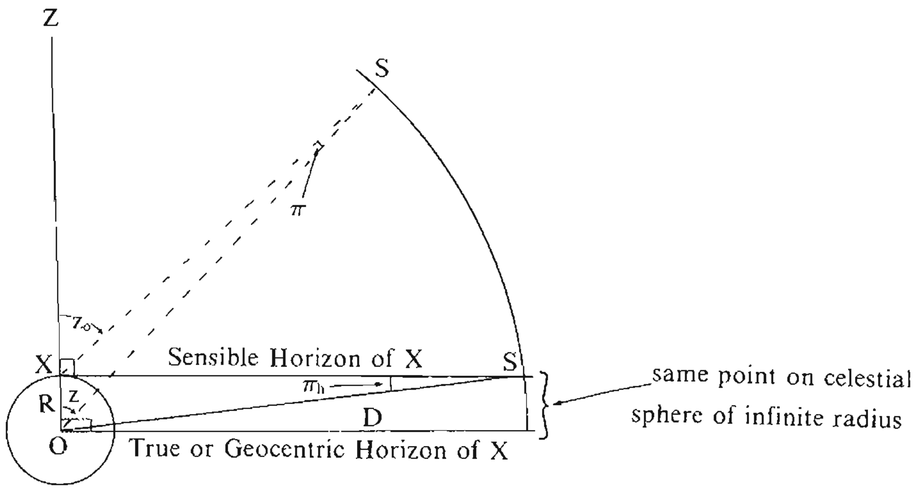
Let O be the centre of the earth of radius R, and S a celestial body, which is distant D from the earth and which is on the sensible or visible horizon of the station. Let S be observed, at the station X, to have a zenith distance of z_o. It is required to find z the corresponding geocentric zenith distance.

From the Sine Rule in the Triangle XOS

$$\frac{\sin \pi}{R} = \frac{\sin(180-z_o)}{D}$$

$$\therefore \qquad \pi = \frac{R}{D}\rho \sin z_o \qquad \text{since } \pi \text{ is a small angle}$$

When z_o is a right angle, i.e. when S is on the sensible horizon of X, π has a maximum value of π_h known as the horizontal parallax



Earth Fig.4.4

$$\therefore \quad \pi_h = \frac{R}{D}\rho$$

$$\pi = \pi_h \sin z_o$$

$$\therefore \quad z = z_o - \pi = z_o - \pi_h \sin z_o$$

in which $\pi_h \sin z_o$ is the correction to z_o for parallax. For a star, other than the sun, $\frac{R}{D}\rho$ is negligible because D is very large. For the sun, however,

$$\pi_h = \frac{6380}{150\times10^6} \cdot 206265'' = 8.8''$$

This quantity is not quite a constant, but varies slightly, because D varies throughout the year.

TWO TYPES OF ERROR

IN field astronomy the elimination of errors plays an important part in devising the various observation techniques. Although some mention has already been made of errors and their effects, a more detailed explanation should now be given.

All observed quantities are subject to errors, however skilled or meticulous the observer may be. For example, such errors would be due to the observer's inability to point exactly to a star or estimate precisely the value of a theodolite circle reading. These errors are for the most part small ones. In addition to these errors, there are others of a systematic nature, whose behaviour, if known, can be allowed for. Even if the values of these latter errors are not known their effects on the end result may be minimised by using special observing and instrumental techniques. A particular class of systematic error, which occurs frequently in astronomical work, is the constant error, i.e. an error which remains constant in size and sign.

4.61 If an observer is well trained and competent, his observational errors tend to a "random" pattern, which in survey observations almost always follows a "normal distribution". This distribution displays the following features:-

(i) The frequency of the occurrence of a random error of a certain magnitude, irrespective of sign, is inversely related to its magnitude.
(ii) The number of such positive errors will be nearly equal to the number of negative ones.
(iii) The arithmetic mean of a set of observations is more likely to be near the truth than any single value, provided that a large number of observations is made.

If small numbers of observations are made, the statistical quantities obtained from these quantities do not have great reliability and become only estimates of the precision of the results obtained.

4.62 Neglecting to correct observations for instrumental constants causes systematic errors in the quantities sought. Very often an instrumental constant can be determined by observation. For example, the vertical circle index correction of a theodolite can be readily determined by face left and right observations. If some time later these observations are repeated it may be found that a slightly different value of the index correction results. This difference may be entirely due to random errors of observation or there may have been a small change in the index correction in the intervening time between observations. Thus an instrumental constant, supposedly of constant value, may be affected by both random and systematic errors and it may not be possible to separate these two components. To guard against these possibilities, observations, including those made for determining instrumental constants, should be made over as short a time interval as possible.

If instrumental constants are not determined and therefore suitable corrections not applied to the observations, the routine of observation should be such as to exclude their effects from the final results, such as observing on both theodolite faces. Alternatively these constants may be included as additional unknowns in the final solution for the main unknown(s).

THE DETECTION OF POOR OBSERVATIONS OR BLUNDERS IN A SET OF OBSERVATIONS

4.71 WHEN sets of corresponding pairs of quantities, linked by some mathematical relationship, are observed, they should be tested, so that the presence of poor observations or blunders may be detected.

A simple way of doing this is a graphical one, which consists in plotting one member of each pair as the ordinate and the other as the abscissa. The points so plotted should then show up as a smooth curve. This will of course not show up exactly because of the presence of unavoidable small random errors, which are present in even the best observations. If however, any point deviates considerably from the smooth curve drawn through the points plotted, the observations producing this deviant point should be scrutinised to determine the cause of the discrepancy.

The error may be due to a blunder, the cause of which can often be surmised, if it is known from experience or from the type of equipment being used, what kinds of blunders are commonly made. Some of these are the misreading of observed values by whole units or sets of units. Examples are the misreading of clock times by whole minutes or of a theodolite circle by ten minutes of arc. In addition, when an observation is made on a quantity, which is near the end of a unit, the fraction of the unit is correctly read but the next whole unit above is often mistakenly read in place of the correct value. An example is 47°17'58" read as 47°18'58", because the 8' value is visible at the same time as the 50" and 00" values. A clock value of $6^h42^m59.1^s$ can likewise be easily misread as $6^h43^m59.1^s$, because great concentration is given to obtaining the correct value of the small unit of the seconds and, by the time the minute hand is read, it is on the next minute value of 43^m; this mistake is more easily made if the minute hand is not properly set into coincidence with the minute mark, when the second hand is at the zero second reading.

If the circle left values of the observation are plotted they should form a straight line, provided the observations have been made in a short period of time, say of the order of a few minutes. If this period is longer a curvature in the line may be discernible depending upon the type of observation made. The corresponding circle right observations will define another line with similar slope but not necessarily collinear with the first, unless the instrument is in perfect adjustment.

For vertical circle observations the index correction is usually determined prior to the main observations and this when applied should make all observation points collinear. Additional information relating to the slope of this line is also available from a knowledge of the time rate of change of altitude given in section 2.75 as

$$\frac{dh}{dt} = \cos\phi \sin A$$

in which the values of latitude ϕ and azimuth A need not be known precisely. The azimuth value may be obtained from prediction information or from oriented horizontal circle readings made at the time of observation.

The graphical method mentioned above may be impractical because a large scale may need to be used for the plotting in order to show up the errors. Since the observations often extend over some time, the graphical method would require large plotting sheets.

Instead of plotting, the investigation can be done by calculation. A simple way is to displace each observation, by means of the given slope, to a selected value of one of the variables. If the slope is the average for the set of observations the value of the variable selected is best taken near the middle of the observations.

4.72 Example. Time altitude observations on the Star Betelgeuse (No. 162)

Latitude	33°55'S	Temperature	22°C
Clock Mean Time		Pressure	1016 mb
Mean Azimuth	314°12'	Vertical Circle Index	-1'45"

Observations and preliminary reductions

Observed Clock Time	Observed Vertical Circle Reading	Index Corrn.	Refrn.	Reduced Altitude h_o
$3^h52^m59.5^s$	52°51'21"	-1'45"	+1'14"	37°09'10"
53 31.6	52 56 07	-1 45	+1 14	04 24
54 00.4	53 00 24	-1 45	+1 14	37 00 07
54 38.5	53 06 01	-1 45	+1 14	36 54 30
55 55.5	306 45 59	-1 45	-1 15	36 42 59
56 33.3	40 18	-1 45	-1 15	37 18
56 58.7	36 37	-1 45	-1 15	33 37
3 57 25.4	306 32 55	-1 45	-1 15	36 29 55

The time rate of change of altitude in arc seconds per second of mean time will be

$$\frac{dh}{dt} = 15\,F \cos\phi \sin A = -8.948$$

where F = 1.0027379

The reduced altitudes are now displaced to a common fictitious clock time of observation of $3^h55^m00^s$ using the relationship

$$h_c = h_o + \Delta T \frac{dh}{dt}$$

where ΔT is the difference between the selected clock time and the observed clock time. The agreement between individual values of h_c is the criterion of the quality of the set of observations.

CT	ΔT	Δh	h_o	h_c
$3^h52^m59.5^s$	$+2^m00.5^s$	-17'58"	37°09'10"	36°51'12"
53 31.6	+1 28.4	-13 11	04 24	13
54 00.4	+0 59.6	- 8 53	37 00 07	14
54 38.5	+0 21.5	- 3 12	36 54 30	18
3 55 00				
55 55.5	-0 55.5	+ 8 17	36 42 59	16
56 33.3	-1 33.3	+13 55	37 18	13
56 58.7	-1 58.7	+17 42	33 37	19
3 57 25.4	-2 25.4	+21 41	36 29 55	36 51 36

It can be seen here that the results appear to be satisfactory except for the last one which lies about 20" away from the others.

If a value of h_c is substantially different from the others and this difference is say 10', then this could be attributed to a misreading of the vertical circle and the observation corrected. On the other hand a blunder could have been made in the associated clock time of observation which would be

$10'\frac{dt}{dh} = 1^m07^s$; such an odd misreading would be highly unlikely.

Checking of Calculations

4.73 Whatever methods are used for calculation the correct answer must be produced and the calculation must therefore be checked. Care must be exercised and the task of computing carried out in a systematic and objective manner. Input data, such as the station position, the star coordinates and the like must be carefully checked and the corrections of the input must be monitored. The person performing the calculations should be well trained and the importance of correct computing must continually be stressed.

The ideal method is to have the field book handed to a computer, who obtains any additional data required and then carries out the computation. This procedure is repeated by a second computer, who should remain unaware of the identity of the first one. The computations are then compared with each other. In this way, independence is achieved. The independence of the process is its chief guarantee of correctness.

If only one person does the computation, he should be shown where mistakes can easily occur, so that he can guard against them. Different methods of solution should be used to check the possibility of errors in manipulation or arithmetic. Checks should be incorporated in the field work to eliminate the possibility of blunders.

If a programmable calculator is used, the programme should be *thoroughly* tested against data beforehand to guard against mistakes in logic and execution. Checks against the entry of incorrect input data should be provided. The results should be checked when they are transcribed from the calculator. If a printer attachment is used and the programme has been thoroughly tested, the input data can be checked very positively because the results are available in the printed form.

Finally, the computer must keep watch over the process with his critical faculties always alert. When he has any uneasy feeling, even if it is only a momentary one, he is well advised to find the reason for this feeling. It very often has a valid reason behind it.

5
Determination of Latitude

INTRODUCTION

THE latitude ϕ of a station is its angular distance, from the equator, along the meridian of this station and it is defined as a positive value, if the station is north of the equator and negative, if south.

An observer, using a theodolite, has two methods available for the determination of latitude; he may time the passage of a known star across a known altitude circle or across a known azimuth circle. If these times are correlated with Greenwich Time by means of radio time signals, it is possible to determine the station's latitude. The first method will be dealt with, but the second, which is nowadays not used in practice, will not be pursued further. A reference to it may be found in W. Chauvenet, *A Manual of Spherical and Practical Astronomy*. Philadelphia. 1863, which has been reprinted by Dover Publications, Inc., New York, 1960.

LATITUDE FROM TIMED ALTITUDE OBSERVATIONS

5.11 IN this method, a known star is sighted and an altitude is observed. The clock time, at which this observation was made, is noted. The clock correction with respect to Greenwich Time is observed. The corresponding hour angle t is found from the observed clock time, the observed clock correction, the longitude of the station and the right ascension of the star. The observed altitude is corrected for index error and refraction to give the altitude h of the star at the moment of observation.

The latitude ϕ is now to be determined from t, h and δ, the known declination of the star observed, by means of the Cosine Formula, which relates these four elements, in the form

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t \qquad \ldots 5.1$$

Variations dt, dh and $d\delta$ will produce a variation $d\phi$. These four quantities are related by means of differentiation of Equation 5.1 and the result may be obtained from the differential relationships, summarised in section 2.75 as

$$dh = \cos\phi \sin A \, dt + \cos A \, d\phi + \cos\omega \, d\delta \qquad \ldots 5.2$$

This is manipulated to show $d\phi$ in terms of dt, dh and $d\delta$ as

$$d\phi = \sec A \, dh - \cos\phi \tan A \, dt - \sec A \cos\omega \, d\delta \qquad \ldots 5.3$$

For the change $d\phi$, produced by specific changes dh, dt and $d\delta$, to be as small as possible, the coefficients of these quantities should be made as small as possible.

5.12 At this point, something must be said about the possible magnitude of $d\delta$, the uncertainty in the star's declination. For all, but the most precise geodetic observations, the declination values published in reliable catalogues may be considered irreproachable and $d\delta$ may therefore be taken as zero (see also sections 6.12, and 7.31).

$$\therefore \quad d\phi = \sec A \, dh - \cos\phi \tan A \, dt \qquad \text{...5.4}$$

5.13 The observed altitude h and the derived hour angle t are known to be subject to errors dh and dt. If, in the first instance, these are taken to be random errors (see section 4.61), consideration should be given to the minimising of their coefficients. If the star is sighted so that the azimuth A is either 0° or 180°, then $\sec A$ will have its smallest value of ± 1 and $\tan A$ likewise its smallest value of zero. If the star is therefore observed on the meridian, the effect of the error dt will be zero and that of dh will show up fully in the derived latitude ϕ.

5.14 If the errors dh and dt are now taken to be systematic errors (see section 4.62), their effect $d\phi$ on the latitude sought must be considered. The longitude λ and the clock correction with respect to Greenwich Time, as adopted, may not have the exactly correct value. Therefore, the hour angle derived for *any* star will be incorrect by a constant, but unknown, amount dt. Similarly, the observed altitudes may also *all* be incorrect by a constant, but unknown, amount dh. This may result, for instance, from the refraction correction, taken from refraction tables, not representing correctly the conditions ruling at the instant of observation. Another example is that of vertical circle index error not being equal to the true value. This effect can be removed by taking the arithmetic mean of the latitude values computed from the face left observations, likewise for the face right observations and then taking the grand mean. The systematic error in refraction, however, cannot be thus removed because it affects the observations on both faces in the same way and not in opposite ways.

The relationship of equation 5.4 suggests that the effects of the systematic errors dh and dt may be eliminated, if two stars are observed such that the coefficients $\sec A$ and $\cos\phi \tan A$ for one star are equal in magnitude, but opposite in sign, to those for the second star.

If this is done, the observations on the first star will result in a derived value ϕ_1 of the latitude. Similarly, those for the second star will produce a value ϕ_2. The first, however, will be in error by $d\phi_1$ and the second by $d\phi_2$, such that the value ϕ of the latitude is given by

$$\phi = \phi_1 + d\phi_1$$

and

$$\phi = \phi_2 + d\phi_2$$

$$\therefore \quad \phi = \tfrac{1}{2}(\phi_1 + \phi_2) + \tfrac{1}{2}(d\phi_1 + d\phi_2)$$

$$\therefore \quad \phi = \tfrac{1}{2}(\phi_1 + \phi_2) \text{ only if } (d\phi_1 + d\phi_2) = 0$$

This requirement implies that $\sec A_1$ and $\tan A_1$ must be equal to $-\sec A_2$ and $-\tan A_2$ respectively and that both conditions must be satisfied *simultaneously*. This occurs when $A_1 + A_2 = 180^{\circ}$, i.e. when the two azimuths are symmetrical with respect to the prime vertical and when the two stars are at similar altitudes (see section 4.52 et seq). When this is done, the effects of systematic altitude and time errors are eliminated. However, for determination of latitude, stars in the vicinity of the prime vertical should be avoided, because sec A and tan A then tend towards very large values.

5.15 Exact balance in azimuth is hardly ever achieved, but any deviation from exact balance must be such that no significant error is introduced into the result obtained. Tolerable limits to such imbalance must, therefore, be determined. When the two azimuths are close to the meridian, this amount of imbalance will have small effect. It will, however, have a much larger effect, if the azimuths are far from the meridian, because the secant and the tangent are larger and are then varying at a faster rate. Therefore, to reduce the effect, the limit in imbalance must be reduced and balancing done more carefully.

THE CALCULATION OF LATITUDE FROM TIME ALTITUDE OBSERVATIONS

The General Method

5.21 EQUATION 5.1 gives the latitude ϕ, which is the unknown being sought from the altitude h, the hour angle t and the declination δ, in the form of an implicit equation, which can be solved by means of auxiliary angles. From section A.41 in the appendix, the following relationships are abstracted for this purpose:-

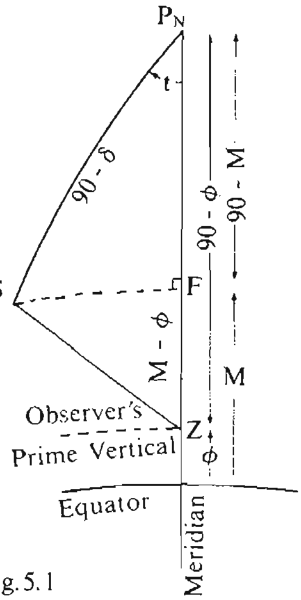


Fig.5.1

$$\tan M = \frac{\tan \delta}{\cos t} \qquad \ldots 5.5$$

and $$\cos(M-\phi) = \sin h \sin M \operatorname{cosec} \delta \qquad \ldots 5.6$$

Equation 5.5 gives M without ambiguity, if the signs of numerator and denominator are followed but equation 5.6 gives an ambiguity for $(M-\phi)$, because $\cos(M-\phi) = \cos(\phi-M)$. The angle obtained from the cosine is first of all chosen, according to its sign, to lie in either the first or second quadrant. If the star was observed to the north of the prime vertical this angle would be equal to $M-\phi$. However, for a star to the south of the prime vertical this angle would be equal to $\phi-M$. The latitude ϕ is then calculated using the value of M obtained from Equation 5.5.

Alternatively, if N is defined as

$$\cos N = \sin h \sin M \operatorname{cosec} \delta \qquad \ldots 5.7$$

and, if N is then given a sign, positive if the observed star is north of the observer's Prime Vertical and negative if south, the *general* relationship

$$\phi = M \sim N \qquad \ldots 5.8$$

will hold. Attention is drawn to the similarity that Equation 5.8 has to Equation 5.10.

Meridian Methods

5.31 These are methods, prompted by simplicity and backed up by the theory, which brings out that the star should be observed on the meridian for best results. In observatories, a telescope is permanently mounted so that its line of sight defines the meridian very closely; also, in the precise determination of latitude to geodetic accuracy at field stations, a large theodolite is set to define the local meridian and to observe stars as they cross the meridian. These observations are made on or very close to the meridian on sets of very accurately balanced star pairs. Generally only a single pointing is made on each star, but a large number of pairs is observed to achieve the precision required.

For latitude determinations of less than geodetic precision, a smaller theodolite is used. The meridian observation is still favoured and the tyro, preparing to make his first astronomical determination, may well decide to adhere to the meridian method. He may, of course, be influenced in making this choice by the fact that the prediction, the observing and the computing are very simple and straightforward. He will soon, however, find that the restriction to an observation on the meridian gives the poor return of only one observation per star. His "Star Almanac for Land Surveyors" will soon be found to have insufficient stars for his purpose. Also, he knows that the theodolite should be used to make observations on both faces and to make multiple observations to reduce the *effects* of random errors of observation. If the need to stay *exactly* in the meridian is slightly relaxed, such observations can be made. These considerations, therefore, lead directly to the circum-meridian method of latitude determination, in which the star is observed in the vicinity of the meridian, before during and after its transit.

5.32 The determination of latitude from timed meridian altitudes or zenith distances utilizes the best position for observing the star. Fig. 5.2 shows meridian sections, with the south pole as the elevated pole in Fig 5.2(a) and with the north pole as the elevated one in Fig 5.2(b). Stars are shown in

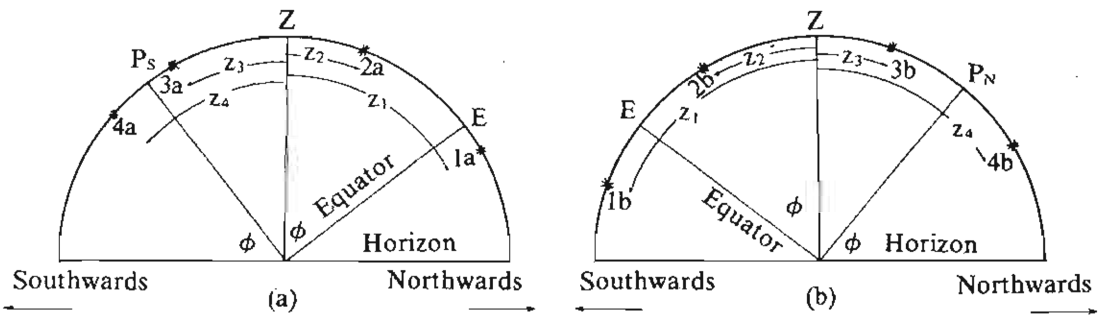


Fig. 5.2

four salient positions in each of these figures. Table 5.1 provides the numerical data for all the situations illustrated. By inspection, the latitude ϕ can be deduced.

Table 5.1

	Star	δ	z	ϕ
Upper Transit	(1a	8° N	60° N	52° S
	(2a	32 S	20 N	52 S
	(3a	82 S	30 S	52 S
Lower Transit	4a	78 S	50 S	52 S
Upper Transit	(1b	20 S	70 S	50 N
	(2b	20 N	30 S	50 N
	(3b	70 N	20 N	50 N
Lower Transit	4b	70 N	60 N	50 N

Declination δ is a quantity already defined. It starts from zero at the equator and goes to +90° at the north pole and to -90° at the south pole and is therefore restricted to the first or the fourth quadrant. Zenith distance z is also already defined and is restricted to first and second quadrant. In Fig 5.2 the zenith distances are meridian zenith distances, which are zenith distances measured either northward or southward. Such zenith distances, which will be denoted by z_M, may now be considered to have a sign, positive to the north of the observer's zenith from zero at this point and negative to the south. This, in effect, makes meridian zenith distance an angle going right round the local meridian through four quadrants. This corresponds to the vertical circle graduations on many of the modern theodolites. Table 5.2 shows up this information.

Table 5.2

	Star	δ	z_M	ϕ
Upper Transit	(1a	+ 8°	+60°	-52°
	(2a	-32	+20	-52
	(3a	-82	-30	-52
Lower Transit	4a	-78	-50	-52
Upper Transit	(1b	-20	-70	+50
	(2b	+20	-30	+50
	(3b	+70	+20	+50
Lower Transit	4b	+70	+60	+50

From this it can be seen that the latitude from an *upper* transit sight can be found from the relationship

$$\phi = \delta - z_M \qquad ...5.9$$

in which the subtraction is done algebraically.

5.33 Consideration will now be given to developing a relationship, which will include Lower Transit cases also. An observer's meridian is a local one, which may be split into an upper and a lower branch, each a semi-circle. The former is defined as that one containing the observer's zenith and the latter as that containing his nadir. The point E, at which the equator cuts the *upper* branch of the observer's meridian, may now be taken as the starting point for declination, *when it is being used in the meridian*, where it will be denoted as δ_M. This *meridian declination* may then be taken as a full four quadrant system, with its zero at the point E and with values, increasing from this point positively northwards or negatively southwards.

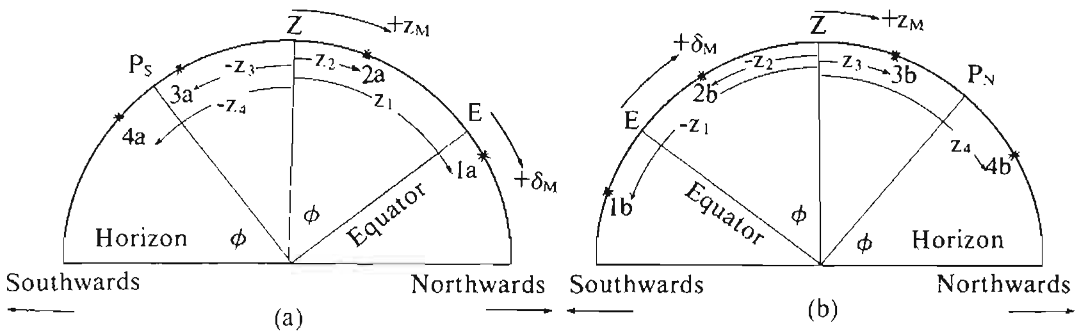


Fig.5.3

For a catalogue declination δ, it can be seen from Fig 5.3 that $\delta_M = \delta$ for a celestial body on the upper branch of the observer's meridian and that $\delta_M = 180 - \delta$ for one on the lower branch, the subtraction being carried out algebraically. Also, 360° may be added or subtracted as required for convenience.

Table 5.3 has been drawn up to show the detail of these manipulations.

Table 5.3

	Star	δ_M	z_M	$\phi=\delta_M-z_M$
Upper Transit	(1a	+ 8°	+60°	-52°
	(2a	-32	+20	-52
	(3a	-82	-30	-52
Lower Transit	4a	-102(258)	-50	-52
Upper Transit	(1b	-20	-70	+50
	(2b	+20	-30	+50
	(3b	+70	+20	+50
Lower Transit	4b	+110	+60	+50

The above shows that the extended conventions for meridian declinations and meridian zenith distances produce the general relationship for the latitude, from either upper or lower transit sights, as

$$\phi = \delta_M - z_M \qquad ...5.10$$

5.34 If the meridian observations are solved by the general method of section 5.21, it will be found that the auxiliary angles M and $(M-\phi)$, for this special case, are equal to δ_M and z_M respectively. This shows that Equation 5.10, which had long been suspected as holding in practice, has now been justified in theory.

Near Meridian Methods

5.41 Because the meridian observation for latitude determination is so restricted (see section 5.31), circum-meridian methods have been developed. These are methods, in which time-altitude observations are made on stars not exactly on the meridian, but on stars as they approach, pass over and leave the meridian. During this period their azimuths are not very different from that of the meridian. Such observations are therefore made when the star is only slightly away from the very best position for latitude determination. They are suited to theodolite observation, because observations can be made on both faces. Also they enable multiple observations to be made on each star, so that precise results can be obtained.

The two pole stars are available for circum-polar observations, because they are so close to the pole that their azimuth, whatever the hour angle, never deviates very far from that of the meridian, unless the observer is in very high latitudes indeed. These stars can, therefore, be observed at any hour angle.

5.42 In section 5.32 the generalized concept of meridian zenith distance was introduced. Since the zenith distances, observed in the near meridian methods lie on a great circle, which departs only very slightly from that, in which the meridian zenith distances lie, the circum-meridian zenith distances can be generalized in the same manner to gain the same advantages. This kind of zenith distance, designated as z_{CM}, is likewise taken to start from zero at the observer's zenith and to increase positively northwards from this point, or negatively southwards so that circum-meridian distances can be used as a full circle four quadrant system. Furthermore, when the meridian zenith distance z_M and the circum-meridian zenith distance z_{CM} are being used, the meridian declination value δ_M will also be used.

5.43 The relationships for reduction of near meridian zenith distances to meridian zenith distances have been developed in section A.71 in the appendix. In all these, it is necessary to have a value for the unknown latitude sought before solution can be achieved. If the observations are circum-meridian ones, a reasonably close preliminary value of the latitude can easily be obtained from the observations themselves. An iterative process may then be used to obtain the unknown to the accuracy of the observations made, but, in practice, the need to iterate is hardly ever found necessary, because the preliminary value of the latitude, as obtained from the observations themselves, is fairly close to the truth (see section 5.47 for justification). In the circum-meridian observation method, the set of observations made is usually one, balanced about the point of transit. The uncertainty $\Delta\lambda$ in longitude produces an effect $\cos\phi \tan A \, d\lambda$ in the derived latitude. Over transit, $\tan A$ changes sign and the effects of $\Delta\lambda$ tend to cancel. In practice, however, the observer, who has assumed a poor longitude, knows what his clock time of transit is and tends to make his balance using time differences from this point instead of making balance on obtaining equal altitudes on the star on each side of the meridian (see section 5.45).

Sometimes this balance is not carried out exactly, for various reasons, which should be avoided if the programme is properly predicted. If there is any choice, the star on the equator side should be balanced properly and the one on the pole side of the observer should be the one not balanced properly, because the effect of $\Delta\lambda$ on this star is smaller because its azimuth is departing from that of the meridian at a slower rate (see section 10.32).

Sometimes, one of the pole stars may be observed (see section 5.50). This observation is made very often when this star is not at or near transit. There is thus no balancing on the opposite side of the meridian and the uncertainty $\Delta\lambda$ may produce an unacceptable error in the derived latitude (see section 5.512). In practice, a value of longitude of sufficient accuracy will often be known.

Circum-Meridian Stars

5.44 There are many stars available for latitude determination by circum-meridian methods, in which the stars are observed when they are close to meridian transit (see section 5.41).

In this position, the meridian zenith distance z_M and the circum-meridian zenith distance z_{CM} do not differ much from each other; so that special methods, in place of the general one, were developed in the past in the form of a series (see section A.72 in the appendix). Several methods of derivation have been used there in order to illustrate the possible lines of approach in such derivations. The relevance of such methods of computing is nowadays disappearing, because of the modern facilities for computing direct, instead of indirect, solutions.

The observer arranges his programme in such a way that the circum-meridian star is located in the field of view of the telescope about ten minutes before the time of its transit. Prior to this, he has determined his clock correction with respect to Greenwich Time and also the index correction of the vertical circle readings. He then proceeds to make, say, six timed altitude observations before transit. He finishes these with time enough to transit and find the star on the other face of the theodolite and he then makes another six such observations at points approximately symmetrical with respect to the previous six.

5.45 If there is any doubt about the value of longitude assumed for determining the latitude from such sets of observations, the period, over which the observations are made should be extended so that it is fairly obvious that the observed zenith distances have actually decreased to a minimum value and have thereafter increased by the same amount. This is easy to see by inspection, if the vertical circle index error has been reduced to a small value by adjustment, which, if necessary, can easily be carried out in the preparatory period before observing is to start. Before computation, the reduced meridian zenith distances should be plotted on thin paper against the observed clock time. The sheet of paper is now folded about a line perpendicular to the time base, so that the rising and the falling sections of the curve joining the plotted points are superimposed on each other. The fold line then will give a position on the clock time base corresponding to the correct clock time of transit.

In practice, this plotting method gives a better value of the clock time of transit from the observations on the star on the equatorial side of the observer than from those on its balanced partner on the pole side. This device should be seldom resorted to, as it is definitely much better to avoid this method by obtaining a reliable longitude for use in determining the latitude. This can be very simply done by making a few time altitude observations to stars, one to the east and the other to the west near the prime vertical and so obtaining a fairly good value of the longitude (see section 5.491).

5.46 Individual observations are normally calculated separately. From the clock time, combined with longitude and clock correction, the local sidereal time of the observation is determined. The local hour angle is found from this, combined with the star's right ascension. The vertical circle reading is reduced for index and refraction to give the circum-meridian zenith distance.

5.47 The latitude is now computed from these, together with the star's declination, by means of the general method of determination of the latitude (see section 5.21) or by means of the special circum-meridian methods (see section A.72 in the appendix). The latter, in full, is given as

$$z_M = z_{CM} - Am + Bn - Cs \quad . \; . \; . \; . \qquad \ldots 5.11$$

in which $A = \cos\phi \cos\delta_M \operatorname{cosec} z_M$ and $m = 2\sin^2(\tfrac{1}{2}t')\rho$

$B = A^2 \cot z_M$ and $n = 2\sin^4(\tfrac{1}{2}t')\rho$

$C = \tfrac{2}{3}A^3(1 + 3\cot^2 z_M)$ and $s = 2\sin^6(\tfrac{1}{2}t')\rho$

and in which ρ is in the same angular units as those used for all the other quantities and is equal to the number of these units in a radian.

The quantities m, n and s are related to each other as follows:-

$$m = 2\sin^2(\tfrac{1}{2}t')\rho$$

$$n = \frac{m^2}{2\rho}$$

and

$$s = \frac{m^3}{4\rho^2} = \frac{nm}{2\rho}$$

in which t' is defined as

$t' = t$ near upper transit

$t' = t - 180$ near lower transit

If m is expressed in sexagesimal seconds, then

$$m'' = 2\sin^2(\tfrac{1}{2}t')\rho''$$

This quantity is tabulated on page 68 of the Star Almanac for Land Surveyors to the nearest whole second of arc.

If the interval t' is expressed in minutes of time as Δt^m, then an approximate value of

$$m'' = \frac{900^2}{2\rho''}(\Delta t^m)^2 = 1.9635''(\Delta t^m)^2$$

$$n'' = \frac{(m'')^2}{2\rho''}$$

$$s'' = \frac{(m'')^3}{(2\rho'')^2} = \frac{n''\,m''}{2\rho''}$$

Table 5.4 gives numerical values for these over a range, which should not normally be exceeded in any circum-meridian observations, which aim at single second accuracy in the reduction by the above series expansion.

Table 5.4

Angle t'	m"	Truncated value of m" from Star Almanac	Approx m" from (Δt^m)	n"	s"
4^m	31.41"	31"	31.42"	0.00"	0.00"
8	125.65	126	125.66	0.04	0.00
12	282.68	283	282.74	0.19	0.00
16	502.45	502	502.66	0.61	0.00
20	784.90	785	785.40	1.49	0.00
24	1129.94	-	1130.98	3.10	0.01

If the near meridian zenith distance is not less than 30° and if the star observed is within twenty minutes of the time of its transit, and the preliminary value of the latitude ϕ is known to within one minute of arc, then the error in the term Am will be less than one arc second. In a set of such observations, this error will be significantly less than this amount because closer to the meridian this error diminishes very quickly and the mean result from the set is always taken (see section 5.43).

If a star has been observed in any latitude at a near-meridian zenith distance not less than 30° and the star was within ten minutes (in time) of transit, the correction Bn" is not greater than 0.6 arc seconds. The correcting term Cs is even smaller.

5.481 An example of a circum-meridian observation made for determining latitude will now be given in order to illustrate the calculation procedures. First the general method of section 5.21 and then the circum-meridian method of section 5.47 will be used. Because, it is a computing example only, a single observation only is provided for this purpose. Subsequently, in section 5.491 a full set of observations will be worked out in detail.

The data for the above example represents one observation from each of two sets of observations made on a balanced pair of circum-meridian latitude stars.

Star No.	Aspect	Right Ascension	Declination	Observed Local Sidereal Time	Circum-Meridian Zenith Distance *	Remarks
338	S	$12^h40^m23^s$	S 1°18'43"	$12^h49^m22^s$	48°26'40"	Near Upper Transit
BS0285	N	01 04 53	N 86 07 03	12 52 47	46 47 41	Near Lower Transit

*These have been corrected for refraction and index

Aspect	Star South	Star North
Declination δ	- 1°18'43"	+86°07'03"
Altitude h	41 33 20	43 12 19
Local Hour Angle t = LST-RA	$0^h 8^m59^s$	$11^h47^m54^s$
	2°14'45"	176°58'30"
$\tan M = \dfrac{\tan \delta}{\cos t}$	$\dfrac{-\ 0.022\ 9017}{+\ 0.999\ 2319}$	$\dfrac{+14.734\ 8504}{-\ 0.998\ 6066}$
M	- 1°18'46.6"	93°52'37.6"
$\cos(M-\phi) = \dfrac{\sin h \sin M}{\sin \delta}$	0.663 8515	0.684 6186
$M-\phi$	±48°24'20.6"	±46°47'39.8"
N	-48 24 20.6	+46 47 39.8
ϕ = M-N	+47 05 34.0	+47 04 57.8
Latitude from pair of Observations	47°05'15.9" North	

5.482 The example of section 5.481 will now be reduced by the circum-meridian method of section 5.47, in which Equation 5.11, less the term Cs, gives the relationship as

$$z_M = z_{CM} - Am + Bn$$

The factor A requires a preliminary value of the latitude ϕ as well as one of the meridian zenith distance z_M, both of which are being sought. Both are easily found from the observations themselves, provided the observations have been made in the vicinity of the point of the star's transit across the meridian.

Preliminaries	Star South at U.T.	Star North at L.T.
$\delta_M = \delta$ for Upper Transit U.T. $= 180-\delta$ for Lower Transit L.T.	$-1°18'43"$	$+93°52'57"$
$t = LST-RA$	$0^h\ 8^m59^s$	$11^h47^m54^s$
$t' = t$ for U.T. $= 12^h - t$ for L.T.	$0^h\ 8^m59^s$	$0^h12^m06^s$
Approx $z_M = z_{CM}$ minimum at U.T. maximum at L.T.	$-48°26'40"$	$+46°47'41"$
Approx $\phi = \delta_M - z_{CM}$	+47 07 57	+47 05 16
Mean approx. ϕ	47°06'36" North	

Accurate Determination

Approx $z_M = \delta_M - \phi_{approx}$	$+48°25'19"$	$-46°46'21"$
$A = \dfrac{\cos\phi\ \cos\delta_M}{\sin z_M} = \dfrac{1}{\tan\delta_M - \tan\phi}$	-0.9096	-0.0632
$m" = 2\sin^2(\frac{1}{2}t').\rho"$	+158.4"	+287.4"
Am	$-00°02'24"$	$-00°00'18"$
$z_M = z_{CM} - A_M$	$-48\ 24\ 16$	$+46\ 47\ 59$
$\phi = \delta_M - z_M$	$+47\ 05\ 33$	$+47\ 04\ 58$
Latitude from pair of observations	47°05'15.5" North	

Iteration is seldom required if, in the first instance, a reasonably correct value of the meridian zenith distance can be obtained from the observations themselves. This is normally achieved with well balanced multiple observations over the point of transit or close to it. If iteration is carried out in the above example, no change occurs. The second term Bn of Equation 5.11 should be taken out to test its magnitude. In the above, this term amounts to 0.04" for the star south and to 0.00" for the star north, and they can therefore be neglected.

5.49 A full example, in which the observations were made to determine both latitude and longitude, will now be set out.

Multiple observations are provided for one pair of circum-meridian latitude stars and for one pair of longitude stars observed near the prime vertical. A preliminary value of the latitude will be determined from meridian distances found by inspection. The longitude, to be used in the next step of calculating an accurate value of the latitude, is provided. The longitude from this set of observations has been calculated in section 6.222.

5.491 Example

Place UNB Fredericton, New Brunswick, Canada
Date Thursday evening 9th October 1969
Zone 3^h W
R_o for 9th October 1969 $1^h09^m44.5^s$
R_{18} for 9th October 1969 $1^h12^m42.0^s$

Theodolite Wild T2
Clock Mean Time
Vertical Index Subtract 30" from all V⊙ Rdgs
Barometer 1019.4 mb
Temperature 14.4°C

Latitude Star North at Lower Transit
Star No. BS2609 RA $7^h27^m04.0^s$ δ 87°05'05" N

V⊙ Rdg	Obs CT	V⊙ Rdg	Obs CT
313°03'24"	$19^h33^m47^s$	46°57'19"	$19^h41^m33^s$
CR 26	35 05	CL 18	43 06
313 03 25	19 35 27	46 57 21	19 44 14
Clock was 39^s ahead of Zone Time			

Latitude Star South at Upper Transit
Star No. 549 RA $19^h50^m56.6^s$ δ 0°55'37" N

V⊙ Rdg	Obs CT	V⊙ Rdg	Obs CT
314°58'25"	$19^h57^m45^s$	45°00'45"	$20^h03^m52^s$
CR 59 01	19 59 30	CL 00 46	05 40
314 59 32	20 01 16	45 01 00	20 07 13
Clock was 39^s ahead of Zone Time			

Longitude Star West
Star No. 449 RA $16^h40^m08.0^s$ δ 31°39'28" N

V⊙ Rdg	Obs CT	V⊙ Rdg	Obs CT
45°17'22"	$20^h39^m58.2^s$	314°07'57"	$20^h43^m20.4^s$
CL 24 08	40 37.3	CR 313 54 30	44 37.8
45 27 58	20 40 59.3	313 49 08	20 45 08.6
Clock was 39.6^s ahead of Zone Time			

Longitude Star East
Star No. 12 RA $0^h37^m43.5^s$ δ 30°41'56" N

V⊙ Rdg	Obs CT	V⊙ Rdg	Obs CT
48°05'51"	$20^h52^m07.2^s$	312°43'51"	$20^h56^m49.1^s$
CL 47 58 33	52 49.0	CR 312 54 59	57 53.1
47 46 12	20 54 00.0	313 11 59	20 59 31.2
Clock was 39.6^s ahead of Zone Time			

5.492 To find a preliminary value of the latitude from inspection

		Star N		Star S
Min or Max V⊙ Rdg	CR	313°03'24"	CR	314°59'32"
ZD		46 56 36		45 00 28
Min or Max V⊙ Rdg=ZD	CL	46 57 21	CL	45 00 45
Mean Observed ZD		46 56 58		45 00 36
Refraction		1 02		58
Preliminary z_M		+46 58 00		-45 01 34
Declination		+87 05 05		+ 55 37
Meridian Declination δ_M		+92 54 55		+ 55 37
$\phi = \delta_M - z_M$		+45 56 55		+45 57 11
Preliminary value of the latitude				+45°57'03"

5.493 To determine an accurate value of the latitude from Star North

Determination of the clock time of lower transit of Star North
Star No. BS 2609 RA $7^h27^m04.0^s$ δ 87°05'05" N
Preliminary value of the longitude λ $4^h26^m35^s$W (see sections 5.45 and 6.222)

LST of Local Lower Transit = RA + 12^h	$19^h27^m04^s$	9th October
Local Longitude	4 26 35 W	
GST of Local Lower Transit	23 53 39	9th October
R_o for 9th October	1 09 44	
Sidereal Time Interval since GMT 0^h	22 43 55	

Conversion of this interval	$00^h 03^m 44^s$
Mean Time Interval since GMT 0^h equals GMT of Local Lower Transit	22 40 11
Zone Longitude	3 W
Zone Time of Local Lower Transit	19 40 11
Clock Correction	39 fast on Zone Time
Clock Time of Local Lower Transit	19 40 50

Evaluation of constant $A = \frac{\cos\phi \cos\delta_M}{\sin z_M} = \frac{1}{\tan\delta_M - \tan\phi} = -0.0484$

Reduction of the latitude from $z_M = z_{CM} - Am''$ (see section A.72)

	h m s	h m s	h m s	h m s	h m s	h m s
CT of Obs	19 33 47	19 35 05	19 35 27	19 41 33	19 43 06	19 44 14
t' MT units	- 7 03	- 5 45	- 5 23	+ 0 43	+ 2 16	+ 3 24
t' ST units	- 7 04	- 5 46	- 5 24	+ 0 43	+ 2 16	+ 3 25
m"	+98	+65	+57	+ 1	+10	+23
	° ' "	° ' "	° ' "	° ' "	° ' "	° ' "
V⊙ Rdg	313 03 24	313 03 26	313 03 25	46 57 19	46 57 18	46 57 21
Index	-30	-30	-30	-30	-30	-30
Corrected V⊙ Rdg	313 02 54	313 02 56	313 02 55	46 56 49	46 56 48	46 56 51
Obs ZD	46 57 06	46 57 04	46 57 05	46 56 49	46 56 48	46 56 51
Refraction	1 01	1 01	1 01	1 01	1 01	1 01
Corrected ZD z_{CM}	+46 58 07	+46 58 05	+46 58 06	+46 57 50	+46 57 49	+46 57 52
Am"	- 5	- 3	- 3	- 0	- 0	- 1
Meridian ZD z_M	+46 58 12	+46 58 08	+46 58 09	+46 57 50	+46 57 49	+46 57 53
δ_M=180- δ at L.T.	+92 54 55	+92 54 55	+92 54 55	+92 54 55	+92 54 55	+92 54 55
$\phi = \delta_M - z_M$	+45 56 43	+45 56 47	+45 56 46	+45 57 05	+45 57 06	+45 57 02
	Mean CR +45°56'45.3"			Mean CL +45°57'04.3"		
	Mean Value of the Latitude from Star North = +45°56'54.8"					

5.494 To determine an accurate value of the latitude from Star South

Determination of the clock time of upper transit of star south

Star No. 549 RA $19^h 50^m 56.6^s$ δ $0^\circ 55' 37''$ N

LST of Local Upper Transit = RA	$19^h 50^m 57^s$	9th October
Local Longitude	4 26 35	W
GST of Local Upper Transit	17 32	9th October
R_o for 9th October	1 09 44	
Sidereal Time Interval since GMT 0^h	23 07 48	
Conversion for this interval	3 48	
Mean Time Interval since GMT 0^h equals GMT of Local Upper Transit	23 04 00	
Zone Longitude	3	W
Zone Time of Local Upper Transit	20 04 00	
Clock Correction	39	fast on Zone Time
Clock Time of Local Upper Transit	20 04 39	

Evaluation of Constant $A = \frac{\cos\phi \cos\delta_M}{\sin z_M} = \frac{1}{\tan\delta_M - \tan\phi} = -0.9827$

Reduction of the latitude from $z_M = z_{CM} - Am''$(see section A.72)

	h m s	h m s	h m s	h m s	h m s	h m s
CT of Obs	19 57 45	19 59 30	20 01 16	20 03 52	20 05 40	20 07 13
t' MT units	- 6 54	- 5 09	- 3 23	- 0 47	+ 1 01	+ 2 34
t' ST units	- 6 55	- 5 10	- 3 24	- 0 47	+ 1 01	+ 2 34
m"	+94	+52	+23	+ 1	+ 2	+13
	° ' "	° ' "				
V⊙ Rdg	314 58 25	314 59 01	314 59 32	45 00 45	45 00 46	45 01 00
Index	-30	-30	-30	-30	-30	-30
Corrected V⊙ Rdg	314 57 55	314 58 31	314 59 02	45 00 15	45 00 16	45 00 30
Obs ZD	45 02 05	45 01 29	45 00 58	45 00 15	45 00 16	45 00 30
Refraction	57	57	57	57	57	57
Corrected ZD z_{CM}	-45 03 02	-45 02 26	-45 01 55	-45 01 12	-45 01 13	-45 01 27
Am"	- 1 32	-51	-23	- 1	- 2	-13
Meridian ZD z_M	-45 01 30	-45 01 35	-45 01 32	-45 01 11	-45 01 11	-45 01 14
$\delta_M = \delta$ for U.T.	+00 55 37	+00 55 37	+00 55 37	+00 55 37	+00 55 37	+00 55 37
$\phi = \delta_M - z_M$	+45 57 07	+45 57 12	+45 57 09	+45 56 48	+45 56 48	+45 56 51
	Mean CR	+45°57'09.3"		Mean CL	+45°56'49.0"	

Mean value of Latitude from Star South	= +45°56'59.2" N
Mean value of Latitude from pair	= +45°56'57.0" N

This example has been computed with the aid of the tables on pages 62 and 68 of the Star Almanac for Land Surveyors (SALS). It should be seen that the use of δ_M simplifies the calculation, particularly with the lower transit calculation, because the sign of the factor A is found automatically.

Circum-Polar Stars

5.50 The two pole stars, Polaris (α Ursae Minoris) in the northern hemisphere and σ Octantis in the southern hemisphere, are available, for the determination of latitude, provided the observer is not at a station close to the equator. At such latitudes, the two pole stars are difficult to see because the line of sight must traverse a long path through the dense atmosphere layers, close to the earth's surface. This is particularly so for the southern star, which is quite faint.

These two stars are within one degree of the pole and their azimuths, in all but high polar latitudes, never depart much from the meridian azimuth, so that they may be observed at any hour angle. Polaris, of magnitude 2.1, is easily visible to the naked eye, but σ Octantis, of magnitude 5.5, is not easily seen by the naked eye observer, unless he is experienced, and therefore, for most people, a telescope is needed.

5.511 An example of latitude determination from an observation on a circum-polar star will be worked by a number of methods. These provide alternative methods for checking calculations.

Example. The southern pole star of declination 89°04'00" S was observed from a station to have a zenith distance, corrected for refraction and index error, of 56°10'45.6", when the star's hour angle was $6^h40^m00.0^s$. Determine an accurate value of the latitude of this station, if an approximate value of 34¼° was scaled off an atlas map.

5.512 The meridian distance z_M at upper transit will first be determined with a relatively rough value of 34¼°S as the preliminary value for the latitude. An iterative process will then be carried out until the value of the latitude stabilises. From section A.71 in the appendix, the relevant relationship is the following:-

$$\cos z_M = \cos z_{CM} + 2 \cos \delta_M \cos \phi \sin^2(\tfrac{1}{2}t') \qquad ...5.12$$

$$\delta_M = -89°04'00" \text{ for a star at upper transit}$$

$$t' = t = 6^h40^m00^s = 100°00'00"$$

$$z_{CM} = 56°10'45.6" \text{ South} = -56°10'45.6"$$

$$\therefore \quad \cos z_M = 0.556\ 595\ 32 + 0.019\ 117\ 58 \cos \phi \qquad ...5.13$$

Table 5.5 is obtained from a series of iterations from this relationship:-

Table 5.5

Preliminary ϕ	z_{CM}	z_M	$\phi = \delta_M - z_M$
-34°15'00"	-56°10'45.6"	-55°04'56.65"	-33°59'03.35"
-33 59 03.3		-55 04 44.15	-33 59 15.85
-33 59 15.8		-55 04 44.31	-33 59 15.69
-33 59 15.7		-55 04 44.31	-33 59 15.69

The solution converges very quickly even though the preliminary latitude is very inaccurate. If the longitude is equally inaccurate its effect $\Delta\phi$ on the derived latitude is equal to $-\Delta\lambda \cos\phi \tan A$. For the above example $A = 181.1°$ and with $\Delta\lambda = \tfrac{1}{4}°$.

$$\Delta\phi = -\Delta\lambda \cos \phi \tan A = -900" \cos 34° \tan 181.1° = -14"$$

This discrepancy is not eliminated because in this type of observation the star is not observed on both sides of the meridian. The hour angle has been deliberately chosen to give a large error. However, when the star is near transit this error is very small and in addition the iterative solution for latitude converges more quickly.

Instead of using the very rough value of latitude scaled from a small scale map, a better preliminary value can be calculated from the actual observation itself. This is obtained from the relationship of section A.51 in the appendix. This is

$$\begin{aligned} |\phi| &= h - p" \cos t \ ... \qquad ...5.14 \\ &= 33°49'14.4" - 3360" \cos 100° \\ &= 33°49'14" + 0°09'43" \\ \phi &= 33°58'57" \text{ South} \end{aligned}$$

With this value as the preliminary one, the first computation from Equation 5.13 gives the latitude as -33°59'15.9". Comparison with the values of Table 5.5 shows that the refined preliminary value produces only one less iteration than the rough value.

5.513 Section A.71 in the appendix provides the following alternative relation for this type of solution

$$\sin \tfrac{1}{2}(z_M - z_{CM}) = - \frac{\cos \phi \cos \delta_M}{\sin \bar{z}} \sin^2 (\tfrac{1}{2}t') \qquad ...5.15$$

in which $\bar{z} = \tfrac{1}{2}(z_M + z_{CM})$

The computation will be illustrated with the data of section 5.511, but it will be treated as a lower transit example. The value of 33°59' South for the latitude as obtained from Equation 5.13, in the previous section will be used. The preliminary meridian distance z_M is obtained quite easily.

$$\begin{aligned} \text{Preliminary } \phi &= -33°59' \\ \delta_M &= 269°04' \text{ for lower transit} \\ \text{or } \delta_M &= -90°56' \\ z_M &= -56°57' \text{ from } \phi = \delta_M - z_M \\ z_{CM} &= -56°10'45.6" \\ \bar{z} &= -56°33'52.8" \end{aligned}$$

$$\therefore \quad t' = t - 180 = -80°00' \text{ for lower transit}$$

Substitution in Equation 5.15 gives

$$\begin{aligned} z_M - z_{CM} &= -00°45'58.8" \\ z_{CM} &= -56\ 10\ 45.6 \\ z_M &= -56\ 56\ 44.4 \\ \delta_M &= -90\ 56\ 00 \\ \phi &= -33°59'15.6" \end{aligned}$$

Further iteration will lead to results similar to those of section 5.512.

5.52 This computation has been done by two methods to illustrate the theory in some detail and also to show the complete generality of the methods. The above relationship of Equation 5.15 suggests the use of a power series as a means of solution. This is given in the appendix, where the theory has been developed for a power series for the solution of both latitude and azimuth (see section A.51 and A.52).

These series are the basis for the widely used tables for the reduction of latitude and azimuth observations made on Polaris, which is such an outstanding mark in the sky of the Northern Hemisphere.

5.53 The data of section 5.511 will be used for this purpose. Equation A.52 in the appendix gives the relation needed for this solution as

$$|\phi| = h - p \cos t + \frac{p^2}{2\rho} \sin^2 t \tan h - \frac{p^3}{3\rho^2} \cos t \sin^2 t + \frac{p^4}{24\rho^3} \sin^2 t \tan h (3 \sin^2 t \tan^2 h + 9 \sin^2 t - 4).. \qquad ...5.16$$

in which p is the polar distance from the adjacent pole and ρ is in the same units as p.

The numerical values for this solution are tabulated as:-

$$\begin{aligned} h &= 90° - z = 33°49'14.4" \\ p &= 00°56'00" = 3360" \\ t &= 100°00'00" \\ \rho &= 206265" \\ -\ \text{1st term} &= +583.46" \\ +\ \text{2nd term} &= +\ 17.78 \\ -\ \text{3rd term} &= +\ \ 0.05 \\ +\ \text{4th term} &= +\ \ 0.00 \\ \text{sum} &= +601.29 = 00°10'01.29" \\ h &= 33\ 49\ 14.4 \\ |\phi| &= 33\ 59\ 15.7 \end{aligned}$$

Since the star observed was σ Octantis the latitude is south

$$\therefore \quad \phi = -33°59'15.7"$$

5.54 The data for this example is that from section 5.511 and is given below. The method of solution is that of section 5.21.

Star South

$$\begin{aligned} \delta &= -89°04'00" \\ t &= 100\ 00\ 00 \\ h &= 33\ 49\ 14.4 \end{aligned}$$

$\tan M = \dfrac{\tan \delta}{\cos t}$		M	269°50'16.5"
$\cos N = \sin h \sin M \operatorname{cosec} \delta$		N	- 56 10 27.8
		$M - N$	326 00 44.3
	$\therefore$	ϕ	- 33°59'15.7"

Of all the methods used to illustrate the calculation of latitude from a close circum-polar star, this last method is the simplest and most direct. The method is applicable to stars of any declination and therefore is highly recommended.

5.55 The two pole stars may be observed at any hour angle. The error $\Delta\lambda$ is usually not large and its effect $\cos\phi \tan A \, d\lambda$ is further diminished for such a star because the azimuth is always close to that of the meridian, unless the observer happens to be at a station very close to one of the earth's poles. While the pole stars are very useful for observations in mid latitudes, many other stars are available adjacent to the meridian and hence the circum-meridian methods of section 5.44 are far more often used.

THE DETERMINATION OF THE UNKNOWNS AND THEIR PRECISION FROM BALANCED OBSERVATIONS

5.61 IF a set of latitude stars has been properly predicted and properly observed, the latitude should be rigorously computed from these observations. If this can be done simply, so much the better. In the past, it has been usual to assess the results, obtained from each star, separately instead of inspecting the results from individual observations on both stars and obtaining from this inspection, all the relevant unknowns, (instrumental and physical included), as well as the precisions.

5.62 It is assumed that a set of timed altitude observations has been made by means of a theodolite on a pair of well-balanced circum-meridian stars. This gives rise to correction equations of the form:-

$$\phi = \phi' \pm C\frac{d\phi}{dh} + \Delta r\frac{d\phi}{dh} + \Delta\lambda\frac{d\phi}{d\lambda} + v$$

in which ϕ is the adjusted value of the latitude,
ϕ' is the computed value of the latitude,
C is the vertical circle index correction, with the positive sign applying to its use with one face of the theodolite and the negative sign applying to the other,
Δr is an unknown systematic error in the refraction values taken from the refraction tables,
$\Delta\lambda$ is an unknown systematic value of the longitude used in the computation and is a small quantity
and v is the correction to be applied to the computed value of the latitude to obtain the adjusted value.

In this method, the quantities ϕ, δ, t and h are linked by the Cosine Formula and the above coefficients are obtained by differentiation of this relationship. This gives

$$\frac{d\phi}{dh} = \sec A \qquad \text{and} \qquad \frac{d\phi}{d\lambda} = -\cos\phi \tan A$$

Since these are circum-meridian observations, A is close to either 0° or 180° and therefore $\frac{d\phi}{dh} = \pm 1$ and $\frac{d\phi}{d\lambda} = 0$. The above correction equations, therefore, become,

$$\phi = \phi' \pm C \pm \Delta r + v$$

and, since these observations are made on a star to the north on both faces of the theodolite and also on a star to the south on both faces, four sets of correction equations of the following type arise

$$\phi - C - \Delta r = \phi'_{NL} + v_{NL} \qquad \text{n observations}$$
$$\phi + C - \Delta r = \phi'_{NR} + v_{NR} \qquad \text{"}$$
$$\phi + C + \Delta r = \phi'_{SL} + v_{SL} \qquad \text{"}$$
$$\phi - C + \Delta r = \phi'_{SR} + v_{SR} \qquad \text{"}$$

This is so because the index correction C is eliminated in the mean from circle left and circle right observations on each star, whereas a constant refraction effect Δr is eliminated only if the results from both a north and a south star are meaned.

These produce the following Normal Equations, shown here in the detached coefficient form

ϕ	C	Δr	=	L (the absolute term)
N	O	O	=	$\Sigma\phi'_{NL} + \Sigma\phi'_{NR} + \Sigma\phi'_{SL} + \Sigma\phi'_{SR}$
O	N	O	=	$-\Sigma\phi'_{NL} + \Sigma\phi'_{NR} + \Sigma\phi'_{SL} - \Sigma\phi'_{SR}$
O	O	N	=	$-\Sigma\phi'_{NL} - \Sigma\phi'_{NR} + \Sigma\phi'_{SL} + \Sigma\phi'_{SR}$

Provided equal numbers of observations have been made on each face i.e. $N = 4n$. The solution of the unknowns is then,

$$\phi = \tfrac{1}{4}(\bar{\phi}_{NL} + \bar{\phi}_{NR} + \bar{\phi}_{SL} + \bar{\phi}_{SR})$$
$$C = \tfrac{1}{4}(-\bar{\phi}_{NL} + \bar{\phi}_{NR} + \bar{\phi}_{SL} - \bar{\phi}_{SR})$$
$$\Delta r = \tfrac{1}{4}(-\bar{\phi}_{NL} - \bar{\phi}_{NR} + \bar{\phi}_{SL} + \bar{\phi}_{SR})$$

in which $\bar{\phi}_{NL}$, $\bar{\phi}_{SL}$ etc. are the means computed in each set.

The values of the v corrections may be obtained by back substitution in the correction equations. The standard deviation of a single observation then is given as

$$\sigma_{so} = \sqrt{\frac{\Sigma vv}{N-3}}$$

Since the Normal Equation matrix is a diagonal one, the standard deviations of the unknowns are given as

$$\sigma_{\phi} = \sigma_{C} = \sigma_{\Delta r} = \frac{\sigma_{so}}{\sqrt{N}}$$

An alternative and convenient method of determining the v values consists of first computing the differences u from the means for each set of n observations as

$$u_{NL_i} = \bar{\phi}_{NL} - \phi'_{NL_i}$$

and similarly for the other three sets and then relating these to the v values.

From the first equation

$$\begin{aligned}
v_{NL_i} &= \phi - C - \Delta r - \phi'_{NL_i} \\
&= \phi - C - \Delta r - \bar{\phi}_{NL} + u_{NL_i} \\
&= \tfrac{1}{4}(\bar{\phi}_{NL} + \bar{\phi}_{NR} + \bar{\phi}_{SL} + \bar{\phi}_{SR} \\
&\quad +\bar{\phi}_{NL} - \bar{\phi}_{NR} - \bar{\phi}_{SL} + \bar{\phi}_{SR} \\
&\quad +\bar{\phi}_{NL} + \bar{\phi}_{NR} - \bar{\phi}_{SL} - \bar{\phi}_{SR} \\
&\quad -4\bar{\phi}_{NL} \quad) + u_{NL_i} \\
&= \tfrac{1}{4}(-\bar{\phi}_{NL} + \bar{\phi}_{NR} - \bar{\phi}_{SL} + \bar{\phi}_{SR}) + u_{NL_i} \\
&= D + u_{NL_i}
\end{aligned}$$

(see also further explanation in section 5.63)

The constant D may be conveniently evaluated when solving for ϕ, C and Δr.

Similar treatment gives the following results for the other sets

$$v_{NR_i} = -D + u_{NR_i}$$

$$v_{SL_i} = D + u_{SL_i}$$

$$v_{SR_i} = -D + u_{SR_i}$$

When observations are made by skilled observers, the required number of observations is usually obtained with only a few, if any rejections. If this does occur, a slight imbalance results and the weights of the mean values $\bar{\phi}_{NL}$ etc. are no longer exactly the same. However, if the imbalance is only a slight one, the use of this method will provide answers, which do not vary significantly from the correct ones.

5.63 Example. The following observations were made at the University of New South Wales for the purpose of determining its latitude.

Local Date: Wednesday evening 5th May 1976
Observer: K. Gillies
Recorder: P. Ritchie

Approximate Longitude $10^h04^m56^s$ E
Time Zone 10^h East
Clock Correction on Zone Time $+18^h18^m04.1^s$
R_o for local date 14 51 57.9

Theodolite Zeiss 010
Clock Mean Time
Pressure 1021 mb
Temp. 16.5°C

Relationships used

Refraction $r° = \dfrac{0.0045\,P}{273.2+T}(\tan z - 0.0012 \tan z \sec^2 z)$

$t° = 15\{\lambda - RA + R_o + F(CT + CC_{ZT} - \text{Zone})\}$

$F = 1.002\ 7379$

R_o for Greenwich Date, equal to Local Date, must be used

$\Sigma u = 0$

$v = u \pm D$ and $\Sigma v = \Sigma u \pm nD = \pm nD$

$\tan M = \dfrac{\tan \delta}{\cos t}$

$\cos N = \sin h \sin M \operatorname{cosec} \delta$

with N positive north, negative south

$\phi = M - N$

Latitude Star North No. 319 RA $12^h04^m01.7^s$ δ 8°51'43.9 N

Observed Vertical Circle Reading	Observed Clock Time	Calculated Latitude	u	±D	v
42°50'26"	$2^h36^m50^s$	-33°55'17.12"	+0.54"	+0.10"	+0.64"
49 55	37 25	14.18	-2.40		-2.30
49 29	38 02	16.09	-0.49		-0.39
49 04	38 36	15.15	-1.43		-1.33
CL 48 47	39 02	15.52	-1.06		-0.96
48 31	39 31	17.85	+1.27		+1.37
48 01	40 25	19.01	+2.43		+2.53
47 48	40 47	17.60	+1.02		+1.12
47 33	41 14	15.94	-0.64		-0.54
42 47 16	2 41 55	-33 55 17.38	+0.80	+0.10	+0.90
	$\bar{\phi}_{NL}$	-33 55 16.58	Σ+0.04		Σ +1.04
317 13 56	2 46 32	-33 55 11.67	+0.95	-0.10	+0.85
55	47 16	13.86	+3.14		+3.04
57	47 42	11.36	+0.64		+0.54
55	48 22	10.84	+0.12		+0.02
CR 53	48 46	10.31	-0.41		-0.51
46	49 43	08.25	-2.47		-2.57
37	50 14	10.52	-0.20		-0.30
29	50 48	09.68	-1.04		-1.14
19	51 16	11.25	+0.53		+0.43
317 13 12	2 51 42	-33 55 09.49	-1.23	-0.10	-1.33
	$\bar{\phi}_{NR}$	-33 55 10.72	Σ + .03		Σ -0.97
					Σvv 39.641

Latitude Star South No. 325 RA $12^h17^m03.8^s$ δ 79°11'09.2 S

Observed Vertical Circle Reading	Observed Clock Time	Calculated Latitude	u	±D	v
314°44'50"	$2^h54^m28^s$	-33°55'16.08"	+0.05"	-0.10"	-0.05"
44 52	54 52	16.15	+0.12		+0.02
44 53	55 26	14.66	-1.37		-1.47
44 54	55 55	13.75	-2.28		-2.38
CR 44 57	56 24	15.04	-0.99		-1.09
45 01	57 14	16.58	+0.55		+0.45
45 03	57 37	17.65	+1.62		+1.52
45 04	58 04	17.71	+1.68		+1.58
314 45 04	2 58 46	-33 55 16.61	+0.58	-0.10	+0.48
	$\overline{\phi}_{SR}$	-33 55 16.03	Σ-0.04		Σ -0.94
45 15 02	3 00 16	-33 55 09.67	-0.92	+0.10	-0.82
14 59	00 51	12.83	+2.24		+2.34
15 03	01 13	09.08	-1.51		-1.41
15 03	01 38	09.51	-1.08		-0.98
CL 15 01	02 10	12.27	+1.68		+1.78
15 04	02 33	09.98	-0.61		-0.51
15 08	03 53	09.41	-1.18		-1.08
15 19	06 52	11.66	+1.07		+1.17
15 23	07 20	10.43	-0.16		-0.06
45 15 25	3 07 45	-33 55 11.06	+0.47	+0.10	+0.57
	$\overline{\phi}_{SL}$	-33 55 10.59	Σ 0.00		Σ +1.00
					Σvv 29.645

$\overline{\phi}_{NL}$ -33°55'16.58" (1): $\phi = \frac{1}{4}\{(1)+(2)+(3)+(4)\}$ = $-33^{\circ}55'13.48''$

$\overline{\phi}_{NR}$ 10.72 (2): $C = \frac{1}{4}\{-(1)+(2)+(3)-(4)\}$ = + 2.82

$\overline{\phi}_{SL}$ 10.59 (3): $\Delta r = \frac{1}{4}\{-(1)-(2)+(3)+(4)\}$ = + 0.17

$\overline{\phi}_{SR}$ -33 55 16.03 (4): $D = \frac{1}{4}\{-(1)+(2)-(3)+(4)\}$ = + 0.10

$$\sigma_{so} = \sqrt{\frac{\Sigma vv}{N-3}} = \sqrt{\frac{69.286}{39-3}} = \pm 1.39'' \qquad \sigma_{\phi} = \sigma_C = \sigma_{\Delta r} = \frac{\sigma_{so}}{\sqrt{N}} = \frac{\pm 1.92}{\sqrt{39}} = \pm 0.22''$$

$$\phi = -33^{\circ}55'13.48'' \quad \pm 0.22''$$

This example has been calculated to a greater accuracy than is normally warranted, purely for purposes of illustration.

In this and later examples the technique of calculating the quantity u, which may be thought of as initial estimate of the error v, has the advantage of providing a preliminary assessment of the quality of the observations before embarking on the least squares solution. The quantity D in the aforegoing calculation represents half the variation in the vertical collimation of the theodolite as determined from the results of observation on each star.

$$C_N = \frac{\overline{\phi}_{NR} - \overline{\phi}_{NL}}{2} \quad \& \quad C_S = \frac{\overline{\phi}_{SL} - \overline{\phi}_{SR}}{2}\ ; \quad C = \frac{C_N + C_S}{2} \quad \& \quad D = \frac{C_N - C_S}{2}$$

Therefore when all observations are considered the additional criterion that D should be small, otherwise $v = u \pm D$ will be large, will be helpful in analysing the results and assessing their precision.

6
Determination of Longitude

INTRODUCTION

THE longitude λ of a station is equal to the angle at the pole measured from the Greenwich to the local meridian. This is taken to be positive eastward (see section 2.21). Therefore, to determine longitude, it is necessary to determine the local time of a certain instant as well as the Greenwich time of this instant, both times being in the same system.

An observer, using a theodolite, has two methods available. He may time the passage of a star across either a known altitude or azimuth circle. Then if the latitude is known and the observed times are correlated with known radio time signals, the longitude can be determined. Of these two, only the first method will be dealt with in detail, the second method not being within the scope of this book.

LONGITUDE FROM TIMED ALTITUDES

6.11 IN this method, the longitude is obtained by means of the formula, which links the four quantities, ϕ, t, δ and h in the form:

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t$$

or

$$\cos t = \sec\phi \sec\delta \sin h - \tan\phi \tan\delta \qquad \ldots 6.1$$

The hour angle t is determined from this and from it the Local Time is determined. The corresponding Greenwich Time is found from the observed clock time and the clock correction with respect to Greenwich Time. The longitude is then found from

$$\lambda = \text{Local Time} - \text{Greenwich Time}$$

It is desirable to determine where a star for determining longitude should be observed to give the best value for the quantity sought. The argument in this section is very similar to that used in section 5.11. The effect dt on the derived longitude is found from equation 5.2, which, on making dt the subject, gives

$$dt = \sec\phi \operatorname{cosec} A\, dh - \sec\phi \cot A\, d\phi - \sec\phi \operatorname{cosec} A \cos\omega\, d\delta \qquad \ldots 6.2$$

6.12 The considerations of section 5.12 hold here, i.e. the declinations taken from reliable catalogues may, for all but geodetic quality work, be considered error free. The effect, on the longitude sought, of errors in the data may therefore be taken as

$$d\lambda = dt = \sec\phi \operatorname{cosec} A\, dh - \sec\phi \cot A\, d\phi \qquad \ldots 6.3$$

or

$$D\lambda = d\lambda \cos\phi = \operatorname{cosec} A\, dh - \cot A\, d\phi \qquad \ldots 6.4$$

The quantity $D\lambda$ (see also section 9.51) is a more meaningful quantity to the practical man, who wishes, in most cases, to know what the uncertainty in his results represents in terms of distance on the ground. It follows, therefore, that $d\lambda$ can assume quite a sizeable amount in polar regions and yet the corresponding east-west distance on the ground $D\lambda$ will still remain small. In fact, it is fallacious to think that one cannot determine ones east-west position in high latitudes with great accuracy.

6.13 If the errors dh and dt are taken to be random errors of observation, consideration should be given to determining where a star should be observed so that these errors have a minimum effect. From equation 6.3, it appears that, if the azimuth A is made either 90° or 270°, then cosec A will have its smallest numerical value of unity and cot A likewise its smallest value of zero. The effect of an error dh will then enter directly into the result, while that of the error $d\phi$ will have no effect. These deductions should be compared with those of section 5.13.

6.14 Now the effects of these errors, if they are taken to be systematic errors, should be investigated. The value of the latitude ϕ adopted in the solution may or may not be the exactly correct value; also the altitudes may be incorrect, as indicated in section 5.14.

The effect $d\lambda$ of the systematic errors dh and $d\phi$ on the derived quantity λ are therefore given by equation 6.3, which is

$$d\lambda = \sec\phi \operatorname{cosec} A \, dh - \sec\phi \cot A \, d\phi$$

This relationship suggests that it might be possible to eliminate these effects, if a balanced pair of stars is observed. If, therefore, observations, for determining longitude from timed altitudes, are made on a balanced pair of stars, the derived results will be λ_1 from one star and λ_2 from the other with unknown discrepancies $d\lambda_1$ and $d\lambda_2$ such that

$$\lambda = \lambda_1 + d\lambda_1$$

and

$$\lambda = \lambda_2 + d\lambda_2$$

$$\lambda = \tfrac{1}{2}(\lambda_1 + \lambda_2) + \tfrac{1}{2}(d\lambda_1 + d\lambda_2)$$

$$\lambda = \tfrac{1}{2}(\lambda_1 + \lambda_2) \quad \text{only if} \quad d\lambda_1 + d\lambda_2 = 0$$

This requirement implies that cosec A_1=-cosec A_2 and cot A_1=-cot A_2 *simultaneously*. This occurs when $A_1 + A_2 = 360^{\circ}$, i.e. when the two azimuths of the star pair are symmetrical with respect to the meridian and when the two stars are at similar altitudes, see section 4.52 et seq. When this is done, the effects of the systematic errors dh and $d\phi$ will be eliminated. Deviations from exact balance must be decreased as the azimuths depart from the prime vertical, because the coefficients become larger as this occurs. Therefore, stars should be selected as close as possible to the prime vertical to obtain the best from observations made on balanced pairs of stars.

Star pairs are finally selected with similar values for their declinations, so that they reach similar altitudes in the vicinity of the prime vertical. Moreover, the instants, at which balance is reached, should be neither so close in time that observations cannot be fitted in nor so far apart that observing conditions may change between the two sets of observations.

6.15 In this method of longitude determination, systematic errors, which cannot be eliminated by means of observations on balanced pairs of stars, are those made in observing the clock times and the clock corrections with respect to Greenwich Time. These errors enter fully into the longitude sought and explain why it is quickly noticed that, if observing methods of the same precision are used for the determination of latitude and longitude, better precision is obtained for the former, in which timing does not play a critical part.

Thus extra precautions must be taken in the timing arrangements for longitude, if like precisions are desired. The timing arrangements must be further refined to eliminate constant errors in the timing and observers should be properly trained. Even then, an experienced observer's results may still be subject to a characteristic error, known as Personal Equation, due to an inherent anticipation or delay in timing his observations *and this cannot* *be eliminated*, but may possibly be allowed for.

THE CALCULATION OF LONGITUDE FROM TIMED ALTITUDES

6.21 THIS is a fairly straightforward procedure. The observed vertical circle reading on a star is corrected for vertical circle index error and for refraction to give a corrected zenith distance or altitude. From this, with the latitude and the declination, the hour angle is calculated from the Cosine Formula of equation 6.1. At this point the value computed is set into its correct quadrant to provide the generalized hour angle of the astronomical triangle. This value is then added algebraically to the Right Ascension to give the Local Sidereal Time LST as

$$LST = RA + t$$

The Greenwich Sidereal Time GST of the instant of the observation is next determined. If the clock, being used, is running at, or nearly at, the sidereal rate, the clock correction with respect to Greenwich Sidereal Time is determined as CC_{GST}. The Greenwich Sidereal Time of observation is then found by adding the clock time of the observation to this clock correction

$$\therefore \quad GST = CT + CC_{GST}$$

$$\therefore \quad \lambda = LST - GST = RA + t - CT - CC_{GST}$$

If the clock, being used, is running at, or nearly at, the mean time rate, the clock correction with respect to Zone Time or with respect to Greenwich Mean Time is determined as CC_{GMT}. The Greenwich Mean Time of observation is then found by applying this clock correction.

$$GMT = CT + CC_{GMT}$$

$$\text{and} \quad LST = RA + t$$

Before the longitude can be found, the GMT must be converted to the corresponding GST as

$$GST = GMT + R = CT + CC_{GMT} + R_o + dR$$

in which R_o is the GST at $UT0^h$ for the appropriate date and dR is the gain of sidereal time in the period GMT i.e. from midnight. (see section 3.415 for use of R_{n6}).

$$\therefore \quad GST = R_o + GMT \times F = (CT + CC_{GMT})\, F + R_o$$

where F is the ratio between Mean Time and Sidereal Time units and is equal to 1.0027379. This last relationship is useful with a calculator for determining GST from the corresponding value of GMT.

The above refers to observations made to a star. A slightly different procedure of reduction is necessary to compute the longitude from sun observations. This is dealt with in Chapter 8.

6.221 The following observations were made for determining longitude.

Local Date	:	Monday evening, 22nd June, 1959			
Station	:	Δ Mooifontein	Theodolite	:	Wild T2
Observer	:	O.H. Meyer	Index Corrn	:	-11"
Recorder	:	J.G. Freislich	Clock	:	Mercer, Sidereal
Latitude	:	26°03'13" S			CC_{GST} - $4^h\ 52^m\ 37.1^s$
Met. Readings	:	P 845 mb T 8°C			

Star East No. 430 RA $16^h03^m06.1^s$ δ 19°41'41"S		Star West No. 285 RA $10^h24^m07.6^s$ δ 16°37'54"S	
Observed Clock Time	Observed Vertical Circle Reading	Observed Clock Time	Observed Vertical Circle Reading
$16^h08^m45.5^s$	CL 40°23'54"	$16^h23^m39.5^s$	CL 42°41'56"
16 10 23.8	CR 319 58 37	16 25 19.5	CR 316 56 02

Relationship used $\cos t = \dfrac{\sin h - \sin \phi \sin \delta}{\cos \phi \cos \delta}$

Obs Vert.Circle Rdg	40°23'54"	319°58'37"	42°41'56"	316°56'02"
Index Correction	-11	-11	-11	-11
	40 23 43	319 58 26	42 41 45	316 55 51
Obs Altitude	49 36 17	49 58 26	47 18 15	46 55 51
Refraction	42	42	45	46
Altitude h	49 35 35	49 57 44	47 17 30	46 55 05
φ	-26 03 13	-26 03 13	-26 03 13	-26 03 13
δ	-19 41 41	-19 41 41	-16 37 54	-16 37 54
Hour angle t	-43 30 36	-43 05 57	44 57 35	45 22 34
t	$- 2^h54^m02.4^s$	$- 2^h52^m23.8^s$	$2^h59^m50.3^s$	$3^h01^m30.3^s$
RA	16 03 06.1	16 03 06.1	10 24 07.6	10 24 07.6
LST = RA + t	13 09 03.7	13 10 42.3	13 23 57.9	13 25 37.9
Obs Clock Time	16 08 45.5	16 10 23.8	16 23 39.5	16 25 19.5
CC_{GST}	- 4 52 37.1	- 4 52 37.1	- 4 52 37.1	- 4 52 37.1
GST	11 16 08.4	11 17 46.7	11 31 02.4	11 32 42.4
λ = LST-GST	$+ 1^h52^m55.3^s$	$+ 1^h52^m55.6^s$	$+ 1^h52^m55.5^s$	$+ 1^h52^m55.5^s$
	$1^h52^m55.45^s$ E		$1^h52^m55.50^s$ E	

Mean Longitude from pair $1^h52^m55.48^s$ East

6.222 Calculation of the Longitude

Note (i) The data for this example is given in section 5.491 and the latitude value of 45°56'57" N is the final value determined in section 5.494.

(ii) The Star Almanac for Land Surveyors has been used for this calculation.

Star West No. 449 RA $16^h40^m08.0^s$ ϕ 45°56'57" N δ 31°39'28" N Relationship used $\cos t = \dfrac{\sin h - \sin\phi \sin\delta}{\cos\phi \cos\delta}$

Reduction of the Clock Times of Observation to Greenwich Sidereal Times of Observation

CT of Obs	CC_{ZT}	ZT of Obs	Zone	GMT of Obs	R_{18}	dR	GST of Obs
$20^h39^m58.2^s$	-39.6^s	$20^h39^m18.6^s$	3^h W	$23^h39^m18.6^s$	$1^h12^m42.0^s$	55.7^s	$0^h52^m56.3^s$
40 37.3		39 57.7		39 57.7		55.8	53 35.5
40 59.3		40 19.7		40 19.7		55.9	53 57.6
43 20.4		42 40.8		42 40.8		56.3	56 19.1
44 37.8		43 58.2		43 58.2		56.5	57 36.7
20 45 08.6	-39.6	20 44 29.0	3 W	23 44 29.0	1 12 42.0	56.6	0 58 07.6

Reduction of Observed Altitude and Computation of Local Hour Angle, Local Sidereal Times of Observation and the Longitude

Obs V☉ Rdg	Index	Corr. V☉ Rdg	Obs Alt.	Refr	Obs Alt h	Computed Local HA t	Right Ascension	LST of Obs	GST of Obs	Longitude
45°17'22"	-30"	45°16'52"	44°43'08"	59"	44°42'09"	$3^h46^m12.5^s$	$16^h40^m08.0^s$	$20^h26^m20.5^s$	$24^h52^m56.3^s$	$-4^h26^m35.8^s$
45 24 08		45 23 38	44 36 22	59	44 35 23	46 51.5		26 59.5	53 35.5	36.0
45 27 58		45 27 28	44 32 32	59	44 31 33	47 13.5		27 21.5	53 57.6	36.1
314 07 57		314 07 27	44 07 27	60	44 06 27	49 38.1		29 46.1	56 19.1	33.0
313 54 30		313 54 00	43 54 00	60	43 53 00	50 55.6		31 03.6	57 36.7	33.1
313 49 08	-30	313 48 38	43 48 38	61	43 47 37	3 51 26.6	16 40 08.0	20 31 34.6	24 58 07.6	-4 26 33.0

Longitude from CL Obs $4^h26^m35.97^s$ W Mean Longitude $4^h26^m34.50^s$ W

CR Obs 4 26 33.03 W

Star East No. 12 RA $0^h37^m43.5^s$ ϕ 45°56'57" N δ 30°41'56" N

Reduction of the Clock Times of Observation to Greenwich Sidereal Times of Observation

CT of Obs	CC_{ZT}	ZT of Obs	Zone	GMT of Obs	R_{18}	dR	GST of Obs
$20^h52^m07.2^s$	-39.6^s	$20^h51^m27.6^s$	3^h W	$23^h51^m27.6^s$	$1^h12^m42.0^s$	57.7^s	$1^h05^m07.3^s$
52 49.0		52 09.4		52 09.4		57.8	05 49.2
54 00.0		53 20.4		53 20.4		58.0	07 00.4
56 49.1		56 09.5		56 09.5		58.5	09 50.0
57 53.1		57 13.5		57 13.5		58.7	10 54.2
20 59 31.2	-39.6	20 58 51.6	3 W	23 58 51.6	1 12 42.0	59.0	1 12 32.6

Reduction of Observed Altitude and Computation of Local Hour Angle, Local Sidereal Times of Observation and the Longitude

Obs V⊙ Rdg	Index	Corr V⊙ Rdg	Obs Alt.	Refr	Obs Alt h	Computed Local HA t	Right Ascension	LST of Obs	GST of Obs	Longitude
48°05'51"	-30"	48°05'21"	41°54'39"	1'04"	41°53'35"	$-3^h59^m09.9^s$	$0^h37^m43.5^s$	$20^h38^m33.6^s$	$1^h05^m07.3^s$	$-4^h26^m33.7^s$
47 58 33		47 58 03	42 01 57	04	42 00 53	58 27.8		39 15.7	05 49.2	33.5
47 46 12		47 45 42	14 18	04	13 14	57 16.7		40 26.8	07 00.4	33.6
312 43 51		312 43 21	43 21	03	42 18	54 29.2		43 14.3	09 50.0	35.7
312 54 59		312 54 29	42 54 29	02	42 53 27	53 25.0		44 18.5	10 54.2	35.7
313 11 59	-30	313 11 29	43 11 29	1 02	43 10 27	-3 51 47.2	0 37 43.5	20 45 56.3	1 12 32.6	-4 26 36.3

Longitude from CL Obs $4^h26^m33.60^s$ W

from CR Obs 4 26 35.90 W

Mean Longitude $4^h26^m34.75^s$ W

Mean Longitude from pair of stars $4^h26^m34.62^s$ West

6.23 The longitude example of section 6.222 is linked with the latitude example of 5.493. The observations for these determinations are given in section 5.491. When latitude and longitude are to be determined, it is usual to make observations, over the same observing period, to determine both components of the position fix. The computing procedure carried out in the following sequence is necessary because values of both latitude and longitude must be known approximately before accurate values can be determined:-

(i) Determination of a preliminary value of the latitude, as shown in section 5.492.
(ii) Determination of a preliminary value of the longitude by means of the preliminary latitude value with some of the longitude observations.
(iii) Determination of an accurate value of the latitude by means of the preliminary latitude and longitude values with all the observations for the latitude.
(iv) Determination of an accurate value of the longitude by means of the accurately determined latitude with all the observations for the longitude.
(v) Determination of the statistical precision of the fix.

The strength of the principle of observing balanced pairs of celestial bodies lies in the ability to make use of a preliminary, and not necessarily very accurate, value of one of the elements and still to obtain an accurate value of the unknown being sought.

THE DETERMINATION OF THE UNKNOWNS AND THEIR PRECISION FROM BALANCED OBSERVATIONS

6.31 IT is assumed that sets of timed altitude observations n on each face for each star, have been made by means of a theodolite on a pair of well balanced stars at nearly equal altitudes, one star east and the other west and both near the prime vertical. These give rise to correction equations of the form

$$\lambda = \lambda' \pm C \frac{d\lambda}{dh} + \Delta r \frac{d\lambda}{dh} + \Delta\phi \frac{d\lambda}{d\phi} + v$$

in which λ is the adjusted value of the longitude
λ' is the computed value of the longitude
C is the vertical circle index correction, with the positive sign applying to its use with one face and the negative sign applying to the other face.
Δr is an unknown systematic error in the refraction values taken from the refraction tables.
$\Delta\phi$ is an unknown systematic error in the value of the latitude used in the computation
and v is the correction to be applied to the computed value λ' of the longitude to provide the adjusted value λ .

The quantities ϕ, δ, t and h are linked together in the Cosine Formula, from which the differential coefficients, required above, are obtained as

$$\frac{d\lambda}{dh} = \frac{dt}{dh} = \frac{1}{\cos\phi \sin A} \quad \text{and} \quad \frac{d\lambda}{d\phi} = \frac{dt}{d\phi} = -\frac{1}{\cos\phi \tan A}$$

The correction equations above are therefore of the form

$$\lambda = \lambda' \pm C \sec\phi \operatorname{cosec} A + \Delta r \sec\phi \operatorname{cosec} A - \Delta\phi \sec\phi \cot A + v$$

If the balancing of the star pair has been carefully done $\operatorname{cosec} A_E = -\operatorname{cosec} A_W = 1$ very nearly; also $\cot A_E = -\cot A_W$ and numerically these quantities are in the vicinity of zero. This then leads to a family of four sets of correction equations, as follows:-

$$\lambda = \lambda'_{EL} + C \sec\phi + \Delta r \sec\phi - \Delta\phi \sec\phi \cot A_E + v_{EL}$$ East star on face left

$$\lambda = \lambda'_{ER} - C \sec\phi + \Delta r \sec\phi - \Delta\phi \sec\phi \cot A_E + v_{ER}$$ East star on face right

$$\lambda = \lambda'_{WL} - C \sec\phi - \Delta r \sec\phi - \Delta\phi \sec\phi \cot A_W + v_{WL}$$ West star on face left

$$\lambda = \lambda'_{WR} + C \sec\phi - \Delta r \sec\phi - \Delta\phi \sec\phi \cot A_W + v_{WR}$$ West star on face right

in which each set comprises n equations.

$$\lambda = \lambda'_{EL} + C \sec\phi + \Delta r \sec\phi - \Delta\phi \sec\phi |\cot A_E| + v_{EL}$$
$$\lambda = \lambda'_{ER} - C \sec\phi + \Delta r \sec\phi - \Delta\phi \sec\phi |\cot A_E| + v_{ER}$$
$$\lambda = \lambda'_{WL} - C \sec\phi - \Delta r \sec\phi + \Delta\phi \sec\phi |\cot A_W| + v_{WL}$$
$$\lambda = \lambda'_{WR} + C \sec\phi - \Delta r \sec\phi + \Delta\phi \sec\phi |\cot A_W| + v_{WR}$$

$$\therefore \quad \lambda - C' - \Delta H = \lambda'_{EL} + v_{EL}$$
$$\lambda + C' - \Delta H = \lambda'_{ER} + v_{ER}$$
$$\lambda + C' + \Delta H = \lambda'_{WL} + v_{WL}$$
$$\lambda - C' + \Delta H = \lambda'_{WR} + v_{WL}$$

in which $C' = C \sec\phi$

$\Delta H = \Delta r \sec\phi - \Delta\phi \sec\phi |\cot A_{EorW}|$

in which it is not possible to separate the small $\Delta\phi$ and Δr effects because both act in unison.

These equations lead to the same form of Normal Equations as those in section 5.62. The Normal Equations, in the detached coefficient form are

λ	C'	ΔH	=	L (the absolute term)
N	O	O	=	$\Sigma\lambda'_{EL} + \Sigma\lambda'_{ER} + \Sigma\lambda'_{WL} + \Sigma\lambda'_{WR}$
O	N	O	=	$-\Sigma\lambda'_{EL} + \Sigma\lambda'_{ER} + \Sigma\lambda'_{WL} - \Sigma\lambda'_{WR}$
O	O	N	=	$-\Sigma\lambda'_{EL} - \Sigma\lambda'_{ER} + \Sigma\lambda'_{WL} + \Sigma\lambda'_{WR}$

provided equal numbers of observations have been made on each face i.e. N = 4n. The solution of the unknowns is then,

$$\lambda = \tfrac{1}{4}(\bar\lambda_{EL} + \bar\lambda_{ER} + \bar\lambda_{WL} + \bar\lambda_{WR})$$
$$C' = \tfrac{1}{4}(- \bar\lambda_{EL} + \bar\lambda_{ER} + \bar\lambda_{WL} - \bar\lambda_{WR})$$
$$\Delta H = \tfrac{1}{4}(- \bar\lambda_{EL} - \bar\lambda_{ER} + \bar\lambda_{WL} + \bar\lambda_{WR})$$

in which $\bar\lambda_{EL}$ etc. are the mean values computed in each set.

The values of the v corrections may be obtained by back substitution in the correction equations. The standard deviation of a single observation then is given as

$$\sigma_{so} = \sqrt{\frac{\Sigma vv}{N-3}}$$

Since the matrix of the Normal Equations is a diagonal one, the standard deviations of the unknowns are given as

$$\sigma_\lambda = \sigma_{C'} = \sigma_{\Delta H} = \frac{\sigma_{so}}{\sqrt{N}}$$

An alternative method of determining the v values consists in first computing the differences u from the means for each set of n observations as

$$u_{EL_i} = \bar\lambda_{EL} - \lambda'_{EL_i} \qquad \text{etc.}$$

and then relating the v values to the u values.

From the first equation

$$
\begin{aligned}
v_{EL_i} &= \lambda - C' - \Delta H - \lambda'_{EL_i} \\
&= \lambda - C' - \Delta H - \overline{\lambda}_{EL} + u_{EL_i} \\
&= \tfrac{1}{4}(\overline{\lambda}_{EL} + \overline{\lambda}_{ER} + \overline{\lambda}_{WL} + \overline{\lambda}_{WR} \\
&\quad +\overline{\lambda}_{EL} - \overline{\lambda}_{ER} - \overline{\lambda}_{WL} + \overline{\lambda}_{WR} \\
&\quad +\overline{\lambda}_{EL} + \overline{\lambda}_{ER} - \overline{\lambda}_{WL} - \overline{\lambda}_{WR} \\
&\quad -4\overline{\lambda}_{EL}) + u_{EL_i} \\
&= \tfrac{1}{4}(-\overline{\lambda}_{EL} + \overline{\lambda}_{ER} - \overline{\lambda}_{WL} + \overline{\lambda}_{WR}) + u_{EL_i} \\
&= D + u_{EL_i}
\end{aligned}
$$

The constant D may be conveniently evaluated when solving for λ, C' and ΔH. Similar treatment of the other three equations gives the following

$$
\begin{aligned}
v_{ER_i} &= -D + u_{ER_i} \\
v_{WL_i} &= D + u_{WL_i} \\
v_{WR_i} &= -D + u_{WR_i}
\end{aligned}
$$

When observations are made by skilled observers, the required number of observations is usually obtained with only a few, if any, rejections. If this does occur, a slight imbalance results and the weights of the mean values $\overline{\lambda}_{EL}$ etc. are no longer exactly the same. However, if the imbalance is only a slight one, the use of this method will provide answers, which do not vary significantly from the correct ones.

6.32 Example. The following observations were made at the University of New South Wales for the purpose of determining longitude.

Local Date	Wednesday evening 26th May 1976	Latitude 33°55'13" S		Theodolite	Zeiss 010
		Time Zone 10^h East		Clock	Mean Time
Observer	K. Gillies	R_o for local date		Pressure	1018 mb
Recorder	P. Ritchie	$16^h 14^m 45^s.6$		Temperature	16.0°C

Relationships used

1) Refraction $r° = \dfrac{0.0045\ P}{273.2 + T}(\tan z - 0.0012 \tan z \sec^2 z)$

2) Hour Angle $t^h = \dfrac{1}{15} \operatorname{arc}\cos(\sec\phi \sec\delta \cos z - \tan\phi \tan\delta)$

3) GST = (WT + WC - Zone) F + R_o
in which F = 1.0027379, WT is the observed clock time and WC is the clock correction with respect to Zone Time ZT.

The value of R_o at Greenwich date, which is the same as the local date, must be used.

4) $\Sigma u = 0$

5) $v = u \pm D$ and $\Sigma v = \Sigma u \pm nD = \pm n D$

Longitude Star East No. 393 RA $15^h02^m43.34^s$ δ 25°11'28.5" S

Watch Correction on Zone Time $18^h15^m05.53^s$

Observed Vertical Circle Reading	Observed Clock Time	Calculated Longitude	u	D	v
51°23'54"	$0^h30^m17.7^s$	$10^h04^m54.73^s$	-0.03^s	0.04^s	$+0.01^s$
15 40	30 57.7	54.64	+0.06		+0.10
08 51	31 31.1	54.27	+0.43		+0.47
51 00 09	32 12.9	54.59	+0.11		+0.15
50 43 03	33 35.4	54.88	-0.18		-0.14
36 29	34 07.3	54.78	-0.08		-0.04
CL 28 43	34 44.9	54.78	-0.08		-0.04
18 04	35 36.4	54.81	-0.11		-0.07
09 52	36 16.2	54.70	0.00		+0.04
50 02 22	36 52.4	54.79	-0.09		-0.05
49 54 38	0 37 29.9	10 04 54.68	+0.02	0.04	+0.06
	(1) $\overline{\lambda}_{EL}$	10 04 54.70	Σ +0.05		Σ +0.49
					Σvv 0.2889
310 43 10	0 40 30.2	10 04 57.17	-0.06	0.04	-0.10
50 22	41 05.3	56.88	+0.23		+0.19
310 58 57	41 46.7	56.95	+0.16		+0.12
311 08 02	42 30.2	57.34	-0.23		-0.27
15 53	43 08.3	57.16	-0.05		-0.09
CR 25 29	43 54.8	57.04	+0.07		+0.03
35 41	44 43.9	57.21	-0.10		-0.14
42 53	45 18.9	56.97	+0.14		+0.10
311 51 30	46 00.3	57.18	-0.07		-0.11
312 02 51	46 55.3	56.97	+0.14		+0.10
312 11 37	0 47 37.2	10 04 57.39	-0.28	0.04	-0.32
	(2) $\overline{\lambda}_{ER}$	10 04 57.11	Σ -0.05		Σ -0.49
					Σvv 0.2965

Longitude Star West No. 196 RA $7^h07^m25.28^s$ δ 26°21'34.5" S

Watch Correction on Zone Time $18^h15^m05.28^s$

Observed Vertical Circle Reading	Observed Clock Time	Calculated Longitude	u	D	v
47°21'02"	$0^h10^m05.8^s$	$10^h04^m56.74^s$	$+0.26^s$	0.04^s	$+0.30^s$
30 36	10 51.9	56.93	+0.07		+0.11
38 28	11 29.9	57.00	0.00		+0.04
44 54	12 01.2	56.84	+0.16		+0.20
47 53 48	12 44.1	57.02	-0.02		+0.02
48 02 40	13 26.6	57.45	-0.45		-0.41
CL 11 25	14 09.5	56.93	+0.07		+0.11
17 23	14 38.2	57.14	-0.14		-0.10
24 15	15 11.7	56.89	+0.11		+0.15
35 17	16 05.1	56.94	+0.06		+0.10
48 44 59	0 16 51.9	57.15	-0.15	0.04	-0.11
	(3) $\overline{\lambda}_{WL}$	10 04 57.00	Σ-0.03		Σ +0.41
				Σvv	0.3789

310 36 55	0 19 59.5	10 04 54.28	+0.47	0.04	+0.43
29 48	20 33.4	54.89	-0.14		-0.18
21 51	21 12.1	54.77	-0.02		-0.06
CR 13 47	21 51.3	54.71	+0.04		0.00
310 06 49	22 25.1	54.72	+0.03		-0.01
309 58 54	23 03.5	54.76	-0.01		-0.05
49 39	23 48.5	54.66	+0.09		+0.05
37 39	24 46.9	54.54	+0.21		+0.17
30 36	25 20.7	54.98	-0.23		-0.27
18 06	26 21.2	55.21	-0.46		-0.50
309 08 40	0 27 07.5	10 04 54.75	0.00	0.04	-0.04
	(4) $\overline{\lambda}_{WR}$	10 04 54.75	Σ-0.02		Σ -0.46
				Σvv	0.5794

Solution and estimates of precision

$\overline{\lambda}_{EL}$	$10^h04^m54.70^s$	(1):	¼((1)+(2)+(3)+(4))	= λ =	$10^h04^m55.89^s$ E
$\overline{\lambda}_{ER}$	57.11^s	(2):	¼(-(1)+(2)+(3)-(4))	= C' =	+ 1.17
$\overline{\lambda}_{WL}$	57.00^s	(3):	¼(-(1)-(2)+(3)+(4))	= ΔH =	- 0.02
$\overline{\lambda}_{WR}$	10 04 54.75	(4):	¼(-(1)+(2)-(3)+(4))	= D =	+ 0.04

$$\sigma_{so} = \sqrt{\frac{\Sigma vv}{N-3}} = \sqrt{\frac{1.5437}{44-3}} = \pm 0.19^s \quad \sigma_\lambda = \sigma_{C'} = \sigma_{\Delta H} = \frac{\sigma_{so}}{\sqrt{N}} = \frac{\pm 0.194}{\sqrt{44}} = \pm 0.03^s$$

$$\lambda = 10^h04^m55.89^s \text{ E} \quad \pm 0.03^s$$

7
Determination of Azimuth

INTRODUCTION

AZIMUTH determinations are required for the purpose of orienting surveys or checking extended surveys to ensure that their orientation is being maintained. Examples of these applications are given in Chapter 1.

An astronomical determination of azimuth consists basically in measuring a horizontal angle at the instrument station between a distant reference object RO and a star. Once the star's azimuth has been established, it becomes a simple matter to determine the azimuth of the RO.

Two methods of observation to the star are available for the determination of its azimuth. In the first or Time Azimuth Method, the time, at which the horizontal pointing to the star is made, is recorded. In the second or Alt-azimuth Method, a vertical circle observation is made instead of a time observation.

Precautions to be observed in Azimuth Determinations

7.11 Great care must be taken, as in any horizontal direction observations, to set the theodolite up on a stable base and to centre the theodolite precisely over the mark. The RO should preferably be at a distance such that the stellar focus, required for accurate sighting of the star, needs no alteration when the RO is sighted. In addition, the target at the RO should be carefully centred and should present to the observer an image which is capable of being accurately bisected in the vertical sense. For night observations, an ideal object on which to sight is a light source, which gives the appearance of a third or fourth magnitude star.

The line of sight to the RO is seldom inclined to any great degree but that to the star often has a considerable inclination. If the theodolite is imperfectly levelled, a correction, which is proportional to the tangent of the altitude and to the component of the inclination of the vertical axis of the theodolite at right angles to the direction sighted, is applied to the horizontal circle reading. Therefore the levelling of the theodolite should be carefully carried out, preferably between each arc of horizontal readings. This levelling procedure is described in section 4.12 and every endeavour should be made to do this accurately. If this procedure is followed, the residual errors of levelling should be small and of a random nature and their contribution to the final azimuth result should likewise be small. On some theodolites, the inclination of the vertical axis can be determined by means of a striding level and the appropriate correction applied to the horizontal circle readings.

It will be seen later that, in high latitudes, the line of sight to some stars will be at an altitude which is nearly equal to, or greater, than that of the elevated pole. For these high sights, it may be necessary to use special instruments or special attachments for the theodolite.

The Design of an Observation Series

7.21 As was stated before, the determination of astronomical azimuth requires the measurement of a horizontal angle, a process which is familiar to all surveyors. However, the various techniques, which have been devised for these observations in normal surveying practice, may need to be modified for astronomical work because of the following considerations,

(1) The limited period, during which a star is favourably located in an observation programme, restricts the number of observations that can be made.
(2) The stars sighted are often dim and well elevated above the horizon and therefore horizontal and vertical circle settings are needed to locate them.
(3) Great attention must be paid to the levelling of the theodolite throughout the observation series.

Therefore, a great economy in observing time can be effected, whilst still maintaining precision in the final result, by making multiple pointings on the RO and on the star and changing the theodolite face less frequently during the observation period. It will be seen in the examples later in this chapter that various observing techniques have been used.

A single azimuth value is obtained from single observations made on one face to both RO and star. Likewise only a single value of the azimuth is forthcoming, when multiple observations are made on one face to both RO and star, because the individual observations to both objects sighted cannot be specifically paired, one with the other. Each provides the information for one correction equation in the adjustment process and may therefore be considered to provide the statistical unit in this process (see sections 7.43, 7.44, 7.45, 7.47 and 7.62).

There are some, however, who consider that where multiple observations have been made on one face, certain RO observations can be paired with certain star observations to provide a value of azimuth of the RO. The decision as to how these observations are to be paired is one for the observer who may wish to take into account the stability of the instrument over the observation period. In order to preserve a uniform approach in this book, a single method suitable for all cases has been used throughout.

AZIMUTH FROM TIME AZIMUTH OBSERVATIONS

IN this method, the horizontal circle reading is obtained from an observation to the RO and then to a known star. The time of observation to the star is read off a clock, whose correction can be deduced from a knowledge of the station's longitude and of clock comparisons, made with respect to a radio time signal.

7.31 The azimuth A to the star is computed from the known latitude ϕ, the known star's declination δ and the hour angle t deduced from the observed clock time. These four quantities are linked by the Four Parts Formula

$$\cot A = \sin\phi \cot t - \tan\delta \cos\phi \operatorname{cosec} t \qquad \ldots 7.1$$

This, when differentiated, gives the relationship between the small changes dA, $d\phi$, $d\delta$ and dt as

$$dA = \sec h \cos\omega \cos\delta\, dt + \tan h \sin A\, d\phi + \sec h \sin\omega\, d\delta \qquad \ldots 7.2$$

The declinations, taken from reliable catalogues, may, for all but geodetic quality work, be considered error free (see section 5.12). The effect, on the azimuth sought, of errors in the data may therefore be taken as

$$dA = \sec h \cos\omega \cos\delta\, dt + \tan h \sin A\, d\phi$$

On substitution for $\cos\omega \cos\delta$ from the Five Parts Formula

$$dA = \sec h (\sin\phi \cos h - \cos\phi \sin h \cos A) dt + \tan h \sin A\, d\phi$$

$$dA = \cos\phi (\tan\phi - \tan h \cos A) dt + \tan h \sin A\, d\phi \qquad \ldots 7.3$$

In this relationship, $d\phi$ must be considered entirely as a systematic error, because its value is not known exactly and it is not an observed quantity. The error dt is partly systematic and partly random.

The systematic component of dt is due to the error in the assumed value of longitude and to the systematic errors present in the timing system being used. The random component results from the observer's inability to make perfect observations.

7.32 If now a single star is to be observed, it should be at meridian transit (A = 0° or 180°) and also at elongation (ω = 90° or 270°) to eliminate the $d\phi$ and dt components respectively.

When a star is at elongation, its motion is entirely in a vertical sense and thus it is ideally situated for making accurate horizontal pointings on it. The conditions of meridian transit and elongation can only be achieved simultaneously when a star is either at the pole or in the zenith. The latter position is of no interest since azimuth then becomes indeterminate. There is, however, no star exactly at either pole. But there is a star within one degree of each celestial pole, the northern one being bright and easily visible to the naked eye and the southern one being faint and usually needing a telescope to be seen. Since any star at a very low altitude is difficult to see, because the line of sight is traversing a long part of its path through the lower layers of the earth's atmosphere, which are often not very clear, the pole stars will be difficult to see from stations closer to the equator than about 15°.

Table 7.1 shows the minimum and the maximum values of the rates $\frac{dA}{dt}$ and $\frac{dA}{d\phi}$ for these pole stars.

Table 7.1

	$\frac{dA}{dt}$		$\frac{dA}{d\phi}$	
Latitude	Maximum Value (at Upper Transit)	Minimum Value (at Elongation)	Minimum Value (at Transit)	Maximum Value (at Elongation)
15°	0.02	0	0	0.00
30	0.02	0	0	0.01
45	0.03	0	0	0.02
60	0.04	0	0	0.06

From this it is seen that, if the timing is *correct even to the nearest* second of time and the latitude *to the nearest* fifteen seconds of arc, the azimuth can be obtained to a very high accuracy. (see Table 7.3 for a comparison) These facts account for the widespread use of the pole star for azimuth determination, especially in the northern hemisphere, in which the pole star is easily seen, although the southern pole star can easily be found in the theodolite telescope's field of view, if precomputation is used. (see Section 10.12) Two examples of pole star observations are given in Sections 7.43 and 7.44.

For low latitudes, however, other methods for azimuth determination must be investigated. These are the circum-meridian and circum-elongation methods. If only one star is observed, neither method will give azimuth results free of the effects of the systematic errors $d\phi$ and dt. If, however, two stars are used, they can be balanced to achieve this, whilst still keeping the effects of the random errors of observation small.

7.33 For meridian observations in low latitudes, stars will be visible at upper transit only if they are above a certain minimum altitude, which depends on the star's magnitude and the atmosphere's clarity and which, for practical purposes, will be assumed to be 15°. The $d\phi$ coefficient is zero for a star on the meridian, is numerically small for low altitude sights close to the meridian and changes sign as the star crosses the meridian. The dt coefficient on the meridian at *upper* transit is

$$\pm \sec h \cos \delta$$

because the parallactic angle ω at transit is 0° or 180° and as $\delta_M = \delta$ at upper transit

$$\frac{dA}{dt} \text{ at upper transit} = -\frac{\cos\delta}{\sin z_M} \qquad \ldots 7.4$$

in which the subscript M denotes a meridian value (see sections 5.32 and 5.33 for conventions and signs). This coefficient does not change sign as the star crosses the meridian, but its sign is positive for a star on the meridian to the south and negative for one on the meridian to the north. Therefore a balanced pair of stars must consist of one star north and one star south observed close to the meridian with the two values of $\frac{dA}{dt}$ equal in magnitude and opposite in sign. In addition, the coefficients should be small to keep the effects of the random errors small.

Table 7.2 shows rates of change of azimuth with respect to time on the meridian for a star at upper transit, with the correspondingly balanced values on the opposite side of the zenith.

Table 7.2

ϕ	z_M	δ	$\frac{dA}{dt}$	δ	z_M	$\frac{dA}{dt}$
0°	+75°	+75°	-0.268	-75°	-75	+0.268
-5	+75	+70	-0.354	-71.1	-66.1	+0.354
-10	+75	+65	-0.438	-68.2	-58.2	+0.438
-15	+75	+60	-0.518	-66.2	-51.2	+0.518

Note: All signs in this table must be reversed for northern latitudes.

To achieve this balance for a pair of stars

$$\frac{dA}{dt} \text{ at upper transit to the south} = -\frac{dA}{dt} \text{ at upper transit to the north}$$

where $$\frac{dA}{dt} = \frac{-\cos\delta}{\sin z_M} = \frac{-\cos\delta}{\sin(\delta-\phi)}$$

$$\therefore \quad \frac{-\cos\delta}{\sin(\delta-\phi)} = \frac{-\cos\delta}{\sin\delta\cos\phi - \sin\phi\cos\delta} = \frac{-1}{\tan\delta\cos\phi - \sin\phi}$$

$$\therefore \quad \tan\delta_S\cos\phi - \sin\phi = -(\tan\delta_N\cos\phi - \sin\phi)$$

$$\therefore \quad \tan\delta_N + \tan\delta_S = 2\tan\phi \qquad \ldots 7.5$$

7.34 The above method is usually confined to equatorial latitudes and it is stressed that, unlike many other astronomical methods, the stars of a matched pair are selected not to transit at equal altitudes, but at such altitudes as will produce equal rates of change of azimuth with respect to time. Very faint stars should be avoided, as they may be difficult to see at the postulated minimum altitude. An example of such a pair of stars is given in section 7.45. If the stars of a pair are well matched and if each of the two stars is observed by means of sights well balanced about the point of upper transit, the effects of systematic error will be eliminated. Random errors in timing are not greatly minimised, because the coefficient of dt is not small and therefore careful attention should be paid to this aspect of the observations.

7.35 The observation of matched pairs of stars at elongation (see section 7.32) will now be further investigated. In this method, the effect of the systematic error $d\phi$ can be eliminated, if the stars are at elongation at points symmetrically disposed about the meridian. The effect of the systematic component of the error dt can be eliminated by means of sights well balanced about the point of elongation on each star, because the coefficient of dt changes its sign, as the star crosses this point. The random component in the error dt is rendered negligible in such observations, if they are made close to the point of elongation, because the coefficient is then very small in magnitude.

In low latitudes, stars may elongate at such low altitudes that they cannot be seen. Fig 7.1 shows the elongation locus for stars, when the observer is in such latitudes. For an observer exactly on the equator every star on the horizon and every star on the prime vertical is at elongation. In these latitudes, a star will elongate at an altitude greater than the minimum at which it can be seen, only when it is well away from the meridian in azimuth. It will then also be changing altitude fast. The prediction techniques for such stars are dealt with in section 10.51.

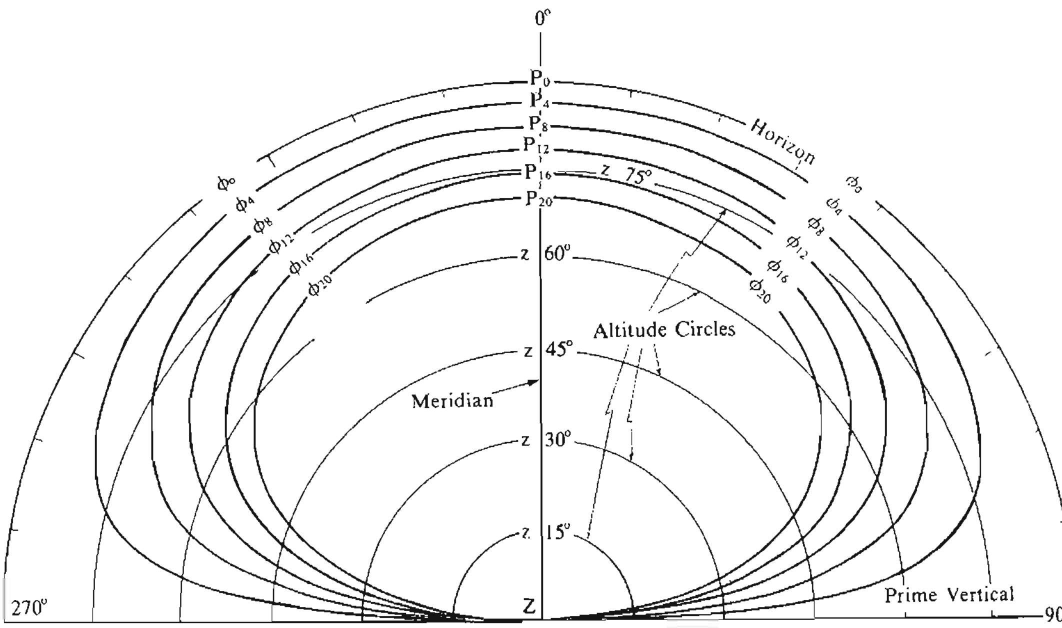


Fig. 7.1 The Elongation Locus at Low Latitudes

7.36 The method of circum-elongation azimuth observations is not confined to equatorial latitudes (see section 7.35). The matching of the stars of a pair can usually be accurately achieved. The errors in azimuth, coming from an uncertainty in the latitude adopted and from any systematic error in the longitude and the timing, are then eliminated in the mean of the azimuths from the star pair. The effects of random timing errors are greatly minimised, because the coefficient of dt is very small indeed, if the observations are made close to the point of elongation. In practice, it is usually quite sufficient to read the observed times to the nearest second and the timing arrangements can be much simplified. An example of such a pair of star observations is given in section 7.47.

Calculation of Azimuth from Time Azimuth Observations

7.41 This calculation can be unequivocally determined by means of the Four Parts Formula in the following form:-

$$\tan A = \frac{-\sin t}{\tan \delta \cos \phi - \sin \phi \cos t} \qquad \ldots 7.6$$

and is recommended for use for *all* Time Azimuth reductions.

Alternatively, the Transformation Formulae (see section A.41 in the appendix) may be used. These are

$$\tan M = \frac{\tan \delta}{\cos t}$$

and

$$\tan A = \frac{-\tan t \cos M}{\sin(M-\phi)} \qquad \ldots 7.7$$

Note: Where a quantity is obtained from a tangent function in the above formulae, the numerator and the denominator are evaluated separately in order to place the quantity in its correct angular quadrant.

For observations on either Polaris or Sigma Octantis, at any hour angle, the following relationships, from section A.52 in the appendix, may be used:-

(i) For the northern pole star, Polaris, α Ursae Minoris

$$A = -p \sin t \sec \phi - \frac{p^2}{\rho} \sin t \cos t \sec \phi \tan \phi - \frac{1}{3}\frac{p^3}{\rho^2} \sin t \sec \phi (1 + 3 \tan^2\phi \cos^2 t - \sin^2 t \sec^2 \phi) \ldots$$

or

(ii) for the southern pole star Sigma Octantis

$$A = 180° + p \sin t \sec \phi - \frac{p^2}{\rho} \sin t \cos t \sec \phi \tan \phi + \frac{1}{3}\frac{p^3}{\rho^2} \sin t \sec \phi (1 + 3 \tan^2\phi \cos^2 t - \sin^2 t \sec^2 \phi) \ldots \qquad \ldots 7.8$$

in which p is the positive angular distance of the star from the adjacent pole and the units of p and ρ are in accordance with one another. These last relationships, in addition to providing an alternative check calculation, can be conveniently used for predictions and approximate reductions from the first term in the series.

For observations on circum-meridian stars, the following relationship from section A.74 of the appendix may be used:-

$$A = A_o - \frac{\cos \delta}{\sin z_M} t' + \frac{1}{6} \cos \phi \cos \delta_M \operatorname{cosec}^3 z_M \{\cos z_M \cos \delta_M + \cos \phi\} t'^3 \ldots \qquad \ldots 7.9$$

in which $A_o = 0°$ for star north and 180° for star south.

For observations on circum-elongation stars, the following relationship from section A.81 of the appendix may be used:-

$$A = A_e - \frac{C}{2\rho} (\Delta t)^2 + \frac{C \cot t_e}{2\rho^2} \Delta t^3 \ldots \qquad \ldots 7.10$$

in which A_e is the azimuth of the star at elongation, $C = \sin^2\delta \tan A_e$ and $\Delta t = t - t_e$, where t_e is the hour angle at elongation. The units of Δt and ρ are in accordance with one another.

7.42 The examples given below aim to show detail enough for a student to follow them through. They have been computed by means of a small calculator. Some of the details of this calculation have been shown for the sake of illustration, although in practice this would be avoided because transcription is so liable to mistakes.

7.43 The following time azimuth observations were made on Polaris.

Station	München Technische Universität Roof Station	ϕ 48°09'05"North λ $0^h46^m16^s.7$East	Theodolite Wild T2 No. 35712
RO	Red light on Olympic Tower		Clock Mean Time split hand stop watch
Date	Monday 26th June 1972		
Observer	G.G. Bennett		
Recorder	S. Fajnor	R_o for date	$18^h16^m51.7^s$
		R_{18} for date	18 19 49.1
		CC_{GMT}	-1 00 00.4

Observations

Pole Star Polaris RA $2^h04^m42.2^s$ δ 89°08'05."8 N

	Horizontal Circle Readings	Observed Clock Times	Horizontal Circle Readings	Observed Clock Times
	Arc I		Arc II	
RO	CR 158°30'42"		CL 68°36'02"	
Star	180 25 25	$21^h19^m32^s$	90 31 50	$21^h24^m54^s$
Star	CL 00 25 31	21 21 17	CR 270 32 38	21 25 43
RO	338 30 53		248 35 52	

7.431 Solution by the direct relationship

Relationships used

$$t^\circ = 15\,\{\lambda + R_o - RA + F\,(CT + CC_{GMT})\}$$

$$F = 1.0027379$$

$$\tan A = \frac{-\sin t}{\cos\phi\tan\delta - \sin\phi\cos t}$$

To determine the azimuth

Obs CT	$21^h19^m32^s$	$21^h21^m17^s$	$21^h24^m54^s$	$21^h25^m43^s$
A_{Star}	0°26'36.2"	0°27'09.3"	0°28'17.2"	0°28'32.5"
H_{Star}	180 25 25	0 25 31	90 31 50	270 32 38
Orienting Correction	+180 01 11.2	+ 0 01 38.3	-90 03 32.8	-270 04 05.5
H_{RO}	158 30 42	338 30 53	68 36 02	248 35 52
A_{RO}	338 31 53.2	338 32 31.3	338 32 29.2	338 31 46.5
	Arc I Mean 338°32'12.3"		Arc II Mean 338°32'07.8"	

Mean Azimuth to RO 338°32'10.0"

7.432 Solution by means of the Transformation Formulae

Relationships used

$$\tan M = \frac{\tan\delta}{\cos t}$$

$$\tan A = \frac{-\tan t\cos M}{\sin(M-\phi)}$$

To determine the local hour angle and the azimuth

CT of Obs	$21^h19^m32^s$	$21^h21^m17^s$	$21^h24^m54^s$	$21^h25^m43^s$
CC_{GMT}	-1 00 00.4	-1 00 00.4	-1 00 00.4	-1 00 00.4
GMT of Obs	20 19 31.6	20 21 16.6	20 24 53.6	20 25 42.6
R_{18}	18 19 49.1	18 19 49.1	18 19 49.1	18 19 49.1
dR	22.9	23.2	23.8	23.9
GST of Obs	14 39 43.6	14 41 28.9	14 45 06.5	14 45 55.6
λ	0 46 16.7E	0 46 16.7E	0 46 16.7E	0 46 16.7E
LST of Obs	15 26 00.3	15 27 45.6	15 31 23.2	15 32 12.3
RA	2 04 42.2	2 04 42.2	2 04 42.2	2 04 42.2
Hour angle t	13 21 18.1	13 23 03.4	13 26 41.0	13 27 30.1
M	90°48'40.3"	90°48'32.0"	90°48'14.1"	90°48'10.0"
ϕ	48 09 05	48 09 05	48 09 05	48 09 05
M-ϕ	42 39 35.3	42 39 27.0	42 39 09.1	42 39 05.0
A	0°26'36."2	0°27'09."3	0 28 17.2	0 28 32.5
Clockwise Angle	338 05 17	338 05 22	338 04 12	338 03 14
Azimuth RO	338 31 53.2	338 32 31.3	338 32 29.2	338 31 46.5
	338°32'12.3"		338°32'07.8"	

Azimuth to RO 338°32'10.0"

7.44 Example. Time azimuth determination from the southern pole star.

Place Pillar 5 Civ.Eng.Bldg UNSW
Latitude 33°55'12" South
Longitude $10^h04^m55.9^s$ East
Date Wednesday 29th January 1975
Reference Object Flashing red light on Harbour Bridge
Observer G.G. Bennett
Recorder J.G. Freislich

Theodolite Wild T2 No. 148423
Watch Heuer Stop Watch (Mean)
R_o for date $8^h30^m29.8^s$
R_6 for date 8 31 28.9
Clock Correction with respect to Zone Time $+18^h40^m07.5^s$
Time Zone 11^h East

Observations on σ Octantis RA $20^h43^m22.9^s$ δ 89°03'06"S

Arc I	Horizontal Circle Reading	Face	Observed Watch Time (WT)	Horizontal Circle Reading	Face	Observed Watch Time (WT)
RO	344°27'53"			164°27'58"		
Star	180 46 22	CL	$3^h43^m16.5^s$	0 45 30	CR	$3^h47^m53^s$
Star	180 46 17		3 43 43.5	0 45 07		3 49 47
RO	344 27 51			164 28 01		
Arc II						
RO	224°31'01"			44°30'55"		
Star	60 47 42	CR	$3^h51^m38.5^s$	240 46 47	CL	$3^h54^m12.5^s$
Star	60 47 38		3 51 57.5	240 46 45		3 54 31.5
RO	224 31 00			44 30 53		
Arc III						
RO	104°34'10"			284°34'19"		
Star	300 49 31	CL	$3^h56^m39^s$	120 49 16	CR	$3^h58^m57^s$
Star	300 49 28		3 56 57	120 49 11		3 59 19
RO	104 34 10			284 34 18		

7.441 Relationships used $t° = 15\{(\lambda + R_o - RA + F(WT - Z + WC)\}$

in which WT is the Observed Watch Time
Z is the Time Zone
WC is the Watch Correction with respect to Zone Time
and F = 1.0027379

$$\tan A = \frac{-\sin t}{\cos\phi \tan\delta - \sin\phi \cos t}$$

Arc I	CL		CR	
Observed WT	$3^h43^m16.5^s$	$3^h43^m43.5^s$	$3^h47^m53^s$	$3^h49^m47^s$
Local Hour Angle t	139°19'46"	139°26'32"	140°29'05"	140°57'40"
A_{star}	180 44 18.9	180 44 12.7	180 43 15.6	180 42 49.2
H_{star}	180 46 22	180 46 17	0 45 30	0 45 07
Orienting Corrn OC	- 0 02 03.1	- 0 02 04.3	-180 02 14.4	-180 02 17.8
Diffs. from Mean	- 0.6	+ 0.6	- 1.7	+ 1.7
$\overline{OC}$	- 0°02'03.7"		-180°02'16.1"	
H_{RO}	344 27 53	344 27 51	164 27 58	164 28 01
Diffs from Mean	- 1.0	+ 1.0	+ 1.5	- 1.5
$\overline{H}_{RO}$	344 27 52.0		164 27 59.5	
A_{RO}	344 25 48.3		344 25 43.4	
	Mean Arc I 344°25'45.8"			

The results from similar calculations for Arcs II and III are,

Arc II

A_{RO} 344°25'52.8" 344°25'41.4"

Mean Arc II 344°25'47.1"

Arc III

A_{RO} 344 25 50.7 344 25 41.7

Mean Arc III 344 25 46.2

Mean Azimuth from Pillar 5 to RO	344°25'46.4"
Eccentric Correction to Geodetic Pillar	+ 2 08.7
Azimuth, Geodetic Pillar to RO	344°27'55.1"

7.442 Example. The alternative method of reduction by means of the series developed in section A.52 will be used. The local hour angle will be computed by means of the value of R_6 instead of R_O as in section 7.441. As will be seen the results of the computations for Arc I are very close to those obtained in section 7.441.

Relationship used

For the southern pole star $A = 180^{\circ} + p \sin t \sec\phi\{1 - \frac{p}{\rho}\cos t \tan\phi\} \ldots$

δ = 89°03'06 S p = 0°56'54" = 3414"

Arc I	CL		CR	
WT of Obs	$3^h43^m16.5^s$	$3^h43^m43.5^s$	$3^h47^m53.0^s$	$3^h49^m47.0^s$
CC_{GMT}	+7 40 07.5	+7 40 07.5	+7 40 07.5	+7 40 07.5
GMT of Obs	11 23 24.0	11 23 51.0	11 28 00.5	11 29 54.5
R_6	8 31 28.9	8 31 28.9	8 31 28.9	8 31 28.9
ΔR	53.1	53.2	53.9	54.2
GST of Obs	19 55 46.0	19 56 13.1	20 00 23.3	20 02 17.6
λ	10 04 55.9E	10 04 55.9E	10 04 55.9E	10 04 55.9E
LST of Obs	6 00 41.9	6 01 09.0	6 05 19.2	6 07 13.5
RA	20 43 22.9	20 43 22.9	20 43 22.9	20 43 22.9
LHA	139°19'45"	139°26'32"	140°29'04"	140°57'39"
A_{star}	180 44 18.6	180 44 12.5	180 43 15.3	180 42 48.9
H_{star}	180 46 22	180 46 17	0 45 30	0 45 07
Orienting Corrn OC	- 0 02 03.4	- 0 02 04.5	-180 02 14.7	-180 02 18.1
Diffs from Mean	- 0.5	+ 0.6	- 1.7	+ 1.7
$\overline{OC}$	-0°02'03.9"		-180°02'16.4"	
H_{RO}	344 27 53	344 27 51	164 27 58	164 28 01
Diffs from Mean	- 1.0	+ 1.0	+ 1.5	- 1.5
$\overline{H}_{RO}$	344 27 52.0		164 27 59.5	
A_{RO}	344 25 48.1		344 25 43.1	
	Mean Arc I	344°25'45.6"		

7.443 Example. Least Squares Solution for a close circum-polar star.

The correction equations for this situation are as follows:-

$$v = A \pm\{C'\sec h + i'\tan h\} - A'$$

$$= A \pm C - A'$$

in which A is the adjusted value of the azimuth to the RO
A' is the calculated value of the azimuth to the RO
C is the combined effect of the horizontal collimation C' and the inclination of the trunnion axis i'
v is the correction to be applied to the calculated azimuth to obtain the adjusted value.

The correction equations for the three arcs of observations in section 7.441 consist of the following:-

$$v_{IL} = A + C - A'_{IL} = A + C - 48.3''$$
$$v_{IR} = A - C - A'_{IR} = A - C - 43.4$$
$$v_{IIL} = A + C - A'_{IIL} = A + C - 52.8$$
$$v_{IIR} = A - C - A'_{IIR} = A - C - 41.4$$
$$v_{IIIL} = A + C - A'_{IIIL} = A + C - 50.7$$
$$v_{IIIR} = A - C - A'_{IIIR} = A - C - 41.7$$

These give the following Normal Equations:-

A		C	-	L	=	0
6	+	0	-	$(\Sigma A'_L + \Sigma A'_R)$	=	0
		6	-	$(\Sigma A'_L - \Sigma A'_R)$	=	0

$$\therefore \quad A = \tfrac{1}{6}(\Sigma A'_L + \Sigma A'_R) = \tfrac{1}{2}(\bar{A}_L + \bar{A}_R) = \tfrac{1}{2}(50.6 + 42.2) = 46.4''$$

$$C = \tfrac{1}{6}(\Sigma A'_L - \Sigma A'_R) = \tfrac{1}{2}(\bar{A}_L - \bar{A}_R) = \tfrac{1}{2}(50.6 - 42.2) = 4.2$$

Back substitution gives the V values as

$v_{IL} = +2.3''$ $\quad v_{IR} = -1.2''$ $\quad \Sigma v = +0.1 \quad \Sigma vv = 12.47$

$v_{IIL} = -2.2$ $\quad v_{IIR} = +0.8$ $\quad \sigma_{so} = \sqrt{\frac{12.47}{6-2}} = \pm 1.77''$

$v_{IIIL} = -0.1$ $\quad v_{IIIR} = +0.5$ $\quad \sigma_A = \sigma_C = \sqrt{\frac{12.47}{(6-2)6}} = \pm 0.72''$

Azimuth to RO from Pillar 5 = 344°25'46.4" ±0.72"

Circum-Meridian Time Azimuths

These are observations suited to azimuth determinations in equatorial latitudes (see sections 7.33 and 7.34).

7.45 Example. The following observations were made in Port Moresby, Papua New Guinea.

Place Mark on roof of CGO Building
Date Wednesday 9th November 1977
R_0 for this date $3^h12^m12.6^s$
Observer B.J. Forester
Recorder I.F. Jarvies

Theodolite Wild T2 No. 145852
Chronometer Mercer MT No. 24950
Reference Object Red light on Mast
DUT1 correction for date -0.2^s
Time Signal VNG Time Zone 10^h E

Latitude ϕ 9°26'22" South
Longitude λ $9^h48^m43.3^s$ East

Four clock correction values were each determined as the mean of eight comparisons with respect to the time signal. These gave the following results:-

Signal Time	DUT1	Zone Time	Observed Clock Time	Clock Correction with respect to GMT
$18^h57^m30.00^s$	-0.2^s	$18^h57^m29.80^s$	$18^h57^m39.10^s$	$-10^h00^m09.30^s$
20 28 30.00		20 28 29.80	20 28 37.75	-10 00 07.95
22 00 30.00		22 00 29.80	22 00 36.09	-10 00 06.29
23 10 30.00	-0.2	23 10 29.80	23 10 35.06	-10 00 05.26

Star North No. 618 RA $22^h30^m23.9^s$ δ 50°10'28" N

Observations

	Hor.Circle Reading	Observed Clock Time	Stop Watch	Hor.Circle Reading	Observed Clock Time	Stop Watch
	CL		Arc I		CR	
RO	141°28'22"			321°28'13"		
RO	141 28 23			321 28 13		
Star	1 41 49	$19^h19^m00^s$	4.7^s	180 51 00	$19^h23^m35^s$	7.7^s
Star	34 19	19 45	9.2	44 03	24 12	7.5
Star	28 06	20 15	5.8	38 21	24 42	6.9
Star	1 21 45	19 20 50	6.5	180 32 58	19 25 10	5.9
RO	141 28 24			321 28 10		
RO	141 28 20			321 28 15		
	CL		Arc II		CR	
RO	231 33 24			51 33 15		
RO	231 33 24			51 33 17		
Star	89 55 08	19 29 05	9.0	269 00 15	19 33 55	5.8
Star	48 16	29 40	7.3	268 53 50	34 30	6.4
Star	41 12	30 17	6.5	48 36	35 00	8.1
Star	89 35 26	19 30 50	8.5	268 42 13	19 35 35	8.9
RO	231 33 17			51 33 11		
RO	231 33 21			51 33 12		

Note The stopwatch was started as the star observation was made and then stopped at the whole second of clock time recorded as Observed Clock Time

Star South No. 641   RA $23^h16^m09.1^s$ δ 58°21'35" S

Observations

	Hor.Circle Reading	Observed Clock Time	Stop Watch	Hor.Circle Reading	Observed Clock Time	Stop Watch
	CL	Arc I		CR		
RO	141°28'26"			321°28'19"		
RO	141 28 26			321 28 18		
Star	178 04 22	$20^h02^m45^s$	7.5^s	358 56 45	$20^h07^m50^s$	10.2^s
Star	09 45	03 15	7.3	359 03 41	08 30	9.6
Star	14 19	03 42	7.7	09 31	09 05	11.3
Star	178 19 36	20 04 13	8.8	359 15 51	20 09 40	10.0
RO	141 28 25			321 28 18		
RO	141 28 25			321 28 18		
	CL	Arc II		CR		
RO	231 33 22			51 33 12		
RO	231 33 24			51 33 14		
Star	270 11 14	20 14 25	8.6	91 03 11	20 19 25	8.0
Star	17 46	15 00	6.0	09 15	20 00	8.9
Star	22 56	15 32	8.3	15 43	20 35	6.3
Star	270 29 52	20 16 10	6.3	91 21 32	20 21 10	8.0
RO	231 33 23			51 33 15		
RO	231 33 23			51 33 15		

7.451 Determination of Azimuth

Relationships used

$$t = \lambda - RA + R_O + F(CC_{GMT} + CT)$$

$$\tan A = \frac{-\sin t}{\cos\phi \tan\delta - \sin\phi \cos t}$$

Star North No. 618

Arc I CL

CC_{GMT}	$-10^h00^m09.04^s$			
CT	$19^h18^m55.3^s$	$19^h19^m35.8^s$	$19^h20^m09.2^s$	$19^h20^m43.5^s$
t	-2°17'29.2"	-2°07'20.1"	-1°58'57.7"	-1°50'21.8"
A_{star}	1 42 01.7	1 34 30.0	1 28 17.4	1 21 54.8
H_{star}	1 41 49	1 34 19	1 28 06	1 21 45
Orienting Corrn	+12.7	+11.0	+11.4	+ 9.8
Diffs from Mean	- 1.5	+ 0.2	- 0.2	+ 1.4
Mean $\overline{OC}$	+11.2"			
H_{RO}	141 28 22	141 28 23	141 28 24	141 28 20
Diffs from Mean	+ 0.2	- 0.8	- 1.8	+ 2.2
Mean $\overline{H}_{RO}$	141°28'22.2"			
A_{RO}	141 28 33.4			

Arc I CR

CC_{GMT}	$-10^h00^m08.94^s$			
CT	$19^h23^m27.3^s$	$19^h24^m04.5^s$	$19^h24^m35.1^s$	$19^h25^m04.1^s$
t	-1°09'16.6"	-0°59'57.0"	-0°52'16.8"	-0°45'00.6"
A_{star}	0 51 25.6	0 44 30.3	0 38 48.7	0 33 24.9
H_{star}	180 51 00	180 44 03	180 38 21	180 32 58
Orienting Corrn	180 00 25.6	180 00 27.3	180 00 27.7	180 00 26.9
Diffs from Mean	+ 1.3	- 0.4	- 0.8	+ 0.0
Mean $\overline{OC}$	180°00'26.9"			
H_{RO}	321 28 13	321 28 13	321 28 10	321 28 15
Diffs from Mean	- 0.2	- 0.2	+ 2.8	- 2.2
Mean $\overline{H}_{RO}$	321°28'12.8"			
A_{RO}	141 28 39.7			
	Arc I Mean	141 28 36.6		

The results from similar calculations for Arc II are,

Arc II	CL	A_{RO}	141 28 30.3
	CR	A_{RO}	141 28 38.8
	Arc II Mean		141 28 34.5
Mean A_{RO}	North Star		141°28'35.5"

Star South No. 641

Arc I CL

CC_{GMT}	$-10^h00^m08.34^s$			
CT	$20^h02^m37.5^s$	$20^h03^m07.7^s$	$20^h03^m34.3^s$	$20^h04^m04.2^s$
t	-2°46'16.0"	-2°38'41.8"	-2°32'01.7"	-2°24'32.0"
A_{star}	178 04 23.7	178 09 39.0	178 14 16.8	178 19 29.1
H_{star}	178 04 22	178 09 45	178 14 19	178 19 36
Orienting Corrn	+ 1.7	- 6.0	- 2.2	- 6.9
Diffs from Mean	- 5.1	+ 2.6	- 1.2	+ 3.5
Mean $\overline{OC}$	- 3.4"			
H_{RO}	141 28 26	141 28 26	141 28 25	141 28 25
Diffs from Mean	- 0.5	- 0.5	+ 0.5	+ 0.5
Mean $\overline{H}_{RO}$	141°28'25.5"			
A_{RO}	141 28 22.1			

Arc I CR

CC_{GMT}	$-10^h00^m08.24^s$			
CT	$20^h07^m39.8^s$	$20^h08^m20.4^s$	$20^h08^m53.7^s$	$20^h09^m30.0^s$
t	-1°30'27.6"	-1°20'16.9"	-1°11'56.0"	-1°02'50.1"
A_{star}	178 57 03.8	179 04 08.5	179 09 56.9	179 16 16.7
H_{star}	358 56 45	359 03 41	359 09 31	359 15 51
Orienting Corrn	+180 00 18.8	+180 00 27.5	+180 00 25.9	+180 00 25.7
Diffs from Mean	+ 5.7	- 3.0	- 1.4	- 1.2
Mean $\overline{OC}$	180°00'24.5"			
H_{RO}	321 28 19	321 28 18	321 28 18	321 28 18
Diffs from Mean	- 0.8	+ 0.2	+ 0.2	+ 0.2
Mean $\overline{H}_{RO}$	321°28'18.2"			
A_{RO}	141 28 42.7			
	Arc I Mean	141 28 32.4		

The results from similar calculations for Arc II are,

Arc II	CL	A_{RO}	141 28 26.5
	CR	A_{RO}	141 28 41.9
	Arc II Mean		141 28 34.2
Mean A_{RO} South Star			141 28 33.3
Mean A_{RO} North Star			141 28 35.5
Mean A_{RO} from pair			141°28'34.4"

7.452 For the sake of illustration, some of the observations of the first arc of the star to the south will be computed by means of the power series derived for this purpose in section A.74. It is a series, which contains only the odd powers and which converges very rapidly. In this computation, the first two terms give results which differ from the rigorous results of section 7.451 by only very small amounts.

The method of approach should be compared with that of section 7.451, as well as with the solutions for circum-meridian latitudes of section 5.493 and for circum-elongation azimuths of section 7.472.

Determination of the clock time of passage of the south star over the local meridian

Star South No. 641 RA $23^h16^m09.1^s$	δ	58°21'35" S
LST of local upper transit = RA		$23^h16^m09.1^s$
λ		9 48 43.3 E
GST of this instant		13 27 25.8
GST at UTO^h = R_O		3 12 12.6
Sidereal Time interval since UTO^h		10 15 13.2
Conversion Sidereal to Mean		-1 40.8
Mean Time interval since UTO^h equal to GMT of instant		10 13 32.4

Relationships used (see section A.74)

$$\phi = \delta_M - z_M$$

$$A = A_o - \frac{\cos \delta_M}{\sin z_M} t' + \frac{1}{6} \cos \phi \cos \delta_M \operatorname{cosec}^3 z_M \{\cos z_M \cos \delta_M + \cos \phi\} t'^3 ..$$

For calculation purposes the coefficients of t' and t'^3 are constant.

in which $A_o = 0°$ for star north and 180° for star south

ϕ -9°26'22" δ -58°21'35" z_M -48°55'13"

Determination of the hour angle for each observation and then of the azimuth

Obs CT	$20^h02^m37.5^s$	$20^h04^m04.2^s$	$20^h07^m39.8^s$	$20^h09^m30.0^s$
CC_{GMT}	-10 00 08.3	-10 00 08.3	-10 00 08.2	-10 00 08.2
GMT of Obs	10 02 29.2	10 03 55.9	10 07 31.6	10 09 21.8
GMT of Transit	10 13 32.4	10 13 32.4	10 13 32.4	10 13 32.4
MT Diff.	-11 03.2	- 9 36.5	- 6 00.8	- 4 10.6
Conversion	1.8	1.6	1.0	0.7
ST Diff=LHA star t	- 11 05.0	- 9 38.1	- 6 01.8	- 4 11.3
LHA t°	-2°46'15.0"	-2°24'31.5"	-1°30'27.0"	-1°02'49.5"
t"	-9975.0"	-8671.5"	-5427.0"	-3769.5"
First Term	-1°55'41.8"	-1°40'34.7"	-1°02'56.8"	-0°43'43.3"
Second Term	+ 6.3	+ 4.1	+ 1.0	+ 0.3
Total Correction	-1 55 35.5	-1 40 30.6	-1 02 55.8	-0 43 43.0
A_{star}	178 04 24.5	178 19 29.4	178 57 04.0	179 16 17.0
H_{star}	178 04 22	178 19 36	358 56 45	359 15 51
Orienting Corrn	+ 2.5	- 6.6	+180 00 19.0	+180 00 26.0
H_{RO}	141 28 26	141 28 25	321 28 19	321 28 18
A_{RO}	141°28'28.5"	141°28'18.4"	141°28'38.0"	141°28'44.0"

This calculation has been carried to greater accuracy than is warranted to show the excellent agreement of these results with those of the rigorous solution.

The Assessment of Precision of Circum-Meridian Time Azimuth Observations

7.46 It is assumed that n observations have been made face left and n face right on each star of a balanced pair. Furthermore, it is assumed that sights on the two stars have been made during an observing period on the same night by the same observer, using the same equipment.

The correction equations for such a situation are expressed as follows:-

$$A = A' \pm C' \sec h \pm i' \tan h + \frac{dA}{dt} \Delta\lambda + \frac{dA}{d\phi} \Delta\phi + v$$

in which
- A is the adjusted value of the azimuth to the RO
- A' is its calculated value
- C' is the theodolite horizontal collimation
- i' is the theodolite horizontal axis inclination
- h is the altitude of the star
- $\Delta\lambda$ and $\Delta\phi$ are uncertainties in the values assumed for the position of the observing station,

and v is the correction to be applied to the calculated value of the azimuth to give the adjusted value.

It should be noted that the effects C'sec h and i'tan h cannot be separated from one another because each changes sign with change of face. Also their effects on the two stars of a pair are not the same in this method of determining azimuth, because the altitude h_N to the star north is not quite the same as the altitude h_S to the star south. Allowance for this must therefore be made in the solution (see section 7.48 for comparison) and correcting terms

$$Y_S = (C' \sec h_S + i' \tan h_S) \quad \text{for the south star}$$

and

$$Y_N = (C' \sec h_N + i' \tan h_N) \quad \text{for the north star}$$

will be included as unknowns in the correction equations.

The differential coefficients $\frac{dA}{dt}$ and $\frac{dA}{d\phi}$ for circum-meridian azimuth observations are given as

$$\frac{dA}{dt} = -\frac{\cos \delta_M}{\sin z_M}$$

and

$$\frac{dA}{d\phi} = \tan h \sin A$$

This latter coefficient $\frac{dA}{d\phi}$ is a small quantity, because altitudes are kept low and observations are made close to the meridian so that its value is easily kept less than 0.05. If also the observer's latitude is reasonably well known so that $\Delta\phi$ is small, the effect of the discrepancy produced in the azimuth may be considered negligible.

The former coefficient will be used in a term in the correction equations to allow for a systematic error in the longitude or timing system as

$$X = \left| \frac{\cos \delta_M}{\sin z_M} \right| \Delta\lambda$$

Random errors of observation will be present in the pointings and horizontal circle readings to the star and reference object and also in the timing of the star across the vertical hair. One is reasonably well justified in assuming that these random errors will have similar magnitudes and distributions for both stars of the pair, considering that the altitudes of the stars are not greatly dissimilar, see Table 7.2. Thus the correction v represents the combined effects of all these errors.

The correction equations will now take the following form:

For star south

Face left $\quad v_{SL} = A + Y_S + X - A'_{SL}$

Face right $\quad v_{SR} = A - Y_S + X - A'_{SR}$

For star north

Face left $\quad v_{NL} = A + Y_N - X - A'_{NL}$

Face right $\quad v_{NR} = A - Y_N - X - A'_{NR}$

If there are n equations in each of the typical correction equation forms, these will give rise to the following Normal Equations shown in detached coefficient form.

A	Y_S	Y_N	X	Absolute Term	
4n	0	0	0	$- \Sigma A'_{SL} - \Sigma A'_{SR} - \Sigma A'_{NL} - \Sigma A'_{NR}$	= 0
	2n	0	0	$- \Sigma A'_{SL} + \Sigma A'_{SR}$	= 0
		2n	0	$- \Sigma A'_{NL} + \Sigma A'_{NR}$	= 0
			4n	$- \Sigma A'_{SL} - \Sigma A'_{SR} + \Sigma A'_{NL} + \Sigma A'_{NR}$	= 0

The solution of these equations yields the unknowns

$$A = \tfrac{1}{4}(\bar{A}_{SL} + \bar{A}_{SR} + \bar{A}_{NL} + \bar{A}_{NR})$$
$$Y_S = \tfrac{1}{2}(\bar{A}_{SL} - \bar{A}_{SR})$$
$$Y_N = \tfrac{1}{2}(\bar{A}_{NL} - \bar{A}_{NR})$$
$$X = \tfrac{1}{4}(\bar{A}_{SL} + \bar{A}_{SR} - \bar{A}_{NL} - \bar{A}_{NR})$$

in which $\bar{A}_{SL}$ etc. are the means of the individual observations of that type.

7.461 Example, making use of the data of section 7.451

$\bar{A}_{SL}$	141°28'24.3	(1)	$:\tfrac{1}{4}((1) + (2) + (3) + (4))$	$= A$	=	141°28'34.4"
$\bar{A}_{SR}$	42.3	(2)	$:\tfrac{1}{2}((1) - (2))$	$= Y_S$	=	-9.0
$\bar{A}_{NL}$	31.8	(3)	$:\tfrac{1}{2}((3) - (4))$	$= Y_N$	=	-3.7
$\bar{A}_{NR}$	39.2	(4)	$:\tfrac{1}{4}((1) + (2) - (3) - (4))$	$= X$	=	-1.1

Arc					
I	$v_{SL} = (A + Y_S + X) - A'_{SL}$ =	141°28'24.3" -	141°28'22.1"	=	+2.2"
II	$v_{SL} = (A + Y_S + X) - A'_{SL}$ =	24.3" -	26.5"	=	-2.2"
I	$v_{SR} = (A - Y_S + X) - A'_{SR}$ =	42.3" -	42.7"	=	-0.4
II	$v_{SR} = (A - Y_S + X) - A'_{SR}$ =	42.3" -	41.9"	=	+0.4
I	$v_{NL} = (A + Y_N - X) - A'_{NL}$ =	31.8" -	33.4"	=	-1.6
II	$v_{NL} = (A + Y_N - X) - A'_{NL}$ =	31.8" -	30.3"	=	+1.5
I	$v_{NR} = (A - Y_N - X) - A'_{NR}$ =	39.2" -	39.7"	=	-0.5
II	$v_{NR} = (A - Y_N - X) - A'_{NR}$ =	39.2" -	38.8"	=	+0.4
			Σvv	=	15.22

Standard Deviation of single observation $\sigma_o = \sqrt{\frac{\Sigma vv}{N-4}} = \sqrt{\frac{15.22}{8-4}} = \pm 1.9$ "

Standard Deviation $\sigma_A = \sigma_X = \sqrt{\frac{\Sigma vv}{N(N-4)}} = \sqrt{\frac{15.22}{32}} = \pm 0.7$

$$\sigma_{Y_S} = \sigma_{Y_N} = \sqrt{\frac{\Sigma vv}{N/2(N-4)}} = \sqrt{\frac{15.22}{16}} = \pm 1.0$$

Azimuth to the RO 141°28'34.4" ±0.7 "

It should be noticed here that the values v are small because the values inserted in the correction equations are each derived from four pointings to the RO and star (see section 7.21).

Circum-Elongation Time Azimuths

7.47 Example. The following observations were made on a balanced pair of circum-elongation stars for the determination of azimuth.

Station ▲ Mooifontein ϕ 26°03'14" S
λ $1^h52^m55.7^s$E
RO Red light on Kempton Park Water Tower
Date Monday 22nd June 1959
Observer O.H. Meyer
Recorder J.G. Freislich

Theodolite Wild T2
Clock Mercer Sidereal
Clock Correction with respect to GST $+7^m22.9^s$

		Azimuth Star East χ Octantis RA $18^{h}31^{m}54^{s}.3$ δ 87°38'34".9S		Azimuth Star West 7 G Octantis RA $7^{h}01^{m}16^{s}.0$ δ 86°58'34".3 S	
		Horizontal Circle Reading	Observed Clock Time	Horizontal Circle Reading	Observed Clock Time
	Mark RO	349°04'03" CL		79°08'55" CL	
	Star	357 16 00	$10^{h}28^{m}13\frac{1}{2}^{s}$	93 20 00	$10^{h}47^{m}37\frac{1}{2}^{s}$
I	Star	177 15 50 CR	10 32 34	I 273 19 57 CR	10 51 21
	Mark RO	169 04 01		259 08 50	
	Mark RO	259 09 06 CR		349 03 52 CL	
	Star	267 20 51	10 39 00	3 15 05	10 57 47½
II	Star	87 21 02 CL	10 42 24	II 183 14 57 CR	11 01 45
	Mark RO	79 09 09		169 03 51	

7.471 Solution of some of the observations by means of the general relationship

$$\tan A = \frac{-\sin t}{\cos\phi\tan\delta - \sin\phi\cos t}$$

	Star East Arc I		Star West Arc II	
CT of Obs	$10^{h}28^{m}13^{s}.5$	$10^{h}32^{m}34^{s}$	$10^{h}57^{m}47^{s}.5$	$11^{h}01^{m}45^{s}$
CC_{GST}	+7 22.9	+7 22.9	+7 22.9	+7 22.9
GST of Obs	10 35 36.4	10 39 56.9	11 05 10.4	11 09 07.9
λ	1 52 55.7E	1 52 55.7E	1 52 55.7E	1 52 55.7E
LST of Obs	12 28 32.1	12 32 52.6	12 58 06.1	13 02 03.6
RA	18 31 54.3	18 31 54.3	7 01 16.0	7 01 16.0
Hour Angle	-6 03 22.2	-5 59 01.7	5 56 50.1	6 00 47.6
A	177°22'40.2"	177°22'35.7"	183°21'57.5"	183°21'53.2"
H⊙R to star	357 16 00	177 15 50	3 15 05	183 14 57
Orienting Corrn	+180 06 40.2	+ 0 06 45.7	+180 06 52.5	+ 0 06 56.2
H⊙R to RO	349 04 03	169 04 01	349 03 52	169 03 51
Azimuth to RO	169 10 43.2	169 10 46.7	169 10 44.5	169 10 47.2

Results of calculations made from all observations:-

	Star E	Star W
Arc I	169°10'43.2"	169°10'47.1"
	46.7	49.8
Arc II	50.2	44.5
	45.0	47.2

Mean Azimuth to RO 169°10'46.7"

7.472 Reduction of the same data of section 7.471 by means of the circum-elongation series.

Relationships used for determining hour angle t_e and azimuth A_e at elongation

$\cos t_e = \dfrac{\tan \phi}{\tan \delta}$ from Four Parts Formula linking ϕ, δ, ω and t

$\sin A_e = \pm \dfrac{\cos \delta}{\cos \phi}$ from Sine Formula linking ϕ, δ, ω and A

In which it should be noted that t_e and A_e must be assigned to their correct quadrants.

Star East

$t_e = -88°50'49''.0 = -5^h55^m23^s.3$

$A_e = 2°37'25''.5$ East of South

$= 177\ 22\ 34.5$

Star West

$t_e = 88°31'12''.5 = 5^h54^m04^s.8$

$A_e = 3°21'58''.4$ West of South

$= 183\ 21\ 58.4$

To determine the clock time of elongation

	Star East	Star West
Hour Angle t_e	$-5^h55^m23^s.3$	$+ 5^h54^m04^s.8$
RA	18 31 54.3	7 01 16.0
LST of Elongation	12 36 31.0	12 55 20.8
λ	1 52 55.7 E	1 52 55.7 E
GST of Elongation	10 43 35.3	11 02 25.1
CC_{GST}	+07 22.9	+07 22.9
CT of Elongation T_e	$10^h36^m12^s.4$	$10^h55^m02^s.2$

Relationship used (see section A.81)

$$A = A_e - \sin^2\delta \tan A_e \ 1.9635'' (\Delta t^m)^2 \ \{1 - \frac{(\Delta t^m)}{229.2} \cot t_e\}\ldots$$

	Star East		Star West	
T_e	$10^h36^m12.4^s$	$10^h36^m12.4^s$	$10^h55^m02.2^s$	$10^h55^m02.2^s$
T	10 28 13.5	10 32 34	10 57 47.5	11 01 45
$\Delta T = T - T_e$	-0 07 58.9	-0 03 38.4	0 02 45.3	0 06 42.8
Δt^m	-7.9817	-3.6400	2.7550	6.7133
$1.9635''(\Delta t^m)^2$	125.09"	26.02"	14.90"	88.49"
1st term	-5.7	-1.2	+0.9	+5.2
2nd term	-0.0	-0.0	+0.0	+0.0
A_e	177°22'34".5	177°22'34".5	183°21'58".4	183°21'58".4
A	177 22 40.2	177 22 35.7	183 21 57.5	183 21 53.2
H⊙R Star	357 16 00	177 15 50	3 15 05	183 14 57
Or. Corr	+180 06 40.2	+ 0 06 45.7	+180 06 52.5	+ 0 06 56.2
H⊙R RO	349 04 03	169 04 01	349 03 52	169 03 51
Azimuth RO	169 10 43.2	169 10 46.7	169 10 44.5	169 10 47.2

It will be seen that these results are in agreement with those obtained in section 7.471.

The Assessment of Precision of Circum-Elongation Time Azimuth Observations

7.48 It is assumed that n observations have been made face left and n face right on two stars forming a well balanced pair of azimuth stars. The stars are very close to being symmetrically disposed with respect to the meridian and each observed close to and symmetrically about its point of elongation. In addition, both stars have been observed during the observing period on the same night by the same observer using the same theodolite.

The correction equations are expressed as

$$A = A' \pm C' \sec h \pm i' \tan h + \frac{dA}{dt}\Delta\lambda + \frac{dA}{d\phi}\Delta\phi + v \qquad \ldots 7.10$$

in which A is the adjusted value of the azimuth of the RO
A' is the calculated value of the azimuth of the RO
C' is the horizontal collimation correction
i' is the horizontal axis inclination
h is the mean altitude of the balanced pair of stars
$\Delta\lambda$ and $\Delta\phi$ are uncertainties in the values of the position of the observing station
and v is the correction to be applied to the computed value of the azimuth to obtain the adjusted value.

It should be noticed that the two terms containing C' and i' above cannot be separated because their effects $C' \sec h$ and $i' \tan h$ both change sign at the same time, ie. when face is changed on the theodolite.

The differential coefficients are

$$\frac{dA}{dt} = \sec h \cos \delta \cos \omega$$
$$= -\sin A . \cot \omega$$

and

$$\frac{dA}{d\phi} = \tan h \sin A$$

Under the conditions assumed above, each of the above coefficients is a small quantity. The first one is very small because each of the two components is small. Each coefficient changes sign from east to west of the meridian and also the first one changes sign on opposite sides of the point of elongation. The first one will therefore be taken as zero and the second one as $\tan h \sin A$.

The correction equations for each set of observations then become

$$v_{EL} + A'_{EL} = A - (C' \sec h + i' \tan h) + (\tan h \overline{\sin A})\Delta\phi = A - C + D\phi$$
$$v_{ER} + A'_{ER} = A + (C' \sec h + i' \tan h) + (\tan h \overline{\sin A})\Delta\phi = A + C + D\phi$$
$$v_{WL} + A'_{WL} = A - (C' \sec h + i' \tan h) - (\tan h \overline{\sin A})\Delta\phi = A - C - D\phi$$
$$v_{WR} + A'_{WR} = A + (C' \sec h + i' \tan h) - (\tan h \overline{\sin A})\Delta\phi = A + C - D\phi$$

in which each set comprises n observations, $\overline{\sin A} = \frac{1}{2}(\sin A_E - \sin A_W)$,
$C = (C' \sec h + i' \tan h)$ and $D\phi = (\tan h \overline{\sin A})\Delta\phi$.

These produce the following Normal Equations, shown in detached coefficient form:-

A	C	Dϕ	=	L
4n	0	0	=	$\Sigma A'_{EL} + \Sigma A'_{ER} + \Sigma A'_{WL} + \Sigma A'_{WR}$
	4n	0	=	$-\Sigma A'_{EL} + \Sigma A'_{ER} - \Sigma A'_{WL} + \Sigma A'_{WR}$
		4n	=	$\Sigma A'_{EL} + \Sigma A'_{ER} - \Sigma A'_{WL} - \Sigma A'_{WR}$

$$\therefore \quad A = \tfrac{1}{4}\{\overline{A}_{EL} + \overline{A}_{ER} + \overline{A}_{WL} + \overline{A}_{WR}\}$$
$$C = \tfrac{1}{4}\{-\overline{A}_{EL} + \overline{A}_{ER} - \overline{A}_{WL} + \overline{A}_{WR}\}$$
$$D\phi = \tfrac{1}{4}\{\overline{A}_{EL} + \overline{A}_{ER} - \overline{A}_{WL} - \overline{A}_{WR}\}$$

in which $\overline{A}_{EL}$, $\overline{A}_{ER}$, etc. are the mean values of each set.

If u_{ELi} is defined as

$$u_{ELi} = \bar{A}_{EL} - A'_{ELi} \qquad \text{i.e.} \qquad -A'_{ELi} = -\bar{A}_{EL} + u_{ELi}$$

it becomes possible to derive the corresponding corrections v_{ELi} from u_{ELi}.
From the correction equations of the first set

$$\begin{aligned} v_{ELi} &= A - C + D\phi - A'_{ELi} \\ &= \tfrac{1}{4}(\bar{A}_{EL} + \bar{A}_{ER} + \bar{A}_{WL} + \bar{A}_{WR} \\ &\qquad + \bar{A}_{EL} - \bar{A}_{ER} + \bar{A}_{WL} - \bar{A}_{WR} \\ &\qquad + \bar{A}_{EL} + \bar{A}_{ER} - \bar{A}_{WL} - \bar{A}_{WR} \\ &\qquad -4\bar{A}'_{EL}) + u_{ELi} \\ &= \tfrac{1}{4}(- \bar{A}_{EL} + \bar{A}_{ER} + \bar{A}_{WL} - \bar{A}_{WR}) + u_{ELi} \\ &= D + u_{ELi} \end{aligned}$$

From a similar treatment

$$\begin{aligned} v_{ERi} &= -D + u_{ERi} \\ v_{WLi} &= -D + u_{WLi} \\ v_{WRi} &= D + u_{WRi} \end{aligned}$$

Example 7.481 The full results of an azimuth determination are given below.

	Azimuth to RO	u	D	v
A'_{ELi}	336°42'42.7"	+1.5"	+0.3"	+1.8"
	47.1	-2.9		-2.6
	42.3	+1.9		+2.2
	44.9	-0.7		-0.4
$\bar{A}_{EL}$	336 42 44.2	Σ -0.2	Σ +1.2	Σ +1.0 ✓
A'_{ERi}	336 42 53.0	-3.6	+0.3	-3.9
	47.0	+2.4		+2.1
	50.2	-0.8		-1.1
	47.4	+2.0		+1.7
$\bar{A}_{ER}$	336 42 49.4	Σ +0.0	Σ +1.2	Σ -1.2 ✓
A'_{WLi}	336 42 41.6	+2.7	+0.3	+2.4
	43.2	+1.1		+0.8
	47.3	-3.0		-3.3
	44.9	-0.6		-0.9
$\bar{A}_{WL}$	336 42 44.3	Σ +0.2	Σ +1.2	Σ -1.0 ✓
A'_{WRi}	336 42 48.1	+0.3	+0.3	+0.6
	49.7	-1.3		-1.0
	51.2	-2.8		-2.5
	44.5	+3.9		+4.2
$\bar{A}_{WR}$	336 42 48.4	Σ +0.1	Σ +1.2	Σ +1.3 ✓

$$\Sigma vv = 82.07$$

$$\sigma_{SO} = \sqrt{\frac{82.07}{16-3}} = \pm 2.5''$$

$$\sigma_A = \sigma_C = \sigma_{D\phi} = \sqrt{\frac{82.07}{(16-3)16}} = \pm 0.6''$$

$\bar{A}_{EL}$	336°42'44.2"	(1) : ¼((1) + (2) + (3) + (4))	= A =	336°42'46.6"
$\bar{A}_{ER}$	49.4	(2) : ¼(-(1)+ (2) - (3) + (4))	= C =	+ 2.3"
$\bar{A}_{WL}$	44.3	(3) : ¼((1) + (2) - (3) - (4))	= Dφ =	+ 0.2"
$\bar{A}_{WR}$	48.4	(4) : ¼(-(1)+ (2) + (3) - (4))	= D =	+ 0.3"

Azimuth 336°42'46.6" ±0.6"

AZIMUTH FROM ALTAZIMUTH OBSERVATIONS

IN this method of observing, the RO is sighted and the horizontal circle reading observed, then a known star is sighted at the intersection of the crosshairs in the field of view of the telescope, the altitude bubble, if fitted, is centred and both vertical and horizontal circles read. (see section 4.23)

7.51 The observed altitude is corrected for index and refraction. The azimuth A is calculated from this reduced altitude h, the assumed value ϕ for the latitude of the station and the star's declination δ.

These quantities are linked together in the Cosine Formula

$$\sin\delta = \sin\phi \sin h + \cos\phi \cos h \cos A \qquad ...7.11$$

On differentiation, this gives the relationship linking the small changes dA, dh, dϕ, and dδ as

$$d\delta = \cos\phi \sin t\, dA + \cos t\, d\phi + \cos\omega\, dh$$

Declinations taken from reliable catalogues may, for all but work of geodetic quality, be considered to be free of error and therefore the effects of the errors dϕ and dh on the azimuth sought may be taken as

$$dA = -\sec\phi \cot t\, d\phi - \sec\phi \operatorname{cosec} t \cos\omega\, dh$$

On substitution for $\sec\phi \operatorname{cosec} t$ from the Sine Rule

$$dA = -\sec\phi \cot t\, d\phi - \sec h \cot\omega\, dh \qquad ...7.12$$

In this relationship dϕ must be considered entirely as a systematic error, because its value is an assumed one not known exactly. The error dh is partly systematic and partly random. The systematic component of dh is due to the uncertainty in the refraction corrections taken from tables. The random component results from the observer's inability to make perfect observations.

If now a single star is to be observed, it should be when $t = 90°$ or $270°$ to eliminate the dϕ component and also it should be at elongation ($\omega = 90°$ or $270°$) to eliminate dh. These conditions can only be achieved simultaneously when a star is at the pole. However, there is no star exactly at this point, but observations can be made on one or other of the two pole stars. It should be noted that if observations are made to such a star near its meridian transit ($t = 0°$ or $180°$ and $\omega = 0°$ or $180°$) the coefficients of both dϕ and dh become infinite and therefore observations should be well removed in time from meridian transit and be confined to a period near elongation, which also occurs when the hour angle is near 6^h or 18^h. In addition, observations to these stars may be difficult for stations near the equator, see section 7.32.

Table 7.3 shows the value of the rates $\frac{dA}{dh}$ and $\frac{dA}{d\phi}$ in the vicinity of elongation for the pole star.

Table 7.3

ϕ	15°		30°		45°		60°	
t	$\frac{dA}{dh}$	$\frac{dA}{d\phi}$	$\frac{dA}{dh}$	$\frac{dA}{d\phi}$	$\frac{dA}{dh}$	$\frac{dA}{d\phi}$	$\frac{dA}{dh}$	$\frac{dA}{d\phi}$
2^h	+1.8	-1.8	+2.0	-2.0	+2.4	-2.5	+3.4	-3.5
3	+1.0	-1.0	+1.2	-1.2	+1.4	-1.4	+2.0	-2.0
4	+0.6	-0.6	+0.7	-0.7	+0.8	-0.8	+1.1	-1.2
5	+0.3	-0.3	+0.3	-0.3	+0.4	-0.4	+0.5	-0.5
6	-0.0	0.0	-0.0	0.0	-0.0	0.0	-0.1	0.0
7	-0.3	+0.3	-0.3	+0.3	-0.4	+0.4	-0.6	+0.5
8	-0.6	+0.6	-0.7	+0.7	-0.8	+0.8	-1.2	+1.2
9	-1.0	+1.0	-1.2	+1.2	-1.4	+1.4	-2.0	+2.0
10	-1.8	+1.8	-2.0	+2.0	-2.5	+2.5	-3.5	+3.5

From this, it is seen that if such a star is observed only one hour from the point of elongation, there will be from one third to one half of each of the uncertainties dh and $d\phi$ affecting the derived azimuth. This should be compared with the very much more accurate values, obtainable over the *whole* range of hour angles from time azimuth observations on the pole star (see Table 7.1).

7.52 The effects of the error $d\phi$ and the systematic component in the error dh can be eliminated from the derived azimuth, if the technique of observing balanced pairs is used, because the coefficients of $d\phi$ and dh change sign on opposite sides of the meridian and their magnitudes can be made equal if the two stars are observed symmetrically about the meridian. Thus, the two following conditions can then be achieved simultaneously:-

$$\sec h_E \cot \omega_E = -\sec h_W \cot \omega_W$$

and

$$\sec \phi \cot t_E = -\sec \phi \cot t_W$$

or

$$\cot t_E = -\cot t_W$$

The next requirement is that the above coefficients of the errors dh and $d\phi$ should, if possible, be kept small, so that some imbalance between the members of a selected pair of stars can be tolerated. If observations are made about the point of elongation, the coefficient of dh is small in magnitude and it also changes sign across this point. Thus the effects of the systematic component of dh are eliminated and the effects of the random errors in the observed altitudes minimised. The systematic error $d\phi$, however, cannot be eliminated from observations made near elongation, although its effect can be considerably reduced if the stars are chosen near the elevated pole because for a close circum-polar star at elongation the hour angle is close to $\pm 90^\circ$. For these reasons, therefore, pairs of balanced stars are used. No real difficulty in predicting suitable azimuth stars in latitudes from 15° to 55° is encountered, because suitable high declination stars are given in the Star Almanac for Land Surveyors in a set of Supplementary Stars as well as a set of Circum-Polar Stars. Most of these stars are fairly faint and precomputation will be necessary, as well as a means of orienting the horizontal circle of the theodolite, in order to locate them. In equatorial latitudes, these circum-polar stars cannot be used because of the problem of visibility. But the technique of observing stars near elongation can still be used, although suitable stars will not be at the desired hour angle of 90° or 270° when they are at a suitable altitude for observation. An example of a pair of circum-elongation stars observed by the Altazimuth method is given in section 7.62.

Calculation of the Azimuth from Altazimuth Observations

7.61 The direct calculation of the azimuth from these observations can always be used and each observation can conveniently be computed from the relationship of Equation 7.11, set out as

$$\cos A = \sec \phi \sin \delta \sec h - \tan \phi \tan h \qquad \ldots 7.13$$

in which it is noted that $\sec \phi \sin \delta$ and $\tan \phi$ are constants for any specific example. This is a general relationship, but further information is required to resolve the double answer obtained from $\cos A$. Whether the star was east or west of the meridian is known from the field book or the prediction and the ambiguity resolved.

The provision of a check computation is not easy for the above relationship, if more than simply a duplication of the computation is considered necessary. The Tangent Half Angle Formulae provide a means of checking. They are however clumsy to use for the individual observations. If calculations from the arithmetic means are carried out in this way, the allowance for second order correction must be made (see section A.62).

For observations on circum-elongation stars, the following relationship, from section A.82 in the appendix, may be used:-

$$A = A_e - \frac{C}{2\rho} (\Delta h)^2 - \frac{C \tan h_e}{2\rho^2} (\Delta h)^3 \ldots \qquad \ldots 7.14$$

in which A_e is the azimuth of the star at elongation, $C = \sec^2 h_e \cot A_e$ and $\Delta h = h - h_e$ where h_e is the altitude at elongation. The units of Δh and ρ are in accordance with one another.

7.62 Example of an altazimuth circum-elongation determination for azimuth.

Place Peg G, Survey Camp ϕ −33°27'27"
Bathurst, NSW Approx. λ $9^h 58^m$ E
Date Thursday 17th November 1977
Observer G.G. Bennett
Recorder J.C. Trinder

Theodolite Wild T2
RO Navigation Light
Vert. Circle Index Corrn. +39"

Met. Readings

	T	P
Star East	14°C	929.7mb
Star West	12.5	929.5

Abstract from Field Book
Star East No. 672

Arc I

	CL		CR	
	Hor.Circle Rdg.	Vert.Circle Rdg	Hor.Circle Rdg.	Vert.Circle Rdg.
RO	0°00'34"		180°00'39"	
Star	119 38 05	55°42'47"	299 37 13	304°31'52"
Star	119 37 56	55 39 46	299 37 06	304 35 23
RO	0 00 29		180 00 35	

Arc II

	CR		CL	
RO	270 05 03		90 04 47	
Star	29 41 17	304 50 22	209 41 37	54 50 05
Star	29 41 16	304 53 47	209 41 41	54 46 21
RO	270 05 01		90 04 50	

Star West No. 684

	Arc I CL		Arc I CR	
	Hor.Circle Rdg.	Vert.Circle Rdg.	Hor.Circle Rdg.	Vert.Circle Rdg.
RO	0°00'52"		180°01'00"	
Star	153 29 25	54°51'14"	333 30 01	304°50'40"
Star	153 29 48	54 56 54	333 30 09	304 47 52
RO	0 00 48		180 01 01	

	Arc II CR		Arc II CL	
RO	270 05 16		90 05 04	
Star	63 34 40	304 33 27	243 34 53	55 44 33
Star	63 34 42	304 28 53	243 34 46	55 47 08
RO	270 05 14		90 05 03	

7.621 Solution by the general relationship

$$\cos A = \frac{\sin\delta - \sin\phi \sin h}{\cos\phi \cos h}$$

Star East No. 672 δ 74°44'43" S

Arc I

Vert.Circle Rdg	55°42'47"	55°39'46"	304°31'52"	304°35'23"
Index	+39	+39	+39	+39
Corrected Reading	55 43 26	55 40 25	304 32 31	304 36 02
Observed Altitude h_o	34 16 34	34 19 35	34 32 31	34 36 02
Refraction	1 17	1 17	1 16	1 16
Altitude h	34 15 17	34 18 18	34 31 15	34 34 46
Calcd azimuth A_{star}	161 37 50.5	161 37 42.6	161 37 16.2	161 37 11.3
H_{star}	119 38 05	119 37 56	299 37 13	299 37 06
Orienting Corrn	41 59 45.5	41 59 46.6	222 00 03.2	222 00 05.3
Diffs from Mean	+ 0.5	- 0.6	+ 1.0	- 1.0
Mean $\overline{OC}$	41°59'46.0"		222°00'04.3"	
H_{RO}	0 00 34	0 00 29	180 00 39	180 00 35
Diffs from Mean	- 2.5	+ 2.5	- 2.0	+ 2.0
Mean $\overline{H}_{RO}$	0 00 31.5		180 00 37.0	
A_{RO}	42 00 17.5		42 00 41.3	
	Arc I Mean	42°00'29.4"		

The results from similar calculations for Arc II are,

Arc II	CL	A_{RO}	42°00'24.3"
	CR	A_{RO}	42 00 46.5
Arc II	Mean		42 00 35.4
Mean A_{RO}	East Star		42 00 32.4

Star West No. 684 δ 77°06'46" S

Arc I

Vert.Circle Reading	54°51'14"	54°56'54"	304°50'40"	304°47'52"
Index	+39	+39	+39	+39
Corrected Reading	54 51 53	54 57 33	304 51 19	304 48 31
Observed Altitude h_o	35 08 07	35 02 27	34 51 19	34 48 31
Refraction	1 15	1 16	1 16	1 16
Altitude h	35 06 52	35 01 11	34 50 03	34 47 15
Calcd Azimuth A_{star}	195 29 03.7	195 29 23.7	195 29 54.0	195 29 59.8
H_{star}	153 29 25	153 29 48	333 30 01	333 30 09
Orienting Corrn	+41 59 38.7	+41 59 35.7	+221 59 53.0	+221 59 50.8
Diffs from Mean	- 1.5	+ 1.5	- 1.1	+ 1.1
Mean $\overline{OC}$	41°59'37.2"		221°59'51.9"	
H_{RO}	0 00 52	0 00 48	180 01 00	180 01 01
Diffs from Mean	- 2.0	+ 2.0	+ 0.5	- 0.5
Mean $\overline{H}_{RO}$	0 00 50.0		180 01 00.5	
A_{RO}	42 00 27.2		42 00 52.4	
	Arc I Mean	42°00'39.8"		

The results from similar calculations for Arc II are,

Arc II	CL	A_{RO}	42°00'24.3"
	CR	A_{RO}	42 00 52.7
	Arc II Mean		42 00 38.5
Mean	A_{RO}	West Star	42 00 39.2
Mean	A_{RO}	East Star	42 00 32.4
Mean	A_{RO}	from pair	42 00 35.8

7.622 Solution by means of a series

Relationships used

$$\sin A_e = \pm \frac{\cos \delta}{\cos \phi}$$ from the Sine Formula linking ϕ δ ω and A

$$\sin h_e = \frac{\sin \phi}{\sin \delta}$$ from the Cosine Formula linking ϕ δ ω and h

From section A.82 in the appendix

$$A = A_e - \sec^2 h_e \cot A_e \frac{(h-h_e)^2}{2\rho} - \sec^2 h_e \cot A_e \tan h_e \frac{(h-h_e)^3}{2\rho^2} \ldots$$

$$= A_e - C \left[\frac{(h-h_e)^2}{2\rho} + D \frac{(h-h_e)^3}{2\rho^2} \right] \ldots$$

in which $C = \sec^2 h_e \cot A_e$ and $D = \tan h_e$.

ϕ = -33°27'27"

Star East No. 672 δ = -74°44'43"

A_e = 18°22'59.1" East of South h_e 34°51'06"

A_e =161 37 00.9 Refraction 1 16

constants C = -4.468 215 h_{eo} 34 52 22

D = 0.696 355 in which h_{eo} is the altitude at which the star at elongation would be seen.

Arc I

h_o	34°16'34"	34°19'35"	34°32'31"	34°36'02"
h_{eo}	34 52 22	34 52 22	34 52 22	34 52 22
$h - h_e$	-0 35 48	-0 32 47	-0 19 51	-0 16 20
$h - h_e$	-2148"	-1967"	-1191"	-980"
First term	-49.97	-41.91	-15.36	-10.40
Second term	+ 0.36	+ 0.28	+ 0.06	+ 0.03
Correction	+49.61	+41.63	+15.30	+10.37
A_{star}	161°37'50.5"	161°37'42.5"	161°37'16.2"	161°37'11.3"
H_{star}	119 38 05	119 37 56	299 37 13	299 37 06
Orienting Corrn	+41 59 45.5	+41 59 46.5	+222 00 03.2	+222 00 05.3
Diffs from Mean	+ 0.5	- 0.5	+ 1.1	- 1.0
Mean $\overline{OC}$	+41°59'46.0"		+222°00'04.3"	
H_{RO}	0 00 34	0 00 29	180 00 39	180 00 35
Diffs from Mean	- 2.5	+ 2.5	- 2.0	+ 2.0
Mean $\overline{H}_{RO}$	0 00 31.5		180 00 37.0	
A_{RO}	42 00 17.5		42 00 41.3	
	Arc I Mean	42°00'29.4"		

The results from similar calculations for Arc II by this method agree exactly with those of section 7.621.

Star West No. 684 δ = -77°06'46"

A_e 15°30'19.8" West of South

A_e 195 30 19.8

C 5.299 724

D 0.685 778

h_e	34°26'29"
Refraction	1 16
h_{eo}	34 27 45

Arc I

h_o	35°08'07"	35°02'27"	34°51'19"	34°48'31"
h_{eo}	34 27 45	34 27 45	34 27 45	34 27 45
$h - h_e$	+0 40 22	+0 34 42	+0 23 34	+0 20 46
$h - h_e$	+2422"	+2082"	+1414"	+1246"
First term	+75.36	+55.69	+25.69	+19.94
Second term	+ 0.61	+ 0.39	+ 0.12	+ 0.08
Correction	- 1 15.97	-56.08	-25.81	-20.02
A_{star}	195 29 03.8	195 29 23.7	195 29 54.0	195 29 59.8
H_{star}	153 29 25	153 29 48	333 30 01	333 30 09
Orienting Corrn	+41 59 38.8	+41 59 35.7	+221 59 53.0	+221 59 50.8
Diffs from Mean	- 1.5	+ 1.6	- 1.1	+ 1.1
Mean $\overline{OC}$	+41°59'37.3"		+221°59'51.9"	
H_{RO}	0 00 52	0 00 48	180 01 00	180 01 01
Diffs from Mean	- 2.0	+ 2.0	+ 0.5	- 0.5
Mean $\overline{H}_{RO}$	0 00 50.0		180 01 00.5	
A_{RO}	42 00 27.3		42 00 52.4	
	Arc I Mean	42°00'39.8"		

The results from similar calculations for Arc II by this method agree exactly with those of section 7.621.

The Assessment of Precision of Circum-elongation Altazimuth Observations

7.63 The situation and the conditions of section 7.48 are the same here.

The correction equations are expressed as

$$A = A' \pm (C'\sec h + i'\tan h) + \frac{dA}{dh}\Delta C + \frac{dA}{dh}\Delta r + \frac{dA}{d\phi}\Delta\phi + v$$

in which
- A is the adjusted value of the azimuth of the RO
- A' is the calculated value of the azimuth of the RO
- C' is the horizontal collimation correction
- i' is the horizontal axis inclination
- h is the mean altitude at elongation of the balanced pair of stars
- ΔC is the vertical circle index correction
- Δr is the uncertainty in the refraction value assumed
- $\Delta\phi$ is the uncertainty in the value of latitude adopted for the observing station

and
- v is the correction to be applied to the computed value of the azimuth to give the adjusted value.

The effects of C' and i' cannot be separated from one another because their separate effects both change at the same time, i.e. when face is changed on the theodolite. The differential coefficients are

$$\frac{dA}{dh} = -\sec h \cot \omega$$

and

$$\frac{dA}{d\phi} = -\sec h \operatorname{cosec} \omega \cos t$$

The effect of the uncertainty Δr in refraction may be considered negligible because both Δr and the coefficient $\frac{dA}{dh}$ at elongation are small quantities.

Provided an index correction is applied so that ΔC can be considered small, its effect can likewise be considered negligible. The uncertainty $\Delta\phi$ will be treated as being an unknown.

The correction equations then become

$$A = A' \pm (C'\sec h + i'\tan h) \pm \left|\overline{\frac{dA}{d\phi}}\right| d\phi + v$$

$$= A' \pm Y \pm X + v$$

in which $Y = C'\sec h + i'\tan h$

and $X = \left|\overline{\frac{dA}{d\phi}}\right| \Delta\phi$

with $\overline{\frac{dA}{d\phi}}$ being the mean of the values at elongation for the two stars.

The effect Y changes sign with change of face whereas the effect X changes sign between the two stars.

The individual correction equations then become

$$v_{WL} = A + Y + X - A'_{WL}$$
$$v_{WR} = A - Y + X - A'_{WR}$$
$$v_{EL} = A + Y - X - A'_{EL}$$
$$v_{ER} = A - Y - X - A'_{ER}$$

with n equations in each set.

This gives the following Normal Equations, shown in the detached coefficient form as

A	Y	X	L					
4n	0	0	$-\Sigma A'_{WL}$	$-\Sigma A'_{WR}$	$-\Sigma A'_{EL}$	$-\Sigma A'_{ER}$	=	0
	4n	0	$-\Sigma A'_{WL}$	$+\Sigma A'_{WR}$	$-\Sigma A'_{EL}$	$+\Sigma A'_{ER}$	=	0
		4n	$-\Sigma A'_{WL}$	$-\Sigma A'_{WR}$	$+\Sigma A'_{EL}$	$+\Sigma A'_{ER}$	=	0

$$\therefore \quad A = \tfrac{1}{4}\{\bar{A}_{WL} + \bar{A}_{WR} + \bar{A}_{EL} + \bar{A}_{ER}\}$$
$$Y = \tfrac{1}{4}\{\bar{A}_{WL} - \bar{A}_{WR} + \bar{A}_{EL} - \bar{A}_{ER}\}$$
$$X = \tfrac{1}{4}\{\bar{A}_{WL} + \bar{A}_{WR} - \bar{A}_{EL} - \bar{A}_{ER}\}$$

in which the $\bar{A}_{WL}$ etc values are the means of the corresponding A' values. The corrections v may now be determined by a process of back substitution in the correction equations.

7.631 The values computed in the example of 7.621 will be used to determine the precision of the result.

Star No.	Aspect	Arc		CL		CR
				A'_{WL}		A'_{WR}
684	W	I		42°00'27.2"		42°00'52.4"
		II		24.3		52.7
			$\bar{A}_{WL}$	42 00 25.7	$\bar{A}_{WR}$	42 00 52.5
				A'_{EL}		A'_{ER}
672	E	I		42 00 17.5		42 00 41.3
		II		24.3		46.5
			$\bar{A}_{EL}$	42 00 20.9	$\bar{A}_{ER}$	42 00 43.9

Solution

$\bar{A}_{WL}$	42°00'25.7"	(1)	:	¼((1) + (2) + (3) + (4))	=	A	=	42°00'35.8"
$\bar{A}_{WR}$	52.5	(2)	:	¼((1) − (2) + (3) − (4))	=	Y	=	−12.4
$\bar{A}_{EL}$	20.9	(3)	:	¼((1) + (2) − (3) − (4))	=	X	=	+ 3.3
$\bar{A}_{ER}$	43.9	(4)						

Arc								
I	v_{WL}	=	$(A + Y + X) - A'_{WL}$	=	42°00 26.7" −	42°00'27.2"	=	−0.5"
II	v_{WL}	=	$(A + Y + X) - A'_{WL}$	=	26.7" −	24.3	=	+2.4
I	v_{WR}	=	$(A - Y + X) - A'_{WR}$	=	51.5" −	52.4"	=	−0.9
II	v_{WR}	=	$(A - Y + X) - A'_{WR}$	=	51.5" −	52.7"	=	−1.2
I	v_{EL}	=	$(A + Y - X) - A'_{EL}$	=	20.1" −	17.5"	=	+2.6
II	v_{EL}	=	$(A + Y - X) - A'_{EI}$	=	20.1" −	24.3"	=	−4.2
I	v_{ER}	=	$(A - Y - X) - A'_{ER}$	=	44.9" −	41.3"	=	+3.6
II	v_{ER}	=	$(A - Y - X) - A'_{ER}$	=	44.9" −	46.5"	=	−1.6
						Σvv	=	48.18

Standard Deviation of a single observation $\sigma_{SO} = \sqrt{\frac{\Sigma vv}{N-3}} = \sqrt{\frac{48.18}{8-3}} = \pm 3.1"$

Standard Deviations $\sigma_A = \sigma_Y = \sigma_X = \sqrt{\frac{\Sigma vv}{N(N-3)}} = \sqrt{\frac{48.18}{40}} = \pm 1.1$

Azimuth of the RO 42°00'35.8" ±1.1"

COMPARISON OF THE TIME AZIMUTH AND THE ALTAZIMUTH METHODS

7.71 THE superiority of the Time Azimuth method over the Altazimuth method has been referred to in Chapter 8 in which it is seen that the Altazimuth method requires simultaneous pointings to be made on widely separated limbs of the sun, unless a special theodolite attachment is used, whereas in the Time Azimuth Method only a single pointing is needed. The difficulty in making such a double pointing can be easily verified from a casual observation. With star observations this difficulty does not arise, although it will be found that with the timing method the observer can concentrate his attention on making an accurate pointing with the vertical hair without the distraction of perfecting the altitude pointing as well. For work of geodetic quality, it is generally conceded that the Time Azimuth Method is superior.

It may be considered that errors of refraction in altazimuth observations produce a greater discrepancy in the derived azimuth than do errors of time in time azimuth observations. However, if stars are observed in positions where their rates of change of azimuth with respect to either time or altitude are small i.e. at elongation, either method should yield a result of comparable accuracy, because a timing error or an altitude error has only a very small effect on the azimuth derived from a circum-elongation sight. Even if quite large refraction changes occurred over a relatively long observing period, hardly any azimuth error would result. Therefore, it is unnecessary to observe the individual stars of a circum-elongation pair quickly one after the other. The reader may well find it instructive to investigate the consequences of dispensing with both thermometer and barometer in the Altazimuth Method.

If prediction is used to ensure that careful balancing is achieved for the star pairs and if observations are well balanced about the point of elongation, it appears that altazimuth observations have several advantages, because the observer can concentrate on the procedures of pointing and circle reading; procedures which are very familiar to a surveyor. He has all the observing under his own control and his recorder is responsible only for noting the observations, as they are read out to him. No timing equipment or radio is required but a thermometer and barometer is necessary for accurate determinations.

8
Sun Observations

INTRODUCTION

THERE is a surprisingly large number of celestial bodies available for observation with a theodolite during the hours of daylight. Such bodies are the sun, moon, the nearer planets Venus, Mars, Jupiter and Saturn and some bright stars, 20 of which are given a signifying letter 'd' in the Star Almanac for Land Surveyors. The planets and stars referred to will not normally be visible to the unaided eye and, to locate them with a theodolite, one must calculate their altitudes and azimuths and have some prior knowledge of azimuth of a line for the orientation of the horizontal circle. The moon, although bright and easy to identify, has a large horizontal parallax (about 1° compared with 9" for the sun), is often below the horizon during the day and will seldom present a completely illuminated disc to the observer in the daytime. It is because of these drawbacks that the moon is seldom, if ever, used by the surveyor. Thus surveyors have concentrated their attention on the sun, a body which is conveniently available for observation in normal working hours and whose singularity and brightness cause no confusion with other celestial bodies.

8.11 Star observations at night for latitude, longitude and azimuth are relatively easy to predict, make and compute and the effects of systematic and random errors may be minimised by a suitable selection of stars from the catalogue. This flexibility of choice does not exist for the sun, whose restricted celestial path often prevents observations from being made in an optimum position (e.g. elongation, etc.). The attainment of a balanced pair of observations on the sun for azimuth and longitude may require a long time interval between observations and, for a determination of latitude, one must be content with an unbalanced observation. It is mainly because of the aforegoing reasons and other practical considerations to be examined later, that the results of sun observations are less accurate than those obtained from stellar observations. Nevertheless, with care, good results, which will suffice for many surveying purposes, can be obtained.

The only additional special piece of equipment required for sun observations is a dark glass, which is fitted over the eyepiece of the telescope. Without such an attachment, direct observation to the sun is impossible, except when a veil of cloud substantially reduces the intensity of the image. On no account should even a short glimpse of the sun be attempted in clear weather, otherwise permanent serious damage to the eye may result. If a dark glass is not available, the sun can be located and the telescope roughly aligned by silhouetting the telescope sight on a page of the field book. Without varying the main telescope focus, which must have been previously set to infinity, sharp images of the sun and the crosshairs can be obtained on this page by suitable adjustment of the eyepiece focussing.

SPECIAL CONSIDERATIONS

8.21 IT is impossible to point accurately to the centre of the sun, whose disc subtends an angle somewhat over half a degree to an observer. To overcome this difficulty, pointings are made in an eccentric manner by noting the instant or perfecting a pointing when the cross hairs are tangential to edges (limbs) of the disc. It is also customary to take a mean of observations made, in quick succession, to opposite limbs of the sun, and if, in addition, these eccentric observations are made on opposite theodolite faces the mean will be almost free of the effects of eccentric pointings and theodolite misadjustments. If observations are not made on opposite limbs or if pointings are calculated individually, then it will be necessary to apply corrections for eccentricity.

Eccentric Pointings

8.22 To effect the correction for a pointing made to the upper or the lower limb of the sun, the value of the semi-diameter (SD) obtained from the bi-monthly tabulation in the Star Almanac for Land Surveyors is added to or subtracted from the observed zenith distance (or altitude), which has been previously corrected for vertical circle index error and refraction. If the SD correction is applied first and the *resulting* zenith distance used as the argument for calculating the refraction correction, then an incorrect value of the corrected zenith distance results.

To assist in applying the SD correction with the correct sign, it is usual to note in the field book the relative position of the cross hairs and the sun *as seen* in the telescope at the time of observation. In this way, one can allow for the effect of the inversion of the image (if applicable) at the later stage of calculation.

Unlike the previous correction (Δh) to the altitude, the correction (ΔA) to the horizontal circle reading resulting from a pointing to a lateral limb will vary depending on the altitude of observation. This, strictly speaking, should be the observed altitude corrected for index only. The situation is shown in Fig. 8.1

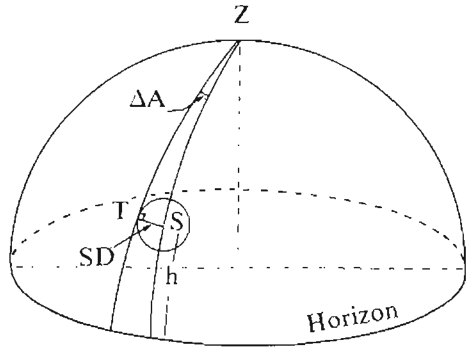


Fig.8.1

An expression for ΔA may be derived by applying the Sine Formula to the right angled triangle ZST, from which one obtains

$$\sin \Delta A = \frac{\sin SD}{\cos h}$$

However the SD is small and the resulting ΔA is also small provided h is not large and therefore a good approximation to ΔA is given by

$$\Delta A = \frac{SD}{\cos h}$$

This expression is convenient to use when altitudes have been measured such as in the determination of azimuth by the altitude method. However for the hour angle method of azimuth determination, the altitude is neither observed nor required to be calculated and a more convenient expression can be obtained by a simple substitution from the sine rule

$$\cos h = \frac{-\cos\delta \sin t}{\sin A}$$

giving

$$\Delta A = \frac{-SD \sin A}{\cos\delta \sin t}$$

Both this and the previous expression for ΔA are correct to 0".5 when compared with the rigorous expressions for ΔA up to an altitude of 80°.

8.23 If sun observations are a regular feature of a surveyor's work, then consideration should be given to equipping the theodolite with a special attachment, which will improve the pointing accuracy to the sun and obviate the need for applying semi-diameter corrections. Such a device has been designed by Professor R. Roelofs and consists of overlapping thin prisms mounted in an attachment, which fits over the objective of the theodolite. Four images of the sun are seen in the field of view, which overlap and provide a bright central "cross" on which one may point with great accuracy. A sun filter is incorporated within the attachment, which has the advantage of reducing the heat falling on the reticule. The attachment is hinged on one side to allow the theodolite to be sighted to the R.O.

Parallax

8.24 In section 4.54 the vertical displacement of a celestial body on the celestial sphere due to the body not being infinitely distant from the earth has been shown to be

$$\pi = \pi_h \sin z_o$$

Because of the elliptical nature of the earth's orbit around the sun, π_h varies between about 8.6" and 9.0" in the course of a year; a variation which can, for all practical purposes, be neglected and π_h taken as 8.8" and therefore

$$\pi'' = 8.8 \sin z_o$$

It should be noted that this parallax correction is only to be applied to vertical measurement and that no horizontal displacement of the celestial body occurs. The sign of this correction is opposite to that of refraction and it is immaterial whether one uses the observed zenith distance or the value found after applying refraction and semi-diameter corrections in the formula for π.

Declination and E

8.25 The right ascensions and declinations of stars vary so slowly throughout the course of a year that tabulation at monthly intervals is quite sufficient for obtaining intermediate values to an accuracy of about 0.1^s and 1" respectively, by means of relatively coarse interpolation. Unlike their stellar counterparts, E and the declination of the sun show considerable variation, which requires them to be tabulated at much closer intervals to permit accurate interpolation. The maximum possible change in E and declination per hour is about $1\frac{1}{2}^s$ and 1' respectively and therefore a rough knowledge of the zone time of observation (the nearest minute will be quite sufficient) is necessary, even though a timing observation method is not being employed. Values of E and declination to the nearest 0.1^s and 0.1' are to be found in the Star Almanac for Land Surveyors at 0^h, 6^h, 12^h and 18^h UT each day.

Changes in the sun's co-ordinates introduce a slight complication in the calculation process, because, if observations extend over a long period of time, one should allow for this by introducing a series of values for E and declination. If this is not done and one value of E and declination is taken for the mean epoch of the observational period then, although the effects of these co-ordinate changes are substantially reduced by taking the mean of all the individual results of the calculations, the individual results may show variations, which are not entirely due to observational errors.

8.26 From 1977 onwards, the Star Almanac for Land Surveyors will include monthly sets of polynomial coefficients as an alternative to the main tabulation of R, E and the sun's declination. This is one of the first steps towards the eventual publication of ephemeral data in a form, which can be stored in an electronic calculator, and the required data at any instant of time can be evaluated without interpolation tables.

Practical Considerations

8.27 One of the chief difficulties to overcome with sun observations is the effect of the exposure of the theodolite to the sun's rays. Thermal gradients are set up in the instrument and these can be noticed at once from the erratic behaviour of exposed bubbles. In modern theodolites, the alidade bubble is usually enclosed within one of the theodolite standards, or, in the latest models, a gravity dependent compensating device automatically corrects the vertical circle reading for dislevelment. For these instruments, the effect of thermal gradients is diminished.

8.28 For azimuth determinations, it is essential that the vertical rotational axis be vertical at the time of observation or, if this is not so, then the component of the dislevelment of this axis at right angles to the direction of pointing should be determined from readings of the plate bubble and the horizontal circle reading corrected. It will be found that plate bubble readings are completely unreliable unless the instrument is shaded and the bubble allowed to assume a stationary position before it is read. These latter remarks also apply to instruments fitted with alidade bubbles, which are exposed. It is recommended that, for azimuth work, the instrument be levelled between arcs throughout the observation series. This may be done by means of the alidade bubble or with reference to vertical circle readings if the instrument is fitted with a compensator (see section 4.12). During this levelling process the instrument should be fully shaded.

8.29 A suggestion for the observer, who wants to improve the accuracy of pointing to the sun's limb, is that he should observe only that limb, which is about to leave the cross hair. Otherwise it will be found that when the other limb is observed the cross hair is usually invisible and the observer must of necessity either anticipate the tangency of the disc and the hair or observe slightly late. For azimuth determinations, if the observer always selects the limb, which is about to leave the vertical hair, then provided he is observing outside the tropics, he will be automatically observing the left hand limb in the northern hemisphere and the right hand limb in the southern hemisphere and thus there will be no doubt about the sign of the semi-diameter correction. If he intends to observe only on one limb, he should include one "dummy" pointing to the other limb. This will enable him to verify that his semi-diameter correction has been applied with the correct sign.

SUN OBSERVATIONS

Latitude

8.31 AS has been noted before, it will not be possible to make a balanced observation for the determination of latitude by observing to the north and to the south near the local meridian at approximately the same altitude. In addition, in some latitudes and at certain times of the year, the sun will be too low or inconveniently high for this observation. In the latter case, the difficulty may be overcome by the use of an eyepiece attachment, which will allow observations to be made up to the zenith, although, when using such an attachment there is a considerable decrease in magnification, which will render the pointings less precise.

8.32 Before attempting this observation, it will be convenient to pre-calculate the standard time when the sun transits the local meridian so that observations may be made over this optimum time, which is termed Local Apparent Noon (LAN).

At LAN $\quad LAT = 12^h$

and as $\quad E = 12^h + LAT - LMT$

$$LMT = 24^h - E$$

the standard time of LAN will therefore be

$$24^h - E - \lambda + \text{Time Zone}$$

When reducing observations by the circum-meridian reduction formula (see section 8.714) it will be found convenient to evaluate the standard time of LAN for the calculation of the individual hour angles. An example of this calculation is given in sections 3.432 and 8.713.

Longitude

8.41 Ideally this observation should be made when the celestial body is on the prime vertical, but, for the sun, this will only occur during half of the year and even then, for part of this time, the sun will be too low to allow accurate observation. Once again the observer may have to be content with observations taken in less than ideal circumstances and have to choose between either taking an observation at a low altitude near the prime vertical or one that is sufficiently high yet somewhat removed from the prime vertical. In either case, it is advisable to take balanced morning and afternoon observations in order to minimise the effects of some systematic errors, notably in the assumed value of latitude used in the calculation and in the observed altitudes. The appropriate differential coefficients to consider are

$$\frac{dt}{d\phi} = -\frac{1}{\cos\phi \tan A} \quad \text{and} \quad \frac{dt}{dh} = \frac{1}{\cos\phi \sin A}$$

from which one notes that for a balanced pair of observations, $A_E + A_W = 360°$ and the magnitudes of each coefficient remain unchanged but the sign reverses. Thus the mean of morning and afternoon observations will be substantially free of the effects of systematic errors, excepting those errors, which arise from anomalous refraction.

8.42 After the Astronomical Triangle has been solved for the hour angle t from the elements ϕ, δ and h, one may evaluate

$$LAT = 12^h + t$$

and since $\quad LAT - LMT = E - 12^h$

$$LMT = t - E$$

Since longitude is defined as

$$\lambda = LMT - GMT$$

$$\lambda = t - E - (CT + CC_{ZT} - Z)$$

in which $\quad CT$ = Watch Time of Observation

CC_{ZT} = Watch Correction on Zone Time

Z = Time Zone

Azimuth

8.51 By far the most important of all sun observations, applied to surveying and mapping, is that of the determination of azimuth. Such an observation is not only convenient for approximate orientation and as a preliminary for more exact methods of azimuth determination but, with care, one can obtain the azimuth of a terrestrial line to within 20", which will be found to be extremely useful for the orientation of isolated surveys or checking of long traverses, which are remote from previously established control stations.

The surveyor will have a choice of two methods of observation

(a) azimuth from altitudes and
(b) azimuth by hour angles

For neither method will it be possible to observe the sun at or near its ideal position, elongation, except if the station is situated within the tropics and even in these situations the sun may be too low or too high to obtain good results. As a general rule, the best results are obtained from early morning and late afternoon observations. For the Altazimuth Method, observations should be made when the sun has attained an altitude of at least 15°. For the Time Azimuth Method this restriction does not apply.

The Altazimuth Method

8.52 The equipment required, in addition to a theodolite and dark glass, is a barometer, thermometer and a watch whose correction (to the nearest minute) to standard time is known. A barometer may be dispensed with if the height of the station above mean sea level is known. The Star Almanac for Land Surveyors gives correcting factors for values of mean refraction for variations from a standard temperature and pressure with arguments for pressure expressed in either millibars or height (metres). Two errors may be present in the estimated refraction when the station height is used as argument:

(i) an error in the estimate of the station height, and
(ii) a local variation in pressure, from that which corresponds to the station height, i.e. a deviation from the pressure height relationships given by the standard atmosphere on which the refraction tables are based.

An error of 100 m in station height and a variation of 10 mb in local pressure, a value which should seldom be exceeded, except under abnormal meteorological conditions, will each contribute to about a 1% variation in the value of the estimated refraction. Errors of this magnitude may be safely neglected with sun observations.

The Time Azimuth Method

8.53 The equipment required, in addition to a theodolite and dark glass, is a watch capable of being read to preferably better than 1^s and a radio receiver for obtaining the watch correction to standard time. Continuous time signal transmissions on short wave may be picked up on a small transistorised radio in most parts of the world. Of lesser convenience, are the hourly 6 'pip' broadcasts from medium wave stations. Apart from the inconvenience of having long gaps between the signals, medium wave transmissions can only be received over a limited distance from the transmitter.

Choice of Method

8.54 The main criterion to be used for a comparison between these observation methods is a study of the propagation of the systematic and random errors affecting the observations and their reduction. It is a well established fact that, unless a special attachment is used, such as the Roelofs' solar prism, the simultaneous pointing to the horizontal and vertical limbs, required by the Altazimuth Method, is very inaccurate. On the other hand for the Time Azimuth Method, the observer can give his undivided attention to pointing on the one limb. In addition, it is not generally known that vertical angles, measured with some types of theodolite in common use, may suffer from serious errors due to the eccentricity of the vertical circle. The instruments referred to are those, whose readings are not obtained from diametrically opposite parts of the vertical circle. Horizontal angles, however, measured with these instruments remain free of this type of error, provided that the mean of face left and face right observations is taken, because in effect the change of face allows diametrically opposite readings to be taken. Other than calibrating the instrument and correcting the observed altitudes, the only way to eliminate this error will be to take the mean of balanced morning and afternoon observations, made with the same instrument.

8.55 The effects of small systematic errors in the assumed and observed quantities to be used in the calculations, for both methods of azimuth observation, can be conveniently determined from an examination of the behaviour of the appropriate first order differential coefficients. The variation in the values of these differential coefficients over a range of latitudes, altitudes and declinations is quite complicated. Such an investigation can be found in *The Australian Surveyor*, March 1974 Vol. 26 No. 1. In this the latitudes and the altitudes were not greater than 45°. If one examines these variations for both methods it will be found that

(i) In equatorial latitudes, at all altitudes and at all times of the year, both methods show that the systematic errors are well controlled i.e. the coefficients are not big.

(ii) As the latitude increases the propagation of the systematic errors for the Altazimuth Method increase considerably unless either the altitude is kept low or the observations are confined to the summer months (middle of the year, northern hemisphere; end and beginning of the year, southern hemisphere).

(iii) For the Time Azimuth Method the coefficients are never large regardless of latitude, season or altitude.

(iv) In nearly all cases the coefficients of the systematic errors for the Altazimuth Method are larger than the corresponding values for the Time Azimuth Method. This indicates the superiority of the latter method.

(v) The Time Azimuth Method allows the surveyor to take observations over a greater time range than the Altazimuth Method; observations using the Time Azimuth Method, even at noon, should be quite satisfactory thus permitting observations to be made throughout the daylight hours in other than equatorial altitudes throughout the winter months. However, when observations for azimuth are made at high latitudes and altitudes, extra care should be exercised in the control of systematic and random errors.

Examples of Sun Observations

In the sun observation examples, which follow, no other statistics, beside the Arithmetic Means, have been evaluated, because the sample sizes are small and the estimates of the variances are therefore not reliable.

8.61 Example of sun observations for the determination of longitude and of azimuth by the Time Azimuth or Hour Angle Method

Place: N Pillar, UNB Eng. Building
Date: Thursday afternoon 11th September 1969 (ϕ 45°57'10" N)

Theodolite: M.O.M.(10") Zero of the vertical circle at the nadir.
Watch: Omega with sweep second hand (Mean Time)
Index Correction -4'29"
Pressure: 30.1 inches = 1020 mb
Time Zone: 3^h W
Temperature: 68°F = 20°C
Reference Object: Radio Mast.

Note: All observations were made using an eyepiece prism which gave an erect vertical image and an inverted horizontal image. The crosshair was shown as it appeared in the field of view with respect to the sun's disc.

Clock Comparisons

Time Signal	Corresponding Watch Time	Watch Correction on Zone Time
$16^h25^m00^s$	$4^h23^m41.9^s$	$+12^h01^m18.1^s$
16 51 00	4 49 41.0	+12 01 19.0
17 20 00	5 18 40.2	+12 01 19.8

Face	Object	Watch	Observations Vertical Circle	Horizontal Circle
CR	RO			301°55'50"
	|O	$4^h33^m21.8^s$		94 39 50
	O|	54 49.3 *		98 53 40
	Ō	55 58.7	241°56'40"	
	O̲	4 57 47.0	242 45 30	
	RO			301 55 50
CL	RO			121 55 50
	Ō	5 01 29.6	117 19 50	
	O̲	05 04.0	116 13 55	
	|O	06 50.0		282 04 50
	O|	5 07 58.9		281 44 05
	RO			121 55 50

* Observations delayed by passing cloud

Extract from Star Almanac for Land Surveyors, 1969
11th September

UT	δ	E	SD
18^h	4°25!7 N	$12^h03^m27.0^s$	15!9
24	4 20.0	12 03 32.3	

Watch Rating

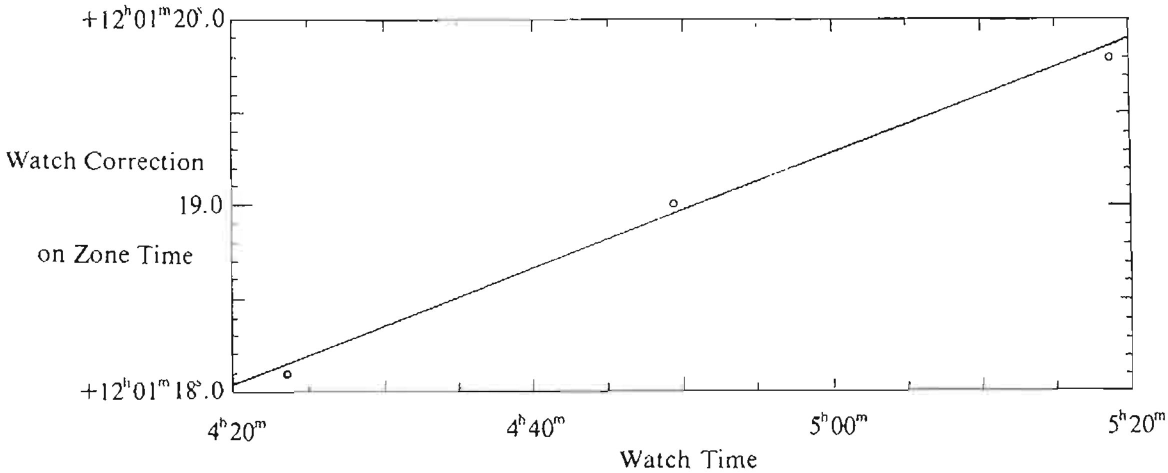


Fig.8.2

8.611 Longitude Calculation

These observations were made within a short period of time (approximately 10^m) and therefore the small changes in δ, E and Watch Correction will be ignored.

Watch Time of Observation	UT of Observation	δ	E	Watch Correction
5^h00^m	20^h01^m	4°23'47" N	$12^h03^m28.8^s$	$+12^h01^m19.3^s$

Reduction of Vertical Circle readings

Reading	Index	Corrected readings	Observed Altitude	Refn.	Par^x	SD	Reduced Altitude
241°56'40"	-4'29"	241°52'11"	28°07'49"	-1'45"	+8"	-15'54"	27°50'18"
242 45 30		242 41 01	27 18 59	1 49		+15 54	27 33 12
117 19 50		117 15 21	27 15 21	1 49		-15 54	26 57 46
116 13 55		116 09 26	26 09 26	1 54		+15 54	26 23 34

Required relationships

$$\cos t = \frac{\sin h - \sin \phi \sin \delta}{\cos \phi \cos \delta}$$

λ = LMT - GMT = (t - E) - (Watch Time + Watch Correction - Time Zone)

λ = (t - Watch Time) + (-E - Watch Correction + Time Zone)

λ = (t - Watch Time) + X

where X = -E - Watch Correction + Time Zone = $-3^h04^m48.1^s$

h	27°50'18"	27°33'12"	26°57'46"	26°23'34"
t	$3^h34^m11.2^s$	$3^h35^m59.5^s$	$3^h39^m42.9^s$	$3^h43^m17.1^s$
Watch Time	4 55 58.7	4 57 47.0	5 01 29.6	5 05 04.0
t - Watch Time	-1 21 47.5	-1 21 47.5	-1 21 46.7	-1 21 46.9
X	-3 04 48.1	-3 04 48.1	-3 04 48.1	-3 04 48.1
λ	-4 26 35.6	-4 26 35.6	-4 26 34.8	-4 26 35.0
	Mean Longitude	$4^h26^m35.2^s$ W		

8.612 Azimuth Calculations

The observations, given in section 8.61, were made over an extended period of time and therefore the changes in δ, E and Watch Correction will be taken into account

Watch Time of Observation	UT of Observation	δ	E	Watch Correction
4^h33^m	19^h34^m	+4°24'13"	$12^h03^m28.4^s$	$+12^h01^m18.4^s$
4 55	19 56	+4 23 52	12 03 28.7	12 01 19.1
5 07	20 08	+4 23 40	12 03 28.9	12 01 19.5
5 08	20 09	+4 23 39	12 03 28.9	12 01 19.5

Required relationships

$$\tan A = \frac{-\sin t}{\cos \phi \tan \delta - \sin \phi \cos t}$$

Semidiameter correction $\Delta A = \frac{-SD \sin A}{\cos \delta \sin t}$

$\lambda = 4^h26^m35.2^s$ W (see previous calculation)

t = E + Watch Time + Watch Correction - Time Zone + Longitude

E	$12^h 03^m 28.4^s$	$12^h 03^m 28.7^s$	$12^h 03^m 28.9^s$	$12^h 03^m 28.9^s$
Watch Time	4 33 21.8	4 54 49.3	5 06 50.0	5 07 58.9
Watch Correction	12 01 18.4	12 01 19.1	12 01 19.5	12 01 19.5
Sum	4 38 08.6	4 59 37.1	5 11 38.4	5 12 47.3
Longitude-Time Zone	-1 26 35.2	-1 26 35.2	-1 26 35.2	-1 26 35.2
t	3 11 33.4	3 33 01.9	3 45 03.2	3 46 12.1
Azimuth of Sun	239°59'32"	244°50'01"	247°25'25"	247°40'01"
ΔA	+18 37	-18 01	+17 42	-17 41
Azimuth of Limb	240 18 09	244 32 00	247 43 07	247 22 20
Hor. reading to Limb	94 39 50	98 53 40	282 04 50	281 44 05
Orienting Corrn.	145 38 19	145 38 20	325 38 17	325 38 15
Hor. reading to RO	301 55 50	301 55 50	121 55 50	121 55 50
Azimuth of RO	87 34 09	87 34 10	87 34 07	87 34 05
Mean Azimuth to RO 87°34'08"				

8.71 Example of sun observations for the determination of latitude and of azimuth by the Altazimuth Method

Place: Pillar 2, Civil Eng. Building UNSW λ $10^h 04^m 56^s$ E
Date: Monday, 20th September 1976 (Morning for Azimuth: Noon for Latitude)
Observer: G.G. Bennett   Theodolites: Wild T2 (erect image)
Recorder: J.G. Freislich
Reference Object RO: Finial on spire of Monastery Church   Watch: Heuer split hand stop watch (Mean Time)
Time Zone: 10^h East   NOTE: Two different instruments

Note: The crosshair was shown as it appeared in the field of view with respect to the sun's disc.

8.711 Latitude observations and calculations

These will be calculated first as the latitude is needed for determining the azimuth.

Note: The sun was observed to the north of the zenith near upper transit.

Watch Correction on Zone Time $11^h 30^m 00.0^s$
Temperature: 21.8°C   Pressure: 1014 mb
Vertical Circle Index Correction: +11"

Face	Object	Watch	Vertical Circle
CL	-Ω-	$10^m 56^s$	34°47'34"
	-Ω-	11 55	46 54
	-Ω-	12 38	46 22
	-Ω-	13 16	46 06
	-Ω-	13 53	45 45
	-Ω-	14 26	34 45 34
CR	-℧-	17 10	324 42 54
	-℧-	18 08	43 09
	-℧-	18 43	43 09
	-℧-	19 13	43 04
	-℧-	19 46	43 05
	-℧-	20 18	324 43 05

The diagonal eyepiece used for these observations produced an inverted image.

8.712 Calculation of declination using polynomial coefficients (see section 8.716 for constants etc.)

Watch Time of Observation	UT of Observation	x	δ
$0^h 11^m$	$1^h 41^m$	0.595 9418	1°06'16" N
12	42	9635	15
13	43	9852	14
13	43	0.595 9852	14
14	44	0.596 0069	13
14	44	0069	13
17	47	0720	10
18	48	0938	09
19	49	1155	08
19	49	1155	08
20	50	1372	07
0 20	1 50	0.596 1372	1 06 07

E = $12^h 06^m 31^s$ Semidiameter 16'00"

8.713 Calculation of the Watch Time of transit

LMT of transit = 24^h - E(see section 8.32)	$11^h 53^m 29^s$	
λ	10 04 56	
UT	1 48 33	
Zone	10	E
Zone Time	11 48 33	
Watch Correction	11 30 00	
Watch Time of transit	18 33	

The observation period is short and close to Local Apparent Noon and therefore only the first term of the circum-meridian reduction formula (see section A.71) will be used.

8.714 Calculation of the Latitude

$$A = \frac{\cos\phi \cos\delta_M}{\sin z_M}$$

$$A = +1.4458$$

Calculation of an approximate latitude

Observed zenith distance closest to transit	35°16'40" N	$z_M = z_{CM} - A.m''$
Refn., Par^x, SD	-15 26	$\phi = \delta_M - z_M$
z_M	+35 01 14	
δ_M	+ 1 06 08	
Approx. ϕ	-33 55 06	

Calculation of an accurate latitude

Watch Time of Observation	t	m"	Vertical ☉ reading	Index	Observed ZD	Refn	Par^x	SD	-Am	Meridian ZD z_M	δ	ϕ
10^m56^s	-7^m37^s	1'54"	34°47'34"	+11"	34°47'45"	+39"	-5"	+16'00"	-2'45"	+35°01'34"	+1°06'16"	-33°55'18"
11 55	6 38	1 26	46 54		47 05				-2 04	35	15	20
12 38	5 55	1 09	46 22		46 33				-1 40	27	14	13
13 16	5 17	0 55	46 06		46 17				-1 20	31	14	17
13 53	4 40	43	45 45		45 56				-1 02	28	13	15
14 26	4 07	33	34 45 34		34 45 45			+16 00	-0 48	+35 01 31	+1 06 13	-33 55 18
17 10	1 23	04	324 42 54	+11	35 16 55	+39	-5	-16 00	-0 06	+35 01 23	+1 06 10	-33 55 13
18 08	-0 25	00	43 09		16 40				-0 00	14	09	05
18 43	+0 10	00	43 09		16 40				-0 00	14	08	06
19 13	0 40	01	43 04		16 45				-0 01	18	08	10
19 46	1 13	03	43 05		16 44				-0 04	14	07	07
20 18	1 45	0 06	324 43 05		35 16 44			-16 00	-0 09	+35 01 09	+1 06 07	02

Mean Latitude 33°55'12" S

8.715 The Altazimuth Observations

Watch Correction on Zone Time $+6^h40^m$

Temperature: 16.6°C Pressure: 1015 mb

Vertical Circle Index Correction: -40" NOTE: Diagonal eyepiece not used

Face	Object	Watch	Vertical Circle	Horizontal Circle
CL	RO			0°10'37"
	$\underline{\odot\vert}$	0^h51^m	70°01'52"	142 52 33
CR	$\overline{\vert\odot}$	0 52	290 48 16	322 04 34
	RO			180 10 21
CL	RO			45 12 48
	$\underline{\odot\vert}$	0 57	68 46 17	186 53 36
CR	$\overline{\vert\odot}$	0 58	291 59 28	6 08 22
	RO			225 12 32
CR	RO			270 15 03
	$\underline{\odot\vert}$	1 02	292 17 40	51 03 36
CL	$\overline{\vert\odot}$	1 04	66 54 57	230 15 14
	RO			90 15 18
CL	RO			135 17 36
	$\underline{\odot\vert}$	1 07	66 45 07	275 16 31
CR	$\overline{\vert\odot}$	1 09	294 06 20	94 25 58
	RO			315 17 19

8.716 Calculation of declination by means of the polynomial coefficients provided in the Star Almanac for Land Surveyors

$$\delta = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

where $a_0 = 8.31516$)
$a_1 = -11.59793$)
$a_2 = -1.11388$) for September 1976
$a_3 = 0.43207$)
$a_4 = 0.03983$)

δ is the declination of the sun expressed in degrees and decimals

x is the Greenwich time of observation, expressed in days and decimals less one day and divided by 32. (see footnote)

Watch Time of Observation	UT of Observation	x	δ
0^h51^m	21^h31^m *	0.590 5165	1°10'18" N
0 52	32	5382	18
0 57	37	6467	13
0 58	38	6684	12
1 02	42	7552	08
1 04	44	7986	06
1 07	47	8637	03
1 09	21 49	0.590 9071	1 10 01

* Note change of date

Semidiameter 16'00"

From 1977 onwards x is defined as the sum of the Greenwich day of the month and the decimal of the day all divided by 32.

8.717 Reduction of Vertical Circle Readings

Reading	Index	Corrected reading	Observed Altitude	Refn	Par^{x}	SD	Reduced Altitude
70°01'52"	-40"	70°01'12"	19°58'48"	-2'35"	+8"	+16'00"	20°12'21"
290 48 16		290 47 36	20 47 36	28		-16 00	20 29 16
68 46 17		68 45 37	21 14 23	25		+16 00	21 28 06
291 59 28		291 58 48	21 58 48	19		-16 00	21 40 37
292 17 40		292 17 00	22 17 00	17		+16 00	22 30 51
66 54 57		66 54 17	23 05 43	12		-16 00	22 47 39
66 45 07		66 44 27	23 15 33	11		+16 00	23 29 30
294 06 20		294 05 40	24 05 40	-2 06		-16 00	23 47 42

Required relationships:

$$\cos A = \frac{\sin \delta - \sin h \sin \phi}{\cos h \cos \phi}$$

Semidiameter correction $\Delta A = \frac{SD}{\cos h_o}$ *

From section 8.714 $\phi = 33°55'12"$ S

8.718 Calculation of the Azimuth

Face	CL	CR	CL	CR
Computed Azimuth of Sun	74°06'43"	73°53'05"	73°05'14"	72°54'57"
ΔA	+17 01	-17 07	+17 10	-17 15
Azimuth of Limb	74 23 44	73 35 58	73 22 24	72 37 42
H⊙ Rdg to Limb	142 52 33	322 04 34	186 53 36	6 08 22
Orienting Corrn	+291 31 11	+111 31 24	+246 28 48	+66 29 20
H⊙R to RO	0 10 37	180 10 21	45 12 48	225 12 32
Azimuth to RO	291 41 48	291 41 45	291 41 36	291 41 52
	291°41'46"		291°41'44"	
Face	CR	CL	CL	CR
Computed Azimuth of Sun	72 13 13	71 59 08	71 23 38	71 08 04
ΔA	+17 17	-17 24	+17 25	-17 32
Azimuth of Limb	72 30 30	71 41 44	71 41 03	70 50 32
H⊙ Rdg to Limb	51 03 36	230 15 14	275 16 31	94 25 58
Orienting Corrn	+21 26 54	+201 26 30	+156 24 32	+336 24 34
H⊙R to RO	270 15 03	90 15 18	135 17 36	315 17 19
Azimuth to RO	291 41 57	291 41 48	291 42 08	291 41 53
	291°41'52"		291°42'00"	
	Mean Azimuth to Mark		291°41'50"	

* Where h_o is the observed altitude corrected for index only.

9

The Simultaneous Determination of Latitude and Longitude

INTRODUCTION

IN Chapters 5 and 6, methods of determining Latitude and Longitude independently of each other have been dealt with. The observation method used in both types of observation employs timed altitudes and it was found that the best circumstances, in which to make such observations, were on or near the local meridian and the prime vertical respectively. Under these circumstances, the effects of systematic and random errors on the quantities sought were kept to a minimum.

In the methods to be described, the same observational technique will be used but instead of making separate observations for latitude and longitude, values of both of the unknowns will be deduced from a consideration of all observations to all stars.

As with independent observations for latitude and longitude, it is possible to use horizontal circle observations, but this latter method requires that observation be made near to the zenith and therefore these observations introduce some practical difficulties when a theodolite is used.

Much of the theory and the semi-graphical treatment, which follows runs parallel with that used by the air and marine navigator. He however, is normally satisfied with an accuracy much less than that required by the land surveyor. The semi-graphic solution and its interpretation, and the concept of position circle and position line arise from the original discovery, which was made in 1837 by T.H. Sumner, the captain of an American merchant vessel.

A full and interesting account of the circumstances leading up to this discovery will be found in "The American Practical Navigator" by Nathaniel Bowditch. Sumner's original technique is now seldom used and today a variation of his method is generally used. This was suggested by the French Admiral Adolphe-Laurent-Anatole Marcq de Blonde de Saint-Hilaire (1832-1889) and is known as the "Method of Zenith Distance Intercepts" or the "Marcq St. Hilaire Method".

Specialised instrumentation has also been developed for these observations, both for use by surveyors and in fixed observatories.

THE DETERMINATION OF POSITION FROM OBSERVATIONS TO TWO STARS

9.11 IF time altitude observations have been made on two stars, which are separated in azimuth by an angle, that is *neither greatly acute nor greatly obtuse*, the latitude and longitude of the observer's position can be deduced, provided that the observations are not affected by appreciable systematic errors. The significance of these conditions will be appreciated later in this chapter (see section 9.11 Step 4). The solution, which follows, is quite general and does not require the person, performing the calculations, to know the relative positions of the two stars in the sky.

The information available for the solution for each of the two stars is,

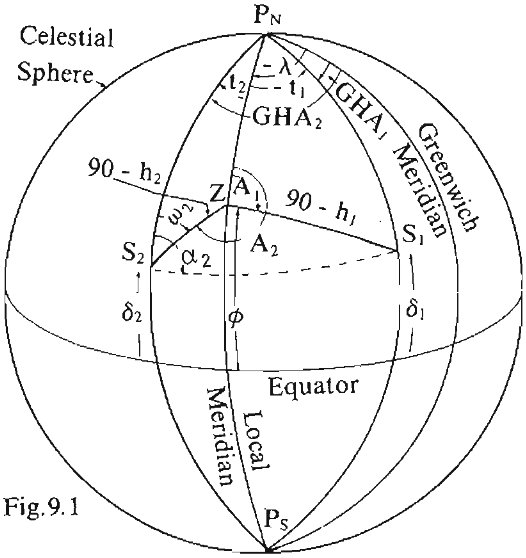


Fig.9.1

(a) the observed altitude, suitably corrected for index error and refraction,
(b) the declination and right ascension of each star, and
(c) the GHA, which is obtained from the clock time of observation after converting this to the corresponding instant of GST and then using the following relationship,

$$GHA = GST - RA$$

The situation is shown in Fig 9.1, where S_1 and S_2 are the positions of the two stars and Z is the observer's zenith.

The relationships for the computations required may be derived from those of section 2.62, together with the principles enunciated in this section for the solution process. If the angles α_2 and ω_2 of Fig 9.1 are defined in the same way as the parallactic angle ω in section 2.73 is defined, then the solution below is completely general.

The reader is invited to do this derivation and solution for himself with, for instance the positions S_1 and S_2 reversed.

1. From the Cosine Formula in triangle $S_1S_2P_N$, S_1S_2 is obtained from

$$\cos S_1S_2 = \sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(GHA_2 - GHA_1)$$

2. From the Four Parts Formula in the same triangle, α_2 is obtained from

$$\tan\alpha_2 = \frac{\sin(GHA_2 - GHA_1)}{\cos\delta_2 \tan\delta_1 - \sin\delta_2 \cos(GHA_2 - GHA_1)}$$

in which α_2 is placed in its correct angular quadrant by taking heed of the signs of the numerator and denominator.

3. From the Cosine Formula in triangle S_1S_2Z, $(\alpha_2 - \omega_2)$ is obtained from

$$\cos(\alpha_2 - \omega_2) = \frac{\sin h_1 - \sin h_2 \cos S_1S_2}{\cos h_2 \sin S_1S_2}$$

The solution for $(\alpha_2 - \omega_2)$ is ambiguous, because Z may lie inside or outside the triangle $S_1S_2P_N$. Therefore the two values of $(\alpha_1 - \omega_2)$, which result, will give two values for ω_2.

4. From the Cosine Formula in triangle S_2ZP_N, ϕ is obtained from

$$\sin\phi = \sin h_2 \sin\delta_2 + \cos h_2 \cos\delta_2 \cos\omega_2$$

From the two values of ϕ, which result (see step 3), the appropriate one of the two values is selected. In practice, this decision presents no problem provided the difference between the azimuths to the two stars is well away from 0° or 180° (see beginning of section 9.11).

5. From the Four Parts Formula in the same triangle, the hour angle t_2 is obtained from

$$\tan t_2 = \frac{\sin \omega_2}{\tan h_2 \cos \delta_2 - \sin \delta_2 \cos \omega_2}$$

6. The longitude λ of the observer's position is then found from

$$\lambda = t_2 - GHA_2$$

and the hour angle t_1 from

$$t_1 = GHA_1 + \lambda$$

7. Finally from the Cosine Formula in triangle $S_1 Z P_N$, a check on the calculation is obtained from

$$\sin h_1 = \sin \phi \sin \delta_1 + \cos \phi \cos \delta_1 \cos t_1$$

9.12 To demonstrate this computation, the following data will be used in order to determine the preliminary values $\phi_a \lambda_a$ of a station, at which these observations were made and which was in South Africa.

Star No.	369	328	564	548
Name	Arcturus	α Crucis	α Pavonis	Altair
Aspect	NW	SW	SE	NE
Right Ascension	$14^h13^m24.8^s$	$12^h23^m48.5^s$	$20^h21^m47.6^s$	$19^h48^m23.3^s$
Declination	19°26'15" N	62°49'48" S	56°53'44" S	8°44'04" N
Corrected Observed Altitude	35°52'26"	36°07'08"	39°53'25"	34°01'53"
GST of Observation	$14^h21^m18.2^s$	$14^h35^m53.6^s$	$14^h44^m23.5^s$	$14^h55^m32.2^s$

1 SW $GHA_1 = GST_1 - RA_1 = 2^h12^m05.1^s$ $\quad \delta_1$ -62°49'48" $\quad h_1$ 36°07'08"

2 SE $GHA_2 = GST_2 - RA_2 = 18^h22^m35.9^s$ $\quad \delta_2$ -56 53 44 $\quad h_2$ 39 53 25

$$GHA_2 - GHA_1 = 16^h10^m30.8^s = 242°37'42"$$

$$\cos S_1S_2 = \sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2 \cos(GHA_2 - GHA_1)$$

$$S_1 - S_2 = 50.907\ 0489°$$

$$\tan \alpha_2 = \frac{\sin(GHA_2 - GHA_1)}{\tan \delta_1 \cos \delta_2 - \sin \delta_2 \cos(GHA_2 - GHA_1)}$$

$$\alpha_2 = 211.498\ 7238°$$

$$\cos(\alpha_2 - \omega_2) = \frac{\sin h_1 - \sin h_2 \cos S_1S_2}{\cos h_2 \sin S_1S_2}$$

$$(\alpha_2 - \omega_2) = \pm 71.894\ 7833°$$

$$\omega_2 = \alpha_2 - (\alpha_2 - \omega_2)$$

$$\omega_2 = 283.393\ 5071° \text{ or } 139.603\ 9406°$$

$$\sin \phi = \sin h_2 \sin \delta_2 + \cos h_2 \cos \delta_2 \cos \omega_2$$

$$\phi = -26.113\ 3327° \text{ or } -58.911\ 0644$$

$$= -26°06'48.00"$$

This value is accepted because the station is known to be in South Africa.

$$\tan t_2 = \frac{\sin \omega_2}{\tan h_2 \cos \delta_2 - \sin \delta_2 \cos \omega_2}$$

$t_2 \; = \; -56.227\,9855° \; = \; -3^h 44^m 54.72^s$

$\lambda \; = \; t_2 - GHA_2 \; = \; -3^h 44^m 54.72^s - 18^h 22^m 35.9^s$

$\; = \; -22^h 07^m 30.62^s \; = \; 1^h 52^m 29.38^s$ E

$t_1 \; = \; GHA_1 + \lambda \; = \; 2^h 12^m 05.1^s + 1^h 52^m 29.38^s$

$\; = \; 4^h 04^m 34.48^s$

Final check

$\sin h_1 = \sin \phi \sin \delta_1 + \cos \phi \cos \delta_1 \cos t_1$

$+0.589\,463 = +0.589\,463$

∴ The values for the preliminary position $\phi_a \lambda_a$ are

ϕ_a 26°06'48.0" South

λ_a $1^h 52^m 29.4^s$ East

It is suggested that various combinations of data from pairs of stars, given at the beginning of this section, be used as additional examples of this calculation. The resulting latitudes and longitudes will not agree exactly because of the presence of small observation errors.

9.13 Either star may be set out as the first star in the above layout, provided the generalized conventions of the Astronomical Triangle are adhered to and provided that the angles α_2 and ω_2 above use the same convention as the parallactic angle ω in this triangle (see section 2.73). The azimuth quadrant, in which each star is observed, is usually noted in the field book. This information often makes it possible to select the required value of the two values obtained for the angle ω_2. The above solution is a general one, which is checked by means of the final equation.

Several points emerge from this. The calculation is easily carried out with modern computing aids. The various methods of calculation for determining position from astronomical position lines all require approximate values of latitude and/or longitude for the observer's station. This information can often be found, with sufficient accuracy, from a map even if its scale is small, but sometimes this is not possible and then the calculation above may well be used, in order to determine preliminary values ϕ_a λ_a directly from the observed values themselves. This calculation uses observations from only two stars. If more than two stars had been observed, it would be an exceedingly complicated process to determine final values of latitude and longitude, by means of this calculation technique, because all the observed data should be used to obtain the final result.

THE CONCEPT OF POSITION CIRCLE AND POSITON LINE

9.21 A simple illustration of what has been done by calculation above can be obtained by plotting the positions of the two stars S_1 and S_2 on a small sphere, such as a plastic ball, and then by drawing on it a circle with S_1 as centre and a spherical radius of $(90 - h_1)$ and another circle with S_2 as centre and $(90 - h_2)$ as radius. The intersections of these two circles gives the required position, provided it can be determined which intersection is the desired one.

If only one star had been observed, then although the position of Z could not be found uniquely, it would be known that the observer's position must lie somewhere along this circle, that is, an identical altitude to the star could have been observed simultaneously by any number of observers, whose zeniths (geographical position) lay on this circle. For this reason this circle is called a position circle.

It is often convenient to consider this position circle as being situated on the earth's surface and, under these circumstances, the centre of the position circle (S_1 on the celestial sphere) has geographical co-ordinates

$\phi = \delta$ and $\lambda = 24^h - GHA$. This point is called the sub-stellar point (SSP) and it is obvious that an observer stationed at the SSP at the instant of observation would see the star in question in his zenith. The concept of the SSP and position circle drawn on the earth's surface is a very useful one indeed, although it must be realised that, under these circumstances, the earth must be considered to be spherical in shape. Fig 9.2 illustrates this latter interpretation, where the terrestrial position P corresponds to the celestial position Z of Fig 9.1.

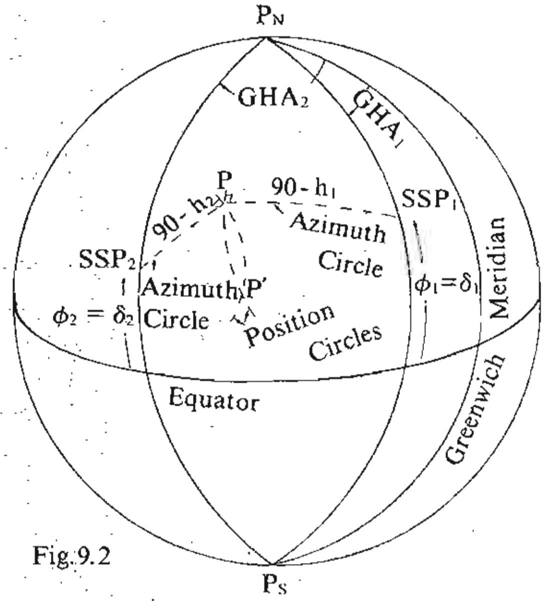


Fig.9.2

It is usually necessary in practice to deal with more than two position line circles and, under these circumstances, it is convenient to be able to plot these circles in the vicinity of P at a large scale (see Fig 9.2). Unless the observations are perfect, the resulting position lines will not intersect at a point but will form a network of intersecting lines, from which the best estimate of the position P must be made. It would be convenient to represent this small area in the vicinity of P orthomorphically on a map or a plotting sheet. This requires its scale to be uniform and then the map will also be angle true.

In addition, it would be convenient to use a graticule of latitude and longitude lines which comprise an orthogonal set of lines. Over this very small area of the earth's surface, these may be taken, without appreciable error, to be a set of orthogonal straight lines.

Let an elementary rectangle on the earth's surface lie between two parallels ϕ and $\phi+d\alpha$ and two meridians λ and $\lambda+d\alpha$. The corresponding figure on the map will also be a rectangle, which lies between two graticule lines, representing these two parallels and spaced at map distance D, as well as two other orthogonal graticule lines, representing the two meridians and spaced at map distance d.

$$\text{Point scale along parallel} = \frac{\text{Element of map distance along parallel}}{\text{Corresponding element of ground distance along parallel}}$$

$$= \frac{d}{R\cos\phi\, d\alpha}$$

$$\text{and point scale along meridian} = \frac{\text{Element of map distance along meridian}}{\text{Corresponding element of ground distance along meridian}}$$

$$= \frac{D}{R\, d\alpha}$$

in which R is the earth's radius.

But the map represents a very small area and also the scale is independent of bearing. Therefore the two scales above are equal to one another.

$$\therefore \qquad \frac{d}{\cos\phi} = D$$

$$\therefore \qquad \frac{d}{D} = \cos\phi$$

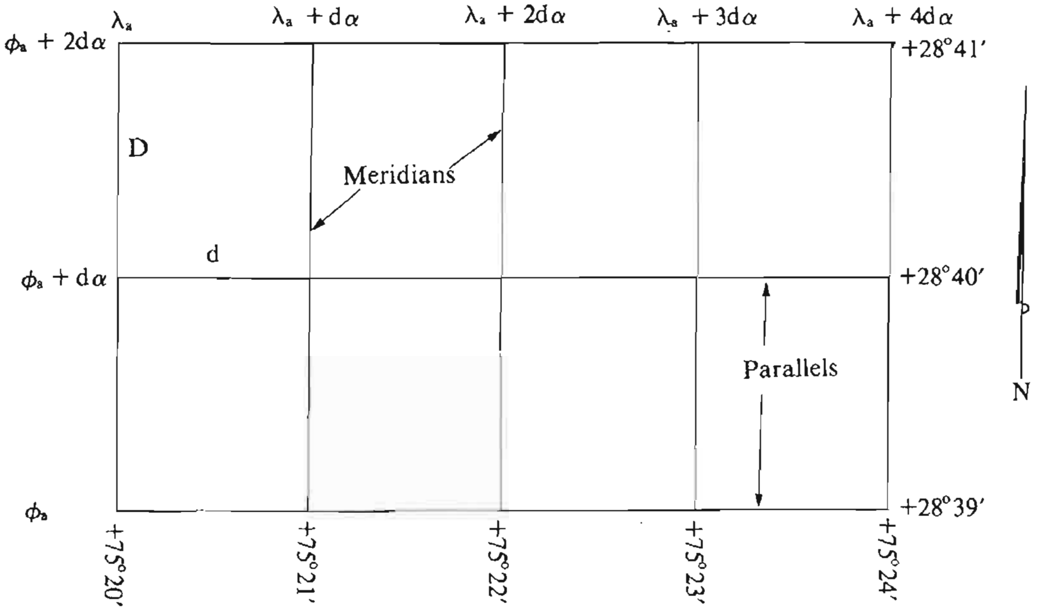


Fig.9.3

The graticule may now be drawn as a set of orthogonal straight lines with the lines representing parallels spaced at intervals D and those representing meridians spaced at intervals d where $d = D\cos\phi$ and D is the map distance corresponding to an angular unit $d\alpha$. These lines are labelled λ_a, $\lambda_a + d\alpha$, $\lambda_a + 2d\alpha$, ... $\lambda_a + nd\alpha$ increasing towards the east and ϕ_a, $\phi_a + d\alpha$, $\phi_a + 2d\alpha$, ... $\phi_a + md\alpha$ increasing towards the north (see Fig 9.3).

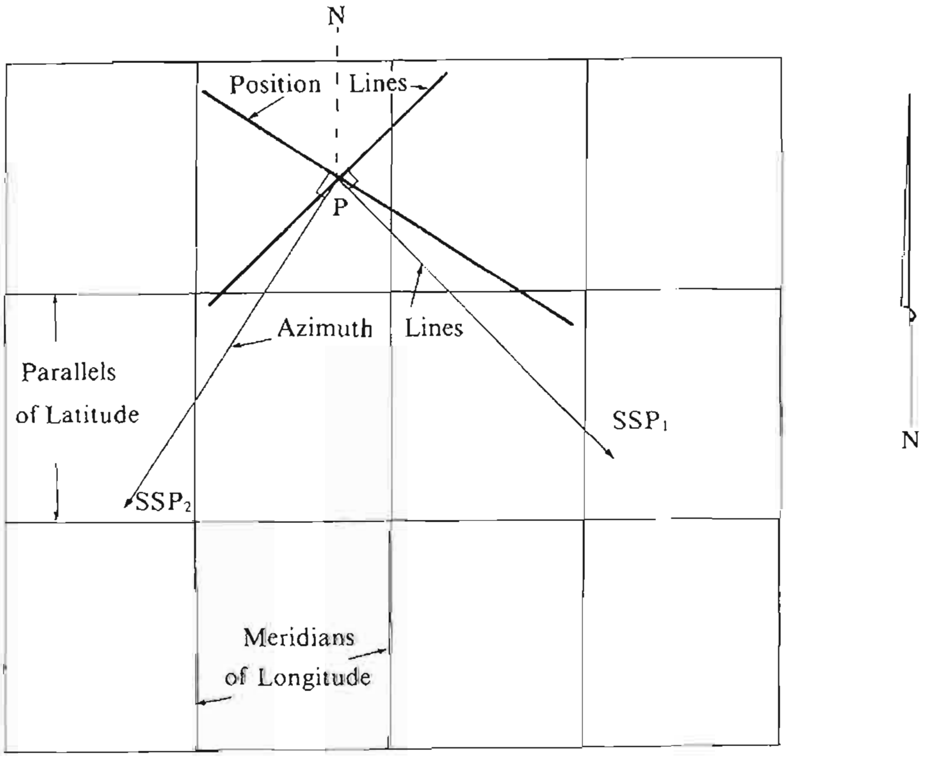


Fig.9.4

Fig 9.4 represents the position circles through P and the azimuth circles from P to the substellar points.

The short arcs shown on the plot have been drawn as straight lines, an approximation, which is both convenient and sufficiently accurate for our purpose. It will be noted that the position line and the azimuth line for a star are mutually perpendicular and also that one is able to plot their directions from a knowledge of the azimuth of the star (see also Figs 9.1 and 9.2).

The concept of position lines is not new for surveyors, who use them in many different forms, in everyday work. If, for instance, a point lies on a river or on a path and these features are represented by lines on a map, then these lines are position lines on which the point, whose position on the map is required, is situated. If this point lies both on the river and on the path represented, then the point's position on the map lies at an intersection of these two position lines. Position lines on a map may be irregular, straight or curved lines, because they represent natural features, such as rivers, streams, shore lines, etc. or artificial ones, such as roads, railway lines, fences etc. On the other hand, position lines may result from some condition, e.g. numbered grid or graticule lines on the map, or from some measurement made on the ground. In the case under consideration, the position lines are small parts of circles of measured altitude which, when drawn on a map, are known to intersect at a point representing the station at which they were observed.

THE CALCULATION AND PLOTTING OF A POSITION LINE

9.22 THE determination of the position of P by calculation, as stated in section 9.13, would be extremely complicated if more than two stars were observed and, for this reason, a semi-graphic process is often used to provide a simpler solution. The individual position lines may be located on the plot in a number of ways, two of which are shown in Fig 9.5.

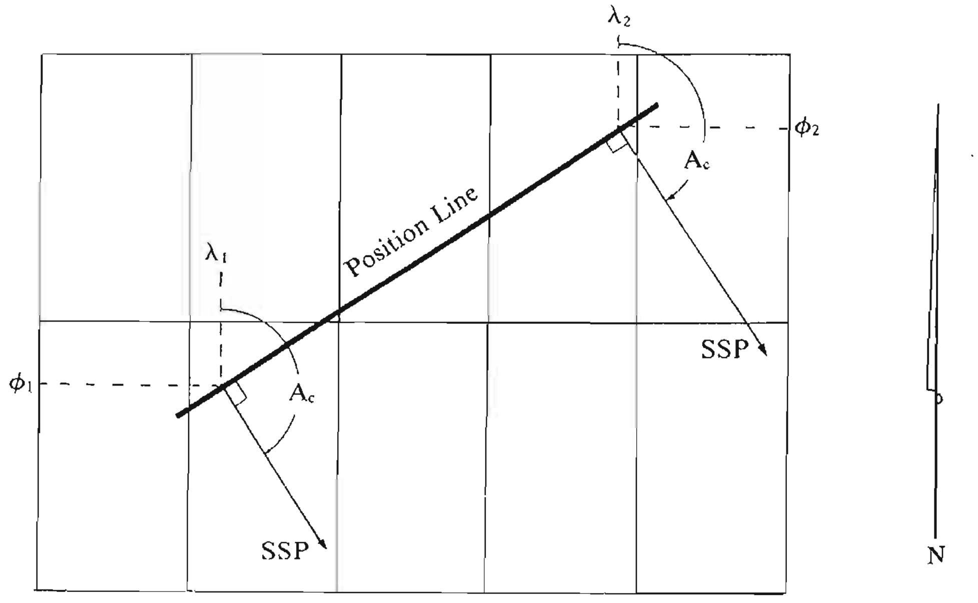


Fig.9.5

In the first method, a value of latitude ϕ_1 near to the position of observation, is chosen and this, together with the star's altitude and declination, enables one to solve for the hour angle. This hour angle, combined with the RA, gives LST. The clock time of observation furnishes a value of GST and thus a value of longitude λ_1 is obtained.

In the second method, a value of longitude λ_2, also near to the position of observation, is chosen. From this longitude value, one may deduce the star's hour angle from consideration of the RA and the GST of observation.

This hour angle together with the stars declination and altitude enables one to solve for the latitude .

For both methods, other points (ϕ,λ) along the position line may be calculated in order to draw the line, but a simpler way is that of calculating the azimuth of the star A_c, which need only be done with a low accuracy (say 0°.1) and then drawing the position line from (ϕ_1, λ_1) or (ϕ_2, λ_2) in a direction at right angles to the calculated azimuth.

The first method is often referred to as the "Longitude Intercept Method" or "Modified Sumner Method", while the second is known as the "Latitude Intercept Method". Neither of these methods finds much favour today, because of the difficulties of locating points on the position lines, when stars are observed either close to the meridian or close to the prime vertical respectively, because the point (ϕ_1, λ_1) or (ϕ_2, λ_2) may fall outside the limited plotting area. A third method (see section 9.31) does not suffer from this disadvantage and, for this reason, it is recommended for use exclusively and is dealt with in detail.

Calculation of the Marcq St. Hilaire Position Line

9.31 This method, "The Method of Zenith Distance Intercepts" or "The Marcq. St. Hilaire Method", utilises an approximate value for both latitude and longitude. This seems a more rational approach than the previous methods, because the purpose of the observations is to refine the approximate values of latitude and longitude. However, in this approach, one has four values of either observed or assumed data to deal with in the astronomical triangle, in which three are sufficient for a solution. The preliminary calculation is therefore done as follows. An hour angle is calculated by means of an assumed value of longitude λ_a, and together with an assumed value of latitude ϕ_a and the star's declination, an altitude h_c is then calculated from the astronomical triangle. As with the previous methods, an azimuth A_c of sufficient accuracy for plotting purposes is also obtained. In most cases, the calculated and observed values of altitude are not the same. They would be so only if the position line passed through the assumed position of observation ϕ_a, λ_a.

The Plotting of St. Hilaire Position Lines

9.32 The astronomical position line shown in Fig 9.6, is portion of the locus of an observed altitude circle. This may be represented by means of a small circle on the spherical earth. Similarly for the calculated altitude, there is a locus, which has the same SSP at its centre. These two loci therefore

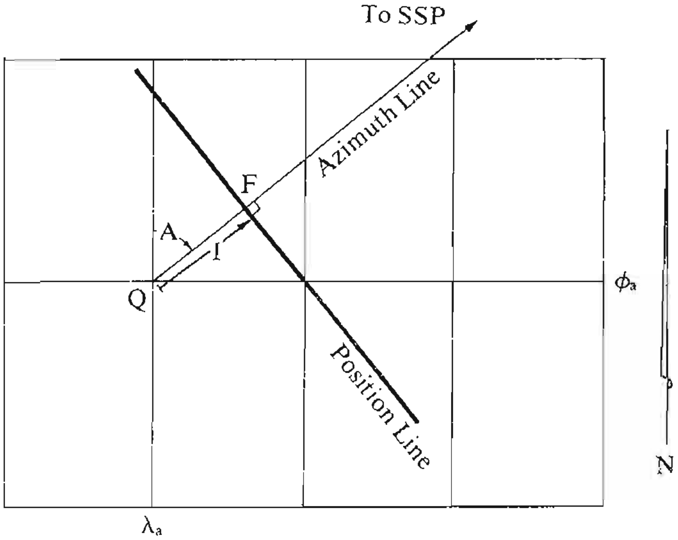


Fig. 9.6

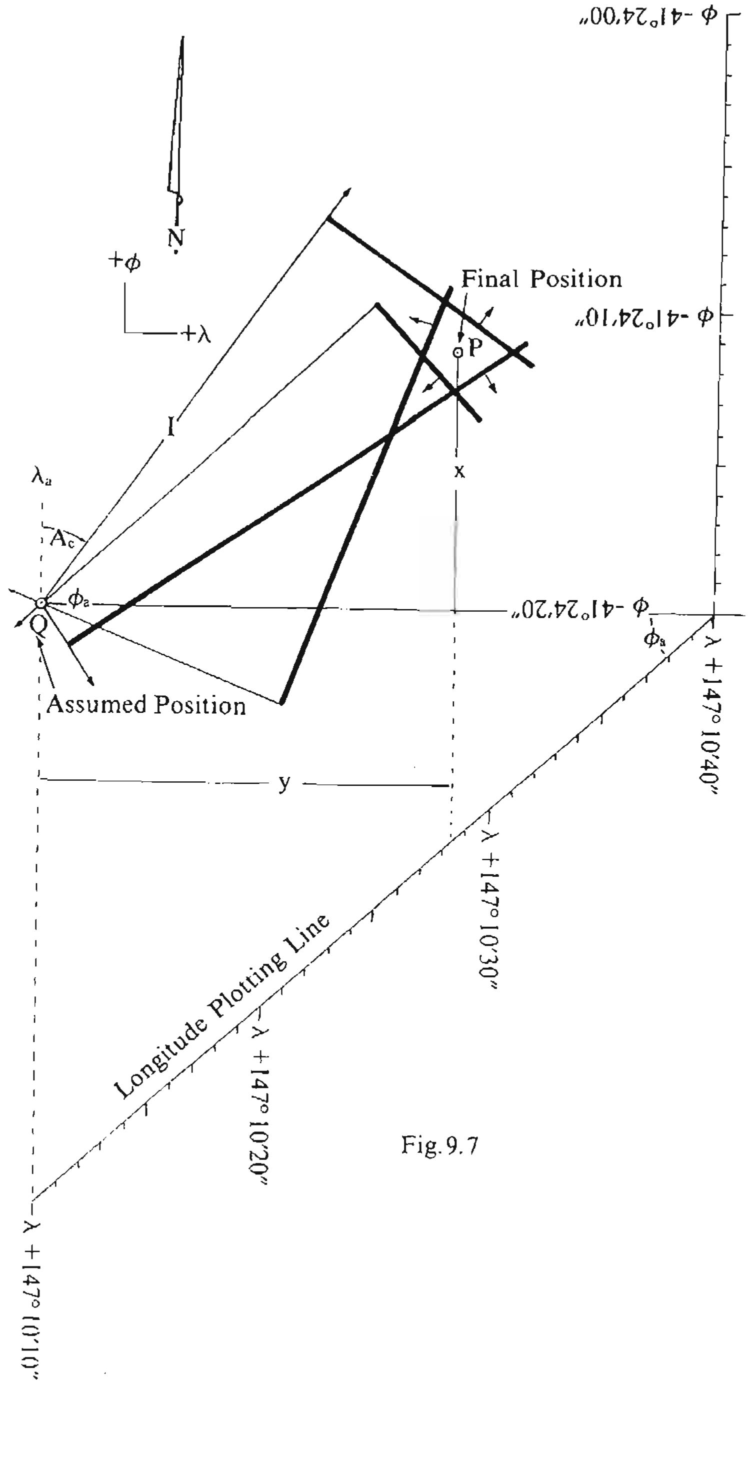

N
$+\phi$
$+\lambda$
Final Position
P
I
x
λ_a
A_c
ϕ_a
Q
Assumed Position
y
ϕ -41°24'00"
ϕ -41°24'10"
ϕ -41°24'20"
ϕ_a
λ +147° 10'40"
λ +147° 10'30"
λ +147° 10'20"
λ +147° 10'10"
Longitude Plotting Line

Fig. 9.7

would plot as two parallel straight lines, with the latter one passing through the assumed position Q at $\phi_a \lambda_a$. The distance between these parallel lines is given by the difference

$$I = h_o - h_c$$

where I is called the intercept and h_o and h_c are the observed and calculated altitudes respectively.

The intercept I is given a sign, which depends upon whether h_o is greater or less than h_c. This sign will indicate whether the position line lies between Q and SSP (as in Fig 9.6) or on the other side of Q away from SSP.

The distance from SSP to Q is $(90^\circ - h_c)$ and from SSP to the position line is $(90^\circ - h_o)$ and therefore, if h_o is greater than h_c, I is positive and the position line is plotted between Q and SSP. This is said to be *towards* and the converse *away*. A handy mnemomic, used by both navigators and surveyors, is

G O A T

the initial letters of Greater Observed Altitude Towards, meaning that if the observed altitude is the greater, then the position line is plotted towards the SSP and vice versa.

Fig 9.6 illustrates how the intercept and its position line are plotted. The azimuth line is first set out and drawn and then the intercept is measured along it from Q, either towards or away from the SSP as required, in units the same as those along the meridian line. These units are those to be used for measuring distances along any great circle, e.g. differences in latitude along the meridian great circle or differences in altitude along any azimuth circle, which is also a great circle.

When several position lines are to be plotted on the diagram, the multiplicity of azimuth lines and position lines may cause some confusion and, to avoid cluttering up the plot, it is suggested that only the footpoint F of the intercept on the position line (see Fig 9.6) be plotted and the position line drawn through this point. The geographical co-ordinates $\phi_F \lambda_F$ of this footpoint F will be

$$\phi_F = \phi_a + I \cos A_c$$

$$\lambda_F = \lambda_a + I \sin A_c \sec \phi_a$$

because $$\Delta\phi_F = I \cos A_c$$

and $$D\lambda_F = \Delta\lambda_F \cos \phi_F = I \sin A_c \quad \text{(see Fig 9.6)}$$

9.33 A handy graphical way of plotting or reading off the geographical coordinates consists in plotting a line at an angle of ϕ_a with respect to a parallel of latitude (see Fig 9.7) and subdividing this Longitude Plotting Line according to the chosen scale. The meridian is likewise subdivided. It is suggested that the labelling of the latitude and longitude lines be done so that the numerical values, irrespective of sign, face in the direction, in which they are increasing. This may produce numbers, which are upside down on the plot,but this arrangement tends to prevent mirror image plotting and interpolation in the wrong direction. The latitude and longitude values of any point may now be plotted or read off directly from the graduations along the meridian and the longitude plotting lines.

If observations have been made to a number of stars and such measurements are *not* influenced by the presence of systematic errors, then a position line for each star may be drawn using a convenient scale (say 1 cm representing 5") for the plotting sheets. The position lines so plotted will form a network of lines which, because of the presence of random errors in the timing and measurement of the star altitudes, will not intersect at a point.

P, the best estimate of the observer's position, is a point whose position is such that *the sum of the squares of the distances from P to each position line is a minimum*, provided that each position line has the same weight. (see section 9.65 for a fuller discussion of this aspect).

The position of P can, if required, be found numerically from Fig 9.7 as follows:

$$\phi_P = \phi_a + \Delta\phi = \phi_a + x$$

$$\lambda_P = \lambda_a + \Delta\lambda = \lambda_a + \frac{D\lambda}{\cos\phi_a} = \lambda_a + \frac{y}{\cos\phi_a}$$

where x and y are scaled from the plotting sheet (see Fig 9.7).

The Influence of Systematic Errors on Position Lines

9.41 Two types of systematic error may be present in the observations under consideration. Firstly, there may be a systematic error present in the timing system being used and the observer may always make his observations a little early or a little late depending upon his personal reactions. These errors will have a marked effect on those position lines, which are oriented north south, and little or no effect on those oriented east west. This is not surprising, since it is known that systematic errors of this nature have their worst effect, when observations have been made near the prime vertical, and have little or no effect for near meridian observations. In other words, the final position for latitude is unaffected, but the longitude value will be displaced by an amount equal in size to that of the systematic error and this uncertainty in the final position cannot be discovered or allowed for, unless it can be independently determined.

The second type of systematic error, which can occur, is an error in the altitude, in which uncertainties in the corrections for refraction and index may be present. In the treatment which follows, it will be found desirable to assume that such errors are constant in character and that their presence or absence may be deduced from an examination of the position line plot. In order to maintain the assumption, that these systematic errors are constant, observations to stars should be made in quick succession and at about the same altitude, a condition which has been shown to be necessary for the methods of determining latitude and longitude independently of each other.

A full discussion of this aspect is given in sections 5.14 and 6.14. If such errors are present, then it is apparent that *each* position line will be displaced parallel to itself by an amount equal in size to the error and in a direction, which will shift *all* position lines either towards or away from their SSPs. The truth of this latter assertion is established from the fact that altitudes always increase in a direction along the azimuth line towards the SSP.

If two stars are observed and the resulting position lines are plotted, the

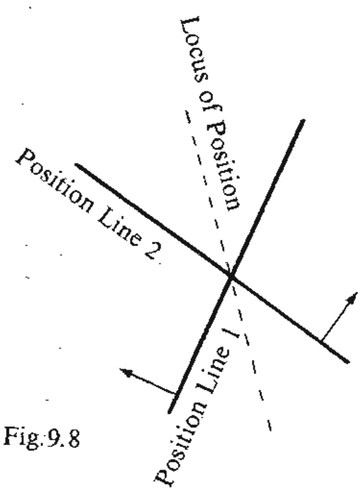


Fig.9.8

observer's position lies at their point of intersection only if no systematic error in altitude is present. If a systematic error is present however, his position will lie somewhere on the line, bisecting the angle between the two

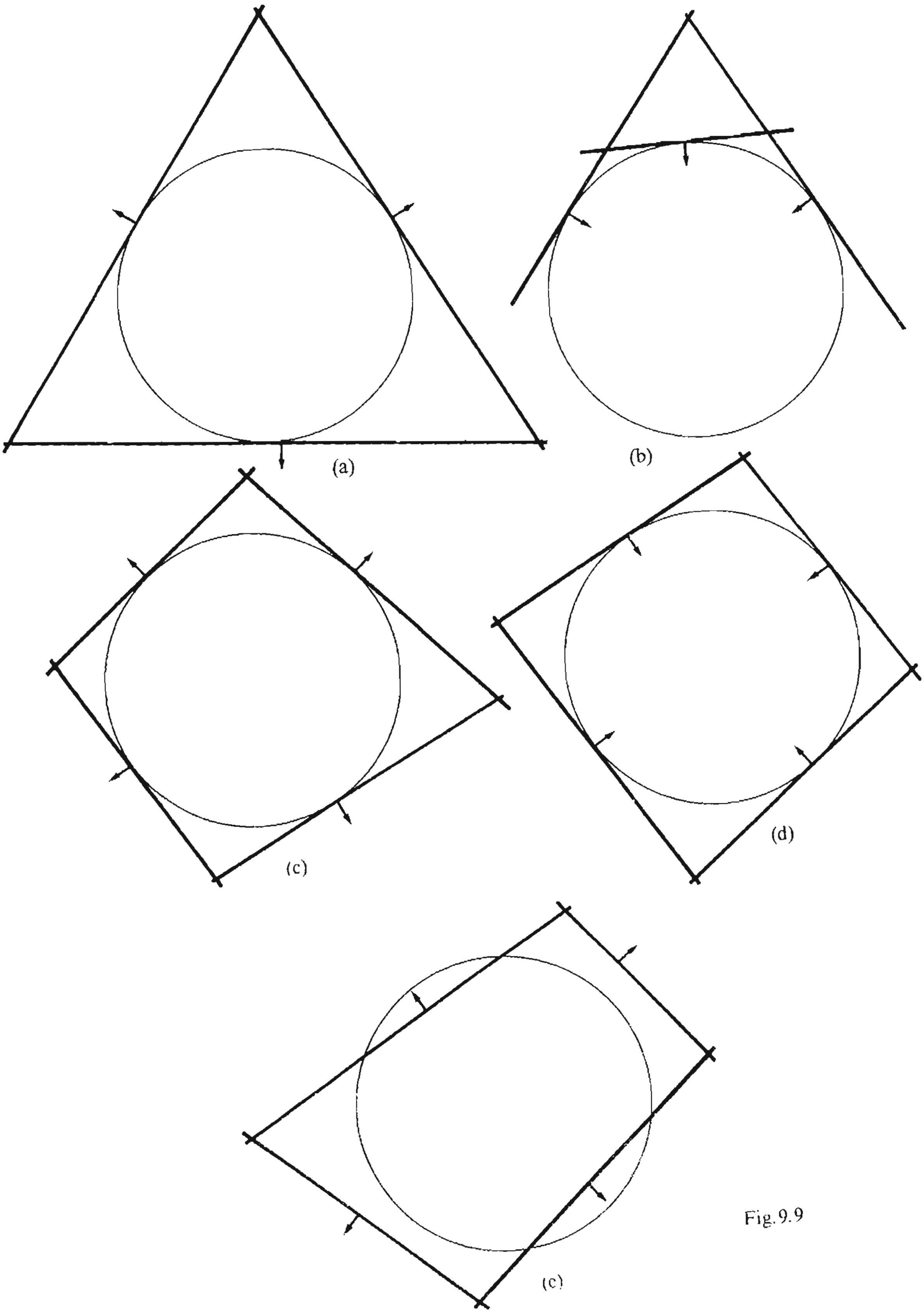


Fig. 9.9

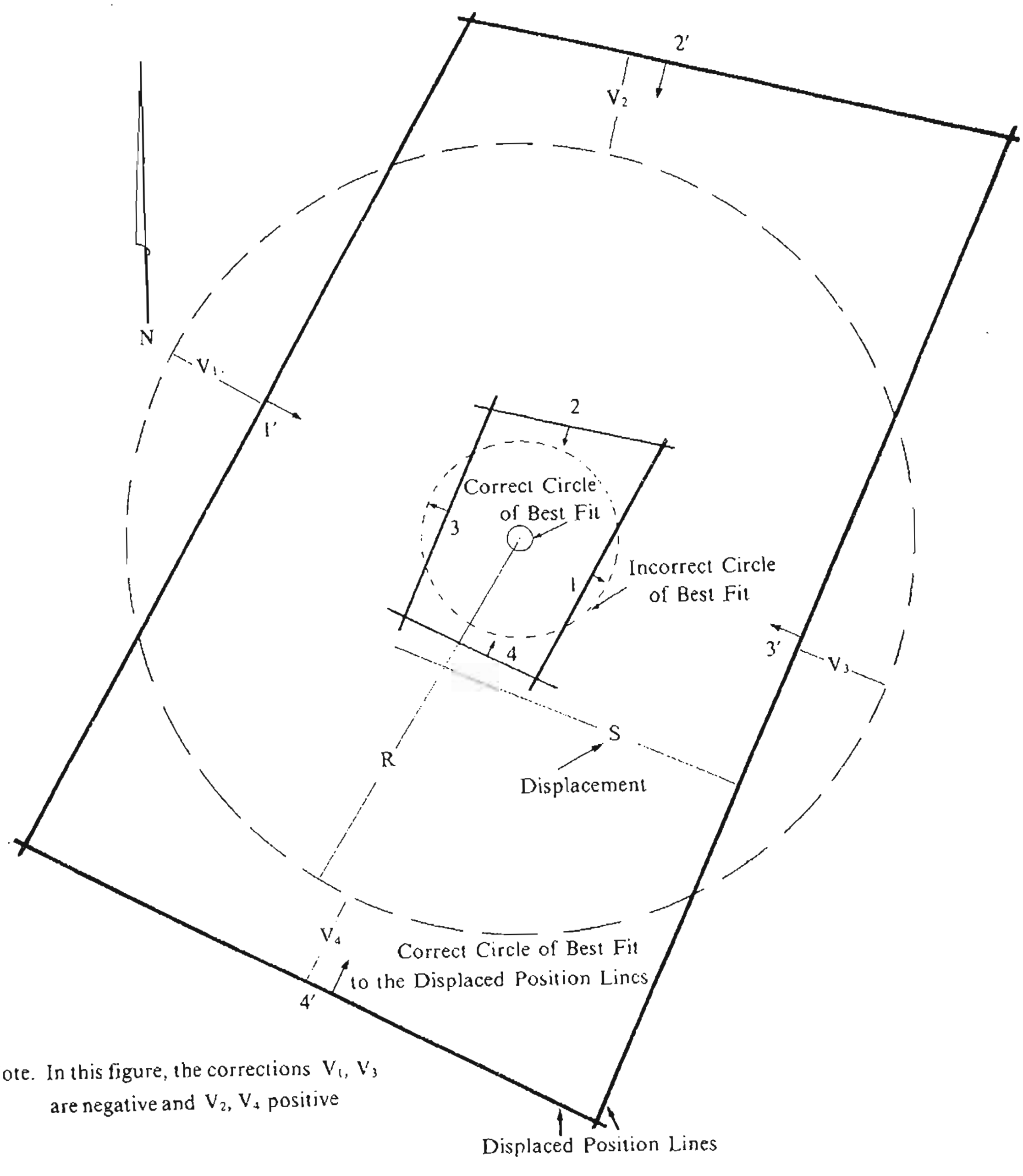
position lines, whose arrows show the directions towards their SSPs (see Fig 9.8). Thus, there are now three unknowns to be determined, the coordinates of the observer's position and the systematic error Δh, affecting the altitude observations. One can therefore conclude that, if such a systematic error is suspected or known to exist, then observations to three stars, suitably situated, will be required for a unique solution. In practice, a fourth star is usually observed to provide, by its redundancy, a check on the results obtained and the means of estimating the precision of the unknowns. Fig 9.9 shows the results of plotting position lines from observations made to sets of three and sets of four stars. Each position line is marked with a small arrow to indicate the direction towards its SSP. In Fig 9.9(a), a circle has been drawn to touch each of the position lines. The centre of this circle gives the required position and the radius of this circle is the value of Δh. It will be noticed that *all* the arrows point away from P, indicating that if *each* position line is displaced *away* from its SSP by an amount equal to the radius of this circle, then the three position lines would pass through the point P. In Fig 9.9(b), the circle has been drawn outside the triangle formed by the three position lines but, once again, it will be seen that, by displacing *all* position lines towards their SSPs by the radius value, the position lines will intersect at P.

When three stars have been observed, which is not a recommended practice, P may be located on the plot by bisecting the appropriate angles at the apex points of the triangle formed by the position lines. It will be noticed that, in Fig 9.9(a), the stars have been observed in azimuths well separated from one another, whilst in Fig 9.9(b) the stars are confined to a limited azimuth sector. If this sector is less than 180° the position of P will always be external to the triangle and vice versa.

9.42 Figs 9.9(c) and (d) show the ideal situation for four stars which have been observed in azimuths, which are approximately A, A+90°, A+180° and A+270°. Not only is the point P located in an unambiguous manner, but it appears that because the inscribed circle exactly fits the rectangle, the observations, which gave rise to the position line, are error free. However, the situation, which normally occurs, is shown in Fig 9.9(e) in which it is not possible to inscribe a circle to touch each line. Instead, one may, by bisecting angles or drawing bisecting parallel lines etc., construct a *circle of best fit*, i.e. one where the distances from the circle to the adjacent position line conform to the principle of Least Squares. If the situation occurring in Fig 9.9(c) or (d) is encountered, it must not be thought that the observations were error free, because each position line is normally derived from a mean of a number of observations. If one were to plot the individual position lines from a star, the uncertainty of this mean value would then become apparent.

9.43 It sometimes happens (see Fig 9.10), especially if the plotting scale is large and Δh small, that the arrows on one pair of nearly parallel position lines point inwards and those on the other pair point outwards. In this anomalous case, a circle of best fit should not be drawn to these lines, although the centre of such a circle would give a position very close to that of P, but the diagram should be transformed to look like the situation shown in Fig 9.9(e), i.e. the arrows on the position lines should either all point outwards or all point inwards. This may be effected quite simply by displacing each position line, parallel to itself, by a constant amount in the same direction with respect to its SSP. The shift S is to be of sufficient size to make the arrows of the displaced position lines all point inwards or all outwards. The circle of best fit is drawn and the errors of observation are now shown distinctly, whereas before this transformation, a quite erroneous estimation of the precision of the fix would have been made.

The shifting of position lines by a constant amount (see Fig 9.10) either all towards, or all away from, their SSPs does not invalidate the solution of the unknowns, because the application of such a shift means nothing more than changing the size of the constant unknown Δh. The true value of Δh may be



Note. In this figure, the corrections V_1, V_3 are negative and V_2, V_4 positive

Fig.9.10 The Anomalous Case

found by taking the algebraic sum of the shift and the radius of the circle using the following conventions:

(a) The shift S is positive, if it is made in a direction towards the sub-stellar points.

(b) The radius R of the displaced circle is positive if the arrows of the displaced position lines point inwards.

(c) The residual V is positive, if the arrow of the displaced position line points towards the circumference of the displaced circle. Then

$$\Delta h = R + S$$

(see Figs 9.10 and 9.12 for further notes about signs).

Example of a Semi-Graphical Solution

9.44 The data for this example is the same as that used in sections 9.62, 9.63 and 9.81. The mean value of the intercepts and that of the computed azimuths for each star are used in this solution.

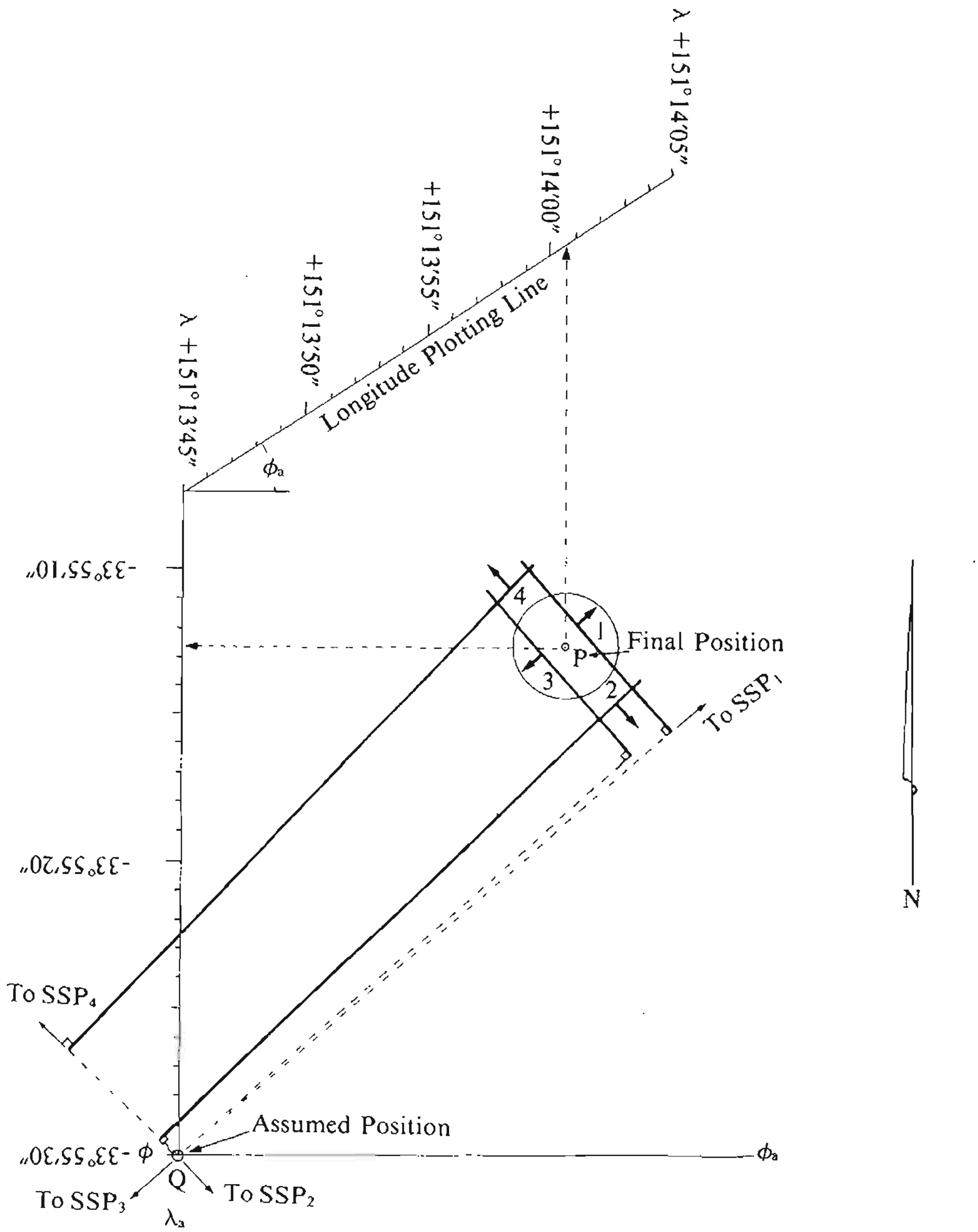


Fig.9.11

Preliminary Position ϕ_a $33^\circ 55'30''$ South

λ_a $10^h 04^m 55^s$ ($151^\circ 13'45''$) East

Plotting Data

Star	Aspect	Mean Intercept	Mean Calculated Azimuth
1	NE	+22.2"	48°42'
2	SE	- 0.8	136 12
3	SW	-20.6	228 16
4	NW	+ 5.2	313 10

Final Position ϕ $33^\circ 55'12.5''$ South

λ $10^h 04^m 56^s.05$ ($151^\circ 14'00''.7$) East

$\Delta h = -1.7''$ $v_1 = v_3 = +0.9''$

$v_2 = v_4 = -0.9''$

In Fig 9.11, use has been made of a semi-graphic technique, which should be familiar to those surveyors, who have used similar methods in triangulation breakdown procedures for minor order control etc. The method has the advantage of showing a clear picture, at a large scale, of the relationships between many quantities. Such relationships would be very hard to visualise, if they were given in numerical form only. The reader may reduce the data of section 9.12 as a further example. As with all semi-graphic procedures there is an analytical counterpart.

THE ANALYTICAL SOLUTION OF A ST.HILAIRE POSITION LINE FIX

The Derivation of the Analytical Relationships

9.51 The unknown position ϕ, λ of a station P is sought from a set of position line observations. An approximate position ϕ_a λ_a has been obtained for this station, where

$$\phi = \phi_a + \Delta\phi$$

and

$$\lambda = \lambda_a + \Delta\lambda$$

in which $\Delta\phi$ and $\Delta\lambda$ are small quantities, which now become two of the unknowns sought.

Using the above values of ϕ_a and λ_a, h_c and A_c are calculated from the Cosine Formula

$$\sin h_c = \sin \phi_a \sin \delta + \cos \phi_a \cos \delta \cos t_c$$

and the Four Parts Formula

$$\tan A_c = \frac{- \sin t_c}{\tan \delta \cos \phi_a - \sin \phi_a \cos t_c}$$

where $t_c = \lambda_a + GST_{obs} - RA$

Likewise the true local hour angle t with respect to the observer's meridian is given by

$$t = \lambda + GST_{obs} + v_T - RA$$

in which the GST of observation is shown to be subject to a small correction v_T.

$$\therefore \quad t = t_c + v_T + \Delta\lambda$$

In addition the true altitude

$$h = h_o + v_h + \Delta h \pm C$$

where the observed altitude is subject to a small correction v_h.

Δh is a small unknown systematic quantity affecting all measured altitudes and C is the theodolite index correction, whose sign will depend on the theodolite face used for that observation. The intercept I is defined by

$$I = h_o \pm C' - h_c$$

where C' is a close approximation of C and

$$C = C' + \Delta C$$

$$\therefore \quad h = h_c + I + v_h + \Delta h \pm \Delta C$$

One may now substitute for h, ϕ and t in the Cosine Formula

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t$$

to give

$$\sin(h_c + I + v_h + \Delta h \pm \Delta C) = \sin(\phi_a + \Delta\phi) \sin\delta + \cos(\phi_a + \Delta\phi) \cos\delta \cos(t_c + v_T + \Delta\lambda)$$

Expanding this expression by means of Taylor's theorem and retaining only first order terms gives

$$\sin h_c + (I + v_h + \Delta h \pm \Delta C)\cos h_c = \sin\phi_a \sin\delta + \Delta\phi \cos\phi_a \sin\delta + \cos\phi_a \cos\delta \cos t_c - \Delta\phi \sin\phi_a \cos\delta \cos t_c - (v_T + \Delta\lambda) \cos\phi_a \cos\delta \sin t_c$$

but from the Cosine Formula

$$\sin h_c = \sin\phi_a \sin\delta + \cos\phi_a \cos\delta \cos t_c$$

$$\therefore \quad (I + v_h + \Delta h \pm \Delta C) \cos h_c = \Delta\phi (\cos\phi_a \sin\delta - \sin\phi_a \cos\delta \cos t_c) - (v_T + \Delta\lambda) \cos\phi_a \cos\delta \sin t_c$$

From the Five Parts Formula

$$\cos A_c \cos h_c = \cos\phi_a \sin\delta - \sin\phi_a \cos\delta \cos t_c$$

$$\therefore \quad (I + v_h + \Delta h \pm \Delta C) \cos h_c = \Delta\phi \cos A_c \cos h_c - (v_T + \Delta\lambda) \cos\phi_a \cos\delta \sin t_c$$

Dividing by $\cos h_c$ and substituting from the sine formula

$$\sin A_c = \frac{-\cos\delta \sin t_c}{\cos h_c}$$

$$I + v_h + \Delta h \pm \Delta C = \Delta\phi \cos A_c + (v_T + \Delta\lambda) \cos\phi_a \sin A_c$$

or $$-\Delta h \pm \Delta C + D\lambda \sin A_c + \Delta\phi \cos A_c - I = v_h - v_T \cos\phi_a \sin A_c = v$$

where $$D\lambda = \Delta\lambda \cos\phi_a$$

This equation, involving all the unknown quantities sought, with the exception of the index correction, may now be reconciled with the semi-graphic treatment described previously.

From Fig 9.12 one may deduce the identical relationship.

Numerical Methods of Position Line Solution

9.61 Each observation to a star yields a position line from the intercept and azimuth calculated. For each set of observations, the mean of these intercepts and azimuths may be taken and used in calculating a single position line for each star. This procedure has the virtue of eliminating the effect of the unknown index correction, but it will not provide the means for determining estimates of precision. However, these estimates can, if required, be obtained by further treatment, as shown in section 9.72. The analytical relationship, for each set of observations, then simplifies to four equations

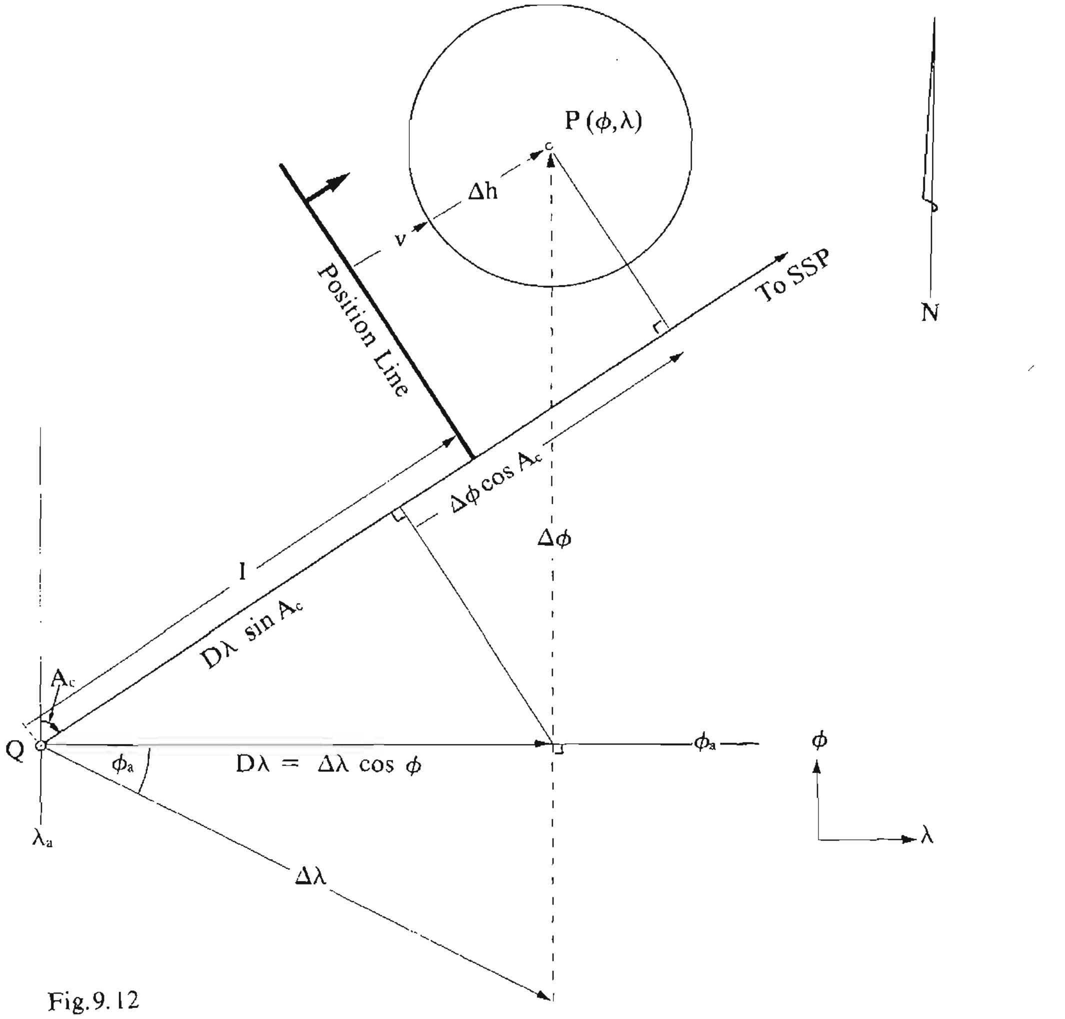


Fig.9.12

Note. The quantities I, Δφ, Δλ, Dλ, v and Δh, all shown with arrows are positive quantities in this figure

of the following form:-

$$- \Delta h + D\lambda \sin \bar{A}_c + \Delta\phi \cos \bar{A}_c - \bar{I} = V$$

in which $\bar{A}_c$ and $\bar{I}$ are the means of the individual values of the azimuth and the intercept and V is the required correction necessary to satisfy the equation.

If the four stars are chosen such that they are situated near the centre of each azimuth quadrant, the four equations of a set can be manipulated to give a solution in which each unknown is derived almost independently, one from the other. This can be done rigorously by means of a Least Squares solution or non-rigorously by means of an approximate method.

A Non-Rigorous Solution for a Position Line Fix

9.62 This will be best shown by means of an example. If the four stars are always set out in the same sequence of azimuth quadrants, irrespective of the time sequence of observation, a uniform method of calculation can be carried out by means of simple calculating aids.

The data is that of section 9.81 and the Observation or Parametric Equations, in detached coefficient form, are as follows:-

Δh	$D\lambda$	$\Delta\phi$	$-\bar{I}$	= 0
-1	sin 48°42'	cos 48°42'	-22.2"	...1
-1	sin 136°12'	cos 136°12'	+ 0.8	...2
-1	sin 228°16'	cos 228°16'	+20.6	...3
-1	sin 313°10'	cos 313°10'	- 5.2	...4
-1	+0.7513	+0.6600	-22.2	...1
-1	+0.6921	-0.7218	+ 0.8	...2
-1	-0.7463	-0.6657	+20.6	...3
-1	-0.7294	+0.6841	- 5.2	...4
-4	-0.0323	-0.0434	- 6.0	...5=1+2+3+4

Eliminating Δh gives

	+0.0592	+1.3818	-23.0	...1' = 1-2
	-0.0169	-1.3498	+25.8	...2' = 3-4
	+1.4807	-0.0241	-17.0	...3' = 1-4
	+1.4384	-0.0561	-19.8	...4' = 2-3
	+0.0761	+2.7316	-48.8	...1" = 1'-2'
	+2.9191	-0.0802	-36.8	...2" = 3'+4'
	-0.0761	+0.0021	+0.9594	...3" = 2" x $\frac{-0.0761}{2.9191}$
	+0.0761	+2.7316	-48.8	...1"

Back Solution

$$+2.7337 \ \Delta\phi \ -47.8406 = 0 \quad ...4" = 1"+3"$$

$$\Delta\phi = -\frac{-47.8406}{2.7337} = +17.5"$$

$$D\lambda = \frac{+36.8 + 0.0802 \times \Delta\phi}{2.9191} = +13.1"$$

From Equation 5 above, $$\Delta h = \frac{+6.0 + 0.0323\ D\lambda + 0.0434\ \Delta\phi}{-4} = -1.8"$$

$V_1 = +1.0"$ $\qquad V_2 = -1.0"$

$V_3 = +1.0$ $\qquad V_4 = -1.0$

Final Position $\phi = -33°55'30" + 17.5" = 33°55'12.5"$ South

$\lambda = +10^h04^m55^s + \frac{13.1}{15 \cos \phi_a} = 10^h04^m56.05^s$ East

The Least Squares Solution for a Position Line Fix

9.63 The parametric equations of section 9.62 are used for the following least squares solution.

Normal Equations:

Δh	$D\lambda$	$\Delta\phi$	$-\bar{I}$	= 0
4	+0.0322	+0.0433	+6.0000	= 0
	2.1323	-0.0059	-27.7044	= 0
		1.8677	-32.4996	= 0

Solution:

$$\Delta h = -1.8''$$

$$D\lambda = 13.1 \qquad \Delta\lambda = \frac{13.1''}{\cos\phi_a} = 15.8'' = 1.05^s$$

$$\Delta\phi = 17.5$$

$$V_1 = +1.0'' \qquad V_2 = -1.0'' \qquad V_3 = +1.0'' \qquad V_4 = -1.0''$$

$$\therefore \quad \phi = -33^\circ 55' 30'' + 17.5'' = -33^\circ 55' 12.5''$$

$$\lambda = +151\ 13\ 45 + 15.8 = +151\ 14\ 00.8$$

$$= +\ 10^h 04^m 56.05^s$$

9.64 The details of the solution of the normal equations etc. have been purposely omitted. With modern computing aids, intermediate steps in this type of calculation are seldom written down, because of the risk of making transcription mistakes. After the formation and solution of the normal equations and the determination of the V's, a useful check is provided as follows:-

$$\Sigma v_i = \Sigma v_i \sin A_i = \Sigma v_i \cos A_i = 0$$

One need not form and solve the normal equations if one is satisfied with other non-rigorous forms of numerical solution (the semi-graphic process is one). It will be seen that the normal equations for properly planned position line observations have dominant diagonal terms. This indicates that the unknowns have very little mathematical dependence and are therefore amenable to non-rigorous methods of solution.

The least squares solution given in section 9.63 has not taken into account the unknown index correction C, which should be included when the observations have been made with a theodolite (see sections 9.51 and 9.71). However if one of the specialised instruments or attachments, referred to in section 9.91, has been used, then the least squares solution of section 9.63 is appropriate.

The results of the example solved by all three methods (see sections 9.44, 9.62 and 9.63) are in remarkable agreement.

Weighting of the Least Squares Solution

9.65 So far, all observations have been treated as having equal weight in the least squares solution, a situation that arises only when certain conditions are imposed on the azimuth distribution of the stars to be observed.

The contribution of each star to the overall solution, i.e. weight, will depend upon the magnitude and propagation characteristics of the random errors of observation. In the case under consideration, random errors of observation occur in the timing and measurement of altitudes. If the relative or absolute estimates of the variances of these measurements, σ_h^2 and σ_T^2, are known, say from previous experience, their combined effect may be found from examining the v term in the analytical solution, where

$$v = v_h - v_T \cos\phi_a \sin A_c$$

Provided the variances are uncorrelated, which is generally considered to be the case,

$$\sigma_v^2 = \sigma_h^2 + \sigma_T^2 \cos^2\phi_a \sin^2 A_c$$

and, as weight W is inversely proportional to the variance, individual weights W may be assigned to the parametric equations, where W is proportional to

$$\frac{1}{\sigma_h^2 + \sigma_T^2 \cos^2\phi_a \sin^2 A_c}$$

The Azimith Distribution of Stars for a Set of Position Lines

9.66 So far, the restrictions imposed on the selection of stars in a set should be that each star should be observed at about the same altitude and that the observations be made over as short a time interval as possible. If six observations on each face to each of four stars are made, this should be accomplished in about an hour. These considerations will now be used as a basis for further examination of how these stars should be distributed in azimuth.

It has been demonstrated, both graphically and analytically, that position lines intersecting at right angles give a clear indication of where the final position of P should be located, i.e. stars should have azimuths of A, $A+90^\circ$, $A+180^\circ$ and $A+270^\circ$, where A may have any value. Only two cases will be considered in detail, because the remaining cases lead to complications involving a prior knowledge of relative weights.

Cardinal Position Lines

9.67 For this case, the basic azimuth value A is taken to be zero, so that the stars are observed near the local meridian north and south and near the prime vertical east and west. The parametric equations will be of the form

	Δh	$D\lambda$	$\Delta\phi$	$-\bar{I}$	= 0	Weight
N	-1	0	1	$-\bar{I}_N$		W_M
S	-1	0	-1	$-\bar{I}_S$		W_M
E	-1	1	0	$-\bar{I}_E$		W_P
W	-1	-1	0	$-\bar{I}_W$		W_P

and the normal equations

Δh	$D\lambda$	$\Delta\phi$	Absolute Term	= 0
$2(W_M + W_P)$	0	0	$W_M(\bar{I}_N+\bar{I}_S)+W_P(\bar{I}_E+\bar{I}_W)$	
	$2W_P$	0	$W_P(\bar{I}_W-\bar{I}_E)$	
		$2W_M$	$W_M(\bar{I}_S-\bar{I}_N)$	

where W_M and W_P are the weights of observations made in the meridian and prime vertical respectively.

From the structure of the normal equations, it is clear that $D\lambda$ and $\Delta\phi$ may be solved independently of one another and also independently of the assigned weights. However, Δh cannot be found, unless the relative values of the weights are known. This latter aspect is not of great concern, because $D\lambda$ and $\Delta\phi$ are the principal unknowns sought and one has only a marginal interest in the value of Δh. Furthermore, there is little to recommend making observations in this way when one can predict, observe and compute pairs of stars quite independently for latitude and longitude without having the additional restrictions that all four stars should be observed in quick succession and all four at nearly the same altitudes.

Mid-quadrant Position Lines

9.68 For this case, the basic azimuth value A is taken to be 45°. From an examination of the expression for the weight W of section 9.65, it will be seen that for this situation, equal weights may be assigned to the parametric equations, which will be of the form

	Δh	$D\lambda$	$\Delta\phi$	$-\bar{I}$	$= 0$
NE	-1	k	k	$-\bar{I}_{NE}$	
SE	-1	k	-k	$-\bar{I}_{SE}$	equal weights
SW	-1	-k	-k	$-\bar{I}_{SW}$	
NW	-1	-k	k	$-\bar{I}_{NW}$	

where $k = \frac{\sqrt{2}}{2}$

and the normal equations

Δh	$D\lambda$	$\Delta\phi$	Absolute Term $= 0$
4	0	0	$(\bar{I}_{NE}+\bar{I}_{SE}+\bar{I}_{SW}+\bar{I}_{NW})$
	2	0	$\frac{\sqrt{2}}{2}(-\bar{I}_{NW}-\bar{I}_{SE}+\bar{I}_{SW}+\bar{I}_{NW})$
		2	$\frac{\sqrt{2}}{2}(-\bar{I}_{NE}+\bar{I}_{SE}+\bar{I}_{SW}-\bar{I}_{NW})$

The structure of these normal equations is ideal, because each unknown may be solved for quite independently of the others, with the additional advantage that one need not know the relative weighting. Also $\Delta\phi$ and $D\lambda$ are determined with equal precision, i.e. the uncertainty of the position of P on the ground is the same in any direction and may be represented by an error circle.

Comparison of Cardinal and Mid-quadrant Position Lines

9.69 The question now is which of the two methods selected is the better. The answer to this is not absolutely clear unless one can make *reliable* estimates of the relative sizes of the random errors involved in the measurements, which will largely depend upon the skills of individual observers and the type of equipment used.

The expression for the weight W shows that, in the first method, the latitude observations are subject to random errors of altitude measurement but free of those of time measurement and have a weight inversely proportional to σ_h^2, whereas the longitude observations are subject to errors in both altitude and time measurement and have a weight inversely proportional to $(\sigma_h^2+\sigma_T^2 \cos^2\phi)$. Thus the latitude is obtained with a higher precision than the longitude. Each component is obtained independently from two stars.

In the second method, however, each star contributes information for the solution of both components, which have the same precision because each equation has the same weight, which lies between the two values above and is inversely proportional to $(\sigma_h^2+\sigma_T^2 \cos^2\phi \sin^2 45)$.

The Least Squares Solution including Vertical Index Correction

9.71 The complete parametric equation for a position line observation,

$$-\Delta h \pm \Delta C + D\lambda \sin A_C + \Delta\phi \cos A_C - I = v,$$

and its application to 2n observations on each of four stars should now be considered. Half of the observations on each star are made in the face left and the other half in the face right position. In all, there are N=8n observations. A representative set of such equations for one star is as follows,

	Δh	ΔC	$D\lambda$	$\Delta\phi$	Absolute term $= 0$	
CL	-1	1	$\sin A_c$	$\cos A_c$	$-I_{L1}$	n
	.	.	.	.	.	equations
	.	.	.	.	.	
	-1	1	$\sin A_c$	$\cos A_c$	$-I_{Ln}$	

	Δh	ΔC	ΔD	$\Delta\phi$	Absolute term = 0	
CR	-1	-1	$\sin A_C$	$\cos A_C$	$-I_{R_1}$	
	.	.	.	.	.	n equations
	.	.	.	.	.	
	-1	-1	$\sin A_C$	$\cos A_C$	$-I_{R_n}$	

in which a common value of A_C, the mean, has been used. This is a reasonable approximation provided the observations are made in a short period of time. The normal equations resulting from four such sets, regarded as having equal weight and each being an observation near a quadrant centre, are

Δh	ΔC	$D\lambda$	$\Delta\phi$	Absolute term = 0
N	0	$-[\sin A_C]$	$-[\cos A_C]$	$+[I]$
	N	0	0	$[I_R] - [I_L]$
		$[\sin^2 A_C]$	$[\sin A_C \cos A_C]$	$-[I \sin A_C]$
			$[\cos^2 A_C]$	$-[I \cos A_C]$

where the square brackets indicate summation. It will be noticed that the ΔC term can be excluded from the normal equations and be solved for, *exactly*, from

$$\Delta C = \frac{[I_L] - [I_R]}{N}$$

The three remaining normal equations may now be solved in exactly the same way as given in section 9.63. As observations are confined to the vicinity of the quadrant centres, the terms $[\sin A_C]$, $[\cos A_C]$ and $[\sin A_C \cos A_C]$ are very small and the inverse of the matrix of normal equations will be similar in character to the matrix of normal equations i.e. each has dominant diagonal terms. The reciprocal values of the diagonal terms of the matrix of normal equations are the corresponding diagonal terms of its inverse. Thus, one can write simple expressions for the estimates of precision of the unknowns, as follows

Standard Deviation of a single observation $\sigma_o = \sqrt{\frac{\Sigma vv}{N-4}}$

$$\sigma_{\Delta\phi} = \sigma_{D\lambda} = \sigma_{\Delta\lambda\cos\phi_a} = \sqrt{\frac{2\Sigma vv}{N(N-4)}} \quad \text{and} \quad \sigma_{\Delta h} = \sigma_{\Delta C} = \sqrt{\frac{\Sigma vv}{N(N-4)}}$$

because

$$[\sin^2 A_C] \simeq [\cos^2 A_C] \simeq \frac{N}{2}$$

The adjusted values of the unknowns may now be substituted back into each parametric equation to obtain values of v for the calculation of precision.

Alternative Method of Solution from Mean Values $\bar{I}$ of the Intercepts

9.72 The tempting expedient of taking the means of observed altitudes and times and calculating results from these treated as a single observation can be used. This, however, is dangerous, because the relationships used in this and other types of calculation are not linear ones and to preserve accuracy, higher order effects must be allowed for (see section A.61). In addition, mistakes may be hidden in the means and go undiscovered. It is, therefore, a recommended practice, in this and other calculations, to derive results from individual observations which, for the method under consideration, will be intercepts and azimuths. It is also preferable to use rigorous adjustment procedures with appropriate checks.

The rigorous Least Squares solution by means of modern calculating aids is not an arduous task. Nowadays it would be most uncommon to find a surveyor, who does not have an electronic calculator for his sole use, and a large proportion of these calculators would have programmable features of some kind.

If the same number of multiple observations has been made on each star, equally divided between each face and the stars have been observed in accordance with the principles previously established, a calculation technique is available for making a preliminary analysis of the intercepts and, with a little extra manipulation, estimates of precision of the unknowns can also be found, whilst the full rigour of the Least Squares solution is maintained. The mean values of the intercepts for each star on each face are first calculated. An examination of the differences, $u = \bar{I} - I$, will give the opportunity of making a preliminary assessment of the consistency of the observations and of also revealing the presence of a mistaken observation or calculation. The elementary check $\Sigma u = 0$ should be made.

The unknown ΔC may now be conveniently calculated from a consideration of the following,

$$\Delta C = \frac{\Sigma I_L - \Sigma I_R}{N} \qquad \text{(see section 9.71)}$$

$$\Delta C = \frac{\Sigma \bar{I}_L - \Sigma \bar{I}_R}{8}$$

$$\Delta C = \frac{\Delta C_1 + \Delta C_2 + \Delta C_3 + \Delta C_4}{4}$$

where $\Delta C_1 = \dfrac{\bar{I}_L - \bar{I}_R}{2}$ for Star No. 1 etc.

The next stage of the process is to solve for the unknowns $D\lambda$, $\Delta\phi$, Δh and the V's by any one of the methods previously described.

9.73 All that now remains, is the calculation of the individual v's, and for this process one can take advantage of the arithmetic already completed for the calculation of the u's and the ΔC's.

For a single observation in face left on one star

$$-\Delta h + \Delta C + D\lambda \sin A_C + \Delta\phi \cos A_C - I_L = v_L$$

and for all observations on that star in both face left and face right

$$-\Delta h + D\lambda \sin A_C + \Delta\phi \cos A_C - \left(\frac{\bar{I}_L + \bar{I}_R}{2}\right) = V_1$$

The difference between these equations is

$$\Delta C + \frac{\bar{I}_L + \bar{I}_R}{2} - I_L + V_1 = v_L$$

but $\Delta C_1 = \dfrac{\bar{I}_L - \bar{I}_R}{2}$ and $u_L = \bar{I}_L - I_L$

$$\therefore \quad u_L - \Delta C_1 = \frac{\bar{I}_L + \bar{I}_R}{2} - I_L$$

$$\therefore \quad \Delta C - \Delta C_1 + u_L + V_1 = v_L$$

and, for an observation on face right, it may be similarly proved that

$$\Delta C_1 - \Delta C + u_R + V_1 = v_R$$

with identical expressions for the other stars in the set.

All the quantities on the LHS of the equations for v_L and v_R have been previously calculated and therefore it is a simple matter to complete the calculation for the v's and thence the estimates of precision of the unknowns.

9.74 It is apparent that the success of this calculation technique depends upon the balance of numbers of face left and face right observations on both individual stars and all stars in the set. Occasionally it may happen, such as when a mistaken observation must be rejected, that this balance is disturbed. Under these circumstances, it would seem that there is no choice other than to perform a Least Squares solution, based on a set of parametric equations containing one equation for each observation, if rigour is to be maintained. This will require a solution of 4 normal equations, because now all terms are non-zero in the matrix of normal equations and therefore one cannot solve for ΔC independently of the other unknowns.

If one is satisfied with a solution, which is only *minutely different* from the rigorous one, then *the same technique* as that outlined in section 9.72 may be used. The justification for this is based on the fact that when Least Squares adjustments are made using equations of different, but not markedly dissimilar, weights, the difference between solutions using equal and unequal weights is seldom of practical significance. In the technique outlined previously, the 4 parametric equations which had been derived from the mean of the intercepts (and the mean azimuth) from each star, had weights of $2n$. In the example given, $2n = 12$, and if, for example, one observation had been rejected, the relative weights would be 11, 12, 12 and 12, which are not markedly different from equal weights.

Practical Considerations

9.75 It will be appreciated that the quality of the determination of position by the position line method of observation depends to a great extent upon how well the planning and execution of the observations have been made.

If only rough determinations are needed, e.g. the navigation of an expedition in unmapped or featureless country, then one may be content with a few observations made to readily identifiable bright stars, selected in positions such that the position lines give reasonable intersections. As the ideal conditions for selection of stars are relaxed, then so must the precision and accuracy of the determination of position decrease.

Therefore, if the best available accuracy is required, each set of four stars should be predicted for observation near the mid quadrant positions at nearly equal altitudes within about an hour. These observations can be made by means of an astrolabe (see section 9.91) or a theodolite. If a theodolite is to be used, then this position line method must be compared with the methods of Chapters 5 and 6 for determining latitude and longitude independently of each other. Such stars should likewise be predicted. The prediction for a pair of longitude stars in the vicinity of the prime vertical is similar to that for the position line stars, but a pair of latitude stars at transit is much more easily predicted. Also these two pairs do not have to be observed within say an hour, because it is only necessary to observe the members of each pair with an interval of less than half an hour between them.

The method of independent determination of the two elements, latitude and longitude, by means of the same techniques and under the same circumstances gives results of different precision for these elements. This agrees with the weighting theory and shows up clearly in practice. If the same techniques are used under the same circumstances for the simultaneous determination of latitude and longitude from position lines, derived from observations to four mid-quadrant stars, the precisions obtained for the two elements are equal, because the weights for each star are equal. This precision should lie between the two obtained in the method of independent determination.

9.81 Example of a Set of Position Line Observations

Station	Pillar 5, University of NSW	Approx. Latitude	$\phi = 33^{\circ}55'30''$ S
Date	Wed. evening, 29th Jan. 1975	Approx. Longitude	$\lambda = 10^{h}04^{m}55^{s}$ E
Observer	G.G. Bennett	Theodolite	Wild T2 No. 148423
Recorder	J.G. Freislich	Watch	Heuer No. 23 (Mean Time)
Signal	VNG DUT1 = $+0.6^{s}$	Temperature	$19^{\circ}C$ Pressure 1020 mb
Zone	11^{h} E	R_6 for date	$= 8^{h}31^{m}28.9^{s}$

Star Observations

Star No. 198 (NE) RA $7^h10^m37.8^s$ δ $00^\circ27'12''$ S
Watch Correction $+18^h40^m05.4^s$

Watch (CL)	V⊙R (CL)	H⊙R	Watch (CR)	V⊙R (CR)
$2^h39^m47.6^s$	$45^\circ47'35''$		$2^h46^m05.6^s$	$315^\circ12'38''$
40 50.3	37 33		46 58.7	20 56
42 05.8	25 30	CL	47 34.8	26 26
42 47.2	19 01		48 13.3	32 23
43 43.8	10 03		49 02.8	40 05
2 44 22.9	45 03 55	$48^\circ55'$	2 49 39.6	315 45 43

Star No. 258 (SE) RA $9^h21^m23.6^s$ δ $54^\circ54'21''$ S
Watch Correction $+18^h40^m06.1^s$

Watch (CR)	V⊙R (CR)	H⊙R	Watch (CL)	V⊙R (CL)
$3^h05^m51.5^s$	$314^\circ18'45''$		$3^h10^m29.6^s$	$45^\circ01'50''$
06 27.7	23 57		11 02.7	44 57 02
07 02.8	29 00	CR	11 42.0	51 23
07 41.1	34 34		12 18.9	46 01
08 18.2	39 50		12 53.2	41 05
3 08 55.9	314 45 18	$316^\circ13'$	3 13 35.2	44 35 03

Star No. 82 (NW) RA $3^h35^m37.4^s$ δ $00^\circ19'21''$ N
Watch Correction $+18^h40^m06.5^s$

Watch (CL)	V⊙R (CL)	H⊙R	Watch (CR)	V⊙R (CR)
$3^h18^m18.7^s$	$44^\circ18'38''$		$3^h23^m02.7^s$	$314^\circ59'12''$
19 01.9	25 01		23 46.0	52 37
19 32.3	29 35	CL	24 31.0	45 45
20 04.7	34 24		24 59.3	41 24
20 55.4	42 02		25 40.3	35 04
3 21 27.6	44 46 57	$313^\circ15'$	3 26 03.8	314 31 24

Star No. 40 (SW) RA $1^h54^m59.6^s$ δ $51^\circ44'11''$ S
Watch Correction $+18^h40^m06.8^s$

Watch (CR)	V⊙R (CR)	H⊙R	Watch (CL)	V⊙R (CL)
$3^h27^m36.7^s$	$315^\circ36'18''$		$3^h32^m06.9^s$	$45^\circ06'14''$
28 05.2	31 53		32 36.3	10 44
28 36.8	26 56	CR	33 13.7	16 34
29 09.3	21 55		33 44.2	21 21
29 41.5	16 53		34 26.7	27 56
3 30 08.2	315 12 47	$48^\circ20'$	3 34 56.1	45 32 29

Reduction in full of the first Observation of each Star:

Star No.	198	258	82	40	
Quadrant	NE	SE	NW	SW	
Observed Watch	$2^h39^m47.6^s$	$3^h05^m51.5^s$	$3^h18^m18.7^s$	$3^h27^m36.7^s$	
Watch Corrn	18 40 05.4	18 40 06.1	18 40 06.5	18 40 06.8	
Corrected Watch	21 19 53.0	21 45 57.6	21 58 25.2	22 07 43.5	
Zone	11 E	11 E	11 E	11 E	
U T	10 19 53.0	10 45 57.6	10 58 25.2	11 07 43.5	
R_6	8 31 28.9	8 31 28.9	8 31 28.9	8 31 28.9	
dR	42.7	47.0	49.0	50.6	
λ_a	10 04 55.0E	10 04 55.0E	10 04 55.0E	10 04 55.0 E	
LST_a	4 56 59.6	5 23 08.5	5 35 38.1	5 44 58.0	
RA	7 10 37.8	9 21 23.6	3 35 37.4	1 54 59.6	
t_c	21 46 21.8	20 01 44.9	2 00 00.7	3 49 58.4	
δ	$-0^\circ27'12''$	$-54^\circ54'21''$	$0^\circ19'21''$	$-51^\circ44'11''$	
ϕ_a	-33 55 30	-33 55 30	-33 55 30	-33 55 30	
Calculated z	45 48 33	45 42 28	44 19 18	44 24 34	(1)
Calculated A	50 09 55	136 10 15	314 18 04	228 16 19	(2)
LHS Five Parts	0.834 734	0.291 253	0.865 986	0.332 800	(3)
RHS Five Parts	0.834 735	0.291 252	0.865 987	0.332 801	(3)
Observed z	$45^\circ47'35''$	$45^\circ41'15''$	$44^\circ18'38''$	$44^\circ23'42''$	
T = 19 P = 1020 f	0.97	0.97	0.97	0.97	
r_o r	60 58	60 58	57 55	57 55	
Corrected z Intercept	45 48 33 OA	45 42 13 15T	44 19 33 15A	44 24 37 3A	(4)

Refs: (1) $\cos z_c = \sin\phi_a \sin\delta + \cos\phi_a \cos\delta \cos t_c$

(2) $$\tan A_c = \frac{-\sin t_c}{\tan\delta \cos\phi_a - \sin\phi_a \cos t_c}$$

The azimuth of the star is only required for plotting purposes or for evaluating the coefficients of the unknowns in a least squares solution and therefore a low accuracy (0°1 or 3 decimal places) is quite sufficient. However in order to provide a check on the accurate calculation of the zenith distance it will be necessary to calculate the azimuth accurately.

(3) Check $\cos\delta \cos t_c = \cos z_c \cos\phi_a - \sin z_c \sin\phi_a \cos A_c$ (Five Parts)

(4) Intercept = Calculated z - Observed z
= Observed h - Calculated h

positive, towards (T); negative, away (A)

No. 198 (NE)						No. 258 (SE)					
I_L	u	v	I_R	u	v	I_L	u	v	I_R	u	v
+0.2"	+4.7"	+6.1"	+38.3"	+1.1"	+1.7"	-20.8"	+3.5"	+2.1"	+15.7"	-0.1"	-0.7"
+2.8	+2.1	+3.5	+41.6	-2.2	-1.6	-18.5	+1.2	-0.2	+15.0	+0.6	0
+7.5	-2.6	-1.2	+36.4	+3.0	+3.6	-18.7	+1.4	0	+14.9	+0.7	+0.1
+4.1	+0.8	+2.2	+36.9	+2.5	+3.1	-15.2	-2.1	-3.5	+18.1	-2.5	-3.1
+7.5	-2.6	-1.2	+42.0	-2.6	-2.0	-15.1	-2.2	-3.6	+13.7	+1.9	+1.3
+7.3	-2.4	-1.0	+41.5	-2.1	-1.5	-15.5	-1.8	-3.2	+16.2	-0.6	-1.2
$\bar{I}_L$	Σ	Σ	$\bar{I}_R$	Σ	Σ	$\bar{I}_L$	Σ	Σ	$\bar{I}_R$	Σ	Σ
+4.9	0✓	+8.4✓	+39.4	-0.3✓	+3.3✓	-17.3	0✓	-8.4✓	+15.6	0✓	-3.6✓

No. 198 (NE):

Intercept $= \frac{\bar{I}_L + \bar{I}_R}{2} = +22''.2, \bar{A} = 48°42'$

$\Delta C - \Delta C_1 = +0.4''$ $\quad\Delta C_1 - \Delta C = -0.4''$

$V_1 = +1.0$ $\quad V_1 = +1.0$

Sum $\Delta u = +1.4$ $\quad \Delta u = +0.6$

No. 258 (SE):

Intercept $= \frac{\bar{I}_L + \bar{I}_R}{2} = -0''.8, \bar{A} = 136°12'$

$\Delta C - \Delta C_2 = -0.4''$ $\quad\Delta C_2 - \Delta C = +0.4''$

$V_2 = -1.0$ $\quad V_2 = -1.0$

Sum $\Delta u = -1.4$ $\quad \Delta u = -0.6$

No. 82 (NW)						No. 40 (SW)					
I_L	u	v	I_R	u	v	I_L	u	v	I_R	u	v
-13.2"	+2.3"	+0.5"	+21.2"	0.0"	-0.2"	-38.7"	+0.7"	+2.3"	-2.1"	-1.1"	-0.7"
- 9.9	-1.0	-2.8	+22.2	-1.0	-1.2	-35.0	-3.0	-1.4	-1.8	-1.4	-1.0
-11.2	+0.3	-1.5	+23.1	-1.9	-2.1	-36.8	-1.2	+0.4	-4.6	+1.4	+1.8
- 8.8	-2.1	-3.9	+22.4	-1.2	-1.4	-39.9	+1.9	+3.5	-3.0	-0.2	+0.2
- 9.4	-1.5	-3.3	+20.6	+0.6	+0.4	-39.3	+1.3	+2.9	-5.3	+2.1	+2.5
-12.9	+2.0	+0.2	+17.9	+3.3	+3.1	-38.6	+0.6	+2.2	-2.7	-0.5	-0.1
$\bar{I}_L$	Σ	Σ	$\bar{I}_R$	Σ	Σ	$\bar{I}_L$	Σ	Σ	$\bar{I}_R$	Σ	Σ
-10.9	0	-10.8✓	+21.2	-0.2✓	-1.4✓	-38.0	+0.3✓	+9.9✓	-3.2	+0.3✓	+2.7

No. 82 (NW):

Intercept $= \frac{\bar{I}_L + \bar{I}_R}{2} = +5''.2, \bar{A} = 313°10'$

$\Delta C - \Delta C_3 = -0.8''$ $\quad\Delta C_3 - \Delta C = +0.8''$

$V_3 = -1.0$ $\quad V_3 = -1.0$

Sum $\Delta u = -1.8$ $\quad \Delta u = -0.2$

No. 40 (SW):

Intercept $= \frac{\bar{I}_L + \bar{I}_R}{2} = -20''.6, \bar{A} = 228°16'$

$\Delta C - \Delta C_4 = +0.6''$ $\quad\Delta C_4 - \Delta C = -0.6''$

$V_4 = +1.0$ $\quad V_4 = +1.0$

Sum $\Delta u = +1.6$ $\quad \Delta u = +0.4$

Note: $u + \Delta u = v$ checks: $\Sigma u \simeq 0$ & $\Sigma v \simeq n\Delta u$ (in this case n=6)

Calculation of Collimation Correction:

$\Delta C_1 = \frac{\bar{I}_L - \bar{I}_R}{2} = \frac{4''.9 - 39''.4}{2} = -17.2''$

$\Delta C_2 = \frac{-17.3-15.6}{2} = -16.4$

$\Delta C_3 = \frac{-10.9-21.2}{2} = -16.0$

$\Delta C_4 = \frac{-38.0 + 3.2}{2} = -17.4$

$\Delta C = \frac{\Delta C_1 + \Delta C_2 + \Delta C_3 + \Delta C_4}{4} = -16.8$

$\Delta C - C_1 = +0.4''$

$\Delta C - C_2 = -0.4$

$\Delta C - C_3 = -0.8$

$\Delta C - C_4 = +0.6$

Check $\simeq 0$ Σ -0.2 ✓

Δh, ϕ and λ may be solved by any one of the methods described in sections 9.44, 9.62 and 9.63, in which the star sequence numbers 3 and 4 are the reverse of those used in this section.

$$\Delta h = -1.8''$$

$$\phi = -33^{\circ}55'12.5'' \qquad C' = 0$$

$$\lambda = +10^{h}04^{m}56.05^{s} \qquad \therefore \quad C = \Delta C = -16.8''$$

$$\Sigma v^2 = 243.63 \qquad N = 48$$

$$\sigma_{\Delta\phi} = \sigma_{\Delta\lambda\cos\phi_a} = \sqrt{\frac{2\Sigma vv}{N(N-4)}} = \pm 0.48''$$

$$\sigma_{\Delta C} = \sigma_{\Delta h} = \sqrt{\frac{\Sigma vv}{N(N-4)}} = \pm 0.34''$$

EQUAL ALTITUDE OBSERVATIONS FOR POSITION LINES

THE initiative for the development of specialised instrumentation and observing methods for this work was given by Gauss, who devised a procedure in which stars are observed at a fixed altitude for the simultaneous determination of latitude and longitude. The principal advantage of this method is that the value of the fixed altitude is *not measured* but may be solved for together with the station's latitude and longitude, *provided* the altitude remains *constant* during the course of the observing period. A theodolite would serve this purpose, but any variations in this altitude over the observing period would have to be corrected or allowed for by readings or settings of the vertical circle and, when fitted, the altitude bubble.

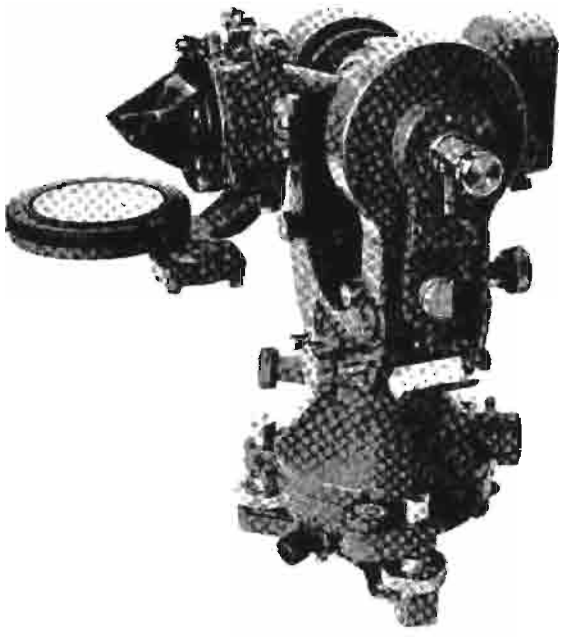
The Astrolabe

9.91 The special instruments used for this work are generally called astrolabes, which may be reserved entirely for this purpose or may consist of an attachment to a theodolite or level. The first of such instruments, the 60° prismatic astrolabe, was developed by MM. Claude and Driencourt of the French Bureau of Longitudes and Hydrographic Department of the French Navy respectively. An equilateral prism is mounted in front of the objective of a horizontal telescope with the rear face of the prism at right angles to the optical axis of the telescope. Two images of a star are formed, one directly through the prism and the other via a mercury pool placed beneath the prism.

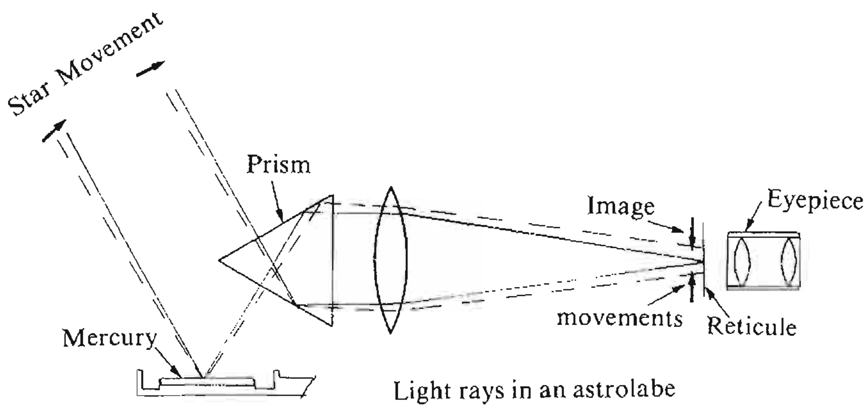
The two images are seen moving in opposite directions in the eyepiece and when the images are in line horizontally, time is recorded and the star is at an altitude equal to that of the apex angle of the prism. The success of the technique depends upon the fact that this altitude is unaffected by small variations in the horizontality of the telescope. The prism and mercury receptacle have been designed as an objective attachment to a theodolite. Fig 9.13 shows this attachment for a Wild T2 theodolite.

9.92 A variation of this original form, the 45° prismatic astrolabe, was developed by Captain T.Y. Baker of the British Navy in 1930. In this instrument, a pentagonal prism is used to create a fixed altitude of 45°. The advantage of such an instrument, when compared with the previous one, is that more stars cross the 45° almucantar than the 60° almucantar. This makes it possible to observe a greater number of stars on the 45° circle of equal altitude. A weak duplicating prism is mounted above half the surface of the pentagonal prism to duplicate the direct image of the star laterally and the observational instant occurs when the reflected image is in line with the two direct images. In addition, a series of deflecting prisms is placed in turn in the path of the reflected image. In this way, multiple observations are obtained with a significant increase in the precision of the results obtained from observation of a star.

Mercury is not an ideal substance to use under field conditions. In the



Wild astrolabe attachment on a T2 theodolite



Light rays in an astrolabe

Fig.9.13

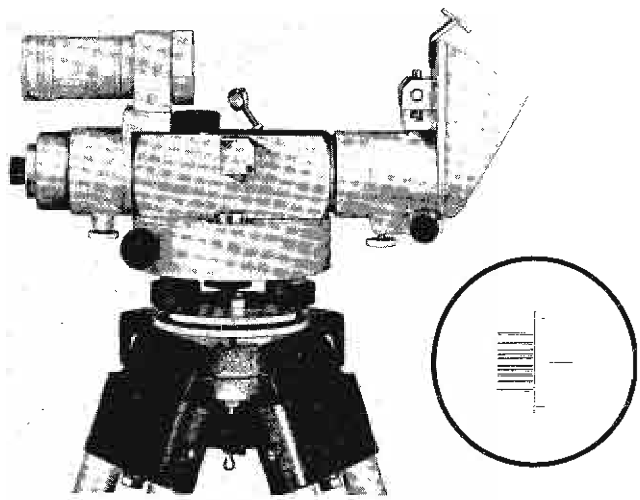
60° and 45° prismatic astrolabes described above, the mercury pool is formed on the surface of an amalgamated copper plate. It is essential that the mercury surface be kept absolutely clean in order to provide a bright reflected image. Also the thickness of the mercury layer is critical; too thick a layer is sensitive to small vibrations,whilst too thin a layer may form a surface, which is not truly horizontal.

9.93 Another form of astrolabe, which uses a different gravity dependent device, an essential feature of all astrolabes, is the pendulum astrolabe. In this instrument, which was invented by the American astronomer Willis, stars are viewed at a nominal altitude of 60°. Compensation for small dislevelments of the instrument is effected by a horizontal metallic mirror attached to an air-damped pendulum. The mirror is interposed between the objective and eyepiece of the telescope thus requiring the optics to be "broken" at the point of reflection.

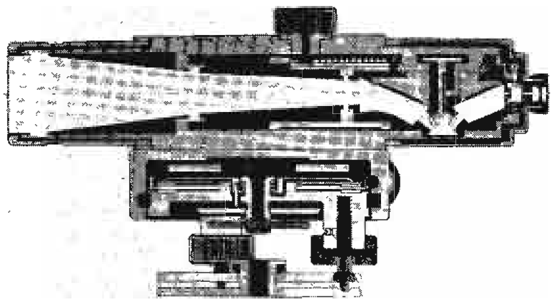
Only one image of the star is formed and this is observed over a series of reticule lines. The pendulum astrolabe may be said to be the forerunner of instruments, incorporating dislevelment compensating devices, which are in almost universal use in modern theodolites and levels.

9.94 In 1950 Zeiss (Oberkochen) devised a level, in which the line of sight is automatically corrected for small dislevelments of the instrument. Essentially the compensator consists of three prisms, two of which are fixed and the third is suspended by four wires. The accuracy of compensation of this device is exceptionally high and thus the instrument was suitable for adaptation as an astrolabe.

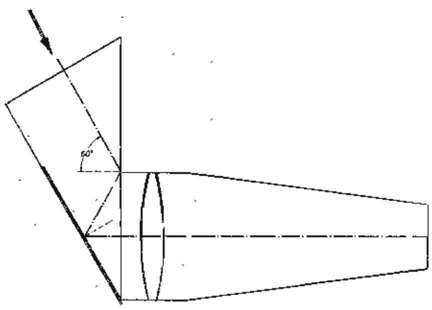
The light from the star is deviated through 60° by a right angle objective



General view with astrolabe reticule in the telescope's field of view also shown.



Light path through Ni2 Level.



Light path through astrolabe prism.

Fig. 9.14 The Zeiss Ni2 Level Astrolabe

prism to provide a near horizontal ray; the constancy of this deviation is maintained as a result of the principle of double reflection. Ten observations are made on each star. The time is observed as the single image of the star reaches the midpoint of each one of the set of double reticule lines (see Fig 9.14).

This type of astrolabe combines features of both the prismatic and pendulum instruments previously described. The principal advantage of this astrolabe is that the basic instrument may be used for its primary purpose, i.e. levelling, when not being used for astronomical observations.

Observations made by means of the astrolabe are much simpler than those made by means of the theodolite, mainly because there are no circle or bubble readings to be taken and only a relatively coarse levelling procedure is necessary. On the other hand, the theodolite may be used for this and other surveying operations, whereas the astrolabe is a single purpose instrument. The accuracy claimed for astrolabe determinations is superior to that obtained by means of a theodolite of comparable optics.

The above list of astrolabes is not an exhaustive list of all types, but the principal features of these special instruments have been highlighted. Details may be obtained from manufacturers' handbooks.

The Reduction of Astrolabe Observations

9.95 The technique of reduction for the astrolabe is very similar to that used for theodolite observations (see section 9.81). When only a single observation has been taken, as with the 60° astrolabe, the computations are identical; the value of the observed altitude is the best known value of the apex angle of the prism, corrected for refraction. This may be conveniently determined from a trial set of observations.

When multiple observations have been made on a star by means of either deflecting prisms, as with the 45° astrolabe, or a series of reticule lines, then several methods of reduction are possible.

If all reticule lines or all deflecting prisms have been used on all stars, the mean of the clock times for individual stars can be used to calculate intercepts. However, before calculating intercepts it will be necessary to apply a second order correction to *either* these means *or* the value of the fixed altitude (see section A.61). It is tempting to assume that the effects of the second order correction will be eliminated from such a series of observations, especially if the stars in a set are symmetrically disposed in azimuth. However, this is not the case, because the correction for stars on the pole side of the prime vertical is significantly different for stars on the equatorial side. To illustrate this point, the second order corrections to the means of the clock times for the stars shown in section 9.97 have been calculated as follows:-

Star No. (Aspect)	FK604 (SW)	FK761 (NE)	FK1461 (NW)	FK796 (SE)
Correction	-0.03^{s}	$+0.08^{s}$	-0.13^{s}	$+0.06^{s}$

This correction, in seconds of time, is given by

$$\frac{1}{2n\rho} \cdot \frac{\Sigma\Delta h^2}{15} \cdot \frac{d^2t}{dh^2}$$

where
$$\frac{d^2t}{dh^2} = -\sec\phi \cot A \operatorname{cosec} A(\tan\phi \operatorname{cosec} A - \tan h \cot A)$$
$$= -\frac{dt}{dh}\left(\cot t \frac{dt}{dh} + \tan h\right)$$

where n is the number of observations,
Δh is the altitude difference between the observed reticule line and the reticule centre,
and Δh and ρ are in seconds of arc and for a full set of observations $\Sigma\Delta h^2$ is a constant quantity.

It will be seen that one needs the individual values of Δh for the evaluation of these corrections. More often than not these intervals are known, but if this is not the case they can be derived, with sufficient accuracy for this purpose, from the observations themselves from the time rate of change of altitude as follows,

$$\Delta h = \Delta T \cos\phi \sin A$$

where ΔT is the time interval corresponding to Δh.

The final assessment of the three unknowns $\Delta\phi$, $\Delta\lambda$ & Δh is very unreliable if only the mean values of the intercepts are used, especially if a small number of stars has been observed, because the estimates of precision are based on the V and not on the v quantities. In fact one can obtain what appears to be a perfect solution with each $V=0$ (see Figs 9.9(c) and (d)).

Ideally one should know the precise values of the prism deflections or reticule line spacing in order to obtain estimates of precision based on individual observations. The evaluation of the spacing may be obtained from a large number of star observations, but these values may only be applicable for a particular focal setting and therefore not constant for all observers.

With the 45° astrolabe, advantage can be taken of the fact that the same prism, or combination of prisms, is used to an observation on either side of the nominal altitude. Therefore the mean of intercepts obtained from a pair of observations symmetrical about the centre should be identical. These mean intercepts may then be treated as if they had been obtained from a single observation at the centre and the calculation of the unknowns and the estimates of precision may proceed in the usual way.

This situation may also exist in astrolabes, which are fitted with reticules having multiple hairs for observations on a single star image. The manufacturers of modern instruments can usually ensure that the linear spacing of such lines is symmetrical about the centre point. The small departures from the manufacturer's nominal angular values of the line spacing are caused by the small variations in the nominal values of the focal lengths of the telescope lenses in a serial production process.

If, as sometimes happens, an observation is missed then two courses of action are open. Either the corresponding observation from all other star observations may be rejected or, if the spacing between reticule lines is known precisely, all observations may be used.

9.96 An example of a set of astrolabe observations and their full reduction is given in this section. The observations were made by means of a Zeiss Ni2 level fitted with an astrolabe attachment. Times of passage of each star through the centre of each of the double reticule lines were recorded. The calculation of the intercepts, I, was carried through in exactly the same way as those for the theodolite observations in section 9.81, on the assumption that the reticule line spacing was equal to the manufacturer's stated values of ±11'00", ±7'30", ±5'00", ±3'00" and ±1'30" from the centre and the combined prism angle and collimation error of the level was 59°59'30". It was known, from a very large number of observations made with this instrument, that the reticule line spacing was not exactly equal to those nominal values but that the lines were symmetrical about the centre to a very high degree of accuracy. Therefore secondary intercepts, I', were derived by taking the mean of the intercepts obtained from the corresponding pair of symmetrical reticule lines. These secondary intercepts were then treated as though they had been derived from a single observation and the remainder of the calculation was performed in exactly the same way as the example given in section 9.81. It will be noted that this method of reduction avoids the need for second order corrections.

Station: Δ Razorback ϕ_a 34°08'20" S
λ_a $10^h02^m40^s$ E

Observer: G.J. Hoar
Recorder: K.I. Groenhout
Date: 14th July, 1977
R_0: $19^h26^m59.28^s$

Clock: Omega Printing Chronograph (Mean Time)
Correction on 10^h E Zone Time $+18^h00^m00.40^s$
Instrument: Zeiss Ni2 level with astrolabe attachment.
Hairs: ±11'00", ±7'30", ±5'00", ±3'00", ±1'30" (nominal values)
Combined prism angle and collimation 59°59'30"
Temperature: 6.9°C Pressure: 976.3 mb

FK4 604 (SW)
RA $16^h18^m11.51^s$ δ $-50^\circ06'08.7''$

Clock Time	I	I'	u	v
$5^h04^m50.24^s$	0.6"			
05 14.10	8.2			
05 30.24	5.4			
05 43.43	5.8			
05 53.64	9.0			
06 13.46	10.0	9.5"	-0.8"	+0.2"
06 23.26	9.5	7.6	+1.1	+2.1
06 36.60	11.5	8.4	+0.3	+1.3
06 52.98	11.2	9.7	-1.0	0
5 07 16.49	16.3	8.4	+0.3	+1.3
Mean	8.8	8.7	Σ-0.1	Σ+4.9

$\bar{A}_1 = 227^\circ.2$ $\quad u + V_1 = v$

FK4 761 (NE)
RA $20^h16^m50.18^s$ δ $-12^\circ36'42.5''$

Clock Time	I	I'	u	v
$5^h11^m42.97^s$	9.2"			
12 05.32	5.8			
12 21.34	3.1			
12 34.14	1.3			
12 43.58	1.5			
13 02.54	1.4	1.4"	+1.2"	+2.1"
13 11.98	1.9	1.6	+1.0	+1.9
13 24.64	2.0	2.6	0	+0.9
13 40.54	1.5	3.6	-1.0	-0.1
5 14 03.08	-1.4	3.9	-1.3	-0.4
Mean	2.6	2.6	Σ-0.1	Σ 4.4

$\bar{A}_2 = 49^\circ.7$ $\quad u + V_2 = v$

FK4 1461 (NW)
RA $17^h33^m32.93^s$ δ $-11^\circ13'30.5''$

Clock Time	I	I'	u	v
$5^h25^m27.14^s$	3.9"			
25 51.12	4.5			
26 08.23	5.1			
26 22.09	7.4			
26 32.33	7.8			
26 52.46	5.8	6.8"	+0.1"	-0.8"
27 02.76	7.1	7.2	-0.3	-1.2
27 16.34	7.6	6.4	+0.5	-0.4
27 33.33	8.6	6.6	+0.3	-0.6
5 27 57.14	10.7	7.3	-0.4	-1.3
Mean	6.8	6.9	Σ+0.2	Σ-4.3

$\bar{A}_3 = 314^\circ.7$ $\quad u + V_3 = v$

FK4 796 (SE)
RA $21^h14^m11.92^s$ δ $-53^\circ21'07.9''$

Clock Time	I	I'	u	v
$5^h30^m00.74^s$	13.0"			
30 26.48	15.7			
30 45.87	9.7			
31 00.72	10.4			
31 12.02	9.6			
31 35.00	5.1	7.4"	+1.1"	+0.2"
31 46.05	6.4	8.4	+0.1	-0.8
32 01.28	4.4	7.0	+1.5	+0.6
32 19.64	7.3	11.5	-3.0	-3.9
5 32 46.32	3.8	8.4	+0.1	-0.8
Mean	8.5	8.5	Σ-0.2	Σ-4.7

$\bar{A}_4 = 139^\circ.8$ $\quad u + V_4 = v$

Checks $\Sigma I = \Sigma I'$; $\Sigma u = 0$; $\Sigma v = nV$ where $n=5$

Parametric Equations:

$-\Delta h$	$D\lambda$	$\Delta\phi$	$-I$	$= 0$
1	sin $227^\circ.2$	cos $227^\circ.2$	-8.8"	
1	sin 49.7	cos 49.7	-2.6	
1	sin 314.7	cos 314.7	-6.8	
1	sin 139.8	cos 139.8	-8.5	

Normal Equations:

$-\Delta h$	$D\lambda$	$\Delta\phi$	$-I$	$= 0$
4	-0.0364	-0.0931	-26.7000	
	2.0419	-0.0012	3.8209	
		1.9581	6.0066	

Solution:

Δh	-6.6"	V_1	1.0"
$D\lambda$	-1.8	V_2	0.9
$\Delta\phi$	-2.8	V_3	-0.9
		V_4	-0.9

Final Position

ϕ_a	$-34^\circ 08' 20.0''$	λ_a	$+10^h 02^m 40.00^s$
$\Delta\phi$	-2.8	$\Delta\lambda$	-0.14
ϕ	- 34 08 22.8	λ	+10 02 39.86

$\Sigma vv = 38.01 \qquad N = 20$

$$\sigma_{\Delta\phi} = \sigma_{D\lambda} = \sqrt{\frac{2\Sigma vv}{N(N-3)}} = \pm 0.47''$$

$$\sigma_{\Delta h} = \sqrt{\frac{\Sigma vv}{N(N-3)}} = \pm 0.33''$$

The semi-graphic solution is given as well for completeness. (see Fig 9.15)

9.97 The semi-graphic method of solution is as follows:-

Preliminary Position

$\phi_a \quad -34^\circ 08' 20''$

$\lambda_a \quad +10^h 02^m 40^s$

Plotting Data			
Star No.		Calculated Azimuth A_C	Mean Intercept $\bar{I}$
1	FK_4 604	227°.2	8.8"
2	FK_4 761	49.7	2.6
3	FK_4 1461	314.7	6.8
4	FK_4 796	139.8	8.5

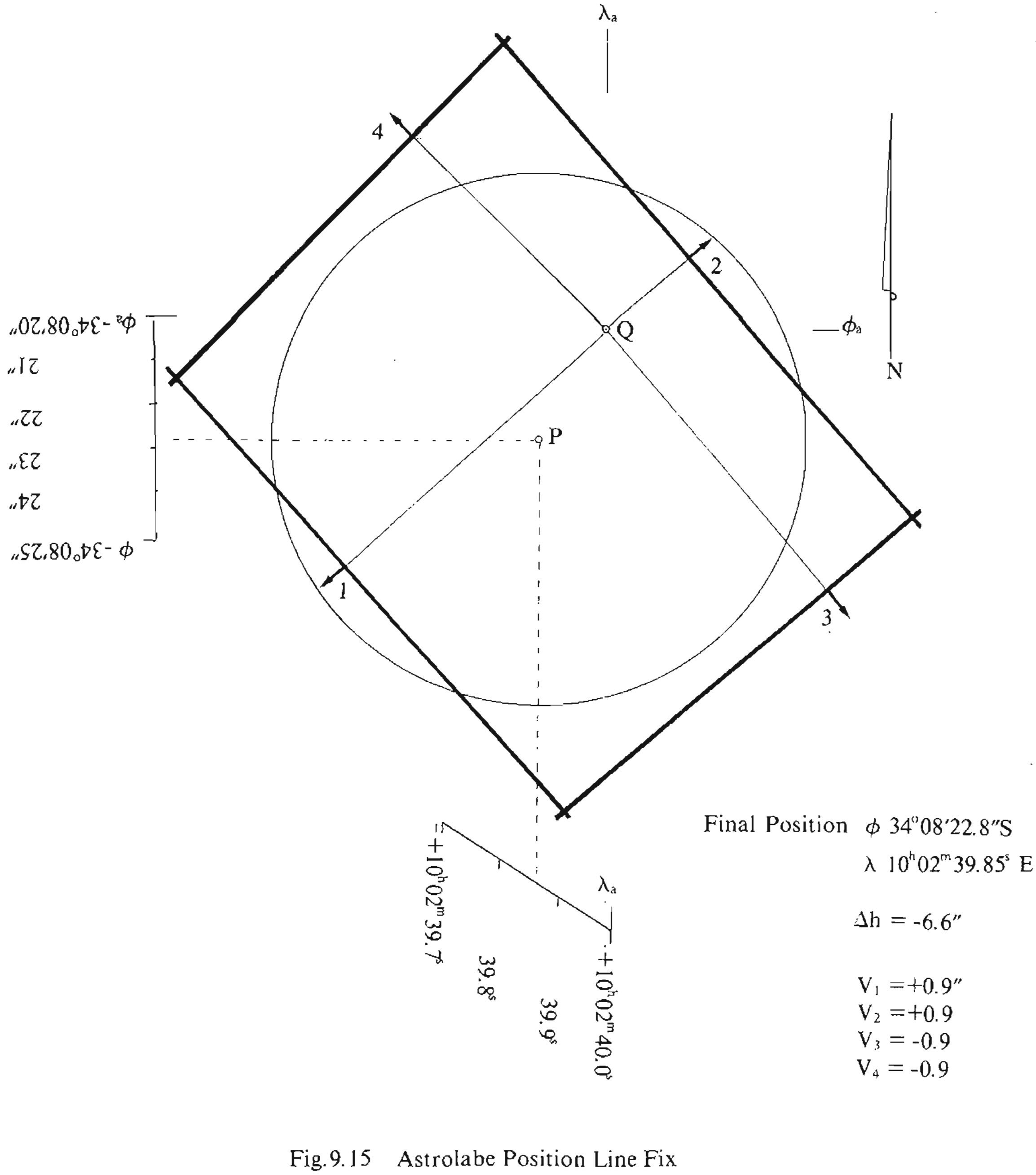


Fig.9.15 Astrolabe Position Line Fix

10
Prediction

INTRODUCTION

IT has been shown in Chapters 5, 6 and 7 that, in order to eliminate or minimise the effect of certain systematic and random errors, which may be present in the best known values of assumed data and observed quantities, observations for latitude, longitude and azimuth should be made on stars, which comply with certain conditions regarding their position in the sky and, when altitudes are being measured, the time separating the observations between individual stars. If one were to attempt to take such observations without the assistance of a list of predicted instrument settings and clock times, it would be only by chance that the observations would comply with the conditions, which should have applied. The requirements for such a working list are predicted settings, which will enable the observer to pick up the stars prior to observation. Once the star has been found, it is a relatively easy matter to follow its movement in the field of view of the instrument. However, in the event of a delay, such as that caused by changing face or by an interruption, caused perhaps by passing cloud, it is essential that the assistant be able to give to the observer new instrumental settings for locating the star again. Thus it will be necessary to provide a series of times and settings for each star at intervals, which are close enough to allow a quick visual estimation of the star's position without recourse to calculation.

The calculations for the prediction need not be made with high accuracy, because the angular width of the field of view of most theodolites is about one degree, and if a star corresponding in magnitude to the one listed in the programme appears within a few tenths of a degree from the cross hairs, then the observer is reasonably assured that he will be observing the right star. The location of the star in azimuth from values in the working list presupposes that an approximate orientation of the instrument is known. An illuminated reference object (R.O.) is required for convenience in night observing, even though the observer may not be making observations for azimuth. He may then conveniently check the orientation of the horizontal circle at intervals throughout the observing programme.

ORIENTATION

10.11 ORIENTATION may be obtained from a knowledge of the azimuth of a line in a local survey or, if time permits, a sun observation made during the day prior to the night's observation will satisfy this requirement. Alternatively an orienting star, usually one of the bright well known ones, may be pre-computed for a time earlier than the start of the predicted programme.

A necessary pre-requisite for finding an approximate orientation is a rough knowledge of the station's position and therefore one must investigate the effect of errors in the values of the station's latitude and longitude

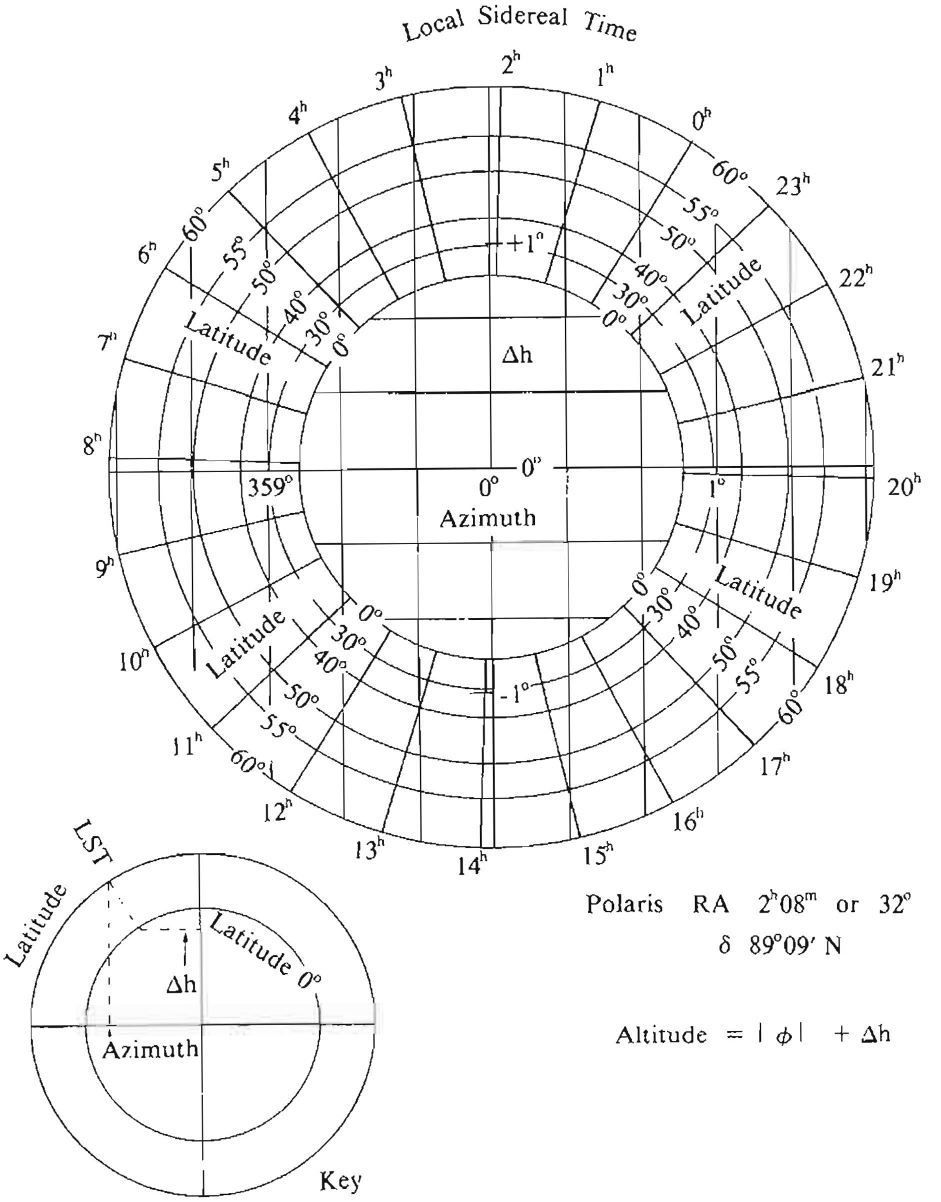


Fig. 10.1(a) Altitude and Azimuth of Polaris

on the calculated azimuth. For a more extensive treatment of these effects, the reader is referred to Chapter 7. The differential coefficients to consider are

$$\frac{dA}{d\phi} = \frac{-\tan h \sin \omega \cos \delta}{\cos \phi} \quad \& \quad \frac{dA}{d\lambda} = \frac{\cos \omega \cos \delta}{\cos h},$$

both of which will be small for stars of high declination, especially for close circum-polar stars, whose movement in azimuth is slow. Polaris (m=2.1), in the northern hemisphere, and σ Octantis (m=5.5), in the southern hemisphere, are both within about a degree of the pole and for the former, which is a bright star, the Star Almanac for Land Surveyors provides tables, by means of which the altitude and azimuth for any hour angle can be readily calculated.

10.12 Example of a Polaris calculation

Standard Time	$21^h 30^m$	(29th June 1975)
Zone	5 W	
UT	2 30	(30th June 1975)
R	18 30	
GST	21 00	
λ	5 04 W	
LST	15 56	

Altitude of Polaris — latitude $- a_o$

Azimuth of Polaris — $b_o \sec \phi$

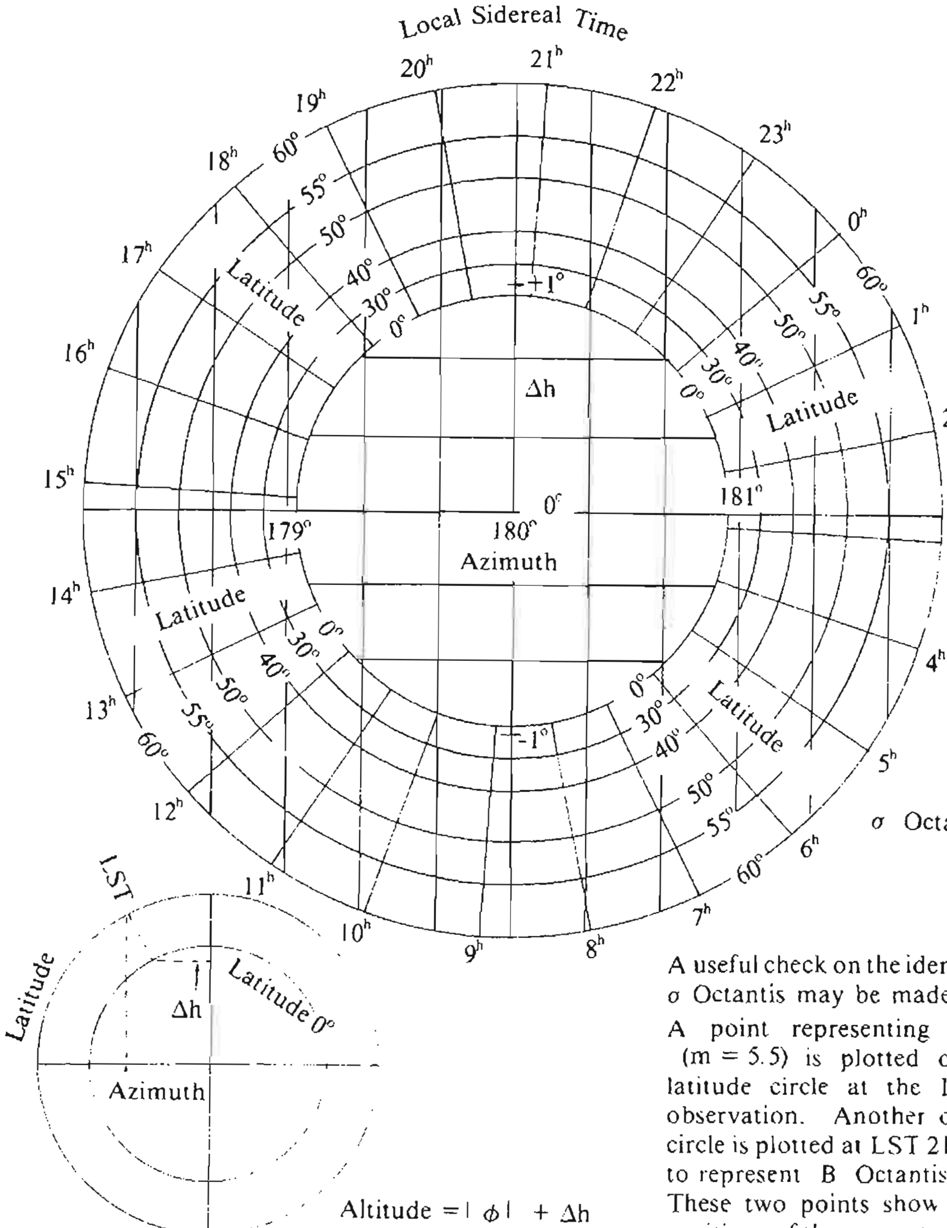
Terms involving a_1, a_2, b_1 and b_2 may be safely neglected.

Latitude ϕ	$45^\circ 20'$ N	$\sec \phi$	1.42
a_o	+46	b_o	+22'
Altitude	44 34	Azimuth	$0^\circ 31'$

	Star	RO
Observed Horizontal Circle reading	$1^\circ 43'$	$298^\circ 17'$
Azimuth	0 31	297 05
Orientation	1 12	1 12

In the southern hemisphere σ Octantis is a dim star, difficult to locate without the aid of approximate preliminary settings. One method of locating this star is to calculate its altitude and to set this on the vertical circle (refraction being ignored). The instrument is pointed roughly to the South, clamped in azimuth and one then proceeds to scan the sky at degree settings of the horizontal circle on either side of the initial setting. If a star corresponding in magnitude to σ Octantis appears close to the cross hairs, then approximately 20' distant from it, one should find a fainter star, B Octantis (m=6.5). The relative position of the two stars in the field of view may be found from Fig 10.1(b). This will confirm the identification. The cross hair illumination may need to be extinguished in order to discern this fainter star.

10.13 Example of σ Octantis calculation.



σ Octantis RA 20^h45^m
or 311°15′
δ 89°03′S

A useful check on the identification of σ Octantis may be made as follows: A point representing σ Octantis (m = 5.5) is plotted on the zero latitude circle at the LST of the observation. Another on the same circle is plotted at LST 2 hours earlier to represent B Octantis (m = 6.5). These two points show the relative positions of these two stars in the field of view of an erecting telescope. For an inverting telescope, the diagram should be turned upside down.

Fig. 10.1(b) Altitude and Azimuth of σ Octantis

Standard Time	$19^h 25^m$	(3rd August 1975)
Zone	10 E	
UT	9 25	
R	20 45	
λ	10 05 E	
LST	16 15	
RA	20 46	$\delta = -89^\circ 03'$, $p = 57'$
t	19 29	
t	$292^\circ 15'$	$\sin t = -0.93$, $\cos t = 0.38$

Altitude of σ Octantis $\simeq |\phi| + p \cos t$

Azimuth of σ Octantis $\simeq 180^\circ + p \sin t \sec \phi$

Latitude ϕ	$37^\circ 48'$ S	$\sec \phi$	1.27
$p \cos t$	+22	$p \sin t \sec \phi$	$-1^\circ 07'$
Altitude	38 10	Azimuth	$178^\circ 53'$

	Star	RO
Observed Horizontal Circle reading	$178^\circ 21'$	$92^\circ 52'$
Azimuth	178 53	93 24
Orientation	-32	-32

As an alternative to some of the previous calculations, a handy graphical solution of the altitude and azimuth of Polaris and σ Octantis can be obtained from Figs 10.1(a) and (b).

In equatorial latitudes, it may be difficult to sight close circum-polar stars, because of the poor transparency of the atmosphere near the horizon. In this case, one may use either a star at transit or an extra-meridian star, preferably at low altitude near the prime vertical. The use of a star at transit for orientation need not be confined to equatorial latitudes; in fact, it is one of the easiest ways of determining orientation and may even be used as a preliminary to find σ Octantis, which, when found, may then in turn be used for a greater refinement of the orientation. It should be stressed again that stars of high declination are to be preferred, i.e. stars as close as possible to the visible pole, otherwise an error in the assumed value of the station's longitude will have an adverse effect on the orientation.

Example of a transit calculation

10.14 To use this method, one should calculate the LST of the start of the observing programme and then select from the almanac a bright high declination star, whose RA = LST (Upper Transit) or RA = LST $\pm 12^h$ (Lower Transit); the RA of this star should be such as to allow sufficient time to complete the orientation before the beginning of the programme. The theodolite altitude to be set is then calculated from the simple meridian formula (see Section 5.33), from which it is seen that whatever error is present in the assumed value of the station's latitude, this will be present in the altitude set out. The star is then identified, the theodolite swung on to it and the star followed up to the calculated time of transit, when the final pointing is made and the horizontal circle read. Immediately after this the RO is sighted and the horizontal circle read. It is a simple matter to calculate the azimuth of this star at say 5^m before and after transit using the appropriate differential relationship of section 10.27. These additional values will be useful for checking purposes.

Standard Time of the start of the programme	$20^h 15^m$	(19th Jan 1975)
Zone	1	E
UT	19 15	
R_{18}	7 54 01.6^s	
dR	12.3	
λ	1 13 45	E
LST	4 22 58.9	
Star No. 91(m=3.2) RA	3 47 38.2	
Difference (sidereal units)	35 20.7	
Conversion sidereal to mean	- 5.8	
Difference (mean units)	35 14.9	
Standard Time of the start of the programme	20 15	
Standard Time of transit	19 39 45.1	

Star No. 91 δ_M	$-74^o 19'$
Latitude, ϕ	-33 56
Zenith distance, z_M	-40 23

	Star	RO
Observed Horizontal Circle reading	$177^o 32'$	$233^o 18'$
Azimuth	180	235 46
Orientation	- 2 28	- 2 28

Example of an extra meridian calculation

10.15 Before the start of the programme a bright star is seen to the south and identified as Fomalhaut, Star No. 632 in the Star Almanac for Surveyors. The observer may then choose either to observe the star and then calculate the azimuth, or to calculate the azimuth for a particular time and then make the observation at this time. The following calculation corresponds to the latter method.

Standard Time of observation	20^h	(19th Jan 1975)
Zone	1	E
UT	19	
R_{18}	7 $54^m 01.6^s$	
dR	9.9	
λ	1 13 45	E
LST	4 07 56.5	
RA of Fomalhaut	22 56 16.5	
t	5 11 40.0	
t	$77^o 55' 00''$	
δ of Fomalhaut	-29 45 20	
ϕ	-33 56 00	

Formulae: $$\tan A = \frac{-\sin t}{\tan \delta \cos \phi - \sin \phi \cos t}$$

$$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t$$

check $$\cos \delta \cos t = \sin h \cos \phi - \cos h \sin \phi \cos A$$

Solution: $A = 249^{\circ}55'$ $h = 25^{\circ}20'$

	Star	RO
Observed Horizontal Circle reading	$249^{\circ}23'$	$92^{\circ}52'$
Azimuth	249 55	93 24
Orientation	-32	-32

PREPARATIONS FOR PREDICTION

10.21 THE preparation of a predicted programme for star observations can be undertaken in a systematic manner with the minimum of subjective judgement being exercised, provided the constraints such as catalogue to be used, position of the observation station, limits of altitude etc., are stated quite specifically beforehand. The following examples have been chosen to illustrate the techniques, assuming that the observations are to be made with a single second theodolite on stars selected from the Star Almanac for Land Surveyors. The principles used in the examples which follow are equally applicable to observations made with more sophisticated equipment and catalogues containing more stars than those in the Star Almanac for Land Surveyors. The routines have been devised so that there is a minimum amount of calculation; however, it is assumed that a small calculator is available for preparing such a predicted programme. The end product of the prediction procedure is a working list to be used as a guide for the observer. How detailed this working list is will depend to a large extent upon the skill and experience of the observer and his assistant.

Time Rates of Change of Zenith Distance and Azimuth

10.22 Zenith distances and azimuths for times shortly before and after the central predicted value may be conveniently calculated from the appropriate differential coefficients. However, care should be exercised in applying these small changes with their correct signs. Alternatively, the azimuths and zenith distances to the star may be computed at the start and the finish of the predicted interval. Intermediate values between these two points may then be determined with sufficient accuracy by linear interpolation.

The differential coefficients to be used are as follows:-

$$\frac{dz}{dt} = -\cos \phi \sin A \qquad \text{...10.1}$$

$$\frac{dA}{dt} = \frac{\cos \omega \cos \delta}{\cos h} \qquad \text{...10.2}$$

Equation 10.2 is not in a convenient form, because it contains the parallactic angle ω, but it can be transformed by using the Five Parts Formula

$$\cos \delta \cos \omega = \sin \phi \cos h - \cos \phi \sin h \cos A$$

then

$$\frac{\cos \delta \cos \omega}{\cos h} = \sin \phi - \cos \phi \tan h \cos A$$

$$\therefore \quad \frac{dA}{dt} = \sin \phi - \cos \phi \cot z \cos A \qquad \text{...10.3}$$

Equations 10.1 and 10.3, or modifications of them, may now be used for the following observations.

Latitude Determinations

10.23 These observations are made near the meridian, i.e. $A \simeq 0^\circ$ and 180° and therefore

$$\frac{dz}{dt} = 0 \quad \& \quad \frac{dA}{dt} = \sin\phi \pm \cos\phi \cot z \quad \begin{matrix} +\text{ Star South} \\ -\text{ Star North} \end{matrix}$$

A more convenient expression for $\frac{dA}{dt}$ is obtained by transforming equation 10.2 where $\cos\omega = \pm 1$. With the notation and conventions postulated in Chapter 2, this becomes

$$\frac{dA}{dt} = -\frac{\cos\delta_M}{\sin z_M}$$

If one of the stars forming the pair matches a close circum-polar star, then the zenith distance of the former will be approximately equal in magnitude to the co-latitude, and thus

in the northern hemisphere, $A = 180^\circ$ and $z = 90^\circ - \phi$
and in the southern hemisphere, $A = 0^\circ$ and $z = 90^\circ + \phi$,
which, when substituted in equation 10.3, gives for both cases

$$\frac{dA}{dt} = 2\sin\phi$$

Longitude Determinations

10.24 These observations are made near the prime vertical, ie. $A \simeq 90^\circ$ and 270° and therefore

$$\frac{dz}{dt} = \pm\cos\phi \quad \begin{matrix} +\text{ West} \\ -\text{ East} \end{matrix} \quad \text{and} \quad \frac{dA}{dt} = \sin\phi$$

The above form of the differential $\frac{dz}{dt}$ is very accurate even for observations away from the prime vertical, because $\sin A$ does not change rapidly in the vicinity of $A=90^\circ$ and 270°. However the differential $\frac{dA}{dt}$ should not be used in the above form, unless the observations are made close to the prime vertical.

Azimuth Determinations

10.25 These observations may be made when the star is in the vicinity of elongation, ie. $\omega=90^\circ$ and 270°.

Substituting the Sine Formula $\sin A = \frac{-\cos\delta\sin\omega}{\cos\phi}$ into equation 10.1 which is

$$\frac{dz}{dt} = -\cos\phi\sin A, \text{ one obtains}$$

$$\frac{dz}{dt} = \cos\delta\sin\omega$$

or

$$\frac{dz}{dt} = \pm\cos\delta \quad \begin{matrix} +\text{ West} \\ -\text{ East} \end{matrix}$$

and from equation 10.2

$$\frac{dA}{dt} = \frac{\cos\omega\cos\delta}{\cos h} = 0$$

Position Line Determinations

10.26 These observations are often made on extra-meridian stars at the same altitude and, if this altitude is 45°, then

$$\frac{dz}{dt} = -\cos\phi \sin A \qquad \text{and} \qquad \frac{dA}{dt} = \sin\phi - \cos\phi \cos A$$

Furthermore, if the observations are made near the centres of the azimuth quadrants, sin A and cos A have the same magnitude, but of course may vary in sign.

10.27 Summary

Observation	$\frac{dz}{dt}$	$\frac{dA}{dt}$
Latitude (circum-meridian)	0	$\frac{-\cos\delta_M}{\sin z_M}$
Latitude (star matching a circum-polar)	0	$2\sin\phi$
Longitude (near prime vertical)	$\pm\cos\phi$ +West −East	$\sin\phi - \cos\phi\cot z\cos A$ or $\sin\phi$ close to prime vertical
Azimuth (circum-elongation)	$\pm\cos\delta$ +West −East	0
Position Lines (45° altitude)	$-\cos\phi\sin A$	$\sin\phi - \cos\phi\cos A$

PREDICTED PROGRAMMES

Latitude from Circum-Meridian Observations

10.31 An example of the preparation of a predicted programme for this method for the following circumstances:-

Station position — Latitude 41°30' N
Longitude 5^h26^m W (Time Zone 5^h W)
Date: 10th May 1975
Programme: Start at about 20^h Standard Time. Duration about 2^h.
Altitude range: 20° to 50° (zenith distance range 70° to 40°)
Altitude balance: ±5°
Maximum time between stars forming a pair: About 30^m
Duration of observation on each star: About 10^m

Calculation of the range of RA's

Standard Time of the start of the programme	20^h00^m	(10th May 1975)
Zone	5 W	
UT	1 00	(11th May 1975)
R	15 13	
GST	16 13	
λ	5 26 W	
LST of the start of the programme	10 47	
LST of the finish of the programme	12 47	

Ranges of RA: Upper transit 10^h47^m to 12^h47^m
Lower transit 22 47 to 0 47

Calculation of the range of declinations:

$$\delta_M = \phi + z_M \qquad \phi = +41^\circ 30'$$

Table 10.1

Circumstance	z_M	δ_M	δ
Upper transit South	-70° -40	$-28^\circ 30'$ $+ 1\ 30$	$28^\circ 30'$ S 1 30 N
Upper transit North	+40 +48 30	+81 30 +90 00	81 30 N 90 00 N
Lower Transit North	+48 30 +70	+90 00 +111 30	90 00 N 68 30 N

A local meridian section diagram may be used in lieu of Table 10.1 (see Fig 10.2).

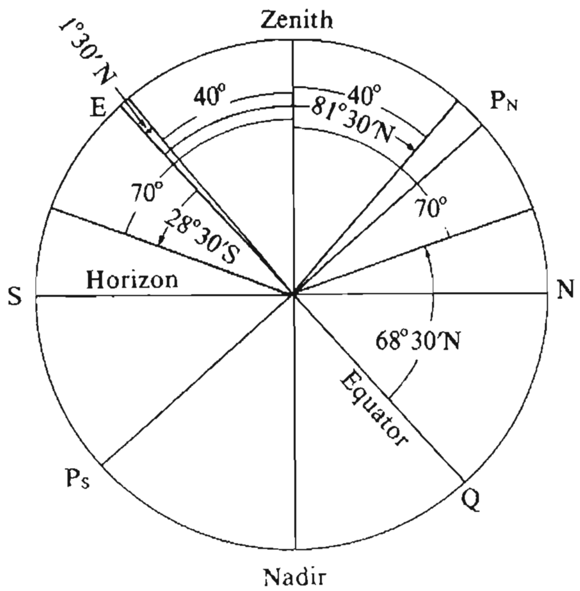


Fig. 10.2

Stars which conform to the ranges of RA & δ calculated, may now be selected from the catalogue. When the Star Almanac for Land Surveyors is used, it should be noted that stars may be selected not only from the main list, but also from the supplementary list and the circum-polar list. It will be found that there are fewer stars available for observation on the pole side of the zenith than on the equatorial side. This is due to the convergence of the celestial meridians. If stars were distributed equally over the celestial sphere, then a rectangle bounded by say 10° of declination and 1^{h} of right ascension near the equator would contain more stars than a corresponding rectangle near the pole. Because there are fewer stars available for observation on the pole side, the selection of stars and their matching to form pairs can often be done simultaneously when one is looking up the almanac. First of all a star on the pole side is selected and then a companion star or stars are sought in the RA range of the equatorial side stars.

10.32 Table 10.2 contains *all* stars available within the chosen limits of RA and declination. The above method of selection of pairs of stars will be seen to apply.

Table 10.2 Available Stars

North							South				
No.	m	RA	LST	δ	δ_M	z_M	No.	m	RA=LST	$\delta=\delta_M$	z_M
668	4.8	$22^h 48^m$	$10^h 48^m$	$83^{\circ}01'$N	$+96^{\circ}59'$	$+55^{\circ}29'$	293	3.3	$10^h 48^m$	$-16^{\circ}04'$	$-57^{\circ}34'$
669	4.5	23 07	11 07	75 15 N	+104 45	+63 15	296	4.2	10 59	-18 10	-59 40
647	3.4	23 38	11 38	77 30 N	+102 30	+61 00	305	3.8	11 18	-14 39	-56 09
							308	4.1	11 24	-17 33	-59 03
							321	3.2	12 09	-22 29	-63 59
							324	2.8	12 15	-17 25	-58 55
							326	4.0	12 19	- 0 32	-42 02
							330	3.1	12 29	-16 23	-57 53
							335	2.8	12 33	-23 16	-64 46
							338	2.9	12 40	- 1 19	-42 49

To assist in finding matching pairs, it is convenient to calculate individual zenith distances and then select stars of a pair, which, in our case, are to have zenith distances differing by not more than 5°. It is also possible to match stars from the declinations alone. From the meridian relationship

$$\phi = \delta_M - z_M$$

and the fact that z_M for a north and for a south star will be of opposite sign,

$$2\phi = \delta_M(\text{N Star}) + \delta_M(\text{S Star})$$

Thus, in our example, the RHS of this equation should be within the limits of $2\phi \pm 5^{\circ}$ i.e. 78° to 88°, except when the stars forming a pair lie at, or close to, the extremities of the postulated altitude range. In such cases it may be found that the altitude of the companion star falls outside this range.

Table 10.3 Selected Latitude Stars

Pair No.	Aspect & Star No.	LST	Diff	Standard Time on Prediction Date	Az change for $+5^m$
Start of Prediction Period		10^h47^m		20^h00^m	
			1^m		
1	N 668	10 48		20 01	$+0^\circ 11'$
			11		
1	S 296	10 59		20 12	+1 23
			8		
2	N 669	11 07		20 20	+0 21
			17		
2	S 308	11 24		20 37	+1 23
			14		
3	N 647	11 38		20 51	+0 19
			31		
3	S 321	12 09		21 22	+1 17
		$\Delta\ 1^h22^m$	$\Sigma 1^h22^m$ ✓	$\Delta\ 1^h22^m$ ✓	

Note Δ is the difference between first and last times

Table 10.4 Working List

Pair	Aspect Star No.	Mag.	Vertical Circle CL	Vertical Circle CR	LST	Standard Time	Horizontal Circle CL	Horizontal Circle CR
1	N 668	4.8	$55^\circ 29'$	$304^\circ 31'$	10^h43^m	19^h56^m	$359^\circ 49'$	$179^\circ 49'$
					10 48	20 01	0 00	180 00
					10 53	20 06	0 11	180 11
	S 296	4.2	59 40	300 20	10 54	20 07	178 37	358 37
					10 59	20 12	180 00	0 00
					11 04	20 17	181 23	1 23
2	N* 669	4.5	63 15	296 45	11 02	20 15	359 39	179 39
					11 07	20 20	0 00	180 00
					11 12	20 25	0 21	180 21
	S 308	4.1	59 03	300 57	11 19	20 32	178 37	358 37
					11 24	20 37	180 00	0 00
					11 29	20 42	181 23	1 23
3	N 647	3.4	61 00	299 00	11 33	20 46	359 41	179 41
					11 38	20 51	0 00	180 00
					11 43	20 56	0 19	180 19
	S 321	3.2	63 59	296 01	12 04	21 17	178 43	358 43
					12 09	21 22	180 00	0 00
					12 14	21 27	181 17	1 17

* Observations on this star could be started a little later if necessary.

It will be seen that, in the working list, corresponding values of LST and Standard Time are given. The values of LST remain invariable for star observations, but those of standard time will vary from one night to the next by about 3^m56^s. If observations are not made on the predicted date, the Standard Time values of the working list must be altered. However, the use of an auxiliary watch, set to read LST approximately, will be found a convenient substitute for calculating standard times for different nights.

The working list shows that the difference between the times of transit of Stars Nos. 296 and 669 is only 8 minutes, which means that, if the observations on the first star lasted the anticipated full 10 minutes, the observer would have to start his observations late on Star No. 669 and might therefore, not be able to distribute his observations on this star evenly on either side of transit. This imbalance is not a serious disadvantage for stars on the pole side of the zenith, because it is known that systematic errors have a lesser effect on the latitude derived from observations on these stars, (see section 5.15). Thus, if one finds that the times of transit of a pair of well balanced stars are too close to each other to allow for a full set of observations, one can adopt the following procedure. The observations should be arranged, in such a way, that a full set on either side of transit is obtained for the star on the *equatorial* side of zenith. This will mean that observations to the star on the pole side of the zenith will have to be made either slightly earlier or later depending upon the order in which they transit.

Latitude from Observations on the Circum-Polar Star

10.33 A simple and convenient method of determining latitude may be used if one takes advantage of the fact that, in medium to high latitude, observations on a close circum-polar star need not be made in the vicinity of transit but at any hour angle, because the time rate of change of altitude on such a star will never be large and therefore, one may obtain highly accurate observations at any hour angle. Such a star may be coupled with one on the equatorial side to form a balanced pair.

The zenith distance of the star, which is on the equatorial side of the zenith and which will balance that of the circum-polar star, will be approximately equal to the magnitude of the co-latitude and this fact makes the prediction procedure very simple. Then from the meridian formula

$$\phi = \delta_M - z_M$$

one can deduce that the declination of the matching star will be

(a) for the northern hemisphere

$$\delta = 2\phi - 90^\circ$$

and (b) for the southern hemisphere

$$\delta = 2\phi + 90^\circ$$

This method of latitude determination has the advantage that the observer can make observations on the star on the equatorial side of the zenith knowing that the circum-polar star will be available at any time either before or after these observations.

10.34 To illustrate this technique, such a pair will be predicted for the same circumstances as those of Section 10.31. These observations are to start after the finish of the circum-meridian observations (see Table 10.3), i.e. at about LST = 12^h20^m (Standard Time 21^h33^m).

Declination of the south star $\delta = 2\phi - 90^\circ = 83^\circ - 90^\circ = -7^\circ$

From the catalogue, star No. 348 is found as a suitable one.

No.	Mag.	RA=LST	$\delta = \delta_M$	z_M	dA (dt = 5^m)
348	4.5	13^h09^m	$-5^\circ25'$	$-46^\circ55'$	$+1^\circ42'$

Polaris may then be observed at about LST = 12^h50^m.

From the Polaris diagram Fig 10.1(a) (or Pole Star Tables) the following is obtained:-

$A = 359^\circ37'$ $\Delta h = -48'$ and then $z = 49^\circ18'$

Table 10.5 Working List

Aspect & Star No.	Mag.	Vertical Circle		LST	Standard Time	Horizontal Circle	
		CL	CR			CL	CR
N Pol.	2.1	$49^{\circ}18'$	$310^{\circ}42'$	12^h50^m	22^h03^m	$359^{\circ}37'$	$179^{\circ}37'$
S 348	4.5	46 55	313 05	13 04	22 17	178 18	358 18
				13 09	22 22	180 00	0 00
				13 14	22 27	181 42	1 42

Longitude from Near Prime Vertical Observations

10.41 An example of the preparation of a predicted programme for the determination of longitude for the following circumstances:-

Station position: Latitude $41^{\circ}30'$ N
Longitude 5^h26^m W (Time Zone 5^h W)
Date: 10th May 1975
Programme: Start at about 22^h30^m Standard Time
Duration about 2^h
Altitude range: 30° to 50° (zenith distance range 60° to 40°)
Declination balance: $\pm 2^{\circ}$
Duration of observation on each star: about 10^m

Calculation of the LST of the start and finish of the programme.

Standard Time of the start of the programme	22^h30^m	(10th May 75)
Zone	5 W	
UT	3 30	(11th May 75)
R	15 13	
GST	18 43	
λ	5 26 W	
LST of start of the programme	13 17	
LST of the finish of the programme	15 17	

To assist in the initial selection of stars from the catalogue, a diagram has been constructed (see Fig 10.3), from which the limits of declination and hour angle can be read off without preliminary calculations. For the preparations of this diagram, declinations have been limited to a band 10° wide centred on a point, which has a zenith distance of 50° (ie. middle of the range of zenith distances) on the prime vertical. From these limits of declination, hour angle limits have been calculated to correspond to the limits of zenith distance (ie. 40° and 60°). Thus if stars, which conform to these limits of declination and hour angle, are observed, the observations will lie within the chosen range of zenith distance and close to the prime vertical.

From the diagram for $\phi = 41^{\circ}30'$ N the following is obtained:-

Hour angle range: West 3^h12^m to 4^h29^m Declination range: 20°N to 30°N
East 19 31 to 20 48

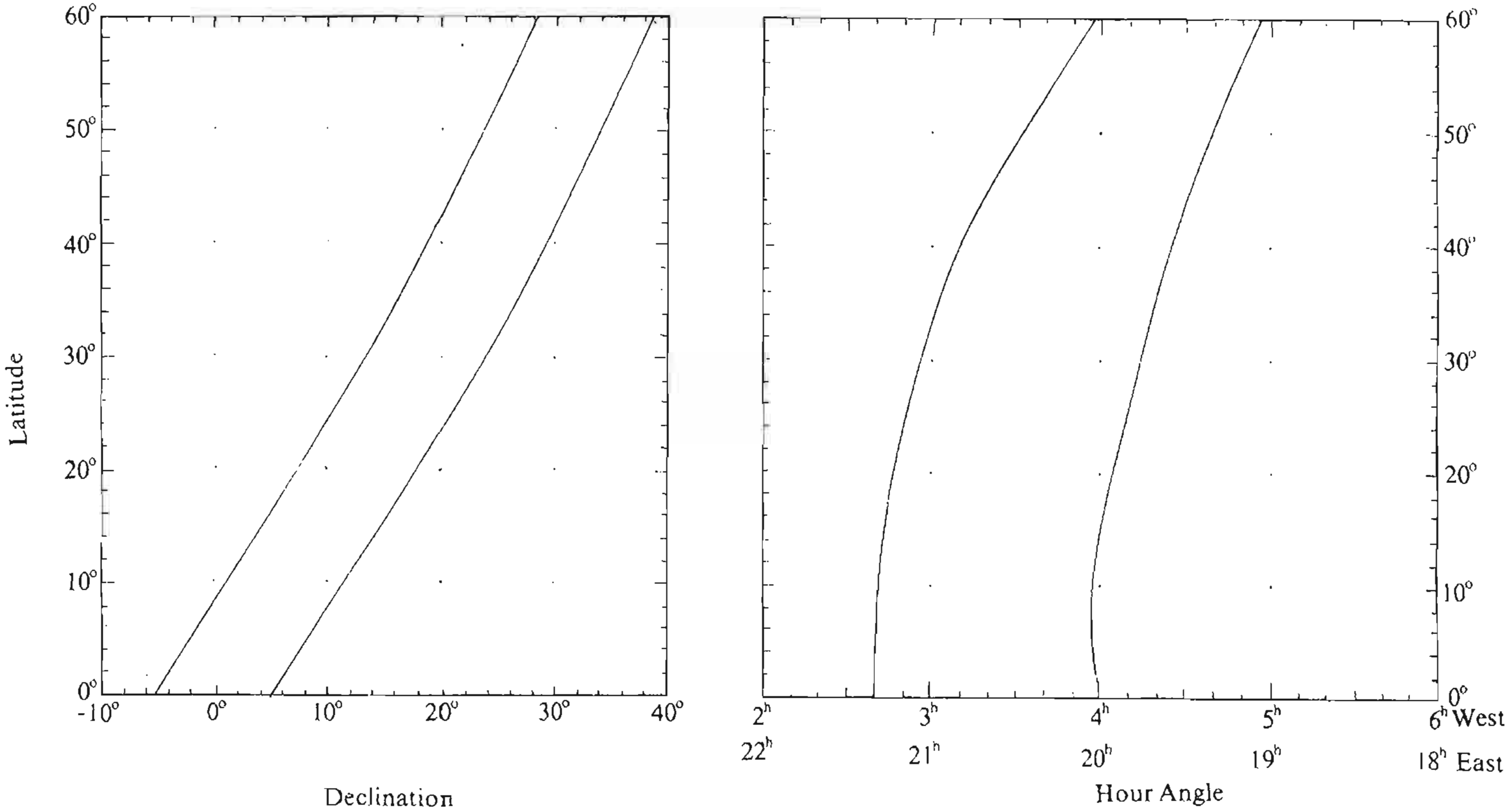
10.42 When the process of selecting stars is started, limits of altitude and azimuth are set out to define a suitable area in which the required observations should be made. Such limits define a quasi-rectangle, which may be referred to as a Sky Window. With the passage of time, a certain band of stars will appear to move through the sky window, which may be considered to be stationary. Such a star band is one bounded by specific limits of right ascension and declination.

Fig. 10.3 Limits of declination and hour angle for near prime vertical observations

Declination band 10° wide

Zenith distance range 40° to 60°

For stations in south latitude, signs of latitude and declination are changed.



The star band and the sky window normally are, as shown in Fig 10.4, inclined with respect to each other. At the start of the observing period, the leading edge of the star band is situated at the corner of the sky window. It will be seen, from this figure, that some of the stars in the star band have already passed through the sky window and are not available for observation, and also that stars at the declination boundaries of the star band appear only fleetingly within the sky window, whilst those stars nearer the centre of the band remain for some time. In addition, at the end of the observing period, some stars within the star band will not have reached the sky window. Comparison with sections 10.52 and 10.62 is instructive.

10.43 From the limits of hour angle, previously determined at the end of

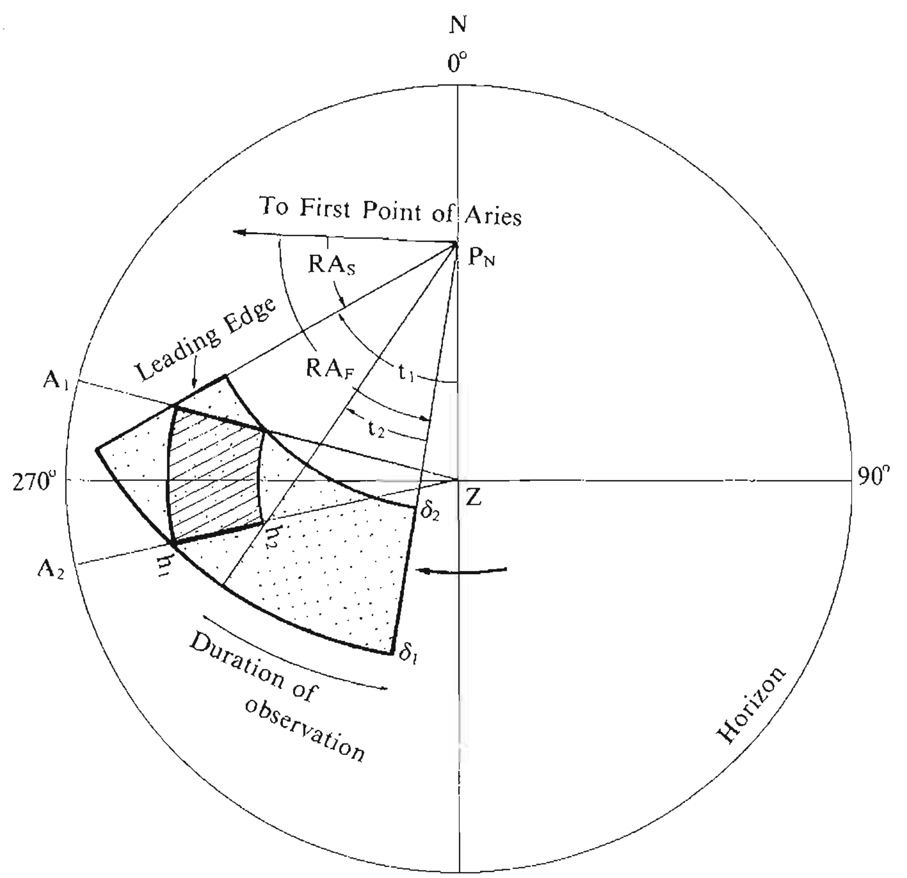


Fig. 10.4 This figure illustrates the situation at the start of the observing period.

section 10.41, the ranges of RA may be determined as follows:-

	East	West
LST of the start of the programme	13 17	13 17
Largest hour angle (see section 10.41)	20 48	4 29
RA_S*	16 29	8 48
Duration of the programme	2	2
Hour angle difference $(t_1 - t_2)$ **	1 17	1 17
RA_F*	19 46	12 05

* RA_S and RA_F are the catalogue limits of RA between which stars are selected.
**t_1 and t_2 are the extremes of hour angle over the sky window.

From the list of available stars, one now selects pairs of stars, each comprising one east and one west star. The ideal stars in a pair would have identical declinations and, if it were possible to observe them simultaneously, they would be symmetrically placed on either side of the meridian, ie. $t_E + t_W = 24^h$, $A_E + A_W = 360^o$ and $z_E = z_W$, thus satisfying the optimum conditions of the observation. Now if one were to observe on one of the stars for a short period before the instant when these conditions are fulfilled and for the same period on the other star immediately afterwards, then the average zenith distances of each star would be equal, thus still maintaining all the conditions. In practice, a compromise must be made and pairs of stars are so selected that they have similar declinations (in our case within 2^o). Now the condition, that $t_E + t_W = 24^h$ (or $t_E = -t_W$) will be applied. This will simplify computation and, at the same time, *partially* satisfy the other two conditions concerning azimuth and zenith distance.

From the relationships,

$$\text{LST of observation} = RA_E + t_E = RA_W + t_W$$

one obtains, for practical computation,

$$\text{LST of observation} = \frac{RA_E + RA_W}{2} \quad (\text{if } RA_W > RA_E \text{ add } 12^h)$$

$$t_W = \frac{RA_E - RA_W}{2}$$

$$t_E = \frac{RA_W - RA_E + 24}{2}$$

These relationships may now be used to pair off the stars because the ranges of LST and RA have already been calculated.

Condition	Range
$2LST = RA_E + RA_W$	$26^h 34^m$ to $30^h 34^m$
$2t_W = RA_E - RA_W$	$6^h 24^m$ to $8^h 58^m$

Those stars, which satisfy these conditions for pairing, are given in the following table of available stars.

Table 10.6 Available Stars

East					West				
No.	Mag.	RA	δ	Pair	No.	Mag.	RA	δ	Pair
442	2.8	$16^h 29^m$	$21^\circ 32'$ N		266	3.1	$9^h 44^m$	$23^\circ 53'$ N	1
467	3.2	17 14	24 52 N	1,2,3	270	4.1	9 51	26 07 N	2,4,5
477	4.5	17 30	26 08 N	4	277	3.6	10 15	23 32 N	3,6
488	3.5	17 46	27 44 N	5	301	2.6	11 13	20 39 N	7,8
495	3.8	17 57	29 15 N						
501	3.8	18 07	28 45 N						
510	3.9	18 23	21 45 N	6,7					
517	4.3	18 45	20 31 N	8					
542	3.2	19 30	27 54 N						

The small line diagram below will now be found to be of assistance in the final selection of stars.

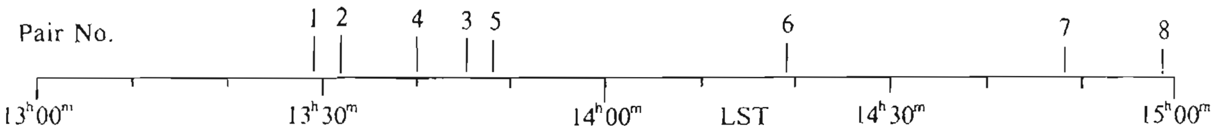


Fig 10.5

Table 10.7 Selected Longitude Stars

Pair No.	Aspect & Star No.	LST	Diff	Standard Time on Prediction Date	Hour Angle t	Zenith Distance z	Azimuth A	ZD Change for $+10^m$	Az Change for $+10^m$
Start of Prediction Period		$13^h 17^m$		$22^h 30^m$					
1	E 467	13 29	12^m	22 42	$-3^h 45^m$	$49^\circ 00'$	$91^\circ 27'$	$-1^\circ 52'$	$+1^\circ 42'$
1	W 266	13 29	0	22 42	+3 45	49 33	267 29	+1 52	+1 44
5	E 488	13 48	19	23 01	-3 58	49 51	86 11	-1 52	+1 33
5	W 270	13 48	0	23 01	+3 58	50 44	272 03	+1 52	+1 36
6	E 510	14 19	31	23 32	-4 04	54 21	91 28	-1 52	+1 41
6	W 277	14 19	0	23 32	+4 04	53 19	270 19	+1 52	+1 39
		$\Delta\ 1^h 02^m$	$\Sigma\ 1^h 02^m$ ✓	$\Delta\ 1^h 02^m$ ✓					

Formulae $\cos z = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t$

$$\tan A = \frac{-\sin t}{\tan\delta \cos\phi - \sin\phi \cos t}$$

check $\cos\delta \cos t = \cos z \cos\phi - \sin z \sin\phi \cos A$

Table 10.8 Working List

EAST							LST	Standard Time	WEST					
Pair	No.	Mag	Vertical Circle		Horizontal Circle				Vertical Circle		Horizontal Circle		Mag	No
			CL	CR	CL	CR			CL	CR	CL	CR		
1	467	3.2	50°52'	309°08'	89°45'	269°45'	13h19m	22h32m	47°41'	312°19'	265°45'	85°45'	3.1	266
			49 56	310 04	90 36	270 36	13 24	22 37	48 37	311 23	266 37	86 37		
			49 00	311 00	91 27	271 27	13 29	22 42	49 33	310 27	267 29	87 29		
			48 04	311 56	92 18	272 18	13 34	22 47	50 29	309 31	268 21	88 21		
			47 08	312 52	93 09	273 09	13 39	22 52	51 25	308 35	269 13	89 13		
2	488	3.5	51 43	308 17	84 38	264 38	13 38	22 51	48 52	311 08	270 27	90 27	4.1	270
			50 47	309 13	85 25	265 25	13 43	22 56	49 48	310 12	271 15	91 15		
			49 51	310 09	86 11	266 11	13 48	23 01	50 44	309 16	272 03	92 03		
			48 55	311 05	86 57	266 57	13 53	23 06	51 40	308 20	272 51	92 51		
			47 59	312 01	87 44	267 44	13 58	23 11	52 36	307 24	273 39	93 39		
3	510	3.9	56 13	303 47	89 47	269 47	14 09	23 22	51 27	308 33	268 40	88 40	3.6	277
			55 17	304 43	90 38	270 38	14 14	23 27	52 23	307 37	269 29	89 29		
			54 21	305 39	91 28	271 28	14 19	23 32	53 19	306 41	270 19	90 19		
			53 25	306 35	92 18	272 18	14 24	23 37	54 15	305 45	271 09	91 09		
			52 29	307 31	93 09	273 09	14 29	23 42	55 11	304 49	271 58	91 58		

Note. (1) One may choose to observe either an east or a west star first and then change over to the other star at the tabulated central value of the LST or Standard Time.

(2) Another pair Nos. 507 and 301 is also available.

[redacted]

Azimuth from Circum-Elongation Observations

10.51 An example of the preparation of a predicted programme for the determination of azimuth for the following circumstances:-

Station Position: Latitude $41^{\circ}30'$ N
Longitude $5^{h}26^{m}$ W (Time Zone 5^{h} W)
Date: 10th May 1975
Programme: Start at about 20^{h} Standard Time
Duration about $4^{h}30^{m}$
Declination range: 75° N to 90° N
Declination balance: $\pm 5^{\circ}$
Duration of observation on each star: about 20^{m}

Calculation of the LST of the start and finish of the programme

Standard Time of the start of the programme	$20^{h}00^{m}$	(10th May 1975)
Zone	5 W	
UT	1 00	(11th May 1975)
R	15 13	
GST	16 13	
λ	5 26 W	
LST of the start of the programme	10 47	
LST of the finish of the programme	15 17	

To assist in the initial selection of stars from the catalogue, a diagram has been constructed (see Fig 10.6). From this, hour angles, and for later purposes azimuths and altitudes at elongation, may be read off without preliminary calculation.

From this diagram for $\phi = 41^{\circ}30'$ N, for the limits of declination from 75° to 90° as shown, the corresponding hour angle ranges are:-

West $5^{h}07^{m}$ to $6^{h}00^{m}$

East 18 00 to 18 53

10.52 For azimuth determination from circum-elongation observation, the sky window, as defined in section 10.42, becomes a curved line, which is a portion of the elongation locus and, therefore does not enclose an area on the celestial sphere. This portion of the locus, comprising the sky window for the azimuth case, extends from the pole to the lower limiting declination of 75° N. The situation at the start of the observing period is illustrated in Fig 10.7. The star band here, unlike that of section 10.42, consists of a sector with the pole at the apex.

At the start of the observing period, the leading edge of the star band is situated tangential to the elongation locus at the pole, i.e. it lies at right angles to the meridian. Figure 10.7 shows some of the stars in the star band have already passed over this locus and are not available for observation. In addition, at the end of the observing period, some stars in the star band have not reached the locus of elongation. Comparison with the situation in section 10.42 is instructive.

10.53 From the limits of hour angle, previously determined at the end of section 10.51 and in the same manner as that of section 10.43, the ranges of RA may be determined as follows:-

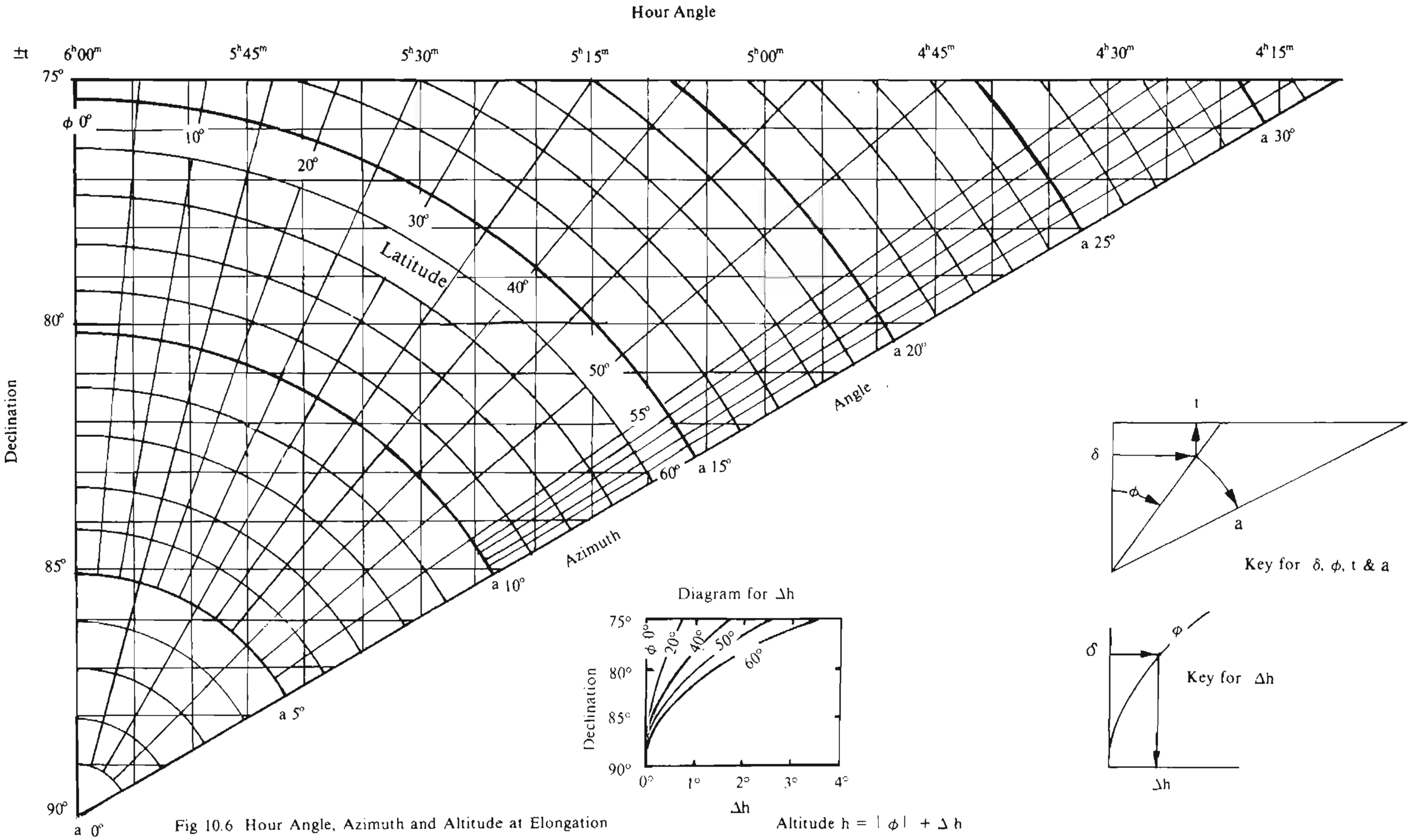


Fig 10.6 Hour Angle, Azimuth and Altitude at Elongation

Altitude h = | φ | + Δ h

Key for δ, φ, t & a

Key for Δh

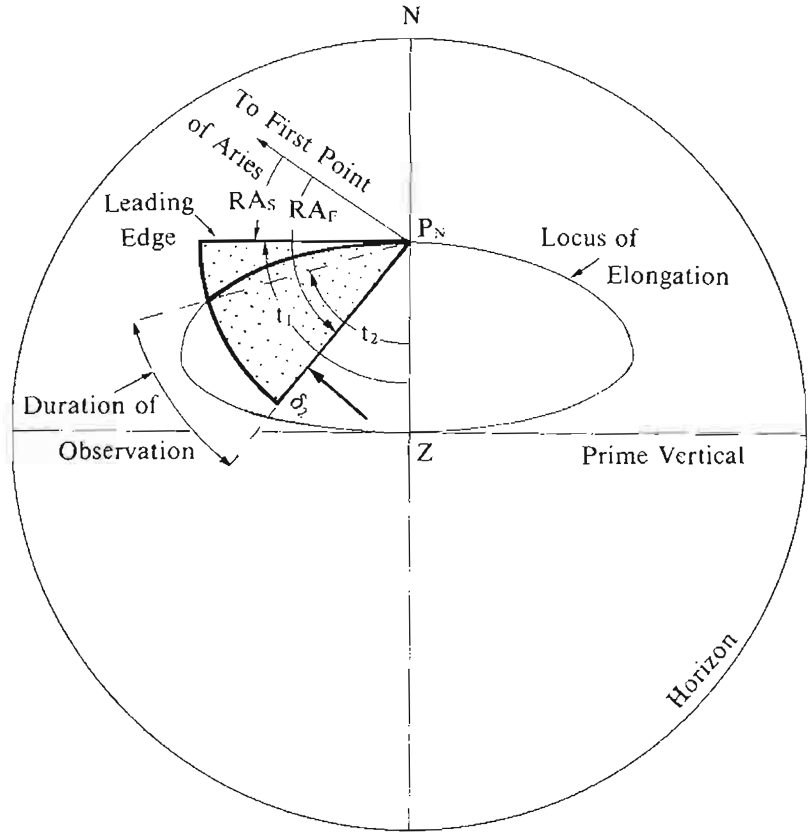


Fig 10.7 This figure illustrates the situation at the start of the observing period

	East	West
LST of the start of the programme	$10^h 47^m$	$10^h 47^m$
Largest hour angle (see section 10.51)	18 53	6 00
RA_S	15 54	4 47
Duration of the programme	4 30	4 30
Hour angle difference (t_1 - t_2)	53	53
RA_F	21 17	10 10

Table 10.9 Available Stars

East						West					
No.	Mag	RA	δ	t	LST	No.	Mag	RA	δ	t	LST
664	5.0	16^h18^m	$+75°49'$	18^h52^m	11^h10^m	655	5.1	4^h56^m	$+81°10'$	5^h28^m	10^h24^m*
453	4.4	16 49	+82 05	18 28	11 17	656	5.2	5 18	+79 12	5 21	10 39 *
665	5.0	17 51	+76 58	18 47	12 38	657	4.7	6 56	+77 01	5 13	12 09
666	5.1	19 10	+76 31	18 49	13 59	658	5.3	8 01	+79 33	5 22	13 23
557	4.4	20 10	+77 38	18 45	14 55	659	4.6	9 34	+81 26	5 29	15 03

* Some stars with certain combinations of hour angle and declination do not elongate within the given range of LST's. These stars (marked *) may then be rejected, but it should be noted that, by using the aforegoing procedure, we include in our list of available stars *all* which elongate between the programme time limits (see section 10.52).

Table 10.10 Selected Azimuth Stars

Pair No.	Aspect & Star No.	LST	Diff	Standard Time on Prediction Date	Zenith Distance	ZD Change for $+10^m$	Azimuth
Start of Prediction Period		10^h47^m		20^h00^m			
			30^m				
1	NE 453	11 17		20 30	$48°01'$	$-0°21'$	$10°36'$
			52				
2	NW 657	12 09		21 22	47 09	+0 34	342 33
			29				
2	NE 665	12 38		21 51	47 09	-0 34	17 31
			45				
3	NW 658	13 23		22 36	47 38	+0 27	345 59
			36				
3	NE 666	13 59		23 12	47 03	-0 35	18 08
			1 04				
1	NW 659	15 03		00 16	47 56	+0 22	348 32
		$\Delta\ 4^h16^m$	$\Sigma 4^h16^m$✓	$\Delta\ 4^h16^m$✓			

Table 10.11 Working List

Pair	Aspect & Star No.	Mag	Vertical Circle		LST	Standard Time	Horizontal Circle	
			CL	CR			CL	CR
1	NE 453	4.4	$48°22'$	$311°38'$	11^h07^m	20^h20^m	$10°36'$	$190°36'$
			48 01	311 59	11 17	20 30		
			47 40	312 20	11 27	20 40		
2	NW 657	4.7	46 35	313 25	11 59	21 12	342 33	162 33
			47 09	312 51	12 09	21 22		
			47 43	312 17	12 19	21 32		
2	NE 665	5.0	47 43	312 17	12 28	21 41	17 31	197 31
			47 09	312 51	12 38	21 51		
			46 35	313 25	12 48	22 01		

Table 10.11 (contd)

Pair	Aspect & Star No.	Mag	Vertical Circle		LST	Standard Time	Horizontal Circle	
			CL	CR			CL	CR
3	NW 658	5.3	47°11'	312°49'	13^h13^m	22^h26^m	345°59'	165°59'
			47 38	312 22	13 23	22 36		
			48 05	311 55	13 33	22 46		
3	NE 666	5.1	47 38	312 22	13 49	23 02	18 08	198 08
			47 03	312 57	13 59	23 12		
			46 28	313 32	14 09	23 22		
1	NW 659	4.6	47 34	312 26	14 53	0 06	348 32	168 32
			47 56	312 04	15 03	0 16		
			48 18	311 42	15 13	0 26		

10.54 Star pairs should be selected mainly by means of declination balance. In addition, sufficient time should be allowed for the purpose of completing the observations on each star. This requires a period of approximately twenty minutes between any two stars, observed in succession. It is not necessary for a maximum period between the two stars of a balanced pair to be specified, as is required for the longitude and the latitude pairs. This is so, whether time azimuth or altazimuth observations are made, provided they are both made on stars near their points of elongation. (see section 7.71)

10.55 An example of the preparation of a predicted programme for the determination of azimuth, for a station situated in a low latitude, for the following circumstances:-

Station Position: Latitude 9°27' S
Longitude 9^h49^m E (Time Zone 10^h E)
Date: 10th May 1975
Programme: Start at about 20^h Standard Time
Duration about 2^h
Minimum altitude: 15°
Maximum altitude: 30°
Declination balance: ±2°
Duration of observation on each star: about 20^m

Calculation of the LST of the start and finish of the programme.

Standard Time of the start of the programme	20^h00^m	
Zone	10	E
UT	10 00	
R	15 10	
GST	1 10	
λ	9 49	E
LST of the start of the programme	10 59	
LST of the finish of the programme	12 59	

To ensure that observations are kept within the altitude limits of 15° and 30°, which have been imposed from considerations of atmospheric transparency and the minimisation of the effects of transverse dislevelment respectively, additional arbitrary altitude limits of 20° and 25° have been set. Declinations and hour angles may now be computed corresponding to the limits from

$$\sin \delta_e = \frac{\sin \phi}{\sin h_e} \qquad \text{and} \qquad \sin t_e = \pm\frac{\cos h_e}{\cos \phi}$$

The results of this computation are:-

Declination	$-22^{\circ}51'$	to	$-28^{\circ}41'$
Hour Angle	$\pm 4^h27^m$	to	$\pm 4^h49^m$

Using the principles stated in section 10.51, the ranges of RA may be calculated as follows:-

	East	West
LST of the start of the programme	10^h59^m	10^h59^m
Largest hour angle (see section 10.51)	19 33	4 49
RA_S	15 26	6 10
Duration of programme	2 00	2 00
Hour angle difference $(t_1 - t_2)$	22	22
RA_F	17 48	8 32

Table 10.12 Available Stars

East						West					
No.	Mag	RA	δ	t	LST	No.	Mag	RA	δ	t	LST
413	3.8	15^h36^m	$-28^{\circ}03'$	-4^h47^m	10^h49^m*	192	3.7	7^h01^m	$-27^{\circ}54'$	4^h47^m	11^h48^m
426	3.0	15 57	-26 03	-4 40	11 17	193	3.1	7 02	-23 48	4 31	11 33
438	3.1	16 20	-25 32	-4 38	11 42	196	2.0	7 07	-26 21	4 41	11 48
441	1.2	16 28	-26 23	-4 42	11 46	200	3.8	7 14	-26 44	4 43	11 57
447	2.9	16 34	-28 10	-4 48	11 46	218	3.5	7 48	-24 48	4 36	12 24
469	3.4	17 21	-24 59	-4 36	12 45	222	2.9	8 06	-24 14	4 33	12 39
472	4.3	17 25	-24 09	-4 33	12 52						

* See note section 10.53

For the calculation in Table 10.12 and later, the following relationships are obtained from the right angled triangle at elongation,

$$\cos t_e = \frac{\tan \phi}{\tan \delta} \qquad \cos z_e = \frac{\sin \phi}{\sin \delta} \qquad \sin A_e = \pm \frac{\cos \delta}{\cos \phi}$$

For the final selection of star pairs the reader is referred to section 10.54.

Table 10.13 Selected Azimuth Stars

Pair No.	Aspect & Star No.	LST	Diff	Standard Time on Prediction Date	Zenith Distance	ZD Change for $+10^m$	Azimuth
Start of Prediction Period		10^h59^m		20^h00^m			
			18^m				
1	SE 426	11 17		20 18	$68^{\circ}02'$	$-2^{\circ}15'$	$114^{\circ}24'$
			31				
1	SW 196	11 48		20 49	68 17	+2 14	245 17
			36				
2	SW 218	12 24		21 25	66 57	+2 16	246 58
			28				
2	SE 472	12 52		21 53	66 20	-2 17	112 20
		$\Delta\ 1^h53^m$	$\Sigma 1^h53^m$✓	$\Delta\ 1^h53^m$✓			

Table 10.14 Working List

Pair	Aspect & Star No.	Mag	Vertical Circle CL	Vertical Circle CR	LST	Standard Time	Horizontal Circle CL	Horizontal Circle CR
1	SE 426	3.0	70°17'	289°43'	11^h07^m	20^h08^m	114°24'	294°24'
			68 02	291 58	11 17	20 18		
			65 47	294 13	11 27	20 28		
1	SW 196	2.0	66 03	293 57	11 38	20 39	245 17	65 17
			68 17	291 43	11 48	20 49		
			70 31	289 29	11 58	20 59		
2	SW 218	3.5	64 41	295 19	12 14	21 15	246 58	66 58
			66 57	293 03	12 24	21 25		
			69 13	290 47	12 34	21 35		
2	SE 472	4.3	68 37	291 23	12 42	21 43	112 20	292 20
			66 20	293 40	12 52	21 53		
			64 03	295 57	13 02	22 03		

Azimuth from Circum-Meridian Observations

10.56 An example of the preparation of a predicted programme for the determination of azimuth in equatorial latitudes from circum-meridian observations is given below for the same circumstances as those of section 10.55. Theory is given in section 7.31 and discussion in section 7.33.

In this method, pairs of stars, each of which comprises one star to the north and one to the south, are observed at upper transit. The stars forming a pair have balanced time rates of change of azimuth, see section 7.33 and 7.34. This balance of rates is achieved when the following condition is satisfied,

$$\tan \delta_N + \tan \delta_S = 2 \tan \phi$$

The altitude of a star observed on the equatorial side of the zenith is always smaller than that of the other star of the pair. For the former stars, a minimum of 15° and a maximum of 30° will be assigned as their limits of altitude. From these limits, the corresponding values of declination for both north and south stars may then be evaluated as follows,

For $\phi = -9°27'$

At upper transit $\phi = \delta - z_M$

North Stars

$h = 15°$ $z_M = +75°$ $\delta_N = +65°33'$)
$h = 30$ $z_M = +60$ $\delta_N = +50\ 33$) Declination limits

South Stars

$$\tan \delta_N + \tan \delta_S = 2 \tan \phi$$

For $\delta_N = +65°33'$ $\delta_S = -68°27'$)
For $\delta_N = +50\ 33$ $\delta_S = -57\ 08$) Declination limits

From these limits of declination and from the limits of LST, which here equal those of the RA values, a set of available stars is taken out in Table 10.15, in which the $\frac{dA}{dt}$ rates are also shown. From these, by balancing rates and allowing sufficient time for the observation to be made, Table 10.16 shows the stars selected and Table 10.17 the final working list.

Table 10.15 Available Stars

North

No.	Mag	RA	δ_N	z_M	$\frac{dA}{dt}$	Pair
298	1.9	11^h02^m	$+61^\circ52'$	$+71^\circ19'$	-0.50	1,2,3
318	2.5	11 52	+53 49	+63 16	-0.66	4,5,6,7
323	3.4	12 14	+57 10	+66 37	-0.59	8,9,10
342	1.7	12 53	+56 05	+65 32	-0.61	11,12,13

South

No.	Mag	RA	δ_S	z_M	$\frac{dA}{dt}$	Pair
299	4.0	11^h07^m	$-58^\circ50'$	$-49^\circ23'$	+0.68	4
311	3.3	11 35	-62 53	-53 26	+0.57	8,11
312	3.8	11 45	-66 35	-57 08	+0.47	1
322	3.1	12 14	-58 37	-49 10	+0.69	5
327	3.6	12 20	-60 15	-50 48	+0.64	6,12
328	1.6	12 25	-62 57	-53 30	+0.57	2,9
339	3.3	12 45	-67 58	-58 31	+0.44	3
340	1.5	12 46	-59 33	-50 06	+0.66	7,10,13

Table 10.16 Selected Stars

Pair No.	Aspect & Star No.	LST	Diff	Standard Time on Prediction Date	Zenith Distance	Azimuth	Azimuth Change for $+10^m$
Start of Prediction Period		10^h59^m		20^h00^m			
4	S 299	11 07	8^m	20 08	$-49^\circ23'$	$180^\circ00'$	$+1^\circ42'$
8	S 311	11 35	28	20 36	-53 26	180 00	+1 25
4	N 318	11 52	17	20 53	+63 16	00 00	-1 40
8	N 323	12 14	22	21 15	+66 37	00 00	-1 29
		$\Delta\ 1^h15^m$	$\Sigma 1^h15^m$	$\Delta\ 1^h15^m$			

Table 10.17 Working List

Pair	Aspect & Star No.	Mag.	Vertical Circle CL	Vertical Circle CR	LST	Standard Time	Horizontal Circle CL	Horizontal Circle CR
4	S 299	4.0	$49^\circ23'$	$310^\circ37'$	10^h57^m	19^h58^m	$178^\circ18'$	$358^\circ18'$
					11 07	20 08	180 00	00 00
					11 17	20 18	181 42	1 42
8	S 311	3.3	53 26	306 34	11 25	20 26	178 35	358 35
					11 35	20 36	180 00	00 00
					11 45	20 46	181 25	1 25
					A slight overlap here			

Table 10.17 (contd)

Pair	Aspect & Star No.	Mag	Vertical Circle		LST	Standard Time	Horizontal Circle	
			CL	CR			CL	CR
4	N 318	2.5	$63^\circ16'$	$296^\circ44'$	11^h42^m	20^h43^m	$1^\circ40'$	$181^\circ40'$
					11 52	20 53	00 00	180 00
					12 02	21 03	358 20	178 20
8	N 323	3.4	66 37	293 23	12 04	21 05	1 29	181 29
					12 14	21 15	00 00	180 00
					12 24	21 25	358 31	178 31

LATITUDE AND LONGITUDE FROM POSITION LINE OBSERVATIONS

10.61 AN example of the preparation of a predicted programme for the simultaneous determination of latitude and longitude for the following circumstances:-

Station position: Latitude $41^\circ30'$ N
Longitude 5^h26^m W (Time Zone 5^h W)

Date: 10th May 1975
Programme: Start at about 20^h Standard Time
Duration about 2^h
Altitude: 45°
Azimuth: Close to the azimuth quadrant centres
Duration of observation on each star: About 10^m

Calculation of the LST of the start and finish of the programme:

This is the same as that in section 10.31, where

LST of the start of the programme 10^h47^m

LST of the finish of the programme 12^h47^m

To assist in the initial selection of stars from the catalogue, two diagrams have been constructed (Figs 10.8(a) and (b)) from which the limits of declination and hour angle can be read off without preliminary calculations. For the preparation of these diagrams, declinations have been limited to a band 10° wide centred on points, which have zenith distances of 45° in the centres of the azimuth quadrants. From these limits of declination corresponding values of the hour angle have been calculated.

From Figs 10.8(a) and (b) for $\phi = 41^\circ30'$ N, the following is read off:

Quadrant	NE	&	NW	SE	&	SW
Declination	$+52^\circ.5$	to	$+62^\circ.5$	$+0^\circ.4$	to	$+10^\circ.4$
Hour Angle	$\pm 4^h26^m$	to	$\pm 4^h39^m$	$\pm1^h21^m$	to	$\pm 2^h28^m$

From these limits of hour angle the ranges of RA may be calculated.

10.62 For position line observations, the sky window of section 10.42 becomes a line, as in section 10.52. The star band is the same as that of section 10.42. At the start of the observing period, the leading edge of the star band is shown in Fig 10.9 to be situated at the end of the line representing the sky window. It also shows that some stars in the star band have already crossed the almucantar and are not available for observation. In addition, at the end of the observing period, it is clear that some stars in the star band have not reached this line. Comparison with sections 10.42 and 10.52 is instructive.

Fig 10.8(a) Limits of Hour Angle for Position Line Observations

	Latitude North	Latitude South	Legend
Quadrants	SE & SW	NE & NW	———
	NE & NW	SE & SW	— — —

Zenith Distance = 45°

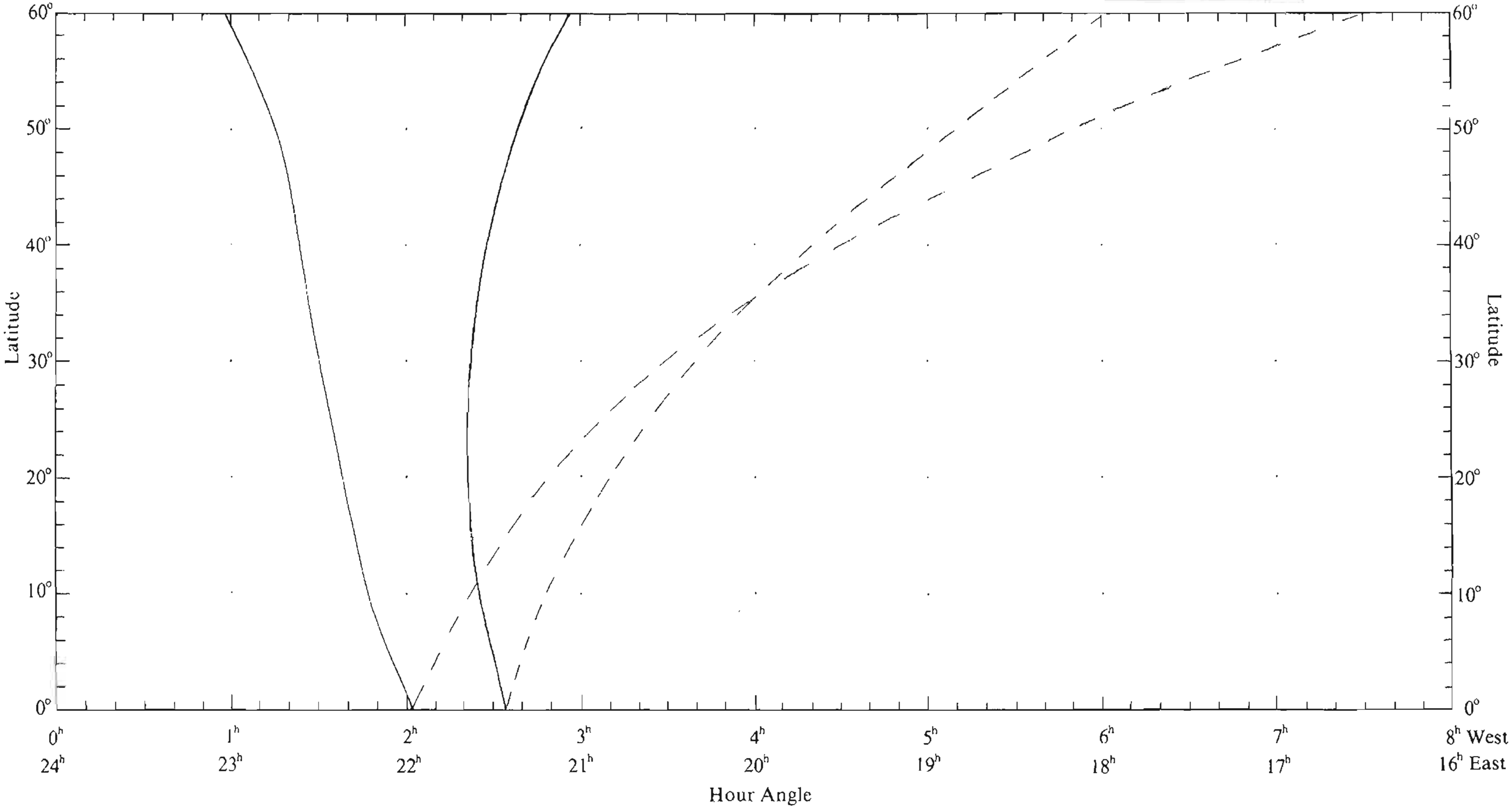


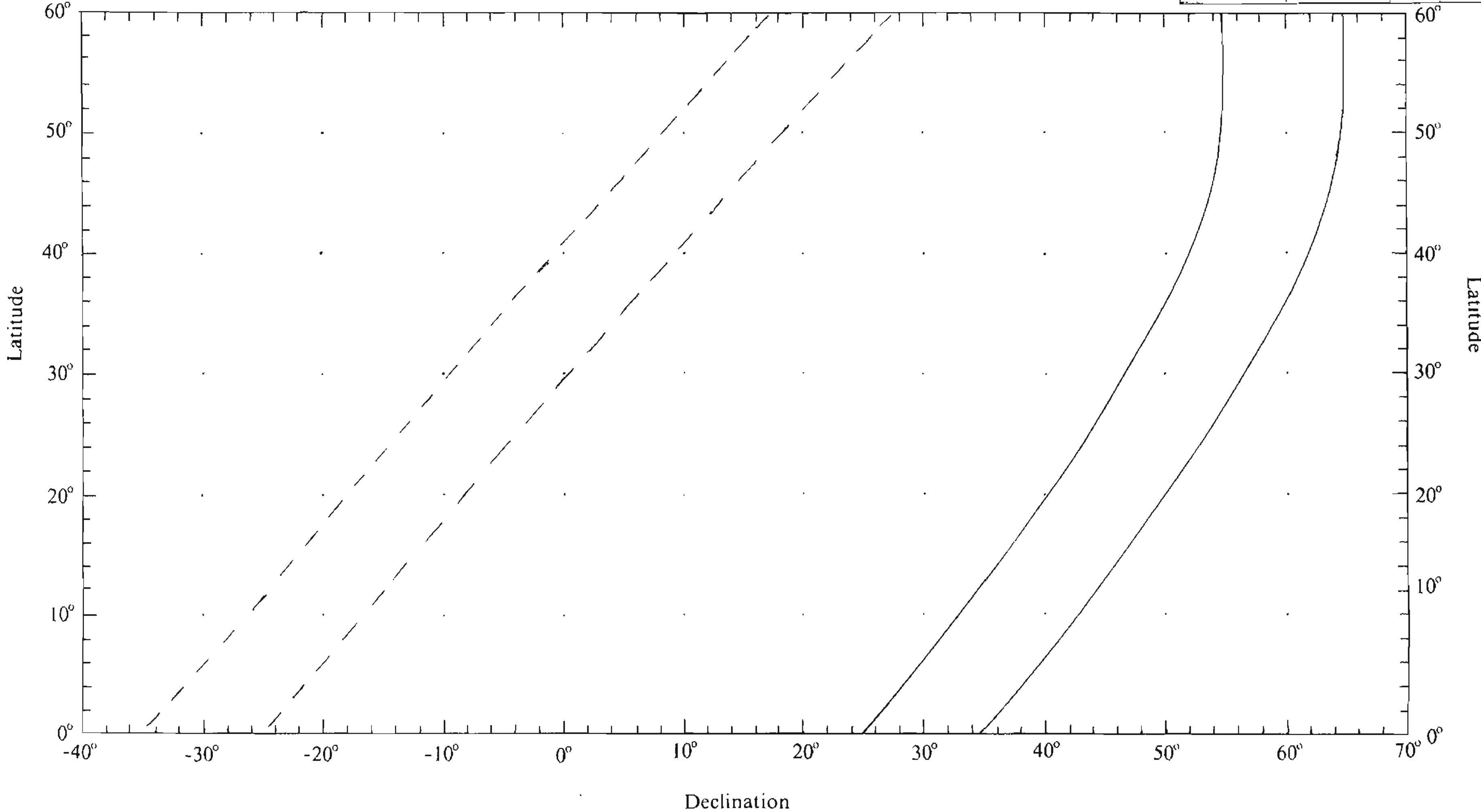
Fig 10.8(b) Limits of Declination for Position Line Observations

Declination band is 10° wide

Zenith Distance = 45°

For stations in south latitude, signs of latitude and declination are changed

	Latitude		Legend
	North	South	
Quad-rants	SE&SW	NE&NW	———
	NE&NW	SE&SW	— — —



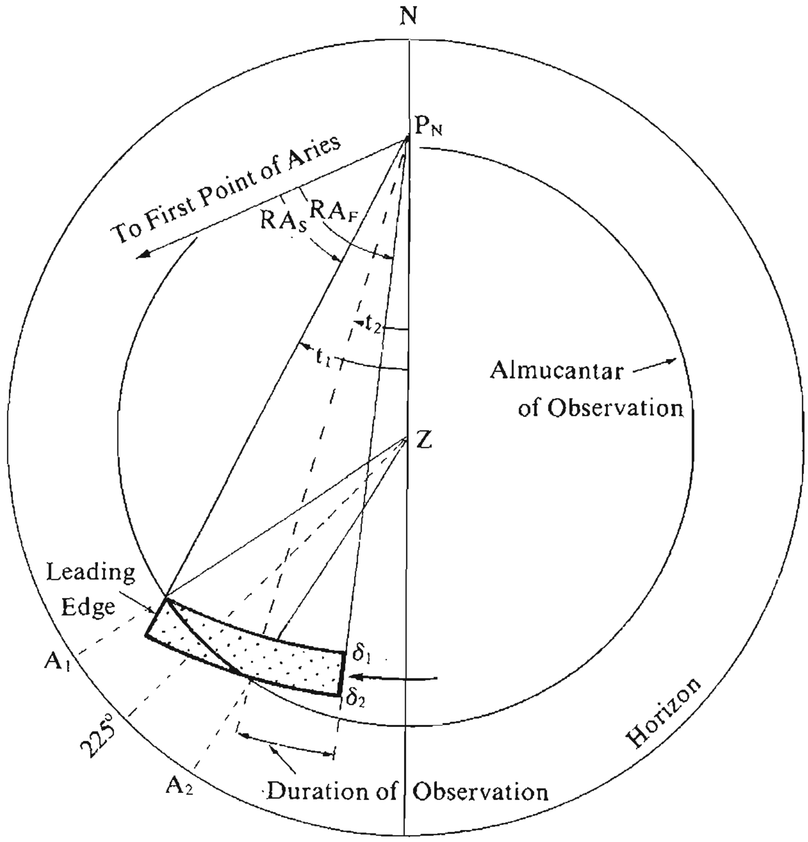


Fig 10.9 This figure illustrates the situation at the start of the observations

10.63 From the limits of hour angle, previously determined at the end of section 10.61 and in the same manner as that of section 10.43 the ranges of RA may be determined as follows:-

	NE	NW	SE	SW
LST of the start of the programme	10^h47^m	10^h47^m	10^h47^m	10^h47^m
Largest hour angle (see section 10.61)	19 34	4 39	22 39	2 28
RA_S	15 13	6 08	12 08	8 19
Duration of Programme	2	2	2	2
Hour angle difference (t_1 - t_2)	13	13	1 07	1 07
RA_F	17 26	8 21	15 15	11 26

Table 10.18 Available Stars

NE						NW					
No.	Mag	RA	δ	t	LST	No.	Mag	RA	δ	t	LST
405	3.5	15^h24^m	$59^\circ03'$N	19^h25^m	10^h49^m	173	4.4	6^h17^m	$59^\circ01'$N	4^h35^m	10^h52^m
429	4.1	16 01	58 38 N	19 25	11 26						
440	2.9	16 24	61 34 N	19 22	11 46						
SE						SW					
343	3.7	12 54	3 32 N	22 12	11 06	232	4.2	8 36	5 47 N	2 03	10 39 *
363	4.3	14 00	1 40 N	22 27	12 27	240	3.5	8 45	6 30 N	2 07	10 52
387	3.8	14 45	2 00 N	22 24	13 09*	242	3.3	8 54	6 02 N	2 05	10 59
						254	3.8	9 13	2 25 N	1 39	10 52
						265	3.8	9 40	10 00 N	2 26	12 06
						288	3.8	10 32	9 26 N	2 24	12 56 *
						306	4.1	11 20	6 10 N	2 05	13 25 *

*Some stars with certain combinations of hour angle and declination will not cross the 45° almucantar within the given range of LST's. These stars (marked *) may then be rejected but it should be noted that, by using the aforegoing procedure, there are included in our list of available stars, *all* which will cross this almucantar between the programme time limits. (see section 10.62)

One now selects 4 stars, one in each of the quadrants, such that it is possible to observe them in the shortest possible period.

Table 10.19 Selected Position Line Stars

Aspect & Star No.	LST	Diff	Standard Time on Prediction Date	Azimuth A	ZD Change for $+5^m$	Az Change for $+5^m$
Start of Prediction Period	10^h47^m		20^h00^m			
NW 173	10 52	5^m	20 05	$317^\circ14'$	$+0^\circ38'$	$+0^\circ08'$
NE 429	11 26	34	20 39	43 19	-0 39	+0 09
SW 265	12 06	40	21 19	236 10	+0 47	+1 21
SE 363	12 27	21	21 40	146 05	-0 31	+1 36
	$\Delta\ 1^h40^m$	$\Sigma 1^h40^m$	$\Delta 1^h40^m$			

Table 10.20 Working List

Aspect	No.	Mag	Vertical Circle		LST	Standard Time	Horizontal Circle	
			CL	CR			CL	CR
NW	173	4.4	$44^\circ22'$	$315^\circ38'$	10^h47^m	20^h00^m	$317^\circ06'$	$137^\circ06'$
			45 00	315 00	10 52	20 05	317 14	137 14
			45 38	314 22	10 57	20 10	317 22	137 22

Table 10.20 (contd)

Aspect	No.	Mag	Vertical Circle		LST	Standard Time	Horizontal Circle	
			CL	CR			CL	CR
NE	429	4.1	45°39'	314°21'	11^h21^m	20^h34^m	43°10'	223°10'
			45 00	315 00	11 26	20 39	43 19	223 19
			44 21	315 39	11 31	20 44	43 28	223 28
SW	265	3.8	44 13	315 47	12 01	21 14	234 49	54 49
			45 00	315 00	12 06	21 19	236 10	56 10
			45 47	314 13	12 11	21 24	237 31	57 31
SE	363	4.3	45 31	314 29	12 22	21 35	144 29	324 29
			45 00	315 00	12 27	21 40	146 05	326 05
			44 29	315 31	12 32	21 45	147 41	327 41

Combined Observation Programmes

10.71 The previous examples illustrate methods of prediction for the separate determination of latitude, longitude and azimuth. Sometimes it may be necessary to compile a programme for the determination of all three elements together (position lines excluded) and, in this case, one can still use the same techniques, although the order in which the prediction for each element is done, can be critical. In most cases, it will be found that if one restricts oneself to observing close circumpolar stars, there are only a few stars available for azimuth determination, and therefore one should predict for this element first. Now, because the latitude observations may also require high declination stars, one should endeavour to fill in the gaps between the times of the azimuth stars with latitude star pairs. After this, the remaining gaps may be filled with longitude star pairs, which are often quite numerous, because these stars are chosen from the lower declination ranges. In practice, there may arise situations where this order of prediction leads to difficulties, which should not be construed as being a failure of the principle but as being due to a situation brought about by irregular star distribution, a fact which is easy to verify from a casual look at the night sky.

PREDICTION AIDS

10.81 THE prediction procedures used in the previous examples can all be described in strict mathematical terms, which may then be translated into computer programmes. These programmes, in conjunction with a data set made up from a star catalogue, can then be used for the preparation of working lists. The preparation of such programmes may be quite difficult and time consuming, notwithstanding the apparent simplicity of the prediction process (eg. meridian transit observations require a large number of complicated logical branching statements) and, therefore, there should be a continuing need for such programmes to justify their compilation.

Aids to prediction, other than the computer may be divided into two broad classes,

(a) those, which assist in the calculation of ranges of hour angle and declination before the stars are selected from the catalogue
(b) those, from which the stars for observation are directly selected.

Some of the former class of aids have been used for the preparation of the previous examples, where it can be seen that no great accuracy is required in the construction of these graphs or in the results obtained from them. If

Table 10.21 Balanced Pairs of Longitude Stars

West Star							LST		East Star						
Cat No.	Mag	Z.D.		AZ					AZ		Z.D.		Mag	Cat No.
		Lat40	Lat45	Lat40	Lat45	Hrs	Min	Lat40	Lat45	Lat40	Lat45		
254	3.2	40.1	41.0	262.2	256.5	9	42	89.0	94.2	44.3	44.4	4.3	492
		42.0	42.7	264.1	258.7	9	52	90.6	96.1	42.4	42.7		
		43.9	44.5	265.9	260.8	10	02	92.3	98.1	40.5	40.9		
1158	4.5	51.3	51.0	275.9	271.9	9	57	77.0	80.4	55.4	54.4	3.8	534
		53.3	52.8	277.4	273.6	10	7	78.3	82.0	53.5	52.7		
		55.1	54.5	278.8	275.3	10	17	79.6	83.6	51.6	50.9		
1168	4.5	48.9	48.5	276.9	272.6	10	11	78.8	82.6	52.8	52.0	3.8	534
		50.8	50.2	278.3	274.2	10	21	80.2	84.3	50.9	50.2		
		52.7	52.0	279.7	275.9	10	31	81.5	85.9	49.0	48.4		
1168	4.5	49.3	48.8	277.2	272.9	10	13	79.4	83.2	53.2	52.5	4.5	1380
		51.2	50.6	278.6	274.6	10	23	80.8	84.8	51.4	50.7		
		53.0	52.3	280.0	276.2	10	33	82.1	86.4	49.5	49.0		

Table 10.22 Balanced Sets of Position Line Stars

NW QUADRANT									NE QUADRANT								
Cat No.	Mag	AZIMUTH			Z.D.			LST	Cat No.	Mag	AZIMUTH			Z.D.			LST
		Lat-26	A	Lat-24	Lat-26	B	Lat-24	Hrs Min			Lat-26	A	Lat-24	Lat-26	B	Lat-24	Hrs Min
1603	4.7	316.5	2.7	315.1	45.8	-1.6	44.4	1 06	1058	4.5	44.5	2.7	45.9	45.7	1.6	44.3	0 10
1606	5.2	315.0	2.6	313.6	45.8	-1.6	44.4	1 14	85	4.3	45.4	2.6	46.9	45.7	1.6	44.3	0 23
28	4.6	312.6	2.6	311.2	45.7	-1.7	44.4	2 56	98	4.4	41.8	2.7	43.1	45.7	1.5	44.3	0 48
36	4.5	313.3	2.6	311.8	45.8	-1.7	44.4	3 09	1083	4.7	44.4	2.7	45.8	45.7	1.6	44.3	0 57

SW QUADRANT									SE QUADRANT								
Cat No.	Mag	AZIMUTH			Z.D.			LST	Cat No.	Mag	AZIMUTH			Z.D.			LST
		Lat-26	A	Lat-24	Lat-26	B	Lat-24	Hrs Min			Lat-26	A	Lat-24	Lat-26	B	Lat-24	Hrs Min
829	2.2	228.2	-0.5	226.8	44.4	-1.7	45.8	1 27	187	4.9	135.5	-0.6	136.9	44.3	1.6	45.8	1 48
860	3.7	221.8	-0.7	220.4	44.2	-1.5	45.7	2 00	199	5.5	137.1	-0.6	138.4	44.2	1.5	45.7	2 05
856	2.2	228.3	-0.5	226.8	44.3	-1.7	45.7	2 01	1152	5.5	131.8	-0.5	133.3	44.3	1.7	45.7	2 10
15	4.9	225.5	-0.5	224.1	44.4	-1.6	45.8	3 48	263	2.8	137.0	-0.6	138.3	44.3	1.5	45.8	3 35

The values of azimuth and zenith distance 10 minutes earlier than those tabulated may be found by adding the A factor to the azimuth and the B factor to the zenith distance.

one wishes to use aids from the latter class, there are two principal ways of representing this information. The first of these is to use tables such as "Star prediction tables for the fixing of position" prepared by G.G. Bennett, J.G. Freislich and M. Maughan and published as a monograph by the School of Surveying, University of New South Wales, Australia. These tables extend over a latitude range from 60°S to 60°N and contain stars selected from the Apparent Places of Fundamental Stars (FK4) catalogue. The tables are so constructed that a working list of near prime vertical star pairs and sets of position line stars can be prepared quite quickly by simple linear interpolation between the tabulated values. Extracts from these tables appear in Tables 10.21 and 10.22.

The second way of obtaining star information directly is to use a planisphere or similar device, which often has some mechanical movement. Planispheres have their origins in antiquity, when the term astrolabe was applied to observing instruments, which incorporated a star chart. Modern planispheres usually consist of a central circular disc, upon which the stars have been plotted on either a stereographic or a polar equidistant projection. A transparent cursor or mask, upon which altitude and azimuth lines have been plotted, rotates around the centre of the chart (the visible pole) thus simulating the daily motion of the stars. If the planisphere is to be used over a range of latitudes, then it is necessary to have a number of cursors for the various latitudes, because each cursor represents only the situation for the latitude used in the calculation of its azimuth and altitude lines. This is the chief limitation of the planisphere, because, if one wishes to make accurate readings from it, then it must be large and with it there must be provided a considerable number of cursors. For a teaching situation, where instruction is given in one latitude, it is an ideal aid and it can be used not only for prediction purposes but also for demonstrating sun and star movements in a very clear and easily understandable manner. A photograph of such a planisphere appears in Fig 10.10. A small planisphere called the "Star Finder and Identifier", based upon the principles of Rude and Collins, often called simply the "Rude Star Identifer" is produced by the U.S. Naval Oceanographic Office (for reference when ordering, No. 2101-D). The star chart of this planisphere is approximately 21 cm in diameter and has plotted on it the 57 bright stars listed in the Air and Nautical Almanacs. It is also possible to plot on this chart the position of other celestial bodies, such as the sun and the planets. Transparent cursors, 9 in number are also supplied. These have latitude values between 85°S and 85°N in steps of 10°. The planisphere is intended for use in sea and air navigation and although small, is also very useful for star observations taken on land.

For position line observations, especially when an astrolabe is employed, the prediction process is simplified considerably because one of the variables viz. altitude, is fixed. A variety of tables, diagrams and charts have been produced to assist in the prediction process for this method.

A star globe is usually found where any astronomy is being taught. This can be made into an efficient means of star prediction. An excellent type is one, supported in a cradle, which allows the globe to rotate about the polar axis and also allows the axis to be set at various angles of inclination, which correspond to the latitude. A yoke or similar device may then be placed over the globe to allow the altitude and azimuth of the stars to be read off for any desired time. An illustration of two such globes appears in Fig. 10.11. These globes are normally employed for teaching purposes, because their bulk renders them inconvenient for field use. Both planispheres and star globes may be used for star identification purposes, when often only a rough knowledge of the RA and declination of a star is sufficient to identify the star in the catalogue.

Fig 10.10 A Planisphere

STAR IDENTIFICATION

10.91 SOMETIMES when observations are computed, it is found that the star observed was not the one, which it was intended to observe. This may result from one or more of the following causes:-

(i) Observations were undertaken with a sketchy or incorrect working list or even without any working list at all.
(ii) The horizontal circle was incorrectly oriented.
(iii) Observations were made on the incorrect one of two or more stars in the field of view of the theodolite telescope.

If this situation arises and the surveyor considers it essential to include these observations in his work, he must then identify the star in a star catalogue from a calculation of its right ascension and declination. For this purpose it will be assumed that the following information is available.

(i) The station's position (latitude and longitude).
(ii) An observed altitude of the body.
(iii) The clock time of the observation and further information so that the Local Sidereal Time of the observation may be determined.
(iv) An observed azimuth of the body.

These quantities need not be known accurately, since an error of one tenth of an arc-degree can usually be comfortably tolerated.

For this calculation, one may use the Cosine and the Four Parts Formulae of section 2.75, namely

$$\sin \delta = \sin h \sin \phi + \cos h \cos \phi \cos A \qquad \text{...10.4}$$

$$\tan t = \frac{- \sin A}{\cos \phi \tan h - \sin \phi \cos A} \qquad \text{...10.5}$$

with a check from the Five Parts Formula given as

$$\cos \delta \cos t = \sin h \cos \phi - \cos h \sin \phi \cos A \qquad \text{...10.6}$$

Also

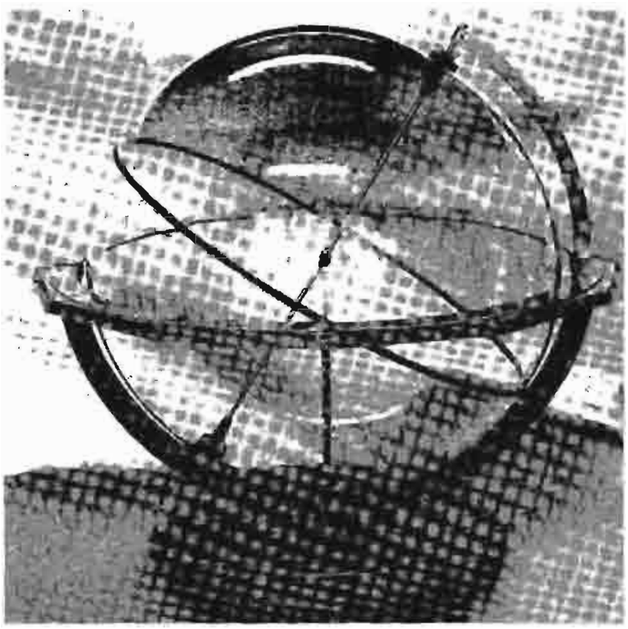
$$RA = LST - t \qquad \text{...10.7}$$

The catalogue may then be searched for a star, having the corresponding RA and δ . If this is successful, the computation can then be completed.

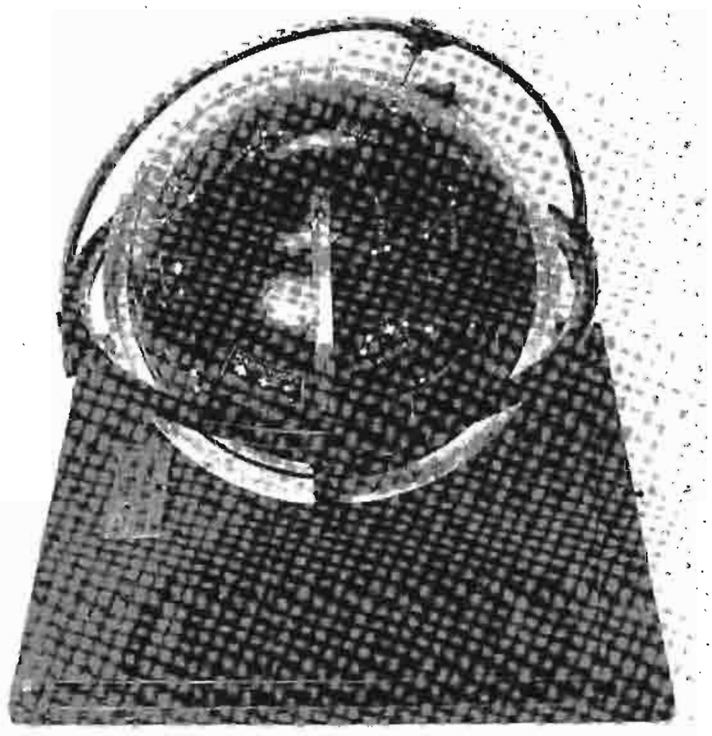
Sometimes, however, the search may be fruitless. This is likely to occur if the catalogue used contains the positions of only a limited number of stars. The principal catalogues used by surveyors are:-

Name	Number of Stars Listed
The Star Almanac for Land Surveyors	685
The Apparent Places of Fundamental Stars (FK4)	1 535
The Supplementary Catalogue to the FK4	1 990
The Boss General Catalogue	33 342
The Smithsonian Astrophysical Observatory Catalogue	258 997

The Star Almanac for Land Surveyors, an annual publication, gives apparent places of all stars brighter than magnitude 4 and some additional stars of high declination with magnitudes to 5.5. The FK4, also an annual publication, lists stars up to magnitude 7.9, but does not include all stars up to this limiting magnitude, because the catalogue is intended to provide a general coverage. Where a cluster of bright stars appears, the positions of only some of these stars will be given. The remaining catalogues in the list are not so convenient to use for two reasons. Firstly, the coordinates are not published annually but for a specific epoch, e.g. the Boss Catalogue contains



A clear plastic star globe



A McCormick star globe

Fig 10.11 Star Globes

the mean places of stars for the epoch 1950. This necessitates a special calculation for obtaining the apparent place at the epoch of observation. Secondly, these catalogues are not as readily available as the first two, because copies, either in book form or on computer files, are generally held only by survey or mapping authorities, observatories and some teaching institutions. These inconveniences should be kept in mind, when observing programmes and working lists are being prepared; otherwise, if attention is not given to these aspects in the preparatory phase of the task, additional calculations may be necessary and delays may be incurred in the subsequent processing of the observations.

The calculation of the RA and declination, referred to before, may not lead to a successful identification, even if the larger catalogues are used. In many instances, this failure to identify the star is due to a poor orientation of the horizontal circle and therefore the azimuth assumed in the calculation is in doubt. The identification may be effected by duplicating the calculation of the RA and declination with a value of azimuth a few degrees different from that used in the original calculation and plotting the star positions resulting from the calculations on a star chart, such as that given in Norton's Star Atlas or on a Modified Mercator Chart similar to that used for plotting astronomical position lines (see section 9.21 and Figs 9.3 and 10.12). It should be kept in mind that, in this case, the chart is being plotted on a small scale and a large area of the sky is shown. Thus the scale is not uniform over the area shown and, as a result, projection distortions, considered negligible in the very large scale chart on which only a small area is represented, can no longer be considered negligible. Several points should therefore be computed for plotting to justify the drawing of a straight line between adjacent points. Such duplicate calculation is easily carried out. An example of such a plot is shown in Fig 10.12. Data for this example is found in section 10.93.

If a horizontal circle reading is not included in the observations and the star observed is mis-identified, it is often possible to obtain a preliminary value of the azimuth, as a starting value for the identification search, from the observed rate of change of altitude with time. This is given by

$$\frac{dh}{dt} = \cos\phi \sin A$$

so that

$$\sin A = \frac{dh}{dt} \sec\phi$$

10.92 Example. The data of section 9.81 will be used. The star was known to lie in the north western sky. The latitude was $33^\circ 55'$S and the time rate of change of altitude obtained from the first and last observation was found to be -8.86 arc-seconds per clock second. From this the azimuth is found to be 225.2° or 314.8°. The ambiguity here is resolved, because the star was observed to the north west. In fact, this ambiguity raises difficulty only when the star observed lies near the Prime Vertical. The identification process is now started with this value of 314.8° as a preliminary value of the azimuth.

10.93 Example of a star identification

The following information was available from an observation made on an unknown star towards the south east.

Latitude of the Station	$33^\circ 55'$ S
Observed Altitude	$44^\circ 18'$
Local Sidereal Time of the observation	$5^h 23^m 08^s$

An approximate azimuth of 140° was assumed as a starting value.

With this information the following was obtained by means of Equations 10.4 to 10.7.

Azimuth 140° t $-3^h57^m01^s$ RA $9^h20^m09^s$ δ $-57°38'$

A search in the Star Almanac for Land Surveyors gave the following stars clustered in this vicinity:-

Star No.	Magnitude	RA	δ
252	3.6	9^h10^m	58°52' S
255	2.2	9 16½	59 10 S
258	2.6	9 21½	54 54 S
262	3.0	9 30½	56 55 S

None of these values agrees with the calculated value. With the same data and varying values of azimuth, the following results were obtained:-

Point No.	Azimuth	RA	δ
I	135°	$9^h21.5^m$	-54°04'
II	137.5	9 21.0	-55 51
III	140	9 20.2	-57 38
IV	142.5	9 18.8	-59 25
V	145	9 16.8	-61 11

These have been plotted on the Modified Mercator Chart of Fig 10.12 where it can be seen that if only the azimuth is in doubt, the star No. 258 is a likely one to accept as the unknown star.

Finally a knowledge of approximate magnitudes is very useful as this can prove to give strong corroborative evidence of identification. This is easily learnt with a little experience. It should always be remembered that a star's magnitude is not altered by the magnifying property of the theodolite's telescope, because the star is a point of light from a distance, which may be justifiably considered to be infinite.

It is good practice, when two or more stars appear near the centre of the field of view of the telescope, to note their relative positions. If there is doubt, about which is the correct star to be observed, the observations should be made on the brightest one. Also if the orientation of the horizontal circle is unknown, or even only in doubt, then an identification sight should be included among the set of observations made to any star. The identification sights made to the known stars then serve to orientate the horizontal circle readings observed and to provide an azimuth for identifying the unknown stars observed.

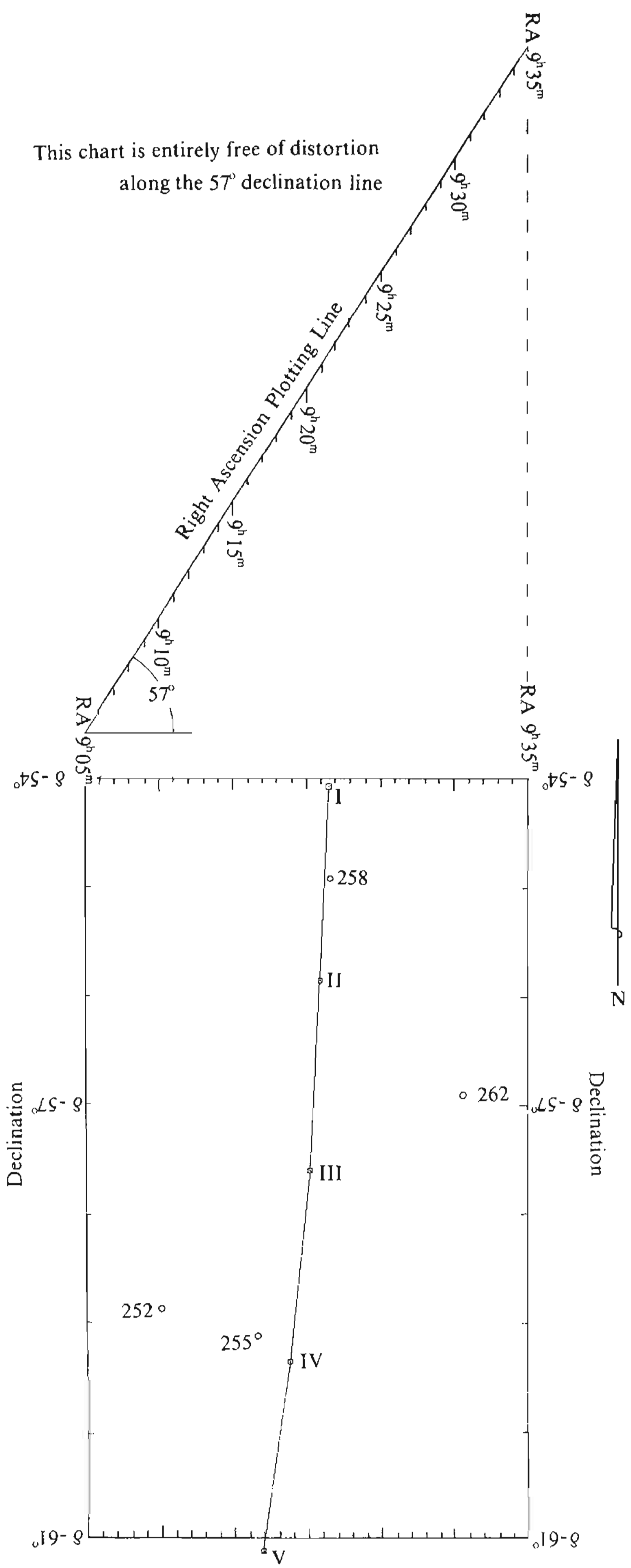


Fig. 10.12 A Chart of a Portion of the Celestial Sphere

Appendix

USEFUL FORMULAE AND RELATIONSHIPS

A.11 Trigonometrical Relationships

$$1 = \sin^2 A + \cos^2 A \qquad \sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$1 = \sec^2 A - \tan^2 A \qquad \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$1 = \operatorname{cosec}^2 A - \cot^2 A \qquad \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A \qquad \sin A \pm \sin B = 2 \sin\tfrac{1}{2}(A \pm B) \cos\tfrac{1}{2}(A \mp B)$$

$$\cos 2A = 2 \cos^2 A - 1 \qquad \cos A + \cos B = 2 \cos\tfrac{1}{2}(A+B) \cos\tfrac{1}{2}(A-B)$$

$$= 1 - 2 \sin^2 A \qquad \cos A - \cos B = -2 \sin\tfrac{1}{2}(A+B) \sin\tfrac{1}{2}(A-B)$$

$$= \cos^2 A - \sin^2 A \qquad \tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A \cos B}$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A \qquad \sin^2 A - \sin^2 B = \sin(A+B) \sin(A-B)$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A \qquad \cos^2 A - \cos^2 B = -\sin(A+B) \sin(A-B)$$

$$\cos^2 A - \sin^2 B = \cos(A+B) \cos(A-B)$$

$$\sin A \sin B = -\tfrac{1}{2}\cos(A+B) + \tfrac{1}{2}\cos(A-B)$$

$$\cos A \cos B = \tfrac{1}{2}\cos(A+B) + \tfrac{1}{2}\cos(A-B)$$

$$\sin A \cos B = \tfrac{1}{2}\sin(A+B) + \tfrac{1}{2}\sin(A-B)$$

A.12 Power Series Expansions

Binomial Series

$$(x+a)^n = x^n + n\,x^{n-1}\,a + \frac{1}{2!}\,n(n-1)x^{n-2}\,a^2 + \frac{1}{3!}\,n(n-1)(n-2)x^{n-3}\,a^3 \ldots$$

Exponential Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$a^x = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \frac{X^4}{4!} + \ldots$$

in which $X = x \log_e a$, $x^2 < \infty$ and $e = 2.718\ 281\ 828\ 5 \ldots$

A.13 Logarithmic Series

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad (-1 < x \leqslant 1)$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \qquad (-1 \leqslant x < 1)$$

$$\log_{10} N = \log_e N \log_{10} e = 0.434\ 294\ 481\ 9 \log_e N$$

$$\log_e N = \log_{10} N \log_e 10 = 2.302\ 585\ 093\ 0 \log_{10} N$$

A.14 Trigonometrical Series

In these expressions the angle θ must be given in radians

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots \qquad (\theta^2 < \infty)$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots \qquad (\theta^2 < \infty)$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + \frac{62\theta^9}{2835} + \ldots \qquad (\theta^2 < \tfrac{1}{4}\pi^2)$$

$$\cot\theta = \frac{1}{\theta} - \frac{\theta}{3} - \frac{\theta^3}{45} - \frac{2\theta^5}{945} - \frac{\theta^7}{4725} + \ldots \qquad (\theta^2 < \pi^2)$$

$$\sec\theta = 1 + \frac{\theta^2}{2} + \frac{5\theta^4}{24} + \frac{61\theta^6}{720} + \ldots \qquad (\theta^2 < \tfrac{1}{4}\pi^2)$$

$$\operatorname{cosec}\theta = \frac{1}{\theta} + \frac{\theta}{6} + \frac{7\theta^3}{360} + \frac{31\theta^5}{15120} + \ldots \qquad (\theta^2 < \pi^2)$$

$$\text{arc}\sin\theta = \theta + \frac{1}{2}\frac{\theta^3}{3} + \frac{1}{2}\frac{3}{4}\frac{\theta^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{\theta^7}{7} + \ldots \qquad (\theta^2 \leqslant 1)$$

$$\text{arc}\cos\theta = \tfrac{1}{2}\pi - \text{arc}\sin\theta$$

$$\text{arc}\tan\theta = \theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \frac{\theta^7}{7} + \ldots \qquad (\theta^2 \leqslant 1)$$

$$= \tfrac{1}{2}\pi - \frac{1}{\theta} + \frac{1}{3\theta^3} - \frac{1}{5\theta^5} + \frac{1}{7\theta^7} \ldots \qquad (\theta^2 \geqslant 1)$$

$$\text{arc}\cot\theta = \tfrac{1}{2}\pi - \text{arc}\tan\theta$$

$$\text{arc}\sec\theta = \tfrac{1}{2}\pi - \frac{1}{\theta} - \frac{1}{2}\frac{1}{3\theta^3} - \frac{1}{2}\frac{3}{4}\frac{1}{5\theta^5} - \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{1}{7\theta^7} + \ldots \qquad (\theta^2 > 1)$$

$$\text{arc}\operatorname{cosec}\theta = \tfrac{1}{2}\pi - \text{arc}\sec\theta$$

A.15 Taylor and Maclaurin Series

If $f(x)$ is a continuous function with successive derivatives f_1, f_2, f_3 ...

then $f(x + \Delta x)$ can be expressed as a Taylor Series as follows:-

$$f(x + \Delta x) = f(x) + f_1(x)\frac{\Delta x}{1!} + f_2(x)\frac{(\Delta x)^2}{2!} + f_3(x)\frac{(\Delta x)^3}{3!} + \ldots$$

The special case when $x = 0$ gives the Maclaurin Series as follows:-

$$f(\Delta x) = f(0) + f_1(0)\frac{\Delta x}{1!} + f_2(0)\frac{(\Delta x)^2}{2!} + f_3(0)\frac{(\Delta x)^3}{3!} + \ldots$$

A.16 Inversion of a Power Series

If y can be expressed as a power series in terms of x, with each successive term being smaller than its predecessor, then if $y = ax + bx^2 + cx^3 + dx^4 + ex^5 \ldots$

$$x = \frac{1}{a}y - \frac{b}{a^3}y^2 + \frac{2b^2 - ac}{a^5}y^3 - \frac{5b^3 - 5abc + a^2d}{a^7}y^4$$

and if $y = px + qx^3 + rx^5 + sx^7 \ldots$

$$x = \frac{1}{p}y - \frac{q}{p^4}y^3 + \frac{3q^2 - pr}{p^7}y^5 - \frac{12q^3 - 8pqr + p^2s}{p^{10}}y^7 \ldots$$

THE MANIPULATION OF A TRIGONOMETRICAL FUNCTION AND ITS EVALUATION

Experience has shown that for many there is some uncertainty in the manipulation and evaluation of trigonometrical functions of an angle, which does not lie in the first angular quadrant, i.e. between 0 and 90. Such problems frequently occurred in the past, when trigonometrical function values and their inverses were obtained from tables, often arranged in a semi-quadrantal form. Nowadays the problems occur less frequently because of the widespread use of the electronic calculator, but it is often convenient when manipulating trigonometrical functions, in a formula for instance, to express a result such as sin(-185) in the simpler form of sin 5, etc.

It is therefore considered desirable here to set down simple rules and guide lines, which, if properly learnt, will eliminate any uncertainty entirely. This will be done without formal proof, as the use of such rules is all that is necessary in practice.

A.21 As a preliminary, the signs of the various trigonometrical functions will be determined from an inspection of Fig.A.1 which shows the vector diagram with its angle of rotation γ increasing in a

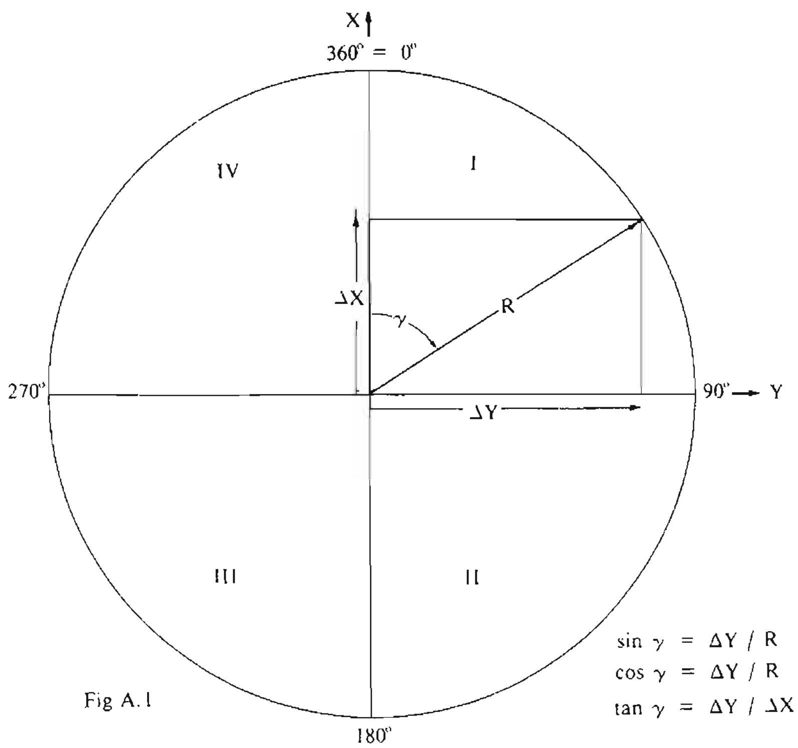


Fig A.1

clockwise direction as is usual in survey practice. This convention is opposite to that used in cartesian coordinate geometry in mathematics. From this diagram, the signs of the trigonometrical functions of any value of γ can be determined by inspection. Table A.1 embodies the results of such inspection.

Table A.1						
Quadrant and Range			Function sin	cos	tan	Positive Trigonometrical Function
First	0 →	90	+	+	+	ALL
Second	90 →	180	+	−	−	SIN
Third	180 →	270	−	−	+	TAN
Fourth	270 →	360	−	+	−	COS

There are several mnemonics, which are commonly used to remind the user which of the trigonometrical functions are positive and which negative in the quadrant under consideration. This table provides the rule, which in section A.22 has been called Rule 2.

A.22 It is required to manipulate $F(\theta)$, where θ is an angle of any magnitude and F is one of the trigonometrIcal functions.

Step 1 Rule 1

Multiples of 360 are applied to θ *until there is left a remainder* γ, *which lies within the function range of 0 to 360.*

i.e. $F(\theta) = F(360N + \gamma) = F(\gamma)$

in which N is an integer and $0 < \gamma < 360$.

Step 2

The angle γ is divided into n right angles *plus or minus* a basic angle α

i.e. $F(\gamma) = F(n90 \pm \alpha) = \pm f(\alpha)$

in which n is an integer and $0 \leqslant \alpha < 90$

Step 3 Rule 2

"All, Sin, Tan, Cos" is used to give the sign to be used in front of the function f from the quadrant sign of the function F.

Step 4 Rule 3

If n is even, f is the same as F.

If n is odd, f becomes the co-function of F.

Examples

F(θ)	F(γ)	F(n90+α)	±f(α)	F(θ)	F(γ)	F(n90−α)	±f(α)
sin 1030	sin 310	sin (3x90+40)	−cos40	sin 470	sin110	sin(2x90−70)	+sin70
cos −1640	cos 160	cos (1x90+70)	−sin70	tan−320	tan 40	tan(1x90−50)	+cot50
tan 590	tan 230	tan (2x90+50)	+tan50	cos1270	cos190	cos(3x90−80)	−sin80
cos 40	cos 40	cos (0x90+40)	+cos40	tan− 60	tan300	tan(4x90−60)	−tan60

A.23 The above procedure can be generalized to deal with angles γ when they are expressed in the algebraic form of $(n90+\alpha)$, such as $180-\omega$, $A-180$, $-t$, $90+\delta$, etc. The procedure of section A.22 is carried through in exactly the same way, with the algebraic portion α of the angle γ regarded here also as if it lies in the first quadrant.

Examples

$F(\theta)$	$F(\gamma)$	$F(n90\pm\varepsilon)$	$\pm f(\varepsilon)$
$\tan(180-\omega)$	$\tan(180-\omega)$	$\tan(2x90-\omega)$	$-\tan\ \omega$
$\cos(90+\delta)$	$\cos(90+\delta)$	$\cos(1x90+\delta)$	$-\sin\ \delta$
$\sin(-t)$	$\sin(360-t)$	$\sin(4x90-t)$	$-\sin\ t$
$\cos(A-180)$	$\cos(180+A)$	$\cos(2x90+A)$	$-\cos\ A$
$\cot(90+\delta)$	$\cot(90+\delta)$	$\cot(1x90+\delta)$	$-\tan\ \delta$
$\sec(180-A)$	$\sec(180-A)$	$\sec(2x90-A)$	$-\sec\ A$
$\cot(\omega-180)$	$\cot(180+\omega)$	$\cot(2x90+\omega)$	$+\cot\ \omega$

Examples of the need for this kind of manipulation can be found in the text e.g. in sections 2.62, 2.74 and 2.76 as well as in the derivation of the generalized relationships of section 2.75 from those of section 2.62.

A.24 The reverse process consists in finding γ if the value of $F(\gamma)$ is known and a set of trigonometrical tables, restricted to first quadrant angles η, is available. If the rule established in table A.1 in section A.21 is used, two possible values of γ will result, except in the case of the tangent function, derived from a ratio, in which the signs of numerator and denominator are both known. In this case, γ is known without ambiguity.

The value of η is abstracted from the table at the point corresponding to the modulus value of the function. The two quadrants in which γ may lie are determined from Table A.1 or the associated mnemonic. The possible values of γ will then be the relevant pair from $\pm\eta$ or $180 \pm \eta$. This can be verified from the vector diagram in Fig.A.1

For example, if $\sin\gamma = +\frac{1}{2}$, then $\eta = 30$. The sine is positive in quadrants 1 and 2 so that γ can be either 30 or 180-30 = 150. Sometimes one has some other piece of information, which enables one to select the correct value from the two possible ones.

If the signs of more than one function are known, the angle γ can be found by a simple process of elimination. For example, if $\sin\gamma$ was known to be $+\frac{1}{2}$ and $\tan\gamma$ was known to be negative, two possible values of γ, namely 30 or 150, are obtained from the sine. Of these two, only 150 has a negative tangent. Therefore γ is 150 in this case.

DIFFERENTIAL RELATIONSHIPS

The derivation of two commonly used differential relationships will be carried out to illustrate the process:-

A.31 Differentiation of the Cosine Formula

$$\cos w = \cos x \cos y + \sin x \sin y \cos W$$

gives

$$
\begin{aligned}
-\sin w\, dw &= -\cos x \sin y\, dy - \sin x \cos y\, dx + \cos x \sin y \cos W\, dx \\
&\quad +\sin x \cos y \cos W\, dy - \sin x \sin y \sin W\, dW \\
&= -(\cos y \sin x - \cos x \sin y \cos W)\, dx \\
&\quad -(\cos x \sin y - \sin x \cos y \cos W)\, dy \\
&\quad -(\sin x \sin y \sin W)\, dW \qquad \text{...A.31}
\end{aligned}
$$

From the Five Parts Formula

$$\sin w \cos X = \cos x \sin y - \sin x \cos y \cos W$$

and $\sin w \cos Y = \cos y \sin x - \sin y \cos x \cos W$

and from the Sine Formula

$$\sin w \sin Y = \sin y \sin W$$

Substitution of these in equation A.31 gives

$$
\begin{aligned}
-\sin w\, dw &= -\sin w \cos Y\, dx - \sin w \cos X\, dy - \sin w \sin x \sin Y\, dW \\
dw &= \cos Y\, dx + \cos X\, dy + \sin x \sin Y\, dW \qquad \text{...A.32}
\end{aligned}
$$

A.32 Differentiation of the Four Parts Formula

$$\cot y \sin x = \cot Y \sin W + \cos x \cos W$$

gives

$$
\begin{aligned}
&-\operatorname{cosec}^2 y \sin x\, dy + \cot y \cos x\, dx \\
&\qquad = -\operatorname{cosec}^2 Y \sin W\, dY + \cot Y \cos W\, dW \\
&\qquad\quad -\cos x \sin W\, dW - \sin x \cos W\, dx
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ -\frac{1}{\sin y}\frac{\sin x}{\sin y}\, dy + \left(\frac{\cos y}{\sin y} \cos x + \frac{\sin y}{\sin y} \sin x \cos W\right) dx \\
= -\frac{1}{\sin Y}\frac{\sin W}{\sin Y}\, dY + \left(\frac{\cos Y}{\sin Y} \cos W - \cos x \sin W \frac{\sin Y}{\sin Y}\right) dW
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ -\frac{1}{\sin y}\left[\frac{\sin x}{\sin y}\, dy - (\cos y \cos x + \sin y \sin x \cos W)\, dx\right] \\
= -\frac{1}{\sin Y}\left[\frac{\sin W}{\sin Y}\, dY - (\cos Y \cos W - \sin Y \sin W \cos x)\, dW\right] \text{...A.33}
\end{aligned}
$$

From the Sine Formula

$$\frac{\sin x}{\sin y} = \frac{\sin X}{\sin Y} \qquad \text{and} \qquad \frac{\sin W}{\sin Y} = \frac{\sin w}{\sin y}$$

and from the Cosine Formula

$$\cos w = \cos x \cos y + \sin x \sin y \cos W$$

and from the Polar Cosine Formula

$$-\cos X = \cos W \cos Y - \sin W \sin Y \cos x$$

Substitution in equation A.33 gives

$$
\begin{aligned}
-\frac{1}{\sin y}\left[\frac{\sin X}{\sin Y}\, dy - \frac{\sin Y}{\sin Y} . \cos w\, dx\right] \\
= -\frac{1}{\sin Y}\left[\frac{\sin w}{\sin y}\, dY + \frac{\sin y}{\sin y} \cos X\, dW\right]
\end{aligned}
$$

$$\therefore \quad \sin w \, dY = \sin X \, dy - \cos w \sin Y \, dx - \sin y \cos X \, dW \qquad \text{...A.34}$$

Needless to say, the above approach was not achieved at the first attempt and also hindsight helped to anticipate some of the difficulties in the manipulation.

A.33 Two useful second order differential coefficients (see section 2.75) are derived as follows:-

If the latitude ϕ and the declination δ are held constant in the astronomical triangle,

$$\frac{dh}{dt} = \cos\phi \sin A = -\cos\phi \cos\delta \sin t \sec h$$

since $\quad \sin A = -\dfrac{\cos\delta}{\cos h} \sin t$

$$\therefore \quad \frac{d_2 h}{dt^2} = -\cos\phi\cos\delta \frac{d}{dt}(\sin t \sec h)$$

$$= -\cos\phi\cos\delta(\cos t \sec h + \sin t \sec h \tan h \frac{dh}{dt})$$

$$= -\cos\phi\cos\delta \sec h \sin t (\cot t + \tan h \frac{dh}{dt})$$

$$= \frac{dh}{dt}(\cot t + \tan h \frac{dh}{dt}) \qquad \text{...A.35}$$

Similarly

$$\frac{dt}{dh} = -\sec\phi\sec\delta \operatorname{cosec} t \cos h$$

$$\therefore \quad \frac{d_2 t}{dh^2} = -\sec\phi\sec\delta(-\cot t \operatorname{cosec} t \frac{dt}{dh} \cos h - \operatorname{cosec} t \sin h)$$

$$= -\sec\phi\sec\delta \operatorname{cosec} t \cos h(-\cot t \frac{dt}{dh} - \tan h)$$

$$= -\frac{dt}{dh}(\cot t \frac{dt}{dh} + \tan h) \qquad \text{...A.36}$$

THE TRANSFORMATION FORMULAE

These relationships, about to be derived, give a method of conversion between the astronomical coordinates of hour angle and declination and the terrestrial coordinates of azimuth and altitude. In addition, a general solution for the latitude from timed altitude observations is also provided (see section 5.21). A summary of the transformation formulae is set for comparison beside the relationships used for the methods of direct solution for the unknowns.

A.41 The astronomical triangle $P_N ZS$ is shown in Fig.A.2 with its elements as they have been conventionalized (see also Fig.2.9 and section 2.73), the quantities A and h and t and ϕ having already been defined. To effect a transformation between these systems, two new quantities M and m may be defined in a similar way. M is the arc measured along the local meridian from the equator towards the north pole to the footpoint of a great circle that passes through the west point and the star. It will be seen that M can also be measured at the west point between the equator and the great circle referred to. The distance from this footpoint to the star is defined as m, positive towards the west point. Thus it will be seen that M may exist in any angular quadrant but m can only exist in the first or fourth quadrants.

In the triangle $P_N FS$, the Cosine and the Five Parts Formulae give the following:-

$$\cos(90-\delta) = \cos m \cos(90-M) + \sin m \sin(90-M) \cos F$$

and $\quad \sin(90-\delta)\cos t = \cos m \sin(90-M) - \sin m \cos(90-M) \cos F$

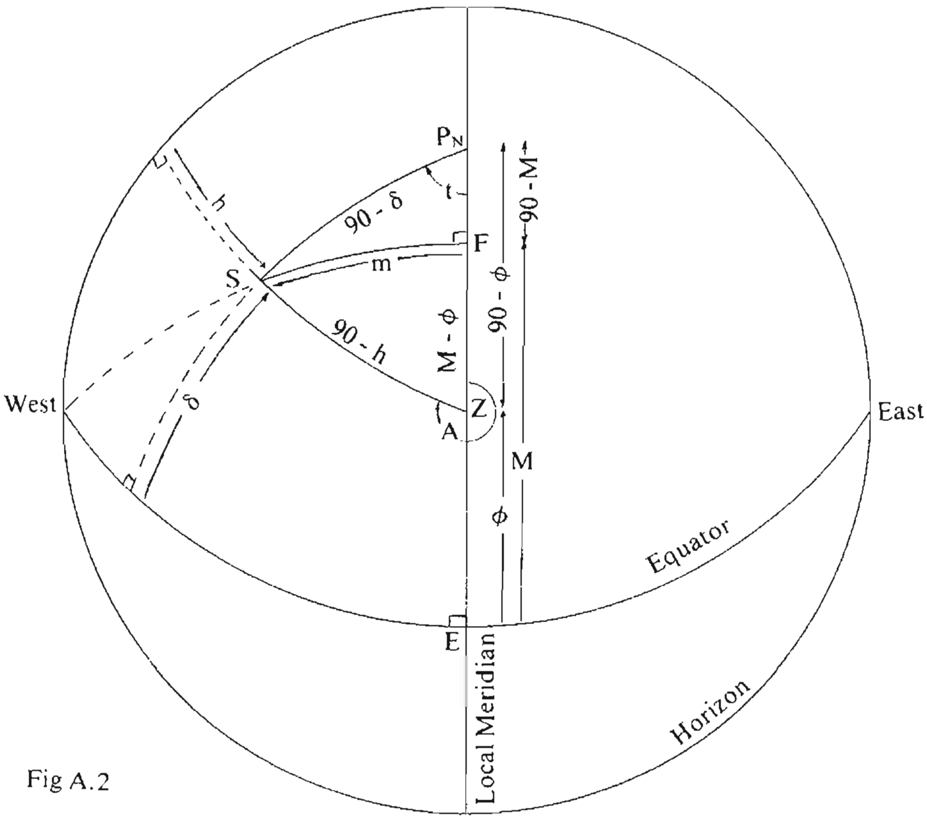


Fig A.2

In this triangle $F=90^{\circ}$ or 270° and in each case, therefore, cos F is zero and so the last term in each of these relationships goes to zero, because the zero function cos F is multiplied by quantities, which never exceed unity.

$$\therefore \quad \sin\delta = \cos m \sin M \qquad \text{...A.411}$$

$$\text{and} \quad \cos t \cos\delta = \cos m \cos M \qquad \text{...A.412}$$

$$\therefore \quad \tan M = \frac{\tan\delta}{\cos t} \qquad \text{...A.413}$$

Exactly the same treatment applied in the triangle ZFS gives

$$\sin h = \cos m \cos (M-\phi) \qquad \text{...A.414}$$

$$\text{and} \quad \cos A \cos h = \cos m \sin (M-\phi) \qquad \text{...A.415}$$

$$\therefore \quad \tan(M-\phi) = \frac{\cos A}{\tan h} \qquad \text{...A.416}$$

A.42 Transformation between the δ, t system and the A, h system is achieved as follows:

The Sine Formula in the astronomical triangle gives

$$\cos h \sin A = -\cos\delta \sin t \qquad \text{...A.421}$$

From equation A.415

$$\cos h \cos A = \cos m \sin (M-\phi)$$

Substituting cos δ cos t sec M for cos m from equation A.412 in the above gives

$$\cos h \cos A = \cos\delta \cos t \sec M \sin (M-\phi)$$

$$\therefore \quad \frac{\sin A}{\cos A} = \frac{-\cos\delta \sin t}{\cos\delta \cos t \sec M \sin(M-\phi)}$$

But as cos δ, cos h and cos m are all positive quantities, because by definition δ, h and m can lie only in the first and fourth quadrants, and thus also cos t sec M is always positive (see equation A.412), therefore

$$\tan A = \frac{-\tan t \cos M}{\sin (M-\phi)}$$

In summary

$$\tan M = \frac{\tan \delta}{\cos t} \quad : \quad \tan A = \frac{-\tan t \cos M}{\sin(M-\phi)} \quad : \quad \tan h = \cos A \cot (M-\phi) \qquad \text{...A.422}$$

Similarly it can be proved that

$$\tan(M-\phi) = \frac{\cos A}{\tan h} \quad : \quad \tan t = \frac{-\tan A \sin(M-\phi)}{\cos M} \quad : \quad \tan \delta = \tan M \cos t \qquad \text{...A.423}$$

It should be noted that those quantities computed from the tangent expressed as a ratio are unambiguously determined in one of the four quadrants, while those computed from a tangent not expressed as a ratio are quantities which exist in only the first or fourth quadrants and the sign of the tangent in these cases also determines the quantity sought, without ambiguity.

The equivalent direct solutions are

$$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t$$

$$\tan A = \frac{-\sin t}{\cos \phi \tan \delta - \sin \phi \cos t}$$

and

$$\sin \delta = \sin \phi \sin h + \cos \phi \cos h \cos A$$

$$\tan t = \frac{-\sin A}{\cos \phi \tan h - \sin \phi \cos A}$$

A.43 For the determination of A and ϕ from h, δ and t, the azimuth may be determined from the Sine Formula, which from equation A.421 gives

$$\sin A = \frac{-\cos \delta \sin t}{\cos h}$$

The ambiguity in this relationship is easily resolved provided observations are not made on a star in the vicinity of the prime vertical, because then inspection by the observer determines the azimuth quadrant in which the observation is being made. After this has been settled, M and $(M-\phi)$ can both be unambiguously determined from equations A.413 and A.416 as

$$\tan M = \frac{\tan \delta}{\cos t}$$

$$\tan (M-\phi) = \frac{\cos A}{\tan h}$$

and ϕ is then determined very simply.

If however a solution for latitude only is sought (see section 5.21), M may be calculated unambiguously as above and $(M-\phi)$ from its cosine given in equation A.414 as

$$\cos(M-\phi) = \sin h \sec m$$

From equation A.411

$$\sec m = \sin M \operatorname{cosec} \delta$$

$$\therefore \cos(M-\phi) = \sin h \sin M \operatorname{cosec} \delta \qquad \text{...A.431}$$

Since $\cos(\phi-M) = \cos(M-\phi)$ ambiguity results, but from Fig. A.2 it can be seen that the footpoint F coincides with the zenith Z if the star lies on the observer's prime vertical. If the star were observed south of the prime vertical, the distance FZ would be $(\phi-M)$; if north, it would be $(M-\phi)$. If therefore a quantity N were defined as

$$N = |\phi - M| \qquad \text{...A.432}$$

and N were signed positive if the star had been north of the prime vertical and negative if south, then

$$\phi = M - N \qquad \text{...A.433}$$

The transformation formulae were developed for use in logarithmic computation, for which they were admirably suited. In modern computing direct and general methods of solution are used but the derivation of the transformation formulae are given here for completeness.

RELATIONSHIPS FOR A CLOSE CIRCUM-POLAR STAR

Latitude from Observations on a Close Circum-Polar Star

A.51 The altitude of a close circum polar star is very nearly numerically equal to the latitude of the place of observation. This fact will be used to develop a power series, from which the altitude can be accurately determined from a time altitude observation made on such a star.

From Equation 5.1

$$\sin h = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t$$

Let $|\phi| = h - x$ and $p = 90 - |\delta|$

in which h = the observed altitude corrected for vertical circle index error and refraction

p = the polar distance of the star from the adjacent celestial pole.

It should be noticed that

$$x \leqslant p$$

and that both x and p are small quantities.

Equation 5.1 may then be written as

$$\sin h = \sin(h-x) \cos p + \cos(h-x) \sin p \cos t$$

which is still applicable to both hemispheres. Upon expansion and re-arrangement, this relationship becomes

$$\tan h(1 - \cos p \cos x - \cos t \sin p \sin x) - \cos t \sin p \cos x$$
$$+ \cos p \sin x = 0 \qquad \text{...A.51}$$

In addition x may be expressed as a power series in terms of p as follows:-

$$x = a_1 p + a_2 p^2 + a_3 p^3 + a_4 p^4 \ldots$$

in which attention is drawn to the absence of a constant term, due to the fact that when $p = 0$, $x = 0$ and therefore $|\phi|$ then equals h.

Expansion of Equation A.51, up to terms of the fourth degree, gives

$$\tan h\,[1 - (1 - \tfrac{1}{2}p + \frac{1}{24}p^4)(1 - \tfrac{1}{2}x^2 + \frac{1}{24}x^4) - \cos t(p - \frac{1}{6}p^3)(x - \frac{1}{6}x^3)]$$

$$- \cos t\,(p - \frac{1}{6}p^3)(1 - \tfrac{1}{2}x^2 + \frac{1}{24}x^4) + (1 - \tfrac{1}{2}p^2 + \frac{1}{24}p^4)(x - \frac{1}{6}x^3) = 0$$

$$\therefore \tan h\,[\tfrac{1}{2}(p^2 + x^2) - \tfrac{1}{4}p^2x^2 - \frac{1}{24}(p^4 + x^4) - \cos t(px - \frac{1}{6}px^3 - \frac{1}{6}p^3x)]$$

$$- \cos t(p - \tfrac{1}{2}px^2 - \frac{1}{6}p^3) + x - \tfrac{1}{2}p^2 x - \frac{1}{6}x^3 = 0$$

Substitution for x in this, from the power series in p, gives

$$\tan h\{\tfrac{1}{2}p^2(1 + a_1^2) + a_1a_2p^3 + p^4[a_1a_3 + \tfrac{1}{2}a_2^2 - \tfrac{1}{4}a_1^2 - \frac{1}{24}(1 + a_1^4)]$$

$$- \cos t\,[a_1p^2 + a_2p^3 + p^4(a_3 - \frac{1}{6}a_1 - \frac{1}{6}a_1^3)]\}$$

$$- \cos t\,[p - p^3(\frac{1}{6} + \tfrac{1}{2}a_1^2) - a_1a_2p^4\ ]$$

$$+ a_1p + a_2p^2 + p^3(a_3 - \tfrac{1}{2}a_1 - \frac{1}{6}a_1^3) + p^4(a_4 - \tfrac{1}{2}a_2 - \tfrac{1}{2}a_1^2a_2) = 0$$

This equation will be true whatever the value of p and hence the coefficients of p^n ($n = 1$ to ∞) must each vanish.

$\therefore$ Equating each of the coefficients of p^n to zero gives

$\therefore$ for $n = 1$ $\quad a_1 = \cos t$

for $n = 2$ $\quad \tan h\,\{\tfrac{1}{2}(1 + a_1^2) - a_1\cos t\} + a_2 = 0$

$\therefore \quad a_2 = -\tfrac{1}{2}\sin^2 t \tan h$

for $n = 3$

$$\tan h\,(a_1a_2 - a_2\cos t) + \frac{1}{6}\cos t(1 + 3a_1^2) + \frac{1}{6}(6a_3 - 3a_1 - a_1^3) = 0$$

$\therefore \quad a_3 = \frac{1}{3}\cos t \sin^2 t$

for $n = 4$

$$\tan h\,[a_1a_3 + \tfrac{1}{2}a_2^2 - \tfrac{1}{4}a_1^2 - \frac{1}{24}(1 + a_1^4) - \cos t(a_3 - \frac{1}{6}a_1 - \frac{1}{6}a_1^3)]$$

$$+ a_1a_2\cos t + a_4 - \tfrac{1}{2}a_2 - \tfrac{1}{2}a_1^2 a_2 = 0$$

$$\therefore \quad a_4 = -\frac{1}{24}\sin^2 t \tan h\,(3\sin^2 t \tan^2 h + 9\sin^2 t - 4)$$

Finally, this gives

$$|\phi| = h - p \cos t + \frac{p^2}{2\rho} \sin^2 t \tan h - \frac{p^3}{3\rho^2} \cos t \sin^2 t$$

$$+ \frac{p^4}{24\rho^3} \sin^2 t \tan h \, (3 \sin^2 t \tan^2 h + 9 \sin^2 t - 4) \qquad \text{...A.52}$$

In practice, care should be exercised in judging where to truncate this series, because sometimes the terms do not diminish progressively.

Azimuth from Observations on a Close Circum Polar Star

A.52 The general formula for determination of azimuth from timed horizontal circle readings is given by

$$\tan A = \frac{-\sin t}{\tan \delta \cos \phi - \sin \phi \cos t}$$

This can be expressed in the form

$$\tan A = \frac{-\sin t \sec \phi \cot \delta}{1 - \tan \phi \cos t \cot \delta} \qquad \text{...A.53}$$

which is convenient for expansion into a series.

The derivation will be given in two parts, one for each of the two hemispheres.

A.521 For the northern hemisphere, the northern polar distance p is given by

$$p = 90 - \delta$$

which, because δ for the northern pole star is near 90°, will be a small angle.

Furthermore, if the latitude is not very high, the term $\tan \phi \cos t \cot \delta$ is not large and equation A.53 may therefore conveniently be expanded by means of the binomial theorem (see section A.12 in the appendix) to give:-

$$\tan A = -\sin t \sec \phi \tan p \, (1 - \tan \phi \cos t \tan p)^{-1}$$

$$= -\sin t \sec \phi \tan p \, (1 + \tan \phi \cos t \tan p + \tan^2\phi \cos^2 t \tan^2 p \,..)$$

But $\tan p = p + \frac{1}{3} p^3 \ldots$

$$\therefore \tan A = -\sin t \sec \phi (p + \tfrac{1}{3} p^3 \ldots) \{1 + \tan \phi \cos t (p + \tfrac{1}{3} p^3 \ldots) +$$

$$\tan^2\phi \cos^2 t \, (p + \tfrac{1}{3} p^3 \ldots)^2 \ldots \}$$

This simplifies to

$$\tan A = -p \sin t \sec \phi - p^2 \sin t \cos t \sec \phi \tan \phi$$

$$- \frac{p^3}{3} \sin t \sec \phi \, (1 + 3 \tan^2\phi \cos^2 t) \ldots \qquad \text{A.54}$$

But $\tan A = A + \frac{1}{3} A^3 \ldots$

and this may be inverted (see equation A.16 in this appendix) to give

$$A = -p \sin t \sec \phi - p^2 \sin t \cos t \sec \phi \tan \phi$$

$$- \frac{1}{3} p^3 \sin t \sec \phi (1 + 3 \tan^2\phi \cos^2 t - \sin^2 t \sec^2\phi) \quad \ldots A.55$$

A.522 For the southern hemisphere, the southern polar distance p is given by

$$p = 90 + \delta$$

which, because δ is nearly -90°, will also be a small angle.

Substitution in equation A.53 gives

$$\tan A = \frac{\sin t \sec \phi \tan p}{1 + \tan \phi \cos t \tan p}$$

By means of a similar technique to that given above, the following is obtained

$$\tan A = p \sin t \sec \phi - p^2 \sin t \cos t \sec \phi \tan \phi$$

$$+ \frac{1}{3} p^3 \sin t \sec \phi (1 + 3 \tan^2\phi \cos^2 t) \quad \ldots A.56$$

Here, however, p is a small angle about the south pole and thus the azimuth A may be given as

$$A = \pi + A^*$$

in which A^* is a small quantity.

Inversion therefore produces

$$A = \pi + p \sin t \sec \phi - p^2 \sin t \cos t \sec \phi \tan \phi$$

$$+ \frac{1}{3} p^3 \sin t \sec \phi (1 + 3 \tan^2\phi \cos^2 t - \sin^2 t \sec^2\phi) \ldots$$

$$A = 180^\circ + p \sin t \sec \phi - \frac{p^2}{\rho} \sin t \cos t \sec \phi \tan \phi$$

$$+ \frac{p^3}{3\rho^2} \sin t \sec \phi (1 + 3 \tan^2\phi \cos^2 t - \sin^2 t \sec^2\phi) \quad \ldots A.57$$

in which the first is in radians and the second in sexagesimal units with p and ρ in accord as far as units are concerned.

A summary of these relations is given as follows:-

$$|\phi| = h - p \cos t + \frac{p^2}{2\rho} \sin^2 t \tan h - \frac{p^3}{3\rho^2} \cos t \sin^2 t$$

$$+ \frac{p^4}{24\rho^3} \sin^2 t \tan h (3 \sin^2 t \tan^2 h + 9 \sin^2 t - 4) \quad \ldots A.58$$

For observations on the northern pole star α Ursae Minoris:-

$$A = - p \sin t \sec \phi - \frac{p^2}{\rho} \sin t \cos t \sec \phi \tan \phi$$

$$- \frac{p^3}{3\rho^2} \sin t \sec \phi (1 + 3 \tan^2\phi \cos^2 t - \sin^2 t \sec^2\phi) \quad \ldots A.59$$

and for observations on the southern pole star σ Octantis

$$A = 180 + p \sin t \sec \phi - \frac{p^2}{\rho} \sin t \cos t \sec \phi \tan \phi$$

$$+ \frac{p^3}{3\rho^2} \sin t \sec \phi (1 + 3 \tan^2\phi \cos^2 t - \sin^2 t \sec^2 \phi) \quad \text{...A.60}$$

The truncated form of these to the first term only is useful for rough prediction and computation.

SECOND ORDER CORRECTIONS TO LINKED QUANTITIES COMPUTED FROM MEANS

A.61 Sometimes it may be necessary to calculate a quantity desired from the mean of a set of values. If the function used for this is not a linear one, the result obtained will not be the correct one. It is then required to determine a correction to this computed value to give the correct value.

If $y = f(x)$, y_i is obtained by substitution of x_i in the function $f(x)$ and evaluating. If a set of such values x_i are so computed and then means are taken out, the following results are obtained:-

$$\bar{x}_i = \frac{1}{n}(x_1 + x_2 + x_3 \ldots . x_n)$$

and $$\bar{y}_i = \frac{1}{n}(y_1 + y_2 + y_3 \ldots . y_n) = \frac{1}{n}(f(x_1) + f(x_2) + \ldots f(x_n)) = \overline{f(x_i)}$$

If $\bar{x}_i$ is substituted in the function $f(x)$ and evaluated, a value $Y = f(\bar{x}_i)$ will be obtained. It is required to find the relationship between these two values

y_i corresponds to x_i and Y corresponds to $\bar{x}_i$

$$\therefore \Delta y_i = y_i - Y \text{ corresponds to } x_i - \bar{x}_i = \Delta x_i$$

or $$y_i = Y + \Delta y_i \text{ corresponds to } x_i = \bar{x}_i + \Delta x_i$$

But $$y_i = f(x_i) = f(\bar{x}_i + \Delta x_i) = f(\bar{x}_i) + \Delta x_i \, f_1(\bar{x}_i) + \tfrac{1}{2} \Delta x_i^2 \, f_2(\bar{x}_i)$$

$$\therefore y_i = Y + (x_i - \bar{x}_i) \, f_1(\bar{x}_i) + \tfrac{1}{2}(x_i - \bar{x}_i)^2 \, f_2(\bar{x}_i) \ldots$$

If now all n members of this family of equations are summed and meaned, then

$$\bar{y}_i = Y + \Sigma(x_i - \bar{x}_i) \frac{f_1(\bar{x}_i)}{n} + \frac{1}{2n} \Sigma(x_i - \bar{x}_i)^2 \, f_2(\bar{x}_i) \ldots$$

$$\therefore \bar{y}_i = Y + \frac{f_2(\bar{x}_i)}{2n} \Sigma(x_i - \bar{x}_i)^2 = Y + \text{the second order correction} \quad \text{...A.61}$$

since $\Sigma(x_i - \bar{x}_i)$ the sum of the differences from the mean equals zero.

A.62 In field astronomy, an observation made on a star consists of the observation of a pair of linked quantities. Such a pair consists usually of either (i) a circle reading and an associated clock reading or (ii) a circle reading on each of the two circles of a theodolite.

In addition, ancillary observations may be made, at suitable times during the observing period, for determining clock corrections required, and for determining refraction corrections or reasonably accurate values of instrumental errors needed or for referring circle readings to a reference object.

The unknown sought is then set up in an equation involving the observed quantities, one of which is used to determine an element in the astronomical triangle.
Two brief examples will be given to illustrate the process:-

For longitude determination from timed altitudes, the observation pairs are altitudes h_i and the corresponding observed values T_i of Greenwich Sidereal Time. Each pair of quantities gives a value λ_i of the longitude as

$$\lambda_i = LST_i - GST_i$$

$$= t_i + RA - GST_i$$

$$= f(h_i) + L_i$$

in which $t_i = \arccos(\sec\phi \sec\delta \sin h_i - \tan\phi \tan\delta)$

If λ_c is the value computed from $\bar{h}_i$ the mean altitude and $\bar{T}_i$ the mean of the observed times, then

$$\bar{\lambda}_i = \lambda_c + \frac{1}{2n\rho} f_2(\bar{h}_i) \Sigma(h_i - \bar{h}_i)^2$$

$$= \lambda_c - \frac{1}{2n\rho} \frac{dt}{dh} (\cot \bar{t} \frac{dt}{dh} + \tan \bar{h}) \Sigma(h_i - \bar{h}_i)^2$$

in which λ and Δh are in the same units and ρ is the number of these units in one radian; also $f_2(\bar{h}_i) = \frac{d_2t}{dh^2}$ comes from equation A.36 or section 2.75.

For azimuth determination by the altazimuth method, the observed pairs are horizontal circle readings $Hstar_i$ and observed altitudes h_i on the star with an ancillary observation $Hmark_i$, a horizontal circle reading observed on the mark RO. Each observed set gives the following value A_i for the azimuth to the mark (see section 7.62).

$$A_i = Astar_i + Hmark_i - Hstar_i$$

$$= F(h_i) + \alpha_i$$

in which $Astar_i = F(h_i) = \arccos(\sec\phi \sin\delta \sec h_i - \tan\phi \tan h_i)$

If A_c is the value computed from $\bar{h}_i$ the mean altitude and the means of the horizontal circle readings, then

$$\bar{A}_i = A_c + \frac{1}{2n\rho} F_2(\bar{h}_i) \Sigma(h_i - \bar{h}_i)^2$$

$$= A_c - \frac{1}{2n\rho} \sec^2 \bar{h}_i \cot \bar{\omega}(\sin \bar{h} + 2 \cot A_c \operatorname{cosec} 2\bar{\omega}) \Sigma(h_i - \bar{h}_i)^2$$

in which A and Δh are in the same units and ρ is the number of these units in one radian; also $F_2(\bar{h}_i) = \frac{d_2A}{dh^2}$ comes from section 2.75.

CIRCUM-MERIDIAN AND CIRCUM-ELONGATION RELATIONSHIPS

Introduction

Meridian observations for latitude determination lead directly to the circum-meridian method (see sections 5.41 et seq.). Similarly the meridian observations for azimuth determination in low latitudes lead to the circum-meridian observations (see section 7.33). In each case, the star is in a favoured position and symmetry about the meridian occurs. Such symmetry makes the use of a power series very attractive for evaluation, because, in each case, alternate terms vanish and the series usually converges very rapidly.

The elongation position for the determination of azimuth is also a favoured position, at which point the azimuth value reaches a minimum or a maximum so that the star being sighted then has no horizontal component and only a small one in the vicinity of this point (see section 7.32). The star's movement in azimuth about the point of elongation is not symmetrical and the power series consists of all ascending powers except the first one, which is zero.

Circum-Meridian and Meridian Zenith Distances

A.71 The Cosine Formula gives the following general relationship

$$\cos z = \sin\phi \sin\delta + \cos\phi \cos\delta \cos t$$

In section 5.42 the handling of the circum-meridian situation postulates the use of a generalized zenith distance z_{CM} with that of a generalized meridian declination δ_M, defined as

$$\delta_M = \delta \quad \text{for an upper transit}$$

$$\text{and} \quad \delta_M = 180 - \delta \text{ for a lower transit}$$

The hour angle for a circum-meridian situation lies in the vicinity of 0° for an upper transit one and in the vicinity of 180° for a lower transit one. It is proposed to use a circum-meridian hour angle t' for convenience. This is defined as

$$t' = t \quad \text{in the vicinity of an upper transit}$$

$$\text{and} \quad t' = t-180 \quad \text{in the vicinity of a lower transit}$$

Substitution of these quantities in the general relationship immediately above gives rise to the general relationship

$$\cos z_{CM} = \sin\phi \sin\delta_M + \cos\phi \cos\delta_M \cos t' \qquad \text{...A.71}$$

which holds for both upper and lower transit circum-meridian situations.

From this, the special case of the meridian zenith distance z_M, which occurs when t' is zero is

$$\cos z_M = \sin\phi \sin\delta_M + \cos\phi \cos\delta_M \qquad \text{...A.72}$$

Subtracting equation A.71 from equation A.72 gives

$$\cos z_M = \cos z_{CM} + \cos\phi \cos\delta_M (1 - \cos t')$$

$$= \cos z_{CM} + \cos\phi \cos\delta_M \, 2\sin^2(\tfrac{1}{2}t') \text{...A.73}$$

$$\therefore \cos z_M - \cos z_{CM} = \cos\phi \cos\delta_M \, 2\sin^2(\tfrac{1}{2}t')$$

$$\therefore -2 \sin \{\tfrac{1}{2}(z_M - z_{CM})\} \sin \{\tfrac{1}{2}(z_M + z_{CM})\} = \cos\phi \cos\delta_M \, 2\sin^2(\tfrac{1}{2}t')$$

$$\sin \{\tfrac{1}{2}(z_M - z_{CM})\} = -\frac{\cos\phi\cos\delta_M}{\sin\{\tfrac{1}{2}(z_M + z_{CM})\}} \sin^2(\tfrac{1}{2}t')$$

$$\sin \{\tfrac{1}{2}(z_M - z_{CM})\} = -\frac{\cos\phi\cos\delta_M}{\sin\bar{\bar{z}}} \sin^2(\tfrac{1}{2}t') \qquad \text{...A.74}$$

All the relationships in this section are rigorous.

A.72 The derivation of a very well known power series for reducing an observed circum-meridian zenith distance to the corresponding meridian zenith distance will be carried out by means of two methods for the purposes of illustration.

The following relationship is derived from equation A.73 above:-

$$z_{CM} = \text{arc cos } \{\cos z_M - \cos\phi\cos\delta_M \, 2\sin^2(\tfrac{1}{2}t')\}$$

$$= \text{arc cos } (k + x) = F(x)$$

in which $k = \cos z_M$ which is a constant for a particular station and a particular celestial body,

and $x = -\cos\phi\cos\delta_M \, 2.\sin^2(\tfrac{1}{2}t')$, which is a variable.

The function $z_{CM} = F(x)$ can be expanded as a Maclaurin Series, which

gives $z_{CM} = F(0) + x\,F_1(0) + \tfrac{1}{2}x^2\,F_1(0) + \frac{1}{6}x^3 F_3(0) \quad \ldots.$

These differential coefficients are evaluated as follows:-

$$z_{CM} = \text{arc cos } (k + x)$$

$$\therefore \quad x + k = \cos z_{CM}$$

$$= \frac{dx}{dz_{CM}} = -\sin z_{CM}$$

$$F_1(x) = \frac{dz_{CM}}{dx} = -\text{cosec } z_{CM}$$

$$F_2(x) = \frac{dF_1}{dx} = +\cot z_{CM} \text{ cosec } z_{CM} \frac{dz_{CM}}{dx} = -\cot z_{CM} \text{ cosec}^2 z_{CM}$$

$$F_3(x) = \frac{dF_2}{dx} = -\cot z_{CM} \{2 \text{ cosec } z_{CM} (-\cot z_{CM} \text{ cosec } z_{CM} \frac{dz_{CM}}{dx})\}$$

$$-\text{cosec}^2 z_{CM} \{-\text{cosec}^2 z_{CM} \frac{dz_{CM}}{dx}\}$$

$$= -\text{cosec}^3 z_{CM} \{2\cot^2 z_{CM} + \text{cosec}^2 z_{CM}\}$$

$$= -\text{cosec}^3 z_{CM} (1 + 3\cot^2 z_{CM})$$

$$z_{CM} = F(0) = x\,F_1(0) + \tfrac{1}{2}x^2\,F_2(0) + \frac{1}{6}x^3\,F_3(0) \ldots$$

$$= \text{arc cos}(k) + \{-\cos\phi\cos\delta_M \, 2\sin^2(\tfrac{1}{2}t')\}(-\text{cosec } z_M)$$

$$+\tfrac{1}{2}\{-\cos\phi\cos\delta_M \, 2\sin^2(\tfrac{1}{2}t')\}^2(-\cot z_M \text{ cosec}^2 z_M)$$

$$+\frac{1}{6}\{-\cos\phi\cos\delta_M \, 2\sin^2(\tfrac{1}{2}t')\}^3\{-\text{cosec}^3 z_M(1+3\cot^2 z_M)\}$$

$$\begin{aligned} z_{CM} = z_M &+ A\,2\sin^2(\tfrac{1}{2}t')\,\rho \\ &- A^2 2\sin^4(\tfrac{1}{2}t')\,\rho\cot z_M \\ &+ \tfrac{2}{3}A^3 2\sin^6(\tfrac{1}{2}t')\,\rho\,(1+3\cot^2 z_M) \end{aligned} \qquad \text{or} \qquad \begin{aligned} z_{CM} = z_M &+ A\,m \\ &- B\,n \\ &+ C\,s \end{aligned}$$

in which ρ is in accord with the angular units used.

$$\therefore \qquad z_M = z_{CM} - Am + Bn - Cs$$

in which

$$\begin{aligned} A &= \cos\phi\cos\delta_M\operatorname{cosec} z_M && \text{and} & m &= 2\sin^2(\tfrac{1}{2}t')\,\rho \\ B &= A^2\cot z_M && \text{"} & n &= 2\sin^4(\tfrac{1}{2}t')\,\rho \\ C &= \tfrac{2}{3}A^3\,(1+3\cot^2 z_M) && \text{"} & s &= 2\sin^6(\tfrac{1}{2}t')\,\rho \qquad \text{...A.75} \end{aligned}$$

A.73 The alternative method of approach uses the Taylor Series expansion of section A.15 in this appendix followed by the power series inversion of section A.16.
From equation A.73

$$\cos z_M - \cos\phi\cos\delta_M\,2\sin^2(\tfrac{1}{2}t') = \cos z_{CM}$$

If Δz is defined as

$$\Delta z = z_{CM} - z_M \qquad \text{i.e.} \quad z_{CM} = z_M + \Delta z$$

$$\cos(z_M + \Delta z) = \cos z_M - \cos\phi\cos\delta_M\,2\sin^2(\tfrac{1}{2}t')$$

Expanding LHS, simplifying and dividing through by $\sin z_M$ gives

$$\begin{aligned} &\cos\phi\cos\delta_M\operatorname{cosec} z_M\,2\sin^2(\tfrac{1}{2}t') \\ &= \Delta z + \tfrac{1}{2}\cot z_M\,\Delta z^2 - \frac{1}{6}\Delta z^3 \;.\;.\;.\;. \end{aligned}$$

Let $\qquad Y = \cos\phi\cos\delta_M\operatorname{cosec} z_M\,2\sin^2(\tfrac{1}{2}t')$

then $\qquad Y = \Delta z + \tfrac{1}{2}\cot z_M\,\Delta z^2 - \frac{1}{6}\Delta z^3 \;.\;.\;.\;.$

In this expression Y is given in terms of a power series in Δz. It is required that Δz be expressed as a power series in Y (see section A.16).

$$\begin{aligned} \therefore \qquad \Delta z &= Y - (\tfrac{1}{2}\cot z_M)\,Y^2 + \{2(\tfrac{1}{2}\cot z_M)^2 - (1)\,(-\tfrac{1}{6})\}\,Y^3 \\ &= Y - \tfrac{1}{2}\cot z_M\,Y^2 + \{\tfrac{1}{2}\cot^2 z_M + \frac{1}{6}\}\,Y^3 \end{aligned}$$

$$\begin{aligned} z_M = z_{CM} &- (\cos\phi\cos\delta_M\operatorname{cosec} z_M)\,2\sin^2(\tfrac{1}{2}t')\,\rho \\ &+ (\cos\phi\cos\delta_M\operatorname{cosec} z_M)^2\,2\sin^4(\tfrac{1}{2}t')\,\rho\cot z_M \\ &- (\cos\phi\cos\delta_M\operatorname{cosec} z_M)^3\,2\sin^6(\tfrac{1}{2}t')\,\rho\,\frac{2}{3}(1+3\cot^2 z_M) \end{aligned}$$

$$z_M = z_{CM} - Am + Bn - Cs$$

The same result is produced here as in the alternative method immediately above.

A.74 It is desired to derive a power series expressing the azimuth in the terms of ascending powers of the hour angle for a star in the vicinity of transit. Large numerical values of the hour angle for a star near transit may be avoided by the use of the circum-meridian hour angle t' which is defined as

$$t' = t \quad \text{in the vicinity of upper transit}$$

and

$$t' = t - 180^\circ \quad \text{in the vicinity of lower transit}$$

with the proviso that t' is given its smallest numerical value. In this way, values of the ascending powers of t' will decrease numerically.

From the set of differential relationships of section 2.75, the following holds when the latitude and declination are held constant:-

$$A = f(t')$$

$$f_1(t') = \frac{dA}{dt'} = \operatorname{cosec} z \cos \omega \cos \delta,$$

$$\frac{d\omega}{dt'} = -\operatorname{cosec} z \cos A \cos \phi$$

and

$$\frac{dz}{dt'} = -\cos \phi \sin A$$

These coefficients at transit may be simplified by the use of the meridian zenith distance z_M and the meridian declination δ_M, which are signed quantities defined in sections 5.32 and 5.33, and are linked by the general relationship

$$\phi = \delta_M - z_M$$

From an examination of all possible meridian transit situations, these coefficients become

$$f_1(0) = \frac{DA}{Dt'} = \frac{-\cos \delta_M}{\sin z_M},$$

$$\frac{D\omega}{Dt'} = \frac{-\cos \phi}{\sin z_M}$$

and

$$\frac{Dz}{Dt'} = 0 \qquad (\sin A = 0)$$

Further differentiation gives $f_2(t')$ as

$$f_2(t') = \frac{d_2A}{dt'^2} = \cos \delta \{-\cot z \operatorname{cosec} z \frac{dz}{dt'} \cos \omega - \operatorname{cosec} z \sin \omega \frac{d\omega}{dt'}\}$$

but at transit $\frac{dz}{dt'} = 0$ and $\sin \omega = 0$, and z in practice is never such that cot z and cosec z become very large

$$\therefore \quad f_2(0) = \frac{D_2A}{Dt'^2} = 0$$

Further differentiation gives $f_3(t')$ as

$$f_3(t') = \frac{d_3A}{dt'^3} = \cos \phi \cos \delta \{(-\operatorname{cosec}^3 z - \cot^2 z \operatorname{cosec} z)(\frac{dz}{dt'})^2 \sin A \cos \omega$$
$$+ \cot z \operatorname{cosec} z \cos A \frac{dA}{dt'} \cos \omega$$
$$- \cot z \operatorname{cosec} z \sin A \sin \omega \frac{d\omega}{dt'}$$

$$- 2 \operatorname{cosec}^2 z \cot z \frac{dz}{dt'} \sin \omega \cos A$$

$$+ \operatorname{cosec}^2 z \cos \omega \frac{d\omega}{dt'} \cos A$$

$$- \operatorname{cosec}^2 z \sin \omega \sin A \frac{dA}{dt'} \}$$

At transit, all terms, except the second and fifth, become zero because $\sin A$ and $\frac{dz}{dt'}$ are then zero. Substituting for $\frac{DA}{Dt'}$ and $\frac{D\omega}{Dt'}$ one obtains

$$f_3 (t') = \frac{d_3A}{dt'^3} = \cos \phi \cos \delta \ \{\cos z \operatorname{cosec}^2 z \cos A \left(\frac{-\cos \delta_M}{\sin \delta_M} \right) \cos \omega$$

$$+ \operatorname{cosec}^2 z \cos \omega \left(\frac{- \cos \phi}{\sin z_M}\right) \cos A\}$$

It can also be shown from an examination of all possible meridian transit situations that

$$\cos \delta \cos \omega \cos A = -\cos \delta_M$$

and as $\operatorname{cosec}^2 z = \operatorname{cosec}^2 z_M$

$$f_3 (0) = \frac{D_3A}{Dt'^3} = \cos \phi \cos \delta_M \operatorname{cosec}^3 z_M (\cos z_M \cos \delta_M + \cos \phi)$$

These coefficients can now be used in a Maclaurin series (see section A.15) to expand the function $A = f(t')$ as follows:-

$$A = f(t') = f(0) + f_1(0)\ t' + \tfrac{1}{2} f_2(0)\ t'^2 + \frac{1}{6} f_3 (0)\ t'^3 \ldots.$$

$$= A_0 - \cos \delta_M \operatorname{cosec} z_M\ t'$$

$$+ \frac{1}{6} \cos \phi \cos \delta_M \operatorname{cosec}^3 z_M \{\cos z_M \cos \delta_M + \cos \phi\} t'^3$$

...A.76

in which $A_0 = f(0) = 0^\circ$ for a star north and 180° for a star south.

Circum-Elongation Time Azimuths

A.81 It is required to express the azimuth A at hour angle t in terms of the elongation azimuth A_e at hour angle t_e.

The general relationship for azimuth from time observation gives the following:-

$$\cot A = \frac{\sin \phi \cos t}{\sin t} - \frac{\tan \delta \cos \phi}{\sin t}$$

From the Four Parts Formula

$$\sin \delta \cos t = \cos \delta \tan \phi - \sin t \cot \omega$$

At elongation this becomes

$$\sin \delta \cos t_e = \cos \delta \tan \phi - \sin t_e \cot \omega_e$$

$= \cos \delta \tan \phi$ because $\omega_e = 90^\circ$ or 270° and $\sin t_e$ never exceeds unity in magnitude

$$\therefore \quad \tan \delta \cos \phi = \frac{\sin \phi}{\cos t_e}$$

...A.81

$$\therefore \qquad \cot A = \frac{\sin\phi\cos t}{\sin t}\quad\frac{\cos t_e}{\cos t_e} - \frac{\sin\phi}{\sin t\cos t_e}$$

From the Polar Cosine Formula

$$-\cos\omega = \cos t\cos A + \sin t\sin A\sin\phi$$

At elongation this becomes

$$0 = \cos t_e\cos A_e + \sin t_e\sin A_e\sin\phi$$

$$\therefore \qquad \cot A_e = -\frac{\sin t_e\sin\phi}{\cos t_e}$$

$$\therefore \cot A - \cot A_e = \frac{\sin\phi\cos t}{\sin t}\quad\frac{\cos t_e}{\cos t_e} - \frac{\sin\phi}{\sin t\cos t_e} + \frac{\sin\phi\sin t_e}{\cos t_e}\quad\frac{\sin t}{\sin t}$$

$$= \frac{\sin\phi}{\sin t\cos t_e}(\cos t\cos t_e - 1 + \sin t_e\sin t)$$

$$= -\frac{\sin\phi}{\sin t\cos t_e}\{1 - \cos(t - t_e)\}$$

$$= -\frac{\sin\phi}{\sin t\cos t_e}\,2\sin^2\tfrac{1}{2}(t - t_e) \qquad \text{...A.82}$$

which is a rigorous expression

Now let $\Delta A = A - A_e$ and $\Delta t = t - t_e$

$$\therefore \qquad A = A_e + \Delta A \quad\text{and}\quad t = t_e + \Delta t$$

Expanding LHS as a Taylor's Series gives

$$\cot A - \cot A_e = \cot(A_e + \Delta A) - \cot A_e$$

$$= -\operatorname{cosec}^2 A_e\ \Delta A + \operatorname{cosec}^2 A_e\cot A_e\ \Delta A^2\ \ldots.$$

$$\therefore \quad \operatorname{cosec}^2 A_e\ \Delta A - \operatorname{cosec}^2 A_e\cot A_e\ \Delta A^2\ \ldots = \frac{\sin\phi}{\sin t\cos t_e}\,2\sin^2\tfrac{1}{2}\Delta t = Y$$

Here Y is expressed as a series in ascending powers of ΔA. This can be inverted to give ΔA as a series in ascending powers of Y (see section A.16 in the appendix)

$$\Delta A = \sin^2 A_e\,Y + \sin^6 A_e\operatorname{cosec}^2 A_e\cot A_e\,Y^2$$

$$= \frac{\sin\phi\sin^2 A_e}{\sin t\cos t_e}\,2\sin^2\tfrac{1}{2}\Delta t + \frac{\sin^2\phi\sin^4 A_e}{\sin^2 t\cos^2 t_e}\cot A_e\,4\sin^4\tfrac{1}{2}\Delta t \qquad \text{...A.83}$$

It is convenient to replace $\sin t$ by $\sin t_e$ in the above expression.

$$\sin t = \sin(t_e + \Delta t) = \sin t_e\cos\Delta t + \cos t_e\sin\Delta t$$

$$= \sin t_e(1 - 2\sin^2\tfrac{1}{2}\Delta t) + \cos t_e\,2\sin\tfrac{1}{2}\Delta t(1 - \sin^2\tfrac{1}{2}\Delta t)^{\frac{1}{2}}$$

$$= \sin t_e\{1 - 2\sin^2\tfrac{1}{2}\Delta t + 2\cot t_e\sin\tfrac{1}{2}\Delta t - 2\cot t_e\,\tfrac{1}{2}\sin^3\tfrac{1}{2}\Delta t\}$$

$$\therefore \operatorname{cosec} t = \operatorname{cosec} t_e \{1 - (2\sin^2 \tfrac{1}{2}\Delta t - 2 \cot t_e \sin \tfrac{1}{2}\Delta t + \cot t_e \sin^3 \tfrac{1}{2}\Delta t\}^{-1}$$

$$= \operatorname{cosec} t_e \{ 1 + 2 \sin^2 \tfrac{1}{2}\Delta t - 2 \cot t_e \sin \tfrac{1}{2}\Delta t + \cot t_e \sin^3 \tfrac{1}{2}\Delta t$$

$$+ 4 \cot^2 t_e \sin^2 \tfrac{1}{2}\Delta t - 8 \sin^3 \tfrac{1}{2}\Delta t \cot t_e$$

$$- 8 \cot^3 t_e \sin^3 \tfrac{1}{2}\Delta t \}$$

$$= \operatorname{cosec} t_e \{1 - 2 \cot t_e \sin \tfrac{1}{2}\Delta t + (1 + 2 \cot^2 t_e) 2 \sin^2 \tfrac{1}{2}\Delta t$$

$$-(7 + 8 \cot^2 t_e) \cot t_e \sin^3 \tfrac{1}{2}\Delta t \}$$

up to $\sin \tfrac{1}{2}\Delta t$ to the third power.

$$= \operatorname{cosec} t_e . E$$

The expression $\dfrac{\sin \phi \sin A_e}{\sin t_e \cos t_e}$ can be simplified

From the Polar Cosine Formula

$$\cos A_e = \sin t_e \sin \omega_e \sin \delta$$

$$\therefore \quad \sin t_e = \cos A_e \operatorname{cosec} \omega_e \operatorname{cosec} \delta$$

From the Sine Formula at elongation

$$\sin A_e = -\cos \delta \sec \phi \sin \omega_e$$

From Equation A.81 above

$$\cos t_e = \tan \phi \cot \delta$$

$$\therefore \quad \frac{\sin \phi \sin^2 A_e}{\sin t_e \cos t_e} = - \sin \phi \sin A_e \cos \delta \sec \phi \sin \omega_e \sec A_e \sin \omega_e \sin \delta \tan \delta \cot \phi$$

$$= - \sin^2 \delta \sin^2 \omega_e \tan A_e$$

$$= - \sin^2 \delta \tan A_e$$

$$= - C$$

Substituting the above expressions into Equation A.83 gives

$$\Delta A = \frac{\sin \phi \sin^2 A_e}{\sin t_e \cos t_e} 2 \sin^2 \tfrac{1}{2}\Delta t \, E + \frac{\sin^2 \phi \sin^4 A_e}{\sin^2 t_e \cos^2 t_e} \cot A_e \, 4 \sin^4 \tfrac{1}{2}\Delta t \, E^2$$

$$= - C \, 2 \sin^2 \tfrac{1}{2}\Delta t \, E + C^2 \cot A_e \, E^2$$

$$= - C \, 2\sin^2 \tfrac{1}{2}\Delta t \{ 1 - 2\cot t_e \sin \tfrac{1}{2}\Delta t + (1 + 2\cot^2 t_e) 2\sin^2 \tfrac{1}{2}\Delta t - (7 + 8\cot^2 t_e) \cot t_e \sin^3 \tfrac{1}{2}\Delta t\}$$

$$+ C^2 \cot A_e \, 4\sin^4 \tfrac{1}{2}\Delta t \{1 - \text{negligible terms}\}$$

$$= - C \, 2\sin^2 \tfrac{1}{2}\Delta t + C \cot t_e \, 4\sin^3 \tfrac{1}{2}\Delta t - C \, 4\sin^4 \tfrac{1}{2}\Delta t (1 + 2\cot^2 t_e) + C^2 \cot A_e \, 4\sin^4 \tfrac{1}{2}\Delta t$$

$$A - A_e = -C\, 2\sin^2 \tfrac{1}{2}\Delta t + C \cot t_e\, 4\sin^3 \tfrac{1}{2}\Delta t$$

$$- C\, 4\sin^4 \tfrac{1}{2}\Delta t(1 + 2\cot^2 t_e - \sin^2 \delta) \qquad \text{...A.84}$$

in which $C = \sin^2\delta \tan A_e$

Substitution of $\sin \tfrac{1}{2}\Delta t = \tfrac{1}{2}\Delta t - \frac{1}{48}\Delta t^3$ gives

$$A - A_e = -\tfrac{1}{2}C\,\Delta t^2 + \tfrac{1}{4}C\,\Delta t^3 \cot t_e - \frac{1}{24}C(12\cot^2 t_e + 6\cos^2\delta - 1)\Delta t^4 \qquad \text{...A.85}$$

The following are the first two terms of this series expansion:-

$$A = A_e - \sin^2\delta \tan A_e\, \tfrac{1}{2}(\Delta t)^2 + \sin^2\delta \tan A_e\, \tfrac{1}{4}(\Delta t)^3 \cot t_e$$

in which $\Delta t = (t - t_e)$ and all quantities are expressed in radians.

$$\therefore \quad A.\rho'' = A_e\,\rho'' - C\,\tfrac{1}{2}\,(\Delta t)^2\,\rho'' + C\,\tfrac{1}{4}\,(\Delta t)^3\rho'' \cot t_e$$

in which $C = \sin^2\delta \tan A_e$.

$$\therefore \quad A'' = A''_e - C\,\tfrac{1}{2}\frac{(\Delta t'')^2}{\rho''} + C\,\tfrac{1}{4}.\frac{(\Delta t'')^2}{\rho''}\,\rho''\,\frac{(\Delta t'')}{\rho''} \cot t_e$$

in which the units are arc seconds.
But

$$\frac{\Delta t''}{\rho''} = \frac{\Delta t''}{\rho''}\,\frac{15}{15} = \frac{15}{\rho''}\Delta t^s = \frac{15}{\rho''}\,\frac{60}{60}\Delta t^s = \frac{900}{\rho''}\,(\Delta t^m)$$

and

$$\tfrac{1}{2}\left(\frac{\Delta t''}{\rho''}\right)^2 \rho'' = \frac{\rho''}{2}\left(\frac{900}{\rho}\Delta t^m\right)^2 = \frac{900^2}{2\rho''}\,(\Delta t^m)^2$$

in which Δt^s and Δt^m are the Δt values expressed in time seconds and minutes.
Substitution above gives

$$A'' = A''_e - C\,\frac{900^2}{2\rho''}(\Delta t^m)^2 + C\,\frac{900^2}{2\,\rho''}(\Delta t^m)^2\,\frac{900}{\rho''}(\Delta t^m) \cot t_e$$

$$= A''_e - C\,\frac{900^2}{2\rho''}(\Delta t^m)^2 \left\{1 - \frac{900}{\rho''}\Delta t^m \cot t_e\right\}$$

$$= A''_e - C\; 1.963\;496''\;(\Delta t^m)^2 \left\{1 - \frac{900}{\rho''}(\Delta t^m) \cot t_e\right\} \qquad \text{...A.86}$$

in which form, the relationship can be very conveniently set up for a calculator.

Under the heading Table for Circum-Meridian Observations the relationship $m = 2\sin^2(\tfrac{1}{2}HA)\operatorname{cosec} 1''$ is tabulated. The quantity $1.9635''(\Delta t^m)$ is a very good approximation to m.

Circum Elongation Altazimuth

A.82 It is required to express the azimuth A at altitude h in terms of the elongation azimuth A_e at altitude h_e, ie. $A = F(h - h_e)$. This can be expressed as a Maclaurin Series as

$$A = F(0) + F_1(0)(h - h_e) + \frac{1}{2!}F_2(0)(h - h_e)^2 + \frac{1}{3!}F_3(0)(h - h_e)^3 \qquad \text{...A.87}$$

in which $F(0) = A_e$, because then h equals h_e, and $F_1(0) = \frac{DA}{Dh}$, $F_2(0) = \frac{D_2A}{Dh^2}$ . . .

$$\therefore \quad A = A_e + \frac{DA}{Dh}(h-h_e) + \frac{1}{2}\frac{D_2A}{Dh^2}(h-h_e)^2 + \frac{1}{6}\frac{D_3A}{Dh^3}(h-h_e)^3 + \frac{1}{24}\frac{D_4A}{Dh^4}(h-h_e)^4 \qquad \text{...A.88}$$

The differential relationship, given in section 2.75 and connecting A, ϕ, δ and h, gives $d\delta = \cos\phi \sin t \, dA + \cos t \, d\phi + \cos\omega \, dh$

$$\therefore \quad \frac{dA}{dh} = -\cos\omega \sec\phi \operatorname{cosec} t \quad \text{when } \phi \text{ and } \delta \text{ are held constant}$$

$$= -\cot\omega \sec h \quad \text{since } \sec\phi \operatorname{cosec} t = \sec h \operatorname{cosec}\omega \text{ from the}$$

Sine Formula.

Further differentiation gives

$$\frac{d_2A}{dh^2} = -\cot\omega \sec h \tan h + \operatorname{cosec}^2\omega \frac{d\omega}{dh} \sec h$$

Differentiation of the Cosine Formula, linking ω, ϕ, δ and h gives

$$\frac{d\omega}{dh} = \cos A \sec\delta \operatorname{cosec} t = -\cot A \sec h$$

when ϕ and δ are held constant and the Sine Formula is used as above.

$$\therefore \quad \frac{d_2A}{dh^2} = -\cot\omega \sec^2 h \sin h - \sec^2 h \operatorname{cosec}^2\omega \cot A$$

Further differentiation gives

$$\frac{d_3A}{dh^3} = +\operatorname{cosec}^2\omega \frac{d\omega}{dh} \sec^2 h \sin h - \cot\omega \, 2\sec^2 h \tan h \sin h$$
$$- \cot\omega \sec^2 h \cos h - 2\sec^2 h \tan h \operatorname{cosec}^2\omega \cot A$$
$$+ \sec^2 h \, 2\operatorname{cosec}\omega \cot\omega \operatorname{cosec}\omega \frac{d\omega}{dh} \cot A + \sec^2 h \operatorname{cosec}^2\omega$$
$$\operatorname{cosec}^2 A \frac{dA}{dh}$$

Substitution for $\frac{d\omega}{dh}$ and $\frac{dA}{dh}$ and simplification gives

$$\frac{d_3A}{dh^3} = -\operatorname{cosec}^2\omega \sec^3 h \sin h \cot A - 2\cot\omega \sec^3 h \sin^2 h - \cot\omega \sec h$$
$$- 2\sec^3 h \sin h \operatorname{cosec}^2\omega \cot A - 2\sec^3 h \operatorname{cosec}^2\omega \cot\omega \cot^2 A$$
$$- \sec^3 h \cot\omega \operatorname{cosec}^2\omega \operatorname{cosec}^2 A$$

At the point of elongation, where $\omega_e = 90^\circ$ or 270, these differential coefficients become

$$\frac{DA}{Dh} = -\cot\omega_e \sec h_e = 0$$

except when $h_e = 90^\circ$, which is a condition never encountered in practice.

$$\frac{D_2A}{Dh^2} = -\cot\omega_e \sec^2 h_e \sin h_e - \sec^2 h_e \operatorname{cosec}^2\omega_e \cot A_e$$

$$= -\cot 90 \sec^2 h_e \sin h_e - \sec^2 h_e \operatorname{cosec}^2 90 \cot A_e$$

$$= -\sec^2 h_e \cot A_e$$

$$\frac{D_3A}{Dh^3} = -3 \operatorname{cosec}^2\omega_e \sec h_e \sin h_e \cot A_e$$ + terms containing $\cot\omega_e$ which reduces each of these terms to zero.

$$= -3 \sec^2 h_e \tan h_e \cot A_e$$

The fourth differential coefficient $\frac{D_4A}{Dh^4}$ at elongation, which is the last one it is proposed to evaluate here, is obtained by further differentiation, which discards any quantity that produces a term containing the factor $\cot\omega$. This produces

$$\frac{D_4A}{Dh^4} = -11 \sec^4 h_e \sin^2 h_e \cot A_e - 4 \sec^2 h_e \cot A_e$$

$$-2 \sec^4 h_e \cot^3 A_e - \sec^4 h_e \cot A_e \operatorname{cosec}^2 A_e$$

$$= -\sec^2 h_e \cot A_e (9 \sec^2 h_e + 3 \sec^2 h_e \operatorname{cosec}^2 A_e - 7)$$

$$\therefore \quad A = A_e - \tfrac{1}{2}\sec^2 h_e \cot A_e (h-h_e)^2 - \tfrac{1}{2}\sec^2 h_e \cot A_e \tan h_e (h-h_e)^3 \ldots$$

$$A'' = A''_e - \sec^2 h_e \cot A_e \frac{(h''-h''_e)^2}{2\rho''} - \sec^2 h_e \cot A_e \tan h_e \frac{(h''-h''_e)^3}{2(\rho'')^2}$$

...A.89

in which ρ is in accord with the units used.

DERIVATION OF THE LAPLACE EQUATION

A.91 AT a Laplace Station (see section 1.53) astronomical values ϕ_A λ_A defining its position and A_A the azimuth to an adjacent station of the geodetic survey are available as well as the corresponding geodetic values $\phi_G \lambda_G$ and A_G, which have been carried forward in the geodetic survey from the Funadmental Station of this survey. Fig A.3 shows the astronomic zenith Z_A and the geodetic zenith Z_G of the Laplace Station with a much exaggerated space between them.

The line, along which the azimuth was observed on the earth from the Laplace Station to the reference object RO defines a vector, which has azimuth A_A and an altitude h_A, in practice a value very close to the horizon. This vector produced will intersect the celestial sphere at the point R, as is shown in Fig A.3. The two triangles pole, zenith and R in this figure are spherical triangles but neither is an astronomical triangle, because the point R is not a star and therefore maintains its position with respect to the local meridians $P_N Z_N$ and $P_N Z_G$, so that the pole angles P_A and P_G, as well as the longitude angles λ_A and λ_G illustrated do not vary with time so that

$$\lambda_A + P_A = \lambda_G + P_G$$

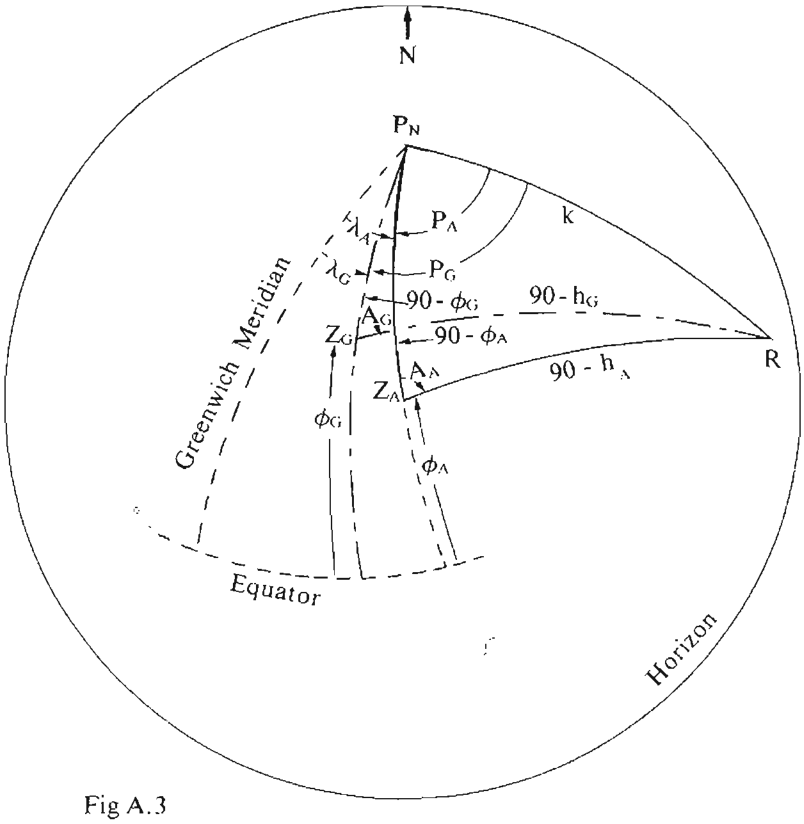


Fig A.3

If $\quad \phi_G = \phi_A + d\phi \quad$ then $\quad d\phi = \phi_G - \phi_A$

and if $\quad P_G = P_A + dP \quad$ then $\quad dP = P_G - P_A = -(\lambda_G - \lambda_A)$

because longitude is defined as increasing eastwards and the pole angles here are shown as increasing westwards.

Also if $\quad A_G = A_A + dA \quad$ then $\quad dA = A_G - A_A$

The side $P_NR = k$ is common to both triangles and is constant in length so that $dk = 0$.

The differential relationship connecting dA, $d\phi$, dP and dk is given in section 2.62 as

$$\sin w \, dY = -\cos X \sin y \, dW - \cos w \sin Y \, dx + \sin X dy$$

Substitution from the spherical triangle P_NZ_AR gives

$$\sin(90-h_A) \, dA = -\cos R \sin k \, dP - \sin h_A \sin A_A \, d\phi + \sin R \, dk$$

$$\cos h_A \, dA = -\cos R \sin k \, dP - \sin h_A \sin A_A \, d\phi$$

with the last term going to zero because k does not vary.

The Five Parts Formula in this triangle gives

$$\sin k \cos R = \cos(90-\phi_A)\ \sin(90-h_A) - \sin(90-\phi_A)\ \cos(90-h_A)\ \cos A_A$$

$$= \sin\phi_A \cos h_A - \cos\phi_A \sin h_A \cos A_A$$

Substitution for $\sin k \cos R$ from here into the previous equation gives

$$\cos h_A\ dA = -(\sin\phi_A \cos h_A - \cos\phi_A \sin h_A \cos A_A)\ dP - \sin h_A \sin A_A\ d\phi$$

$$\therefore \quad dA = (-\sin\phi_A + \cos\phi_A \tan h_A \cos A_A)\ dP - \tan h_A \sin A_A\ d\phi$$

$$\therefore \quad A_G - A_A = \sin\phi_A(\lambda_G-\lambda_A) - \cos\phi_A \tan h_A \cos A_A(\lambda_G-\lambda_A) - \tan h_A \sin A_A(\phi_G-\phi_A)$$

Since h_A in practice is a small angle, $\tan h_A$ is small and therefore to first order accuracy, the Laplace Equation is given by

$$A_G - A_A = (\lambda_G - \lambda_A) \sin\phi$$

in which either value ϕ_G or ϕ_A may be used for ϕ.

CALCULATOR METHODS OF TIME CONVERSION

A.101 THE increasing use of calculators, particularly the programmable type, has rendered former methods of time conversion cumbersome. In particular the selection of the appropriate value of R to be used in these calculations may cause a problem when times are transferred between the Greenwich and the observer's meridian. The following techniques are simple and eliminate any date ambiguity.

The Conversion of an Instant of Standard Time to the Corresponding Instant of LST.

$$LST = (\text{Standard Time} - \text{Zone})\ F + R_o + \lambda$$

Add or subtract multiples of 24^h if the calculated value of LST does not lie in the range $0\text{-}24^h$.

The values of Zone and λ, the longitudes of the standard and observer's meridian, have the convention that they are positive when east and negative when west of the Greenwich meridian.

R_o is GST at $UT0^h$ on the Greenwich date *equal* to the *local* date of observation. Note that dates are *always* associated with the Mean Time and *not* with the Sidereal Time System.

$$F = 1.0027379$$

The Conversion of an Instant of LST to the Corresponding Instant of Standard Time.

$$\text{Standard Time} = \frac{LST - R_o - \lambda}{F} + \text{Zone}$$

Add or subtract multiples of $24^h/F$ if the calculated value of Standard Time does not lie in the range $0\text{-}24^h$.

It will be noted that if the instant of Standard Time lies within a range of $3^m55\overset{s}{.}9$ on either side of midnight, two *identical values* of LST on the *same date* can occur. However, the two values of Standard Time which result are so far removed from one another that the choice of which of the two values is the correct one is obvious.

EXAMPLES

	Local Date	R_o*	Zone	Longitude	Std Time	LST
1	12 Sep. 1977	$23^h23^m32^s.5$	4^hW	$4^h26^m34^s.1$ W	$1^h14^m27^s.3$	?
2	28 Apr. 1977	14 23 24.5	10 E	9 39 51.0 E	8 00 00.0	?
3	16 Jun. 1977	17 36 35.7	2 E	1 13 44.0 E	18 32 43.2	?
4	17 Aug. 1977	21 41 02.1	5 W	5 19 34.5 W	?	$1^h02^m30^s.1$
5	23 Sep. 1977	0 06 54.6	8 E	7 32 18.1 E	?	23 59 42.2
6	21 Dec. 1977	5 57 47.9	12 E	11 21 58.1 E	?	5 20 05.7

* R_o on Greenwich date equal to the local date of observation.

Example 1

Calculated LST	$24^h12^m17^s.4$
24^h	- 24
LST	0 12 17.4

Example 2

Calculated LST = LST	22 02 55.8

Example 3

Calculated LST	35 25 46.0
24^h	- 24
LST	11 25 46.0

Example 4

Calculated Std Time	- 20 16 27.0
24^h/F	+ 23 56 04.1
Std Time	3 39 37.1

Example 5

Calculated Std Time	24 17 48.9
24^h/F	- 23 56 04.1
Std Time	0 21 44.8

Example 6

Calculated Std Time	0 02 17.6
24^h/F	+ 23 56 04.1
Std Time	23 58 21.7

Note that two values of Standard Time i.e. $0^h02^m17^s.6$ and $23^h58^m21^s.7$ have the same corresponding value of LST on this date.

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