A

TREATISE

ON

ASTRONOMY

THEORETICAL AND PRACTICAL.

BY

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CHAP. XVII.

ON THE SOLAR THEORY.

Inequable Motions of the Sun in Right Ascension and Longitude.—The Obliquity of the Ecliptic determined from Observations made near to the Solstices.—The Reduction of Zenith Distances near to the Solstices, to the Solstitial Zenith Distance.—Formula of such Reduction.—Its Application.—Investigation of the Form of the Solar Orbit.—Kepler's Discoveries.—The Computation of the relative Values of the Sun's Distances and of the Angles described round the Earth.—The Solar Orbit an Ellipse.—The Objects of the Elliptical Theory.

IN giving a denomination to the preceding part of this Volume, we have stated it to contain the Theories of the fixed Stars; such theories are, indeed, its essential subjects; but they are not exclusively so. In several parts we have been obliged to encroach on, or to borrow from, *the Solar Theory*; and, in so doing, have been obliged to establish certain points in that theory, or to act as if they had been established.

To go no farther than the terms Right Ascension, Latitude, and Longitude. The right ascension of a star is measured from the *first point of Aries*, which is the technical denomination of the intersection of the equator and ecliptic, the latter term designating the plane of the *Sun's orbit:* the latitude of a star is its angular distance from the last mentioned plane; and the longitude of a star is its distance from the first point of Aries measured along the ecliptic.

The fact then is plain, that the theories of the fixed stars have not been laid down independently of other theories: and it is scarcely worth the while to consider whether or not, for the sake of a purer arrangement, it would have been better to have postponed certain parts of their theories till the theory of the Sun's orbit, and of his motion therein, should have been estalished.

According to our present plan, indeed, (a plan almost always adopted by Astronomical writers) we shall be obliged to go over ground already trodden on. But we shall go over it more carefully and particularly. In those parts of the solar theory which it was necessary to introduce, either for the convenient or the perspicuous treating of the sidereal, we went little beyond approximate results and the description of general methods. For instance, in pages 137, 138, it is directed, and rightly, to find the obliquity of the ecliptic from the greatest northern and southern declinations of the Sun. But the practical method of finding such extreme declinations was not there laid down; and on that, as on other occasions, much detail, essentially necessary indeed, but which would then have embarrassed the investigation, was, for the time, suppressed.

Such detail is now to be given together with other methods, that belong to the solar theory. But it may be right, previously to enumerate some of the results already arrived at.

In Chapter VI, which was on the Sun's Motion and its Path, it was shewn that the Sun possessed a peculiar motion tending, in its general description, from the west towards the east, almost always oblique to the equator, and inequable in its quantity. These results followed, almost immediately, from certain meridional observations made with the transit instrument and mural circle.

By such observations two motions or changes of the Sun's place are determined; one in the direction of the meridian, the other in a direction perpendicular to the meridian. The oblique motion of the Sun, therefore, is, in strictness, merely an inference from the two former motions: or, if we suppose the real to be an oblique motion, its two resolved parts will be those which the transit instrument and mural circle discover to us; neither of which motions (see p. 126.) is an equable one.

But although the two resolved motions are inequable, it does not at once follow that the oblique or compounded motion must be inequable. For, if it were equable, the resolved parts, namely, the motion in right ascension, and the motion in declination, would be inequable. Some computation, therefore, is necessary to settle this point, and a very slight one is sufficient.

Thus, by observations made in 1817,

July 1, $\odot$'s Æ	$6^h\ 40^m\ 1^s.7$	.... Decl.	$23^\circ\ 8'\ 44''$ N.
2, ...	$6\ 44\ 9.7$	....	$23\ 4\ 35$
Jan. 1, ...	$18\ 17\ 2.2$	....	$23\ 1\ 26$ S.
2, ...	$18\ 51\ 27.0$	....	$22\ 56\ 14$

Compute the longitudes of the Sun by means of this formula,

$$1 \times \sin.\ \odot\text{'s long.} = \cos.\ \odot\text{'s dec.} \times \cos.\ \odot\text{'s Æ},$$

and we have

		Difference.
July 1, $\odot$'s long.	$3^s\ 9^\circ\ 11'\ 39''$	....$0^\circ\ 57'\ 11''$,
2, ...	$3\ 10\ 8\ 50$	
Jan. 1, ...	$9\ 10\ 48\ 38$	....$1\ 1\ 10$.
2, ...	$9\ 11\ 49\ 48$	

The oblique daily motions then, instead of being equal, are to one another as 3433 to 3670.

Besides the results relating to the Sun's path and motion, there were obtained, in Chapter VI, other results, such as the obliquity of the ecliptic, and the times of the Sun entering the equator and arriving at the solstitial points. But the methods by which the results were obtained require revision, or rather we should say, that these methods having answered their end, namely, that of forwarding us in the investigation which we were then pursuing, may now be dismissed, and make way for real practical methods.

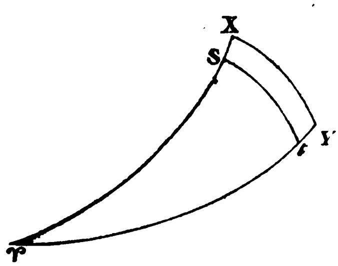
We will turn our attention, in the first place, to the determination of the obliquity of the ecliptic.

If at an Observatory, the Sun arrived at the solstice exactly when he was on the meridian, the observed declination would be the measure of the obliquity. But it is highly improbable that such a case should happen : nor is it, indeed, on the grounds of astronomical utility, much to be desired. A solitary observation, under the above-mentioned predicament, would not be sufficient to establish satisfactorily so important an element as that of the obliquity. It would be necessary to combine with it other observations of the Sun's declination, made on several days before and after the day of the greatest declination, to *reduce*, by computation, such less declinations to the greatest, and then to take their *mean* to represent the value of the obliquity. In such a procedure, it is clearly of little or no consequence, whether the middle declination be itself exactly the greatest, or whether, like the declinations on each side of it, it requires to be similarly reduced to the greatest.

The *reduction* of declinations to the greatest, which is the solstitial declination, is an operation of the same nature, and founded on the same principles, as the reduction of zenith distances observed *out* of the meridian to the meridional zenith distance: the formulæ of which latter reduction, together with their demonstration, were given in pages 418, &c. It is convenient, however, on the present occasion, to modify the result of that demonstration, or to express it by a different formula; which we will now proceed to do.

Let then,

d ($= St$) be the Sun's declination, d' ($= XY$) the solstitial,



$\odot$ ($= \Upsilon S$), $\odot'$ ($= 90^0$) the corresponding longitudes,

w, the obliquity of the ecliptic,

then, by Naper's Rules, we have

$$\sin. d = \sin. \odot . \sin. w,$$

$$\sin. d' = \sin. \odot' . \sin. w;$$

consequently,

$$\sin. d' - \sin. d = \sin. w\, (\sin. 90^0 - \sin. \odot),$$

or (see *Trigonometry*, pp. 32, 42),

$$\sin. \frac{w-d}{2} . \cos. \frac{w+d}{2} = \sin. w . \sin.^2 \frac{u}{2}, \text{ if } u = 90^0 - \odot .$$

Let $w (= d') = d + \delta$, then,

$$\sin. \frac{\delta}{2} . \cos. \left(w - \frac{\delta}{2}\right) = \sin. w . \sin.^2 \frac{u}{2},$$

$$\text{and, } \sin. \frac{\delta}{2} \left\{\cos. w \cos. \frac{\delta}{2} + \sin. w \sin. \frac{\delta}{2}\right\} = \sin. w . \sin.^2 \frac{u}{2}.$$

Substitute, instead of $\sin. \frac{\delta}{2}$, $\cos. \frac{\delta}{2}$, and $\frac{\delta}{2} - \frac{\delta^3}{48}$, and $1 - \frac{\delta^2}{8}$, respectively, (suppressing for the present $\sin. 1''$, $\sin.^3 1''$, $\sin.^2 1''$, by which δ, δ^3, δ^2, ought, respectively, to be multiplied), and we shall have this approximate expression,

$$\left(\frac{\delta}{2} - \frac{\delta^3}{48}\right)\left\{\cos. w - \cos. w \frac{\delta^2}{8} + \sin. w \frac{\delta}{2}\right\} = \sin. w . \sin.^2 \frac{u}{2},$$

$$\text{whence } \frac{\delta}{2} - \frac{\delta^3}{48} = \frac{\tan. w . \sin.^2 \frac{u}{2}}{1 - \frac{\delta^2}{8} + \frac{\delta}{2} \tan. w;};$$

or, nearly,

$$\frac{\delta}{2} = \tan. w . \sin.^2 \frac{u}{2} \left\{1 - \frac{\delta}{2} \tan. w + \frac{1}{2} . \frac{\delta}{2} \frac{\delta}{2} + \frac{\delta}{2} . \frac{\delta}{2} \tan.^2 w\right\}$$

$$+ \frac{1}{6} . \frac{\delta}{2} . \frac{\delta}{2} . \frac{\delta}{2},$$

from which expression, approximate values of $\frac{\delta}{2}$, of sufficient exactness may be obtained: for instance, to obtain a first approximation, neglect the terms on the right hand side of the equation, that involve δ, and

$$\text{(1st value)} \quad \frac{\delta}{2} = \tan. w . \sin.^2 \frac{u}{2}.$$

Again, retain the terms involving δ and neglect those involving δ^2, and

$$\text{(2nd value)*} \quad \frac{\delta}{2} = \tan. w . \sin.^2 \frac{u}{2} \left\{ 1 - \tan. w . \sin.^2 \frac{u}{2} . \tan. w \right\}$$

$$= \tan. w . \sin.^2 \frac{u}{2} - \tan.^3 w . \sin.^4 \frac{u}{2}.$$

Again, substitute this new value, and neglect those terms that involve higher dimensions of $\sin. \frac{u}{2}$ than the 6th, and

$$\frac{\delta}{2} = \tan. w . \sin.^2 \frac{u}{2} - \tan.^3 w . \sin.^4 \frac{u}{2}$$

$$+ 2 \tan.^5 w . \sin.^6 \frac{u}{2} + \frac{1}{6} \tan.^3 w . \sin.^6 \frac{u}{2}.$$

$$\text{But } \sin. \frac{u}{2} = \frac{u}{2} - \frac{1}{2 . 3} . \frac{u^3}{8} + \frac{1}{2 . 3 . 4 . 5} . \frac{u^5}{32} + \&c.$$

From this value find $\sin.^2 \frac{u}{2}$, $\sin.^4 \frac{u}{2}$, &c. and substitute in the preceding expression, and then

$$\dagger\ \delta = \tan. w . \frac{u^2}{2} - \frac{1}{24} . \tan. w (1 + 3 \tan.^2 w) u^4$$

$$+ \frac{1}{720} . \tan. w (1 + 30 \tan.^2 w + 45 \tan.^4 w) u^6,$$

which is sufficiently exact for all practical purposes, since u rarely exceeds 10^0.

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* This is the expression which Biot uses in his *Astronomie*, pp. 31, 336.

† This is the same expression as Delambre's, Tom. II. p. 244, but is differently obtained.

For the purpose of avoiding multiplicity of symbols, the powers of sin. 1″ (see p. 5. l. 12, &c.) were omitted in the preceding investigation. These, however, must be restored in order to render the above expression for δ fit for application. This is easily effected: δ being very small, δ has been written instead of sin. δ: whereas δ . sin. 1″ should have been written; on the right hand of the side, instead of u^2, u^4, u^6, &c. $u^2 . \sin.^2 1''$, $u^4 . \sin.^4 1''$, $u^6 . \sin.^6 1''$, &c. should have been written: supplying then the omitted symbols, and dividing each side of the equation by sin. 1″, we have

$$\delta = \frac{\tan. w}{2} . u^2 . \sin. 1'' - \frac{\tan. w}{24} (1 + 3 \tan.^2 w) u^4 . \sin.^3 1''$$

$$+ \frac{\tan. w}{720} (1 + 30 \tan.^2 . w + 45 \tan.^4 w) u^6 . \sin.^5 1'' ;$$

u is the difference betwen 90^0 (the longitude of the Sun at the solstice), and the Sun's longitude at the time of observation. If the place of observation be Greenwich, u is known by the Nautical Almanack, and from the value therein given, may easily be computed for any other place of observation. Suppose, for instance, the Sun's meridional distance either from the north pole, or from the zenith to have been observed at Greenwich, on June 18, 1812. By the Nautical Almanack,

$$\odot = 2^s\ 27^0\ 0'\ 4'' ;\ \therefore u = 2^0\ 59'\ 56'' = 10796''.$$

In this case the reduction to the solstice (δ) will be expressed with sufficient exactness by the first term. w, then, being taken $= 23^0\ 27'\ 54''$, we have

$$\delta = \frac{1}{2} \tan. 23^\circ\ 27'\ 54'' . \sin. 1'' \times (10796)^2 = 2'\ 2''.6\ *.$$

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* Computation.

Log. tan. 23° 27′ 54″	= 9.6375760
arith. comp. of 2	= 9.6989699
log. sin. 1″	= 4.6855749
2 log. 10796	= 8.0665258
	2.0886466 = log. 122″.64.

If, therefore, the observed meridional zenith distance of the Sun's centre (after being corrected for refraction), were, on the noon of June 18, equal to $28^0\ 3'\ 2''.5$ the reduced zenith *solstitial* distance would be, nearly,

$$28^\circ\ 3'\ 2''.5 - 2'\ 2''.6, \text{ or } 28^\circ\ 0'\ 59''.9.$$

This is an application of the formula to one instance : and like applications to other instances are easily made ; with greater length of computation, indeed, if the Sun should be so far from the solstice, as to render it necessary, by reason of the magnitude of u, to compute the second and third terms of the value of δ. Now the obliquity of the ecliptic being an element of great astronomical importance, the finding it by means of the *reduction* is a frequent operation. It becomes worth the while, then, to construct a Table from the preceding expression, and for every ten minutes of the Sun's distance from the solstice. To obtain this latter end, instead of u write $10'\ u = 600''\ u$, and

$$\delta = \frac{\tan. w}{2} . \sin. 1''. (600)^2. u^2$$

$$- \frac{\tan. w}{24} . (1 + 3 \tan.^2 w) \sin.^3 1''. (600)^4 u^4 + \&c.$$

or, the value of the obliquity being assumed equal to $23^0\ 27'\ 54''$,

$$\delta = 0''.378812\ u^2 - 0''.0000004181\ u^4 + 0''.00000000000006217\ u^6.$$

From this expression a Table may be expeditiously constructed. The values of δ, most easily obtained, are those which belong to u, when its values are, respectively, 1, 2, 3, 4, &c. 10, 20, 30, 100, &c. that is, since the value of the unit of u is $10'$, when the distances from the solstice are $10'$, $20'$, $30'$, $40'$, &c. $1^0\ 40'$, $3^\circ\ 20'$, 5°, &c. $16^0\ 40'$, &c.

For instance,

Distance from Solst.	Values of u.		
$0^0\ 10'$	1	$\delta = 0''.3788$	$0^0\ 0'\ 0''.3788$
20	2	$\delta = 0\ .3788 \times 4$	0 0 1 .515
30	3	$\delta = 0\ .3788 \times 9$	0 0 3 .409
&c.			
1 40	10	$\delta = 37''.881 - .00418$	0 0 37 .876
3 20	20	$\delta = 151.524 - .0668$	0 2 31 .457
16 40	100	$\delta = 3788.12 - 41.81 + .6217$	1 2 26. 93

This is a sample of a Table, to be constructed from the preceding expression. M. Delambre has given such a Table in p. 269, of the second Volume of his Astronomy. In that Table the expressed numerical values of δ belong to an obliquity $= 23^\circ\ 28'$.

Our values belong to an obliquity $= 23^\circ\ 27'\ 54''$, and, therefore, are somewhat smaller, as they need must be, than Delambre's. But a very slight correction will reduce one set of values to the other. And M. Delambre's Table furnishes the means of effecting this: since it contains, in a separate column, a series of corrections due to a variation of $100''$ in the obliquity, and corresponding to the several values of u.

In order to obtain the algebraical expression of the correction just mentioned we must resume the original value of δ, or, which will be sufficient for the occasion, express it by its first term: now, if

$$\delta = \frac{1}{2} \tan.\ w \,.\, \sin.\ 1''.\ u^2,$$

$$\dot{\delta} = \frac{\dot{w}}{2} \,.\, \sin.^2 1''.\ \sec.^2 w \,.\, u^2,$$

$\dot{\delta}$, $\dot{w}$, expressing the corresponding variations of δ and w.

If $\dot{w} = 100''$,

$$\dot{\delta} = .000000001396\ u^2, \text{ the unit of } u \text{ being } 1''.$$

If, as in the former case, we make the unit of u equal to $10'$,

$$\dot{\delta} = .000000001396 \times (600)^2\ u^2 = 0''.000502812\ u^2,$$

and from this expression the column of corrections, to which we alluded at l. 11, may be computed.

We will now give an example of the computation of the obliquity of the ecliptic, from observations of the Sun's meridional zenith distances observed during several days on each side of the solstice.

1812, June.	Refraction.	Zenith Distance by Instrument.		Sun's Semi-diameter.	Zenith Distance of Sun's Centre.	Reduction.	Solstitial Zen. Dist.
	″	° ′ ″		′ ″	° ′ ″	′ ″	° ′ ″
12	30.3	28 1 58.9	U	15 47.2	28 18 16.4	17 15.5	28 1 0.9
14	29.4	27 55 14.4	U	15 47.2	28 11 31.0	10 32.4	28 0 58.6
18	29.6	27 46 46.1	U	15 46.8	28 3 2.5	2 2.6	28 0 59.9
19	29.7	28 17 13.4	L	15 46.8	23 1 56.3	0 57.0	28 0 59.3
20	29.2	27 45 0.2	U	15 46.8	28 1 16.2	0 16.3	28 0 59.7
23	30.4	28 16 58.74	L	15 46.6	28 1 42.56	0 42.7	28 0 59.8
24	29.8	27 46 26.56	U	15 46.6	28 2 42.96	1 41.1	28 1 1.9
25	30.4	28 19 20.76	L	15 46.6	28 4 4.56	3 4.2	28 1 0.4
27	29.7	27 51 58.76	U	15 46.6	28 8 5.06	7 4.7	28 1 0.7
28	30.7	28 25 56.76	L	15 46.6	28 10 40.86	9 41.4	28 0 59.4
29	30.1	27 57 26.66	U	15 46.6	28 13 43.36	12 43.0	28 1 0.4
30	30.7	28 32 24.76	L	15 46.6	28 17 8.86	16 8.9	28 1 0

The refractions in the second column are computed from the heights of the barometer and thermometer, and the zenith distances of the Sun's limb, according to the Rules of Chapter X, (see pp. 247, &c.) The zenith distances of the Sun's centre in the fifth column, are formed by adding the refractions to the zenith distances of the observed limb, and by adding or subtracting (according as the observed limb is an upper or lower limb) the Sun's semi-diameter. The *reductions* in the sixth column are computed by the formulæ of p. 436 *, or may be taken from a Table constructed from such formulæ: the *solstitial* zenith distances of the Sun's centre in the seventh

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* In computing these reductions, the values of u are known by the Nautical Almanack. Thus, we have from that book,

☉'s long. June 12, being $2^s\ 21°\ 16'\ 22''$, $u = 8°\ 43'\ 38''$,
14 ...... 2 23 10 59, ... 6 49 1,
20 ...... 2 28 54 32, ... 1 5 28;

therefore on the 12th $u = 8°\ 43'\ 38'' = 52.3633$, which being substituted in the value of δ (see p. 436, l. 21,)

$$\delta = 17'\ 15''.46,$$

on the 20th $u = 1°\ 5'\ 28'' = 6.54666$, &c. and $\delta = 16''.23$.

column are formed by subtracting the numbers in the sixth from those in the fifth column: the *decimals* being expressed by the figures that most nearly represent their values *.

The sum of the numbers in the last column, is

$$12 \times 28^{0}\ 12'\ 1'',$$

the 12th of which, in the nearest numbers, is

$$28^{0}\ 1'\ 0''.1,$$

which represents the mean solstitial zenith of the Sun's centre deduced from twelve observations. But such zenith distance has been corrected for refraction only. It is, therefore, for reasons abundantly given in the preceding part, an *apparent* zenith distance, and is affected with nutation, parallax, and another inequality arising from the attraction of the planets, and explained in Chapter XXII, of *Physical Astronomy*. With regard to the first inequality, the nutation (the place of the Moon's node being $5^{s}\ 2^{0}\ 9'$) equals (see pp. 375, &c.) $-\ 8''.4$, the parallax also equals $-\ 4''$, and their sum, accordingly, equals $-\ 12''.4$. The value of the third inequality, the *Sun's Latitude*, as it is called, caused by the Sun being drawn from the plane of the ecliptic by the action of the planets, is $-\ 0''.63$.

So that we have (from l. 7,)

Sun's solstitial zenith distance	28^{0}	$1'$	$0''.1$
nutation and parallax		$-$	12.4
Sun's mean solstitial zenith distance....	28	0	47.7
if the co-latitude (ZP) be	38	31	21.5
$ZP + Z\odot$	66	32	9.2
therefore, solstitial declination	23	27	50.8
subtract Sun's latitude			.63
mean obliquity of summer solstice......	23	27	50.17

This is the determination of the obliquity from the summer solstice, and is founded on a knowledge of the latitude of the

* For instance, *decimals* such as .86, .47, &c. would be represented by .9, .5.

place, which knowledge is founded on that of the quantity and law of refraction (see Chapter X.) Now, with regard to this latter point, there is something that remains still to be determined by Astronomers. For, if we suppose the Sun, at the winter solstice, equally distant from the equator as at the summer solstice, the obliquity determined at the former season from the expression,

$$ZP + Z\odot' - 90^0,$$

ought to equal the obliquity determined, as it just has been, from

$$90^0 - \{ZP + Z\odot\};$$

if the theory of refractions were good, and the observations accurately made. Now the fact is, as we have already stated it at p. 138, the two values of the obliquity do not agree, when the respective zenith distances of the Sun are corrected by that formula of refraction which results from a comparison of the observations of circumpolar stars, (see p. 230.)

Let L be the latitude of the place, then, at the summer solstice,

$$\begin{aligned} w &= 90^\circ - \{90^\circ - L + Z\} \\ &= L - Z; \end{aligned}$$

at the winter solstice,

$$\begin{aligned} w &= 90^\circ - L + Z' - 90^\circ \\ &= Z' - L. \end{aligned}$$

In the first case then, (supposing Z, Z', the solstitial zenith distances to be correct)

$$dw = dL, \text{ in the second } dw = -dL.$$

If we suppose then an error in the value of the latitude of the place of observation, the obliquity, determined from the summer solstitial distance, will be increased by it, and, if determined from the winter, equally diminished. If, therefore, we add the two values of the obliquity together, their half sum, or *mean*, may, in a certain sense, be said to be free from the error of latitude; but the *mean*, thus determined, will not necessarily be the true value of the obliquity, since the zenith distances (Z, Z') are corrected by the formula of refraction, and partake of its uncertainties.

To illustrate the formula of the *reduction to the solstice*, and the method of finding the obliquity of the ecliptic, an example was taken of observations made at Greenwich with the mural circle. Like observations made with a mural quadrant, would have answered precisely the same end: and so, indeed, would observations made, as they are made (see pp. 417, &c.) at the Observatory of Trinity College, Dublin, with Ramsden's circle, or by the repeating circle, according to the practice of the French Astronomers. These latter observations, being made *out* of the plane of the meridian, require, in order to be made to bear on the point in question, a previous *reduction to the meridian*, founded, as we have already shewn, (see pp. 418, 432,) on the same principle as the *reduction to the solstice*, and to which the latter, as well as the observations made in the meridian, are equally subject.

There is indeed a peculiarity, belonging to observations made on the Sun with the repeating circle, and instruments so used, which is this. In the interval between the observation and the meridional transit of the Sun, the Sun changes his declination: whereas, in the investigation of the formula of reduction to the meridian, the declination of the observed body is supposed to suffer no change. This change of condition, then, requires some slight correction. Suppose the observations to be made before the Sun has reached the solstice, then, in the interval (h), between the observation and the Sun's meridional transit, the Sun's north polar distance is diminished. The Sun's *real* meridional zenith distance, then, is less than the reduced. Let e be the change of declination answering to one minute of time, then, if such change be uniform, the change in a time h equals he. Consequently, if Z be the zenith distance observed out of the meridian, R the computed reduction (see p. 418, &c.) the meridional zenith distance equals

$$Z - R - he,$$

if Z', R', h', &c. be other zenith distances corresponding reductions and hour angles, the corresponding meridional zenith distances will be

$$Z' - R' - h'e,$$

$$Z'' - R'' - h''e,$$

&c.

After the Sun has passed the meridian, the contrary effect, with regard to the correction for the change of declination, will take place. The *reduced* zenith distance will be less than the real meridional zenith, because, after the passage of the meridian, the Sun's north polar distance (the Sun not having attained the solstice) has decreased. If, therefore, $Z_{\prime}$, $R_{\prime}$, $h_{\prime}$, be the corresponding zenith distances, reduction and hour-angles, the corresponding meridional zenith distance will be

$$Z_{\prime} - R_{\prime} + h_{\prime}e.$$

Hence, if n be the number of observations, the mean meridional zenith distance will be

$$\frac{1}{n}\left\{\begin{array}{c} Z'+Z''+\&c. - (R'+R''+\&c.) + Z_{\prime}+Z_{\prime\prime}+\&c. - (R_{\prime}+R_{\prime\prime}+\&c.) \\ -(h'+h''+\&c.) + (h_{\prime}+h_{\prime\prime}+\&c.)\, e, \end{array}\right\}$$

and, consequently, the last correction of which we have been treating, will be

$$\frac{1}{n}(W - E)\, e,$$

W being the sum of the hour-angles to the west of the meridian, and E of angles to the east, and e being the change of declination in one minute of time.

For instance, suppose the Sun's zenith distances to have been observed on June 15, 1809, eleven times before it reached the meridian, and seventeen times after it had passed, and the sum of the hour-angles of the eleven observations to have been $75^{m}.6$, of the seventeen, $187^{m}.12$. Now, by the Tables, or the Nautical Almanack, it appears that e very nearly equals $0''.1$: consequently,

$$\begin{aligned} \text{since, } W &= 187^{m}.12 \\ E &= 75.60 \\ \hline W - E &= 111.52 \qquad (W-E)\,e = 11''.15, \end{aligned}$$

$$\text{and } \frac{(W-E)\,e}{n} = \frac{11''.15}{28} = 0''.3982.$$

In the preceding matter we have described the method, such as is practised in Observatories, of finding the obliquity of the ecliptic. The parts of that method are founded, all save one, on observation, or, rather we should say, on results that can be deduced from observation. Such a result, for instance, is the quantity of nutation. The *excepted* part of the process of page 439, is the correction for the Sun's latitude, which (see *Physical Astronomy*, Chap. VI, and XXII.) is known from Physical Astronomy.

But this is far from being a solitary instance of the aid of this latter science. The solar theory is mainly founded on it: at least it may be said that the solar Tables are indebted, for their accuracy, to the computed results of planetary perturbation.

Before, however, our attention is called to these results, there are others of much less difficult enquiry, that must be considered. The Sun, as we have seen (pp. 431, &c.) moves in some orbit, the plane of which is inclined to that of the equator, and does not move equably in that orbit. To find the laws of its *inequable* motion, it would seem to be necessary, previously to investigate its form, or the nature of its curvilinear path. And this, in fact, is the enquiry which, two hundred years ago, Kepler instituted, and after many years of incessant study brought to an happy issue. The orbit of the planet Mars was the object of his researches: their result was the *planet Mars moves in an ellipse round the Sun placed in the focus of the ellipse.*

If this result be extended to the other planets, of which the Earth is one, then the Earth moves round the Sun in an ellipse, the Sun being placed in its focus: or, to use the common Astronomical language, the solar orbit is elliptical *.

The elliptical form of a planet's orbit was a truth not easily arrived at. In endeavouring to reach it, Kepler had to strive against, and to overcome, his own prejudices, which were also those of the age. From some vague notions of simplicity the antient Astronomers fancied that the motions of the heavenly bodies must, of necessity, be performed in the most simple curves, and that, for such a reason, a planet must move in a circle. After Kepler had found, by his reasonings on observations, that the orbit of Mars could not be a circle of which the Sun occupied the centre, he did not altogether abandon his former opinions, but tried whether the observations of the planet were consistent with its movements in a circle, the Sun occupying a point within the circle, but not in its centre. This conjecture, like his former ones, proving fallacious, Kepler, at last, hit upon the right one, or found the observed places of Mars consistent with its description of an ellipse of certain dimensions.

This, like many other astronomical results, is now so familiar to us, that we do not properly appreciate Kepler's merit in discovering it. If we view, however, the state of Science, and Kepler's means and the inherent difficulty of the investigation, we must consider it to have been a great discovery. And even now, availing ourselves of all the facilities of modern science, it is not easy, briefly to shew, from a comparison of the observations of the Sun, that the solar orbit is an ellipse.

The two kinds of observations, to be used for the above purpose, are those of distances and angles: the former to be known, as far as their relative values are concerned, from observations of the Sun's diameter: the latter from the Sun's longitudes to be computed from the observed right ascensions of the Sun and the obliquity of the ecliptic.

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* The Earth moves round the Sun, but an observer sees the Sun to move, and to describe a curve similar to that which would be seen if we imagine the observer transferred to the Sun.

With these data we might from a centre set off a series of distances, *Radii Vectores* as they are called, and draw a curve through their extremities, which, being of an oval form, might be guessed to be an ellipse, and would, on trial, be verified as such. This, in fact, was Kepler's way, and modern mathematicians have no other, except they ground their speculation on Physical Astronomy, and shew, on mechanical principles, the necessity of the description of an elliptical orbit.

It has just been said that the relative distances of the Sun from the Earth may be known from the observed diameters of the Sun: for, the Sun being supposed to remain unaltered, the visual angle of his disk will be less, the greater his distance, and in that proportion. But there exists a better method of determining the same thing, founded on a discovery of Kepler's, and which, in time, was antecedent to that of the elliptical form of Mars' orbit. The discovery was, that at the aphelion of the orbit, the area comprehended within the arc described, and two radii vectores, drawn from the extremities of the arc to the Sun, was equal to a similar area at the Perihelion, supposing the two arcs to be described in equal times. A like fact has since been proved to be generally true: that is, areas comprehended, respectively, within their arcs and two radii vectores, are equal, provided the arcs are of such a magnitude as to be described in equal times. Now this fact, or law, as it is now called, enables us easily to compute the relative distances of the Sun from the Earth. For by observing (see Chapter VII.) the transits of the Sun and stars, the right ascension of the former may be determined; from which and the obliquity of the ecliptic the Sun's longitude may be computed. The difference of the Sun's longitudes on two successive noons is the angle described by the Sun in twenty-four hours of apparent solar time, from which (as we shall soon shew) the angle described in twenty-four hours of *mean* solar time (which twenty-four hours represent an invariable quantity) may be computed. Let v represent this latter angle: then the small circular arc which, at the distance r, measures the same angle, is rv, and the corresponding small area will be, nearly, $rv \times \frac{r}{2}$, or $\frac{r^2 v}{2}$. Suppose one of the values of r to be 1, and A to be the corresponding value of v: then the area $= 1 \times \frac{A}{2}$: and from Kepler's Law of the equal description of areas

$$\frac{r^2 v}{2} = \frac{A}{2},$$

$$\text{whence, } r = \sqrt{\frac{A}{v}};$$

and consequently, in order to compute r, we must be able to determine A and v.

A is the angle corresponding to the mean distance 1, and, therefore, in an ellipse of very small eccentricity (and such an ellipse is the solar orbit) is nearly, the *mean* of the greatest and least angular velocities, or has for its measure half the sum of the angles respectively described, in twenty-four hours, at the perigean and apogean distances: which angles, as it has been already explained, are the daily increases of the Sun's longitudes. Now, by examining the longitudes, it will be found that their greatest daily difference takes place at the end of December: their least at the beginning of July: the value of the former is

	1°	1′	9″.94
of the latter		57	11.48
so that their mean is		59	10.7

and, if we take this latter angle to represent the value of A, we have

$$r = \sqrt{\left(\frac{59' \; 10''.7}{v}\right)}.$$

In order to determine v for any particular day, we must first take the difference of the Sun's longitudes on the noon of that day, and on that of the day succeeding, and if (which will almost ever be the case) the interval between the two noons be greater or less than twenty-four mean solar hours, we must, in computing v, allow for such excess: for instance, let d represent the difference of two longitudes of the Sun on two successive noons, and let $24 \pm x$ represent the time elapsed, then, very nearly,

$$d : v :: 24 \pm x : 24;$$

$$\therefore v = \frac{24\,d}{24 \pm x},$$

or, if we wish to express (and it is sometimes convenient so to express it) the time in parts of sidereal time,

$$v = \frac{24^{h}.0657}{24^{h}.0657 \pm x}.d,$$

and accordingly,

$$r = \sqrt{\left(\frac{59'\ 10''.7}{d} \times \frac{24.0657 \pm x}{24.0657}\right)};$$

or, using mean solar time,

$$r = \sqrt{\left(\frac{59'\ 10''.7}{d} \times \frac{24 \pm x}{24}\right)}.$$

It only remains to shew the method of exhibiting the numerical values of r: suppose, then, such values were required on January 12, and April 1775. In order to find the values of d and x on those days, we must have recourse to recorded observations. In those of Greenwich we find, on January 12, the transits of the Sun's first and second limb, and of the stars α Ceti Rigel, β Tauri, α Orionis, α Lyræ: from which (see pp. 102, 103, &c. Chap. VII.) the right ascension of the Sun's centre may be computed: if computed, it will be found to be

$$19^{h}\ 36^{m}\ 2^{s}.7936, \text{ or, in degrees,}$$
$$9^{s}\ 24^{0}\ 0'\ 41''.9.$$

If then we take the obliquity, as it is expressed in the Nautical Almanack, to be equal to $23^{0}\ 27'\ 58''.5$, we shall from this expression,

$$\tan. \odot . \cos. w = \tan. Æ,$$

($\odot$ being the Sun's longitude and w the obliquity),

find ($\odot$), the longitude equal to $9^{s}\ 22^{\circ}\ 13'\ 35''$.

Institute a like process for the next day, January 13, that is, from the observed transits of the Sun and the fixed stars, and the Catalogues and Tables belonging to the latter, deduce (see pp. 102, 103,) the clock's error and rate, and then the Sun's right ascension: which right ascension, in the case we are treating of, would be $9^s\ 25^\circ\ 5'\ 29''.9$: from which the longitude deduced as before (see p. 447,) will be

$$9^s\ 23^\circ\ 14'\ 42'',$$

the difference between which and the Sun's longitude on the 12th (see p. 447, l. 26,) is $1^\circ\ 1'\ 7''$, which accordingly is the value of d. Again, since the difference of the Sun's right ascensions on the 13th and 12th

$$\text{is}\quad 9^s\ 25^\circ\ 5'\ 29''.9 - 9^s\ 24^\circ\ 0'\ 41''.9,$$

$$\text{or}\quad 1^\circ\ 4'\ 48'',\quad \text{or in time,}\quad 4^m\ 19^s.2;$$

consequently, the interval, in sidereal time, of the two transits on the 12th and 13th is $24^h\ 4^m\ 19^s.2$ ($= 24^h.072$) and, accordingly, (see p. 447, l. 7,)

$$r = \sqrt{\left(\frac{59'\ 10''.7}{61'\ 7''} \times \frac{24.072}{24.0657}\right)}$$

$$= .98418.$$

In like manner if we investigate the Sun's right ascensions on April 28, and April 29, and thence compute his longitudes and take their difference, it will be found to be equal to $58'\ 14''.34$, whilst the interval between the transits, in sidereal time, is only $24^h\ 3^m\ 47^s.66$ ($= 24^h.06324$), and therefore less than a mean solar day. In this case then

$$r = \sqrt{\left(\frac{59'\ 10''.7}{58'\ 14''.34} \times \frac{24.06324}{24.0657}\right)}.$$

$$= 1.00798.$$

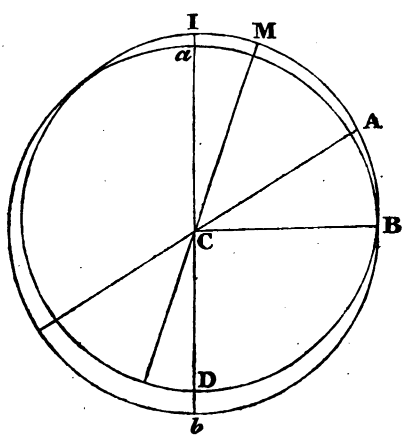
We might thus compute the distance for every pair of successive observations made during the year. The value of r that results from the computation should be made to belong to the *mean* of the two successive longitudes from which it is computed. Thus, the Sun's longitudes being

on January 12, ...	$9^s\ 22^\circ\ 13'\ 35''$
on January 13, ...	9 23 14 42
their mean is	9 22 44 8.5

to which $r = .98418$ belongs; and if we apply this rule, and computations like to the preceding, to certain of the Sun's longitudes computed by M. Delambre from Maskelyne's Observations (of 1775), and inserted by the former Astronomer in the Berlin Acts for 1785, (pp. 206, &c.) we shall have the following results which may be arranged in a Table:

Times of Observation.	Longitudes of Sun.	Distances from the Earth.
Jan. 12 to 13	$9^s\ 22^\circ\ 44'\ 8''.5$	.98418
Feb. 17 to 18	10 29 13 59.7	.98950
March 14 to 15	11 24 15 37.5	.99622
April 28 to 29	1 8 26 20.7	1.00798
May 15 to 16	1 24 51 45.9	1.01234
June 17 to 18	2 26 27 43.4	1.01654
July 1 to 3	3 10 17 38.7	1.01658
August 26 to 27	5 3 27 46.6	1.01042
Sept. 22 to 23	5 29 44 22.7	1.00283
Oct. 24 to 25	7 2 24 24.2	.99303
Nov. 18 to 20	7 28 2 46.4	.98746
Dec. 17 to 18	8 25 58 47.8	.98415

The above Table contains twelve longitudes and twelve corresponding distances. Assume a centre C, and with a radius $= 1$ describe a circle Bab. From a point B in this circle begin



to reckon the longitudes, and then, through the extremities of the arcs proportional to such longitudes draw radii and set them off proportional to their values. Thus, if the angles BCA, BCM, BCI be proportional to

$$1^s\ 8^0\ 26'\ 20'',\quad 1^s\ 24^o\ 51'\ 46'',\quad 2^s\ 26^0\ 27'\ 43'',$$

CA, CM, CI must be made proportional to 1.00798, 1.01234, 1.01654, and accordingly the points A, M, I will fall a little without the circle described with the radius CB.

If the remainder of the figure be formed in a like manner, the points belonging to November, December, January, will fall a little within the circle, so that a curve drawn through all the points will be (very little differing, however from a circle) an oval, most drawn in about D, most going out near I: in other words, in the oval representing the solar orbit, the apogean distance will be near to I, the perigean near to D.

The distances (see the Table of p. 449,) for November 18, December 17, January 12, being .98746, .98415, .98418, the least or perigean distance is evidently between the first and third dates. So, the apogean distance is between June 17, and August 26. In order to discover whether the perigean distance is between June 17, and July 2, or between July 2, and August 26, we must have recourse to the original observations which have already been used in forming the preceding Table; and amongst these we find the following * :

	Sun's Right Ascen.	Sun's Longitude.	Diff. of R. A.	Diff. of Long.
June 30,	$6^h\ 36^m\ 32^s.6$	$3^s\ 8^o\ 23'\ 29''.3$	$4^m\ 7^s.9$	$57'\ 6''.2$
July 1,	6 40 40.5	3 9 20 35.5		

* This is not strictly correct. The right ascensions and longitudes of the text are *not* expressed in the Greenwich Observations, but are deduced from them. We cannot do better, considering the object of this Work, which is to teach the very methods of Astronomical Science, than to subjoin the original observations, and the means of reducing them to those forms under which they appear in the text.

1775.

which are the Sun's right ascensions and longitudes reduced, according to the processes of the subjoined note, from the original observations.

1775.	I.	II.	III.	IV.	V.	*Stars.*
June 29,		$0^m\ 51^s$	$4^h\ 20^m\ 22^s.8$	$0^m\ 54^s.5$		Aldebaran.
June 30,		32 10.6 34 28	6 32 43.9 6 35 1.4	38 17 35 34.4		☉ 1 L 2 L
July 1,		36 17.8 38 35	6 36 51 6 39 8.3	37 23.7 39 41.4		☉ 1 L 2 L
July 2,	16.5 16	0 48 0 46.8	4 20 20 5 40 17.5	0 51.5 0 48	23.3 19.1	Aldebaran. *a* Orionis.

If the intervals of the wires were all equal we could immediately take the *means* of the times, as is done in pages 86, 87, &c.: which *means* would denote the transits of the stars and Sun by the clocks. But we find from Dr. Maskelyne's Introduction to these Observations (see p. iv,) that in the year 1775, the *equatoreal* intervals (see p. 91, of this Work) between the several wires of the Greenwich transit instrument were

$$30^s.40 \mid 30^s.54 \mid 30^s.36 \mid 30^s.55 \mid$$

consequently, (see p. 90,) the intervals of a star, the north polar distance of which is Δ, would be the above intervals multiplied, respectively, into cosec. Δ: and, if t were the time at the middle wire, $t-a$, $t-b$, $t+c$, $t+d$ the times of an equatoreal star at the second, first, fourth, and fifth wire, t, $t-a\,.\,\text{cosec.}\ \Delta$, $t-b\,.\,\text{cosec.}\ \Delta$, &c. would be the times of a star distant from the pole by Δ: hence, the mean transit would be

$$t - \tfrac{1}{5}(a+b-c-d)\ \text{cosec.}\ \Delta = m \text{ (suppose)}$$

$$\text{consequently, } t = m + \tfrac{1}{5}\{(a-d)+(b-c)\}\text{cosec.}\ \Delta;$$

or, the correction to be applied to m the mean of the times, is

$$\tfrac{1}{5}(a-d+b-c)\ \text{cosec.}\ \Delta.$$

In

Hence, since $4^m\ 7^s.9 = 0^h.06887$, we have from the formula of p. 447,

$$r = \sqrt{\left(\frac{59'\ 10''.7}{57'\ 6''.2} \times \frac{24.06887}{24.0657}\right)} = 1.018.$$

In the case before us $a = 30.40 + 30.54 = 60.94$

$d = 30.36 + 30.55 = 60.91$

$a - d = \qquad .03$

$b = 30.40$

$c = 30.36$

$b - c = \quad .04$

therefore the correction, or $\frac{1}{5}(a - d + b - c) = .014$.

In the case of Aldebaran $\Delta = 74°$ nearly, and cosec. $74 = 1.04$

of Orion $\Delta = 82$...............cosec. $= 1.009$

of ⊙ at solstice $\Delta = 6632$...........cosec. $= 1.09$,

and therefore the three corrections are $+ 0^s.0145$, $0^s.0141$, $0^s.0153$.

Hence, the corrected transit of Aldebaran on June 30, is	4°	20′	22″.8
but (pp. 351, 372,) its Æ by the Catalogue and Tables is	4	23	1.74
clock too slow......	0	2	38.94
Again, transit of Aldebaran by the clock on July 2, is	4°	20′	19″.9
by the Catalogue	4	23	1.82
clock too slow......	0	2	41.92
Again, transit of Orion by clock on July 2, is	5^h	40^m	$17^s.53$
by Catalogue	5	42	59.36
clock too slow	0	2	41.83

Hence, by a *mean* of Aldebaran and Orion, the clock was too slow on July 2, at five hours, by .... $2^m\ 41^s.88$

but on the June 30, it was too slow by 2 38.94

(see pp. 103, &c.) clock's loss in two days $23^h\ 20^m$ 0 2.94

and its daily rate was nearly .... − 0.98

Having now ascertained the *error* and *rate* of the clock we can determine the Sun's transit or right ascension.

June 30,

Hence, since the distances June 17, June 30, July 2, August 26, are

1.01654, 1.018, 1.01658, 1.01042,

it is plain that the Sun must arrive at his apogean distance before July 2, and very nearly at that time. In like manner, if we examine the observations and reduce them, we shall find that the Sun's increase of longitude between December 30, and December 31, is $1° \; 1' \; 15''.1$ and the difference, in sidereal time, between the two transits, is $24^h.07397$, we have, therefore, (as before, in pp. 447, &c.)

$$r = \sqrt{\left(\frac{59' \; 10''.7}{61' \; 15''.1} \times \frac{24.07397}{24.0657}\right)} = .98309,$$

which is, very nearly, the least or perigean distance.

If we take the *means* of the longitudes of June 30, and July 1, and of December 30, and December 31, we shall have

---

June 30, transit of Sun's centre by clock	6^h	33^m	$52^s.6$
Error of clock	0	2	40
Sun's right ascension by observation	6	36	32.6
Again, July 1, transit of Sun	6^h	37^m	$59^s.5$
Error of clock	0	2	40.99
Sun's right ascension by observation	6	40	40.5 nearly,

which right ascensions are those which are specified in page 450, at the bottom line.

In order to compute the longitudes, we have the above right ascensions, and an obliquity $= 23° \; 27' \; 59''.5$, from which, and by means of the equation tan. L. cos. w = tan. right ascension, or by the formula or Table of reduction to the ecliptic, the longitudes in the text (see p. 450,) may be computed.

The above process may appear somewhat long; but it is given, on the grounds already assigned in p. 424, &c. because it is the real and practical process by which original observations are *reduced* and made to become results fit for the illustration or establishment of Astronomical Science.

	Mean Longitude.	Distance from Earth.
June 30, July 1,	$3^s\ 8^\circ\ 55'\ 2''.4$,	1.018
Dec. 30, Dec. 31,	$9^s\ 9^\circ\ 14'\ 11''.3$,	0.98309.

The difference of the longitudes is $6^s\ 0^\circ\ 19'\ 8''.9$, differing from 6^s by $19'\ 8''.9$, so that the two distances, which are, nearly, the greatest and least, lie, very nearly, in the same straight line: and consequently there arises a presumption, that the longitudes of the apogean and perigean distances, if exactly found, would exactly differ by 6^s.

Now this is a property of an ellipse. Two lines drawn, respectively, from the *focus* of an ellipse, to the extremities of the axis major are the greatest and least of all lines that can be drawn from the focus to the curve. The solar orbit then having a general resemblance to an ellipse, and one of its properties, may have all: and, on such a presumption, an ellipse would be assumed and compared with the solar orbit.

The dimensions of the ellipse, so to be made trial of, would be assigned by the preceding results. Its eccentricity, which is half the difference of the greatest and least distances, would be equal to $\frac{1}{2}(1.018 - .98309)$, or .01745. The next step would be to compute, from the properties of the ellipse, or by means of analytical* expressions expounding those properties, the relative values of the *Radii Vectores* as they are called, and the angles included between those radii and a fixed line, the axis major, for example. If the relations between these angles and radii should be found to be the same, as the relations which have just been made (see p. 449.), there would be established a proof of the Earth's orbit being an ellipse, the Sun occupying its focus.

Kepler's investigations were directed not towards the Earth's but Mars' orbit. His proof of that orbit being an ellipse rests,

* The analytical expression between the angle (v) and the radius (r) is $r = \dfrac{a.(1 - e^2)}{1 + \cos. v}$.

in fact, on the same principle as the preceding: which is, the agreement of the computed places in an assumed ellipse with the places computed from observations. The process by which Kepler established this proof is very long, and no process, even taking the most simple case, namely, that of the solar orbit, can be very short. Of which assertion, what has just preceded, is some sort of proof.

The proof of the solar orbit being elliptical has been founded on the equable description of areas: and, historically, this latter fact, or Law, as it is called, (only partially established, however, by Kepler,) preceded the former. To the equable description of areas, and the elliptical forms of planetary orbits, Kepler added a third law, according to which the cubes of the greater axes varied as the squares of the periodic times.

We must now consider the astronomical uses of these discoveries. In the first place it is evident, that, since we know the nature of the solar orbit, and one law regulating the motion in that orbit, we have made some approach towards a knowledge of the Sun's real motion in the ecliptic. If the latter motion should be known, the Sun's right ascension and declination would thence be determinable by the Rules of Spherical Trigonometry. The law of a body's motion in an elliptical orbit is the first and essential thing to be determined. Let the body begin to move from one of the apsides of the ellipse, and let the time be reckoned from the beginning of such motion, then, the problem to be solved, is the assigning of the body's place in the ellipse after a certain elapsed time. This, in fact, is *Kepler's* Problem, as it has been called for distinction's sake. And, by its solution, that great Astronomer, laid the first ground-work of Solar Tables.

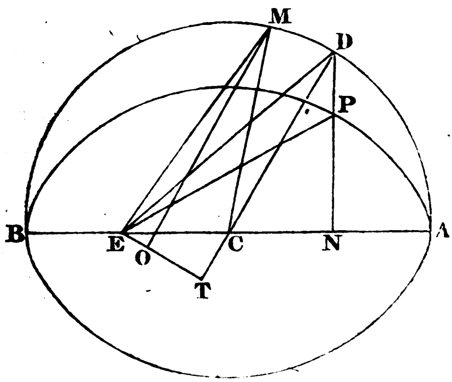
The enquiry, then, in the next Chapter, concerning the best method of solving Kepler's problem, will be purely a mathematical enquiry. A result being attained, the next step will be to apply it. If we begin our reckonings for an apside, we must know where the apsides of the Sun's orbit (which, in other words, are the apogee and perigee) are situated. That is, we must know the longitudes of those points. We indeed, by what has preceded, already know them to a certain degree of exactness, since in page 454, the longitude of the apogee was found to be nearly $3^s\ 8^\circ\ 55'\ 2''.4$. After we have discussed Kepler's problem, we will devise more exact methods for determining the place of the apogee. The place of the apogee being determined, there will arise a question concerning the permanency of that place in the Heavens. In the preceding instance (see p. 447.) the longitude of the apogee was found for the year 1775. Will it be the same for any other epoch? The obvious method of solving this question will be to find, for two different epochs, by the same process, the longitudes of the apogee. The results will shew whether the apogee be stationary, progressive, or regressive.

The place of the apogee being known for any given epoch, and the law of its translation, the place may be determined for any other epoch; and thence, since Kepler's problem determines the body's place in the ellipse, we shall be able to determine the Sun's place or longitude for any assigned epoch. This it is the object of Solar Tables to effect. If their elements be correct, they enable us to assign the Sun's longitude for years that are to come. But the elements of the Tables stand in need of frequent revision: for, the dimensions of the solar ellipse, from the action of the planets, are continually varying, and, which is a reason of a different sort, our means of determining the dimensions become, from the advancement of science and art, progressively better. If, therefore, the construction of solar and planetary Tables be our first object, their correction will be the second.

CHAP. XVIII.

On the Solution of Kepler's Problem, by which a Body's Place is found in an Elliptical Orbit.—Definition of the Anomalies.

LET APB be an ellipse, E the focus occupied by the Sun, round which P the Earth or any other planet is supposed to revolve. Let the time and planet's motion be dated from the



apside or aphelion A. The *condition given*, is the time elapsed from the planet's quitting A; the *result sought* is the place P; to be determined either by finding the value of the angle AEP, or by cutting off, from the whole ellipse, an area AEP bearing the same proportion to the area of the ellipse which the given time bears to the periodic time.

There are some technical terms used in this problem which we will now explain.

Let a circle AMB be described on AB as its diameter, and suppose a point to describe this circle uniformly, and the whole of it, in the same time, as the planet describes the ellipse in: let also t denote the time elapsed during P's motion from A to P: ∴

then if $AM = \frac{t}{\text{period}} \times 2\,AMB$, M will be the place of the point that moves uniformly, whilst P is that of the planet's; the angle ACM is called the *Mean Anomaly*, and the angle AEP is called the *True Anomaly*.

Hence, since the time (t) being given, the angle ACM can always be immediately found (see l. 2.) we may vary the enunciation of Kepler's problem, and state its object to be, *the finding of the true anomaly in terms of the mean.*

Besides the mean and true anomalies, there is a third called the *Eccentric Anomaly,* which is expounded by the angle DCA, and which is always to be found (geometrically) by producing the ordinate NP of the ellipse to the circumference of the circle. This eccentric anomaly has been devised by mathematicians for the purposes of expediting calculation. It holds a mean place between the two other anomalies, and mathematically connects them. There is one equation by which the mean anomaly is expressed in terms of the eccentric: and another equation by which the true anomaly is expressed in terms of the eccentric.

We will now deduce the two equations by which the *eccentric* is expressed, respectively, in terms of the *true* and *mean* anomalies.

Let t = time of describing AP,
P = periodic time in the ellipse,
$a = CA$,
$ae = EC$,
$v = \angle\, PEA$,
$u = \angle\, DCA$; (∴ ET, perpendicular to DT, $= EC \times \sin. u$),
$\rho = PE$,
$\pi = 3.14159$, &c.;

then, by Kepler's law of the equable description of areas,

$$t = P \times \frac{\text{area } PEA}{\text{area of ellip.}} = {}^{*} P \times \frac{\text{area } DEA}{\text{area } \odot} = \frac{P}{\pi a^2}(DEC + DCA)$$

* Vince's *Conics*, p. 15. 4th Ed.

$$= \frac{P}{\pi a^2}\left(\frac{ET.DC}{2} + \frac{AD.DC}{2}\right) = \frac{Pa}{2\pi a^2}(EC.\sin. u + DC.u)$$

$$= \frac{P}{2\pi}(e \sin. u + u): \text{ hence, if we put } \frac{P}{2\pi} = \frac{1}{n},$$

we have

$$nt = e.\sin. u + u \ldots\ldots (a),$$

an equation connecting the mean anomaly nt, and the eccentric u.

In order to find the other equation, that subsists between the true and eccentric anomaly, we must investigate, and equate, two values of the radius vector ρ, or EP.

First value of ρ, in terms of v the true anomaly;

$$\rho = \frac{a.(1 - e^2)}{1 - e.\cos. v} * \ldots\ldots (1).$$

Second, in terms of u the eccentric anomaly,

$$\begin{aligned}
\rho &= a(1 + e.\cos. u) \ldots\ldots\ldots\ldots (2).\\
\text{For, } \rho^2 &= EN^2 + PN^2\\
&= EN^2 + DN^2 \times (1 - e^2)\\
&= (ae + a.\cos. u)^2 + a^2 \sin.^2 u.(1 - e^2)\\
&= a^2\{e^2 + 2e.\cos. u + \cos.^2 u\} + a^2.(1 - e^2)\sin.^2 u\\
&= a^2\{1 + 2e.\cos. u + e^2\cos.^2 u\}.
\end{aligned}$$

Hence, extracting the square root,

$$\rho = a(1 + e.\cos. u).$$

Equating the expressions (1), (2), we have

$$(1 - e^2) = (1 - e.\cos. v).(1 + e\cos. u), \text{ whence,}$$

$$\cos. v = \frac{e + \cos. u}{1 + e.\cos. u},$$ an expression for v in terms of u; but, in order to obtain a formula fitted to logarithmic computation, we must find an expression for $\tan. \frac{v}{2}$: now, (see *Trig.* p. 40.)

* Ibid. p. 23. Bridge, p. 93.

(*b*) $\tan.\frac{v}{2} = \sqrt{\left(\frac{1 - \cos. v}{1 + \cos. v}\right)} = \sqrt{\left(\frac{(1-e)(1-\cos. u)}{(1+e)\cdot(1+\cos. u)}\right)}$

$$= \sqrt{\left(\frac{1-e}{1+e}\right)} \tan.\frac{u}{2}.$$

These two expressions (*a*) and (*b*), that is,

$$nt = e . \sin. u + u,$$

$$\tan.\frac{v}{2} = \sqrt{\left(\frac{1-e}{1+e}\right)} . \tan.\frac{u}{2},$$

analytically resolve the problem, and, from such expressions, by certain formulæ belonging to the higher branches of analysis, may v be expressed in the terms of a series involving nt *.

Instead, however, of this exact but operose and abstruse method of solution, we shall now give an approximate method of expressing the true anomaly in terms of the mean.

MO is drawn parallel to *DC*. (1.) Find the half difference of the angles at the base of the triangle *ECM*, from this expression,

$$\tan.\tfrac{1}{2}(CEM - CME) = \tan.\tfrac{1}{2}(CEM + CME) \times \frac{1-e}{1+e},$$

(see *Trig.* p. 27.) in which, $CEM + CME = ACM$, the mean anomaly.

(2.) Find CEM by adding $\frac{1}{2}(CEM+CME)$ and $\frac{1}{2}(CEM-CME)$ and use this angle as an approximate value to the eccentric anomaly DCA, from which, however, it really differs by $\angle EMO$.

---

* The following is the series for v in terms of nt;

$$v = nt -$$

$$\left(2e - \frac{1}{4}e^3 + \frac{5}{96}e^5\right).\sin. nt + \left(\frac{5}{4}e^2 - \frac{11}{24}.e^4 + \frac{17}{192}.e^6\right)\sin. 2nt$$

$$-\left(\frac{13}{12}.e^5 - \frac{43}{64}.e^7\right).\sin. 3nt + \left(\frac{103}{96}.e^4 - \frac{451}{480}.e^6\right).\sin. 4nt$$

$-\frac{1097}{960}e^5 . \sin. 5nt + \frac{1223}{960}e^6 \sin. 6nt$, in which the approximation is carried to quantities of the order e^6.

(3.) Use this approximate value of $\angle\ DCA = \angle\ ECT$ in computing ET which equals the arc DM: for, since (see p. 458,)

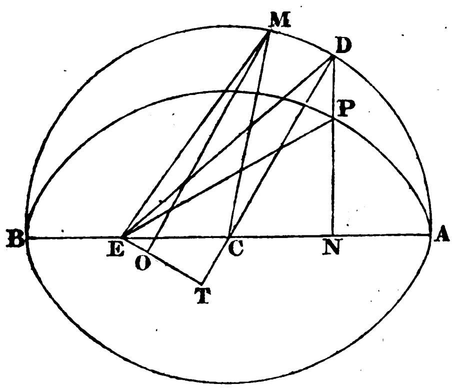
$t = \frac{P}{\text{area}\ \odot} \times DEA$, and (the body being supposed to revolve in the circle ADM) $= \frac{P}{\text{area}\ \odot} \times ACM$; $\therefore$ area AED = area ACM,

or, the area DEC + area ACD = area DCM + area ACD;
consequently, the area DEC = the area DCM,
and, expressing their values,

$$\frac{ET \times DC}{2} = \frac{DM \times DC}{2} \text{ and } \therefore ET = DM.$$

Having then computed $ET = DM$, find the sine of the resulting arc DM, which sine $= OT$: the difference of the arc and sine $(ET - OT)$ gives EO.

(4.) Use EO in computing the angle EMO, the real difference, between the eccentric anomaly DCA, and the $\angle\ MEC$: add



the computed $\angle\ EMO$ to $\angle\ MEC$, in order to obtain $\angle\ DCA$. The result, however, is not the exact value of $\angle\ DCA$, since $\angle\ EMO$ has been computed only approximately; that is, by a process which commenced by assuming $\angle\ MEC$, for the value of the $\angle\ DCA$.

For the purpose of finding the eccentric anomaly, this is the entire description of the process; which, if greater accuracy be required, must be repeated; that is, from the last found value of $\angle\, DCA = \angle\, ECT, ET, EO,$ and $\angle\, EMO$ must be again computed.

(5.) A sufficiently correct value of the eccentric anomaly (u) being found, investigate the true (v), from the formula (b) of p. 460, that is,

$$\tan.\frac{v}{2} = \sqrt{\left(\frac{1-e}{1+e}\right)}\,.\,\tan.\frac{u}{2}.$$

EXAMPLE I.

The Eccentricity of the Earth's Orbit being .01691, *and the Mean Anomaly* = 30°, *it is required to find the Eccentric and the true Anomalies,*

(1.) log. tan. 15	9.4280525
log. $(1-e)$, or log. .98309 ...	$\bar{1}.9925933$
arith. comp. $1+e$, or of 1.01691	$\bar{1}.9927218$
log. tan. $\frac{1}{2}(CEM - CME)$ ..	9.4133676 = log. tan. $14^0\ 31'\ 22''$.

(2.) $\frac{1}{2}(CEM - CME) = 14^0\ 31'\ 22''$
$\frac{1}{2}(CEM + CME) = 15\ \ 0\ \ 0$
$CEM = 29\ \ 31\ \ 22.$ 1st approx^e. value of CDA.

(3.) log. sin. $29^0\ 31'\ 22''$	9.6926438
log. .01691	$\bar{2}.2281436$
+ log. (arc = rad^s.)	5.3144251
log. DM in seconds	3.2352125 = log. 1718.7.

$DM = 28'\ 38''.7$, and its sine expressed in seconds differs from the arc DM by less than half a second.

(4.) The operation prescribed in this number (see p. 461, l. 19, &c.) is, in this case, needless, since the correction for the angle EMC is so small, that the first approximate value of the eccentric anomaly may be stated at $29^0\ 31'\ 22''$.

(5.) log. tan. $\frac{u}{2}$, or log. tan. $14^0\ 45'\ 41''$	9.4207651
$\frac{1}{2}$ log. $(1-e)$, or $\frac{1}{2}$ log. .98309	4.9962966
$\frac{1}{2}$ log. $(1+e)$, or $\frac{1}{2}$ log. 1.01691	4.9963608
log. tan. $\frac{v}{2}$	9.4134225

$= \log. \tan. 14^0\ 31'\ 28''$;

$\therefore$ the true anomaly $= 29^0\ \ 2'\ 56''$.

The difference of the mean and true anomalies, or, as it is called, the *Equation of the Centre*, equals $57'\ 4''$.

If the eccentricity had been assumed $= .016813$, or $.016791$, the equation of the centre would have resulted $= 56'\ 46''.4$, or $= 56'\ 41''.4$, respectively.

EXAMPLE II.

Instead of .01691, *suppose the Eccentricity of the Earth's Orbit be taken at* .016803*, *and the Mean Anomaly, reckoning from Perigee, according to the Plan in the new Solar Tables, be* $10^s\ 12^\circ\ 22'\ 12''.4$.

Taking out 6 signs, we have the mean angular distance from apogee $= 4^s\ 12^0\ 22'\ 12''.4$.

$$
\begin{array}{lr}
\text{(1.) log. tan. } 66^\circ\ 11'\ 6''.2 & 10.3552029 \\
\text{log. } .983197 \ldots\ldots & \bar{1}.9926406 \\
\text{arith. comp. } .1016803 & \bar{1}.9927645 \\
\hline
 & 10.3406080 = \log. \tan. 65^\circ\ 27'\ 56''.4.
\end{array}
$$

$$
\begin{array}{lrrl}
\text{(2.) } \tfrac{1}{2}(CEM - CME) & 65^0\ 27' & 56''.4 & \\
\tfrac{1}{2}(CEM + CME) & 66\ \ 11 & 6.2 & \\
\hline
 & 131\ \ 39 & 2.6 & \text{approx}^e\text{. value of } CDA\ (u)
\end{array}
$$

$$
\begin{array}{lr}
\text{(3.) log. tan. } \frac{u}{2}\text{, or log. tan. } 65^0\ 49'\ 31''.3 \ldots\ldots & 10.3478640 \\
\tfrac{1}{2}\text{ log. } .983197 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 4.9963203 \\
\tfrac{1}{2}\text{ arith. comp. } 1.06803 \ldots\ldots\ldots\ldots\ldots\ldots & 4.9963816 \\
\hline
\text{log. tan. } \frac{v}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & 10.3405659;
\end{array}
$$

$$\therefore\ \frac{v}{2} = 65^0\ 27'\ 49''.2, \text{ and } v = 4^s\ 10^\circ\ 55'\ 38''.4;$$

* In 1750, the eccentricity was 0.016814, and, the secular variation being .000045572, in 1800, it was 0.016791, and in 1810, (for which epoch Delambre's Tables are constructed) .0167866.

∴ the true anomaly, reckoning from perigee, $= 10^s\ 10^\circ\ 55'\ 38''.4$, and difference of the mean and true anomaly $=\ 1^\circ\ 26'\ 34''$.

This difference, or *Equation of the Centre,* is stated, for 1800, in Lalande's Tables, Vol. I. *Astron.* ed. 3. p. 23, at $1^\circ\ 26'\ 38''.6$; but, in the new Tables, Vince, Vol. III. p. 38, at $11^s\ 28^\circ\ 32'\ 44''.4$. Now the difference of this, and of 12 signs, is $1^\circ\ 27'\ 15''.6$, which is still greater than Lalande's result by $45''$. But, it is purposely made greater; for these $45''$ are the sum of the maxima of several very small equations. (See the explanation in Delambre's Introduction, and in Vince's, p. 6.)

In the two preceding Examples, it appears that, by reason of the small eccentricity of the Earth's orbit, the true anomaly and equation of the centre are found by an easy and short process; no second approximation being found necessary. It appears also, by the results, that a small change in the eccentricity makes a variation of several seconds in the equation of the centre. Thus, arranging the results in the preceding Examples:

Mean Anomaly.	Eccentricity.	Equation of Centre.
$30^\circ\ 0'\ 0''$	.016910	$0^\circ\ 57'\ 4$
30 0 0	.016813	0 56 46.4
30 0 0	.016791	0 56 41.4

Now, by observation and theory, it appears, that the eccentricity of the Earth's orbit is diminishing. Hence, Tables of the equation of the Earth's orbit, computed for one epoch, will not immediately suit another: but, they may be made to suit, by appropriating a column to the *secular variation* of the equation of the centre. Thus, in Lalande's Tables, tom. I. ed. 3. p. 18, the equation of the centre is stated to be $56'\ 41''.2$, and in a column by the side, the corresponding secular diminution to be $9''.36$. Now Lalande's Tables were computed for 1800*: (when the eccentricity of the Earth's orbit was .016791) consequently, for the *preceding* epochs of 1750, 1500, the equations of the

* Delambre states, that Lalande's Tables answer better to the epoch of 1809, or 1810, than to 1800. See Introduction to his new Tables.

centre would be $56' \; 41''.2 + 4''.68$, and $56' \; 41''.2 + 23''.44$, that is, $56' \; 45''.9$, and $57' \; 4''.6$ respectively. These are nearly the results previously obtained in p. 463, which they ought to be, since, the secular diminution of the eccentricity being .000045572, the eccentricities corresponding to 1750 and 1560 will be, nearly, .016813 and .016910.

By this mode we may also reconcile the two results in Example 2, in p. 463; for, the equation of the orbit in Lalande's Tables is $1^0 \; 26' \; 30''$, (that is, for an eccentricity, .016791) therefore, for 1760, when the eccentricity was .016803, the equation will be, the secular diminution being $13''.9$, equal to

$$1^0 \; 26' \; 30''.6 + 3''.4, \text{ that is, } 1^0 \; 26' \; 34''.$$

Example III.

The Eccentricity of the Orbit (that of Pallas) *being* 0.259, *the Mean Anomaly* $= 45^0$: *it is required to find the Eccentric and true Anomalies.*

(1.) log. tan. $22^0 \; 30'$ . . . . . . $\bar{9}.6172243$
log. tan. 741 . . . . $\bar{1}.8698182$
arith. comp. 1.259 . . . . 9.8999743

log. tan. $\frac{1}{2}(CEM - CME)$ 9.3870168 = log. tan. $13^0 \; 42' \; 3''.3$.

(2.) $\frac{1}{2}(CEM - CME) = 13^0 \; 42' \; 3''.3$
$\frac{1}{2}(CEM + CME) = 22 \;\; 30 \;\; 0$

$\therefore \; CEM = 36 \;\; 12 \;\; 3 \; .3$ = 1st approx^e. value of $\angle \, CDA$,
and $CME = 8 \;\; 47 \;\; 56.7$

(3.) log. sin. $36^0 \; 12' \; 3''.3$ . . . . $\bar{9}.7713071$
log. .259 . . . . $\bar{1}.4132998$
log. (arc = radius). 5.3144251

log. DM in seconds . . . . 4.4990320 = log. 31552.4;
$\therefore DM = 31552''.4 = 8^0 \; 45' \; 52''.4$;
$\therefore$ log. sin. 9.1829067
log. (arc = rad.) 5.3144251

4.4973318 = log. 31429;

$\therefore$ since $DM = 31552.4$
and sin. $DM = 31429$

$EO = 123.4$

(4.) (*a*) log. .259	$\bar{1}.4132998$
log. sin. 45°	9.8494850
	9.2627848
log. sin. 8° 47′ 56″.7	9.1845968
	.0781880
	5.3144251
	5.3926131
log. r	10
log. 123.4	2.0913152
	12.0913152
(*a*) log. (arc = radius) + log. EM	5.3926131
log. sin. EMO	6.6987021

$$\therefore EMO = 1' \; 43''.1.$$

Hence, since $CDA = 36° \; 12' \; 3''.3$

and $EMO = 0 \quad 1 \; 43.1$

corrected value of $CDA = 36 \; 13 \; 46.4$, the eccentric anomaly.

log. tan. $18^0 \; 6' \; 53''.2$	9.5147282
$\frac{1}{2}$ log. .741	4.9349091
$\frac{1}{2}$ arith. comp. 1.259	4.9499871

log. tan. $\frac{v}{2}$ .... 9.3996244 = log. tan. 14° 5′ 19″;

$\therefore$ the true anomaly is $28^0 \; 10' \; 38''$.

The eccentric and true anomalies being determined, the radius vector ρ may be computed from either of the two expressions, (1) (2) p. 459, but most conveniently from the latter.

Example IV.

Required the Earth's Distance from the Sun, the Mean Anomaly (reckoning from Aphelion) being $4^s \; 12° \; 22' \; 12''.4$, *and the Eccentricity* = .016803. See Ex. 2. p. 463.

$$\rho = 1 + e . \cos. u, \text{ if } a = 1,$$

$$\text{and } u = 131° \; 39' \; 2''.6.$$

log. cos. 131° 39′ 2″.6	$\bar{9}$.8225523
log. .016803	$\bar{2}$.2253868
log. .011167	8.0479391

(since cos. is —), $\rho = 1 - .011167 = .988833$.

Example V.

Required the Distance of Pallas *from the Sun, in the conditions of* Ex. III.

log. cos. 36° 13′ 46″.4	$\bar{9}$.9066881
log. 0.259	1.4132998
log. .208923	9.3199879

$\therefore$ distance = 1.208923

and log. distance = 0.823979.

The knowledge of these distances is useful*, as we shall hereafter see, in computations of the heliocentric longitudes and latitudes of planets. But, in such computations, the *logarithms* of the distances are required. Those can, indeed, be immediately found from the computed distances, by means of the common Tables; with more brevity and facility of computation, however, by taking out, during the process of finding the true anomaly, when the log. sine is taken out, the log. cosine of the eccentric anomaly.

Assume then, $e \,.\, \cos. u = \cos. \theta$, or, $\log. \cos. \theta = \log. e + \log. \cos. u$; thence θ is known: and, lastly,

$$\log. \rho = \log. (1 + e \,.\, \cos. u) - 10 = \log. (1 + \cos. \theta) - 10$$

$$= \log 2 \,.\, \cos.^2 \frac{\theta}{2} - 20$$

$$= \log. 2 + 2 \log. \cos. \frac{\theta}{2} - 20 = 2 \log. \cos. \frac{\theta}{2} - 19.6989700.$$

The sole object of this latter method, is compendium of calculation.

---

* The Nautical Almanack expresses the logarithm of the Sun's distance for every 6th day of the year.

By means of the preceding rule, (see pp. 460, 461,) the true anomaly (as in the Examples) may always be computed from the mean, which is known, by a single proportion from the time. The difference of the true and mean anomalies, is the equation of the centre, or the equation of the orbit. And, the Solar Tables assign to the mean anomaly, as the *argument*, this latter quantity, instead of the true anomaly. It serves then as a *correction* or *equation* to the mean anomaly, by means of which the inequality between the *mean* and *true* places of a planet, at any assigned time, may be compensated. It is additive or subtractive, according as the mean is less or greater than the true anomaly: subtractive, therefore, whilst the body *P* moves, from *A* the aphelion to *B* the perihelion, or, through the first 6 signs of mean anomaly, (reckoning anomaly from the aphelion) and additive, whilst *P* moves, from *B* to *A*, or, through the last 6 signs of mean anomaly.

These circumstances, Lalande's Tables (ed. 3.) used to express, in the common way, by the algebraical signs $-$ and $+$. But the new Solar Tables, (see Delambre's Tables, and Vince's *Astronomy*, Vol. III.) adapted to the operation of addition only, when the mean anomaly exceeds the true, express not the *equation of the centre*, but its *supplement to* 12 *signs* (360^0). The 12 signs, therefore, must be subsequently struck out of the result. This is not the sole difference in the construction of the Tables. In Delambre's last *, the mean anomaly is reckoned from the perihelion, and the equations of the centre are increased by $45''$, the sum of several small inequalities: an arrangement made for the same purpose as the former, l. 20; that of avoiding the operation of subtraction.

The *greatest equation of the centre*, it is plain, can mean nothing else than the greatest difference between the true and mean anomalies; which must happen when the body *P* moves with its mean angular velocity. For, if we conceive a body to move uniformly in a circle round *E* as a centre, with an angular velocity, the mean between the least of *P* at *A*, and its greatest at *B*,

* Both Tables were constructed by Delambre.

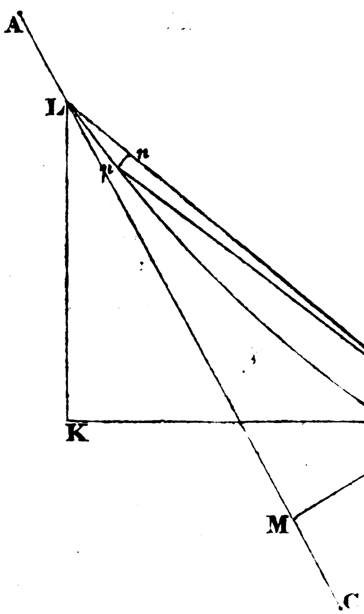
and such, that departing with P from AB the line of the apsides, it shall, in the same time, again arrive at it, together with P; then, it is plain, at the commencement of the motion, the first day, for instance, P moving with its least angular velocity, describes round E a less angle than the fictitious body does: the next day, a greater angle than on the first, but still less than the angle described by the fictitious body: similarly for the third, fourth day, &c.: so that, at the end of any assigned time, the two angular distances of the two bodies from the aphelion, will differ by the accumulation of the daily excesses, of the angular velocity of the fictitious body, above that of the body P. And this accumulation must continue, until P, (always moving, till it reaches B, with an increasing angular velocity), attain its mean angular velocity, or, that velocity with which the body moves in the circle; then, this latter body can, in its daily rate, no longer gain on P; and, past this term, it must lose: exactly at that term, then, the difference of its angular distance from A, or from the line of the apsides, must be the greatest.

The difference of the mean and true anomalies is technically called the *Equation of the Centre*. If we date the planet's motion from the aphelion, then, at the beginning of that motion, the planet moves with its least angular velocity, and consequently the imaginary point, or body that describes the circle with a mean uniform velocity, precedes the planet. The true anomaly then is less than the mean, and consequently the *true* anomaly is equal to the *mean* minus the *equation* of the centre. If the planet's motion had been dated from the perihelion (as it is now the custom in the construction of Tables), then, in a similar position of P, we should have had the true anomaly equal to the mean plus the equation of the centre.

In order to determine this term, or the point in the ellipse, at which the body is moving with the mean velocity, conceive a circle to be described round E as a centre, and to cut the ellipse in some point P, of the figure of p. 457, then such circle will cut the line EA in some point between E and A. Consequently, if the angular velocities be inversely as the squares of the distances from E, the angular velocity in the ellipse from

A to *P* will be, in every intermediate point, less than the angular velocity of the body in the circle, in all points between *EA* and *P*. Now the angular velocities are inversely as the squares of the distances, if *the areas described, respectively, by the body in the ellipse and the body in the circle, be equal* *. This last condition enables us to determine the value of *TP*, or the value of the radius of the intersecting circle. For, if the small areas be equal, the whole areas of the circle and ellipse must be equal, since the whole area $= \frac{\text{area in a given time} \times \text{period}}{\text{given time}}$, and the period, by hypothesis is the same in the ellipse and circle.

* The angle LTp, which expounds the angular velocity, is measured by $\frac{pn}{Tp}$,



and $\frac{pn}{Tp} = \frac{pn \cdot Tp}{Tp^2} \propto \frac{1}{Tp^2}$,

if $pn \cdot Tp$, which is twice the small area LTp, be given.

If, then, x be the sought for value of SP, $2a$ the axis major and ae the eccentricity of the ellipse, we have, by equating the values of the two areas,

$$3.14159 . x^2 = 3.14159 \times a \times a \sqrt{(1 - e^2)};$$

whence,

$$x = a . (1 - e^2)^{\frac{1}{4}}$$

$$= a\left(1 - \frac{e^2}{4} - \frac{3}{32} . e^4\right), \text{ nearly,}$$

$$= a \times .99992942, \text{ nearly,}$$

in the solar orbit.

From the above value of the radius vector, the true and eccentric anomalies, at the time of the greatest equation, may be computed, and by the expressions (1), (2), p. 459, viz.,

$$\rho = \frac{a . (1 - e^2)}{1 - e . \cos. v}, \quad \rho = a (1 + e . \cos. u).$$

Hence, the mean anomaly (nt) is known by the expression

$$nt = u + e . \sin. u,$$

and finally there results the greatest equation of the centre =

$$\pm (v - nt.)$$

EXAMPLE.

In the Earth's orbit, e being very small ($= .016814$),

$$\text{since } (1 - e^2)^{\frac{1}{4}} = 1 + e . \cos. u,$$

$$1 - \frac{e^2}{4} = 1 + e . \cos. u; \therefore \cos. u = -\frac{e}{4},$$

$$\text{and } 1 - \frac{e^2}{4} = (1 - e^2)(1 + e \cos. v); \therefore \cos. v = \frac{3}{4} e;$$

$\therefore$ by the series for the arc in terms of the cosine, and by neglecting the powers of e,

$$nt = \text{quadrant} + \frac{e}{4} + e,$$

$$v = \text{quadrant} - \frac{3}{4} e;$$

$\therefore nt - v$, (the greatest equation) $= \dfrac{8e}{4} = 2e$, and consequently, in the Earth's orbit, the eccentricity $= \frac{1}{2}$ the greatest equation.

This is one method of computing the greatest equation; but it is usually determined from observations. For that purpose we must observe the longitude of the body, when its angular velocity is equal to its mean angular velocity; thus, according to Lacaille,

1751. Oct. 7, ☉'s longitude	6ˢ	13°	47′	13″.7
1752. Mar. 28, ...	0	8	9	25.5
difference of the two longitudes	5	24	22	11.8
The mean motion proportional to the interval of time was	5	20	31	43.2
the diff. or the double of the greatest equation	0	3	50	28.6

Hence, the greatest equation of the centre in the Earth's orbit is 1° 55′ 14″.3: and more nearly, by correcting the above calculation, 1° 55′ 33″.

The difference of the longitudes of the two points in the orbit, at which the real motion nearly equals the mean, is equal to 5ˢ 24° 22′ 11″, or 174° 22′ 11″. This is a very obtuse angle formed by two lines drawn from the above two points to the focus of the solar ellipse. The two points then are not very remote from the extremities of the axis minor; they would be exactly there, if the angle were 178° 4′ 28″. Hence, the greatest equation happens when the body is nearly as its mean distance.

In the Example that has preceded, the Sun's longitude was taken on October 7, and March 28; because, at those times, his daily motions or increases of longitude were equal to his mean motion. That circumstance was ascertained by first taking the Sun's longitudes on two successive days, and then their difference, which is his angular motion. The mean angular motion is nearly 59′ 8″.3: the greatest, about the beginning of January, being 1° 1′ 10″; the least, about the beginning of July, being 57′ 11″.

We shall perceive the use of the equation of the centre, when we treat of the equation of time. Astronomers have used its greatest value in determining the eccentricity of the orbit *. If E be the greatest equation, and $\frac{E}{57^{\circ}.2957795}$ be put $= K$, then the eccentricity, or

$$e = \frac{K}{2} - \frac{11\, K^3}{3.2^8} - \frac{587\, K^5}{3.5.2^{16}} - \&c. \dagger$$

Hence in the case of the Earth's orbit, the eccentricity of which is very small, we have, retaining only the first term of the series, and taking $E = 1^{\circ}\ 55'\ 33''$,

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* See Lacaille, *Mem. Acad.* 1757, p. 123.

† This series was invented by Lambert. The reverse series for the greatest equation is

$$2\,e + \frac{11}{48}\,e^3 + \frac{599}{5120}\,e^5 + \&c.$$

and according to M. Oriani, *Ephes. de Milan.* 1805.

$$\begin{aligned} E = & - \left(2\,e - \frac{1}{2^2}\,e^3 + \frac{5}{2^5\,.\,3}\,e^5 + \&c.\right) \text{sin. } z \\ & + \left(\frac{5}{2^4}\,e^2 - \frac{11}{2^3.\,3}\,e^4 + \&c.\right) \text{sin. } 2\,z, \\ & - \left(\frac{13}{2^3\,.\,3}\,e^3 - \frac{43}{2^6}\,e^5 + \&c.\right) \text{sin. } 3\,z, \\ & \left(\frac{103}{2^5\,.\,3} - e^4\right) \text{sin. } 4\,z \\ & - \frac{1097}{2^6\,.\,3\,.\,5}\,e^5 \text{ sin. } 5\,z, \end{aligned}$$

not extending the series beyond terms containing e^5.

In a Note to page 460, we gave the series expressing the true anomaly in terms of the mean and the eccentricity. The following is Delambre's expression for the equation of the centre, for the year 1810, in terms of the greatest equation and of the mean anomaly z reckoned from the perigee:

$$\begin{aligned} 1^{\circ}\ 55'\ 26''.352 \text{ sin. } z + 1'\ 12''.679 \text{ sin. } 2\,z + 1''.0575 \text{ sin. } 3\,z \\ + 0''.018 \text{ sin. } 4\,z. \end{aligned}$$

$$e = \frac{K}{2} = \frac{1^\circ\ 55'\ 33''}{2 \times 57^\circ.2957795} = .016807.$$

If E be taken $= 1^0\ 55'\ 36''.5$, (the greatest equation in 1750),

$$e = .016814.$$

If E be taken $= 1^\circ\ 55'\ 26''.8$, (the greatest equation in 1800),

$$e = .016791.$$

From these two Examples, the diminution of the greatest equation for 50 years appears to be $9''.7$: and, consequently the *secular diminution* would be $19''.4$. Lalande, in his Tables, states it to be $18''.8$. Delambre, $17''.18$.

In the case of the orbit of *Saturn*, $E = 6^0\ 26'\ 42''$

$$= 6^0.445\ ;\ \therefore K = \frac{6.445}{57.2957795} = .112486,$$

$$\text{and } e = .056243 - 000031 = .056212.$$

We have, in the preceding pages, given only one solution of Kepler's Problem *: which solution is Cassini's, and is an indirect one. But there are several other solutions of the same kind, besides those which may be called direct solutions, and are derived from the simple consideration of the equations of p. 460. The learned Astronomer of the Dublin Observatory, has considered, in a Memoir of the Irish Transactions, these solutions and appreciated their exactness.

In this subject the first object of investigation was strictly a mathematical one. When we apply the result of that investigation to the solar orbit, we find the Sun's place therein cor-

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* The reverse problem, by the solution of which the mean anomaly is found in terms of the true, being of little use, has not been introduced into the text. In order to solve it, find u from v by this expression,

$$\tan.\frac{u}{2} = \sqrt{\left(\frac{1+e}{1-e}\right)} \cdot \tan.\frac{v}{2},$$

and then the mean anomaly $(n\,t)$ from

$$n\,t = e \sin.\ u + u.$$

responding to a given time: and this, as we have stated, is the first step towards the construction of Solar Tables. But it may be asked cannot the investigations of the Sun's elliptical place (which are investigations of no slight intricacy) be superseded by merely registering each day, his longitude. Will not, at the same distance of time from the equinox, the Sun's longitude be the same in 1800 as it was in 1750? Undoubtedly it would be so if the solar ellipse remained fixed in the heavens and of the same dimensions: and in such a case we could dispense, in the solar theory, at least, with Kepler's problem. But if the two preceding circumstances should not take place, if, for instance, the place of the apogee should not remain fixed, the intersection of the equator and ecliptic would not take place in the same point of the solar ellipse. The angular velocity, therefore, of the Sun, in his real orbit, would be variable at that point. It would not be the same in 1800 as in 1750: and, consequently, the Sun's longitude, after the elapsing of a certain time from his departure from the equinox, would not solely depend on such elapsed time. Predicaments similar to these would happen, if the dimensions of the solar orbit (its eccentricity for instance) should be changed. For the above reasons, then, we cannot rely solely on past observations of the Sun's longitude in predicting his future longitudes. Theoretical calculation must be combined with observation. The former will enable us, as we have seen, to assign a body's place in an ellipse when the time from the apside (the mean anomaly, in fact) and the eccentricity of the orbit are given. But, for the purpose of application, we must know the situation of the axis major, or the longitude of one of the apsides. For such knowledge we may have recourse to observation: not indeed to mere observation, but to observation combined with its appropriate method.

The methods then, of so using observations, that from them we may conveniently and exactly deduce the place and motion of the aphelion of a planet's orbit, and the quantity and variation of its eccentricity, will form the subjects of the ensuing Chapters.

CHAP. XIX.

On the Place and Motion of the Aphelion of an Orbit.—Duration of Seasons.—Application of Kepler's Problem to the determination of the Sun's Place.

It follows from what was remarked in p. 445, that the Sun in his perigee being at his least distance, and in his apogee, at his greatest, his apparent diameter in those positions would be respectively the greatest and least. If, therefore, we could, by means of instruments, measure the Sun's apparent diameter with sufficient nicety, so as to determine when it were the least, the Sun's longitude computed for that time, would, in fact, be the longitude of the apogee *.

Or if, computing, day by day, from the observed right ascension and declination, the Sun's longitude, we could determine when the increments of longitude were the least, the Sun's longitude, computed for that time, would be that of the apogee : for, the Sun's angular motion in that point is the least.

The difference of two longitudes thus observed, after an interval of time (t,) would be the angle described by the apogee in that interval. And if the longitudes were not accurately those of the apogee, still, if they belonged to observations, distant from each other by a considerable interval of time, the motion of the apogee would be deduced with tolerable exactness ; since, in such a case, the error would be diffused over a great number of years.

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* *Apogee*, if the Sun be supposed to revolve, *Aphelion*, if the Earth ; and, although, in reality, it is the latter body which revolves, yet, since it affects not the *mathematical* theory, we speak sometimes of one revolving, and sometimes of the other ; and, with a like disregard of precision, we use the terms apogee and aphelion.

Thus, by the observations of Waltherus,

1496. Longitude of the apogee	$3^s\ 3^\circ\ 57'\ 57''$
In 1749, (by Lacaille)	3 8 39 0
∴ progressive motion in 253 years	0 4 41 3

whence the mean annual *progression** results equal to $1'\ 6''$: differing, however, from the result of better observations and methods by more than $1'\ 2''$.

Thus, in the Berlin Memoirs of 1785, M. Delambre, in treating of the Solar Orbit, compares the places of the apogee given by Waltherus (by Lacaille's Calculations) Cocheon King, La Hire, and Flamstead, with Maskelyne's.

Astronomer.	Year.	Longitude of Apogee.	Progression.
Waltherus	1496	$3^s\ 3^\circ\ 57'\ 57''$	$65''.385$
Cocheon King..	1279	3 0 8 0	64.606
La Hire	1684	3 7 28 0	62.116
Flamstead	1690	3 7 35 0	61.584
			4) 253.691
Mean result			63.423

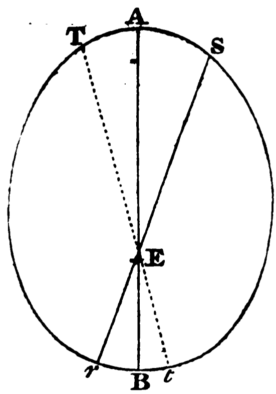
Hence, if the equinoctial year be estimated at $365^d\ 5^h\ 49^m\ 6^s.374$, the anomalistic year, since the time of describing $63''.423$

$$\left(=\frac{63''.423}{59'\ 8''.3}\times 24\right) = 25^m\ 42^s.4, \text{ is } 365^d\ 6^h\ 7^m\ 24^s.307.$$

The more accurate method, however, of determining the progression of the apogee rests upon a very simple principle. Let SEr be a right line, and draw TEt making with the axis major AB of the ellipse, an angle $TEA = SEA$: now, the time through $rBtS$ is less than the time through the remaining arc

* *Progression* is here meant to be used technically: a motion *in consequentia*, or, according to the order of the signs.

$SATr$: for, the equal and similar areas SEt, TEr, are described in equal times, but the area rEt is < area SET; therefore, by



Kepler's law (p. 445,) it is described in less time; therefore $rEt + SEt$, which is equal to the area $SErtS$, is described in less time than $SET + TEr$, which compose the area $SErTS$; therefore the body moves through the arc $rBts$ in less time than through STr. And this property belongs to every line drawn through E, except the line AEB, the major axis, or, the line of the apsides, that line which joins the aphelion and perihelion of the orbit.

Hence it follows, if, on comparing two observations of the Sun at S and at r, (that is, when the difference of the longitudes is 6 signs or 180 degrees) it appears that the time elapsed is not half a year, we may be sure, that the Sun has not been observed in his perigee and apogee. If the interval should be exactly, or nearly, half a year, then we may as certainly conclude, that the Sun was, at the times of observation, exactly, or very nearly, in the line of the apsides.

If the interval of time be nearly half a year, (which is the case that will occur in practice,) then we must find the true position of the apogee by a slight computation, which shall be first algebraically stated, and then exemplified.

The time from r to S = the time from r to B + the time from B to A − the time from S to A;

$\therefore$ time from B to A — time from r to S = ... (a)
time from S to A — time from r to B.

Now the first difference is known, being the difference between half an *anomalistic* year * and the observed interval of observation: and of the second difference, the second term may be expressed by means of the first: thus, let the first term $= t$: then by Kepler's law, (see p. 445,)

$$\text{time from } r \text{ to } B = t \times \frac{\text{area } rEB}{\text{area } SEA}$$

$$= t \times \frac{rB \times EB}{SA \times EA} \text{ (} r \text{ and } S \text{ being near the apsides)}$$

$$= t \times \frac{rB}{EB} \times \frac{EA}{SA} \times \frac{EB^2}{EA^2}$$

$$= t \times \frac{EB^2}{EA^2}\left(\text{since } \frac{rB}{EB} = \angle rEB = \angle SBA = \frac{SA}{EA}\right)$$

$$= t \times \frac{\text{angular velocity at } A}{\text{angular velocity at } B} \text{ (see p. 470.)}$$

Now, the angular velocities at A and B, or the increments of the Sun's longitudes at the apogee and perigee, being known from observation (see p. 431,) and the time from r to B being expressed in terms of those velocities and of t, the quantity t is the only unknown quantity in the equation (a) l. 1, and accordingly may be determined from it. But t being obtained, we can thence determine the exact time when the Sun (S) is at the apogee A: and his longitude, computed for that time, is the longitude of the apogee.

Example.

1743.	Dec. 30, $0^h\ 3^m\ 7^s$ ☉'s longitude . . .	9^s	8^0	$29'$	$12''.5$
1744.	June 30, 0 3 0	3	8	51	1.5
	$\therefore$ difference of 2d and 1st longitudes	6	0	21	49

therefore at the 2d observation June 30th, the Sun was past S.

* The time from the Sun's leaving the apogee to his return to the same.

In order to find when he was exactly at S, that is, when the difference of the longitudes was exactly 6^s; or (supposing the perigee to have been progressive through $31''$), when the difference of longitudes was $6^s\ 0^0\ 0'\ 31''$, we must find the time of describing the difference of $21'\ 49''$, and $31''$, that is, $21'\ 18''$. Now this time, since on June 30, the Sun's daily motion in longitude was $57'\ 12''$, equals $\frac{21'\ 18''}{57'\ 12''} \times 24^h$, or $8^h\ 56^m\ 13^s$: take this from the time (June 30, $0^h\ 3^m$) of the second observation, and there results, June 29, $15^h\ 6^m\ 47^s$, for the time when the difference of the longitudes of the Sun at r and near S was $180^0\ 0'\ 31''$.

The interval between this last time, and Dec. 30, $0^h\ 3^m\ 7^s$, the time of the first observation, is $182^d\ 15^h\ 3^m\ 40^s$, nearly the time from r to S: but, this time is less than half an anomalistic year, which is $182^d\ 15^h\ 7^m\ 1^s$*: and see (*a*) p. 479, l. 1,

$$t - \text{time from } r \text{ to } B = 3^m\ 21^s.$$

But, see the same page, l. 12,

$$\text{the time from } r \text{ to } B = t \times \frac{57'\ 12''}{61'\ 12''};$$

$$\therefore, \text{ substituting, } t \times \frac{4'}{61'\ 12''} = 3^m\ 21^s, \text{ and consequently,}$$

$$t = 47^m\ 54^s.$$

Add this to the time, June 29, $15^h\ 6^m\ 47^s$, when the Sun was at S, and we have, June 29, $15^h\ 54^m\ 41^s$ for the time when the Sun was in the apogee.

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* In this method, which is to determine accurately the given place of the apogee, the motion of the latter, and the length of the anomalistic year are supposed to be known to some degree of accuracy. The one is stated to be $62''$; the other, $365^d\ 6^h\ 14^m\ 2^s$. But, if both be supposed unknown, if we take the difference of the longitudes of r and S to be simply 6^s, and the elapsed time to be half the *tropical* year, still the method will give the place of the apogee very nearly, which may serve as a first approximation to the true place.

The Sun's longitude at that time must be less than his longitude ($3^s\ 8^0\ 51'\ 1''.5$) on June 30, $0^h\ 3^m$ by the difference due to the difference of the times, which is $8^h\ 8^m\ 19^s$: the former difference then is equal (since the increase of longitude in 24 hours was $57'\ 12''$) to

$$\frac{8^h\ 8^m\ 19^s}{24^h} \times 57'\ 12'' = 19'\ 21'';$$

hence the longitude of the apogee $= 3^s\ 8^0\ 51'\ 1''.5 - 19'\ 21'' = 3^s\ 8^0\ 31'\ 40''.5$, or $98^0\ 31'\ 40''.5$, or $8^0\ 31'\ 40''.5$, past the summer solstice.

We will now add another Example, the materials of which are drawn from Delambre's Memoir on the Solar Orbit, inserted in the Berlin Memoirs for 1775.

June 30, 1776.

First Operation—*To find the Error of the Sidereal Clock at the Time of the Sun's passing the Meridian.*

	α Virginis.		Arcturus.
Æ by clock	$13^h\ 10^m\ 9^s.08$	....	$14^h\ 2^m\ 11^s.47$
by catalogue (see p. 352, &c.)	13 13 25.83	....	14 5 27.88
	0 3 16.75		0 3 16.41

clock slow	3 16.75	.... at	$13^h\ 13^m$
ditto	3 16.41	....	14 5
∴ by a mean, clock slow	3 16.58	....	13 39

But the Sun passed seven hours previously.

Second Operation—*Clock's Rate.*

Now clock's rate in 24^h	− 1.716
∴ in 6	− 0.42
in 1	07
in 7	− 0.49

☉'s transit by clock	$6^h\ 36^m\ 23^s.21$
clock's error	{ 0 3 16.58
	− 49
☉'s right ascension	6 39 39.3

Third Operation—*Conversion of the Right Ascension in Time into Space.*

By Zach's Tables, Tab. XXIX, or Vince's, vol. II. p. 297,

6^h	......	3^s 0^0	$0'$	$0''$	
39	......	0 9	45	0	
39	......	0 0	9	45	
3	......	0 0	0	4.5	
		3 9	54	49.5	Sun's right ascension in space.

The obliquity was $23^0\ 28'\ 4''$, from that and the right ascension find the Sun's longitude by Naper's Rule, or thus, by the *Tables of Reduction* to the ecliptic.

Fourth Operation—*Reduction of Equator to the Ecliptic* *.

See Zach's Table XXI, in his *Tabulæ Motuum Solis,* or Vince's Table, *Astronomy,* vol. II, p. 352.

Add 3^s.

	Reduction.	Difference for 1′.	
6^s 9^0 $50'$ $0''.0$	0^s 0^0 $47'$ $57''.45$	$4''.703$	
0 0 4 49.5	0 0 0 22.69	4	
(obliquity being 23^0 $28'$)	0 0 48 20.14	18.812 for	$4'$ $0''.0$
add for $4''$	0 0 0 0.27	3.88	0 49.5
	0 0 48 20.41	22.69	4 49.5
Sun's right ascension..	3 9 54 49.5		
Sun's longitude	3 9 6 29.1		

and this is the whole of the process for the actual finding of the Sun's longitude from his observed right ascension.

By a similar process performed on Maskelyne's observation of the Sun's transit on the December 31, we have

$$\odot\text{'s longitude} = 9^s\ 10^0\ 31'\ 7''.6.$$

Fifth Operation—*Difference of Sun's Longitude found.*

The above are the Sun's longitudes when his centre was on

* See Chapter XXI.

the meridian: they belong, therefore, to *apparent* noon: if, therefore, we add the equations of time (which are $3^m\ 13^s$, $3^m\ 53^s$, respectively) we shall have,

1776, June 30, $0^h\ 3^m\ 13^s$, ☉'s longitude	=	3^s	9^0	$6'$	$29''.1$
Dec. 31, 0 3 53	=	9	10	31	$7.\dot{6}$
difference of Sun's longitudes		6	1	24	38.5

If we take from this $33'$, the half yearly progression of the apogee, we have the difference of the Sun's longitudes equal to

$$6^s\ 1^0\ 24'\ 5''.5\ ;$$

consequently, by reason of the excess $1^0\ 24'\ 5''.5$ above 6^s, or 180^0, the Sun at the times of the two mentioned observations could not occupy, respectively, the extremities of a line drawn through the focus of the orbit. If t were his position on Dec. 31, at $0^h\ 3^m\ 53^s$, T could not have been his position on June 30, at $0^h\ 3^m\ 13^s$: or, if s were his position at the former time, S could not have been his position at the latter.

Suppose a to be the place of the Sun at the former time, then the difference between the longitudes of T and being 6^s, aT will be equal to $1^0\ 24'\ 5''.5$: in order to find the time of describing it, we have from the Solar Tables, or Nautical Almanack, or by the reduction of observations made on the noons of June 30, and July 1,

June 30, Sun's longitude	3^s	8^0	$23'$	$27''$
July 1, ...	3	9	20	40
	0	0	57	13

Hence, in 24 hours, nearly, the Sun moved through $57'\ 13''$, consequently, he described

$$1^\circ\ 24'\ 5''.5 \text{ in } 35^h\ 18^m\ 1^s \left(= 24\ \frac{1^0\ 24'\ 5''.5}{57'\ 13''}\right);$$

and consequently, he was at T on July 1, at $11^h\ 21^m\ 14^s$.

But, the two opposite positions of the Sun, instead of being, as we have supposed them to be, at T and t, might have been at A and B, or at S and s. In order to ascertain this point, we have the difference of the two times (Dec. 31, $0^h\ 3^m\ 53^s$, and

July 1, $11^h\,21^m\,14^s$) equal to $182^d\,12^h\,42^m\,39^s$. Now (see p. 480,) the half of an anomalistic year is $182^d\,15^h\,7^m\,1^s$ *: consequently, the time from t to T is less than the time from A to B, which it ought to be, since (as in p. 478,) the time from T to t = time from A to B − time from A to T + time from B to t = time from A to B − some quantity, whereas, if S and r had been the points, we should have had the time from S to T = time from A to B + time from S to A − time from r to B, = time from A to B + some quantity.

The Sun, therefore, must have been at some such opposite points as T and t, or, in other words, must, on July 1, $11^h\,21^m\,14^s$, have already passed the apogee.

What remains, then, to be done is the computation of the times of describing AT, Bt.

Sixth Operation—*Corrections of the Times of the Sun's passing the Apsides.*

Let t, t', be the times of describing them,

$$\text{then } t = \frac{t' \,.\, \text{area } AET}{\text{area } BEt} \quad \text{(see p. 479,)}$$

$$= \frac{t' \,.\, AT \,.\, AE}{Bt \,.\, BE} \quad \text{(the points } T, t', \text{ being near to the apsides)}$$

$$= \frac{t' \,.\, AE^2}{BE^2} = t' \,.\, \frac{(1+e)^2}{(1-e)^2},$$

e being the eccentricity.

$$\text{Hence, } t - t' = t' \,.\, \frac{4e}{(1-e)^2}, \text{ or } = t\frac{4e}{(1+e)^2};$$

$$\text{consequently, } \quad t = (t - t') \,.\, \frac{(1+e)^2}{4e},$$

$$t' = (t - t') \,.\, \frac{(1-e)^2}{4e},$$

and $t - t'$ = half the anomalistic year − the time from T to t in the case before us = $2^h\,25^m\,3^s$.

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* Or more exactly $182^d\,15^h\,6^m\,59^s.4$.

Hence, if e be the eccentricity for 1776,

$$\log. 2^h\ 25^m\ 3^s = 3.939669 \ldots\ldots\ldots\ldots\ 3.939669$$

$$\log. \frac{(1+e)^2}{4e} = \underline{1.187047} \quad \log. \frac{(1-e)^2}{4e} = \underline{1.157859}$$

$$(\log. 133880.5) \quad 5.126716 \quad (\log. 125718) \quad 5.097528$$

Hence, since

time at T is July 1, $11^h\ 21^m\ 14^s$ .... at t Dec. 31, $0^h\ 3^m\ 53^s$

$t = \underline{37\ \ 11\ \ 20.5}$ .... and $t' = \underline{34\ \ 46\ \ 18}$

∴ time at A June 29, 22 9 53.5 time at B Dec. 29, 13 17 35

which are, respectively, the times of the Sun's passing the apogee and perigee.

The interval of these two times, or the half of an anomalistic year is,

$$182^d\ 15^h\ 7^m\ 41^s.5.$$

The above methods* of determining the place of the apogee are due to Lacaille. That author, on the grounds of simplicity and uniformity, suggested the propriety of reckoning the anomalies from the perihelia of orbits, since, in the case of Comets, they are necessarily reckoned from those points. In the new Solar Tables of Delambre this suggestion is adopted, (see Introduction: also Vince's *Astronomy*, vol. III. Introduction, p. 2.)

In these new Tables the progression of the perigee, and consequently that of the apogee, is made to be about $61''.9$; and the mean longitudes of the perigee for 1750, 1800, 1810, are respectively stated at $9^s\ 8^0\ 37'\ 28''$; $9^s\ 9^0\ 29'\ 3''$; $9^s\ 9^0\ 39'\ 22''$.

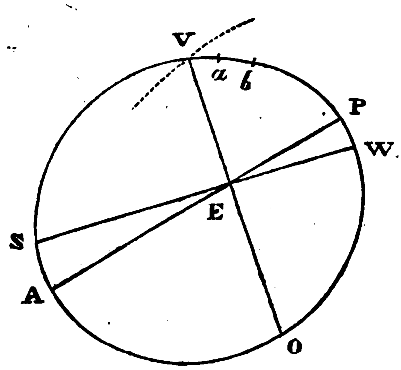
The longitude of the winter solstice is 9^s; therefore in 1810 the perigee was $9^\circ\ 39'\ 22''$ beyond it; at this time, the daily motion of the Sun was $61'\ 11''$; therefore, the solstice happening on December 22, the Sun would be in his perigee about nine days after, or about December 31.

* The method is explained, with singular clearness, by Dalembert, in the historical part (L'Histoire) of the *Memoirs of the Academy of Sciences* for 1742.

From the longitude for any given epoch, and its annual progression, the position of the apogee and of the axis of the solar ellipse, may, by simple proportions, be found for any other epoch. Suppose, for instance, it were enquired when the axis of the solar ellipse was perpendicular to the line of the equinoxes? This, in other words, would be to enquire, when the longitude of the perigee was 270°, or 9^s. Now, its longitude, in 1750, was $9^s\ 8^0\ 37'\ 28''$: the number of years therefore requisite to describe the difference, or $8°\ 37'\ 28''$, taking the annual progression at $62''$, equals $\frac{8^0\ 37'\ 28''}{62''}$, or about 500 years; that is, the major axis was perpendicular to the line of the equinoxes in the year 1250.

The major axis coinciding with the line of the equinoxes the longitude of the perigee was 180°, or 6^s. Between that epoch, therefore, and 1250, the whole quantity of the *progression* of the perigee was $9^s\ 8^0\ 37'\ 28'' - 6^s = 3^s\ 8°\ 37'\ 28''$: and the time of describing it since $\frac{3^s\ 8°\ 37'\ 28''}{62''} = 5720$ was 5720 years. The epoch happened then about 4000 years before the Christian Æra, and is a remarkable one, inasmuch as chronologists consider it to be that of the beginning of the world.

The knowledge of the place of the perigee is necessary to determine the durations of seasons; which are perpetually



varying from its progression. If *W*, *S*, in the Figure, represent the winter and summer solstices, V and O the vernal and autumnal equinoxes, PEA the axis of the solar ellipse; then, in the year 1250, P coincided with W; and, on that account, the time from the autumnal equinox O to the summer solstice W was equal to the time from W to the vernal equinox V. Past that year, P, by reason of its progressive motion, began to separate from W; and in 1800, the separation, measured by the angle PEW, was $9^0\ 29'\ 3''$. By means of this separation, those parts of the elliptical orbit in which the Earth's real motion is the quickest, being thrown nearer to V and away from O, the time from the autumnal equinox O to the solstice W, became gradually greater than the time from W to the vernal equinox: and the time from V to S became less than the time from S to O. In 1800, the following were nearly the lengths of the seasons:

V to S	92^d	21^h	44^m	28^s
S to O	93	13	34	47
O to W	89	16	47	20
W to V	89	1	42	23
length of year	365	5	48	58

This motion of the perigee also, as will be shewn in a subsequent Chapter, continually causes to vary the equation of time.

What has been said concerning the duration, and change of duration, of the Seasons, is, in some degree, digressive; the main object of the Chapter being to explain the method of finding the place, that is, the longitude of the perigee, in order that Kepler's problem might be applied to the determination of the Sun's place.

By Kepler's problem, we are enabled, from the mean anomaly, to assign the true anomaly, or true angular distance, reckoning from perigee*. The mean anomaly of the Sun, is his mean angular distance computed from perigee: in the Figure, if b be the Sun's mean place, it is $\angle\ PEb$. Now,

* The mean anomaly is stated to be reckoned from perigee, since the succeeding extracts are from Delambre's new Solar Tables.

$$\angle PEb = \angle PEV - \angle VEb,$$

and, if V be the first point of *Aries*,

$$\angle PEV = 12^s - \text{mean long. perigee},$$

and $\angle VE\,b = 12^s - \text{mean long. } \odot.$

Hence, the mean anomaly is the difference between the mean longitudes of the Sun and of the perigee. And the Solar Tables assign the mean anomaly by assigning these longitudes. And then, in the same Tables, the mean anomaly is used as an *argument* for finding the equation of the centre. The process may be illustrated by specimens from the Tables, and their application to an Example.

From Table I.

Years.	Mean Longitude of the Sun.	Longitude of Sun's Perigee.
1809.	$9^s\ 10^0\ 42'\ 49''.8$	$9^s\ 9^0\ 38'\ 20''$
1810.	9 10 28 30.2	9 9 39 22
1811.	9 10 14 10.5	9 9 40 24

From Table IV.
Motion for Days. *November*.

Years.		Mean Longitude of the Sun.	Perigee.
Com.	Bissex.		
Days.			
12	11	$10^s\ 10^0\ 28'\ 44''$	$53''.5$
13	12	10 11 27 52.3	53.6
14	13	10 12 27 0.7	53.8

From Table V.
Motion of the Sun for Hours, Minutes, Seconds.

Hours.		Minutes.		Seconds.	
H.	Motion of Sun.	M.	Motion of Sun.	S.	Motion of Sun.
1	2′ 27″.8	1	2″.5	1	0″.0
2	4 55 . 7	2	4 . 9	2	0 . 1
3	7 23 . 5	3	7 . 4	3	0 . 1

From Table VII.
Equation of the Sun's Centre for 1810, with the Secular Variation. (S. V.)

Mean Anomaly.	Equation.	Diff. +	S. V.
10ˢ 12⁰ 0′	11ˢ 28⁰ 32′ 14″.7	13″.5	13″.13
10 12 10	11 28 32 28 . 2	13 . 5	13 . 09
10 12 20	11 28 32 41 . 7	13 . 5	13 . 06
10 12 30	11 28 32 55 . 2	13 . 6	13 . 03

Suppose now the Sun's longitude were required for 1810, November 13, 2^h 3^m 2^s.

Table I. 1st, the mean longitude for the beginning of 1810, is	9ˢ	10⁰	28′	30″.2
Table IV. Nov. 13....................	10	11	27	52 . 3
Table V. 2^h	0	0	4	55 . 7
Table V. 3^m	0	0	0	7 . 4
Table V. 2^s	0	0	0	0 . 1
rejecting 12ˢ, mean long. at time required (*a*)	7	22	1	25 . 7

The longitude of the perigee is to be had from the same Tables; thus:

Table I. Long. at beginning of 1810	9^s	9^0	$39'$	$22''.0$
Table IV. Nov. 13	0	0	0	53.6
longitude of perigee at the time required ...	9	9	40	15.6
subtract this from (*a*) increased by 12 signs, there results the mean anomaly	10	12	21	10.1

With this mean anomaly enter Table VII, and there results

the equation to the centre	11^s	28^0	$32'$	$42''.2$
add to this the mean longitude (*a*)	7	22	1	25.7
	7	20	34	7.9

This result, $7^s\ 20^\circ\ 34'\ 7''.9$, is (if no other corrections are required to be performed) the true longitude reckoned from the *mean* equinox. But, as it has been shewn (pp. 353, &c.), the place of the equinox varies from the inequalities of the Sun's action, and of the Moon's action in causing the precession. Two *equations*, therefore, must be applied to the above longitude, in order to compensate the above inequalities, and so to correct the longitude, that the result shall be the true longitude, reckoned from the *true place of the equinox*. Now, it happens, by mere accident, that, in the above instance, the *lunar* and *solar* nutations are equal to $1''$, but affected with contrary signs. These corrections, therefore, affect not the preceding result. The correction for *aberration* (see p. 307,) has, in fact, been applied; for, since that, in the case of the Sun, must be nearly constant, (and it would be exactly so, if the Sun were always at the same distance from the Earth) the Solar Tables are constructed so as to include, in assigning the mean longitude, the constant aberration ($20''$). The variable part of the aberration (variable on account of the eccentricity of the orbit) is less than the 5th of a second. Let us see then, whether the longitude that has been determined, from a knowledge of the place of the perigee, and from Kepler's problem, expressed by means of Tables, be a true result:

By the Nautical Almanack for 1810; we have

Nov. 13, Sun's longitude	$7^s\ 20^0\ 29'\ 8''$
Nov. 14, ...	7 21 29 36
increase in 24 hours	0 1 , 0 28

Now the Sun's longitude is expressed in the Nautical Almanack for *apparent* time: and the equation of time being $-\ 15^m\ 33^s$, the mean time is $11^h\ 44^m\ 27^s$. Hence, we must find the increase proportional to $2^h\ 18^m\ 35^s$, which is about $5'\ 47''$; consequently the Sun's longitude, on November 13, $2^h\ 3^m\ 2^s$, (mean time) was $7^s\ 20^0\ 34'\ 55''$, which differs from the preceding result, p. 490, l. 11, by about $47''$; consequently, Kepler's problem is not alone sufficient to determine the Sun's place, but some other corrections are requisite to compensate this error of 47 seconds.

Such corrections are to be derived from a new source of inequality; the perturbation of the Earth caused by the attracting force of the Moon and planets; the nature of which will be briefly explained in the ensuing Chapter.

CHAP. XX.

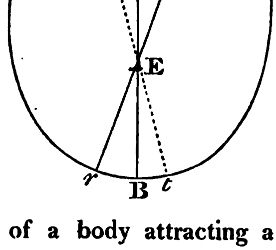
On the Inequalities of the Earth's Orbit and Motion, caused by the Disturbing Forces of the Moon and Planets. On the Methods of determining the Coefficients of the Arguments of the several Equations of Perturbation.

THE discovery of Kepler relative to the form of a Planet's Orbit did not extend beyond the proof of its being an ellipse: and in his *problem* he shewed the method of assigning the planet's place in such an ellipse.

If M be the mean anomaly and E the equation of the centre, then, the planet's elliptical place, or true anomaly is equal to

$$M + E.$$

Newton shewed, on certain conditions and a certain hypothesis, that that must needs take place which Kepler had found to take place. It appears from the third Section of his *Principia*, that if a body, or particle projected, from A perpendicularly to EA,



(E being the place of a body attracting a particle at A, and elsewhere with a force inversely as the square of the distance from *E*), would describe an ellipse, of which *E* would be the focus.

The revolving particle or body *A*, is supposed to be attracted towards *E*, or to be incessantly urged towards *E*, by a *centripetal* force arising from a number of attracting particles, or from an attractive mass, placed at *E*. The centripetal force being the greater, the greater such mass is, and in that proportion.

If in *EA* produced, we place, at an equal distance from *A*, another body of equal mass, and of equal attractive force with the body at *E*, and again suppose the body at *A* to be projected; then, since it is equally urged to describe an ellipse round the new mass, as round that originally placed at *E*, it can describe an ellipse round neither, but must proceed to move in a direction perpendicular to *EA*.

In this extreme case, the elliptical orbit, and the law of elliptical motion would be entirely destroyed.

If now we suppose the mass of the new body to be diminished, or its distance from *A* to be increased; or, if we suppose both circumstances to take place, then, the derangement, or *perturbation*, of the body that is to revolve round *E*, will still continue, but in a less degree. An orbit, or curvilinear path, concave towards *E* in the commencement of motion, will be described; but, neither elliptical, nor of any other class and denomination.

In this latter case, the new body, being supposed less than the body placed at *E*, may be called the *disturbing* body; disturbing, indeed, by no other force than that of attraction, with which the body at *E* is supposed to be endowed; but which latter, from a difference of circumstance merely, is denominated a *Centripetal* force. In the first supposition, of an inequality of mass and distance in the two bodies, from the similarity of circumstance, either body might be pronounced to be equally attracting or equally disturbing.

The disturbing body, whatever be its mass and distance, will always derange the laws of the equable description of areas, and of elliptical motion. If its mass be considerable, and its distance not very great, the derangement will be so much as to render the knowledge of those laws useless in determining the real orbit, and law of motion, of the disturbed body. In such case, Kepler's problem would become one of mere curiosity; and the place of the body would be required to be determined by other means.

If, however, the mass of the disturbing body be, with reference to that of the attracting body, inconsiderable, then the derangements, or perturbations, may be so small, that the orbit shall be nearly, though not strictly, elliptical; and the equable description of areas, nearly, though not exactly, true. Under such circumstances, Kepler's problem will not be nugatory. It may be applied to determine the place of the revolving body, supposing it to revolve, which is not the case, but which is nearly so, in an ellipse. The erroneous supposition, and consequently erroneous results, being afterwards *corrected* by supplying certain small *equations*, that shall compensate the inequalities arising from the disturbing body.

In the predicaments just described, are the bodies of the solar system. The mass of the Sun, round which the Earth revolves, is amazingly greater than that of the Moon *, which disturbs the Earth's motion: greater also, than the masses of the planets, which, like the Moon, must cause perturbations. The Earth, therefore, describes very nearly an ellipse round the Sun.

As a first approximation then, and a very near one, we may, as in the last Chapter, determine the Sun's, or Earth's place, by means of Kepler's problem: and subsequently correct such place, by small equations due to the perturbations of the Moon, and of the planets.

But, how are these small corrections to be computed? By finding, for an assigned time, an expression for the place of a

* The Sun is 1300000 times greater than the Earth, and the Earth more than 68 times greater than the Moon.

body, attracted by one body, and disturbed by another; the masses, distances, and positions, of the bodies being given; that is, by solving what, for distinction, has been called the *Problem of the Three Bodies*.

The consideration of *three* bodies is sufficient: for suppose, by the solution of the problem, the equation, or correction, for the Sun's longitude, to be expressed, by means of the Sun's and Earth's masses, distances, &c., and of other terms denoting the mass, distance, &c., of a third body; then, substituting, for these latter terms, the numbers that, in a specific instance, belong to the Moon, the result will express the perturbation due to the Moon. Instead of the Moon, let the third body be *Jupiter:* substitute, as before, the proper quantities, and the result expresses the perturbation due to *Jupiter:* and similarly for the other planets. The sum of all these corrections, separately computed, will be the correction of the longitude arising from the action of all the planets.

The above corrections are what are necessary to complete the process of finding the Sun's longitude, and to supply the deficiency of several seconds, from the true longitude. The number of corrections which it is necessary to consider, and which the latest Solar Tables enable us to assign, is five; arising from the perturbations of the *Moon, Venus, Mars, Jupiter,* and *Saturn.* Those of *Mercury,* the *Georgium Sidus, Ceres, Juno,* and *Pallas,* are disregarded.

The computation of these perturbations has been attempted in another place (see vol. II. on *Physical Astronomy*), by the approximate solution (all that the case admits of) of the *problem of the three bodies.* Even by the little explanation that has already (see p. 494,) been given, it is plain that the results of that solution are essential to the solar theory, and to the construction of Solar Tables. They are equally essential to the planetary theory. In fact, they are as much a part of Newton's System, as the elliptical forms of planetary orbits, and the laws of the periods of planets. The perturbation of the planetary system is as direct a consequence of the principle of universal attraction, as the regularity of that system would be, on the hypothesis of the abstraction of disturbing forces. The quantities of the perturbations are, indeed, small and not easily discerned: but they are gradually detected as art continues to invent better instruments, and science, better methods, and they so furnish not the most simple proof, perhaps, but the most irrefragable proof of the truth of Newton's Theory.

Observation, it is plain, must furnish numerous results, before the formulæ of perturbations can be numerically exhibited, or, what is the same thing, be reduced into Tables. The positions and distances of the planets must be known: for, without any formal proof, we may perceive, that, according to the position of a planet, the effect of its disturbing force may be to draw the Earth either directly from, or towards, the Sun, or, in some oblique and transverse direction. In fact, the *heliocentric* longitudes of the Earth and the planets form the arguments in the Tables of perturbations.

Having thus explained, in a general way, the theory of perturbations, we will complete the Example of p. 490, by adding certain corrections, computed from that theory, to the Sun's longitude.

By p. 490, ☉'s longitude	7^s	20^0	$34'$	$8''$
correction due to ☽	0	0	0	5.5
to ♀	0	0	0	17.49
to ♂	0	0	0	4.32
to ♃	0	0	0	12.7
to ♄	0	0	0	0.65
∴ Nov. 13, 1810. 2^h 3^m 2^s; ☉'s true long^e. .	7	20	34	48.86*

* This determination of the Sun's longitude is less by about 7 seconds than the longitude as stated in the Nautical Almanack. But, this latter was computed, (see Preface to the Nautical Almanack) from Lalande's Tables, inserted in the 3d Edition of his *Astronomy:* which differ by a few seconds from Delambre's last Solar Tables (Vince's, vol. III,) and from which the numbers in the text were taken.

By computations like these carried on by the aid of Tables (see pp. 490, &c.), the Sun's longitude is computed for every day in the year, and then registered; in the *Nautical Almanack* of Great Britain, the *Connoissance des Tems* of France, and in the *Ephemerides* of Berlin, and of other cities. The use of registering the Sun's longitude is explained in the Nautical Almanack, at p. 163, &c.

In page 495, l. 5, it was said that the problem of the three bodies was sufficient for the computation of all the inequalities. But this is rather, if we may so express ourselves, practically than metaphysically exact: it is founded on this, that, if v and i should be the perturbations of the Sun's *elliptical* longitude (L) by Venus and Jupiter, the resulting longitude will be

$$L + v + i,$$

whereas i ought, in strictness, to be computed for a longitude $L + v$, and v for a longitude $L + i$. The differences in the two cases are, however, insensible: v and i not exceeding $10''$.

We may add too, some farther limitation to the assertion, that the perturbations of the solar orbit (the variations produced in the Sun's elliptical longitude and distance) are to be computed, by means of the *problem of the three bodies*. Theory alone is not adequate to the above purpose. For, if the Earth be displaced from its elliptical orbit (be made *exorbitant*) by the action of a planet, the *displacement*, in a given position, will be the greater, the greater the mass of the disturbing planet. We must, therefore, know that mass, if we would, *a priori*, compute the displacement. Now, although the masses of Jupiter and Saturn are known from the periods of their satellites, the masses of Venus and Mars and Mercury are not. We can, indeed, setting out from certain effects of their action, indirectly approach, and approximate to, their values (see vol. II, p. 477, &c.). But the method is not a sure one; so that, in computing the perturbations of the Earth's orbit (of which that due to Venus from her proximity to the Earth is probably the greatest) we are obliged to look to other aid than that of mere theory.

The method to be pursued on this occasion is similar to that by which the corrections of the epoch, of the greatest equation, and of the longitude of the apogee, will be investigated in a following Chapter. Thus the true longitude, or L is equal to

$$M + E + P,$$

P being the sum of the perturbations, due to the actions of Mercury, Venus, the Moon, &c.: now the *arguments* of the perturbations are the differences between the longitudes of the disturbing planet and the Earth, or multiples of those differences: thus, if the symbols representing the Moon, Sun, Venus, &c. be made to denote their longitudes, the *argument* for the Moon's perturbation will be ☽ − ☉; for Jupiter's ♃ − ☉, 2(♃ − ☉); for Venus's (♀ − ☉), 2(♀ − ☉), &c.: so that, assuming a, b, c, &c. to be the coefficients of the arguments, the lunar perturbation will be denoted by a. sin. (☽ − ☉); Jupiter's by b. sin. (♃ − ☉) + c. sin. (2♃ − 2☉), &c. and accordingly, the whole perturbation or

$P = a$. sin. (☽ − ☉) + b. sin. (♃ − ☉) + c. sin. (2♃ − 2☉)
+ d. sin. (♀ − ☉) + &c.

compute now the Sun's longitude from the elliptical theory, then, (supposing the epoch, greatest equation, &c. to be exact) the computed longitude will differ from the observed by an error C, which error arises from the perturbations of the planets; accordingly,

$C = a$. sin. (☽ − ☉) + b. sin. (♃ − ☉) + &c.
+ d. sin. (♀ − ☉) + &c.

in which ☽, ☉, ♃, the longitudes of the Moon, Jupiter, Venus, &c. are known, since C is the difference between two longitudes, one observed at a *given time*, the other computed for the same time. Repeat the operation: or find C', C'', C''', &c. the differences between certain observed and computed longitudes, and there will arise equations similar to the one that has been just deduced; and, it is plain, we may form as many equations as there are indeterminate coefficients a, b, c, &c. from which, by elimination, the values of a, b, c, &c. may be deduced. Or, we may form several groups or sets of equations, on the principle of formation which with be hereafter explained, and obtain, by addition, equations that shall be, respectively, most favourable for the deductions of the values of a, b, c, &c.*

If the Moon's equation consist of one term, Venus's of two, Jupiter's of two, Mars of two, there will be required, at the least, seven equations for the determination of the seven coefficients. Now the same method, which has been here described for determining these coefficients, will be, in the next Chapter, used for determining the corrections of the *elements* of the solar orbit: which elements are here meant to be, the epoch of the mean longitude, the eccentricity, and the longitude of the apogee. Three equations, therefore, will be required for such purpose: consequently, if, by one and the same operation, we seek to correct the *elements*, and to determine the corrections due to the perturbations of the Moon and the above-mentioned three planets, we must employ, at the least, ten equations. We shall, however, soon see that it is more expedient to employ and to combine one hundred equations, in order to obtain, by virtue of the principle of *mean* results, exact results. No one of the coefficients of the *equations* of perturbations exceeds nine seconds †.

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* The principle is this: if a be the coefficient, select those equations in which the values of the term (a sin. A) is the greatest, make them all positive (by changing, if necessary, the signs of all the other terms of the equation) and add them together for the purpose of forming a new equation.

† If v be the longitude and δv be the *error* or *correction* due to the perturbations of the planets,

$$\delta v = 8''.9 \sin.(☽ - \odot) + 7''.059 . \sin.(♃ - \odot) - 2''.51 . \sin. 2(♃ - \odot)$$
$$+ 5''.29 . \sin. (♀ - \odot) - 6''.1 \sin. 2 (♀ - \odot)$$
$$+ 0''.4 \sin. (♂ - \odot) + 3''.5 \sin. 2 (♂ - \odot).$$

See *Physical Astronomy*, p. 311. M. Delambre, (Berlin *Memoirs*, 1785, p. 248), add one more equation for Jupiter, three for Venus, and three for Mars.

CHAP. XXI.

On the Methods of Correcting the Solar Tables. The Formula of the Reduction of the Ecliptic to the Equator, &c.

We have, in the preceding Chapters, explained and illustrated the method, of finding *a priori*, or by theory and antecedent calculations, the Sun's longitude. The steps of the method are several. The first is to find, from a given epoch and elapsed time, the Sun's mean longitude (L): the next, to find, from the position of the apogee, at a given epoch, and the quantity and law of its *progression*, the longitude (A) of the apogee. The difference of these two angles, or $L - A$ is the mean anomaly (M), which is the third step: the fourth consists in finding (see p. 490,) the equation of the centre (E) corresponding to M. The sixth and last step is to find, at the given time of the required longitude, the sum (P) of the *perturbations* caused by the Moon and planets: the resulting longitude (S) is equal to

$$L - A + E + P,$$

$$\text{or } M + E + P,$$

setting aside the effects of nutation, aberration and parallax.

The results of the preceding methods, (those by which the equation of the centre and the perturbations of the planets are computed,) are registered in *Solar Tables*. From such Tables the national *Ephemerides*, the *Nautical Almanack* of England, the *Connoissance des Temps* of France, are partly computed. The immediate results from the Solar Tables are the Sun's longitudes. The Sun's right ascensions (which occupy the fourth columns of the second page of each mouth) are deduced from the longitudes and the obliquity; not, in practice, by Naper's Rules, but, (because the thing can be so more conveniently effected) by the aid of a Table, entitled the *Reduction of the Ecliptic to the Equator*. The construction of such a Table is effected by means of a formula which it is now our business to investigate.

Let Æ, and ⊙ denote the Sun's right ascension and longitude, and let w be the obliquity of the ecliptic, then

$$\tan. Æ = \cos. w . \tan. \odot,$$

and $\tan. (\odot - Æ) = \dfrac{\tan. \odot - \tan. Æ}{1 + \tan. \odot . \tan. Æ} = \dfrac{\tan. \odot (1 - \cos. w)}{1 + \cos. w . \tan.^2 \odot}$;

but, (*Trig*. p. 39.) $1 - \cos. w (n) = \dfrac{2 \tan.^2 \frac{w}{2}}{1 + \tan.^2 \frac{w}{2}} = \dfrac{2t^2}{1 + t^2}$,

making $t = \tan. \dfrac{w}{2}$.

Hence, $\tan. (\odot - Æ) = \dfrac{1 - n}{1 + n} . \dfrac{\sin. 2 \odot}{1 + \frac{1 - n}{1 + n} \cos. 2 \odot}$

$$= \frac{t^2 . \sin. 2 \odot}{1 + t^2 \cos. 2 \odot},$$

and thence, $\sec.^2 (\odot - Æ) = \dfrac{1 + 2t^2 \cos. 2 \odot + t^4}{(1 + t^2 \cos. 2 \odot)^2}$.

Now $d \{\tan. (\odot - Æ)\} = \dfrac{t^2 \cos. 2 \odot + t^4}{(1 + t^2 \cos. 2 \odot)^2} 2 \delta \odot$;

the symbol d denoting the differential,

but, generally, $d \text{ (arc)} = \dfrac{d \text{ (tangent)}}{\text{(secant)}^2}$;

$$\therefore d (\odot - Æ) = 2 \delta \odot \left(\frac{t^2 . (\cos. 2 \odot + t^4)}{1 + 2t^2 \cos. 2 \odot + t^4}\right).$$

Now, if we assume $2 \cos. 2 \odot = x + \dfrac{1}{x}$,

$$1 + 2t^2 \cos. 2 \odot + t^4 = (1 + t^2 x) \left(1 + \frac{t^2}{x}\right),$$

$$\text{and } \frac{1}{1 + 2t^2 \cos. 2\odot + t^4} =$$

$$\frac{1}{1 - t^4}\{1 - 2t^2 . \cos. 2\odot + 2t^4 \cos. 4\odot - \&c.\};$$

multiply each side of the equation by $t^2 \cos. 2\odot + t^4$,

$$\text{and } \frac{t^2 \cos. 2\odot + t^4}{1 + 2t^2 \cos. 2\odot + t^4}$$

$$= t^2 \cos. 2\odot - t^4 \cos. 4\odot + t^6 \cos. 6\odot - \&c.$$

∴ (see p. 501, l. 16,)

$$\int d(\odot - Æ) = t^2 \sin. 2\odot - \frac{t^4 . \sin 4\odot}{2} + \frac{t^6 . \sin. 6\odot}{3} - \&c.$$

$$\text{or } (\odot - Æ) \sin. 1'' = t^2 \sin. 2\odot - \frac{t^4 \sin. 4\odot}{2} + \frac{t^6 \sin. 6\odot}{3} - \&c.$$

or, very nearly, since $2 \sin. 1'' = \sin. 2''$, &c.

$$\odot - Æ = \tan.^2 \frac{w}{2} \frac{\sin. 2\odot}{\sin. 1''} - \tan.^4 \frac{w}{2} \frac{\sin. 4\odot}{\sin. 2''}$$

$$+ \tan.^6 \frac{w}{2} \frac{\sin. 6\odot}{\sin. 3''} - \&c.$$

In order to express the coefficients numerically, we have, assuming the obliquity equal to 23° 28′,

$$\log. \tan. \frac{w}{2}, \text{ or } \log. t = 9.3174299,$$

whence,

2 log. t	= 18.6348598
and log. sin. 1″	= 4.6855749
log. c	3.9492849
4 log. t	= 37.2697196
log sin. 2″	= 4.9866049
log. c'	2.2831147
6 log. t	= 55.9045794
log. sin. 3″	= 5.1626961
log. c''	0.7418833

we have thus the logarithms of the three coefficients c, c', c'', by means of which it is easy to compute ☉ — Æ, when ☉ is given. The logarithm of the fourth coefficient (log. c''') = 9.24518043.

Hence, the *reduction* (R) = c . sin. 2 ☉ − c' . sin. 4 ☉
+ c'' sin. 6 ☉ − c''' sin. 8 ☉ + &c.

If ☉ = 3^s, sin. 2 ☉, sin. 4 ☉, &c. = 0, and the reduction, as it plainly must, is equal to 0.

If ☉ = 45°, sin. 2 ☉ = 1, sin. 4 ☉ = 0, sin. 6 ☉ = − 1;
∴ the reduction = 8897″.85 − 5″.519 = 8892″.33
= 2° 28′ 12″.33,

and consequently, the right ascension (Æ = ☉ − R) = 42° 31′ 47″.67, or, expressed in time, Æ = 2^h 50^m 7^s.17.

If ☉ = 10°, 2 ☉ = 20°, 4 ☉ = 40°, 6 ☉ = 60°, and, accordingly, we have the following computation,

log. sin. 20 . .	9.5340517		
log. c	3.9492849		
	3.4833366,	No. 3043.24	
log. sin. 40 . .	9.8080675		
log. c'	2.2831147		
	2.0911822,	No.	123″.36
log. sin. 60 . .	9.9375306		
log. c''	.7418333		
	.6793639,	No. 4.778	
log. sin. 80 . .	9.9933515		
log. c'''	9.2518043		
	9.2451558,	No.	.175
		3048.018	123.535
		0123.535	
	Reduction	2924.483	

Hence, the reduction $= 48' \; 44''.483$

and consequently, Æ $= 9^0 \; 11' \; 15''.517$.

In the two former instances the terms of the *reduction* were alternately positive and negative, and the reduction itself subtractive, or the right ascension less than the longitude. The contraries of these circumstances happen in the next instance.

Let $\odot = 9^s \; 5^0 \; 40'$, then

$2 \odot = 18^s \; 11^0 \; 20'$, $\quad$ sin. $2 \odot = -$ sin. $11^0 \; 20'$
$4 \odot = 36 \; 22 \; 40$, $\quad$ sin. $4 \odot =$ sin. $22 \; 40$
$6 \odot = 57 \; 4 \; 0$, $\quad$ sin. $6 \odot = -$ sin. $34 \; 0$
$8 \odot = 73 \; 15 \; 20$, $\quad$ sin. $8 \odot =$ sin. $45 \; 20$

Now,

sin. $11^\circ \; 20'$	9.2933995	
log. c	3.9492849	
	3.2426844	$1748''.55$
sin. $22^\circ \; 40'$	9.5858771	
log. c'	2.2831147	
	1.8689918	73.96
sin. $34^\circ \; 0'$	9.7475617	
log. c''	.7418833	
	.4894450	3.86
sin. $45^0 \; 20'$	9.8519970	
log. c'''........	9.2518043	
	9.1038013	.127
		1826.497

Hence, the *reduction* ($\odot$ − Æ) $= - \; 30' \; 26''.497$,

and consequently, Æ $= 9^s \; 5^0 \; 40' + 30' \; 26''.5$, nearly,
$= 9^s \; 6^0 \; 10' \; 26''.5$

and, in time, $= 18^h \; 24^m \; 41^s.7$.

In the Nautical Almanack for 1775, we have very nearly, this result, since,

$$\text{Dec. } 27, \quad \odot = 9^s\ 5^0\ 39'\ 59'',$$
$$Æ = 18^h\ 24^m\ 41^s.6;$$

but besides the difference of $1''$, between the above longitude and the longitude used in our example, the obliquities are slightly different. On December 27, 1775, the obliquity was $23^\circ\ 27'\ 59''.7$, whereas in the preceding instance it was assumed equal to $23^0\ 28'$.

The *correction* in the above, and in like instances, corresponding to any change in the obliquity is easily obtained: thus, since

$$\odot - Æ = \tan.^2 \frac{w}{2} \cdot \frac{\sin. 2\ \odot}{\sin. 1''} - \&c.$$

$$\delta(\odot - Æ) = \delta w \cdot \tan. \frac{w}{2} \sec.^2 \frac{w}{2} \cdot \sin. 2\ \odot - \&c.$$

which first term will be sufficient.

The Tables of *reduction* (see Zach's Tab. XXI. of his *Tabulæ Motuum Solis*, and Vince's *Astronomy*, Table XXXVII, vol. II.) contain a column of variations for every ten seconds of variation of obliquity.

A Table of *reductions* of the ecliptic to the equator is wanted, when, in constructing a work like the Nautical Almanack, we deduce from the Solar Tables the Sun's longitude, and from such longitude his right ascension. In examining and correcting Solar Tables, or the longitudes deduced from them, by the test of observations, corrections or *reductions* of a contrary nature are requisite. For, since the Sun's right ascension is observed, we stand in need of an easy process for *reducing* it to the longitude, or, we stand in need of a Table of the reduction of the equator to the ecliptic. We will now explain, by what artifice and rule, the preceding formula (see p. 502,) and a Table constructed from it, may be adapted to this latter purpose, since (see p. 501,)

$$\tan. Æ = \cos. w \cdot \tan. \odot,$$
$$\tan. (90^0 - \odot) = \cos. w \cdot \tan. (90^0 - Æ),$$

which equation is precisely of the same form as the preceding one of p. 501, l. 6: consequently, a similar formula must result from it, on changing what ought to be changed; that is, by writing $90^0 - \odot$ instead of Æ, and 90^o − Æ, instead of $\odot$.

Hence,

$$(90^o - Æ) - (90^o - \odot) = t^2 . \frac{\sin. (180^0 - 2 Æ)}{\sin. 1''}$$

$$- \frac{t^4 . \sin. (360^o - 4 Æ)}{\sin. 2''} + \&c.$$

$$\text{or, } \odot - Æ = t^2 \frac{\sin. 2 Æ}{\sin. 1''} + t^4 . \frac{\sin. 4 Æ}{\sin. 2''} + \&c.$$

which is the formula required, and from which, as in the former case, a Table might be constructed. But it is desirable to avail ourselves of the former Table and to adapt it to this latter purpose. In order to find the means of so adapting it, make

$$Æ = a - 90^o,$$

then,

$$\odot - Æ = t^2 . \frac{\sin. (2 a - 180^0)}{\sin. 1''} + t^4 . \frac{\sin. (4 a - 360^o)}{\sin. 2''} + \&c.$$

$$= - t^2 . \frac{\sin. 2 a}{\sin. 1''} + t^4 . \frac{\sin. 4 a}{\sin. 2''} - \&c.$$

$$= - \left(t^2 . \frac{\sin. 2 (Æ + 90^o)}{\sin. 1''} - t^4 . \frac{\sin. 4 . (Æ + 90^o)}{\sin. 2''} + \&c. \right),$$

but, in the former case, see p. 502,

$$\odot - Æ = t^2 . \frac{\sin. 2 \odot}{\sin. 1''} - t^4 . \frac{\sin. 4 \odot}{\sin. 2''} + \&c.$$

the two series then are similar. If two Tables then were constructed, the numbers in each would be the same, in every case in which Æ + 90^0 and $\odot$ should be of equal values: for instance, the number expounding the *reduction to the equator* when $\odot = 113^o\ 4'$, would expound *the reduction to the ecliptic*, when Æ $= 23^0\ 4'$. One Table then, would do instead of two. If the Table of the reduction to the equator be already computed, we may thence deduce the reduction to the ecliptic corresponding to a given right ascension, by this simple rule. *Increase the* Æ *by* 3^s *and take out from the Table the reduction belonging to the angle* 3^s + Æ: *which reduction, with its proper sign, is the reduction to the ecliptic.*

The above-mentioned Table of the *reduction of the ecliptic to the equator* * is not, it is to be noted, necessary, nor, indeed, does it abridge the work of computation. The Trigonometrical process (rating it by the number of figures,) is shorter. But the *Table* is more convenient because it is inserted, in the same volume, with other Solar Tables, and is alone sufficient to effect its purpose.

If D be the declination of the Sun, then

$$1 \times \sin. D = \sin. \odot . \sin. w,$$

accordingly, from the Sun's longitude computed from Solar Tables, and from the obliquity (the apparent) of the ecliptic, the declination may be computed: and, in point of fact, the Sun's declination inserted in the fifth column of the second page (every month) of the Nautical Almanack is so computed: not necessarily, indeed, by the Trigonometrical formula just given: since, as in the former case of the deduction of the right ascension, the declination may be expressed by a series, and, in practice, may be computed by a Table entitled '*The Declination of the Points of the Ecliptic*'. (See Vince's *Astronomy*, Table XXXVIII, vol. II, and Zach's Tab. XXIII, of his *Tabulæ Motuum Solis*).

We will now return from this digression concerning the *reduction of the ecliptic to the equator*, and similar formulæ of reduction, to the main subject of the Chapter, and which indeed

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* The *reduction of the ecliptic to the equator* has been computed from the formula of page 502. But it is plain that *reductions* of like nature, but of different denominations, may be deduced from the same formula. For instance, the longitude of Venus in her orbit may be reduced to her longitude in the ecliptic: in which case w (see p. 502,) will be expounded by the inclination of Venus's orbit (about $3^\circ\ 23'$), and the series will rapidly converge.

is first announced in its title. The subject is the correction of the Solar Tables: or the method of so applying observations, made either before or after the epoch of the computation of the Tables, or hereafter to be made, as to correct, or to make more exact, the conditions or elements of such computation; and, for the more distinctly handling of the subject, we will recapitulate the steps of the process by which the Sun's longitude is taken from the Solar Tables.

(1.) The mean longitude (M) of the Sun is taken out of the Tables.

(2.) The mean longitude of the perigee (π) is also taken from the Tables.

(3.) The difference of the mean longitude of the Sun, and of the mean longitude of the perigee, is then taken, which gives the mean anomaly (A).

(4.) To the mean anomaly thus obtained the corresponding equation (E) of the centre is sought for in the Tables.

(5.) The equation of the centre thus obtained is, according to the position of the Sun in its orbit, added to or subtracted from the Sun's mean longitude, and the result is the Sun's *elliptical* longitude.

(6.) To the last sum or difference is added the sum (P) of the several perturbations of the Moon and planets.

(7.) Lastly, the preceding result must be corrected for aberration, and the two nutations, if the true apparent longitude of the Sun be required.

Any error or errors, therefore, in the steps of this process must, according to their degrees, vitiate the exact determination of the Sun's longitude.

The mean longitude, which is taken in the first step, is not taken *immediately* from the Tables, but is found by adding to the *Epoch*, as it is called, the mean motion during the interval between the epoch and the assigned time of the required longitude.

The *epoch* (O) is the Sun's mean longitude at a certain time. For instance, the *epoch*, or the Sun's mean longitude, on the mean noon of the first of January 1752, is

$$9^s\ 10^\circ\ 31'\ 32''.2,$$

the Sun's mean longitude, therefore, on April 3, 1752, is the above longitude, or epoch, plus the Sun's mean daily motion ($59'\ 8''.33$) multiplied into 93 days, which latter product is

$$3^s\ 1^\circ\ 39'\ 54''.69,$$

so that, the Sun's mean longitude is

$$12^s\ 12^\circ\ 11'\ 26''.89,$$

that is, rejecting the 12 signs,

$$12^\circ\ 11'\ 26''.89,$$

and, if the longitude should be required at any time of the day of April 3, other than its noon, we must add to, or subtract from, the above longitude a proportional part of $59'\ 8''.33$. Thus, if the time should be April 3, $3^h\ 5^m\ 25^s$, we must add to the former longitude

$$7'\ 35''.9\ \left(= \frac{3^h\ 5^m\ 25^s}{24^h} \times 59'\ 8''.33\right),$$

so that the Sun's mean longitude will be

$$12^\circ\ 19'\ 2''.79^*.$$

We must now consider whether there is likely to be any error in the terms that compose the Sun's mean longitude.

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* The Tables from which the Sun's mean longitude, &c. are taken, are constructed for the meridian of Greenwich, but are easily adapted to any other meridian. Thus the epoch of the Sun's mean longitude for 1822, in Vince's Solar Tables, is $9^s\ 10^\circ\ 33'\ 59''.6$: Dublin Observatory (to take an instance) is $25^m\ 20^s$ west of Greenwich, and the Sun's motion in $25^m.20^s$ is equal to $\frac{25^m\ 20^s}{24^h} \times 59'\ 8''.33$, or $1'\ 1''.69$; therefore the epoch for Dublin is $9^s\ 10^\circ\ 35'\ 1''.29$.

The Sun's mean motion is, probably, known to a great degree of accuracy. For, it is determined by comparing together distant observations of the Sun's longitudes and by dividing the difference of the longitudes by the interval of time between them. Any small error, therefore, made in the Sun's longitude will, by reason of the above division, very slightly affect the determination of the Sun's mean motion.

Thus, supposing the mean motion is to be determined by comparing the observations of 1752 and 1802, and the error of Bradley's observations at the former period to have been $5''$, the corresponding error in the difference of the longitudes would amount only to $\frac{5''}{50}$, or $0''.1$.

But the case is somewhat different with the epoch. There is no part of the process in determining it that has an effect, like that we have just described, in lessening its errors. The mean longitude at any epoch, 1752 for example, must depend for its accuracy on individual observations made at that epoch, or, at the most, on the mean of such observations. The Sun's right ascension must be determined (according to the method described in Chapters VII and XVI,) and the Sun's longitude must be thence deduced. The mean longitude, therefore, of the epoch is subject to some uncertainty, and, consequently, the mean longitude of the Sun at the proposed time will be alike subject to the same. Hence, if t be the time elapsed since the epoch, and m be the Sun's mean motion, since

$$M = O + mt,$$

$$dM = dO = z.$$

Suppose, in the next step (see p. 508, l. 9,) the longitude (π) of the perigee to be taken. Now, it is plain, if we revert to pages 477, &c. that there is some uncertainty in that method, or that there may be a probable error of several seconds in the determination of its longitude: such error then will affect the mean anomaly (A), and exactly by its quantity, since

$$A = M - \pi;$$

therefore, if $+ x$ be the error in π, $- x$ will be the corresponding error in A: but (see p. 468,) the equation of the centre (E) depends on A, and, according to the value of A, will be increased or decreased by a given error in A. Now any error in the equation of the centre, will affect, with its exact quantity, the true longitude, since this latter equals $M \pm E$, the effects of planetary perturbation and of the inequalities not being considered.

This is one effect on the longitude produced by an error in the equation of the centre: which error is derived, through the mean anomaly, from the error of the longitude of the perigee. But there is a second source of error of the equation of the centre arising from an uncertainty or error in the determination of the eccentricity, or [since (see p. 473,) the *greatest* equation of the centre is expressed in terms of the eccentricity,] from an error in the greatest equation of the centre. This error, according to the value of the mean anomaly, that is, accordingly as the equation of the centre is to be added to, or subtracted from the mean longitude in the finding of the true longitude, will cause a positive or negative error in the resulting value of the true longitude.

Hence, since the true longitude, or

$$L = M \pm E + P,$$

$$\text{or} \quad = O + mt \pm E + P,$$

$$dL = dO \pm \frac{dE}{d\pi} d\pi \pm \frac{dE}{de} de.$$

Supposing P the sum of the perturbations to be rightly determined, and denoting by $\frac{dE}{d\pi} d\pi$ the error in E, arising from an error ($d\pi$) in the longitude of the perigee, and by $\frac{dE}{de} de$ the error in E, arising from the error in the eccentricity.

What now remains to be done is to find the means of stating these variations (dO, $d\pi$, dE) under a form fitted for arithmetical computation. The error dO may be (see p. 510,) expressed by z, since if z ($5''$ for instance,) be the error in the mean longitude, z

($5''$) will be the corresponding error in the true longitude.

$$\text{Next, } \frac{dE}{d\pi}\,d\pi, \text{ or } x,$$

affects the longitude by altering, through the mean anomaly, the equation of the centre. Since (see p. 473,) we have an expression for the equation of the centre in terms of the mean anomaly, we can find the error in the former corresponding to a given error in the latter: but it is most convenient, for such purpose, to use the Tables already constructed. Suppose then [for it is necessary (see p. 511,) to take an instance] the mean anomaly to be $6^s\ 18^0$; we find in the Solar Tables,

anomaly	$6^s\ 18^0\ 0'$,	equation of centre	$0^0\ 35'\ 43''.4$
anomaly	$6\ \ 18\ \ 10$,	equation	$0\ \ 36\ \ \ 2.2$
	$0\ \ \ 0\ \ 10$		$0\ \ \ 0\ \ 18.8$

Hence, to a variation of $1'$ in the anomaly, there corresponds $1''.88$ in the equation, and, accordingly,

$$60'' : x :: 1''.88 : x \times \frac{18.8}{60} = .0313\,x.$$

We may make a like use of the Solar Tables in finding the numerical value of $\frac{dE}{de}\,de$. If the eccentricity be changed, the greatest equation of the centre is changed. Now in the Solar Tables the *secular* variation of the greatest equation (when the anomaly is of a certain value) is supposed to be $17''.18$, and corresponding to such a variation, the proportional secular variation of the equation of the centre, corresponding to a mean anomaly $= 6^s\ 18^0$, is $5''.15$.

Hence, if y be the variation or error of the greatest equation,

$$17''.18 : 5''.15 :: y : \frac{5''.15}{17''.18} \times y = .2969\,y,$$

which is the corresponding error in the equation of the centre belonging to an anomaly of $6^s\ 18^0$: we have now then, in this instance,

$$dL = z + .0313\,x - .2969\,y,$$

dL is an error of the *computed* longitude arising from errors in the epoch, the place of the perigee and the value of the greatest equation. In order to find its value we must compare the computed with the observed longitude (or rather the longitude computed from an observed right ascension and the obliquity of the ecliptic): the difference of the two longitudes, on the supposition of the exactness of the latter, is dL or C, then

$$C = z + .0313\,x - .2969\,y,$$

and in order to determine z, x and y, there is need of two other similar equations.

In page 482, from observations of the Sun's right ascension and the obliquity of the ecliptic, the Sun's longitude was found equal to

$$3^s\ 9^0\ 6'\ 29''.1\,;$$

whereas, in the Nautical Almanack, the computed longitude is

$$3^s\ 9^0\ 6'\ 43''$$

the error of the Tables, then, or C is $13''.9$.

In the instance we have given, the anomaly was assumed equal to $6^s\ 18^\circ$, and the Solar Tables were, on grounds of convenience, made use of to determine the coefficients of y and z. That was effected by merely taking from the Tables the secular variation corresponding to the given anomaly, or to the corresponding equation of the centre, and the difference or variation of the equation of the centre corresponding to a difference of ten minutes in the anomaly. It is plain, then, the coefficient of y will be the greater, the greater is the secular variation, which is the greater the nearer the proposed anomaly is to that anomaly to which the *greatest* equation of the centre corresponds. Now the greatest equation of the centre happens (see p. 472,) in points near to those of the mean distances. The Sun is at his mean distance in March and September. Hence, if we select from observations those made towards the latter ends of those months, and derive equations similar to the above, the coefficients of y will be, nearly, as great as they can be. The contrary will happen, in such observations, to the coefficients of x: since these depend on the *variation* of the equation of the centre corresponding to a given variation of the mean anomaly, they must needs be the smallest when the former variation is at its least; which happens near to the mean distances, when the equation of the centre is at its maximum. The reverse of this whole case will happen if we select observations made near to the apogee and perigee, the *secular* variation * of the equation of the centre is then the least: but the variation of the equation of the centre, corresponding to a given variation of the anomaly, is the greatest. The coefficients, therefore, of x, in this case, will be as great as they can be, and those of y as small. Hence, if we possess a long series of observations, we have it in our power so to use them, that in the derived equations (such as that of p. 512,) the coefficients of x and y shall be, respectively, as large as possible.

For instance, on March 24, 1775, the Sun's mean anomaly, as it appears by the Tables, was

$$2^s\ 22^\circ\ 42'\ 44''.7.$$

The secular variation is $17''.12$, the *difference* $2''.2$; therefore (see p. 512,) the coefficient of $y = \frac{17.12}{17.18}$ $(= .9965)$, of $x = \frac{2''.2}{60} = .0366$, consequently, if the *error* of the Tables (the difference of the computed and observed longitude) were $-1''.7$, we should have

$$-1''.7 = z + .9965\,y - .0366\,x.$$

Again, (about half a year afterwards, the Sun being again near his mean distance) on September 23, we find anomaly $8^s\ 23^0\ 4'$, secular variation $= 16''.95$, *difference* $= 2''.9$;

* The *secular* variation of the greatest equation of the centre is its variation, (arising from a change in the eccentricity of the orbit) in one hundred years. Its present value is $17''.18$, and whenever the *greatest* equation is changed, every other equation of the centre is changed. If $15''$, or $17''.18$ be the change in the former, there will be, in every case, a proportional and calculable change in the latter. But it is *convenient* to use the change $17''.18$ (denominated for the reasons above specified the *secular*) because, in the Solar Tables, we find the proportional change affixed to every equation of the centre.

therefore the coefficients of y and x are $\frac{16.95}{17.18}$, $\frac{.29}{6}$, and, if the error of the Tables were $5''.4$, we should have this equation

$$5''.4 = z - .9866\, y + .0483\, x,$$

and if we selected fifty observations, half made near to the end of March, the other half near to the end of September, the former would all resemble the first equation, the latter the second; in each the coefficient of y must be large, but in the former the coefficient must be positive, in the latter negative, since, when the mean anomaly is about $2^s\ 20^\circ$, the equation of the centre is additive, when about $8^s\ 20^\circ$, subtractive.

In like manner if we select two observations made near the apsides, on June 25, and December 28, 1784, we have

June 25, anomaly $5^s\ 25^0\ 1'\ 33''$, secular var^n. $1''.44$. *diff.* $19''.6$
Dec. 28, ... 11 28 20 51.6 . . . . 0.5 . . . . 20.5

and accordingly, the coefficients of x and y are

$$\frac{1''.44}{17.18},\ \frac{0''.5}{17.18} \text{ and } \frac{1''.96}{6},\ \frac{2.05}{6},$$

and the two resulting equations, if the errors of the Tables be, respectively, $-\ 3''.4$, $-\ 1''.5$

$$-\ 3''.4 = z + .0838\, y + .3266\, x$$
$$-\ 1''.5 = z - .02913\, y - .3416\, x,$$

and, in all pairs of equations so derived (from observations made near to the apsides and distant from each other by about six signs) the coefficients of x will be as large, as they well can be, and the coefficients of y, as in the former pairs of equations, will be respectively positive and negative.

Suppose then, we had, in all, one hundred equations, fifty derived from observations near the mean distances, fifty from observations near the apsides, and that we added the one hundred equations together: then the coefficient of z would be one hundred, and the coefficients of y and x would be the excesses of the positive coefficients, in the several equations, above the negative: the equation divided by one hundred would be of this form,

$$A = z - ay + bx \ldots\ldots\ldots\ldots (1).$$

In order to obtain a second equation, take the fifty equations derived from observations near to the mean distances, then twenty-five of these equations (see p. 515,) must be of the form,

$$-1''.7 = z + .9965\,y - .0366\,x,$$

twenty-five of the form $5''.4 = z - .9866\,y + .0483\,x$,

change the signs in every one of the latter twenty-five, then there will be twenty-five equations such as

$$-1''.7 = z + .9965\,y - .0366\,x,$$

twenty-five, such as $-5''.4 = -z + .9866\,y - .0483\,x$.

Add now the whole fifty together and the z's will disappear; the coefficient of y will be the *sum* of such quantities as .9965, .9866, &c. the coefficient of x will be result of combining several positive and negative quantities: the resulting equation divided by the sum of .9965, .9866, &c. will be of the form

$$B = y - mx \ldots\ldots\ldots\ldots (2).$$

Proceed in like manner with the fifty equations derived from observations made near to the apsides: that is, since the object is to make the coefficient of x, in the resulting equation, as large as possible, make the coefficients of x, in all equations, such as the one of p. 315, l. 20, positive, by thus writing it,

$$1''.5 = -z + .02913\,y + .3416\,x,$$

then, in all the fifty equations, the coefficients of x will be positive: add together the fifty equations, and the coefficient of x will be the sum of fifty quantities such as .3266, .3416, &c. and the coefficients of y and z will be the *differences* of certain quantities: divide by the coefficient of x, and the resulting equation will be of the form

$$C = pz + qy + x \ldots\ldots\ldots\ldots (3).$$

And it is from these three equations $\{(1), (2), (3),\}$ that the values of x, y, z, are to be derived by elimination.

The principle in the above process of combining sets of equations in order to produce a mean equation is obvious: if x, or y, or z is to be determined, the larger its coefficient the more exact will be its resulting mean value.

In what has preceded, we have, in substance, followed Delambre's method in the Memoirs of the Academy of Berlin for 1786. In these Memoirs, which are on *the Elements of the Solar Orbit*, one hundred equations are used, fifty from observations of the Sun near his mean distances, fifty from observations of the Sun at his greatest and least distance. The results (see *Mem. Acad. Berlin*, 1786, p. 243,) of M. Delambre, are

correction of the epoch .... $= - 0''.4092$,
of the longitude of the apogee ..... $= - 24''.71$,
of the greatest equation of the centre .. $+ 0''.3227$,

which corrections are to be applied to Mayer's Tables, with which Delambre compared Maskelyne's Observations.

By means such as we have described, Mayer's Tables were corrected. The errors of the *corrected* Tables were found not to exceed $9''$. The sum of the hundred errors (of the positive and negative together,) amounted to $318''.3$, and, therefore, the mean error was $3''.183$, which, as the learned author remarks *, is, considering all circumstances, a very small error.

The method of correcting *at one operation* all the elements is what is now generally practised. But, in a preceding volume of the Berlin Memoirs (for 1785,) Delambre corrects the elements individually, by the comparison of particular observations with the results obtained from the Solar Tables. Thus, suppose the longitude of the apogee, or the longitude of the Sun occupying the apogee, to be found, on June 29, at $22^h\ 37^m\ 37^s$ to be

* "Et si l'on se rappele que ces erreurs si peu considerables sont pourtant produites par trois à quatre causes differentes, comme les erreurs des observations, celles des reductions, celles des catalogues d'ètoiles, enfin les quantities negligés ou peu connues dans la theorie, on s'etonnera peut-être que les Geometres et les Astronomes aient pu les renfermer entre des limites aussi etroites, et l'on ne pourra gueres se flatter d'ajouter beaucoup à une parelle precision."

$3^s\ 9^0\ 3'\ 8''$. But, according to Mayer's Tables, the longitude of the apogee was $3^s\ 9^0\ 6'\ 43''$, therefore $-\ 3'\ 35''$ was the correction of such longitude. The same observation corrects also the *mean longitude* of the Tables: for, at the apogee, the mean and true longitudes are the same. The mean longitude, therefore, was $3^s\ 9^0\ 3'\ 8''$: but the Tables gave $3^s\ 9^0\ 3'\ 20''$. The correction, therefore, for the epoch of the Tables, according to the above observation, was $-\ 12''$.

But, whichever be the method employed, it is essential to its accuracy that all the sources of inequality by which the Sun's true longitude is made to differ from its mean, should be known: for, otherwise, the longitude of the apogee, or the equation of the centre, might be wrongly corrected. Before the discoveries of Newton, for instance, those differences of the observed and computed longitudes which are due to planetary perturbation, would, from ignorance of their causes, have been attributed to errors in the epoch, equation of the centre, and longitude of the perigee; and, had such a method of correcting those errors been used as has been already (see pp. 512, &c.) described, its results would have given wrong corrections.

It needs scarcely be observed that the assigning of the laws and quantities of the perturbations caused by the planets is a difficult operation. The *arguments* (see *Physical Astron.* Chap. VII, &c.) may be derived from theory, but their coefficients must be determined from observations. M. Delambre has accomplished these objects, by the comparison of 314 of Maskelyne's observations, and by Laplace's Formulæ. The learned Astronomer in his first correction of the Solar Tables reduced their errors within $15''$, whilst the errors of Mayer's Tables sometimes exceeded $23''$. But, as he found that the computed and observed longitudes could not be brought nearer to each other, and as their differences did not follow a regular course (in which case they might have been, in part, attributable to the errors of observation) he suspected that the solar theory was in fault, or rather, that part of it which assigns the correction of the Sun's elliptical place on account of the perturbations of the planets. In this emergency he had recourse to Laplace, who, from his Theory, derived two equations due to

Mars' action, the sum of which might amount to $6''.7$: the same great mathematician also assigned $6''$ for the value of the principal term of the lunar equation, and $9''.7$ for the maximum of the equation of Venus.

There are also some other points to be attended to in the correction of the Solar Tables: for instance, the value of the obliquity of the ecliptic. For the *observed longitudes* with which the longitudes derived from the Solar Tables are compared, are, in fact, (see p. 513,) computed from the observed right ascension and the obliquity of the ecliptic, and, therefore, their accuracy depends, in part, on that of the obliquity.

In the deduction of the equations of condition, the coefficients of x and y (see pp. 512, &c.) were obtained by the aid of Solar Tables: an operation, as we then stated, of mere convenience and in nowise essential. If we had not been able to avail ourselves of Tables, we should then have been obliged to have gone back to the very formulæ used in constructing the Tables. And this indeed, but with some loss of expedition, would have been the most scientific proceeding.

We subjoin these formulæ, some of which have been already given.

If e be the eccentricity, and E the greatest equation,

$$e = \frac{1}{2} E \sin. 1'' - \frac{11}{768} E^3 \sin.^3 1'' - \frac{587}{983040} E^5 \sin.^5 1''$$

$$- \frac{40583}{2642411520} E^7 \sin.^7 1'' - \&c.$$

If $E = 1^0\ 55'\ 26''.82$ (its value in 1780) $e = 0.016790543$.

If $Z = nt$, be the mean anomaly, the equation of the centre is equal to

$$- 1^\circ\ 55'\ 26''.352 \sin. Z + 1'\ 12''.679 \sin. 2\, Z - 1''.0575 \sin. 3\, Z$$
$$+ 0''.018 \sin. 4\, Z,$$

and the true anomaly (a) is equal to $Z - 1^\circ\ 55'\ 26''.352 \sin. Z + \&c.$ and the *differential* of the true anomaly, or da is equal to

$$dZ - dZ \,.\, \sin. 1'' \times 1^{0}\ 55'\ 26''.352 \,.\, \cos. Z$$
$$+ 2 dZ \,.\, \sin. 1'' \times 1'\ 12''.679 \cos 2Z - \&c.$$

let $dZ = \frac{1}{24}(59'\ 8''.2) = 2'\ 27''.8416$ the Sun's * mean horary *anomalistic* motion : da is the Sun's elliptical horary motion, and
$da = 2'\ 27''.8416 - 4''.9645 \cos. Z + 0''.1042 \cos. 2Z - 0.002 \cos. 3Z.$

In order to obtain the horary motion in longitude on the ecliptic, we must, since $\frac{1}{24}(59'\ 8''.33) = 2'\ 27''.8471$, write in the above value of da, this latter quantity instead of $2'\ 27''.8416$.

If v be the Sun's true anomaly, $Z - v$ is the equation of the centre, and the greatest value of $(Z - v)$

$$= \left(2e + \frac{11}{48}e^3 + \frac{599}{5120}e^5 + \frac{17219}{229376}e^7 + \&c.\right)\frac{1}{\sin. 1''},$$

and (v)

$$= 90^0 - \left(\frac{3}{4}e + \frac{21}{128}e^3 + \frac{3409}{40960}e^5 + \frac{99875}{1835008}e^7 + \&c.\right)\frac{1}{\sin. 1''},$$

and the sum of these two equations gives that value (Z) of the mean anomaly to which the greatest equation belongs, and, accordingly,

$$(Z) = 90^\circ + \left(\frac{5}{4}e + \frac{25}{384}e^3 + \frac{1383}{40960}e^5 + \frac{39877}{256 \times 7168}e^7\right)\frac{1}{\sin. 1''}.$$

If we neglect the terms beyond the second, we have

$$(Z) = 90^0 + \frac{5}{4}\,\frac{e}{\sin. 1''} = 91^0\ 12'\ 9''.5\,\dagger,$$

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* The time and Sun's motion being dated from the perigee, and the perigee being progressive (see p. 486,) at the annual rate of 62″, the horary motion is that same portion of 360° which 1 hour is of the time of the Sun's leaving his perigee, to his return to the same : which time is an anomalistic year.

† Log. $5e = 8.9240351$
log. sin. $1'' = 4.6855749$
$4.2384602 = \log. 4329''.5 = \log. 1^\circ\ 12'\ 9''.5.$

Now,

in the solar orbit, in which, at the epoch of 1780,

$$e = .016790543.$$

Since, in the Earth's orbit (e^4, e^5, &c. being extremely small),

$$E = -\left(2e - \frac{e^3}{2^2}\right)\text{ sin. } Z + \frac{5}{2^2}e^2 \text{ sin. } 2Z - \frac{13\,e^3}{2^2.3}\text{ sin. } 3Z,$$

$$dE = -de\left((2 - .75\,e^2)\text{ sin. } Z + 2.5\,e \text{ sin. } 2Z - \frac{13}{4}e^2 \text{ sin. } 3Z\right);$$

therefore, if we make dE to represent the secular variation of the greatest equation of the centre, we have

$$dE \text{ being } = 17''.18,$$

$$de = -\frac{17''.18}{2\text{ sin. }(Z) - 2.5\,e\text{ sin. }(2Z) + \frac{13}{4}e^2\text{ sin. }(3Z)},$$

(Z) being the anomaly (91° 12′ 9″.5) belonging to the greatest equation.

From this equation the secular variation of the eccentricity may be computed.

The variation of the equation of the centre is to be had from the formula of l. 5, and if, in that formula, we substitute for de the secular variation of the eccentricity, the result will be the secular variation † of the equation of the centre corresponding to the anomaly Z. By such an expression, then, we are able to dispense with the Solar Tables, or, which amounts to the same, to compute what is therein computed.

In the preceding pages of this Chapter frequent mention has been made of the *secular* variation of the eccentricity, and (which

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Now,

log.................................. 360° = 2.5563025
anomalistic year = $365^d.25971$ log. = 2.5626017

9.9937008 = 0°.9856
= 59′ 8″.16,

and $\frac{1}{24}$th = 2 27.84.

* Expressed by 17″.177 sin. Z − 0″.03606 sin. $2Z$ − 0″.0078 sin. $3Z$.

depends upon it) on the secular variation of the greatest equation of the centre. Now these are, as the terms themselves import them to be, the variations effected in one hundred years, and the terms are never applied except to the changes that happen in quantities nearly constant. The method of determining their values, is, in fact, contained in that process (see pp. 511, &c.) by which the elements themselves are determined. Thus, with regard to the greatest equation of the centre, its value ought first to be corrected by comparing the observed longitudes of 1752, for instance, with the computed longitudes. In a second operation, by comparing, for instance, the observed longitudes of 1802, with the computed. The result of each operation would be a corrected value of the greatest equation of the centre. The difference between such values would be the *variation* in fifty years, or would be half the *secular variation*.

There is a method *, other than what has been given, for correcting the elements : it consists in making the sum of the squares of equations like (1), (2), (3), (see p. 515,) a minimum : for instance, using, for illustration, the equations obtained in pp. 515, 516, we should have

$$(1''.7+z+.9965\,y-.0366\,x)^2+(-5''.4+z-.9866\,y+.0483\,x)^2$$
$$+(3''.4+z+.0838\,y+.3266\,x)^2+\&c.=\text{a minimum},$$

and, accordingly, making y to vary,

$$\begin{gathered}.9965\,(1''.7+z+.9965\,y-.0366\,x)\\ -.9866\,(-5''.4+z-.9866\,y+.0483\,x)\\ +\&c.=0.\end{gathered}$$

In like manner, make x to vary, and z to vary, and obtain similar equations : then, from the three resulting equations thus obtained, eliminate x, y and z.

We have explained what ought to be understood by the *secular* variation of an element: and there is, what is called, the *secular* motion of the Sun, which is the excess of the Sun's longitude above 36000° in 100 Julian years : a Julian year con-

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* Laplace, *Sur les Probabilités*, Chap. IV. Biot, *Phys. Astron.* tom. II. Chap. X.

sisting of 365 ¼ days. Now, by comparing together the Sun's mean longitudes at different epochs, it appears that, in 100 Julian years, or in 36525 years, the Sun's motion = $36000° \, 45' \, 45''$, accordingly, in one Julian year of $365^d \, 6^h$, the Sun's motion is $360° \, 0' \, 27''.45$, or $12^s \, 0' \, 27''.45$; accordingly,

in 1 Julian year of $365^d \, 6^h$ the Sun's motion	= $360° \;\; 0' \;\; 27''.45$
and, in 1 common year of 365	= 359 45 40.37
in a Bissextile year of 366	= 360 44 48.697

and, accordingly, to find the epochs of the Sun's mean longitude on years succeeding a given epoch, add, for common years, repeatedly, to the epoch, $11^s \, 29^0 \, 45' \, 40''.37$, and reject the 12^s, or subtract $14' \, 19''.63$.

When a Bissextile year occurs, add

$$12^s \, 44' \, 48''.697, \text{ or } 44' \, 48''.697.$$

	s	°	′	″
Thus, 1781, epoch of Sun's mean longitude	9^s	11^0	$29'$	$9''.5$
	0	0	14	19.63
epoch for 1782	9	11	14	49.87
	0	0	14	19.63
epoch for 1783	9	11	0	30.24
	0	0	14	19.63
epoch for 1784	9	10	46	10.6
1784, is a Bissextile, therefore add	0	0	44	48.697
epoch in 1785	9	11	30	59.3

Thus the epochs are successively formed: but, if we wish to deduce, at once, the epoch of 1821, for instance, from that of 1781, since in the interval of forty years* thirty-one are common,

* The year 1800 divisible by four, and, therefore, according to the common rule, a Leap year, is, however not so, but, as a *complementary* year, a common year of 365 days (see the Chapter on the Calendar).

and nine Bissextile years, we must subtract from the epoch of 1781, the difference between

$31 \times 14' \, 19''.63$, and $9 \times 44' \, 48''.697$, that is, $40' \, 50''.23$,

accordingly, since the epoch of 1781 is ...... $9^s \; 11^0 \; 29' \;\; 9''.5$

epoch of 1821.......... $9 \;\; 10 \;\; 48 \;\; 19.27$

Before we quit this subject we wish to say one word respecting the difference between the French and English Tables of the Sun. The epochs in the former are for the first of January, *mean midnight,* and the meridian of the Paris Observatory: in the latter for the first of January, *mean noon,* and the meridian of Greenwich. Now Paris is $2^0 \, 20' \, 15''$, or in time $9^m \, 21^s$ to the east of Greenwich: consequently, the interval of the two epochs, is $12^h \, 9^m \, 21^s$, in which time, the mean increase of the Sun's longitude ($59' \, 8''.33$ being the increase in a mean solar day,) is $29' \, 57''.2$: consequently, the epochs of the Sun's mean longitudes, for the same years, are greater, in the English Tables, by $29' \, 57''.2$.

The knowledge of the Sun's mean *secular* motion enables us, most correctly, to assign the length of a tropical, or equinoctial year. But this point and others connected with the subject of solar time, will be reserved for the ensuing Chapter.

CHAP. XXII.

On Mean Solar and Apparent Solar Time.—The Methods of mutually converting into each other Solar and Sidereal Time.—The Lengths of the several Kinds of Years deduced.—On the Equation of Time.

It happens with mean solar time, as it does with sidereal time. We cannot obtain their measures immediately from phenomena, but are obliged from phenomena to compute them.

The constant part, the unit, if we may so call it, of sidereal time, is the time of the Earth's rotation round its axis (see pp. 106, &c.): and such time, in our computations respecting portions of sidereal time, or of right ascensions, is supposed to remain unaltered. The phenomena made use of, are the transits of fixed stars over the meridian: but the intervals between successive transits of the sume star, are not (as it has been already explained in pp. 106, &c.) exactly equal: they are, therefore, not sidereal days, if such terms be intended to signify equal portions of absolute time.

Besides the causes that equally affect the fixed stars and the Sun, the proper motion of the latter, inequable from its proper motion in the ecliptic, and inequable by reason of the obliquity of the ecliptic, prevents the intervals between successive transits of the Sun, over the meridian, from being equal portions of solar time. We must consider then, by what means we are able to compute *mean* solar time, and to know whether or not, a clock, going equably, *keeps* mean solar time.

The Sun's motion (see p. 523,) in $365^d.25$, is $360^0\ 0'\ 27''.45$: consequently,

$$\frac{360^\circ\ 0'\ 27''.45}{365.25} = 59'\ 8''.33,$$

is the increase of the Sun's mean longitude in one day, consisting of twenty-four mean solar hours. A mean solar day, therefore, must exceed a sidereal day, by the portion of sidereal time consumed in describing $59'\ 8''.33$. Now 360^0 are described in twenty-four *sidereal* hours;

$$\therefore\ 360^0 : 24^h :: 59'\ 8''.33 : 24 \times \frac{59'\ 8''.33}{360}$$

$$= 236^s.555 = 3^m\ 56^s.555 \text{ of sidereal time:}$$

hence, twenty-four mean solar hours are equal to $24^h\ 3^m\ 56^s.555$ of sidereal time: and a clock will be adjusted to mean solar time, if its index hand makes a circuit, whilst that of the sidereal clock makes one circuit and $3^m\ 56^s.555$ over: or, if each clock beats seconds, the solar clock ought to beat 86400 times whilst the sidereal beats $86636\frac{1}{2}$, nearly.

In order to find the number of solar hours to which a sidereal day of twenty-four hours is equal, we must use this proportion,

$$86636.555 : 24 :: 86400 : 24 \times \frac{86400}{86636.555}$$

$$= 23^h.93447 = 23^h\ 56^m\ 4^s.092 \text{ of mean solar time.}$$

The difference between twenty-four hours and the last time, is $3^m\ 55^s.908$. Hence, subtract from twenty-four hours of sidereal time $3^m\ 55^s.908$, and the remainder is the number of mean solar hours, minutes, seconds, and decimals of seconds, to which twenty-four hours of sidereal time are equal.

Hence, subtract $1^m\ 57^s.954$ from twelve sidereal hours, and the remainder is their value in mean solar time; subtract $0^m\ 58^s.977$ from six sidereal hours, and the remainder is their value in mean solar hours: and these *subtracted* quantities are called the *accelerations* of the stars in mean solar time; a table of which *accelerations* might, as it is plain from what precedes, be easily formed (see Zach's Table XXVI, in his *Nouvelles Tables d'Aberration*, &c.)

By means of these latter results and the Solar Tables, we can now, from the sidereal time, find the mean solar time. Thus, suppose it were required to find the mean solar time at Greenwich, on August 20, 1821, when the corrected sidereal time by the clock was $20^h\ 42^m\ 19^s.4$.

By the Solar Tables,

Sun's epoch for 1821	9^s	10^0	$48'$	$19^s.2$
mean motion to August 20, ...	7	17	41	4.2
mean longitude of Sun on Aug. 20,	16	28	29	23.4

Reject 12^s, and convert the remainder into time, and

$4^s\ 28^0\ 29'\ 23''.4$ .... =	9^h	53^m	$57^s.54$
now equation of equinoxes (see p. 376,)	0	0	0.47
*Sun's mean longitude on the meridian at Greenwich on August 20, 1821, ...	9	53	58.01
but true sidereal time	20	42	19.4
diff. of Æ between Sun and the point of* the heavens on the meridian	10	48	21.39
subtract (see p. 526,) the acceleration, or	0	1	46.216
mean solar time when the sidereal time was $20^h\ 42^m\ 19^s.4$	10	46	35.17

Now one use of this operation (the conversion of time shewn by the transit of a star, or by the sidereal clock, into mean solar time) is the correction, or the means of ascertaining the rate, of chronometers. For instance, in the above case, if the chronometer, at the instant the sidereal time was noted, should mark

* The *corrected* time shewn by the sidereal clock, is technically called the *Right Ascension of the Mid-Heaven.* By means of the transits of known stars, the error and rate of the clock (see pp. 104, 105, &c.) are determined. The clock so corrected, must shew at every point of time, during the sidereal day, the right ascension of a star, (should there be any one) or of a point in the heavens then on the meridian.

* Viz.: of the mean ☉ from ♈ [illegible] mean equinox.

$$\begin{array}{rl} 11^h\ 10^m\ 11^s & \text{of mean solar time, since} \\ \text{(see p. 527, l. 19,)}\ \underline{10\ \ 46\ \ 35.17} & \text{was the true mean solar time,} \\ 0\ \ 23\ \ 35.83 & \text{would be the chronometer's error.} \end{array}$$

If, on the next day, by similar observations and computations,

$$\begin{array}{rl} 11^h\ 12^m\ 13^s & \text{should be the watch's time,} \\ \underline{10\ \ 48\ \ 38.5} & \text{the true mean solar time,} \\ 0\ \ 23\ \ 34.5 & \text{the error.} \end{array}$$

Hence, the watch would be $23^m\ 35^s.83$ too fast the first day, $23^m\ 34^s.5$ too fast the second day, accordingly, in the twenty-four mean solar hours the watch would have lost, nearly, $1^s.33$, or, as far as these two observations shewed, its daily rate would be $-\ 1^s.33$.

In illustrating the use of finding, by the Solar Tables and the sidereal clock, the mean solar time, we have supposed the place of observation to be Greenwich, for which our present Solar Tables (those inserted in the third Volume of Vince's *Astronomy*) are constructed. For any other place of observation, (Dublin Observatory, for instance) we must, in computing the Sun's longitude from the Solar Tables, allow for the difference of the longitudes of the two observations of Greenwich and Dublin. That difference, in time, is $25^m\ 20^s$, and the increase of the Sun's longitude in that time is

$$\frac{25^m\ 20^s}{24^h} \times 59'\ 8''.33 = 4^s.15 \text{ in time,}$$

consequently, we must add $4^s.15$ to the Sun's mean longitude expressed in p. 527, l. 13, which will so become

$$9^h\ 54^m\ 2^s.16.$$

The secular motion of the Sun affords, as it was hinted at the end of the last Chapter, a good method of determining the length of the equinoctial year. Thus, in 36500 days the Sun describes $1200^s\ 0^0\ 45'\ 45''$: but in one hundred equinoctial years the Sun describes only 1200^s: consequently,

$$100 \text{ equinoctial years} = \frac{1200^s}{1200^s\ 0^\circ\ 45'\ 45''} \times 36500^d$$

$$= 36524^d.226396593684,$$

consequently,

$$\text{a mean equinoctial year} = 365^d.242264, \text{ nearly,}$$
$$= 365^d\ 5^h\ 48^m\ 51^s.6.$$

We may hence deduce a sidereal year. In this year a complete circle of 360° is described, whereas, in the equinoctial year, an angle equal to $360^\circ - 50''.1$ (supposing $50''.1$ to be the precession) is described.

Hence,

$$359^\circ\ 59'\ 9''.9 : 360^\circ :: 365^d\ 5^h\ 48^m\ 51^s.6 : 365^d\ 6^h\ 9^m\ 11^s.5,$$

the length of a sidereal year exceeding the equinoctial by $20^m\ 19^s.9$. This is the kind of year which Kepler's Law speaks of (see p. 455.).

The anomalistic year is the period from apogee to apogee. The *progression* of the apogee (its increase of longitude) being $11''.8$, the anomalistic year is completed when the Sun has described $360^\circ\ 0'\ 11''.8$.

Hence, its length

$$= \frac{360^\circ\ 0'\ 11''.8}{360} \times 365^d\ 6^h\ 9^m\ 11^s.5 = 365^d\ 6^h\ 13^m\ 58^s.8,$$

longer than the sidereal by $4^m\ 47^s.3$ and longer than the equinoctial by $25^m\ 7^s.2$.

The use of the anomalistic year consists, as we have seen in p. 477, in finding the exact place of the apogee. The horary motion which we computed at p. 519, is a portion of the *anomalistic* motion.

By means of the preceding results it is easy to convert one species of time into another, and to assign the number of degrees, minutes, &c. which the Sun and a star will respectively describe in a specified portion of sidereal time, or in an equivalent portion of mean solar time. For instance, the Sun describes an entire revolution of 360° in $24^h\ 3^m\ 56^s.5554$ of sidereal time. In one mean solar day the motion of the sphere, or of a star, is $360^{\circ}\ 59'\ 8''.33$, consequently, a star, in one mean solar hour, describes

$$\frac{360^{0}\ 59'\ 8''.33}{24} = 15^{0}\ 2'\ 27''.84708.$$

But hitherto no method has been given of converting either sidereal, or mean solar time, into apparent time, or of computing, from the instants of apparent time, (which instants, as we shall see, are marked by phenomena) the corresponding *mean solar* times and sidereal times.

In apparent solar time, the term *day* means the interval between two successive transits of the Sun over the meridian: which interval (see pp. 431, &c.) is a variable quantity *. There cannot, therefore, be any simple rule for converting apparent solar time into mean: since there cannot be a constant proportion between the two, as there is between sidereal and mean solar time.

The *correction* then to be applied to apparent time, in order to reduce it to mean time, is a variable correction: not to be expressed by a simple term, but by several variable terms that respectively expound the several causes that render inequable, the Sun's motion in right ascension.

This correction, or *equation*, by which apparent time is made equal to mean time, is technically called the *Equation of Time*: and our present concern is with the method of computing it.

For the purpose of elucidating such method, and of guiding us in it, let us feign mean solar time to be measured by a fictitious Sun, moving equably in the equator, with the real Sun's mean motion in right ascension, and consequently, (see p. 526,) at the rate of $59'\ 8''.33$, in twenty-four mean solar hours.

If this motion begin to be dated from the first point of Aries, the right ascension of the fictitious Sun, after an interval of time

* Not only variable according to the time of the year, but, in strictness, variable on the same days of civil reckoning at different places.

equal to t, will be equal to $59' \, 8''.33 \times t$. The right ascension of the real Sun depends upon, or may be computed from, his true longitude, and the true obliquity of the ecliptic, of which latter computation we have given instances in pp. 504, &c. In each case, the reckoning is made from the first point of Aries, and the equable regression of that point is taken account of, when $59' \, 8''.33$ is assigned as the mean increase of the Sun's right ascension in a mean solar day.

In the above case then (that of the equable retrogradation of the equinoctial point), the difference between mean solar time and apparent time, or the equation of time, is equal to the difference between the true right ascension of the real Sun, and the right ascension of the fictitious Sun, or, which is the same thing, between the true right ascension of the Sun, and his mean longitude.

But let us suppose, which indeed is the case, that the equable retrogradation of the equinoctial point is disturbed by a displacement of the pole of the equator (and consequently of the equator itself) such as is caused by *nutation*: then the longitude of the real Sun, and the right ascension of the fictitious Sun describing the equator will both be altered. The right ascension of the latter will no longer be

$$59' \, 8''.33 \times t, \text{ but } 59' \, 8''.33 \times t \pm \text{♈} \, \text{♈}' \times \text{cos. obliquity},$$

(see the figure of p. 357, in which ♈ ♈′ represents the effect of nutation) whilst the longitude of the Sun, no longer measured from ♈ but from ♈′ will be affected with the whole quantity ♈ ♈′. But, wherever the point ♈ be, the true longitude is always measured from it, and from such true longitude the right ascension must be computed. In this latter case, then, the equation of time is the difference of the Sun's right ascension, and of his mean longitude ($59' \, 8''.33 \times t$) $\pm$ ♈ ♈′. cos. obliquity. But this last term (♈ ♈′ cos. obliquity) is the nutation in right ascension of a star in the equator, or, technically, is the *equation of the equinoxes in right ascension;* if, therefore, we use this latter term, *The equation of time is the difference of the Sun's true right ascension, and of his mean longitude corrected by the equation of the equinoxes in right ascension.*

The *equation of the equinoxes in longitude* (the effect of nutation on the Sun's longitude) is (see p. 376,) $= 18''.034 . \sin. ☊$: the equation of the equinoxes in right ascension, (the effect of nutation on the right ascension of the fictitious Sun, which is supposed to describe the equator) is

$18''.034 . \sin. ☊ . \sin.$ obliquity $=$ (see p. 375,) $16''.544 . \sin. ☊$,.

Hence, if

S represent the Sun's true longitude,
A his true right ascension,
M his mean longitude,
E the equation of the centre,
R (see p. 501,) the reduction to the ecliptic,
P (see p. 511,) the effect of the several planetary perturbations,

$$S = M + E + P + 18''.034 . \sin. ☊ ,$$

and, $A = S \mp R = M + E + P \mp R + 18''.034 \sin. ☊$,

but the Æ (A') of the fictitious Sun $= M + 16''.544 \sin. ☊$;

$\therefore A - A'$ (the equation of time) $= E + P \mp R + 1''.49 . \sin. ☊$,

and, expressed in time,

$$\text{the equation of time} = \frac{E + P \mp R}{15} + 0^s.0993 . \sin. ☊ .$$

The cosine of the obliquity (cos. $23^0\ 28'$) is, nearly, equal to $\frac{9173}{10000} = \frac{11}{12}$. Hence, since the equation of time is equal to the Sun's true right ascension, diminished by his mean longitude and the equation of equinoxes in right ascension, we have

$$\text{the equation of time} = A - M \mp 18''.034 . \sin. ☊ \times \frac{11}{12},$$

which, essentially, is the form under which Dr. Maskelyne expressed the equation of time (see *Phil. Trans.* 1764).

Since, the right ascension is derived from the true longitude, which itself depends, in part, on the effect of the planetary perturbations, we cannot, without the aid of Physical Astronomy, compute the equation of time. For such a reason the Astronomers, who lived previously to Newton, were unable to compute it. They could indeed *nearly* assign its value, since the Earth is not considerably disturbed by the action of the planets.

The Solar Tables, of the present day, enable us to compute the effect of the planetary perturbations. They, in fact, assign the Sun's true longitude, when such perturbations are taken account of. They enable us, then, (although this is not the most convenient mode) to compute the equation of time.

Thus, on March 12, 1822,

Sun's mean longitude	11^s	19^0	$33'$	$43''.2$
longitude of perigee	9	9	50	54.9
mean anomaly	2	9	42	48.3
(see p. 468,); ∴ equation of centre (E)	0	1	48	18.2
sum of perturbations (P)	0	0	0	22.18
(see pp. 501, &c.) reduction (R).....	0	0	42	13.3
$E + P + R$	0	2	30	53.68

Hence, the equation of time (see p. 532,)

$$= \frac{2^\circ\ 30'\ 53''.68}{15} + .0993 \text{ sin. } \Omega\ ;$$

$$\text{but, } \Omega = 10^s\ 23^0\ 54' \text{ and sin. } \Omega = -\ .5891\ ;$$

$$\therefore \text{ the equation of time} = 10^m\ 3^s.57 - 0^s.058$$

$$= 10^m\ 3^s.5, \text{ nearly *}.$$

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* The equation of time may be computed from an observed right ascension of the Sun, and from the Sun's mean longitude known from the Tables. For instance, by observations (reduced observations) at Greenwich, June 11, 1787,

	By Clock.	By Cat. (see pp. 371, &c.)	Diff.
Æ of Sun's centre	$5^h\ 17^m\ 9^s.6$		
of Procyon	7 27 15.58	$7^h\ 28^m\ 9^s.820$	$54^s.24$
of β Pollux	7 31 22.82	7 32 17.069	54.249

daily

In the above example, *E*, &c. were computed to a mean anomaly belonging to the *mean* noon of March 12, whereas, in strictness, the computations ought to have been for the *apparent* noon of that day. In other words, since the equation of time is nearly $10^m\ 3^s$, the Sun's true longitude ought to have been computed from the Solar Tables (which are constructed for mean time) for March 11, $23^h\ 49^m\ 57^s$ of mean solar time; since such is nearly the time of apparent noon, on March 12; and the equation of time, on the apparent noon of March 12, is the difference of the Sun's true right ascension at that time and of his mean longitude (corrected by the equation of equinoxes in right ascension) at the same time. The result of the computation, however, thus conducted, will differ, very slightly, from that which has been just obtained.

The equations of time are set down in the Nautical Almanack, and in the foreign Ephemerides, for every day of the year.

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daily rate of clock $0^s.84$; therefore, at the time of the transit of the Sun's centre, the error of the clock was

54.245 − 0.07	0^h	0^m	$54^s.17$
Sun's transit	5	17	9.6
therefore Sun's right ascension	5	18	3.77

Again, Sun's mean longitude 1787,	9^s	11°	$2'$	$20''$			
motion to June 11, ...	5	8	41	21.1			
	2	19	43	41.1	5^h	18^m	$54^s.74$
equation of equinoxes in right ascension					0	0	1.08
					5	18	55.82
					5	18	3.77
					0	0	52.05

The difference then of the true right ascension of the Sun, and of the Sun's mean longitude corrected by the equation of equinoxes in right ascension, on the *mean* noon of June 11, 1787, (for the Tables are constructed for mean time) was $52^s.05$, true or apparent time preceding mean. The mean longitude then at the time of observation, or on true noon, was less by the increase of the mean longitude during $52^s.05$, or by $0^s.142$: consequently, the equation of time was $52^s.05 - 0^s.142$, or $51^s.91$.

They enable us to convert apparent solar time into mean and sidereal time, and also, which is the reverse operation, sidereal time into apparent solar time. We will give some instances of these operations taken from M. Zach.

Example I.

Sidereal Time converted into Mean Solar Time, and true Time. Place of Observation, Greenwich.

Jan. 18, 1787, beginning of a solar eclipse by sid. clock	18^h	$4'$	$59''$
clock too slow	0	0	5
beginning of the eclipse by sidereal time	18	5	4
epoch of Sun's mean longitude for the beginning of 1787, and the meridian of Gothe . .	18^h	$40'$	$5''.895$
Sun's motion to January 18	1	10	57.996
Sun's motion in an interval of time representing the difference ($42'$ $55''$) of the longitudes of Gothe and Greenwich, ...	0	0	7.049
equation of equinoxes in right ascension	0	0	1.06
Sun's mean right ascension	19	51	12
Æ of the mid-heaven or sidereal time	18	5	4
approximate mean solar time.	22	13	52
(see pp. 526, &c.) acceleration	0	3	38.52
mean time	22	10	13.48
equation of time	—	11	15.08
true or apparent time of the beginning of the eclipse.	21	58	58.4

Example II.

Mean Time converted into Sidereal,

Marseilles, 21^m 29^s east of Greenwich,

1787, *mean* time of Venus' transit over the meridian 0^h 17^m $25^s.5$.

By Vince's Tables, epoch of Sun's mean longitude for 1787, ...	9^s	11^o	$2'$	$20''$
motion to January 2, ...	0	0	59	8.33
for $17^m\ 25^s.5$	0	0	0	42.92
	9	12	2	11.25
Sun's motion in $21^m\ 29^s$	0	0	0	52.9
Sun's mean longitude, or Æ of mean Sun	9	12	1	18.35

and in time	18^h	48^m	$5^s.223$
equation of equinoxes	0	0	1.055
Sun's mean Æ from true equinox	18	48	6.278
culmination or transit of ♀	0	17	25.5
sidereal time, or, apparent Æ of ♀ ...	19	5	31.78

or, if we convert time into degrees,

Æ of ♀ .... $286^0\ 22'\ 59''.1$.

Example III.

True or Apparent Time converted into Sidereal.

Greenwich, June 11, 1787, Sun on meridian	0^h	0^m	0^s
equation of time	0	0	52.379
mean solar time of Sun's transit	23	59	7.621

Now, by Tab. I—III, Vince, vol. III, converting the degrees, &c. into time, at the rate of 15^0 for 1^h,

Sun's mean longitude on June 11, 1787, ..	5^h	18^m	$54^s.74$
equation of equinoxes in right ascension ..	0	0	1.08
	5	18	55.82
correction on account of $52^s.379$	0	0	0.14
distance of mean Sun from true equinox ..	5	18	55.68
distance of mean Sun from mid-heaven ...	0	0	52.379
Æ of mid-heaven or sidereal time	5	18	3.3

In computing the equation of time by the methods in the preceding page, we are obliged, in fact, to compute the Sun's true longitude: which is a laborious computation. In order to avoid or to lessen such labor, Tables and approximate methods have been devised (see Delambre's *Astronomy*, vol. II, pp. 207, &c. Vince's *Astronomy*, vol. III, pp. 20, &c.)

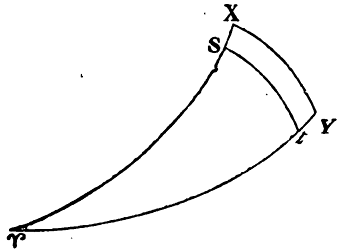
In the preceding reasonings, for the sake of simplicity, we have supposed the noon of mean time to be determined, by the aid of the noon of true or apparent time marked by the phenomenon of the real Sun on the meridian. But, if by means of the Sun's altitude observed out of the meridian, and a knowledge of his declination and of the latitude of the place, or by other means, we compute the hour angle measuring the time from apparent noon, we may, as easily as in the preceding case, compute the equation of time for such time, and thence deduce the corresponding mean solar time.

What has preceded contains the principle and the mode of computing the equation of time; all, therefore, that concerns the practical Astronomer. But if, for the purpose of new and farther illustration, we continue our speculations, we shall find that the equation of time, relatively to its causes, depends on two circumstances; *the obliquity of the ecliptic to the equator,* and *the unequal angular motion of the Sun in its real orbit.*

The Sun moves every day through a certain arc of the ecliptic: which, in other words, is his daily increase of longitude. If we suppose two meridians to pass through the extremities of this arc, they will cut off, in the equator, an arc which is the daily increase of the Sun's right ascension. This latter arc will not remain of the same value, even if the former, that of the ecliptic, be supposed constant. At the solstice it will be larger than at the equinox: the reason is purely a geometrical one: let S ♈ be the ecliptic, and ♈ y the equator, then by Naper's rule, if I be the obliquity, l the longitude, A the right ascension, D the declination, $1 \times \cos. I = \text{cotan.}\ ♈ S \times \tan.\ ♈ t = \frac{\tan. A}{\tan. l}$,

hence, $\tan. l \times \cos. I = \tan. A$, and, taking the differential,

$$dl \cdot \frac{\cos. I}{(\cos. l)^2} = \frac{dA}{(\cos. A)^2}, \text{ or, since } \cos. l = \cos. A \times \cos. D$$



$$dl \cdot \cos. I = dA\,(\cos. D)^2, \text{ or } dA = dl \cdot \cos. I \cdot (\sec. D)^2.$$

Hence, I being the same, dA varies, if dl be given, as $(\sec. D)^2$; ∴ is least at the equinoxes and greatest at the solstices, and its value is easily estimated at the former, for since $D = 0$, $dA = dl \cdot \cos. I$; at the latter, since

$$\sec. D = \frac{1}{\cos. D} = \frac{1}{\cos. I}, \quad dA = \frac{dl}{\cos. I};$$

$$\therefore dA \text{ (equinox)} : dA \text{ (solstice)} :: (\cos. I)^2 : 1$$
$$:: (\cos. 23^0\, 28')^2 : 1^2$$
$$:: 8414 : 10000.$$

Hence, even on the hypothesis of the Sun's equable motion in the ecliptic, the true right ascension will not increase equably; but since, by the very definition of the term, the mean longitude does, the equation of time, which is the difference of the true right ascension and the mean longitude (disregarding the equation of the equinoxes) would be a quantity, throughout the year, continually varying, and vanishing at the solstices.

The hypothesis, however, of the Sun's equable motion is contrary to fact; the Sun moves in an ellipse, and consequently, does not move uniformly, or equably in it. If a fictitious Sun, moving with the Sun's mean angular velocity, be supposed to leave, at the same time with the real Sun, the apogee, they will again come together at the perigee: but, in the interval, the fictitious Sun would constantly precede the real Sun: the latter therefore, would be first brought on the meridian; true noon, therefore, would precede the noon of mean time, supposing, now, mean time to be measured by the imaginary Sun moving uniformly in the ecliptic.

If therefore, we hypothetically annul the first cause of the equation of time, by supposing the ecliptic to coincide with the equator, still from the second, (the elliptical motion of the Sun,) there would exist a difference between true and mean time; in other words, an equation of time, continually varying; vanishing, however, at the apogee and perigee.

But, both causes in nature exist; the Sun moves unequably, and not in the equator. From their combination then, the actual equation of time must depend. It cannot be nothing at the solstices, except the solstitial points coincide with those of the apogee and perigee, but, (see p. 486,) in the solar orbit, there is no such coincidence.

At what conjunctures then, will the equation of time be nothing? We have already, for the purposes of explanation, introduced two fictitious Suns, one moving equably in the ecliptic, the other in the equator; let the former be represented by S'', and the latter by S''', and the true Sun, that which moves unequably in the ecliptic, by S'; then, true time depends on S', and mean time on S'''; and consequently, when the meridian, passing through one, passes also through the other, then is mean time equal to the true, therefore no equation is requisite, or the equation of time is nothing. Let us suppose the two fictitious Suns S'', S''' to move from the autumnal equinox towards the perigee; S''', in this case, must constantly precede S'', till they arrive at the solstice, where the meridian that passes through one will pass through the other*. Hence, the real Sun S', which coincided

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* We shall frequently use the expression of S' *rejoining* S''', or, *coinciding* with it. Nothing farther, however, will be meant by such expression, than that the meridian, which passes through the former in the ecliptic, passes through the latter in the equator; and when S' is said to precede S''', nothing more is meant, than that the point in the equator in which a meridian through S' cuts it, is beyond the place of S''', or, to the eastward of it.

with S'' at the apogee, being constantly behind it (see pp. 469, &c.) till their arrival at the perigee, must certainly be behind it, at and before the solstice, which is previous to the perigee (see p. 485.). Hence, before the winter solstice, the order of the Suns is

$$S' \; S'' \; S'''*.$$

At the solstice $S' \left\{ \begin{matrix} S'' \\ S''' \end{matrix} \right\}$; for S'' then ceases to be preceded by S'''. Immediately after the solstice, S'' takes the lead of S''': therefore, then, the order is

$$S' \; S''' \; S''.$$

But, at the perigee, S' must rejoin S'': it cannot effect that, except by previously passing S''': the moment of passing it is that in which true time is equal to mean time, in which, in other words, the equation of time is nothing.

The equation of time then is nothing, between the winter solstice and the time of the Sun's entering the perigee: and, for the year 1810, (when the longitude of the perigee was $9^s\, 9^0\, 39'\, 22''$) between Dec. 21, and Dec. 30. By the Nautical Almanack the exact time was Dec. 24, at midnight: since the equation for the noon of that day is $-\ 15^s$, and, for the noon of the succeeding day, $+\ 15^s$.

In the year 1250, when the perigee coincided with the winter solstice (see p. 486,) the equation of time was nothing on the shortest day.

Immediately after the passage of the perigee, S', the true Sun, moving with its greatest angular velocity (see p. 469,) precedes S''; therefore, since up to the vernal equinox S'' precedes S''', the order is

$$S''' \; S'' \; S';$$

and this order must continue up to the equinox; consequently, S''' and S' cannot come together: and therefore between Dec. 24, (for 1810,) and March 21, the equation of time cannot equal nothing.

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* The symbol most to the right of the page denotes the preceding Sun.

After the vernal equinox, S''' precedes S'', and the order is

$$S''\ S'''\ S',$$

S'', and S' are then, (see p. 472,) near the point of their greatest separation, but S'' and S''' begin to separate and reach the point of their greatest separation *, about $46^\circ\ 14'$ from the equinox that is, about the 8th of May. Now, this greatest separation, or, technically, greatest *equation*, is $2^\circ\ 28'\ 20''$, or in time $9^m\ 52^s$, whereas the greatest equation of the centre, being only $1^\circ\ 55'\ 33''$, (pp. 473, &c.) the greatest corresponding separation in the equator cannot exceed $2^\circ\ 6'$ †, and that is already past. Hence, before S''

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* $1 \times \cos. I = \tan. A . \cot. l$, by Naper, or $\cos. I \times \tan. l = \tan. A$; $\therefore l > A$; $\therefore$ if Y be supposed the place of S''', so that, $\Upsilon Y = \Upsilon S$, Y is beyond t, and the *separation* is $t\,Y$ (since on that the difference solely depends.)

To find $t\,Y$, is a common problem, (see Simpson's *Fluxions*, vol. II, p. 551. Vince's *Fluxions*, p. 27.) Since $t\,Y = \Upsilon Y - \Upsilon t = l - A$;

$$\therefore \tan. t\,Y = \frac{\tan. l - \tan. A}{1 + \tan. l . \tan. A} = \frac{\tan. A . (\sec. I - 1)}{1 + (\tan. A)^2 . \sec. I}.$$

Hence, since $d\,(t\,Y) = d \tan. t\,Y . (\cos. t\,Y)^2$, which must $= 0$; if we take the differential of the quantity equal to it, make it $= 0$, and reduce it, there results

$$\tan. A = \surd \cos. I = \surd (\cos. 23^\circ\ 27'\ 58'')$$

$$A = 43^\circ\ 43'\ 50'', \text{ and } l \text{ (from equation, l. 2 of Note)} = 46^\circ\ 14',$$

and $l - A$ (in its greatest value) $= 2^\circ\ 28'\ 20''$.

† By p. 538, it appears that the arc of the equator, included between two meridians passing through the extremities of a given arc in the ecliptic, is greatest when the latter arc is at the solstice. The arc of the equator measures the *separation* of the Suns S'', S'''. Hence, putting in the formula of p. 230, $dl = 1^\circ\ 55'\ 33''$, and $D = I$, which it is at the solstice, we have, very nearly,

$$dA = 1^\circ\ 55'\ 33'' \times \sec. 23^\circ\ 27'\ 58'' = 2^\circ\ 5'\ 55''.$$

The two common problems then of the maximum equation of time, are not merely mathematical problems, exercises for the skill of the student, or Examples to a fluxionary rule, but of use in the discussion of the real problem of nature.

is at its greatest separation from S''', it is impossible that the order

$$S'' \; S''' \; S'$$

should not have been changed. S' must have come nearer to S'' than S''' is: consequently, S''' must have passed S': but at the moment of passage, mean and true time are equal, that is, the equation of time is nothing: and this must happen between March 21, and the end of April. In the year 1810, it happened, according to the Nautical Almanack, on April 15, $11^h \; 12^m$.

This second point, at which the equation of time is nothing, being passed, the order of the Suns will become

$$S'' \; S' \; S'''.$$

At the solstice, S'' must rejoin S''': but, previously to the solstice, it cannot effect that by passing S': since S'' does not rejoin S' till their arrival at the apogee, which point is more distant than the solstitial: the coincidence of S'' and S''' then can only take place, by S' previously passing S''': but, as before, the moment of passage, is the time when the equation of time is nothing: that circumstance therefore, must happen, before the summer solstice: therefore, between the middle of April and June 22: and, in 1810, according to the Nautical Almanack, it happened on June 15, 14^h.

In the year 1250, the equation of time was nothing on the longest day.

After this third *evanescence* of the equation of time, the order of the Suns will become

$$S'' \; S''' \; S'.$$

At the solstice on June 22, S'' will rejoin S''': immediately afterwards, the order becomes

$$S''' \; S'' \; S',$$

which will continue to the time of the Sun's entering the apogee: then, S'' rejoins S': and, immediately after, S'' moving with greater angular velocity than S' will precede it, and the order becomes.

$$S''' \; S' \; S''.$$

Now S' cannot rejoin S'' till their arrival at the perigee: but S' will rejoin S''' at the autumnal equinox, consequently, previously to that time, S''' must pass S': but, as before, the moment of passage is, when the equation of time is nothing. It must happen then, between the time of the apogee and the autumnal equinox: between (for 1810) June 30, and September 24; and, by the Nautical Almanack, it happened August 31, 20^h.

It is plain, from the preceding explanation, that the days of the year in which the equation of time is nothing depend on the position, or the longitude of the perigee and apogee: and consequently, since those points are perpetually progressive, the equation of time will not be nothing on the same days of any specified year, as it was, of preceding years: nor, when not nothing, the same in quantity, on the corresponding days of different years.

The preceding statement (beginning at p. 537,) is to be regarded merely as a mode of explaining the subject of the equation of time. It is not essential, and might have been omitted; for, the two causes of inequality are considered and mathematically estimated, in the processes of finding the true longitude and true right ascension. But it has been inserted, since it serves to illustrate more fully, and, under a different point of view, a subject of considerable difficulty and importance.

With regard to results, very little is effected by the preceding statement. Four points are determined, at which, mean time is equal to apparent: in other words, four particular values (evanescent values) of the equation of time. But, according to the process in p. 533, we are enabled to assign its value for every day in the year: and accordingly, in constructing Tables of the equation of time, the above four particular values would be necessarily included amongst the 365 results.

If the question were, merely to determine when the equation was nothing, it would certainly be an operose method of resolution, to deduce all the values of the equation of time, and then, to select the *evanescent* ones. In such case, it would be better to have recourse to considerations like the foregoing (pp. 537, &c.). But, both these methods would be superseded, if, which is not the case*, the equation of time could be expressed by a simple analytical formula.

The mere inspection of such formula, or some easy deduction from it, would enable us to assign the times when the equation of time vanished.

Instead of a formula, we must use a process consisting of several distinct and unconnected steps, for computing the equation of time. And, in point of fact, the process is quite as convenient as a formula could be; since the concern of the Astronomical Computist is not with special, as such, but with the general values of the equation of time.

If special values are sought after, it must be principally on the grounds of curiosity. The method of ascertaining four such values, independently of direct computation, has been already exhibited. And, on like grounds, a similar method might be used in the investigation of other special values: in determining, for instance, when the equation of time is of a mean value; or, when minute, the two causes of inequality counteracting each other; or, when large, the two causes co-operating. We will confine ourselves to two instances.

After the evanescence of the equation of time between the winter solstice and the perigee, the order, as we have seen, (p. 542,) is

$$S''' \; S' \; S'',$$

but S' is gaining fast on S'' in order to rejoin it at the perigee, and S'', after parting with S''' at the solstice, is preceding it, by still greater and greater intervals. Consequently, both causes of inequality conspire to make mean time differ from the true, and the equation of time goes on increasing till the Sun is about 40^0 distant from the vernal equinox, that is, past the point, at which the equation arising from the obliquity is a maximum, (see p. 541,) and before the point at which the equation from the Sun's ano-

* Lagrange, however, although by no direct process, has succeeded in assigning a formula for the equation of time. See *Mem.* Berlin, 1772. So also has M. Schulze, *Mem.* Berlin, 1778. p. 249.

malous motion is a maximum. For the year 1810, the time would be about Feb. 10, and the maximum of the equation is $14^m\ 36^s$.

About the Summer solstice, on the contrary, between that and the apogee, the order is

$$S''' \quad S'' \quad S'.$$

S''' is indeed separating from S'', but S'' is approaching S' to reach it at the apogee: consequently, the two causes of inequality, in some degree, counteract each other, and the equation between the two periods at which it is successively nothing, (June 15, and August 31, for 1810,) never attains to the value of seven seconds.

In a similar way, we may form a tolerably just conjecture of the limits of the quantity of the equation of time, for other parts of the year.

The greatest quantity of the real equation of time can never reach the sum of the greatest equations arising from the separate causes. It must therefore be less than

$$2^0\ 28'\ 29'' + 2^0\ 6', \text{ or } 4^0\ 34'\ 29'',$$

or in time less than $18^m\ 15^s$ of mean solar time.

The equation of time computed for every day in the year, according to the method given in p. 533, or, by some equivalent method, is inserted in the Nautical Almanack; and, for the purpose of deducing mean solar, from apparent time. In order to regulate its application, the words *additive* and *subtractive* are interposed into the column that contains its several values. And, there will be no ambiguity belonging to that application, if we consider, that the equation is to be applied to a certain time marked by some phenomenon: which phenomenon is the real Sun on the meridian: determined to be so, either by a transit telescope, or by a quadrant, or declination circle that enables us to ascertain, when the Sun is at its greatest altitude. Apparent time, then, is what is instrumentally determined; and to such time, the equation, with its concomitant sign, must be applied, in order to deduce mean time, which, it is plain, is indicated by no phenomenon.

Thus, Dec. 31, 1810, the equation of time in the Nautical Almanack is stated to be $3^{m}\ 12^{s}.7$ *additive;* therefore, when the Sun was on the meridian, at its greatest height, on that day the mean solar time was $12^{h}\ 3^{m}\ 12^{s}.7$. Again, Nov. 13, 1810, the equation is stated at $15^{m}\ 33^{s}.2$ *subtractive;* therefore, on that day, the Sun was at its greatest height at $12^{h} - 15^{m}\ 33^{s}.2$, that is, $11^{h}\ 44^{m}\ 26^{s}.8$, mean solar time.

Independently of computation, very simple considerations will shew that this procedure is just. In the first instance, the true Sun precedes the mean; that is, is more to the east, or more to the left hand of a spectator facing the south: consequently, by the rotation of the Earth, from west to east, the meridian of the spectator must first pass through the hinder Sun, which, in this instance, is the mean Sun; 12^{h} therefore of mean time happens before the meridian has reached the true Sun, when it does reach it, then, the time is, in mean time, 12^{h} + the difference of right ascensions, or 12^{h} + the equation of time. In the second instance, the true Sun is behind the fictitious: therefore the meridian of the spectator first passes through the former: true noon therefore, or 12 hours apparent time, happens before the meridian has reached the fictitious mean Sun; before therefore the noon of mean solar time. The time consequently is not 12 hours, but 12 hours — some quantity, which quantity is the equation of time.

What has been given in the latter pages, has been for the purpose of illustration rather than for settling the grounds of, and arranging the method of computing, the equation of time. It may suit some students: others, perhaps, will be satisfied with the investigations that terminate at p. 537.

CHAP. XXIII.

THE PLANETARY THEORY.

On the general Phenomena of the Planets: their Phases, Points of Stations, Retrogradations, &c.

WE have now passed, in our course of enquiry, through the theories of the fixed stars and of the Sun, and are arrived at the Planetary Theory. This latter theory has many points in common with the preceding ones. The planet Venus, by reason of the Earth's rotation, is transferred to the west, as Orion is and as the Sun is. By reason of the same rotation, she *rises* and *sets* as any fixed star is made to rise and set. But the points of the horizon at which Venus rises and sets, do not remain the same, which is a circumstance of distinction between that planet and the fixed stars: and indicative of a peculiar motion in Venus, whether such motion respects, as its centre, the Earth or the Sun.

The question, in truth, is not to be at once reduced to the above alternative. We may *conjecture,* besides the Sun and Earth, other points to be the centres of the planets' revolutions. But we shall here, as we have done before, avail ourselves of the results of previous investigations and restrict the range of our conjectures. Indeed, the restriction will be so close, that we purpose merely to enquire whether the phenomena of the planets (the phenomena of change of place and law of motion) can be explained on the hypothesis of the planets describing elliptical orbits round the Sun as a centre, and of their mutual perturbation.

We at once get rid of the suggested possibility of a simple revolution of the planets round the Earth, on this consideration:

namely, that, in such a case, the motion would take place, and seem to take place, in one and the same direction: whereas, as observation shews, the planets are sometimes stationary and sometimes retrograde.

These apparent quiescences and retrogradations, are some of the phenomena which it will be the business of this, and of the ensuing Chapters to explain, on the principle of the combination of the motions of the planets and of the Earth. In the first place, these phenomena will be explained in a popular way, on the principles of the Earth's rotation round its axis, and of the Earth's and planets' revolutions round the Sun. After this, the phenomena will be more scientifically explained, or the times and circumstances of their happening will be computed. But in order to effect this we must know the *elements*, as they are called, of the planetary orbits: such as their axes, the places of their nodes and of their aphelia, and their inclinations to the plane of the ecliptic. For this end we must have recourse to observations, and, according to modern practice, to observations of right ascensions and declinations. The *elements* being obtained, we may combine them according to Kepler's principles, and by means of his problem and other aids, compute the planet's longitude in his orbit. From such longitude, and a knowledge of the inclination of the orbit, and of the place of the node, we may compute the planet's *ecliptical* longitude and his latitude, and thence compute, by a Trigonometrical process, or by a Table of *reductions* (see p. 501,) the planet's right ascension and declination. The last step in this process, would be to compare these previously computed longitudes and latitudes, with longitudes and latitudes resulting immediately from observed right ascensions and declinations: or, which is in fact the same, the previously computed right ascensions and declinations, with the observed. Such comparisons, as in the Solar Theory, (see pp. 508, &c.) enable us to *correct* the elements of the orbit, from which the planet's longitudes and latitudes are to be computed.

The order then, briefly stated, is this: the explanation of the phenomena: extrication of the elements from observations: the subsequent correction of those elements by a comparison with observations: and, in pursuance of the first of these objects, we will begin with the planet Venus.

This brilliant star when seen in the west, at the time of the setting of the Sun, is called the *Evening* Star *. It will be found, by observing it on successive nights, to vary its distance from the Sun: sometimes apparently moving away from the Sun, until it reaches a certain term of elongation, at other times, having passed such term, approaching the Sun. When the star begins, it continues, to approach: and, at certain epochs, it approaches so nearly to the Sun, as by reason of the Sun's effulgence, to be no longer distinguishable by unassisted vision. There are other epochs, rare, indeed, at which Venus passes over the Sun's disk, and is seen, during such transit, as a black spot on the disk. After either of these two sorts of epochs Venus ceases to be the evening star and will soon become the *morning* star †, and will be seen rising just before the Sun.

On successive mornings, *Venus* will rise still sooner: will continue to be separated from the Sun, till having reached an angular distance of about 45^0, she will again approach, and finally rejoin the Sun. She then again becomes the evening star, and the same appearances, in the same order, are renewed.

These appearances prove, not decisively, that *Venus* describes either an oval, or a circle about the Sun, but that, at least, she oscillates about the Sun: they prove too, that her orbit can neither be round the Earth, as its centre, nor inclusive of the Earth; for, she is never seen in opposition; that is, in the production of a line drawn from the Sun through the Earth.

To the naked sight, or to unassisted vision, the disk of *Venus* appears circular and nearly of the same magnitude. But, the telescope and its micrometer ‡ prove both appearances to be delusive. Viewed through the former, *Venus*, when the evening

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* Εσπερος, Hesperus, Vesper. † Φωσφορος, Lucifer.

‡ An instrument for measuring small angles, and commonly attached to the telescope.

star, at her greatest separation from the Sun, assumes the form of a crescent, the convex illuminated part being towards the Sun, or towards the west. As she approaches the Sun, the crescent diminishes. Having passed the Sun, she appears as the morning star, and the crescent is turned the other way, or towards the east. Day after day, the crescent increases, till it is changed into a full orb, just at the time when *Venus* is about to rejoin the Sun.

In this last situation the disk of *Venus*, though most illuminated, is least in magnitude. It is greatest in magnitude, when the disk is least illuminated, and *Venus* is about to rejoin the Sun. These latter circumstances, relative to the magnitude of the disk, are determined by the micrometer.

This last-mentioned instrument enables us to determine the greatest and least apparent diameters of *Venus* to be about $60''$, and $10''$.

If we now enumerate the circumstances relative to *Venus*, they are as follow:

Venus, whatever be the Sun's place in the ecliptic, always attends on him, and is never separated by a greater angle of elongation, (technically so called) than 45^{0}.

Venus is continually at different distances from the Earth: when at her greatest, that is, when her apparent diameter is the least, she shines with a full orb: when seen at her least distance, that is, when her apparent diameter is the greatest, her crescent is very small; and there are conjunctures, as we have noted, when *Venus* eclipses part of the Sun's disk, and passes over it like a dark spot.

Venus, when the evening star and separating from the Sun, moves from west to east; or according to the order of the signs, or, as the phrase may still be varied, *in consequentia*. Returning towards the Sun, from her greatest elongation, she moves towards the west, that is, *in antecedentia*, contrary to the order of the signs. And, in like manner, she moves, when the morning star, alternately, according and contrary to, the order of the signs.

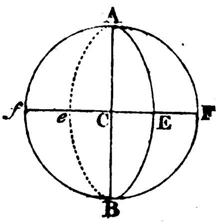
These are the phenomena of observation, that are proposed for explanation, on the grounds of two hypotheses: the first, that *Venus* is an opaque spherical body illuminated by the Sun: the second, that *Venus* revolves round the Sun in an orbit which is interior to the Earth's orbit.

If *Venus* be a sphere, only half of it can be illuminated by the Sun. And the illuminated hemisphere, called, for distinction, the Hemisphere of *Illumination*, is thus to be determined. From the centre of the Sun, to that of *Venus*, conceive a right line to be drawn; perpendicular to this line, and passing through the centre of *Venus*, conceive also a plane to be drawn; then, such plane will divide the body of the planet into two hemispheres, the one luminous, the other dark.

But, a spectator, whatever be his distance from a sphere, can never see more than half of the same. The hemisphere which he sees, called the Hemisphere *of Vision*, is thus to be determined: conceive the eye of the spectator and the centre of the planet to be joined by a right line; a plane perpendicular to this line, passing through the centre of the planet, divides its body into two hemispheres; the one towards the spectator, is that of *vision*.

The two hemispheres, and their boundaries, the circles of illumination and of vision, do not necessarily coincide: indeed, they can coincide only when the Sun, which illuminates the planet, is between it and the spectator on the Earth's surface. In every other situation, part of the planet's illuminated hemisphere is turned away from the spectator; and, when the planet is between the Sun and spectator, wholly turned away: in other words, the planet's disk can either not be seen, or must appear as a dark circle or spot on the Sun's face.

When the spectator, Sun, and *Venus* (for of that planet we are now speaking) lie not in the same right line, the delineation of the illuminated disk, or phase, is reduced to a very simple proposition in orthographic projection. On the plane of projection which is always perpendicular to a line joining the eye of the spectator and the centre of the planet, it is required to delineate the ellipse into which the circular boundary of light and darkness will be projected. The minor axis of the ellipse, will, as it is well known, bear that proportion to the major, which the radius bears to the cosine of the inclination of the planes. The inclination is equal to the angle formed by two lines, one drawn from the Sun to the centre of *Venus*, the other, from that same centre and directly from the spectator. Hence, if *AFBA* represent the



disk, and we take *CF* : *CE* :: rad. : cos. planet's inclination,* then, describing, with the semi-axes *AC*, *CE*, the semi-ellipse *AEB*, we shall have the illuminated disk represented by *AFBEA*.

If *KVuL* be the orbit of *Venus*, *S* the Sun, *E* the Earth; then, the angle of inclination of the planes of illumination, and vision at *V*, is the angle *SVF*, and at *u*, the angle *SuF*. In the latter, the angle is acute, in the former, obtuse; consequently, if *CE* in the above Figure be taken to represent the cosine of the acute angle, to the right of the line *AB*, *Ce* must be taken to the left of the same line, in order to represent the cosine of the obtuse angle *SVF*. At *K*, when the planet is in *superior* conjunction †, the angle *SVF* is equal to two right angles; consequently, the

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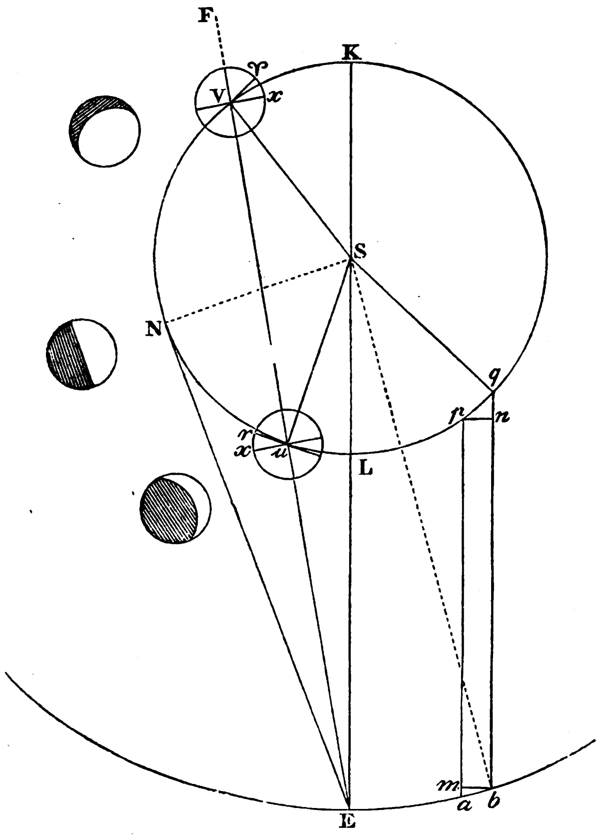
* *ru* is the half of the projection of the circle of illumination, *xu* of vision, and

$\angle rux = \angle Fux - \angle Fur = 90^{0} - \angle Fur = 90^{\circ} - (\angle Sur - \angle SuF)$
$= 90^{\circ} - (90^{\circ} - SuF) \angle = \angle SuF.$

† An inferior planet is in superior conjunction, when it lies in the direction of a line drawn from the Earth to the Sun, and produced beyond the Sun.

cosine (with a negative sign) becomes equal to radius, and the point



E, falls in *f* (Fig. p. 552,); or the whole orb is illuminated. At *L*, when the planet is in inferior conjunction* the angle, such as *SuF*, becomes nothing; therefore the cosine becomes equal to

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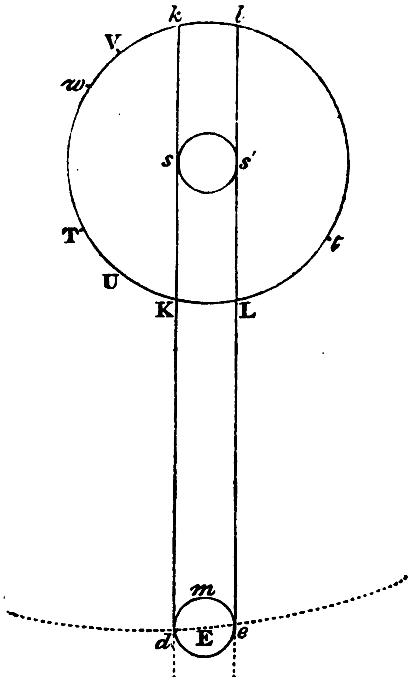
* We have, for simplicity's sake, supposed the ecliptic and the plane of the orbit of Venus to be coincident. Such is not the case in nature. It will happen, and commonly, that the planet at the time of inferior conjunction will be above the Sun, in which case its bright crescent will be visible: and exactly at the time of conjunction, the line joining the horns of the crescent, will be parallel to the ecliptic.

radius, and the point E falls in F: or the whole orb is dark. From K to L, in the intermediate points, *Venus* exhibits all her varieties of phases; the full orb, near K; the half illuminated orb at N, where $SNE = 90^{\circ}$, and then the crescent diminishing, till its extinction at L*.



These phenomena that would happen if *Venus* an opaque spherical body be illuminated by the Sun, and revolve in an orbit round him, are strictly conformable to the phenomena that are observed, and have been described in the preceding pages.

Thus far then the hypothesis of *Venus's* revolution round the Sun is probable, and seems to involve no contradiction; it will be

* The phases which *Venus* at V, N, and u, exhibits to a spectator at E, are represented by the small circular Figures that are, respectively, to the left of the points V, N, and u (see p. 553.)

still farther confirmed, if we can shew, that it affords an adequate explanation of the other phenomena which the planet exhibits.

Suppose *emd* to be the Earth, and two tangents *dsk*, *es′l*, to the points *d* and *e*, to represent the respective horizons to a spectator at *d* and *e**. If the Earth's rotation be according to the order *emd*, when the horizon *dsk* of the spectator at *d* shall touch the Sun's disk, the Sun will *set* to that spectator; the moment after, by the rotation of the Earth, the point *k* will be transferred to some point between *k* and *V*, the line *dsk* will no longer touch the Sun's disk, or, the Sun will be below the horizon. But, *Venus*, if at *V*, will be above the line of the horizon, and above as an evening star, till the Earth, by its farther rotation, shall have so transferred the line *dsk*, that its extremity *k* shall be in some point between *V* and *U*. In the interval between this and the next night, *V* will have moved forward in its orbit to some point *w*; therefore, the line *dsk*, after leaving the Sun's disk, must revolve through a greater angle than it did the preceding evening, before it reaches *V* at *w*. The planet therefore, is now separated from the Sun by a greater angle of elongation: and the elongation on succeeding nights will still continue, till *V* reaches a point *T*, where a line drawn from *E* touches her orbit. Hence from superior conjunction at *k*, to the greatest elongation at *T*, *Venus* is continually separating or *elongating* from the Sun; and, if we refer her place to the fixed stars, will seem to move amongst them in a direction *kVwT*, that is, according to the order of the signs.

From *T* to *L* the inferior conjunction, the line *dsk*, after quitting the Sun's disk, will reach the planet after the description of angles still less and less, and the planet will be found approaching the Sun: but, referred to the fixed stars, will be found to change its place amongst them in a direction from *T* towards *L*, contrary to the direction of the former change of place, and

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* In this explanation, of a popular nature, *Venus's* orbit and the Earth's equator *emd*, are supposed to be projected on the plane of the ecliptic, (represented by the plane of the paper,) and, the spectator is supposed to be placed in the equator.

contrary to the order of the signs. In other words, the planet is now *retrograde*.

Suppose now the planet to have passed the inferior conjunction at *L*. *Day breaks* to a spectator at *e*, when the line *es′l*, representing his horizon, touches the Sun's disk. But, before this has happened, the line *es′l* has passed the planet, or the planet is above the horizon, and has risen as the morning star: on succeeding mornings, the planet having moved forward in its orbit from *L* towards *t*, will rise before the Sun by greater and greater intervals; will continue, to appearance, separating from him, till its arrival at its greatest elongation *t*. From *L* to *t*, the planet will, as from *T* to *L*, still continue retrograde. From *t* to *l*, it will again approach the Sun, and move according to the order of the signs.

These phenomena, then, that would happen if *Venus* revolve either in a circular or elliptical orbit round the Sun, are in strict conformity with the phenomena that are observed, and which have been previously described.

In the preceding explanation of the phases and retrogradations of *Venus*, we have, for the sake of simplicity, supposed the Earth to be at rest at *E*. But, there is one phenomenon, that of the seeming quiescence of *Venus* during several successive days, which cannot be explained, except we depart from that supposition, and combine, according to the actual state of things, the motion of the Earth with that of *Venus*.

If *Venus* be at *L*, and the Earth at *e*, and both describe in the same time (24 hours for instance), two small arcs of their orbits, such arcs will be nearly parallel to each other. If, then, they were equal, during their description, *Venus* would be referred by a spectator on the Earth, to the same point in the heavens. But, *Venus* revolving round the Sun according to the laws of planetary motion (see p. 557, l. 16,) describes a greater arc than the Earth does in the same time. She must, therefore, at the end of the 24 hours, be referred by a spectator on the Earth, to a point in the heavens situated to the right of her former place. But, as *Venus* advances from *L* towards *t* in her orbit, the arcs of her orbit (or tangents to them) will become more and more inclined to the arcs of the Earth's orbit. There will then be somewhere between L and t an arc pq (see Fig. p. 553,) such that, its obliquity compensating its greater length, two lines pa, qb, drawn to the contemporaneously described arc ab of the Earth's orbit, shall be parallel; when that circumstance happens, *Venus* must appear *stationary*.

We may determine the exact time of its happening by computing the angle bSq, which is, in the same time, the excess of the angular motion of *Venus* above that of the Earth *.

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* bSq may be thus computed: (see Fig. p. 553,).

Draw from p and b; pn, bm perpendicular to the parallel lines qb, pa, then $pn = bm$: call Sb, r, and Sq, r';

$$\text{then } pn = pq \,.\, \sin.\, pqn = pq \,.\, \cos.\, Sqb,$$
$$bm = ab \,.\, \cos.\, mba = ab \,.\, \cos.\, Sbq;$$

$$\therefore \frac{\cos.\, Sqb}{\cos.\, Sbq} = \frac{ab}{pq} = \frac{\text{vel.} \oplus}{\text{vel.} ♀} = \frac{\surd r'}{\surd r} \text{ (Newton, Sect. II. Prop. 4. Cor. 6;)}$$

$$\therefore \cos.^2 Sbq = \cos.^2 Sqb \times \frac{r}{r'}.$$

$$\text{But, } \sin.^2 Sbq = \sin.^2 Sqb \times \frac{r'^2}{r^2} \quad (\textit{Trigonometry}, \text{ p. } 16,)$$

$\therefore$ adding these two latter equations, and putting for $\cos.^2 Sqb$, $1 - \sin.^2 Sqb$,

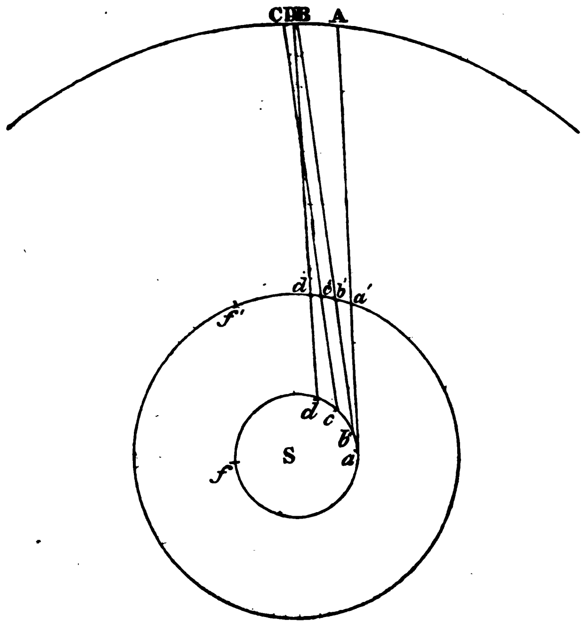
$$1 = \frac{r}{r'}(1 - \sin.^2 Sqb) + \frac{r'^2}{r^2} \sin.^2 Sqb,$$

$$\text{and } \sin.\, Sqb = \sqrt{\left(\frac{r^3 - r^2 r'}{r^3 - r'^3}\right)} = \frac{r}{\surd(r^2 + rr' + r'^2)}:$$

$$\text{Hence, } \sin.\, Sbq = \frac{r'}{\surd(r^2 + rr' + r'^2)}.$$

The two angles Sqb, Sbq, being thus determined, $bSq = 180° - (Sqb + Sbq)$ is known; and thence the time from conjunction at L. Thus, the mean daily motions of *Venus* and the Earth being 1° 36′ 7″.8, and 59′ 8″.33, the daily excess is 36′ 59″.5; therefore, if the angle bSq be 13°, the time from conjunction will be $\frac{13°}{36' 59''.5}$, or about 21 days.

It is plain that *Venus* will be retrograde whilst moving through an arc such as NLt, whether the Earth be supposed to be at rest, or to be in motion. The case however, is different with a superior planet*, which can only be shewn to be retrograde by combining with its motion, the Earth's. Thus, let ab, bc, cd, be three equal arcs in the Earth's orbit, $a'b'$, $b'c'$, $c'd'$, three equal arcs in *Jupiter's* (for instance,) contemporaneously described, but less (see p. 557, l. 16,) let also A, B, C, D, be four points in the imaginary sphere of the fixed stars, to which a', b', c', d' are successively referred by a spectator at a, b, c, d. Now, if ABC be according to the order of the signs, the body in the orbit $a'\,b'\,c'\,d'$, is transferred in that direction or is *progressive;* whilst



the spectator moves from c to d, and the planet from c' to d', the latter, amongst the stars, is transferred from C to D towards B and A, that is, contrary to the order of the signs. During the

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* A superior planet includes within its orbit, the Earth's; an inferior planet's orbit is included within that of the Earth's.

description then of the intermediate arcs $c\,b$, $c'\,b'$, the planet must have been stationary. The retrogradation will continue from c through opposition, where it will be the greatest, to a point f, situated similarly to c; that is, such that the angle made by two lines joining $f'\,f$, $f\,S$ shall $=$ the angle $c'\,c\,S$. From f through conjunction to c, the planet will move according to the order of the signs.

Here then is a material circumstance of distinction, in this part of their theory, between inferior and superior planets. In the explanation of the quiescences and retrogradations of the former, the Earth's motion is not an essential circumstance; it merely modifies their extent and duration. But, with superior planets, the Earth's motion is an indispensable circumstance. The very nature of the explanation depends on its combination with that of the planets.

In speaking of the stations and retrogradations of the planets, we have been obliged to use a language and phrases by no means descriptive of the observations by which those phenomena are ascertained. But, the student must be reminded upon this, as upon other occasions, to attend to the simple facts of observations. When a planet is stationary, the fact of observation is, that the right ascension continues the same: when retrograde, that the right ascension diminishes. The right ascension being determined by the hour, minute, &c. at which the observed body comes on the middle vertical wire of a transit telescope,

Jupiter, in treating of his retrogradations, has been assumed to be a superior planet. One proof of his being such, as well as that *Mars*, *Saturn*, and the *Georgium Sidus* are, is to be derived from their phases; which have not as yet been described.

Now, *Mars* exhibits no such variation of phases as *Venus* does; he is seen, indeed, sometimes a little *gibbous*, but never in the shape of a crescent, nor as a black spot on the Sun's disk. If we add to these circumstances, that he is seen at all angles of elongation from the Sun, we may presume that *Mars* revolves in an orbit round the Sun inclusive of the Earth's; that he is therefore a superior planet. He certainly cannot revolve round the

Earth, for then he would never be stationary, nor retrograde; nor can his orbit pass between the Sun and Earth.

Jupiter, Saturn, and the *Georgium Sidus* do not appear gibbous, but shine, almost constantly, with full orbs.

These phenomena can be accounted for, by supposing *Mars, Jupiter, Saturn,* and the *Georgium Sidus,* to be opaque spherical bodies illuminated by the Sun; and *Mars* to be the least distant: and, if not very distant (relatively to the Earth's distance), his illuminated disk may, in some situations, be so much averted from the spectator, as to give him the appearance of being a little gibbous; and, he will be most gibbous in quadratures: where, however, the breadth of the illuminated part will be to that of the whole disk as 175 to 200.

If we were to increase the distance of *Mars,* the above proportion would approach more nearly to one of equality. Hence the reason, why *Jupiter, Saturn,* and the *Georgium Sidus,* much more distant from the Sun than *Mars,* do not appear gibbous, even in quadratures.

From what has preceded, we may draw this conclusion; that, the adequate explanation of the phases, the stations, and the retrogradations of the planets, on the hypothesis of their revolution round the Sun, renders, at least, that hypothesis probable. But, since the explanation has been one, of obvious and general appearances, and not of phenomena precisely ascertained by accurate observations, the mere fact of a *revolution* has alone been rendered probable, without any determination of the nature of the curve of revolution. It may be either circular or elliptical. The system of Copernicus, therefore, is rather proved to be true, than Kepler's laws, or Newton's theory. Their truth, however, is intended to be shewn, and, that the planets revolve round the Sun in orbits very nearly elliptical: the deviations from the exact elliptical forms being such, as would result from the mutual disturbances of the planets computed according to the law of gravitation. For this end, phenomena, of a different kind from the preceding, must be selected and examined, and explanation, from being general, must become particular, and proceed by calculation. The *elements* of the orbits and the motions of the planets must be deduced from observations; arranged in Tables; again compounded according to theory; and, in this last state, as results, subjected to the test of the nicest observations.

The elements of the orbits of planets depend on certain distances, linear and angular, measured from the Sun. But, the observations, from which these elements are to be deduced, are made at the Earth. The first step then, in the succeeding investigation, must be towards the invention of a method, for transmuting observations made at the Earth, into observations that would be made by a spectator supposed to be placed in the Sun; in technical language, for converting *geocentric* into *heliocentric* angular distances.

This method is necessary for the extrication of the elements. For the examination of the system founded on those elements, the reverse method is required; in other words, we must be possessed of the means of converting *heliocentric* into *geocentric* angular distances.

CHAP. XXIV.

On the Method of reducing Observations, made at the Earth, to Observations that would, at the same time, be made by a Spectator situated at the Sun: or, on the Methods of extricating, from the Geocentric Observations of a Planet's Place, the Elements of the Orbit which it describes round the Sun.

In the theory of the fixed stars, the spectator is supposed to be placed in the centre of that sphere, which revolving, in twenty-four hours, round an axis passing through the poles of the Earth, produces the common phenomena of the risings, settings, and culminations of stars. In the solar system also, the spectator is supposed to be, very nearly, the centre of the solar motions. In both these cases, the observations are of right ascensions and declinations convertible, by rules already laid down, into longitudes and latitudes; in the case of the fixed stars, either geocentric, or heliocentric longitudes and latitudes; in the case of the Sun, its longitudes, seen from the Earth, differ from the longitudes of the Earth, seen from the Sun by the constant difference of 180 degrees.

The case is very different with the planets. These respect the Sun as the centre of their motions, which motions can only be observed at the Earth. It is necessary, then, if we would trace the orbit of a planet described round the Sun, and lay down the laws of its motion, that we should be able, from *geocentric* observations of a planet's place, and change of place, to infer what that place and change of place would be, were the spectator at the centre of the planet's motions.

The first steps, in this process, would be the same as in the sidereal and solar theories. The planet is to be observed on the meridian, with the transit instrument and declination quadrant or circle, and, then, from such observed right ascension and declination, the planet's *geocentric* longitude and latitude are to be computed by the formulæ of Chapter VII, (see pp. 160, &c.).

We will give an instance in the computation of the geocentric latitude and longitude of Venus,

March 13,

♀ Æ $22^h\ 58^m$, declination $2^0\ 43'$ N : obliquity $23°\ 27'\ 54''$,

Æ .... $344^0\ 30'$

Æ $-$ 90 ... 254 30

$\frac{1}{2}$(Æ $-$ 90).. 127 15 .... log. sin. 9.9009142

2

19.8008284

N. P. D. .... $87^0\ 17'\ 0''$ .... log. sin. 9.9995117

I .... 23 27 54 .... log. sin. 9.6000890

$-$ 2 log. r $-$ 20.

2) 19.4014291

M .... 30 7 59 .. (log. sin. M) 9.7007145

$\frac{\text{N.P.D.} + I}{2}$ $55°\ 22'\ 27''$

$\frac{\text{N.P.D.} + I}{2} + M$ 85 30 26 ...... log. sin. 9.9986635

$\frac{\text{N.P.D.} + I}{2} - M$ 25 14 28 ..... log. sin. 9.6298461

2) 19.6285096

(log. sin. .... $40^0\ 41'\ 38''$) 9.8142548

∴ comp. of lat. = 81 23 16

and latitude ... = 8 36 44

To find the longitude,

Δ	$81^\circ\ 23'\ 16''$	log. sin.	9.9950753
I	23 27 54	log. sin.	9.6000890
δ	87 17 0	(d)	19.5951643
2)	192 8 10		
$\frac{1}{2}$ sum	96 4 5	log. sin.	9.9975598
$\frac{1}{2}$ sum $-\delta$	8 47 5	log. sin.	9.1839025
		2 log. r	20
			39.1814623
		(d)	19.5951643
		2)	19.5862980
			9.7931490

Now 9.7931490 is the log. sin. of $38^0\ 23'\ 40''$, &c. and of

$$360^\circ + 38^0\ 23'\ 40'' = 398^0\ 23'\ 40''$$
$$\therefore 90 + L = 796\ \ 47\ \ 20$$
$$L = 706\ \ 47\ \ 20$$
$$= 360^0 + 346^\circ\ 47'\ 20'';$$

$\therefore$ rejecting 360^0,

the geocentric longitude of ♂, or $L = 11^s\ 16^0\ 47'\ 20''$.

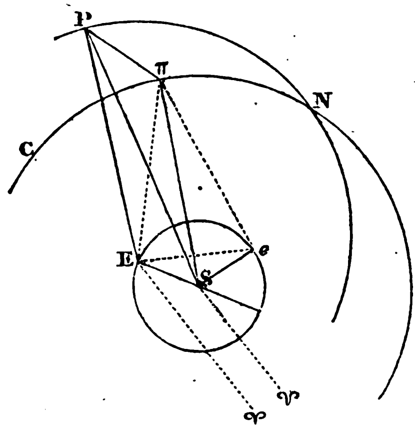
By these means, then, that is, by meridional observations of the planet, and by computations, may its longitude and latitude be determined. Amongst the resulting values of the latitude, there must be some either nothing or very small. Now when the geocentric latitude is nothing, the heliocentric also is nothing, or the planet is in the plane of the Earth's orbit: or, technically, the planet is in its *node:* the node being the intersection of the orbit of a planet, with the plane of the ecliptic. We are able then, by examining the series of the values of the geocentric latitudes, (computed as above) to determine when a planet is in its node, and we also know the geocentric longitude corresponding to such a situation of the planet.

Some values of the latitude will, it has been said, be either nothing, or very small. The latter circumstance is likely to take place: for, it is very improbable that the planet should be, at the same time, on the meridian of the observer, and in the plane of the ecliptic: in the same way, as it is very unlikely to happen that the Sun should be, at once, in the solstice at noon, or in the equinoctial at noon. But the same artifice, or method of computation, which makes amends for the want of coincidence of the two events in the latter case, applies to the one now under consideration. Find, for instance, the longitude and latitude of the planet when just above the ecliptic (to its north) and, the next day, find the like quantities when the planet (supposing it to be descending towards the ecliptic) is just below, or to the south of, the ecliptic. *The Rule of Three*, or some equivalent rule of proportion, will give the longitude corresponding to a latitude that is nothing, or, in other words, will give the geocentric longitude of the descending node.

Before we proceed any farther we will just advert to a point which will soon be more fully discussed. Since we are able to compute the exact time of the planet's entering its node, we are able to determine the interval elapsed in its passage from the *descending* to the *ascending* node, and also the interval of time between two successive returns to the same node. The latter interval must be (supposing the places of the nodes, and the dimensions and positions of the orbit, not to have changed) the periodic time of the planet. The former interval, should it be exactly the half of the latter, would be a proof either that the orbit of the planet was circular, or, if elliptical, so placed as to have its axis major coincident with the line of the nodes.

We will now consider, on what conditions the reduction of geocentric longitudes and latitudes to heliocentric depends: or, what points, relative to the place of a planet, the position and dimensions of its orbit, are necessary to be settled previously to the accomplishment of such reduction.

Let NP be part of the orbit of a planet (superior according to the figure). $N\pi C$ part of the great circle of the ecliptic, E the Earth, S the Sun. Conceive $P\pi$ (part of a great circle) to be drawn from P perpendicularly to the plane of the ecliptic. Now a spectator at E sees P distant from the ecliptic by an



angle $PE\pi$, which is, therefore, the *geocentric latitude* (G), and P, viewed from S, would appear to be distant from the ecliptic by the angle $PS\pi$, which is, therefore, the *heliocentric latitude* (H).

Suppose ♈ to be, what is called, the first point of Aries: then, since such a point is equally distant with the fixed stars, or so distant that the diameter of the Earth's orbit subtends at it an insensible angle, a line drawn from E to ♈ is to be held to be parallel to a line drawn from S to ♈. From this point ♈ longitudes are computed, therefore,

the geocentric longitude of P (L) is $\angle\ \pi E$ ♈,
the heliocentric longitude of P (P) is $\angle\ \pi S$ ♈,
the longitude of the Sun ($\odot$).... is $\angle\ SE$ ♈.

Hence,

$$L = \odot + \angle\ SE\pi = \odot + E;$$

E representing the angle $SE\pi$, which is technically called the angle of *Elongation*.

This is the denomination of one of the angles of the triangle $SE\pi$. The angle $ES\pi$ is called the *Angle of Commutation* (C), the angle $S\pi E$, or rather, the angle SPE (the angle under which the planet sees the radius of the Earth's orbit) is called the *Annual Parallax*.

The examination of the parts of the triangle $SE\pi$, will shew us the conditions necessary for the deduction of heliocentric longitudes and latitudes from geocentric.

In the first place

$$\begin{aligned} \Upsilon S\pi\,(P) &= \angle\, SE\Upsilon + 180^0 - ES\pi \\ &= \odot + 180^\circ - C. \end{aligned}$$

Hence, we can determine P, the heliocentric longitude, if C the angle of commutation be previously determined.

SE is known from the solar theory,

$$SE\pi, \text{ or } E, = L - \odot,$$

is known since (see p. 564,) L the geocentric longitude can be computed, and the Sun's longitude is known from the solar theory: consequently, in order to determine the angle $ES\pi$ and all the other parts of the triangle, it is only necessary to know $S\pi$, which is denominated the *Curtate Distance*.

Now, $S\pi = SP\,.\cos.\,\angle\, PS\pi = r\,.\cos.\, H$,

consequently, in order to determine $S\pi$, we must know the values of r and H.

Let I (equal to the angle $PN\pi$) represent the inclination of the plane of the orbit to the plane of the ecliptic, then, by Naper's Rule for circular parts

$$\begin{aligned} 1 \times \sin.\, N\pi &= \cot.\, I\,.\tan.\, P\pi, \\ \text{or } \sin.\, N\pi\,.\tan.\, I &= \tan.\, H. \end{aligned}$$

In order then to determine H, we must previously know I, the inclination, and $N\pi$, the distance of the *reduced* place of the planet from the node of its orbit, which distance is evidently equal to the longitude of the planet minus the longitude of the node.

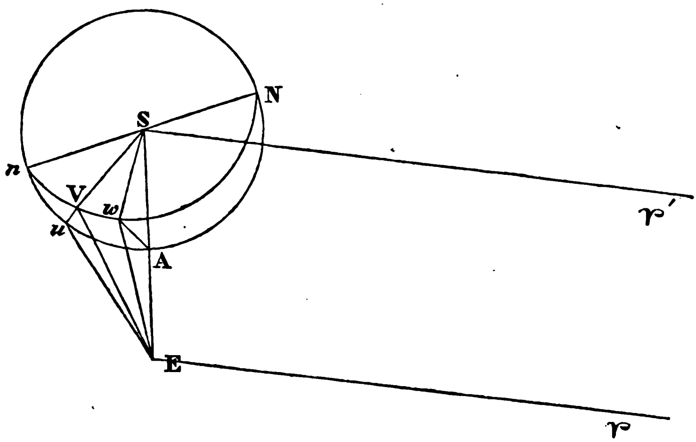
With regard to r (SP), its value may be determined, nearly, (on the supposition of a small eccentricity in the orbit) from

Kepler's law (see p. 455.). It is, however, the *mean distance* which is determined by such law. SP, therefore, is not exactly determined, except P move (which we have no reason to suppose) in a circle. If, therefore, we should be able to determine H exactly, still there would be some uncertainty in determining $S\pi = r \cdot \cos. H$, from the uncertainty respecting r's value, and, accordingly, there would be a corresponding uncertainty respecting the value of the heliocentric longitude determined from the angle $ES\pi$.

For the above reasons, since the heliocentric longitude (we are speaking of the original processes for determining the elements of a planet's orbit) cannot, generally, be exactly found, Astronomers have selected those particular positions of a planet in which its heliocentric longitude is known with certainty. Now such a position, if the planet be an inferior planet, such as Venus and Mercury are, is the superior, or inferior conjunction: in the former the planet's heliocentric longitude is equal to (☉) the Sun's longitude: in the latter, to $180^0 + ☉$. In the case of a superior planet (one whose orbit embraces that of the Earth) its heliocentric longitude, in conjunction, is equal to ☉, and in opposition, equal to $180^0 + ☉$.

In such positions, then, the heliocentric longitude of a planet is known independently of any computation of such a triangle as $SE\pi$, and of a radius SP. It is necessary, indeed, to compute its geocentric longitude by the method of p. 564. Suppose Venus to be the planet, and near to her inferior conjunction, on March 8, 1822. Compute from the passage over the meridian (which will be near to noon) and the declination, the geocentric longitude: it will be found to be greater than the Sun's longitude, which, by the Solar Tables, or the Nautical Almanack, is $11^s\ 17^0\ 23'\ 39''$: on the 9th it will also be greater, on the 10th less: so that, at some time on March 9, (when Venus is on the meridian of some other observer) which is easily found by simple proportions, the geocentric longitude will have the same value which the Sun's longitude has at the same time: and at such a time, the geocentric longitude of the planet is the same as its heliocentric.

The diagram employed in p. 566, belongs to a superior planet: but what has been shewn applies equally to an inferior planet. The angle of *elongation* of the latter can never exceed a certain quantity: thus, if NV represent its orbit, the angle



SEu is the angle of elongation, which is greatest at that point at which a line drawn from E becomes a tangent to NAu.

This greatest elongation is called *Digression:* its value in the case of Venus is about $45^0\ 42'$: not always of the same value, because both the orbits of the Earth and Venus are eccentric, and inclined to each other.

The angle SVE, the *annual parallax,* may in the case of an inferior planet, be of any value between 0 and 180.

When, however, the planet is Mars, or Jupiter, or Saturn, the angle of elongation may be of any value between 0 and 180°: but the annual parallax can never exceed a certain limit: which limit in the case of Mars is .... 53^0

of Jupiter 12

of Saturn 6

of the Georgium Sidus.. 3.

In the preceding disquisition we have endeavoured to bare to the view the real difficulties of the planetary theory, for the purpose of pointing out the way of overcoming them. They are, in many cases, to be got rid of by being eluded: and, indeed, always so to be got rid of when that is the easier way. We here allude to what has been just said respecting the particular positions in which a planet is to be observed, which are those of its conjunctions and oppositions. In such positions, the difficulties of determining the heliocentric longitudes from the geocentric are eluded; or, all cause of uncertainty, respecting the exact values of the former, rescinded. The principle of the method is to be extended to other cases. In determining the inclination of the orbit, its eccentricity, the place of the aphelion, observations of the planet, when it occupies particular positions, are to be selected, or rather, particular positions of the planet and of its orbit: for instance, such would be the observations of a planet in conjunction, and, at the same time, near to the line of its apsides.

But, in these, as in most astronomical processes, there can be prescribed no general and absolute rules. The circumstances of the case must point out the method to be pursued. We must arrive at the end as we can. The simplest way is the best. It is frequently the real triumph of science to elude difficulties that are not easily grappled with.

If we revert to what has been said in pp. 567, &c. we shall easily discern the traces of the route we must pursue. The *nodes*, the *inclination* of the orbit, the *period* with the *mean distance* and *mean motion*, are, in the first place, to be determined approximately, and on the supposition of a circular orbit. In the next place, the *eccentricity* and place of the *aphelion*, are to be determined by a comparison of the mean, with the true longitudes, or, which is the same, by a comparison of the mean with the true motions: the true longitudes being (see p. 568,) what we can obtain, independently of the knowledge of the elements of the orbit, from observations of the planet in its conjunctions or oppositions: the mean longitudes being known from the period of the planet and its longitude at a given epoch.

This, it is plain, is the description of a process which can only give approximate results. But the approximate values of the eccentricity, and of the place of the aphelion being obtained, the approximate value of the radius vector may be obtained, on which, as we shall soon shew, the determination of the place of the node depends. The place of this latter element may, therefore, by repeating the process for finding it, be more accurately found: or the approximate value of the radius vector may be applied to new or other observations for the same purpose. And it is after this manner, and not by the absolute results of any geometrical, or algebraical theorems, that the knowledge of the elements of a planet's orbit are gradually to be arrived at.

We shall proceed to give, under their separate heads, the methods of finding the elements of a planet's orbit.

Method of finding the Periodic Time, Mean Motion, and Mean Distance of a Planet.

From observations of the right ascension, and declination of the planet, compute (see p. 564,) its geocentric latitude and find when its latitude is equal to nothing. The planet is then in its node. Again, observe the planet and find when it next returns to the same node. The interval of the two computed times, is the periodic time of the planet; which may be nearly determined by one such process as has been just described, and exactly, by the mean of several; exactly, if the *retrogradation* of the nodes be not considerable.

The periodic time of Venus, found from the mean of several passages between its nodes, is, nearly, $224^d\ 16^h\ 41^m$.

The periods of Mars, Jupiter, and Saturn, may also be conveniently found by this method. But if we possess only a limited range of observations, the method loses some of its practical exactness, from our not being able to take the mean of several results. It is an excellent method for Venus, but nearly useless in the case of the Georgium Sidus.

This method, if the entrance of the planet into each node be observed, leads to something beyond the mere determination of the periodic time. It shews, whether or not the orbit be eccentric, and to what extent at least it must be eccentric: and this will appear from the following detail, which Delambre has given us for finding the period of Mars.

(1.) July 23, 1807. ♂ in his *descending* node (☋) and his southern latitude increased till December 16. If we assume this latter time to be that of his greatest latitude, and the interval (145 days) between this greatest latitude, and his being in the node, to be $\frac{1}{4}$th of his period, the period will then be equal to 580 days.

(2.) May 21, 1808. ♂ in his *ascending* node (☊), and the interval elapsed in the passage between node and node (between ☋ and ☊) was 302 days. If that interval were half the period, the period would equal 604 days.

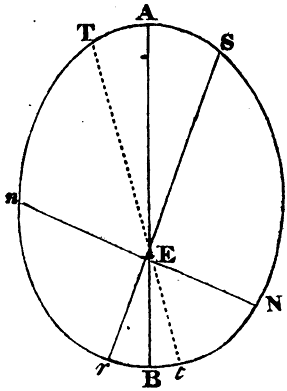
(3.) March 7, 1809. North latitude of Mars was 2° 49′, and on June 8th, was 0: at this latter time Mars had returned to his orbit, after a period of 687 days, which must be, very nearly, its true duration. The mean of several results, obtained as above, makes the period equal to

$$686^d\ 22^h\ 18^m\ 19^s.$$

Now, since the interval between node and node is not half the interval between two successive passages of the planet through the same node, it follows that the orbit is not circular, and, moreover, that the major axis is not coincident with the line of the nodes. Neither can the major axis be perpendicular to the line of the nodes: for, in that case, the planet when at the extremity of the axis, would have been at its greatest latitude, and the time from the node to the greatest latitude, would have been half the interval between node and node: whereas, (see above) the time from ☋ to the greatest latitude, was 145 days, but the time from ☋ to ☊ was 302 days ($= 2 \times 151$). This same result, however, which proves the major axis not to be perpendicular, shews also that it must be nearly so.

But we may draw farther inferences. The time from the descending to the ascending node, (from ☋ to ☊) being less than the other half of the period by the quantity 83 ($= 385 - 302$), we have (supposing Nn to represent the line of the nodes),

$$\frac{NAn - NBn}{NAn} = \frac{83}{385},$$



since the areas are proportional to the times. Now when Nn is perpendicular to AB, the difference between NAn and NBn is the greatest it can be. In such a position

$$\frac{AEN - NEB}{AEN} \text{ would equal } \frac{41}{193}, \text{ nearly,}$$

or, the time from B to N would be nearly 152 days,
and the time from N to A . . . . 193.

Now the period being, nearly, 687 days, in which the planet describes 360^0, the time of describing 90^0 would nearly equal 171 days, supposing the planet to depart from B, and to move with its mean motion: but (see l. 6,) the planet was really at N nineteen days previously: in nineteen days, however, the amount of the mean motion is equal to $360° \times \frac{19}{687}$, or nearly 10°.

At the time, therefore, the real planet was at N, the fictitious planet or body would be, nearly, 10^0 behind. Now this difference, or angular distance is no other (see Chapter XVIII.) than the *equation of the centre*. Such equation, at the point N, is not exactly, although it is nearly so, at its greatest value. *The* *greatest equation of the centre,* then, in Mars' orbit, cannot be less than 10°. In fact, it must be greater, not only from the cause just assigned, but because the difference of the times from B to N, and from N to A, would be greater than observation shews it to be, if Nn were (which it is not) perpendicular to AB the line of the apsides.

The same process for finding the period, and like inferences, relative to the degree of eccentricity, are applicable to Jupiter and Saturn. For instance, we have, according to M. Delambre,

in Oct. 13, 1794, (286 days) ♃ in ☋,
May 18, 1800, (138 days) ♃ in ☊;

therefore $5^y\ 218^d$, or 2043 days is half a revolution.

Again,

1806, 239^d .... ♃ in ☊,
1794, 286................ ♃ in ☋,

$11^y\ 318^d$, or 4335 days is the period of Jupiter.

Hence, the difference between the two half revolutions, is about 249 days: the fourth of which is $62\frac{1}{4}$, in which time Jupiter describes about $5^0\ 4'$ $\left(= 360 \times \frac{62.25}{4335}\right)$. The greatest equation, therefore, of the centre in Jupiter's orbit (see p. 575,) cannot be less than 5° 4′. The axis major of Jupiter's orbit is nearly perpendicular to the line of the nodes; which circumstance, as in the former case (see p. 575,) might be ascertained by an observation of Jupiter, at the time of his greatest latitude.

In the case of Saturn, the two half revolutions from node to node (from ☋ to ☊ and from ☊ to ☋) are nearly equal. The orbit of Saturn, therefore, is either nearly circular, or (which by other methods is proved to be the case) the line of its nodes is coincident with the axis major. We cannot in this case, from observations of the passages of the nodes, determine the quantity, than which the greatest equation cannot be less.

Since the periodic time is an important element, we will give other methods of determining it.

*Second Method of determining the Periodic Time**.

Observe the planet in opposition, then its place, with regard to longitude, is the same as if the observation were made at the Sun. Amongst succeeding oppositions, note that in which the planet is in the same part of the heavens, as at the time of the first opposition. The interval between the two similar oppositions is nearly the periodic time of the planet, or a multiple of the periodic time.

Since the planet, at the last of the two similar oppositions, will not be exactly in the place in which it was at the time of the first, the *error*, or *deviation*, must be corrected and accounted for, by means of a slight computation, similar, in principle, to several preceding computations, and the nature of which will be sufficiently explained by an Example.

Sept. 16, 1701, 2^h ♄'s long. in ☍ $353^\circ\ 21'\ 16''$ S. lat. $2^\circ\ 27'\ 45''$
(2) Sept. 10, 1730, $12^h\ 27^m$ ♄'s long. in ☍ 347 53 57 S. lat. 2 19 6
Interv. $29^y - 5^d\ 13^h\ 33^m$, diff. of long. $5^\circ\ 27'\ 19''$.

Hence, it is plain, we must find the time of describing this difference $5^\circ\ 27'\ 19''$: and the means of finding it may be drawn from other observations of the planet made in September 1731.

(3) Sept. 23, 1731, $15^h\ 51^m$ ♄'s long. in ☍ $0^\circ\ 30'\ 50''$ S. lat. $2^\circ\ 36'\ 55''$
Interval betw. (3) and (2) $1^y\ 13^d\ 3^h\ 24^m$, diff. of long. $= 12^\circ\ 36'\ 53''$

Hence,

$12^\circ\ 36'\ 53'' : 5^\circ\ 27'\ 19'' :: 1^y\ 13^d\ 3^h\ 24^m$: time required, which time $= 163^d\ 12^h\ 41^m$.

Hence, adding this time to the former interval between opposition and opposition, we have

$$\text{♄'s periodic time} = \left\{\begin{array}{rrrrl} 29^y & 7^d & 0^h & 0^m & \quad\text{(7 Bissex.)} \\ + & 163 & 12 & 41 & \\ - & 5 & 13 & 33 & \\ \hline 29 & 164 & 23 & 8 & \end{array}\right.$$

* The periodic times of planets are important elements, and admit of being very exactly determined; and when determined, become the best means of determining the mean distances, which by parallax, or other methods, are very inaccurately found.

And consequently, *Saturn's* mean motion for one year, or mean annual motion $= 360^0 \times \dfrac{1^y}{29^y\ 164^d\ 23^h\ 8^m} = 12^\circ\ 13'\ 23''\ 50'''$.

If the major axis of *Saturn's* orbit be, like that of the Earth's, progressive, then the above determination of the periodic time will not be very exact. And indeed, it ought rather to be regarded as a first approximation, and as the means of obtaining the true value of the periodic time more exactly. Using it therefore as an approximation, we may, by comparing oppositions of the planet, distant from each other by so large an interval of time, that the inequalities of the several revolutions will be mutually balanced and compensated, determine the periodic time to much greater, and indeed, to very great exactness. Thus,

228 *A. C.* March 2, 1^h ♄'s long. in ☋ $98^\circ\ 23'\ 0''$ N. lat. $2^\circ\ 50''$
(2) Feb. 26, 1714, $8^h\ 15^m$ ♄'s long. in ☋ 97 56 46 N. lat. 2 3
* Interval $1943^y\ 105^d\ 7^h\ 15^m$, diff. of long. 26 14.

In order to find the time of describing $26'\ 14''$, as before, p. 575, &c.

(3) March 11, 1715, $16^h\ 55^m$ ♄'s long. in ☋ $111^\circ\ 3'\ 14''$ N. lat. $2^\circ\ 25'$
Interval between (2) and (3) $378^d\ 8^h\ 40^m$; diff. of long. $13^\circ\ 6'\ 28''$

$\therefore$ time of describing $26'\ 4'' = 378^d\ 8^h\ 40^m \times \dfrac{26'\ 4''}{13^0\ 6'\ 28''} = 13^d\ 14^h$.

Adding this to the former interval, we have $1943^y\ 118^d\ 21^h\ 15^m$ for the interval, during which, *Saturn* must have made a complete number of revolutions. Now, if the periodic time ($29^y\ 164^d\ 23^h\ 8^m$) previously determined, had been exactly determined, then, dividing the interval by the periodic time, the result would have been an integer, the exact number of revolutions. But, the period having been only nearly determined, the result of the division (the quotient) will be an integer and some small fraction : still the number of revolutions which can only be denoted by an integer, must be denoted by that same integer. And in the case

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* 11 days are subtracted, in order to reduce it to the stile of the first observation, and 485 days added on account of the Bissextiles.

before us, it will be 66. The number of revolutions then being exactly 66, the exact time of one revolution

$$= \frac{1943^y\ 118^d\ 21^h\ 15^m}{66} = 29^y\ 162^d\ 4^h\ 27^m.$$

Hence, according to this more correct value of the periodic time, the mean annual motion is $12^\circ\ 13'\ 35''\ 14'''$, and the mean daily $2'.0097$.

In the preceding method of determining the periodic time, *Saturn* was reduced to the same *longitude*. And longitude is measured from the first point of *Aries*, which point is continually moving westward $50''.1$ annually, and therefore, in $29^y\ 162^d\ 4^h\ 27^m$ moves through $24'\ 35''$. The period, then, of *Saturn*, which has been determined ($29^y\ 162^d\ 4^h\ 27^m$) belongs to his *tropical* revolution, and is shorter than that of his *sidereal*, by the time requisite to describe $24'\ 35''$, that is, about $12^d\ 7^h$. Hence, *Saturn's* period of sidereal revolution will be $29^y\ 174^d\ 11^h\ 27^m$.

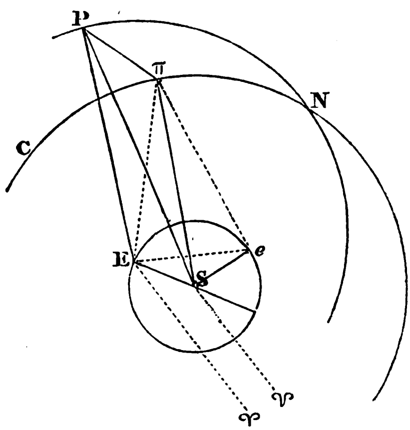
It is equally easy to determine, directly from observations, the period of the sidereal revolution. Since, instead of *reducing Saturn* to the same longitude, we should have so to reduce his place, that it should be at the same distance from a fixed star at the end, as it was at the beginning of the period.

But suppose a new planet to be discovered more distant than Saturn, must we be obliged to wait during a long term of years, to observe the successive returns of the planet to its node, in order to discover its mean period and distance, or, amongst the resources of Astronomical Science, can we find some means of supplying the defect of past observations, or of anticipating the results of observations to be hereafter made? We shall find an answer to this question by merely stating what has taken place with respect to the Georgium Sidus (or Uranus as the French call it). The planet was discovered in 1781, and in 1796 the Tables of its motions were inserted in the Nautical Almanack : indeed, so near the time of its discovery as the year 1782, the elements of its orbit, (as we find by the Memoirs of the Academy of Paris for that year) were computed by Lalaude, and, amongst such elements, that of its period was stated to be 84 years.

This then is a sort of practical answer to the question just stated, and a proof that some method, other than has been described, was resorted to by Astronomers for discovering the period, and other elements of a planet, endowed with so very slow a motion.

The method of Lalande, one of trial and conjecture (of trial indeed, which after a few times was sure of succeeding) will easily be understood by adverting to what was said in pages 566, &c.

The angle of elongation $(E) = L - \odot$, L being the geocentric longitude, and $E\pi E$ the angle of parallax, (π) is the



difference of the heliocentric and geocentric longitude, and, therefore, is equal to $P - L$.

Now E is known from L and $\odot$ (see p. 566.), and since $\sin. \pi = \sin. E \cdot \frac{SE}{S\pi}$ we can find π, and thence $P = L + \pi$, if we can find $S\pi$, or, which is the same thing, if we *assume* a value (r) for $S\pi$ ($S\pi$ and SP are nearly equal) we shall have from the above equation of a corresponding value of π, and thence of P: suppose its value to be P'. Use the same process, with the same assumed radius $S\pi$, with a second and third geocentric longitude, and let the two resulting heliocentric longitudes be P'' and P''', then we have

$$P'' - P',\ P''' - P'',\ \text{and}\ P''' - P',$$

and from knowing the three times of observations (t', t'', t''') we know

$$t'' - t',\ t''' - t'',\ \text{and}\ t''' - t'.$$

Take any one of these three differences, the last, for instance, then

$$P''' - P' : t''' - t' :: 360^0 : \text{period of the planet.}$$

But r is the assumed mean distance, accordingly, by Kepler's law (see p. 455,)

$$1^{\frac{3}{2}} : r^{\frac{3}{2}} :: 365.256384 : \text{planet's period.}$$

The agreement of this value with the former would be a proof that r had been rightly assumed. The disagreement, by its nature and degree, would point out to us the manner and extent of correcting the first assumption of r.

This is a description of the method which Lalande employed. He possessed three geocentric observations of the planet (♅) made in 1781, on April 25, July 31, and December 12, and he found the period (according to the method just described) by means of the first and third observation. The two values of the period (as it was probable they would) were found to disagree. Lalande, therefore, amended his first assumption: and assigned, partly by conjecture, and partly by the guidance of his first trial, a new value of the distance, and then examined it, as the former. By a repetition of like trials and examinations a radius vector was at length obtained, which agreed with all observations *.

* This method of M. Lalande's, is a kind of sample and exemplar of almost all astronomical processes. In these, at first, nothing is determined exactly. Approximate quantities are assumed and substituted, the results derived from them, examined and compared, and then other approximations, probably nearer to the truth, suggested. Astronomy leans for aid on Geometry; but the precision of Geometry does not extend beyond the

We may state somewhat differently, but without any alteration of principle, the above process of approximation.

Should the first, or any observations of the planet shew the angle of elongation ($= L - \odot$) to be obtuse, the planet must be a superior one: in which case, 1 being the mean distance of the Earth, r must be > 1.

Assume $r = 1.5,\ 2,\ 2.5,\ 3,\ 3.5,$ &c.

and form the corresponding values of π from

$$\sin. \pi = \frac{\sin. E}{r}:$$

thence, write down the corresponding values of

$$P = L + \pi.$$

Repeat these operations on succeeding observations, and then, by subtracting the heliocentric longitudes of one day, from those of the preceding day, deduce the heliocentric motions of the planet; suppose dP to represent this motion, and $d\odot$ the Sun's daily motion, then, since the angular velocity

$$= \frac{\text{area described in a given time}}{(\text{dist.})^2}$$

$$= \frac{\text{whole area}}{\text{period}} \times \frac{1}{(\text{dist.})^2},$$

and since the whole areas (if the orbits be circular) vary as the squares of the radii, and the periods vary as $(\text{radii})^{\frac{3}{2}}$, we have

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the limits of its theorems. In Astronomy scarcely one element is presented simple and unmixed with others. Its value when first disengaged, must partake of the uncertainty to which the other elements are subject; and can be supposed to be settled to a tolerable degree of correctness, only after multiplied observations, and many revisions. There are no simple theorems for determining at once the parallax of the Sun, the right ascension of a star, or the heliocentric latitude of a planet.

$$dP : d\odot :: \frac{r^2}{r^{\frac{3}{2}}} \times \frac{1}{r^2} : 1;$$

$$\therefore r = \left(\frac{d\odot}{dP}\right)^{\frac{2}{3}},$$

from which expression, since $d\odot$ is known from the Solar Tables, or the Nautical Almanack, r may be computed, and its several values corresponding to the several values of dP. Of the originally assumed values of r (see p. 580, l. 7,) that which, most nearly, approaches to one of these lastly deduced values of r, is the value nearest to the truth. Thus suppose one of the values from the expression

$$r = \left(\frac{d\odot}{dP}\right)^{\frac{2}{3}},$$

should be 19.3, then, since 19.5 is, of the originally assumed values, nearest to 19.3, we may conclude 19.5 to be nearly the true value, and whether the true value is between 19 and 19.5, or between 19.5 and 20, must be inferred from the two contiguous values of r, namely, from

$$r = \left(\frac{d\odot}{dP}\right)^{\frac{2}{3}}, \text{ and } r' = \left(\frac{d\odot}{dP}\right)^{\frac{2}{3}}.$$

The periodic time of a planet (P) being found, its mean daily motion (M) may be thence derived from this proportion,

$$P : 1 :: 360^{0} : M = \frac{360}{P},$$

P being expressed in days and parts of a day.

Thus, in the case of Venus, P being $225^{d}\ 16^{h}\ 41^{m}$, the mean motion is

$$= \frac{360}{225^{d}\ 16^{h}\ 41^{m}} = \frac{1^{0}}{.62415319} = 1^{0}.6027 = 1^{0}\ 36'\ 9''.7.$$

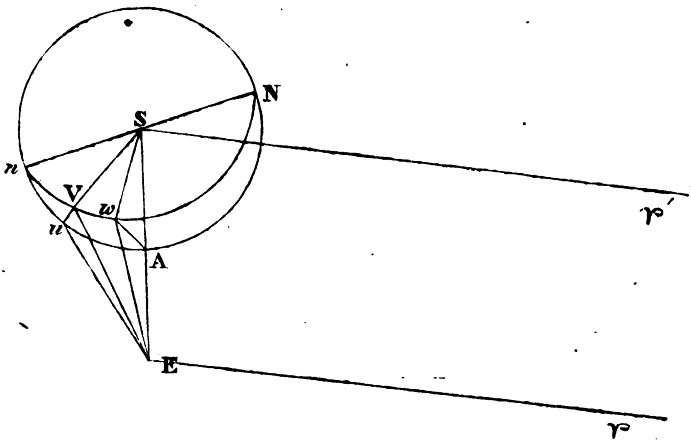
The mean distance (a) may be found by Kepler's law. Thus, 1 representing the Earth's mean distance from the Sun, and $365^{d}.256384$ being the value of the Earth's sidereal period,

$$(365.256384)^{\frac{2}{3}} : p^{\frac{2}{3}} :: 1 : a = \left(\frac{p}{365.256384}\right)^{\frac{2}{3}} *.$$

But although this is the best, it is not the only way of finding the distance of a planet. The distance of Venus may be found from her greatest elongation (technically called her *digression*). Thus, by examining a series of angles of *elongation* (E) formed from the expression

$$E = \pm (L - \odot),$$

it is found, that the greatest value of E is about $45^0\ 42'$, and



when E is the greatest, the angle SuE is a right angle, Eu being a tangent to nuA. In this case, then,

$$\dagger\ Su = SE \,.\, \sin.\ 45^\circ\ 42' = .7157, \text{ if } SE = 1.$$

These *digressions* of Venus would all be of the same value, if Venus and the Earth revolved in circular orbits. But, as we have

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* This is not exactly true: let μ = Sun's mass + the planet's mass,
μ' = Sun's mass + Earth's mass;

$$\text{then } \left(\frac{365.256384}{p}\right)^2 = \frac{\mu}{\mu'} \times \left(\frac{1}{a}\right)^3,$$

is the exact equation from which a is to be deduced, (see *Physical Astronomy*, p. 30.)

† Vu should have been more inclined to SV, and then Su would be a line drawn from S to v.

seen (p. 449,) SE is a variable distance. Still the differences in the values of the *digressions* cannot be accounted for, by estimating the effects of the eccentricity of the Earth's orbit: the inference from which circumstance is, that Venus's orbit is also elliptical.

There are particular conjunctures from which, on the supposition of the orbit of Venus being elliptical, we could determine the value of its eccentricity. Suppose, for instance, we possessed, amongst our observations, two *digressions* (E and E'), one made when Venus was at the aphelion of her orbit, the other at the perihelion; in that case, if e were the eccentricity, R and R' the distances of the Earth from the Sun, we should have (r being the mean distance of Venus),

$$r + re = R \,.\, \text{sin. } E,$$

$$r - re = R' \,.\, \text{sin. } E',$$

$$\text{whence} \quad e = \frac{R \,.\, \text{sin. } E - R' \text{ sin. } E'}{2\,r}$$

$$= \frac{R}{2\,r}(\text{sin. } E - \text{sin. } E'),$$

if we suppose $R = R'$.

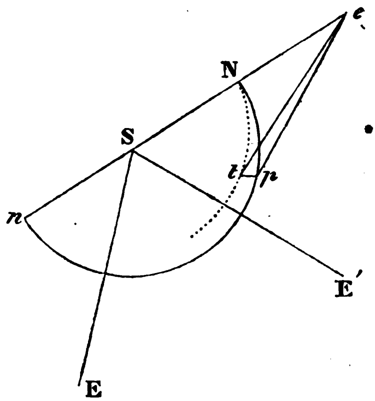
We might also, (could we rely on the accuracy of the measurements) determine the relative values of the radii of the orbits of Venus and the Earth, from the apparent diameters of the former planet, at her greatest and least distances. Thus, should the least and greatest apparent diameters be, respectively, $10''$ and $60''$, we should have

$$\frac{60}{10} = \frac{1 + r}{1 - r}, \text{ and } r = \frac{5}{7}.$$

Method of determining the Nodes of a Planet's Orbit.

The nodes of a planet's orbit, are those two points in it in which it is cut by the ecliptic. The node which the planet quits in ascending towards the north pole of the ecliptic, is called the *Ascending Node*, and its symbol is ☊. The reverse or ☋, is the symbol of the *descending node*, or, of that node from which the planet moves towards the south pole of the ecliptic.

Let N, n represent the nodes; now by observations of the planet's right ascension and north polar distance, we can compute its geocentric latitude (see p. 563,) and thence determine



when the latitude is 0, or when the planet is in its node: let E, E' be the two positions, when the planet is respectively at n and N, then we have (see p. 582,)

$$*SEn = \text{geocentric longitude of planet at } n - \odot,$$
$$SE'N = \odot' - \text{geocentric longitude of planet at } N,$$

and from the last method we know Sn, or SN; thence we can compute, in the triangles SEn, $SE'N$, the angles nSE, SnE, and NSE', SNE': and thence

$$\text{heliocentric long. of } n = \text{geocentric long. of } n + \angle SnE,$$
$$\text{or} = 180^\circ + \odot \ldots\ldots - \angle nSE,$$
$$\text{and helioc}^c\text{. long. of } N = \text{geocentric long. of } N - \angle SNE',$$
$$\text{or} = \odot' - 180^\circ \ldots\ldots + \angle NSE',$$

$\odot$ and $\odot'$ representing the Sun's longitudes at the two times of observation.

The angle ESE' is proportional to the Earth's motion during the planet's passage from n to N.

---

* Conceive two lines drawn from E and E' to n and N, respectively.

Venus, of which the period is less than 225 days, may, in the space of a year, be observed three times in the ecliptic; the longitude of the node is, according to astronomical usage, to be estimated from the mean of a great number of observations at n and N.

In the above method, we have supposed the planet to be successively at n and N: but one observation is sufficient, as far as the principle of the method is concerned, to determine the longitude of the node. For example, in May 14, 1747, Mars was observed to be descending towards, and to be very near to, his descending node. By continuing the observations, and by a computation like that described in p. 575, Mars was found to be in his node on May 14, at $14^h\ 25^m\ 13^s$, whilst his geocentric longitude was computed to equal $7^s\ 6^0\ 13'\ 42''$.

Hence,

$$
\begin{array}{rrrrrr}
 & L = & 7^s & 6^0 & 13' & 42'' \\
\text{by Solar Tables } & \odot = & 1 & 23 & 46 & 47 \\
\hline
\therefore \text{(see p. 584, l. 9,) } & L - \odot \text{ or } E = & 5 & 12 & 26 & 55
\end{array}
$$

$$\text{but sin. } \pi\,(SnE) = \text{sin. } E \times \frac{SE}{Sn};$$

Sn being taken equal to 1.5446, and SE to 1.008;

$$
\begin{array}{rrrrrr}
 & \therefore\ \pi = & 0^s & 11^0 & 22' & 55'' \\
\text{but (see p. 584, l. 10,) } & L = & 7 & 6 & 13 & 42 \\
\hline
\therefore \text{ heliocentric longitude of } n, & \text{or } \pi + L = & 7 & 17 & 36 & 37
\end{array}
$$

which is the longitude of the descending node of Mars, at the time of observation.

The angle π (see l. 19,) depends on the value of $\frac{SE}{Sn}$.

The numerator SE is known from the solar theory: but the preceding method of pages 580, &c. determines solely the *mean distance* of Mars. If, therefore, from original osbervations, we were about to deduce the elements of that planet's orbit, we could only, in the first steps of the deduction, approximate to the longitude of the node: because we should, in such first steps, be obliged to consider the orbit of Mars as circular, or, which is the same thing, we should be obliged to assume for Sn that value of the mean distance, which would result from the expression

$$Sn = \left(\frac{686.979619}{365.256384}\right)^{\frac{2}{3}} = 1.523694.$$

In this case then (see p. 585, l. 19,), we should have

$$\log.\ \sin.\ \pi = \log.\ \sin.\ 5^s\ 12^\circ\ 26'\ 55'' + .00471 - .1828965$$
$$= 9.3011888 = \log.\ \sin.\ 11^\circ\ 32'\ 28''.$$

Hence, the first approximate value of the longitude of the node would be greater than the one deduced by $9'\ 33''$: which is the error caused by supposing Mars' orbit to be circular, for the value of Sn in p. 585, was taken from the Tables of Mars.

When we determine, as above, the longitude of the node, from computing the time of the planet's entering the ecliptic, we do not require to be known the *inclination* of the planet's orbit. In a scientific arrangement, the determination of that element would be placed, after that of the node. But if we suppose the inclination to be known, or (which is the real astronomical usage) if, in performing the circuit of revision and correction, we wish, from an approximate value of the inclination, to correct by means of recorded observations, the elements of the orbit, we may compute the place of the node, by slightly modifying the above method. Thus, in the instance given, the observations of Lacaille were as follow :

May 14, 1749, $10^h\ 50^m\ 43^s$. geo. long. ♂ (L) $7^s\ 6^\circ\ 15'\ 20''$, lat. $25''.5$

Sun's long.	1	23	38	10
E	5	12	37	10
(from $\sin.\ \pi = \sin.\ E \,.\, \frac{1.008}{1.5446}$) π	0	11	16	37
(heliocentric long. ♂); $\therefore\ \pi + L$	7	17	31	57.

But this is the heliocentric longitude of Mars, when his geocentric latitude was $25''.5$. If we could thence find the heliocentric latitude, and knew the inclination of the orbit to the ecliptic, we could thence deduce (see Figure of p. 582,) nu. With regard to the first point, the deduction of the *heliocentric* from the *geocentric* latitude, since Vu is a tangent to the angles VSu, VEu to the respective radii Su, Eu*,

$$Su \,.\, \tan. VSu = Eu \,.\, \tan. VEu,$$

but $\dfrac{Su}{Eu} = \dfrac{\sin. E}{\sin. C}$ (C being ESu the angle of commutation)

$$\begin{array}{rl} \text{and since } E = & 5^s\ 12^\circ\ 37'\ 10'' \\ \pi = & 0\ \ 11\ \ 16\ \ 37 \\ \text{it is necessary that } C = & 0\ \ \ 6\ \ \ 6\ \ 13 \\ \hline & 6\ \ \ 0\ \ \ 0\ \ \ 0 \end{array}$$

Hence, $\tan. VSu = \tan. 25''.5 \times \dfrac{\sin. 6^\circ\ 6'\ 13''}{\sin. 17^\circ\ 22'\ 50''} =$ $\tan. 9''.2$; $\therefore 9''$ is nearly the heliocentric latitude, which being very small, we may consider the right-angled triangle nVu as right-lined, and solve it accordingly: which we can do, if the angle Vnu (the *inclination*) be known. Let it be $1^\circ\ 51'$, then $nu = 4'\ 41''$, nearly, which being added to $7^s\ 17^\circ\ 31'\ 57''$, (the heliocentric longitude of ♂ decending towards and very near to, its node) there results for the heliocentric longitude of the node

$$7^s\ 17^\circ\ 36'\ 38'',$$

which, within one second, is the result of p. 585, l. 23.

In these methods, the determination of the place of the node is the more difficult the less is the inclination of the planet's orbit. For that reason it is difficult to determine the nodes of the orbits of Jupiter and the Georgium Sidus.

Method of determining the Inclination of the Orbit of a Planet to the Plane of the Ecliptic.

The longitude of the node being known by the preceding methods, compute the day on which the Sun's longitude will be the same, or nearly the same. The Earth will then be in the line of the nodes Nn, at some point e (fig. of p. 584,): observe, on that day, the planet's right ascension and north polar distance, and deduce (see pp. 563, &c.) the geocentric latitude (G);

* The lines SV, Vu should have been more bent to each other than they are in the Figure.

$$\text{then } tp = et \,.\, \tan.\, G = St \,.\, \frac{\sin.\, tSe}{\sin.\, Sep} \,.\, \tan.\, G$$

$$= \frac{\sin.\, Nt}{\sin.\, E} \,.\, \tan.\, G,$$

but, in the right-angled triangle Ntp, we have by Naper's Rules,

$$\sin.\, Nt = \cot.\, tNp \,.\, tp, \text{ or } \tan.\, I \,.\, \sin.\, Nt = tp,$$

I denoting the inclination,

$$\text{accordingly, } \tan.\, I \,.\, \sin.\, Nt = \frac{\sin.\, Nt}{\sin.\, E} \,.\, \tan.\, G,$$

$$\text{and } \tan.\, I = \frac{\tan.\, G}{\sin.\, E}.$$

The diagram that has been referred to belongs to an inferior planet: but, a like diagram, and the same process, will apply to a superior planet.

As an instance of the method, suppose we possessd the following observations, on Jan. 12, 1747, $6^h\ 6^m\ 33^s$:

long. ♄	$6^s\ 26^\circ$	$12'$	$52''$,	lat. N. $2^\circ\ 29'\ 18''$
on the above day, ☉, or the Sun's long.	9 21	47	0	
∴ E	2 25	34	8	
Now, by Lalande's Table, ☊ or long. of node ...	9 21	31	0	

or the Earth was, then nearly, in a position such as e.

Hence, from the expression of l. 7,

log. tan. $2^\circ\ 29'\ 18''$	8.6380591
log. sin. 85 34 8	9.9986999
	8.6393592

and this result is the logarithmic tangent of $2^\circ\ 29'\ 44''.8$, which, accordingly, is the value of the inclination of Saturn's orbit from the above observation, and which must be very nearly its true value.

It is not its *exact* value, because the Sun's longitude being greater than the longitude of the node by $15'$, the Sun at the time of observation, had passed the line of the nodes. About 6 hours previously, the Sun was in the line. In order, therefore, to correct the above result, we must correct, proportionally to such time, the geocentric latitude, and the geocentric longitude, and, consequently, (see p. 586, l. 26,) the angle E. The corrected place of the node is then to be deduced from the expression

$$\tan. I = \frac{\tan. G}{\sin. E},$$

G and E being now the corrected values.

But it is plain that this last result will differ very little from the former: for, the angle of elongation being 85° 34′ 8″, and the angle of parallax about 6°, the remaining angle of the triangle formed by the Earth, Sun, and Saturn, or the angle of *commutation*, will be 91° 34′: consequently, Saturn will be nearly at the same distance, both from the Sun and the Earth, and his heliocentric latitude will not differ much from his geocentric: but the latter is 2° 29′ 18″; therefore, since the inclination (which is measured by the greatest heliocentric latitude) is 2° 29′ 44″.8, the planet must be nearly at its greatest heliocentric latitude, and quantities, at or near to their *greatest* values, change very slowly.

The angle of elongation will vary with the geocentric longitude, and accordingly, in the present case, very little: but the inclination (see p. 588,) depends on the sine of the angle, which angle is between 85° and 86°, and consequently not far from that value at which the sine is a maximum. In this case then, as in the former, scarcely any alteration will take place in the new value of the sine of E. Hence, in the expression $\tan. I = \frac{\tan. G}{\sin. E}$, the resulting value of I will be nearly the same whether we use the original or the corrected values of G and E: or, which is the same thing, the inclination was very nearly determined by the first calculation.

---

* Log. sin. E 9.99870
log. ♄'s dist. .... .97949
9.01921 = log. sin. 6°.

The inclination may also be determined from observing the planet at a conjunction, when it has considerable latitude. Thus, suppose the planet to be Venus, at a point w of her orbit, (see fig. of p. 582,) such that A the reduced place in the ecliptic is in the same straight line with E and S: then, as before, we have

$$EA \,.\, \tan.\ AEw = SA \,.\, \tan.\ ASw.$$

$$\text{Let } SE = 1,\ \ SA = \rho,\ \ Sw = r,\ \ ASw = H,$$

$$\text{then } (1 - \rho) \,.\, \tan.\ G = \rho \,.\, \tan.\ H.$$

But in the right-angled triangle Anw (right angled at A),

$$\sin.\ nA \,.\, \tan.\ I = \rho \tan.\ H;$$

$$\therefore\ (1 - \rho)\ \frac{\tan.\ G}{\sin.\ nA} = \tan.\ I.$$

Now nA is the longitude of the planet minus the longitude of the node. The latter quantity is supposed to be known by the preceding methods, and, the planet being in conjunction, its longitude is the same as the Sun's longitude: hence, if ☊ denote the longitude of the node n,

$$\tan.\ I = (1 - \rho)\ \frac{\tan.\ G}{\sin.\ (\odot - ☊)}\ {}^{*},$$

$$\text{but } \rho = r \,.\, \cos.\ I = \frac{r}{\sec.\ I} = \frac{r}{\sqrt{(1 + \tan.^2 I)}} = r \times \left(1 - \frac{1}{2} \tan.^2 I\right)$$

---

* The inclination of the orbit of Venus is about 3° 23′: suppose such an inferior conjunction to be observed, that the planet is 90° from its node: then $\odot - ☊ = 90°$, and

$$\tan.\ G = \frac{\sin.\ 3°\ 23'}{.276} = 214, \text{ nearly, and}$$

$$G = 12°\ 5'.$$

Again, suppose a like superior conjunction to be observed, then

$$\tan.\ G = \frac{\tan.\ I}{1 + \rho} = \frac{\tan.\ 3°\ 23'}{1.723} = .0343,$$

$$\text{and } G = 1°\ 58', \text{ nearly.}$$

Hence, as Delambre observes, it would be necessary, in order that Venus should be always seen in the zodiac, that the breadth of the zodiac should be, at the least, 24°.

if (as is almost always the case) I be very small, hence,

$$\tan. I = \left(1 - r + \frac{r}{2} \tan.^2 I\right) \frac{\tan. G}{\sin. (\odot - \Omega)},$$

from which I may be obtained by approximation, or the solution of a quadratic equation, or, in the expression of p. 590, if we make $\rho = r$, we may thence deduce an approximate value of I, which approximate value being substituted in $\rho = r \cos. I$, we may, from the same equation, obtain a new value of tan. I.

We have now obtained the mean distance, the longitude of the node, and the inclination of the orbit of a planet: but, hitherto, nothing has been determined respecting the form of the orbit: indeed, in some of the previous determinations, we have been obliged to suppose the orbit circular, or to assume for the radius vector of the planet's orbit, its mean distance as it results from Kepler's law. We must now consider whether the steps that have been made good, will enable us to proceed farther, and to find out, what probably, and by analogy, exists, the *eccentricity* of the orbit; and then the place of the aphelion.

We have already seen, in a particular instance, from certain differences in the *digressions* of Venus, that her orbit is eccentric: but our present concern is, with some general method, of ascertaining and valuing the eccentricity and place of the aphelion of the orbit of any planet. It will not be difficult to find out the grounds of such method.

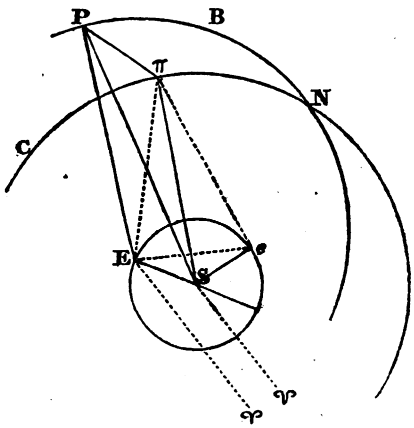
Suppose, for the sake of simplicity, the planet's orbit to lie in the plane of the ecliptic. Since, (see pp. 571, &c.) we know the mean motion, and, by observing the planet in conjunction, or opposition, the planet's true longitude (see p. 568,) we can, after any elapsed time, compute the planet's mean longitude. Let the elapsed time be the interval between two conjunctions: then, if the orbit were circular, the computed mean longitude would agree with the last observed longitude *; but a difference between them would be an indication of the orbit's eccentricity.

* Except, which is highly improbable to happen, the planet, at the times of the two conjunctions, should be in the aphelion, or perihelion of its orbit: for at those points the mean and true anomalies are the same.

This difference must depend both on the eccentricity, and the place of the aphelion. It must depend upon the former, because if, in a given position of the orbit, the eccentricity were increased, the difference between the computed and observed longitudes would also increase. It must depend on the place of the aphelion, because, if the planet be there at the time of the observed conjunction, the true and computed places of the planet will agree. The differences then of the computed and observed longitudes depend on the eccentricity, and the position of the axis major of the orbit, and it is a fit subject of mathematical investigation, to deduce the eccentricity and the place of the aphelion, from such differences.

We will now consider what effect on the preceding reasonings will be produced by restoring to the orbit its inclination.

Let N be the node of the orbit, then its longitude (see p. 583, &c.) is known. The longitude of the planet, when in conjunction, is



known, since it equals $180^0 + \odot$. Hence, deducting the longitude of the planet from the longitude of the node, there remains $N\pi$. Now since the elliptical motion takes place in the orbit NP, it is requisite to know NP, and like distances of the planet in its orbit from the node. But $N\pi$ being known, and the angle $PN\pi$; the distance NP may be determined, either by the solution of the triangle $PN\pi$ (right angled at π) or (see pp. 505, &c.)

by the formula of reduction: for, it is plain, the finding of NP, from $N\pi$ and the angle $PN\pi$, is analogous to the finding of the longitude, from the right ascension and obliquity. In the formula, therefore, of p. 506, l. 8, write NP instead of $\odot$, and $N\pi$ instead of Æ, and let t be the tangent of inclination, then

$$NP = N\pi + t^2 \cdot \frac{\sin. 2\, N\pi}{\sin. 1''} + t^4 \cdot \frac{\sin. 4\, N\pi}{\sin. 2''} + \&c.$$

If we set off, on the orbit of the planet, an arc $(A) = N\gamma$ the longitude of the node, we shall have $A + NP$, which is called the *longitude of the planet on its orbit:* and, accordingly, we shall have as many such longitudes, or as many such distances as NP, as there are observations of the planet in conjunction, or opposition.

Now three such observations are sufficient to determine the two elements of the eccentricity, and place of the aphelion: for, if we have three longitudes on the orbit (V, V', V'') we have, by taking the differences of the second and first, and of the third and second, two *differences* of longitudes, and, since the planet's period is known, we can compute two portions of its mean motion, corresponding to the two noted intervals of time, between the second and first observation, and between the third and second observation. The two differences of real longitudes compared, according to the elliptical theory, with the corresponding portions of mean motion, will give us two equations for determining the eccentricity and place of the aphelion.

Thus, suppose we have three observations of conjunctions or oppositions, then we know the three corresponding longitudes of the planet on the ecliptic, and, deducting from each the longitude of the node, we know three such arcs as $N\pi$, and by the formula of reduction, three such arcs on the orbit as NP, and, lastly, by adding to each the longitude (A) of the node, set off on the orbit, we know three longitudes on the orbit, such as $A + NP$: let these be, respectively, V, V', V'', and let e be the eccentricity (supposed to be very small), ϕ the longitude of the perihelion, the place of which, suppose to be at some point (B)

between N and P: let M, M', M'', be the mean anomalies reckoned from B: then we have (see Chapter XVIII.)

$$BSP = M + 2e \,.\, \sin.\, 2M, \text{ nearly,}$$
$$\text{or } V - \phi = M + 2e \,.\, \sin.\, (V - \phi), \text{ nearly,}$$
$$\text{similarly } V' - \phi = M' + 2e \,.\, \sin.\, (V' - \phi),$$
$$V'' - \phi = M'' + 2e \,.\, \sin.\, (V'' - \phi).$$

Hence, by subtraction

$$V' - V = M' - M + 2e \,.\, \{\sin.\, (V' - \phi) - \sin.\, (V - \phi)\},$$
$$V'' - V' = M'' - M' + 2e \,.\, \{\sin.\, (V'' - \phi) - \sin.\, (V' - \phi)\},$$

or

$$(1)\ (V' - V) - (M' - M)(= a) = 2e\,\{\sin.\, (V' - \phi) - \sin.\, (V - \phi)\},$$
$$(2)\ (V'' - V') - (M'' - M')(= b) = 2e\,\{\sin.\, (V'' - \phi) - \sin.\, (V' - \phi)\}.$$

Now V, V', V'' are known (see p. 568,) and $M' - M$, $M'' - M'$ are known from the period of the planet, and the times elapsed: thus, if t be the interval between the observations of V and V',

$$\text{planet's period} : 360^0 :: t : M' - M = \frac{t}{\text{period}} \times 360^\circ.$$

Hence, since a and b are known, we have two equations for determining e and ϕ.

Divide equation (1) by equation (2), then

$$\frac{a}{b} = \frac{\sin.\, (V' - \phi) - \sin.\, (V - \phi)}{\sin.\, (V'' - \phi) - \sin.\, (V' - \phi)},$$

the numerator of this fraction

$$= \sin.\, (V' - \phi) \,.\, \left(1 - \frac{\sin.\, (V - \phi)}{\sin.\, (V' - \phi)}\right)$$

$$= \sin.\, (V' - \phi) \,.\, \left(1 - \frac{\sin.\, V \cos.\, \phi - \cos.\, V \,.\, \sin.\, \phi}{\sin.\, V' \,.\, \cos.\, \phi - \cos.\, V' \,.\, \sin.\, \phi}\right)$$

$$= \sin.\, (V' - \phi) \,.\, \left(1 - \frac{\sin.\, V - \cos.\, V \,.\, \tan.\, \phi}{\sin.\, V' - \cos.\, V' \,.\, \tan.\, \phi}\right)$$

$$= \sin.\, (V' - \phi) \,.\, \left(\frac{\sin.\, V' - \sin.\, V - \tan.\, \phi\, (\cos.\, V' - \cos.\, V)}{\sin.\, V' - \cos.\, V' \,.\, \tan.\, \phi}\right),$$

similarly, the denominator of the above fraction (l. 21,)

$$= -\sin.(V'-\phi)\cdot\left(\frac{\sin. V' - \sin. V'' - \tan.\phi\,(\cos. V' - \cos. V'')}{\sin. V' - \cos. V'.\tan.\phi}\right).$$

Hence,

$$\frac{a}{b} = \frac{\sin. V' - \sin. V - \tan.\phi\,.(\cos. V' - \cos. V)}{\sin. V'' - \sin. V' - \tan.\phi\,(\cos. V'' - \cos. V')}$$

and, accordingly,

$$\tan.\phi = \frac{a.(\sin. V'' - \sin. V') - b.(\sin. V' - \sin. V)}{a.(\cos. V'' - \cos. V') - b.(\cos. V' - \cos. V)},$$

which is an equation for determining ϕ, the longitude of the perihelion.

In order to determine the eccentricity, we have, ϕ being determined by the preceding equation,

$$e = \frac{a.\sin. 1''}{2.[\sin.(V' - \phi) - \sin.(V - \phi)]}$$

$$= \frac{\frac{1}{4}\,a.\sin. 1''}{\sin.\frac{1}{2}(V' - V).\cos.\left(\frac{V' + V}{2} - \phi\right)}.$$

By these means ϕ and e* are approximately determined: and if we use their approximate values, we may extend the series for $V - \phi$, &c. (see *Physical Astronomy*, p. 32,) and obtain nearer values for $(V' - V) - (M' - M)$, &c. or for a and b, and thence, by means of the equations of l. 5, nearer values of ϕ and e.

The eccentricity (e), the longitude of the perihelion (ϕ), and the axis major ($2a$), being determined, we are able to compute the radius vector (r) from the expression

$$r = \frac{a.(1 - e^2)}{1 + e.\cos.(V - \phi)},$$

---

* The eccentricity and place of the aphelion are often *mathematically* determined by the solution of a problem, of which the conditions are, three given radii vectores, and three given longitudes: but it is plain, from the preceding matter, that the first condition, (that of the *given* radii vectores,) is not easily to be obtained. The knowledge of the period, leads only to the knowledge of the *mean* distance.

and, since the place of the node, and the inclination of the orbit are determined, we are able to compute (see figure of p. 592,) the curtate distance $S\pi$, on the supposition that SP, from which it is deduced, is the radius vector in an elliptical orbit. If, therefore, in any of the processes for determining the elements, the curtate distance $S\pi$ has been supposed to be derived from SP, considered as a mean distance, or constant radius (see p. 567,) we may now, with a truer value of $S\pi$, repeat the processes and correct their results.

The elements of a planet's orbit being now obtained, we will proceed to consider by what means those elements are to be employed in forming Tables of the planets' motions; and, then, by what methods, either recorded or future geocentric observations may be applied to the correction of existing Tables. These subjects will be briefly considered in the ensuing Chapters.

CHAP. XXV.

On the Formation of Tables of the Planets.—The Variations of the Elements of their Motions.—The Processes for deducing the Heliocentric Places of Planets from Tables.

In the planetary theory, as in the solar, the described orbits are supposed to be elliptical. The same process then, which, in the latter theory, gave us the Sun's true anomaly and radius vector from the mean anomaly, will give us (changing what ought to be changed) a planet's true anomaly, whether the planet be Venus, or Saturn.

This regards the elliptical place to be found by Kepler's problem. But the Earth being, according to the doctrine of universal gravitation, disturbed by the action of the Moon and the planets, does not describe an orbit exactly elliptical. By parity of reason, neither Venus nor Saturn can move in orbits exactly elliptical. Each disturbs the other. Their places, therefore, like the Sun's place, require a small correction, or rather several small corrections due to the several planets.

But as in no case these corrections for planetary perturbation are large, so in some they are too small to be worth taking account of. Mercury and Venus are in the above predicament. Their Tables are constructed solely by means of Kepler's problem, and are, therefore, much more easily constructed than the Tables of the other planets. The longitudes of Mercury and Venus are, accordingly, to be had very readily from their Tables. For instance, suppose it were required to find Mercury's longitude in his orbit.

	Longitude.				Aphelion.			
Epoch for 1793,	2^s	28^0	$5'$	$16''$	8^s	14^0	$14'$	$17''$
Mean motion to June 3,	9	0	13	34	0	0	0	24
.... for 5^h	0	0	51	9				
Mean longitude	11	29	9	59	8	14	14	41
Equation of centre	—	23	39	58.5	11	29	9	50
Longitude on orbit	11	5	30	0.5	3	14	55	9
					the mean anomaly.			

This is a process precisely similar to that by which in pp. 489, 490, the Sun's longitude was found: and, to a certain extent, all other processes for computing the longitude of a planet, be it Mars, or Jupiter, or Saturn, must resemble it, inasmuch as Kepler's problem is, in all, the main instrument in procuring a result.

The result by Kepler's problem solely, is the planet's elliptical place: which, in the case of the Earth, Mars, Jupiter, Saturn, and the Georgium Sidus, requires a correction. We will give an instance of Mars' longitude taken from his Tables.

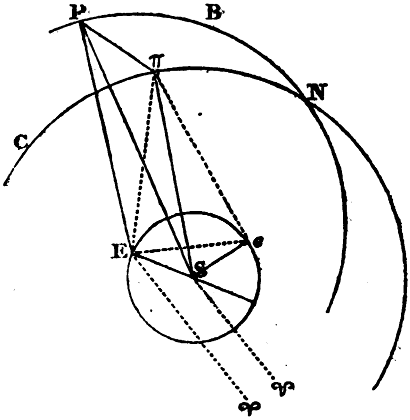
Required the Heliocentric Longitude and Latitude of Mars,
Nov. 13, 1800, 11^h 8^m 20^s.

	Longitude.	*Aphelion.*	*Node.*
Epoch for 1800	7^s 22^0 $34'$ $21''.8$	5^s 2° $23'$ $17''$	1^s 18° $1'$ $1''$
Nov.	5 9 19 3.4	0 0 0 55.8	0 0 0 22.8
13^d	0 6 48 46.5	0 0 0 2.4	0 0 0 1
11^h	0 0 14 24.7		
8^m	0 0 0 10.5		
20^s	0 0 0 0.4		
Mean longitude	1 8 56 47.3	5 2 24 15.2	1 18 1 24.8
(*e*) Sum of equa.	0 10 13 26.9	1 8 56 47.3	1 19 10 14.2
Long. on orbit	1 19 10 14.2	8 6 32 32.1	0 1 8 49.4
Reduction. . . .	0 0 0 − 2.2	the mean anom.	argument of lat.
Heliocen. long.	1 19 10 12		Heliocen. lat. $= 0^\circ$ $2'$ $13''.4$ N.

In this process, *e*, the sum of the equations, contains, besides the equation of the centre ($= 10^0$ $13'$ $13''.5$), three small equations arising from the perturbations of *Venus*, the *Earth*, and *Jupiter*. The sum of these three equations is $13''.4$, which added to the equation of the centre make *e*.

The reduction $-$ $2''.2$, applied to the longitude on the orbit, gives the heliocentric longitude, measured along the ecliptic, and from the mean equinox. If this result be corrected for the effect of nutation, (by applying the equation of the equinoxes) there will be obtained, the longitude measured from the *apparent* equinox.

In the fourth column, the *argument of latitude* is the difference of the longitude on the orbit (1^s 19° $10'$ $14''.2$), and of the longitude of the node (1^s 18^0 $1'$ $24''.8$). It is, in the annexed figure, *NP*: and it is properly called the argument of latitude, because, the inclination of the orbit being given, the latitude depends upon it: for



$$1 \,.\, \sin.\ \text{lat.} = \sin.\ NP \,.\, \sin.\ NP\pi^{*}.$$

There are no direct corrections, from the theory of *perturbation,* of the longitudes of Mercury and Venus, in the Tables of those planets. Still the Tables are not entirely constructed without the aid of such theory. If we revert to p. 598, l. 6, we shall see in the fourth column, under the head of *Aphelion,* 24″ to be added to the epoch of the aphelion, as a quantity due to the change of the aphelion's place, in the interval between January 1, 1793, and June 3, 1793.

Now such a change of place does not obtain in the elliptical theory, but arises from the disturbing forces of the system. Some, therefore, of the results of the theory of perturbation are made use of in constructing the Tables of Mercury and Venus.

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* If the inclination be taken equal to 1° 51′ 4″, we have

log. sin. 1° 51′ 4″	8.5092343
log. sin. 1 8 49.4	8.3014327
	6.8106670

which is the log. sin. of 2′ 13″⅓.

But the changes of the places of the aphelia are phenomena, or laws common to the orbits of all planets. We have another instance in the second Example. These changes are changes of *progression:* and their computation, on the principles of gravitation, was the second great proof of the truth of Newton's System, (see *Physical Astronomy*, Chapters IX, XXII.)

In the second Example there is a small quantity to be added to the place of the node, and indicative of a change of its place in the interval between January 1, and November 13: (see the Chapters above cited).

The accounting for the *progressions* of the aphelia, and the *regressions* of the nodes (for such is the general statement of the laws of their motions), on the principle and law of gravitation, proves, to a certain extent, the truth of such law and principle. But, in determining the exact quantities (and the quantities are very minute) of such progressions and regressions, it is much better to use observations, than computations from theory. And observations are thus to be used: from those that are convenient for the purpose, find for a certain epoch the place of the node: repeat the process for another epoch: the difference of the two places is the change of the node's place in the interval between the two epochs: and the difference divided by the interval (if it be expressed in years and parts of years) will be the mean annual *regression* of the node. A like process will determine the *progression* of the aphelion.

We have now described and illustrated methods of deriving, from observations of right ascension and declination, the elements of a planet's orbit, and the variations and annual changes of those elements. The elliptical theory enables us, then, to form Tables of the planet: from which, at any epoch, its heliocentric longitude and latitude may be computed. The formula or Table of *reduction* to the ecliptic, gives the planet's longitude on the ecliptic. But in order to know at what time, and in what part of the heavens we ought to look for the planet, there is need of a method of deducing the geocentric longitude and latitude from the heliocentric. The geocentric longitude and latitude being known, the right ascension and declination of the planet may be deduced: and, accordingly, if we use instruments placed in the meridian, we know at what time, and at what distance from the zenith, to look for the planet on the meridian. If the predicted, or computed, right ascension and declination should agree with the observed, a presumption would then arise of the Tables being right: and if, in many and various instances, the observed and computed places should be found to agree, a proof would be established of their being right.

But even now, as formerly, there are to be noted some small differences between the observed and Tabular places of the planets: differences, however, too great to be imputed solely to erroneous observation, and which must, therefore, arise, in part, from the errors of the Tables. In order to render the Tables more correct, the noted differences, just spoken of, must be used (as like differences, or errors were used in pages 511, &c.) in forming sets of equations, having indeterminate coefficients that represent the errors of the several elements of the computation. But this and the other matters, previously spoken of in this Chapter, will form the subject of the ensuing.

CHAP. XXVI.

On the Deduction of Geocentric Longitudes and Latitudes from Heliocentric.—Examples of the same: the Method of correcting the Tables of Planets.

In order to attain the objects, pointed out at the conclusion of the last Chapter, it is necessary to be possessed of a formula, or of rules for converting heliocentric longitudes and latitudes, furnished by the planetary Tables, into geocentric.

It is required to determine, from the Heliocentric, the Geocentric Longitude and Latitude of a Planet.

The heliocentric longitude of the planet, and the longitude of the Earth being known, (from the solar theory and Tables) that is, the angles formed by πS, ES, with $S♈$, $E♈$ being known, the angle $ES\pi$, *the angle of commutation*, is known.

Again, from the heliocentric latitude $\angle PS\pi$, and SP, given by the planetary theory, (see p. 595,) the *curtate* distance $S\pi$ may be computed, for

$$S\pi = SP \times \cos. PS\pi.$$

But, SE is also known by the solar theory (see p. 466,) therefore to determine $\angle SE\pi$, the difference of the heliocentric and geocentric longitudes, we have $\angle ES\pi$, SE and $S\pi$.

The angle $SE\pi$ may be thus determined:

Assume (see *Trig*. p. 28, &c.) an angle θ, such, that

$$\tan. \theta = r \times \frac{S\pi}{SE} = r \times \frac{SP \,.\cos. PS\pi}{SE}, \text{ then (see } Trig. \text{ p. 29, 30,)}$$

$$r \times \tan.\left(\frac{SE\pi - S\pi E}{2}\right) = \tan.\frac{ES\pi}{2}\tan.(\theta - 45^\circ)$$

from which formula $SE\pi - S\pi E$ may be computed, and $SE\pi + S\pi E$ being known, the separate angles $SE\pi$, $S\pi E$ may be determined.

The angle $SE\pi$, the angle of elongation, is the difference (see p. 566,) of the geocentric, and of the Sun's longitude. Hence,

geocentric long. planet = longitude of ☉ $\pm$ $\angle$ elongation.

The geocentric latitude may be thus determined,

$$\tan.PE\pi = \frac{P\pi}{E\pi} = \frac{S\pi}{E\pi}.\tan.PS\pi = \frac{\sin.\angle SE\pi}{\sin.\angle ES\pi}\tan.\angle PS\pi,$$

or,

$$\tan.\text{ geocentric lat.} = \frac{\sin.\ \angle\text{ elong}^{n}.}{\sin.\ \angle\text{ commut}^{n}.} \times \tan.\text{ heliocentric lat.}$$

EXAMPLE.

The Heliocentric Longitude and Latitude of Jupiter *being, on July* 11, $5^h\ 48^m\ 39^s$, 1800, $6^s\ 29^0\ 9'\ 14''.3$, *and* $1^\circ\ 13'\ 42''$ *respectively, required the corresponding Geocentric Longitude and Latitude.*

Heliocentric long. ♃	6^s	29^0	$9'$	$14''.3$
(From Solar Tables) long. ☉	3	19	52	28.3
$\angle ES\pi$	3	9	16	46
$\therefore \frac{1}{2} ES\pi$	1	19	38	23

θ computed from $\tan.\theta = r\dfrac{SP.\cos.\text{ heli}^{c}.\text{ lat.}}{SE}$ (p. 603, last line)

From Tables of the planet.	log. SP	.7355821
	log. cos. helio^c. lat.	9.9999001
	arith. comp. SE	9.9928989
	(log. tan. $79^0\ 24'\ 48''$)........	10.7283811 (reject^g. 10)

$\therefore \theta = 79^\circ \; 24' \; 48''$

$\theta - 45^\circ = 34 \;\; 24 \;\; 48$ .... log. tan. 9.8357262

$\frac{1}{2} ES\pi \;\; = 49 \;\; 38 \;\; 23$ .... log. tan. 10.0706464

19.9063726

therefore, rejecting 10, 9.9063726 = log. tan. $\dfrac{SE\pi - S\pi E}{2}$;

$$\therefore \frac{SE\pi - S\pi E}{2} = 38^\circ \; 52' \; 16''.$$

$$\text{But } \frac{SE\pi + S\pi E}{2} = 49 \;\; 38 \;\; 23;$$

$$\therefore SE\pi = 88 \;\; 30 \;\; 39 = 2^s \; 28^\circ \; 30' \; 39''$$

But (p. 604, l. 17,) long. ⊙ .... $= 3 \;\; 19 \;\; 52 \;\; 28.3$

∴ (p. 604, l. 8,) geocen. longitude $= 6 \;\; 18 \;\; 23 \;\; 7.3$

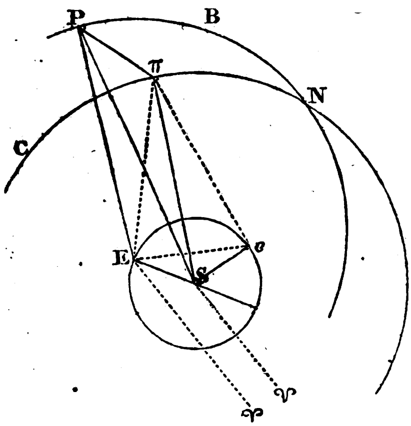
To find the Latitude (from the expression, p. 604, l. 12,)

log. sin. ∠ elon. ($SE\pi = 88^\circ \; 30' \; 39''$) 9.99985

ar. comp. sin. ∠ com. ($ES\pi = 99 \;\; 16 \;\; 46$) 0.00573

log. tan. heliocentric lat. (lat. = 1 13 42) 8.33126

∴ log. tan. geocentric lat. = 8.33684 (reject. 10)



∴ geocentric latitude = $1^\circ \; 14' \; 39''$.

Or, the computation may be effected by the aid of the following formula,

L denotes the geocentric longitude,
P the heliocentric,
λ the heliocentric latitude,
E the angle of elongation,
π the angle of parallax,
r the radius vector SP,
R the radius vector SE,

$$\text{then, } \pi = P - L,$$
$$E = L - \odot,$$

$$\text{then, } \sin. E = \frac{r . \cos. \lambda}{R}, \quad \sin. \pi = \frac{r . \cos. \lambda}{R} \sin. (P - L)$$

$$= \frac{r . \cos. \lambda}{R} (\sin. P \cos. L - \cos. P \sin. L),$$

but also $\sin. E = \sin. (L - \odot) = \sin. L \cos. \odot - \cos. L . \sin. \odot$.

Equate these two values of $\sin. E$, and there results

$$r \cos. \lambda \sin. P \cos. L - r \cos. \lambda \cos. P . \sin. L$$
$$= R \sin. L \cos. \odot - R \cos. L \sin. \odot,$$

$$\text{and thence, } (R \cos. \odot + r \cos. \lambda \cos. P) \sin. L$$
$$= (R \sin. \odot + r \cos. \lambda \sin. P) \cos. L,$$

$$\text{and } \tan. L = \frac{R \sin. \odot + r \cos. \lambda \sin. P}{R \cos. \odot + r \cos. \lambda \cos. P},$$

which is an expression for the geocentric longitude in terms of quantities, given by, or capable of being computed from, the planetary and Solar Tables.

But this expression is not adapted to logarithmic computation. In order to adapt it, thus express the numerator and denominator,

$$\text{the numerator} = \left(\frac{\sin. \odot}{\cos. \odot} + \frac{r}{R} \frac{\cos. \lambda . \sin. P}{\cos. \odot}\right) R . \cos. \odot,$$

$$\text{the denominator} = \left(1 + \frac{r}{R} \cos. \lambda \frac{\cos. P}{\cos. \odot}\right) R . \cos. \odot.$$

Let $\frac{r}{R}\,\frac{\cos.\lambda\,.\sin.P}{\cos.\odot} = \tan.x = \frac{\sin.x}{\cos.x}$;

$$\therefore \frac{r}{R}\cos.\lambda\,\frac{\cos.P}{\cos.\odot} = \tan.x\,.\frac{\cos.P}{\sin.P} = \frac{\sin.x}{\cos.x}\,\frac{\cos.P}{\sin.P};$$

$$\therefore \tan.L = \frac{\sin.\odot\,\cos.x + \cos.\odot\,\sin.x}{\cos.x\,\sin.P + \sin.x\,\cos.P}\cdot\frac{\sin.P}{\cos.\odot}$$

$$= \frac{\sin.(\odot + x)\,.\sin.P}{\sin.(P+x)\cos.\odot}.$$

We will apply this formula to the preceding instance, using the same numbers for r, R, &c.

First Operation. x computed.

log. r	.7355821
arith. comp. R	9.9928989
log. cos. λ	9.9999001
log. sin. P	9.6876697
log. sec. $\odot$	11.4685705
(rejecting 30)......	10.8846213

Second Operation. L computed.

$x =$	82°	34′	8″, nearly,			
$\odot =$	109	52	28.3	......	log. sec ..	11.4685705
	192	26	36.3	......	sin.	9.3333974
$P =$	209	9	14.3	......	sin.	9.6876697
$P+x =$	291	43	22.3	.....	arith. comp.	10.0319914
					(rejecting 30)	10.5216290.

Now 10.5216290 is the log. tangent of 18° 23′ 8″, and of 6^s 18° 23′ 8″, which latter quantity is evidently the true one in the present instance; therefore

$$L = 6^s\ 18^0\ 23'\ 8'',$$

nearly the same result as before.

By these means, then, the geocentric longitudes and latitudes may be computed from the heliocentric, such as the planetary

Tables afford: the next step is to compare the computed geocentric longitudes and latitudes, with the observed, and from such comparison to derive the corrections of the Tables,

Let C be the computed longitude,

L the observed,

O the epoch of the Tables,

m the mean motion,

t the time elapsed since the epoch,

E the equation of the centre, corresponding to a mean anomaly A, then

$$C = O + mt + E;$$

$$\therefore dC = dO + dm.t + dE,$$

but, as in p. 511, E varies both from the variation of the eccentricity, and from the variation $(d\pi)$ of the longitude of the perihelion;

$$\therefore dE = \frac{dE}{de} de + \frac{dE}{d\pi} . d\pi;$$

$$\therefore dC = dO + t.dm + \frac{dE}{de} de + \frac{dE}{d\pi} d\pi.$$

Now dC the variation or error of the computed longitude, may be considered as the difference between the computed and the observed longitude: every comparison, therefore, of the two kinds of longitudes affords an equation like the one of l. 17, and four such equations will be sufficient for the elimination and determination of the errors of the eccentricity, epoch, &c.: but, instead of confining ourselves to a barely sufficient number of equations, it will be expedient to make use of a great number, and by their combination to obtain *mean* results, (see p. 511, &c.)

In the above method of correcting the elements of a planet's orbit, the orbit is supposed to be strictly elliptical: but it must deviate from such form, by the effect of *perturbation*. In order to estimate the parts of such effect, or, in other words, the partial effects of the several planets, it is necessary to assume a series of terms with indeterminate coefficients, and *arguments* depending on the angular distances of the disturbed and disturbing planets (see pp. 498, 519, &c.)

In the next Chapter we will turn our attention to the *synodical* revolutions of planets, and to the means of ascertaining, after what intervals of time, we may expect those rare phenomena of the *transits of Venus and Mercury, over the Sun's disk*: which indeed can only happen at peculiar conjunctions: such that the planet, when it has the same longitude as the Sun, shall be near to the node of its orbit: so near that its geocentric latitude shall either be less than the Sun's semi-diameter, or, in the extreme case, shall scarcely exceed it.

CHAP. XXVII.

On the Synodical Revolutions of Planets.—On the Method of computing the Returns of Planets to the same Point of their Orbit.—Tables of the Elements of the Orbits of the Planets.

In the preceding pages, the conjunctions and oppositions of planets have been spoken of, but hitherto no method has been given of computing the times between successive conjunctions, or successive oppositions.

In the method also of determining the mean motions of planets (see p. 375,) directions were given for observing the planet in the same, or nearly the same point of its orbit, but no process or formula given, of computing the time at which such event would take place.

Towards these points then our attention will be now directed: we shall find that they depend on the same principles, and require, in the business of computation, nearly the same formulæ.

The time between conjunction and conjunction, or between opposition and opposition, is denominated, a *Synodical* period. Suppose we assume, at a given instant, the *Sun, Mercury* and the *Earth* to be in the same right line: then, after any elapsed time (a day for instance,) *Mercury* will have described an angle m, and the Earth an angle M, round the Sun. Now, m is greater than M (p. 581,) therefore at the end of a day, the separation of *Mercury* from the Earth (measuring the separation by an angle formed by two lines drawn from *Mercury* and the Earth to the Sun) will be $m - M$: at the end of two days, (the mean daily motions continuing the same,) the angle of separation will be $2(m-M)$; at the end of three days, $3(m-M)$; at the end of s days, $s(m-M)$.

When the angle of separation then amounts to 360°, that is, when $s\,(m - M) = 360^0$, the *Sun*, *Mercury* and the *Earth* must be again in the same right line, and, in that case,

$$s = \frac{360^0}{m - M} \ldots\ldots\ldots (1).$$

In which expression s denotes the time of a synodical revolution, m and M being taken to denote the mean daily motions, but, as it is plain, m and M may denote any portions, however small, of the mean motions, and s will still be the corresponding time, however reckoned, whether by days, or hours, or seconds.

Let P and p denote the sidereal periods of the Earth and the planet; then, since $1^d : M^\circ :: P : 360^\circ$,

$$\text{and } 1 : m :: p : 360,$$

$$M = \frac{360}{P} \text{ and } m = \frac{360^0}{p}; \ \therefore \text{ substituting}$$

$$s = \frac{360^\circ}{360^0\left(\frac{1}{p} - \frac{1}{P}\right)} = \frac{Pp}{P - p} \ldots\ldots (2),$$

and from either of these expressions, (1), (2), the synodical revolution of the planet may be computed.

We may differently express the synodic period; thus, if 1 be the Earth's mean distance, and r be the planet's mean distance, we have, by Kepler's law

$$P : p :: 1 : r^{\frac{3}{2}}; \ \therefore \frac{P}{p} = r^{-\frac{3}{2}},$$

$$\text{and } s = \frac{P}{r^{-\frac{3}{2}} - 1} = \frac{365^d.256384}{r^{-\frac{3}{2}} - 1},$$

$$\text{or } s = \frac{365^d.256384}{1 - r^{-\frac{3}{2}}}.$$

The first expression belonging to inferior, the second to superior, planets: and from these or the former expressions of l. 4, 14, the synodical periods may be computed.

For instance, in the case of Mercury, $p = 87^d.969$;

$$\therefore\ s = \frac{365.256 \times 87.969}{277.287} = 115^d\ 21^h, \text{ nearly.}$$

In the case of the Moon, $m = 13^\circ.1763$, and M (the Earth's mean daily motion) $= 59'\ 8''.3$;

$$\therefore\ s = \frac{360^0}{m - M} = \frac{360}{12.1906} = 29^d\ 12^h, \text{ nearly,}$$

and the following Table may be formed by substituting in the expression of p. 611, l. 20, the respective values of r,

Planets.	Values of r.	Values of s.
☿	0.3871	$115^d.877$
♀	.7233324	583.920
♂	1.5236927	779.936
?	2.6	479.672
♃	5.202792	398.867
♄	9.5387705	378.090
♅	19.183305	369.656

It is upon this synodical revolution of the Moon, that its phases depend.

$$\text{Since } s = \frac{Pp}{P - p},\quad p = \frac{sP}{s + P};$$

therefore, from the Earth's period (P) known, and the synodic (s) observed, we can determine the periodic time (P) of the planet. This method will not be accurate, if only one synodic period be observed, since that will be affected with all the deviations of the planet's real from its mean motion. To obviate this, the return of the planet to a conjunction nearly in the same part of its orbit, at which a previous one was observed, must be noted; the interval of time divided by the number of synodical revolutions will give the time of a *mean* synodical period. For, in this case, there will take place, very nearly, a mutual compensation of the inequalities arising from the elliptical form of the planet's orbit.

By the above method, the sidereal periods of *Mercury* and *Venus* may be accurately determined.

One reason already assigned for the necessity of knowing those particular conjunctions at which the planet will be nearly in the same part of its orbit, is the mutual compensation that will probably take place of the inequalities (relatively to mean motion) arising from the planet's elliptical motion. Another reason is, that, on such conjunctions, depend observations of great importance in Astronomy; namely, the transits of *Venus* and *Mercury* over the Sun's disk. This will be manifest, if we consider that *Venus*, in order to be seen on the Sun's disk, must not only be in conjunction, but near the node of her orbit: at the next conjunction, after one synodical revolution, she cannot be near her node, and can only be again near, (supposing the motion of the nodes not to be considerable,) when she returns to the same part of her orbit as at the time of the first observation. The importance of knowing these particular conjunctions then is manifest, and we shall be possessed of the means of knowing them, by modifying the formulæ of p. 611, by which the times between successive conjunctions are computed.

The time (t) of a synodical revolution $= \dfrac{Pp}{P-p}$.

At the several times $\dfrac{2Pp}{P-p}$, $\dfrac{3Pp}{P-p}$, $\dfrac{4Pp}{P-p}$ and $\dfrac{nPp}{P-p}$, therefore, the planet is still in conjunction: it will, therefore, be for the first time in conjunction, and, besides, the Earth and planet will be in the same part of their orbits, when $\dfrac{nPp}{P-p} = P$, or when $n = \dfrac{P-p}{p}$. Now, n must be a whole number, but $\dfrac{P-p}{p}$ may not be a whole number; in such a case, therefore, after one revolution of the Earth, the planet cannot be in conjunction, or, if viewed, about that time, in conjunction, it cannot be in the same part of its orbit.

But, the conditions of the planet in conjunction, and in the same part of its orbit, although they cannot take place in 1 or 2 or 3 years ($P = 1$ year), yet they may take place in m years: and if such conditions take place, then must

$$\frac{nPp}{P - p} = mP,$$

$$\text{and } \frac{m}{n} = \frac{p}{P - p},$$

and the question now is purely a mathematical one, namely, that of determining two integer numbers m and n, such, that

$$\frac{m}{n} = \frac{p}{P - p}.$$

Thus, in the case of *Mercury*, whose tropical revolution is $87^d\ 23^h\ 14^m\ 32^s$ (= 87.968),

$$\frac{m}{n} = \frac{87.968}{365.256 - 87.968} = \frac{87.968}{277.288};$$

consequently, in 87968 periods of the Earth, in which will happen 277288 synodic revolutions, *Mercury* will be observed in conjunction, and in the same part of his orbit. But, this result is, on account of the length of the period, practically useless: we must find then the lowest terms of the fraction $\frac{87.968}{277.288}$, and if the lowest terms still give periods too large, we must investigate some integer numbers, which are very nearly in the ratio of 87968 to 277288; so that we may know the periods at which the conditions required will *nearly* take place.

Now, $$\frac{87968}{277288} = \frac{1}{\frac{277288}{87968}} = \frac{1}{3 + \frac{13384}{87968}},$$

$$= \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{5726}{7664}}}}$$

and, by continuing the operation, there is at last obtained a remainder equal nothing, the greatest common measure being 8, and the fraction in its lowest terms $\frac{10996}{34661}$*, which result, for obvious reasons, is of no practical use: we must therefore find two near integer numbers; and this we are enabled to do by the preceding operation, which, as we take more and more terms of the continued fraction, affords fractions alternately less and greater than the proposed $\left(\frac{87968}{277288}\right)$ but, continually, approximating, nearer and nearer, to its true value. Thus, the first approximation is $\frac{1}{3}$: or, in one year, in which happen 3 synodical periods, the planet will not be very distant from conjunction, nor from those parts of its orbit in which it was first observed. Again, the second approximation is $\frac{1}{3+\frac{1}{6}} = \frac{6}{19}$, or in 6 years, in which happen 19 synodical revolutions, the planet will be less distant than it was before, from conjunction, and from those parts of its orbit in which it was in the former instance. The third approximation is $\frac{1}{3+\frac{1}{6+1}} = \frac{7}{22}$, or, in 7 years, in which happen 22 synodical revolutions, the planet will be nearer to conjunction than it was at either of the two preceding points of time, and so on. This follows from the very nature of the process, by which the successive approximations are formed from the continued fraction (see Euler's *Algebra*, tom. II, p. 410, Ed. 1774); but it may be useful to exemplify its truth by means of the instance

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* The operation in finding the *continued* fraction terminates, and gives a greatest common measure, because, since great accuracy is not requisite, we took $\frac{87968}{277288}$ to represent, which it does nearly, but not exactly, the ratio of the mean motions of *Mercury* and the Earth. If we had taken a fraction more exact to the true value, then the operation would not have happened to terminate.

before us. Thus, at the end of 1 year, since the diurnal tropical motion of *Mercury* is $4^\circ\ 5'\ 32''.5 = 4^\circ.092$, nearly, the angle described by that planet is

$$365.25 \times 4^0.092 = 1494^\circ.6, \text{ nearly,}$$

$= 4 \times 360^\circ + 54^0.6$, and consequently, *Mercury* at the end of 1 year, is elongated (reckoning from the Sun) from the line joining the Sun and Earth, and beyond that line, by an angle $= 54^0.6$; again, at the end of 6 years, the angle described by the planet is equal to

$$(4 \times 360^\circ + 54^0.6) \times 6 = (\text{rejecting 24 circumferences})\ 327^0.6;$$

or at the end of 6 years, *Mercury* is elongated from the line joining the Earth and Sun, by $327^\circ.6$, or, not *up* to that line, by an angle $= 32^\circ.4$.

At the end of 7 years, the angle described by *Mercury* is $(4 \times 360 + 54^0.6) \times 7 =$ (rejecting 29 circumferences) $22^0.2$: or *Mercury* is then (observing the analogy of the last expression, l. 12,) *beyond* the line joining the Earth and Sun, by that angle. At the end of 13 years, *Mercury*, (rejecting 54 circumferences) is separated from the line joining the Earth and Sun, and not *up* to that line, by an angle $= 10^0.2$.

The series of fractions, formed as those in p. 614, were formed, is

$$\frac{1}{3},\ \frac{6}{19},\ \frac{7}{22},\ \frac{13}{41},\ \frac{33}{104},\ \frac{46}{145},\ \&c.$$

The denominators denote the number of synodical revolutions, corresponding to the number of years denoted by the numerators: the number of *periods* of the planet must evidently be

$$3 + 1,\quad 6 + 19,\quad 7 + 22,\quad 13 + 41,\quad \&c.$$

that is, 4, 25, 29, 54, &c.

and therefore the series of fractions, in which the denominators are the number of periods of *Mercury*, will be

$$\frac{1}{4},\ \frac{6}{25},\ \frac{7}{29},\ \frac{13}{54},\ \&c.$$

We may, on like grounds, and by like computations, determine the probable epochs, on which we ought to look out for the transits of Venus over the Sun's disk: which are phenomena of more practical importance than the transits of Mercury.

Thus, if Venus's period $(p) = 224^d.7008240$,
the Earth's $(P) = 365.2563835$,

the synodical period, or s, $= \dfrac{Pp}{P - p}$, $= 583^d.92$, nearly*, consequently in one synodical period, the Earth describes an angle equal to

$$360^0 \times \frac{583^d.92}{365.25}, \text{ or } 575^0.51, \text{ nearly,}$$

consequently, in n synodical periods, the Earth describes an angle equal to

$$575^0.51 \times n,$$

and when $575^0.51 \times n$, shall first become a multiple of 360^0, then there will first happen a conjunction of the Earth and Venus, in the same line from which they originally departed. If, therefore, Venus in this original position, was so near to the node of her orbit, that a *transit* took place, a transit will take place when

$$575^0.51 \times n = 360^0 \times m,$$

and we must now find, as before (see p. 614,) the integer values of n and m from the equation

$$\frac{m}{n} = \frac{57551}{36000}.$$

The series of quotients found as before in p. 614, are

$$1, \quad 1, \quad 1, \quad 2, \quad 28, \quad 1, \quad 81,$$

and the series of fractions

---

* Log. P 2.5625977
log. p 2.3516046

4.9142023
log. $(P - p)$ 2.1478477

(log. 583.92)...................... 2.7663545

$$\frac{1}{1},\ \frac{2}{1},\ \frac{3}{2},\ \frac{8}{5},\ \frac{227}{142},\ \frac{235}{147},\ \&c.$$

from which series we are able to tell after what number of synodical periods Venus and the Earth will be nearly in the same parts of their orbits. Thus, taking the fourth fraction $\frac{8}{5}$, after 5 synodical periods, 8 circumferences will be nearly described, and on trial we find $575^{\circ}.51 \times 5 = 2877^{\circ}.55 = 360^{0} \times 8 = 2^{0}.45$, again, taking the next fraction, viz. $\frac{227}{142}$, we infer that, after 142 synodical periods, 227 circumferences will be nearly described; and more nearly described than the former 8 were in 5 synodical periods: or, which is the same thing, 142 synodical periods are nearly equal to 227 years: on trial we find

$$575^{0}.51 \times 142 = 81722^{0}.42 = 360^{0} \times 227 + 2^{0}.42.$$

Again,

$$575^{0}.51 \times 147 = 84599^{0}.97 = 360^{0} \times 235 - 0^{0}.03.$$

Hence, 235 years after a transit of Venus we may confidently expect another; and also after 235 + 8, or 243 years. In these computations, the alteration in the place of the node, that will happen in the interval of the transits, is not taken account of.

But, if we were guided merely by the preceding mathematical results, we should be in danger of missing some transits: for those results are founded on the probability of a transit's happening when Venus and the Earth are nearly in the same parts of their orbits, as they were at the time of a former transit. A transit, however, may happen when the planets are in parts of their orbits diametrically opposite, or, in other words, a transit may happen should there happen to be a conjunction when Venus is, or nearly, in the node of her orbit, opposite to that in which a transit has already happened. In order to find the probable periods at which the transits in the *opposite* node may happen, we must, instead of the equation of p. 617, write this

$$575^{0}.51 \times n = 180^{0} \times (2s - 1),$$

since, it is plain, a transit must happen, whenever, after n synodical periods, the angle described by the Earth shall be either 180°, or, a multiple of 180°. Form then a series of fractions, as before in p. 614, by dividing 57551 by 18000: which, since the successive quotients are

$$3,\quad 5,\quad 14,\quad 2,\quad 40,$$

will be

$$\frac{3}{1},\quad \frac{16}{5},\quad \frac{227}{71},\quad \frac{460}{147},\ \&c.$$

and consequently, beginning with the third, in 71 synodical periods, 227 angles of 180° are described by the Earth: and on trial we find

$$71 \times 575°.51 = 40861°.21 = 180° \times 227 + 1°.21,$$

so that after 71 synodical periods the Earth has described a little more than 227 half circumferences, and, consequently, must be very nearly in the line drawn from the Sun, through the opposite node of Venus's orbit.

Since the Earth describes 227 times 180°, in 113 years and an half, it follows, if a transit happens at the beginning of 8 years, and not at the end, or, happening at the end of 8 does not (from the increase of Venus's latitude) happen at the end of 16 years, that the next period for expecting a transit will be 113 years, and that, agreeably to what has been before said, we ought to examine, or compute the latitudes of Venus at the periods 113 ∓ 8, that is, 105 and 121 years, since transits may happen at these periods.

M. Delambre has calculated the transits of Venus, over the Sun's disk, for 2000 years, some of which are subjoined.

Years.	Months.	Mean time of Conjunction.	Node.
1631	Dec. 6, ...	$17^h\ 28^m\ 49^s$	☊
1639	Dec. 4, ...	6 9 40	☊
1761	June 5, ...	17 44 34	☋
1769	July 3, ...	10 7 54	☋
1874	Dec. 8, ...	16 17 44	☊
1882	Dec. 6, ...	4 25 44	☊
2004	June 7, ...	21 0 4	☊

We now subjoin Tables of the elements of the orbits of planets, principally taken from Laplace, and reduced from the new French measures which he has adopted.

*Sidereal Periods of the Planets**.

Mercury	87^{d}.969258
Venus	224.700824
The Earth	365.256384
Mars	686.079619
Vesta	1335.205
Juno	1590.998
Ceres	1681.539
Pallas	1681.709
Jupiter	4332.596308
Saturn	10758.969840
The Georgian Planet	30688.712687

Movements in 100 *Julian Years of* $365^{d}.25$.

Mercury	415^{c}	2^{s}	14°	$4'$	$20''$
Venus	162	6	19	13	0
The Earth	100	0	0	45	45
Mars	53	2	1	42	10
Jupiter	8	5	6	17	33
Saturn	3	4	23	31	36
The Georgian Planet	1	2	9	51	20

* The tropical periods may be deduced from the sidereal, by deducting the times which the several planets require, respectively, for the description of an arc of longitude equal to the precession.

Mean Distances, or Semi-Axes of the Orbits.

Mercury	0.387098
Venus	0.723332
The Earth	1.000000 *
Mars	1.523694
Vesta	2.373000
Juno	2.667163
Ceres	2.767406
Pallas	2.767592
Jupiter	5.202791
Saturn	9.538770
The Georgian Planet	19.183305

* The Earth's distance is here assumed as a standard and = 1 : its distance from the Sun, in statute miles, is reckoned to be 93, 726, 900.

M. Bode of Berlin discovered the following curious law of the relative distances of the Planets:

Mercury	4	$= 4$
Venus	7	$= 4 + 3.2^0$
Earth	10	$= 4 + 3.2$
Mars	16	$= 4 + 3.2^2$
Ceres	28	$= 4 + 3.2^3$
Jupiter	52	$= 4 + 3.2^4$
Saturn	100	$= 4 + 3.2^5$
The Georgian planet	196	$= 4 + 3.2^6$.

The distances of the next planets (should there be any) according to this law would be

$$388 = 4 + 3.2^7$$

$$722 = 4 + 3.2^8$$

&c. =

We need scarcely mention that this law is empirical. It is not easy to see what led to the conjecturing of it.

Ratio of the Eccentricities (a e) to the Semi-Axes at the beginning of 1801: *with the Secular Variation of the Ratio,* (see p. 464). *The sign — indicates a diminution.*

	Ratio of the Eccentricity.	Secular Variation.
Mercury	0.205514	0.000003867
Venus	0.006853	0.000062711
The Earth	0.016853	0.000041632
Mars	0.093134	0.000090176
Juno	0.254944	not ascertained
Vesta	0.093220	not ascertained
Ceres	0.078349	not ascertained
Pallas	0.245384	not ascertained
Jupiter	0.048178	0.000159350
Saturn	0.056168	0.000312402
The Georgian Planet	0.046670	0.000025072

Mean Longitudes at the beginning of 1801; *reckoned from the Mean Equinox, at the Epoch of the Mean Noon of January* 1, 1801, *Greenwich.*

Mercury	166°	0′	48″.2
Venus	11	33	16.1
The Earth	100	39	10
Mars	64	22	57.5
Vesta	267	31	49
Juno	290	37	16
Ceres	264	51	34
Pallas	252	43	32
Jupiter	112	15	7
Saturn	135	21	32
The Georgian Planet	177	47	38

Mean Longitudes of the Perihelia, for the same Epoch as the above, with the Sidereal and Secular Variations.

	Long. Perihelion.			Sec. Var.	
Mercury	74^0	$21'$	$46''$	$9'$	$43''.5$
Venus	128	37	0.8	−4	28
The Earth	99	30	5	19	39
Mars	332	24	24	26	22
Vesta	249	43	0	not ascertained.	
Juno	53	18	41	not ascertained.	
Ceres	146	39	39	not ascertained.	
Pallas	121	14	1	not ascertained.	
Jupiter	11	8	35	11	4
Saturn	89	8	58	32	17
The Georgian Planet	167	21	42		4

Inclinations of Orbits to the Ecliptic at the beginning of 1801, *with the Secular Variations of the Inclinations to the true Ecliptic.*

	Inclination.			Secular Variation.
Mercury	7^0	$0'$	$1''$	$19''.8$
Venus	3	23	32	− 4.5
The Earth	0	0	0	
Mars	1	51	3.6	− 1.5
Vesta	7	8	46	not ascertained.
Juno	13	3	28	not ascertained.
Ceres	10	37	34	not ascertained.
Pallas	34	37	7.6	not ascertained.
Jupiter	1	18	51	− 23
Saturn	2	29	34.8	− 15.5
The Georgian Planet	0	46	26	3.7

Longitudes of the Ascending Nodes on the Ecliptic, at the beginning of 1801, *with the Sidereal and Secular Motions.*

	Longitude of ☊.	Secular and Sidereal Variation.
Mercury	$45^{0}\ 57'\ 31''$	$13'\ 2''$
Venus	74 52 38.6	— 31 10
The Earth	0 0 0	
Mars	48 14 38	— 38 48
Juno	103 0 6	not ascertained.
Vesta	171 6 37	not ascertained.
Ceres	80 55 2	not ascertained.
Pallas	172 32 35	not ascertained.
Jupiter	98 25 34	— 26 17
Saturn	111 55 46	— 37 54
The Georgian Planet	72 51 14	— 59 57

The use of the secular variation of the eccentricity has been already explained (see p. 464.) The secular variations of the longitudes of the perihelia and the nodes are *sidereal:* consequently, they cannot be immediately applied to find a longitude at an epoch, different from that of the Tables; but, in the first place, the precession of the equinoxes must be added, and then the result will be a variation relatively to the equinoxes, or tropics. Thus, the secular sidereal variation of the longitude of the perihelion of *Mercury's* orbit is stated to be $9'\ 43''.5$; therefore, if we assume the annual precession to be $50''.1$, and consequently the secular to be $1^{\circ}\ 23'\ 30''$, the secular variation, with *regard to the equinoxes,* is $1^{0}\ 33'\ 13''.5$; and, accordingly, the longitude of the perihelion of *Mercury's* orbit, for the beginning of 1901, will be

$$74^{0}\ 21'\ 46'' + 1^{0}\ 33'\ 13''.5 = 75^{0}\ 54'\ 59''.5.$$

For the beginning of 1821, it will be

$$74^{0}\ 21'\ 46'' + 0^{0}\ 18'\ 38''.7 = 74^{0}\ 40'\ 24''.7.$$

Again, the *sidereal* secular variation of the perihelion of *Venus* is stated to be — $4'\ 28''$ (— indicating the motion of the perihelion to be contrary to the order of the signs); therefore the variation with regard to the equinoxes, is

$$1^{\circ}\ 23'\ 30'' - 4'\ 28'' = 1^{\circ}\ 19'\ 2'';$$

and accordingly the longitude of the perihelion for the beginning of 1811, is

$$128^{\circ}\ 37'\ 0''.8 + 0^{\circ}\ 7'\ 54''.5 = 128^{\circ}\ 44'\ 55''.3;$$

and for the beginning of 1781,

$$128^{\circ}\ 37'\ 0''.8 - 0^{\circ}\ 15'\ 49'' = 128^{\circ}\ 21'\ 11''.8.$$

It is easy to see that, both for the nodes and perihelia, a column of the tropical secular variations might be immediately formed from the sidereal by the simple addition of $1^{\circ}\ 23'\ 30''$. The motions of the aphelia and nodes in Lalande's (vol. I. p. 117, &c.) and Mr. Vince's Tables, (vol. III. p. 17, &c.) are motions relative to the equinoxes.

CHAP. XXVIII.

On the Satellites of the Planets.—On Saturn's Ring.

THE planet *Jupiter* is always seen accompanied by four small stars, which are denominated *Satellites,* and sometimes, *Secondary* planets, *Jupiter* being called the primary.

The satellites of *Jupiter* were discovered in 1610, by Galileo: they are discernible by the aid of moderate telescopes, and are of some use in Practical Astronomy. *Saturn* also, and the *Georgian Planet,* are accompanied by satellites, not however, to be seen except through excellent telescopes, and of no practical use to the observer. The number of *Saturn's* satellites is seven, and of the *Georgian's,* six.

The satellites are to their primary planet, what the Moon is with respect to the Earth: they revolve round him, cast a shadow on his disk, and disappear on entering his shadow: phenomena perfectly analogous to solar and lunar eclipses, and which render it probable that the primary and their secondary planets are opaque bodies illuminated by the Sun.

That the satellites when they disappear, are eclipsed by passing into the shadow of their primary, is proved by this circumstance: that the same satellite disappears at different distances from the body of the primary, according to the relative positions of the primary, the Sun, and the Earth, but always towards those parts, and on that side of the disk, where the shadow of the primary caused by the Sun ought, by computation, to be. When the planet is near opposition the eclipses happen close to his disk.

There is an additional confirmation of this fact. The third and the fourth of *Jupiter's* satellites disappear and again appear on the same side of the disk; and the durations of the eclipses are found to correspond exactly to the computed times of passing through the shadow.

The motions of *Jupiter's* satellites are according to the order of the signs. The satellites are observed moving sometimes towards the east, and at other times towards the west: but when they move in this latter direction they are never eclipsed; when the eclipses happen, the satellite is always moving eastward; when the transits over the disk happen, the satellite is always moving westward: the motion therefore towards the east, or, according to the order of the signs, must be the true motion.

By the same proof it is ascertained, that the satellites of *Saturn* perform their motions, round their primary, according to the order of the signs. But the satellites of the *Georgian Planet* may be thought to form an exception; at least, the direction of their motions is ambiguous; for, motions performed in orbits perpendicular to the ecliptic (and such, nearly, are the orbits of the satellites of the *Georgian*) cannot be said to be either direct or retrograde.

The mean motions and periodic times of the satellites are determined by means of their eclipses, and, most accurately, by those eclipses that happen near to opposition.

The middle point of time between the satellite entering and emerging from the shadow of the primary, is the time when the satellite is in the direction, or nearly so, of a line joining the centres of the Sun and the primary. If the latter continued stationary, then the interval between this and the succeeding central eclipse would be the periodic time of the satellite. But, the primary planet moving in its orbit, the interval between two successive eclipses is a *synodic period* (see p. 610.) This synodic period, however, being observed, and the period of the primary being known, the sidereal period of the satellite may be computed*. Instead of two successive eclipses, two, separated from each other by a large interval, and happening when the Earth, satellite, and primary, are in the same position (in the direction of the same right line, for instance,) are chosen, and then the interval of time divided by the number of sidereal periods, will give, to greater accuracy, the mean time of one revolution.

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* Since $\tau = \dfrac{Pp}{P-p}$, $p = \dfrac{P\tau}{P+\tau}$.

The mean motions of the satellites do not differ considerably from their true motions. Hence, the forms of their orbits, must be nearly circular. The orbit, however, of the third satellite of *Jupiter* has a small eccentricity: that of the fourth, a larger.

The distances of the satellites from their primary are ascertained by measuring those distances, by means of a *Micrometer*, at the times of the greatest elongations.

The distance of one satellite being determined, the distances of others, whose periodic times should be known, might be determined by means of Kepler's law, which states the squares of the periodic times to vary as the cubes of the mean distances.

In order to obtain such results, we suppose Kepler's law to be true. But we may adopt a contrary procedure, and, by ascertaining the periodic times and distances of all the satellites according to the preceding methods, determine the above-mentioned law of Kepler to be true. See *Principia Phil. Natur'*. lib. 3tius p. 7, &c. Ed. *La Seur*, &c.

The eclipses of *Jupiter*'s satellites are used in determining the longitudes of places, and, on account of this their practical usefulness, have been studied with the greatest attention. Thence has resulted the curious and important discovery of the *Successive Propagation of Light*, which is the basis of the theory of aberration (see pp. 254, &c.) The phenomenon that led to the discovery of the propagation of light was, that an eclipse of a satellite did not always happen according to the computed time, but later, in proportion as *Jupiter* was farther from the Earth. If, for instance, an eclipse happened, *Jupiter* being in opposition, exactly according to the computed time, then about six months afterwards, when the Earth was more distant from *Jupiter* by a space nearly equal to the diameter of its orbit, an eclipse would happen about 16 minutes later than the computed time. And by similar observations it appeared, that the *retardation* of the time of the eclipse was proportional to the increase of the Earth's distance from *Jupiter*. This fact, the connexion of the retarded eclipse with the Earth's increased distance from *Jupiter*, was first noted by Roemer, a Danish Astronomer, in 1674: who suggested as an hypothesis, and as an adequate cause of the retardation, the successive propagation of light*. Subsequent observations accord so well with this hypothesis, that it is impossible to doubt of its truth: and it receives an additional, although an indirect, confirmation from Bradley's Theory of Aberration which is founded thereon.

The following Table, exhibits the mean distances and sidereal revolutions of the satellites of *Jupiter, Saturn,* and the *Georgium Sidus.*

Mean Distances, (the radius of the planet being = 1.)		According to Laplace, Sidereal Revolutions.	According to Delambre.
Jupiter.		Day.	d h m s
1st. Satellite ..	5.81296	1.7691378	1 18 28 35.94537
2	9.24868	3.5511810	3 13 17 55.73010
3	14.75240	7.1545528	7 3 59 35.82511
4	25.94686	16.6887697	16 18 5 7.02098
Saturn.			d h m s
1st. Satellite ..	3.080	0.94271	0 22 37 32.9
2..........	3.952	1.37024	1 8 53 8.9
3..........	4.893	1.88780	1 21 18 26.2
4..........	6.268	2.73948	2 17 44 51.2
5..........	8.754	4.51749	4 12 25 11.1
6..........	20.295	15.94530	15 22 41 13.1
7..........	59.154	79.32960	79 7 53 42.8
Georgium Sidus.			d h m s
1st Satellite ..	13.120	5.8926	5 21 21 0
2..........	17.022	8.7068	8 17 1 19
3..........	19.845	10.9611	10 23 4
4..........	22.752	13.4559	11 11 5 1.5
5..........	45.507	38.0750	38 1 49
6..........	91.008	107.6944	107 16 40

* Light is propagated through a space equal to the diameter of the Earth's orbit in 16^{m} 26^{s}.

On the Ring of Saturn.

Besides his seven satellites, *Saturn* is surrounded by a flat and thin ring of coherent matter. Dr. Herschel has discovered that the ring instead of being entire is divided into two parts, the two parts lying in the same plane.

The ring is luminous, by reason of the reflected light of the Sun; it is visible to us, therefore, when the faces illuminated by the Sun are turned towards us: invisible, when the opposite faces; invisible also, when the plane of the ring produced passes through the centre of the Earth; since then no light can be reflected to us; invisible also in a third case, when the plane of the ring produced passes through the centre of the Sun, since, in that case, it can receive no light from that luminary. The plane of the ring is inclined to that of the ecliptic in an angle of about $31^0\ 24'$, and revolves round an imaginary axis perpendicular to its plane in $10^h\ 29^m\ 16^s$: and, which is worthy of notice, this period is that in which a satellite, having for its orbit the mean circumference of the ring, would revolve according to Kepler's law *.

We have now gone through another great division of our subject. The Lunar Theory will next occupy our attention, which might, indeed, have taken its place before the Planetary.

* The fact of the squares of the periodic times varying as the cubes of the mean distances, is frequently called, the *Third* law of Kepler.

CHAP. XXIX.

ON THE LUNAR THEORY.

On the Phases of the Moon.—Its Disk.—Its Librations, in Longitude, in Latitude, and Diurnal.

Of all celestial bodies, the Moon is the most important, by reason of its remarkable and obvious phenomena: the intricacy of the theory of its motions; and the usefulness of the practical results derived from such theory.

Some of the phenomena admit of an easy explanation, and require no great nicety of computation. Such are the phases of the Moon. Others, with regard to their general cause, admit also of an easy explanation; but, with regard to the exact time of their appearance and recurrence, require the most accurate knowledge of the lunar motions. Of this latter description, are the eclipses of the Moon.

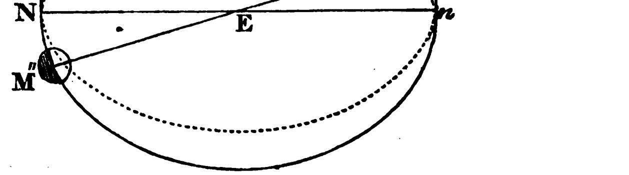
If therefore with a view to simplicity, we arrange the subjects of the ensuing Chapters, we ought first to place the phases of the Moon, next, the elements and form of the orbit, then, the lunar motions and their laws, and lastly, the lunar eclipses.

The explanation of the phases of *Mercury* and *Venus* was founded on the hypothesis, of their being opaque bodies illuminated by the Sun, and, of their revolution round the Sun. A similar explanation, on similar hypotheses, will apply to the Moon. We shall perceive the cause of its phases, if we suppose the Moon to shine by the reflected light of the Sun, and to revolve round the Earth: and, as in the case of the two inferior planets, the explanation does not require a knowledge of the exact curve in which the revolution is performed.

The Moon moves through 12 signs, or 360° degrees of longitude, in about 27 days. This fact is ascertained by observing, each day, on the meridian, its right ascension and declination, and thence deducing, by calculation, (see pp. 158, &c.) the corresponding latitude and longitude. Hence, in a period somewhat more than the preceding, the Moon is on the meridian at all hours of the day, and the angle, formed by two lines drawn from the Moon to the Earth and Sun respectively, passes through all degrees of magnitude. The *exterior* angle therefore, (see p. 553,) on the magnitude of which, the visible illuminated disk depends, passes also through all degrees of magnitude: and the Moon accordingly, like *Venus*, must exhibit all variety of phase; the crescent near conjunction; the *half Moon* in quadratures; and the entire orb illuminated, or the *full Moon* in opposition.

Venus revolves round the Sun, and the Moon round the Earth: but this difference of circumstance, in no wise affects the principle on which the phases depend: they are regulated by the inclination of the planes of the circles of illumination and vision: and their magnitude depends, as it was shewn in p. 553, on the versed sine of the exterior angle at the planet: that is, in Fig. p. 553, on the versed sine of the angle SuF.

The angle, analogous to SuF, in the annexed Figure, will be



contained between a line Ss drawn to the centre of the Moon at M, and a line drawn from E and produced through the same centre. This angle, by reason of the parallelism of the lines drawn from E to the Sun, will equal the interior angle continued between cE and a line drawn from E to the centre of the Moon; which angle, in other words, is the angle of elongation.

Hence, in delineating the Moon's phases, we may use a simpler expression, and state *the visible enlightened part to vary as the versed sine of the Moon's elongation.*

If we suppose the Earth to be illuminated by the Sun, and to serve *as a Moon* to the Moon, the visible illuminated part of the Earth, will to a spectator at the Moon vary as the versed sine of the *Earth's elongation.* Let e be the latter angle, E the former: then by what has just preceded,

$$E + e = 180^0, \text{ nearly};$$

$$\therefore \cos. E = \cos. (180^0 - e) = - \cos. e,$$

$$\text{and } 1 - \cos. E = 1 + \cos. e, \quad 1 + \cos. E = 1 - \cos. e.$$

Hence, when the Moon's phase is ☽'s radius $\times (1 - \cos. E)$, the corresponding phase of the Earth

$\{$⊕'s radius $\times (1 - \cos. e)\}$, is ⊕'s radius $\times (1 + \cos. E)$,

the larger, therefore, the Moon's phase is to us, the smaller, at the same time, is the Earth's phase to an inhabitant of the Moon. Thus, near conjunction when E is nearly 0, the Moon's phase is ☽'s radius $\times (1 - 1)$, nearly, whilst the Earth's phase is ⊕'s radius $\times$ 2, or the Earth is nearly at her *full*, to an inhabitant of the Moon, whilst the Moon is a *new* Moon to us. In such a situation the Earth's light is reflected towards the Moon, falls on its dark disk, and feebly illuminates it, producing the phenomenon called by the French *lumière cendrè.*

When the Moon is in opposition, $E = 180^0$, the Moon's phase is ☽'s radius $\times (1 + 1)$, or the Moon is at her *full*, and the corresponding phase of the Earth is expounded by, ⊕'s radius $\times$ $(1 - 1)$, which being nothing, shews that the dark side of the Earth is then towards the Moon.

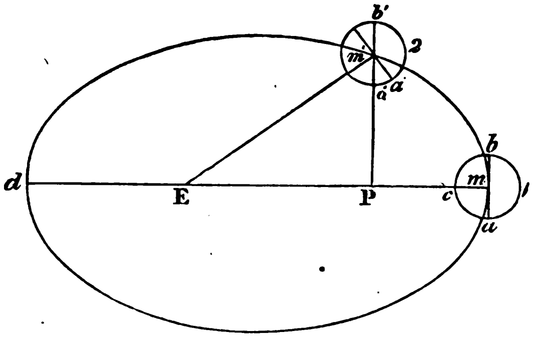
When $E = 90°$, $\cos. E = 0$; $\therefore 1 + \cos. E$, and $1 - \cos. E$, are each $= 1$: consequently, in such a position, the Moon shews half of her illuminated disk to the Earth, while the Earth shews half of her illuminated disk to the Moon.

If $E = 60^0$, cos. $E = \frac{1}{2}$, therefore the Moon's phase is ☽ 's radius $\times$ $\frac{3}{2}$, or the Moon is at her third quarter; the Earth's phase is ⊕'s radius $\times$ $(1 - \frac{1}{2})$, or $\frac{2 \oplus\text{'s radius}}{4}$: or, the Earth, viewed from the Moon, is at her first quarter.

The period of the Moon's phases, or the interval of time which must elapse before the phases, having gone through all their variety, begin to recur, must depend upon the return of the Moon to a situation similar to that which it had, at the beginning of the period. If we date then the beginning of the period from the time of conjunction, (the time of new Moon,) the end of the period must be when the longitudes of the Moon and Sun are again the same. Now the longitude of the Sun is continually increasing; when the Moon therefore has made, from its first position, the circuit of the heavens, it will be distant from the Sun, by the angular space through which, during the Moon's sidereal period, the Sun has moved. In order, then, to rejoin the Sun and to be again in conjunction, it must move through this space, and a little more; and when it does rejoin the Sun, a *synodic* revolution is completed. And the period therefore of the Moon's phases is a *synodic* period. From the inequality of the Moon's motion, this *synodic* period, or *lunation,* is not always of the same length.

If we conceive a plane passing through the centre of the Moon and perpendicular to a line drawn from the Earth to the Moon, then on such a plane the Moon's face will appear to be projected. This face, since the Moon has ever been an object of the attention of Astronomers, has been delineated, and a map made of its seeming Seas, Mountains, and Continents. But, one map of the same hemisphere has always served to represent the Moon's face: in other words, the same face of the Moon is always turned towards us. This is a curious circumstance, and the immediate inference from it is, that the Moon must revolve round its axis, with an angular velocity equal to that with which it revolves round the

Earth. For*, suppose in the position (1) a to be on the verge of the disk, then, if in the position (2) we still see the point a, in the verge, and in the same position, it must have been transferred, by rotation, through an arc $a'a$: since, in the case of



no rotation, $b'a'$, parallel to ba, would have been the position of ba. Now, a being seen on the verge of the Moon's disk, $\angle Em'a =$ a right angle $= \angle Em'a' + \angle a'm'a$. But since EPm' is a right angle, $\angle Em'P + \angle PEm'$ is one also: consequently,

$$\angle Em'a + \angle a'm'a = \angle Em'P (\angle Em'a) + \angle PEm;$$
$$\therefore \angle a'm'a = \angle PEm',$$

and the angle $a'm'a$ measures the rotation of the Moon round its axis that has taken place since it occupied the position (1), and the angle PEm', the angular motion of the Moon round E from the same position.

If the angle PEm', the measure of the Moon's true angular distance from one of the apsides of its orbit, increased uniformly, and the Moon's rotation round her axis were uniform, the above result would always take place; that is, the same face of the Moon ought always to be turned to the spectator: and such phenomenon

* In the Figure, acb is supposed to represent the Moon's equator, and (which is not strictly true) to lie in the plane of the orbit: the axis of rotation, then, is perpendicular at m to that plane: perpendicular, for instance, to the plane of the paper, if the latter be imagined to represent that of the Moon's orbit.

ought constantly to be observed. But since, which is the case, the Moon's true motion differs from the mean, and the angle PEm' does not increase uniformly, the preceding result will not be precisely true, if we suppose, (which is a probable supposition,) the Moon's rotation round her axis to be uniform. If after any time, 3 days for instance, mEm' should measure the Moon's angular distance from the position (1), then, by reason of the Moon's elliptical motion, in 6 days twice the angle mEm' will certainly not measure the Moon's angular distance: but, on the supposition of the Moon's uniform rotation, twice the angle $a'm'a$ would measure the quantity of rotation in 6 days. Hence, if the Moon's angular velocity should be diminishing from the position at (1), at the end of 6 days the point a, previously seen on the verge of the Moon's western limb, would have disappeared, and some points towards the verge of the Moon's eastern limb would be brought into view; and such, by observation, appears to be the case, and the phenomenon is called the Moon's *Libration in Longitude*.

Since this libration in longitude arises from the unequal angular motion of the Moon in her orbit, it must depend on the difference of the true and mean anomalies, in other words, on the equation of the centre, or equation of the orbit; and would be proportional to that equation, and its maximum value would be represented by the greatest equation ($6^{\circ}\ 18'\ 32''$) in case the axis of the Moon's rotation were perpendicular to the plane of its orbit.

In the preceding reasonings, we have supposed the section bca, representing the Moon's equator, to be coincident with $mm'd$ the plane of the orbit: in other words, we have supposed the axis of rotation to be perpendicular to the same plane. Now, the axis is not perpendicular but inclined to the plane at an angle of $5^{\circ}\ 8'\ 49''$; the preceding results therefore will be modified by this circumstance. For, take the extreme case, and suppose the axis of rotation to be parallel to the plane of the orbit, and in the position (1) to be represented by ce*: then it is plain, we should

* e, omitted in the Figure, ought to have been where cm produced cuts the circle eba.

at the position (1), see the pole *c*, and the hemisphere, projected upon a plane passing through *ba* perpendicular to the orbit; and, half a month after, at *d*, we should see the *opposite* pole *e*, and the *opposite* hemisphere, notwithstanding the equality between the Moon's revolution round the Earth, and her rotation round her axis. In intermediate inclinations then of the Moon's axis of rotation, part of this effect must take place, or must modify the preceding results. If in the position (1), the Moon's axis being inclined to the plane of her orbit, we perceive, for instance, the Moon's north pole and not her south, we shall in the opposite position at *d*, after the lapse of half a month, perceive the Moon's south, and not her north pole; and, this effect is precisely of the same nature, as that of the north pole being turned towards the Sun at the summer, and of the south pole at the winter solstice, (see p. 24.) The perpendicularity therefore of the axis of rotation to the plane of the orbit is a condition equally essential, with that of the equality of rotation and revolution, in order that the same face of the Moon should be always turned to the spectator.

This second cause, preventing the same face of the Moon from being always seen, is called, with some violation of the propriety of language, the *Libration in Latitude*. For, it is plain, from the preceding explanation, that there are properly and physically no librations, but librations only seemingly such.

There is a third libration, discovered by Galileo, and called the *Diurnal Libration*. If the two former librations did not exist, the same face of the Moon would be turned, not to a spectator on the surface, but, to an imaginary spectator placed in the centre of the Earth. Now, two lines drawn respectively from the centre and the surface of the Earth to the centre of the Moon, (the directions of two visual rays from the two spectators) form, at that centre, an angle of some magnitude; and, when the Moon is in the horizon, an angle equal to the Moon's horizontal parallax. Hence, when the Moon rises, parts of her surface, situated towards the boundary of her upper limb, are seen by a spectator, which would not be seen from the Earth's centre. As the Moon rises, these parts disappear: but as the Moon, having passed the meridian, declines, other parts, situated near that boundary, which, whilst the Moon was rising, were the lower, are brought into view, and which would not be seen by a spectator placed in the centre of the Earth. The greatest effect of this diurnal libration will be perceived, by observing the Moon first at her rising, and then at her setting.

This last libration, like the two preceding, is purely optical.

The description of general and obvious phenomena requires only popular explanation, which is easily afforded. But the next steps, the accounting for, on principle and by calculation, minute phenomena, (if we may apply that term to effects detected only by the aid and comparison of numerous observations) are more difficult, whether those steps are to be made in the solar, planetary, or lunar theory: and we shall find them peculiarly so in the latter theory.

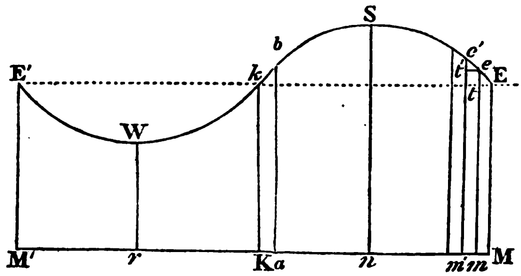
CHAP. XXX.

On the Methods of deducing, from Observations, the Moon's Parallax: the Moon's true Zenith Distance, &c.

According to modern Astronomical usage, the same kind of observations, namely, meridional observations, which are used in determining the places of the fixed stars, and the elements of the orbits of the Sun and the planets, serve also to determine the position and dimensions of the lunar orbit. But, by reason of the proximity of the Moon to the Earth, and the irregularity (if we may use such a term) of her motions, the *reduction* of the Moon's observed right ascensions and declination requires more scientific and longer computations.

The orbits of planets round the Sun, and of secondary planets round their primaries, would, if we abstract the mutual effects of planets, be elliptical. Now the elliptical is a regular motion. It is, therefore, the disturbing forces that render the motions of planets irregular; and, since the mutual influence of planets must be universally felt, there is no planet nor secondary, the motions of which are not, in some degree at least, irregular. The degree of irregularity depends on what may be called the peculiar circumstances of the planet, which are those of the vicinities and magnitudes of other planets. For instance, Jupiter and Saturn, (see *Physical Astronomy*, Chap. XIX.) bodies of great bulk, and, in a certain sense, not very distant from each other, mutually and powerfully disturb each other, or prevent what, according to our theories, would otherwise take place, namely, elliptical motion. In like manner the Earth's motion is rendered *irregular*, but not considerably so (see *Physical Astronomy*, Chap. XVIII.) by the actions of Venus and Jupiter, &c. The Moon is near to the Earth, but then its mass, relatively to the Sun's mass, is very inconsiderable. It is, however, the Sun's mass which is almost the sole cause of the Moon's not describing an ellipse round the Earth, or, which, as we have explained it, prevents her motions from being *regular*, and which, therefore, makes them by reason of its largeness, very irregular.

The irregularities we are speaking of are real ones, and would be observable in the daily changes of right ascensions, and of north polar distances, even if the observer were placed in the centre of the Earth. Or, if from the Moon's right ascensions and north polar distances, her longitudes and latitudes were deduced, and then, on a line such as MM' representing the ecliptic, ordinates ME, me, &c. proportional to the latitudes were erected,



the curve Eee', &c. passing through their extremities would be a curve less *regular* than when (see p. 145,) under similar conditions, it represents the solar orbit. A consequence, or indication of such irregularity would be this, that from me, $m'e'$, &c. representing latitudes, or declinations, computed or observed for equal intervals Mm, mm', &c. an intermediate latitude or declination interpolated, for an intermediate interval, would be less exact in the lunar, than in the solar orbit.

It is plain, when observations are made by means of instruments placed in the meridian, that the north polar distances, and right ascensions of planets can only be known, at times intermediate of their meridional passages, by a species of interpolation. In the case of the Sun, its north polar distance at midnight, on March 1, is nearly the mean of his north polar distances on the noons of March 1 and 2 : and six hours past the noon of March 1, is, nearly, his north polar distance on the noon of March 1, minus the decrease of north polar distance, proportional to six hours. This mode of computation, however, not exact even in the case of the Sun, is less exact when applied to the Moon.

In order to determine the inexactness of the computation, or of any other mode of interpolation, we must observe the heavenly body when it is out of the meridian. In the case of the Sun, for instance, observe its zenith distance, and note the distance in time from noon: then if the co-latitude (PZ) be known, we can from PZ, the horary angle ZPS, and ZS compute PS, and then compare PS, thus computed, with the interpolated value of PS.

But this brings us to the consideration of the second cause of irregularity: that which arises from the proximity of the observed body, and which proximity gives rise to the inequality of *parallax.* In the case of the Sun, its north polar distances, computed according to the above methods, and compared, are found, very nearly, to agree; which agreement is a proof of the smallness of the Sun's parallax. For parallax (see Chap. XII.) affects the zenith distance, and is the larger the greater the zenith distance. The north polar distances, therefore, found by adding to the co-latitude of the place the observed meridional zenith distances, would be incorrect, but would be less so than an intermediate zenith distance, observed out of the meridian. In the case, therefore, of a near heavenly body, it would be impossible that the north polar distances, found according to the above methods, should, on comparison, agree: and this we shall find to be the case with the Moon.

We shall give to this statement greater distinctness, by examining some of the recorded observations of the Sun and Moon.

In the second Volume of the Greenwich Observations, we find the following observations of the zenith distances of the upper and lower limbs of the Sun.

1783.	Barometer.	Thermometer.	Zenith Distance.	Corrected Zenith Distance.
May 4,	29.95	$48\frac{1}{2}$	⊙ L. L. $35^{\circ}\ 43'\ 8''.9$	$35^{\circ}\ 43'\ 50''.76$
			⊙ U. L. 35 11 23.9	35 12 4.92
5,	29.84	$49\frac{1}{2}$	⊙ L. L. 35 25 56.9	35 26 38.34
			⊙ U. L. 34 54 12.4	34 54 52.95
6,	29.81	51	⊙ L. L. 35 9 0.2	35 9 40.82
			⊙ U. L. 34 37 17.3	34 37 57.16
7,	29.9	$47\frac{1}{2}$	⊙ L. L. 34 52 19.4	34 53 0.46
			⊙ U. L. 34 20 37.17	34 21 17.96

The last column contains the zenith distances, corrected or reduced according to the principles and formulæ of Chapter X. If we add together the respective corrected zenith distances of the lower and upper limbs, and take their half sums, the results will be the values of the zenith distances (Z) of the Sun's centre.

	Values of Z.	First Diffs. d'.	Secd. Diffs. d''.	Third Diffs. d'''.
May 4,	$35^{\circ}\ 27'\ 57''.89$	$-17'\ 12''.25$	$+\ 15''.60$	$+\ 1''.27$
5,	35 10 45.64	− 16 56.65	+ 16.87	
6,	34 53 48.99	− 16 39.78		
7,	34 37 9.21			

Here the several differences tend towards an equality, which is a proof (should the several values be represented by the ordinates me, $m'e'$, &c. of a curve Eee', &c.) of the *regularity* of that curve. The use of the Table of differences is to find an intermediate value of Z, and by means of what is called the *Differential Theorem*, (see Appendix to *Trigonometry*.) Thus, the intermediate value of Z corresponding to May 5, 8^{h}, would be, making

$a = 35^{0}\ 10'\ 45''.64,\ d' = -16'\ 56''.65,\ d'' = 16''.87,\ d''' = 1''.27,$

$$x = \frac{8}{24} = \frac{1}{3},$$

$$35^{0}\ 10'\ 45''.64 - 5'\ 48''.88 - 1''.87 + 0''.13 = 35^{0}\ 4'\ 55''.$$

This is not exactly the value of Z, since it has been obtained on the ground, that the interval between two successive meridional zenith distances, is exactly 24^{h} : which, (see Chapter XXII, on the Equation of Time) is not the case. In order to obtain an exact result, we must refer to the Volume of Observations above quoted, and examine the Sun's right ascensions at his transits on the 4th and 5th of May,

1784.	Sun's Right Ascension.	d'	d''.
May 4,	$2^{h}\ 45^{m}\ 53^{s}.9$	$+ 3^{m}\ 51^{s}$	
5,	2 49 44.9	+ 3 51.4	+.4
6,	2 53 36.3	+ 3 51.9	+.5
7,	2 57 28.2		

Here the increase of the Sun's right ascension, between the transits on the 5th and 6th, is $3^{m}\ 51^{s}.4$: if, therefore, the eight hours should be eight hours of sidereal time, we should have

$$x = \frac{8}{24^{h}\ 3^{m}\ 51^{s}.4} = .33244,$$

from which value, as before, (see l. 2, &c.) we may deduce the value of Z, corresponding to eight hours of sidereal time, after the Sun's transit on May 5.

The values of Z are, in fact, meridional zenith distances. But, it is plain, an interpolated value cannot belong to the meridian of the place of observation; it may, however, be conceived to belong to the meridian of some other place, having a different longitude, but the same latitude. In point of fact, the result that has been obtained by the differential theorem is merely a mathematical result. We may, however, by slightly modifying the preceding process, obtain a mathematical result, which, at the same time, shall represent a real quantity. Thus, if to the four values of Z, in the first column of the Table of p. 642, we add the co-latitude of the place, we shall obtain four north polar distances of the Sun, on the noons of the 4th, 5th, 6th, and 7th of May. An interpolated north polar distance is independent of the place of observation: and if we deduce it, as we deduced the value of Z, the deduced north polar distance, must be the same as the co-latitude (PZ) of the place added to that value of Z, because, in each computation, the differences d', d'', d''', are the same: since

$$(PZ + Z) - (PZ + Z') = Z - Z' = d', \text{ \&c.}$$

If, therefore, in the above instance, the place of observation be Greenwich, the co-latitude of which is $38^\circ\ 31'\ 20''$, the Sun's north polar distance, on May 5 at eight hours of sidereal time, is equal to $38^\circ\ 31'\ 20'' + 35^\circ\ 4'\ 53''$, that is, to $73^\circ\ 36'\ 13''$.

But this determination supposes the observed zenith distance to be the same, as if the observer were near to the Earth's centre: in other words, it supposes the angle, subtended by the Earth's radius at the Moon, to be inconsiderable. We shall hereafter, in the Chapter on the Transit of Venus, see that the greatest angle which can be subtended by the Earth's radius, or, the Sun's horizontal parallax, does not exceed $9''$.

A shorter and easier method of proving the smallness of the Sun's parallax has been already described in pp. 326, &c.

If S represent the Sun, Z, the zenith, P the pole, the triangle ZPS can be solved if ZP, PS, and the angle ZPS be given or known. Thus, in the above instance,

$$ZP = 38^\circ\ 31'\ 20'',$$
$$PS = 73\ \ 36\ \ 13,$$

and in order to find the angle ZPS, we have

right ascension of mid-heaven	8^h	0^m	0^s
Sun's right ascension at noon	2	49	44.9
	5	10	15.1
acceleration (see p. 526,)			50.824
	5	9	24.276

ZS computed from these three values, and compared with ZS, found by observations made out of the meridian would shew, by the agreement of the two values, the smallness of the Sun's parallax.

But we shall find results of a different kind, if we examine and compare the Moon's places determined from zenith observations. In the Volume of the Greenwich Observations above referred to, we find

1784.	Barometer.	Thermometer.	Zenith Distance. Moon's Limb.	Right Ascension. Moon's First Limb.
Jan. 31,	30.35	32	L. L. 24^0 48′ 13″.5	4^h 35^m 34^s
Feb. 1,	30.08	$31\frac{1}{2}$	U. L. 23 12 13.3	5 51 35
2,	30.04	32	U. L. 23 33 43.2	6 27 28
3,	30.41	32	U. L. 25 19 0.9	7 22 0
4,	29.99	$33\frac{1}{2}$	U. L. 28 19 27	8 14 20

Correct on account of refraction, as in the former instance, the zenith distances of the upper and lower limbs, and add or subtract the Moon's semi-diameter: the results will be the zenith distances (z) of the Moon's centre, from which zenith distances we may, as before, form a Table of differences.

Values of z.	d'.	d''.	d'''.	d^{iv}.
24^0 33′ 37″				
	-1^0 6′ 2″			
23 27 35		$+1^0$ 27′ 28″		
	+0 21 26		− 3′ 36″	
23 49 1		+1 23 52		− 5′ 1″
	+1 45 18		− 8 37	
25 34 19		+1 15 15		
	+3 0 33			
28 34 52				

Here the differences exhibit considerable irregularities, which arise from two causes: one real; the other, as it may be called, optical, originating, mainly, from the Moon's proximity to the Earth, but varying, in degree, with the Moon's distance from the zenith. But from whatever causes the irregular values of Z arise, they are, as phenomena or results of observation, blended together, and it is necessary to institute an investigation, in order to distinguish the separate causes. Now, the first step in such investigation, is similar to the one made in p. 643, that is, we must find by interpolation, an intermediate value of the Moon's north polar distance, and from it and the horary angle ZPM, and the co-latitude PZ, we must compute the Moon's zenith distance, which is to be compared with the Moon's *observed* zenith distance.

In order to find the value of x, or the interval proportional to eight hours of sidereal time on February 1, we must first deduct the Moon's right ascension on February 1, from her right ascension on January 31: that is, we must take the difference of $5^h\ 31^m\ 35^s$, and $4^h\ 35^m\ 34^s$, which is $56^m\ 1^s$. This $56^m\ 1^s$ is the angle which the meridian, after having passed through the Moon's centre, must describe, in addition to 24^h, before it can again reach the Moon's centre. Unity, therefore, denoting the interval between two successive transits,

$$1 : x :: 24^h\ 56^m\ 1^s : 8^h ; \ \therefore x = .3208.$$

Substitute this value for x, in the differential theorem, and the value of Z corresponding to 8^h (sidereal time) on February 1, is

$$23^\circ\ 27'\ 35'' + (21'\ 26'') \times .3208 + (1^\circ\ 23'\ 52'') \times .3208 \times -.3396$$
$$-8'\ 37'' \times .3208 \times .3396 \times .59304 + 5'\ 1'' \times .3208 \times .3396$$
$$\times .69304 \times .6697 = 23^\circ\ 24'\ 59''.033.$$

Hence the Moon's north polar distance is the above quantity added to $38^\circ\ 31'\ 20''$, or, is nearly equal to $61^\circ\ 56'\ 19''$. It is, however, the Moon's north polar distance, only on the supposition of the non existence of parallax. For if the Moon be so near to the Earth, that the radius of the latter subtends some measurable angle at the former: then (see the Chapter on Parallax) the observed zenith distances are not, in a certain sense, the true zenith distances: but every observed zenith distance will require, proportionally to its sine, a correction to *reduce* it to a true zenith distance.

If from observations contemporaneously made (see p. 325,) in different parts of the Earth, we knew the Moon's horizontal parallax, we could, by means of such a series as is given in p. 324, deduce such correction. But if, without quitting the place of observation, we wish to ascertain the existence and quantity of parallax, we must compute ZM (Z the zenith, M the Moon) from the co-latitude (PZ) an interpolated value of PM, and the horary angle ZPM. Now this horary angle, must, like PM, be obtained by interpolation.

In the case of a fixed star, and only in that case, the horary angle (the angle ZPs) is the difference of the right ascension of the mid-heaven (in other words, the sidereal time) and of the star's right ascension. In the case of the Sun, we must, as we have seen in p. 643, allow for the change of the Sun's right ascension, during his transit over the meridian, and the assigned instant of sidereal time. The computation for a like allowance, in the case of the Moon, is a little more operose. On the 1st of February (see the Table of p. 645,) the Moon's right ascension, at the instant of her transit, was $5^h\ 31^m\ 35^s$, and since her right ascension increases by unequal steps, we must find it at any time, intermediate of her meridional transits, by the differential theorem. If we form then a Table of differences, like the one of p. 645,

R. A. Moon's First Limb.	d'.	d''.	d'''.	d^{iv}.
$4^h\ 35^m\ 34^s$				
	$+56'\ 1''$			
5 31 35		$-0'\ 8''$		
	$+55\ 53$		$-1'\ 13''$	
6 27 28		$-1\ 21$		$+22''$
	$+54\ 32$		$-0\ 51$	
7 22 0		$-2\ 12$		
	$+52\ 20$			
8 14 20				

we have $a = 5^h\ 51^m\ 35^s$, $d' = 55'\ 53''$, $d'' = -1'\ 21''$, $d''' = -51''$, $d^{iv} = 22''$ and (see p. 646,) $x = .32088$, and, accordingly,

$$\text{Æ of } ☽\text{'s 1st L.} = 5^h\ 49^m\ 35^s.48,$$

which is the right ascension of the Moon's preceding limb, eight hours after the Moon's transit of the meridian. But the sidereal time, at the time of the Moon's transit (in other words, the right ascension of the mid-heaven at that time, or the right ascension of the Moon's first limb) was $5^h\ 31^m\ 35^s$; eight hours, therefore, after the sidereal time, or right ascension of the mid-heaven, must be $13^h\ 31^m\ 35^s$, and accordingly, the horary angle must be $13^h\ 31^m\ 35^s - 5^h\ 49^m\ 35^s.48$, or $7^h\ 41^m\ 59^s.2$: from this must be subtracted the angle at the pole, subtended by the Moon's semi-diameter. Now the Moon's semi-diameter is $15'\ 4''$, and the polar distance (see p. 646,) of the Moon's centre is $61^\circ\ 56'\ 19''$; therefore the angle at the pole is

$$\frac{15'\ 4''}{\sin.\ 61^\circ\ 56'\ 19''} = 17'\ 4''.4 = 1^m\ 8^s.297;$$

consequently, the horary angle is $7^h\ 40^m\ 50^s.9$; we have then

$$\begin{array}{lcccc} ZPM = 7^h\ 40^m\ 50^s.9 & = & 115^h & 12^m & 43^s.5 \\ ZP \ldots\ldots\ldots\ldots\ldots & = & 38 & 31 & 20 \\ PM \ldots\ldots\ldots\ldots\ldots & = & 61 & 56 & 19 \end{array}$$

whence, by the solution of a spherical triangle, according to the formula of *Trigonometry*, p. 171, Edit. 3, there results,

$$^* ZM = 82^\circ\ 13'\ 6'', \text{ nearly.}$$

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* See *Trigonometry*, pp. 171, &c.

$$\begin{array}{rlr} \frac{c}{2} = 57^\circ\ 36'\ 21''.7 & \ldots\ldots\ldots\ 2\ \text{log. cos.} & 19.4599294 \\ a = 38\ \ 31\ \ 20 & \ldots\ldots\ldots\ \text{log. sin.} & 9.7943612 \\ b = \underline{61\ \ 56\ \ 19} & \ldots\ldots\ldots\ \text{log. sin.} & \underline{9.9456872} \\ \frac{a}{2}+\frac{b}{2} = 50\ \ 13\ \ 49.5 & & 19.1999778 \\ M = 23\ \ 27\ \ 33.5 & \ldots\ldots\ldots\ (\text{log. sin. } M) & 9.5999889 \\ \frac{a}{2}+\frac{b}{2}+M = 73\ \ 41\ \ 23 & \ldots\ldots\ldots\ \text{log. sin.} & 9.9821604 \\ \frac{a}{2}+\frac{b}{2}-M = 26\ \ 46\ \ 16 & \ldots\ldots\ldots & 9.6536248 \\ \therefore\ \frac{c}{2} = 41\ \ \ 6\ \ 32.8 & \ldots\ldots\ldots\ 2) & 19.6357852 \\ c = 82\ \ 13\ \ \ 5.6 & \ldots\ldots\ \left(\text{log. sin. } \frac{c}{2}\right) & 9.8178926 \end{array}$$

Suppose now the observed zenith distance to be 82° 49′ 10″, then the difference between the two, namely, 36′ 4″, would be an indication of parallax and *partly* its effect. It cannot represent the *whole* effect, because on the supposition of the existence of parallax, the meridional north polar distances, (obtained by adding the co-latitude to the observed meridional zenith distances), from which PM was obtained by interpolation, would be all wrong, and consequently PM, one of the given quantities in the triangle ZPM (see p. 648,) would be so also, and consequently, in the last place, the result of the solution, or the value of ZM would be incorrect. The difference 36′ 4″ then being only in part the effect, and not the measure of parallax, must be considered as a first approximation towards the true value of parallax. Under this point of view, if P (see pp. 323, &c.) should denote the horizontal parallax, we should have (see p. 323,)

$$\sin. P = \frac{\sin. p}{\sin. (D + p)}, \text{ or, nearly,}$$

$$P = \frac{p}{\sin. (D + p)} = \frac{36' \; 4''}{\sin. 82^{0} \; 49' \; 10''} = 36' \; 21''.$$

With this approximate value we may partly correct the observed zenith distances, and obtain more correct values of the north polar distances deduced from such zenith distances. Thus, since $P = 36' \; 21''$, and since the observed zenith distances on Feb. 1, (see p. 645,) was 23° 27′ 35″, we have (see p. 323,) the parallax of the meridional zenith distance

$$= 36' \; 21''. \sin. 23^{\circ} \; 27' \; 35'' = 868''.27 = 14' \; 28'', \text{ nearly.}$$

With this, as a correction, the series of zenith distances should be reduced (see p. 645,) and a new series of meridional polar distances, from which, as before, we may deduce by interpolation, or the differential formula, a more correct value of PM corresponding to 8^{h}. It is plain that this value of PM must be nearly the former value (61° 56′ 19″) minus the parallax on the meridian, that is, 61° 41′ 51″. Instead, therefore, of making $PM = 61^{0} \; 56' \; 19''$, make it, in the formula of solution of p. 648, 61° 41′ 51″, and the resulting value of PM is 82° 1′ 16″: subtract this from 82° 49′ 10″, the observed zenith distance, and the difference, which is the second approximate value of the parallax, is $47' \, 54''$, and, therefore, as before

$$P = \frac{47' \, 54''}{\sin.\, 82^0 \, 49' \, 10''} = 48' \, 16'',$$

and the parallax on the meridian $= 48' \, 16''.\sin.\ 23^0 \, 27' \, 35'' = 19' \, 13''$, and, as before, deducting this from $61^0 \, 56' \, 19''$, the new value of PM is equal to $61^0 \, 37' \, 54''$, with which new value the side ZM is again to be deduced from the formula of p. 648.

The resulting value of ZM, is again to be deducted from the observed zenith distance, in order to obtain new values of p, and P, and after three more approximations, we shall deduce a value of P about $54' \, 10''$: which is nearly that of the Moon's horizontal parallax. This is the description of the process for ascertaining, at the same place of observation, the existence and quantity of the Moon's parallax. But if we knew by means of the method described in pp. 325, &c. and by the result of such observations as were made at the Cape of Good Hope and Berlin, the Moon's horizontal parallax, we could, in the first instance, find the parallaxes corresponding to the several zenith distances, (see p. 645,) correct such distances, and then deduce a series of north polar distances of the Moon, by adding the co-latitude of the place of observation to the zenith distances so corrected.

In what has preceded, we have pointed out and described two methods for determining the Moon's parallax, neither of which can be very conveniently practised. It was a rare occurrence that gave observations, contemporaneously made at places so far distant as the Cape of Good Hope and Berlin, and there are few Observatories provided, for observations out of the meridian, with instruments equally good as their mural quadrants and circles. The quantity and variation of the Moon'sparallax, now well known, has not been so known by one set of observations : but, like other astronomical elements, has been determined by the comparison of numerous observations, and with some small aid from theory.

The large quantity of the Moon's parallax, and its variations arising from the situation of the observer, and the change of distance between the Moon and Earth, render it a subject of considerable astronomical importance. We shall, therefore, continue its discussion before we proceed to deduce the elements of the lunar orbit.

The Moon's horizontal parallax (P), is the angle which the Earth's radius subtends at the Moon. The Moon's apparent semi-diameter (D), is the angle which the Moon's radius subtends at the Earth. Hence,

$$P = \frac{\text{rad. } \oplus}{☽\text{'s dist. from } \oplus},$$

$$D = \frac{☽\text{'s rad.}}{☽\text{'s dist. from } \oplus};$$

$$\therefore \frac{P}{D} = \frac{\oplus\text{'s rad.}}{☽\text{'s rad.}},$$

the ratio, therefore, between the Moon's horizontal parallax and apparent semi-diameter, is a constant ratio, if the Moon and Earth be spheres; and, if the former be a sphere, is a constant ratio at the same place, whatever be the figure of the Earth.

If $P = 57' \, 4''.16844$, and $D = 15' \, 33''.8652$*,

$$\frac{D}{P} = \frac{15' \, 33''.8652}{57' \, 4''.16844} = .27293,$$

or, by the method of continued fractions, is nearly $\frac{3}{11}$. Hence, from the observed apparent semi-diameter of the Moon, we may

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* The ratio of the greatest and least apparent semi-diameters, is the same as the ratio of the perigean and apogean distances of the Moon,

$$\text{and } \frac{\text{the least apparent diameter}}{\text{the greatest apparent diameter}} = \frac{29' \, 30''}{33' \, 30''} = \frac{1 - e}{1 + e},$$

(if e be the eccentricity), whence $e = .0635$, whereas the eccentricity in the solar orbit only $= .0168$. The equation of the centre then, in the lunar orbit, must be about $7° \, 16'$. If, therefore, we set off from a circular motion, and call that the *regular* one, the Moon's motion, besides the causes already assigned (see p. 639,) will be still more irregular than the Sun's.

always deduce the corresponding horizontal parallax by multiplying the former by $\frac{11}{3}$: and *vice versâ*.

The horizontal parallax of the Moon is the angle subtended by the Earth's radius at the Moon. Hence, the Earth not being spherical, the horizontal parallax is not the same*, at the same instant of time, for all places on the Earth's surface. One proof that the Earth is not spherical, is by reversing this inference, namely, that the horizontal parallaxes computed for the same time are found not to be the same. Hence, in speaking of the horizontal parallax it is necessary to specify the place of observation. The Moon's parallax computed for Greenwich is different from the equatoreal parallax. Several corrections therefore, must be applied to an observed parallax, in order to compute, at the time of the observation, the Moon's distance from the centre of the Earth. For, that distance, it is plain, ought to result the same, whatever be the latitude of the place of observation.

The greatest and least horizontal parallaxes of the Moon, computed from observations at Paris, are, according to Lalande, (*Astron.* tom. II, p. 197,) 1° 1′ 28″.9992, and 53′ 49″.728, and the corresponding perigean and apogean distances respectively, 63.8419, 55.9164. The corresponding apparent diameters are 33′ 31″, and 29′ 22″.

The mean diameter, that which is the arithmetical mean between the greatest and least, is 31′ 26″.5; but, the diameter at the mean distance is smaller and equal to 31′ 7″.

Whatever be the quantity, which is the subject of their investigation, Astronomers are accustomed to seek for a constant and mean value of it, from which the true and apparent values are perpetually varying, or, about which they may be conceived to oscillate. In the subjects of time and motion, the search is after

* At the same distance the parallax varies as the radius vector of the spheroid. A Table, therefore, that gives the several values of the radii in a spheroid of a given oblateness, enables us to correct the equatoreal parallax. See Vince's *Astronomy*, vol. III. tab. XLV. p. 173.

mean time and *mean motion*, and by applying corrections or *equations* to deduce the true. The Moon's parallax not only varies in one revolution, from its perigean to its apogean, but the parallaxes which are the greatest and least in one revolution, remain not of the same value, during successive revolutions: they may not be the greatest and least, compared with other perigean and apogean parallaxes. But all may be conceived to oscillate about one fixed and mean parallax, which has been designated by the title of *Constant* Parallax, (*la Constante de la Parallaxe*).

We should obtain no standard of its measure, if we assumed it to be an arithmetical mean between its least and greatest values. For, the eccentricity of the lunar orbit varying, and consequently, the apogean and perigean distances, from the action of the Sun's disturbing force, the greatest parallax, if increased, would not be increased by exactly the quantity of the diminution of the least parallax; the mean of the parallaxes, therefore, would not always be the same constant quantity.

The constant parallax is assumed to be that angle, under which the Earth's radius would be seen by a spectator at the Moon, the Moon being at her mean distance and mean place: such, as would belong to her, when all causes of inequality are subtracted. But then, even by this definition, the *constant* parallax would be represented by the same quantity only at the same place; for, although the Moon's distance remains the same, the radius of the Earth, supposing it spheroidical, would vary with the change of latitude in the place of observation.

In order therefore, to rescind the occasion of ambiguity which might be attached to the phrase of *constant parallax*, Astronomers, in expressing its quantity, are accustomed to state the place for which it was computed. Thus, the equatoreal diameter being greater than the polar, the *constant parallax under the equator* (as it is termed) is greater than the *constant parallax under the pole:* the former, Lalande, by taking a mean of the results obtained by Mayer and Lacaille, states to be $57' \, 5''$, the latter $56' \, 53''.2$; the same author also states the constant parallaxes for Paris, and for the radius of a sphere, equal in volume to the Earth, to be respectively $56' \, 58''.3$, and $57' \, 1''$ (see *Astronomy*, tom. II. p. 315).

M. Laplace, however, proposes to deduce the several constant parallaxes from one alone: and to appropriate the term *constant*, to that parallax, belonging to a latitude, the square of the sine of which is $\frac{1}{3}$*. This parallax, by theory, he has determined to be 57′ 4″.16844, the corresponding apparent semi-diameter of the Moon being 31′ 7″.7304, (= 57′ 4″.16844 × .27293.)

This parallax being reckoned the mean parallax, the true parallax is to be deduced from it; if analytically expressed, to be so, by a series of terms: if arithmetically computed, by the application of certain *equations;* the terms and equations arising, partly, from mere elliptical inequality, and partly, from the perturbation of the Sun.

The terms due to the first source of inequality are easily computed: for, if we call P the horizontal parallax to the mean distance (a), then since we have any distance (ρ) in an ellipse expressed (see p. 459,) by this equation,

$$\rho = \frac{a \cdot (1 - e^2)}{1 \pm e \cdot \cos.\,\theta},$$

and since, the parallax $\times\, \rho = P \times a$, we have the parallax $=$ $P \times \dfrac{1 + e \cdot \cos.\,\theta}{1 - e^2}$, and expanding as far as the terms containing e^3, &c. $= P\,(1 + e \cdot \cos.\,\theta + e^2)$.

The terms due to the theory of perturbation are not easily computed. In the extent of mathematical science, there is no computation of equal importance and greater difficulty †.

The formula for the parallax, in which the constant quantity is 57′ 4″.16844, belongs to a latitude, the square of the sine of which is $\frac{1}{3}$. The corresponding formula for any other latitude is to be

* Laplace chose this parallel, since the attraction of the Earth on the corresponding points of its surface, is very nearly, as at the distance of the Moon, equal to the mass of the Earth, divided by the square of its distance from the centre of gravity. Laplace, *Mec. Cel.* Liv. II, p. 118.

† The difficulty belongs equally to the formulæ for the latitude and longitude. See Lalande, tom. II, pp. 180. 193. 314.

deduced by multiplying the former by $\frac{r}{r'}$, or by applying a correction proportional to $r - r'$; r and r' being the radii corresponding to two latitudes, and computed on the supposition that the Earth is a spheroid with an eccentricity $= \frac{1}{300}$. [See Tables XLV, and XLVI; in the collection (1806) of French Tables, and the Introduction. See also Vince, vol. III, p. 50.]

The Moon's equatoreal horizontal parallax and apparent semidiameter, are inserted in the Nautical Almanack, and, for every 12 hours; the former is computed by the formula that has been mentioned (p. 654): the latter, by multiplying the parallax by .27293.

The Moon's distance may, as it has been already noted, be determined from her parallax; her greatest and least distances from her least and greatest parallaxes; and her mean distance from her mean parallax; and, taking for the value of the latter that determined by Laplace, we shall have

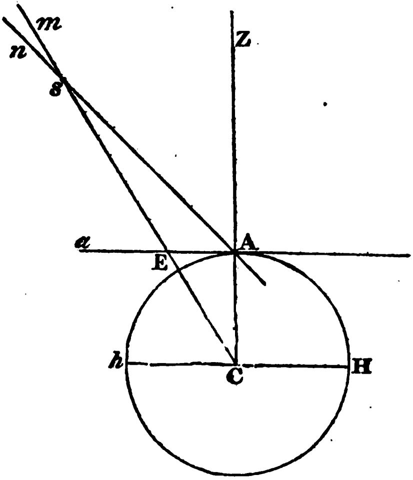
$$☽\text{'s distance} = \frac{57^{\circ}.2957795}{57'\ 4''.16844} \times \text{rad.} \oplus = \frac{57.2957795}{0.9511579} \times \text{rad.} \oplus$$

$= 60.23799 \times \text{rad.} \oplus$; therefore, if we assume the Earth's mean radius to be 3964 miles, the Moon's distance will be about 238783 miles.

The distances of the Sun and of the Moon from the Earth are inversely as their parallaxes. Hence, if the parallax of the former be considered equal to $8''.7$, the distances will be to each other, nearly, as 394 : 1.

Lacaille's method of determining the distance from the parallax applies successfully to the Moon, on account of her proximity to the Earth. It fails, with regard to the Sun, by reason of his distance. That distance is more than 24090 radii of the Earth: consequently, a radius of the Earth bears a very small proportion to it. The Sun's apparent diameter then seen from the surface of the Earth, is nearly the same, as if it were seen from the centre; and his diameter on the meridian cannot be sensibly larger than his horizontal diameter. But, with the Moon, the case is different: since her distance is not much more than 60 radii of the Earth, her apparent diameter at its surface will be one 60th part greater than her diameter viewed from the centre: and as she rises from the horizon, and approaches the spectator, her apparent diameter will increase and be greatest on the meridian. It is easy to assign a formula for its augmentation.

Let s be the Moon, p the parallax represented by the angle



msn, D the ☽'s apparent distance from the zenith, Δ the ☽'s diameter viewed from the Earth's centre, a the augmentation of the diameter, then

$$\text{☽'s real diameter} = \Delta \times Cs = (\Delta + a) \times As;$$

$$\therefore \frac{\Delta + a}{\Delta} = \frac{Cs}{As} = \frac{\sin. CAs}{\sin. ACs} = \frac{\sin. D}{\sin. (D - p)}.$$

$$\text{Hence, } a = \frac{\Delta . \sin. D - \Delta . \sin. (D - p)}{\sin. (D - p)}$$

$$= \frac{2\,\Delta \left\{\sin. \frac{p}{2} . \cos. \left(D - \frac{p}{2}\right)\right\}}{\sin. (D - p)}$$

(see *Trig.* p. 32.)

From this formula, in which $p = P . \sin. D$, (P the horizontal parallax) a may be computed; but, in practice, more easily from a formula, into which, by the known theorems of Trigonometry, the preceding may be expanded. See Table XLIV, in Delambre's Tables; and the Introduction: also Vince, vol. III, p. 49.)

When the Moon is in the horizon, $p = P$, and $D = 90^\circ$;

$$\therefore a = \frac{\Delta\,(1 - \cos. P)}{\cos. P} = \Delta\,.(\sec. P - 1).$$

Hence, the ☽'s horizontal diameter is greater than the diameter (Δ) seen from the centre, in the proportion of the secant of P to radius; that is, if we assume $P = 1^\circ$, in the proportion of 1.0001523 : 1.

With the preceding value of the parallax (1°) the diameter (Δ) see p. 655, will $= 2^\circ \times .27293 = 32'\ 49''.9$, nearly, and accordingly the augmentation $= 32'\ 49''.9 \times (\sec. 1^\circ - 1)$

$$= 32'\ 49''.9 \times .0001523$$
$$= 0''.3, \text{ nearly.}$$

It is plain, independently of any computation, that the Moon's horizontal diameter must appear larger than it would do, if seen from the centre: since the visual ray, in the latter case, is the hypothenuse, in the former, the side of a right-angled triangle. In order to find how much the Moon must be depressed, so that, if it could, it would be seen under the same angle, as when viewed from the Earth's centre, draw a line from the bisection of the radius joining the spectator and the Earth's centre, perpendicularly towards the Moon's orbit: the intersection with the orbit is the Moon's place, and the depression, below the horizon, is, as it is plain, half the Moon's horizontal parallax.

The Moon's parallax is necessary to be known for the purpose of determining, from its observed, its true zenith distance: from the true zenith distance, the Moon's north polar distance is found by adding to it the co-latitude. Lastly, from the north polar distance and right ascension, and the obliquity of the ecliptic, the Moon's longitude and latitude may be computed: and thence the elements of the orbit may be computed, or being computed, may be examined and corrected. This subject of the elements of the lunar orbit, will be briefly treated of in the ensuing Chapter.

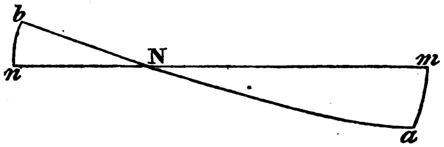
CHAP. XXXI.

On the Elements of the Lunar Orbit; Nodes; Inclination; Mean Distance; Eccentricity; Mean Motion; Apogee; Mean Longitude at a given Epoch.

THE longitudes of the nodes are determined, as in the case of a planet. From the Moon's observed right ascensions and declinations, the corresponding latitudes and longitudes are computed: when the latitude is equal nothing, the Moon is in the ecliptic; in the intersection therefore of the ecliptic and its orbit: or, in other words, in its node: the longitude corresponding to such latitude (= 0) is the longitude of the node.

It will rarely happen (see p. 565,) that the latitude deduced from the meridional right ascensions, and polar distances, is exactly equal nothing: we must then, by proportion, compute the longitude corresponding to such latitude, if it may be called such. The object may be easily arrived at by the following method.

Let N be the place of the node, nNm a portion of the



ecliptic, am, bn (λ, λ') two latitudes, one to the south, the other to the north of the ecliptic: now by Naper's Rules

$$\tan. N = \frac{\tan. \lambda}{\sin. Nm} = \frac{\tan. \lambda'}{\sin. Nn};$$

$$\therefore \frac{\sin. Nm}{\sin. Nn} = \frac{\tan. \lambda}{\tan. \lambda'};$$

$$\therefore \frac{\sin. Nm - \sin. Nn}{\sin. Nm + \sin. Nn} = \frac{\tan. \lambda - \tan. \lambda'}{\tan. \lambda + \tan. \lambda'},$$

$$\text{or, } \frac{\tan. \dfrac{Nm - Nn}{2}}{\tan. \dfrac{Nm + Nn}{2}} = \frac{\sin. (\lambda - \lambda')}{\sin. (\lambda + \lambda')}.$$

$$\text{Hence, } \tan. \frac{Nm - Nn}{2} = \tan. \frac{nm}{2} . \frac{\sin. (\lambda - \lambda')}{\sin. (\lambda + \lambda')},$$

from which expression, $Nm - Nn$ is known, since $Nm + Nn$, the difference of the longitudes on the two succeeding days of observation, is known: and, from the sum and difference of two quantities, we can determine the quantities themselves: in fact

$$Nm = \frac{Nm + Nn}{2} + \frac{Nm - Nn}{2},$$

$$Nn = \frac{Nm + Nn}{2} - \frac{Nm - Nn}{2}.$$

This method is capable of determining, besides the longitude of the node, the inclination of the orbit; for, since

$$\frac{\sin. Nn}{\sin. Nm} + 1 = \frac{\tan. \lambda'}{\tan. \lambda} + 1,$$

$$\frac{\tan. \lambda}{\sin. Nm} = \frac{\tan. \lambda + \tan. \lambda'}{\sin. Nm + \sin. Nn};$$

consequently,

$$\tan. N = \frac{\tan. \lambda}{\sin. Nm} = \frac{\tan. \lambda + \tan. \lambda'}{\sin. Nm + \sin. Nn}$$

$$= \frac{\sin. (\lambda + \lambda')}{\cos. \lambda \cos. \lambda' . 2 . \sin. \dfrac{mn}{2} . \cos. \left(\dfrac{Nm - Nn}{2}\right)}.$$

In which fraction, after the determination of the value of $Nm - Nn$, every thing is known.

In order to determine, whether the place of the node be fixed or not, or, if moveable, the direction and degree of its motion, repeat the above process for finding the longitude, and the difference between the two results will be, during the interval of the two observations, the motion of the node. Thus, if at the end of a month, we make a second computation of the place of the Moon's node, it will be found to have a longitude less than what it had at the beginning, by $1^{\circ}\ 28'$: at the end of two months, a longitude less by $2^{\circ}\ 55'$: and by like computations, or, rather by the comparison of very distant observations, the annual *regression* of the Moon's node, will be found to be $19^{0}\ 19'\ 43''$, and the period of the sidereal revolution of the node will be 6798 days *.

If we take the difference of two longitudes of the same node, we shall have, corresponding to the interval of time, the regression or motion of the node: if the interval be 100 years, the result will be *the secular motion* of the node. But, the mere difference of the two longitudes will not give the whole motion of the node, since the node may have regressed through several entire circuits of the heavens. For instance, in 100 years the mere difference of two longitudes is $4^{s}\ 14^{0}\ 11'\ 15''$: but, since the revolution of the Moon's nodes is performed in about $18^{y}\ 7^{m}$, in 100 years, besides this angle of $4^{s}\ 14^{0}\ 11'\ 15''$, five circumferences must have been described by the node: the proper exponent, therefore, of the secular motion of the node is

* There are certain phenomena which very plainly indicate the regression and its quickness. For instance, the star *Regulus* situated nearly in the ecliptic, (its latitude is about $27'\ 35''$,) was eclipsed by the Moon in 1757: the Moon therefore, must have been nearly in the ecliptic, and consequently, in its node. But, a few years after, the Moon, instead of eclipsing *Regulus*, passed at the distance of 5 degrees from the star. Again, if the Moon be observed at a certain time in conjunction with a star, and passing very near it, after the interval of a month, it will pass the star at a greater distance; after two months, at a still greater distance; and having reached a certain point, it will, in its conjunctions with the star, again approach it, and, at the end of about 19 years, pass it at the same distance, as at the beginning.

$$5 \times 360^{\circ} + 134^{\circ}\ 11'\ 15'' = 1934^{\circ}\ 11'\ 15'', (= 1934^{\circ}.1875.)$$

Hence, the tropical revolution of the node

$$= \frac{36000^{\circ}}{1934.1875} = 6798^{d}.54019 = 6798^{d}\ 12^{h}\ 57^{m}\ 52^{s}.416,$$

and since the equinoctial point in that time has regressed through $15'\ 34''$, the sidereal period is *less* than the former by nearly five days.

The annual regression of the node has been stated to be $19^{\circ}.341875$. This, as is plain from the mode of deducing it, is the *mean* regression. It will differ from the true annual regression, (that which belongs to any particular year, 1810, for instance,) by reason of several inequalities to which it is subject. And, as we shall hereafter see, the regression, besides its periodical inequalities, is affected with a secular inequality, by which its *mean motion* is, from century to century, retarded.

Inclination of the Moon's Orbit.

The inclination may be determined from the expression of p. 659, l. 17: or thus:

Amongst the latitudes computed from the Moon's right ascensions and declinations, the greatest, at the distance of 90° from the node, measures the inclination of the orbit. This, sometimes, is found nearly equal to 5°: at other times, greater than 5°. For instance, the greatest latitude of the new and full Moon, when at 90° from the node, is found equal to 5° nearly: but the greatest latitude when the Moon is in quadrature, and also 90° from the node, is found equal to $5^{\circ}\ 18'$. Hence the inclination of the Moon's orbit is variable: it is greatest in quadratures and least in syzygies.

Major Axis of the Moon's Orbit.

The Moon's distance is to be determined by her parallax. The method of Lacaille, described in Chap. XII, p. 325, (which is inapplicable, in the case of the Sun, on account of his great distance,) applied to the Moon, affords practical results of great exactness.

The degree of exactness is known from the probable error of observation, and the consequent error in the resulting distance: now, a variation of $1''$ in the parallax would cause a difference of about 67 miles in the determination of the distance*: therefore, as the Moon's parallax can certainly be determined within $4''$, the greatest error in the resulting distance cannot exceed 280 miles, out of about 240,000 miles.

Since, generally, the Moon's distance can be determined, her greatest and least may: and consequently, supposing her orbit to be elliptical, the major axis of the ellipse, which is the sum of the greatest and least distances, may be determined.

Eccentricity of the Moon's Orbit.

This is known from the greatest and least distances of the Moon, the apogean and perigean. Or, it may be determined from the greatest equation (see pp. 473, &c.) Its quantity, according to Lalande, (*Astronomy*, tom. II, p. 312,) is 0.055036: which gives for the greatest equation $6^0\ 18'\ 32''.076$, M. Laplace however, states the eccentricity for 1800 to be 0.0548553, which gives the greatest equation of the centre, $6^{\circ}\ 17'\ 54''.492$.

The Moon's Mean Motion.

By p. 611, the time (τ) of a synodic revolution equals $\dfrac{Pp}{P-p}$.

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* Let $p =$ ☽'s parallax, then, see p. 651, ☽'s dist. $= \dfrac{\oplus\text{'s rad.}}{p}$. Let e be the error of parallax, then the corresponding errror in the Moon's distance $= \dfrac{\oplus\text{'s rad.}}{p} - \dfrac{\oplus\text{'s rad.}}{p+e} = \dfrac{\oplus\text{'s rad.}}{p}\left(1 - \dfrac{1}{1+\frac{e}{p}}\right)$

$$= \frac{\oplus\text{'s rad.}}{p}\left(1 - 1 + \frac{e}{p}\right) = \frac{\oplus\text{'s rad.}}{p}\left(\frac{e}{p}\right), \text{ nearly,}$$

(rejecting the terms involving e^2, &c.) Hence, if $e = 1''$, and $p = 1^{\circ}$, and $\dfrac{\oplus\text{'s rad.}}{p}$, or the ☽'s dist. $= 240{,}000$ miles, the error $= \dfrac{1}{60.60} \times 240{,}000 = 67$ miles, nearly. In the case of *Mars*, an error of $1''$ includes in the distance an error of 40,000 miles.

Hence, if τ be computed from observation, since P the Earth's period is known, p, the Moon's, may be computed from the expression

$$p = \frac{P\tau}{P + \tau}.$$

If the Moon and Earth revolved equably in circular orbits, the above method would give accurately the Moon's period; but, since the Moon and Earth are subject to all the inequalities of a disturbed elliptical motion, the result obtained, by the above process, from one observed synodic revolution, would differ considerably from the mean period. In order, therefore, to obtain a mean period, we must observe and compute two conjunctions, or two oppositions, separated from each other by a long interval of time; and then, the interval divided by the number of synodic revolutions will give nearly the length of a mean synodic period, and very nearly indeed, if the Moon's apogee at the time of the second conjunction or opposition should be nearly in the same place in which it was, at the time of the first conjunction or opposition. From this *mean value* of the synodic period (τ), the mean period (p) may be computed from the above expression.

Now the phenomena of eclipses are very convenient for determining certain epochs of oppositions. And great certainty is obtained by their means. For, the recorded time of an eclipse by an antient Astronomer must be nearly the exact time of its happening; whereas, the assigned time of a conjunction or opposition happening long since, might, from the imperfection of instruments and methods, be erroneous, to a very considerable degree.

If we use two oppositions indicated by two eclipses, separated from each other by a short interval, we may deduce, but with no great exactness, (as has been already observed in this page,) the time of a synodic revolution. Thus, according to Cassini, a lunar eclipse happened in Sept. 9, 1718, $8^h\ 4^m$; another eclipse in Aug. 29, 1719, $8^h\ 32^m$. The interval between the two eclipses was $354^d\ 0^h\ 28^m$: and in the interval, 12 synodical revolutions had taken place; consequently, the mean length of one of these twelve, is equal to $\frac{354^d\ 0^h\ 28^m}{12}$, equal to $29^d\ 12^h\ 2^m$.

This result cannot be exact: it is affected by the inequalities of the Moon's elliptical motion: for, independently of other causes, the place of the apogee of the Moon's orbit at the time of the second observation is distant from its place at the first by about 40^0.

In order to obtain a true mean result, we must employ eclipses very distant, in time, from each other. Such are, an eclipse recorded by Ptolemy to have been observed by the Chaldeans in the year 720 before Christ, March 19, $6^h\ 11^m$ (mean time at Paris, according to Lalande,) and an eclipse observed at Paris in 1771, Oct. 23, $4^h\ 28^m$. The interval between the eclipses, is 910044 days minus $1^h\ 43^m$, and expressed in seconds, 78627795420^s. In this interval 30817 synodic revolutions had happened; the mean length of one of these, then,

$$= \frac{78627795420^s}{30817} = 29^d\ 12^h\ 44^m\ 2^s.2.$$ Substituting this value in the expression, p. 663, l. 4, we may obtain the value of p.

The value of the synodic period, computed from different observations, is not always of the same magnitude. Its *mean* length therefore is subject to a variation, arising from a cause called the *Acceleration of the Moon's Mean Motion,* which will be hereafter explained.

According to M. Laplace, the mean length of a synodic revolution of the Moon for the present time, is

$$29^d\ 12^h\ 44^m\ 2^s.8032\ (= 29^d.530588).$$

The periodic revolution of the Moon computed from the expression of p. 663,

$$= \frac{365.242264 \times 29.530588}{365.25 + 29.530588} = 27^d.321582$$

$$= 27^d\ 7^h\ 43^m\ 4^s.6848.$$

This is the *tropical* revolution of the Moon, or the revolution with respect to the equinoxes, since the number which was substituted for P was 365.242264, which expresses the Earth's tropical revolution.

The diurnal tropical movement of the Moon

$$= \frac{360^0}{27.321582} = 13^o.17636 = 13^o\ 10'\ 34''.896.$$

The *sidereal* revolution of the Moon differs from the tropical, for the same reasons, (see p. 198,) as the sidereal year differs from the tropical: and the difference must be computed on similar principles: thus, the mean precession of the equinoxes being 50″.1 in a year, or about 4″ in a month, the sidereal revolution of the Moon will be longer than the tropical, by the time which the Moon, with a mean diurnal motion of $13^0.17636$, takes up in describing 4″: which time is nearly 7^s. The exact length of a sidereal revolution is $27^d\ 7^h\ 43^m\ 11^s.510$, $(= 27^d.321661)$*.

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* We may easily deduce a formula of computation: thus, let p be the Moon's tropical revolution $(= 27^d.321582,)$ and x the sidereal period to be investigated; then, the arc of the precession described in the time

$$= \frac{50''.1 \times x}{365.25},$$

and the time of the Moon's describing it $= \frac{p}{365.25} \times \frac{50''.1}{360^o} \times x.$

Hence, $x = p + \frac{p}{365.25} \times \frac{50''.1}{360^o} \times x$, and thence

$$x = \frac{p}{1 - \frac{p}{365.25} \times \frac{50''.1}{360^0}},$$

= (expanding)

$$p\left\{1 + \frac{p}{365.25} \times \frac{50''.1}{360^o} + \left(\frac{p}{365.25}\right)^2 \times \left(\frac{50''.1}{360^o}\right)^2 + \&c.\right\}$$

in which, since $\frac{p}{365.25} \times \frac{50''.1}{360^o}$ is a very small quantity, two terms will be sufficient to give a value of x sufficiently near.

The same series may be used for determining the length of the sidereal

Since the equinoctial point (from which longitudes are measured) regresses, the Moon departing from a point, where its longitude is $= 0$, returns to a point at which its longitude is again $= 0$, before it has completed a revolution amongst the fixed stars. In like manner, the node of the Moon's orbit regressing, and faster than the equinoctial point, the Moon, quitting a node, will return to the same before completing a revolution amongst the fixed stars, and in a period less than the *tropical.*

This period may be thus found; the diurnal tropical movement of the Moon is $13^{\circ}\ 10'\ 34''.896$, and that of the node (see p. 661,) $= \dfrac{19^{0}.341875}{365.242264} = 3'\ 10''.6386$. Hence, the diurnal separation, which is the sum of the above quantities since the node regresses, $= 13^{\circ}\ 13'\ 45''.535$*: and consequently,

$$13^{\circ}\ 13'\ 45''.535 : 360^{0} :: 1^{d} : 27^{d}\ 5^{h}\ 5^{m}\ 35^{s}.6,$$

the revolution of the Moon with respect to its node.

This latter revolution may also be found by the aid of the formula given in the Note to p. 665.

By like processes, from the ascertained quantity of the apogee of the Moon's orbit, we may determine the *anomalistic* revolution

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sidereal from the tropical year, by substituting for p, $365^{d}.25$: in that case, the length of the sidereal year

$$=365.25\left(1 + \frac{50''.1}{360^{\circ}} + \&c.\right)$$

and a like series would serve to determine the length of an anomalistic year, substituting instead of $50''.1$, the quantity expressing the progression of the apogee.

* The Moon's motion with regard to its node may be found from eclipses; for, when these are of the same magnitude, the Moon is at the same distance from the node. Hipparchus, by comparing the eclipses observed from the time of the Chaldeans to his own, found that in 5458 lunations, the Moon had passed 5923 times through the node of its orbit: thence he deduced the daily motion of the Moon with regard to its node, to be $13^{\circ}\ 13'\ 45''\ 39'''\frac{2}{3}$. See Lalande, tom. II, p. 189.

of the Moon, M. Lalande (*Astronomie*, tom. II, p. 185,) states it to be $27^d\ 13^h\ 18^m\ 33^s.9499$, but M. Delambre, $27^d\ 13^h\ 18^m\ 37^s.44$ ($= 27^d.5546$.)

There is another revolution, of some consequence in the lunar theory, called the *Synodic Revolution of the Node:* this is completed when the Sun departing from the Moon's node first returns to the same. It is to be computed as the preceding periods have been. Thus, since the mean daily increase of the Sun's longitude is $59'\ 8''.33$, and the daily regression of the node is $3'\ 10''.638$, the sum of these quantities, which is the separation of the Sun from the node in a day, is $1^\circ\ 2'\ 18''.96$. Hence, $1^\circ\ 2'\ 18''.96 : 360^\circ :: 1^d : 346^d\ 18^h\ 28^m\ 16^s.032$ ($= 346^d.61963$ *.)

We will now exhibit, under one point of view, the different kinds of lunar periods and motions :

Synodic revolution	29^d	12^h	44^m	$2^s.8032$ =	$29^d.530588$
Tropical	27	7	43	4.6848 ..	27.321582
Sidereal...............	27	7	43	11.5101 ..	27.321661
Anomalistic	27	13	18	37.44	27.5546
Revol^n. in respect of node	27	5	5	35.6	27.212217
Tropical revolu^n. of node	6798^d	12^h	57^m	$52^s.416$	6798.54019
Sidereal	6793	10	6	29.952..	6793.42118

☽'s mean tropical daily motion	13°	$10'$	$34''.896$
☽'s sidereal daily motion	13	10	35.034
☽'s daily motion in respect to the node	13	13	45.534

Place of the Apogee.

The Moon's diameter is least at the apogee, and greatest in the perigee: and since the diameter can be measured by means

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* This and the preceding periods are frequently found on like principles, but by different expressions, from the values of the *secular motions.* Thus, in 100 Julian years, each consisting of $365^d.25$, the *secular motion* of the Sun is $36000^\circ\ 45'\ 45''$ ($36000^\circ.7624998$) and the secular motion of the node (see p. 661,) $1934^\circ.1875$: and the sum of these is $37934^\circ.95$ nearly: thence $37934.95 : 360 :: 100 :$ period $= \dfrac{36000}{37934.95}$.

of a micrometer, or can be computed from the time it takes up in passing the vertical wires of a transit instrument, the times of the least and greatest diameter, or the times when the Moon is in her apogee and perigee, can be ascertained. Instead of endeavouring to ascertain when the Moon's diameter is the least, Lalande, *Astron.* tom. II, p. 162, says, that it is preferable to observe the diameters towards the Moon's mean distances when the diameter is about $31' 30''$. If two observations can be selected when the diameter was of the same quantity, then we may be sure that, at these two observations, the Moon was at equal distances from the apsides of its orbit. The middle time then between the two observations is that in which the Moon was in her apogee.

By finding the places of the apogee, according to the preceding plan, and comparing them, it appears that the apogee of the Moon's orbit is progressive *: completing a sidereal revolution in $3232^d\ 11^h\ 11^m\ 39^s.4$, and a tropical, in $3231^d\ 8^h\ 34^m\ 57^s.1$. Laplace states the sidereal revolution of the apogee to be $3232^d.579$, that is, $3232^d.\ 13^h\ 53^m\ 45^s.6$. (See *Exposition du Systeme du Monde*, Edit. 2, p. 20.)

Mean Longitude of the Moon at an assigned Epoch.

By observations on the meridian, the right ascension and declination of the Moon are known; thence may be computed, the Moon's longitude. This resulting longitude is the true longitude, differing from the mean by the effect of all the inequalities, elliptical, as well as those that arise from the perturbations of the Sun and planets. The *mean* longitude therefore, is the difference of the true longitude and of the sum (mathematically speaking) of the *equations* due to the inequalities. In order, therefore, to determine the mean longitude, the lunar theory must be known to some degree of exactness. Any new inequality discovered will affect the previous determination of the mean motion: and accordingly, keeping pace with the continual improvements in the lunar theory, repeated alterations have been made in the quantity of the mean longitude. In the last Lunar French Tables,

* See *Physical Astronomy*, Chap. XIII.

the epoch of the mean longitude for Jan. 1, 1801, midnight at Paris, is $3^s\ 21^\circ\ 36'\ 30''.6$: which for Greenwich, Jan. 1, at noon, is $3^s\ 28^\circ\ 16'\ 56''.1$.

In order to determine the eccentricity of the Moon's orbit, considered as elliptical, and the deviations from the elliptical form caused by the actions of the Sun and planets, it is necessary to know the angular spaces described by the Moon, in her orbit. Such spaces are not immediately given by observation. We must make several steps to arrive at them. The first is the determination of the Moon's parallax: the second, the observation of the Moon's right ascension and zenith distance: the third, the correction of the zenith distance on account of parallax, in order to obtain the true declination. The fourth, the computation of the Moon's latitude and longitude: the fifth, the *reduction* of the Moon's longitude to a longitude on her orbit, to be effected by the same formula (see pp. 501, &c.) as that of the *reduction of the ecliptic to the equator*.

The comparison of the *reduced* longitudes, or the comparison of the arcs of the Moon's orbit, described in certain times, will shew us how much such arcs, with respect to their forms and laws of description, differ from elliptical arcs. This point will be considered in a subsequent Chapter. In the next we will advert to certain *secular* inequalities (arising, indeed, from the same source as the Moon's periodical inequalities) that affect those elements of the orbit, which we have just considered.

CHAP. XXXII.

On the Secular Equations that affect the Elements of the Lunar Orbit.

THE correction, which is called a *Secular Equation,* is strictly speaking periodical, but requiring a very large period, in order to pass through all its degrees of magnitude before it begins to recur. Its quantity, in general, is very small, and usually expounded by its aggregate in the space of 100 years.

The nodes, the apogee, the eccentricity, the inclination of the Moon's orbit, the Moon's mean motion, are all subject to secular inequalities. And the practical mode of detecting these inequalities is nearly the same in all.

If we subtract the longitude of the Moon's node now, from what it was 500 years ago, the difference is the regression of the node in that interval: the mean annual regression is the above difference divided by 500. If we apply a similar process to an observation of the Moon's node, made now, and to one made 1000 years ago, the result must be called, as before, the mean annual regression of the node; and this last result ought, if the regression were always equable, to agree with the former: if not, (as is the case in nature,) the difference indicates the existence of a *secular inequality,* requiring for its correction a *secular equation.*

By a similar method the motion of the perigee of the Moon's orbit does not appear to be a mean motion, but subject to a secular inequality.

But the most remarkable inequality is that which has been detected in the Moon's mean motion, and which is now known by the title of the *Acceleration of the Moon's mean Motion.* The fact of such acceleration was first ascertained by Halley, from the comparison of observations: the cause of the acceleration has been assigned by Laplace *. Although the method of detecting the existence of these inequalities does not differ, in principle, from methods just described, yet, on account of its importance, we will endeavour to explain it more fully.

As we have before remarked, eclipses are a species of observations on which we may rely with great certainty; quite distinct from merely registered longitudes which must partake of all the imperfections of methods used at the times of their computation. Now, in the year 721 before Christ, with a specified day and hour, Ptolemy records a lunar eclipse to have happened. The Sun's longitude then being known, the Moon's, which must at the time of the eclipse differ from it by six signs, is known also. The Moon's longitude however, computed for the time of the eclipse and by means of the Lunar Tables, does not agree with the former †. In some part or other, then, the Tables are defective, or, without some modification, are not applicable to ages that are past.

The Moon's place computed from the eclipse is advanced beyond the place computed from the Tables by $1^0\ 26'\ 24''$; an error too great to be attributed to any inaccuracies in the coefficients of the equations belonging to the *periodic* inequalities, and which would seem rather to be the aggregate, during many years, of a small error in some reputed constant element, such, for instance, as the Moon's mean motion.

On the hypothesis then of an acceleration in the Moon's motion, or, in other words, if we suppose the Moon now to move more rapidly than it did 2000 years ago, the error of $1^0\ 26'\ 24''$ can be accounted for. With a mean motion too large, we should

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* See Laplace, *Exposition du Syst. du Monde*, Edit. 2, pp. 20, 214, &c. also *Mec. Celeste*, pp. 175, &c. Lalande, tom. II, p. 185: Halley, *Phil. Trans.* Nos. 204, and 218, Newton, p. 481, Ed. 2. and Woodhouse's *Phys. Astron.* Chap. XXII.

† The *true* longitudes are not compared, but the *mean*.

throw the Moon too far back in its orbit. And, with the same motion, but for a point of time less remote than the preceding, we ought, if the hypothesis of the acceleration be true, to throw the Moon less far back in her orbit: for that would produce an error of the *same kind* as the one already stated, (p. 671). Now this is the case, and has been ascertained to be so, by means of an eclipse observed at Cairo by *Ibn Junis*, towards the close of the tenth century.

The acceleration of the Moon's motion therefore, discovered by Halley, may be assumed as established: or, in other words, in the former estimates of the quantity of the Moon's motion, a large secular inequality was included, which it is now necessary to deduct, in order that what remains may be truly a *mean motion*.

The variation in the mean motion of the Moon, will, it is plain, affect the durations of its synodic, tropical, and sidereal revolutions.

With this secular equation in the Moon's mean motion, the equations in the motions of the nodes and of the apogee are connected. The latter are subtractive, whilst the former is positive; and, according to Laplace, *Mec. Celeste*, tom. III, p. 236,) the secular motions of the perigee, of the nodes and mean motion, are to each other, as the numbers 3.00052, 0.735452, and 1.

The mean anomaly of the Moon, which is the difference of her mean longitude and the mean longitude of the apogee, must be subject to a secular equation, which is the difference of the secular equations affecting the longitudes of the Moon and of the apogee.

All quantities, in fact, dependent on the Moon's mean motion, the apogee and nodes, must be modified by their secular equations.

The Moon's distance from the Earth, the eccentricity and inclination of her orbit, are, according to M. Laplace, also affected with secular equations connected with that of the mean motion. But, the major axis is not. (See *Physical Astron.* Chap. XXIII.)

We will, in the next Chapter, explain briefly the origins, quantities, and variations of those inequalities, which during a month, a year, and the periodical revolution of the nodes, render the Moon's true place different from its elliptical, or, more generally speaking, from its mean place.

CHAP. XXXIII.

On the Inequalities affecting the Moon's Orbit.—The Evection.—Variation.—Annual Equation, &c.—The Inequalities of Latitude and Parallax.

By a comparison of the Moon's longitudes and of her distances deduced from her parallaxes, it appears that the lunar orbit is nearly an ellipse with the Earth in one of the foci. It appears also, that the Moon not only wanders from the ellipse which may be traced out as her mean orbit, and transgresses the laws of elliptical motion, but, that the ellipse itself is subject, in its dimensions, to continual variation: at one time, contracted within its mean state, at another, dilated beyond it.

In strictness of speech, neither the Earth's orbit nor the Moon's are to be called ellipses. If they are considered as such, it is purely on the grounds of convenience. It is *mathematically* commodious, or it may be viewed as an artifice of computation, first, to find the approximate place of each body in an assumed elliptical orbit, and then to compensate the error of the assumptions, and to find a truer place, by means of corrections, or, as they are astronomically called, *Equations*.

In a system of two bodies, when forces, denominated centripetal, only act, an accurate ellipse is described by the revolving round the attracting body; and, in such a system, the apsides, the eccentricities, the mean motions, &c., would remain perpetually unchanged. The introduction of a third, or of more bodies, and the consequent introduction of *disturbing forces*, destroys at once the beautiful simplicity of elliptical motion, and puts every element of the system into a state of continual mutation. Yet, the change and the departure from the laws of elliptical motion, are less in some cases than in others. The Earth's orbit approaches much more nearly to the form of an ellipse than the Moon's. The Sun's longitude, as we have seen in p. 496, computed by Kepler's Problem, did not differ from the true place by more than seven seconds: and that quantity, in those circumstances, represented the perturbations of the planets; and, the equations representing the perturbations were only four. But, in the case of the Moon, one inequality alone will require an equation nearly equal to two degrees, and the number of equations amounts to 28.

The quantity of perturbation, and the difficulty of computing it, depend less on the number than on the proximity of the disturbing bodies. In the case of the Sun, one equation suffices for the perturbation of *Venus*, and another for that of *Jupiter*. But, all the equations compensating the inequalities in the Moon's place, arise from different modifications of the Sun's disturbing force. It is not, however, solely the proximity, but the mass of the disturbing body, that gives rise *to equations*. The strictly *mathematical* solution of the problem of the three bodies (see Chap. XX.) is equally difficult, whatever be the mass of the disturbing body. The *practical* difficulty of merely approximating to the true place of the disturbed body, is very considerably lessened by supposing that mass to be small.

If we consider the subject merely in a mathematical point of view, the Moon's place, at any assigned time, results from the compound action of the Earth's centripetal force and the Sun's disturbing force; and the deviation from her place in the exact ellipse, arises entirely from the latter. We are at liberty to call the deviation, or error, one uncompounded effect: yet, since the quantity of the deviation cannot be computed from one single analytical expression, but must be so, by means of several terms, we may separate and resolve the effect into several, (analogous to the above-mentioned terms,) the causes of some of which we may distinctly perceive and trace in certain simple resolutions and obvious operations of the Sun's disturbing force.

Long before Newton's time and the rise of Physical Astronomy, this separation, or resolution of the error of the Moon's place from her elliptical place was, in fact, made. And, the error was said to arise from three inequalities, distinguished by the titles of *Evection*, *Variation*, and *Annual Equation*.

These three inequalities were noted because they rose, under certain circumstances, to a conspicuous magnitude; and, were distinguished from each other, because they were found to have an obvious connexion with certain positions of the Sun and Moon and of the elements of their orbits. Although their real physical cause was not discovered, yet the laws of their variation were ascertained.

The other lunar inequalities have not, like the three preceding, been distinguished by titles. This is owing principally to their want of historical celebrity; they were not detected like the others, by reason of their minuteness and the imperfection of antient instruments and methods.

Some explanation has already been given, (Chaps. XIV, XV,) of the principles and modes of detecting and decompounding inequalities. The difference between an observed and computed place, indicates the operation of causes either not taken account of, or not properly estimated in the previous computation.

Take, for instance, the Moon: her mean place, computed from her mean motion, differs from her observed place; and the difference, if we suppose her to move in an elliptical orbit, is the equation of the centre, or, of the orbit, called the *First Lunar Inequality*.

Compute the Moon's place from a knowledge of her mean motion and of the equation of the centre, and then compare the computed, with the observed, place. In certain situations, a great difference will be noted between the places, ascending in its greatest value to nearly $1^{\circ}\ 18'\ 3''$. This difference is chiefly owing to the *Evection* discovered by Ptolemy, and named the *Second Lunar Inequality*.

In like manner, we may conceive the *Third Lunar Inequality* to be discovered. But, we will now proceed to consider more particularly the second inequality; the mode of ascertaining its maximum; its general effect; the formula expressing the law of its variation; and its cause, reckoning as such, some particular modification of the Sun's disturbing force.

Evection. (See *Physical Astronomy*, pp. 236, &c.)

This inequality has a manifest dependence on the position of the apogee of the Moon's orbit. Let us suppose the Moon to quit the apogee, the line of the apsides to lie in syzygy, and that we wish to compute the Moon's place 7 days after her departure from syzygy, when, in fact, she will be nearly in quadratures. The Moon's place, computed by deducting the equation of the centre*, (then nearly at its greatest value and $= 6^0\ 37'\ 54''.492$,) from the mean anomaly (see Chap. XVIII.) will be found *before* the observed place by more than 80 minutes; in other words, the computed longitude of the Moon is so much greater than the observed longitude. But, if we suppose the apsides to lie in quadratures, the Moon's place, 7 days after quitting her apogee, computed, as before, by subducting the equation of the centre from the mean anomaly, will be found *behind* the observed place by more than 80 minutes; in other words, the computed longitude of the Moon is so much less than the observed.

It is an obvious inference, then, from these two instances, that some inequality, besides that of the elliptic anomaly, and having a marked connexion with the longitude of the lunar apogee, affects the Moon's motion.

What, from the two preceding instances, would be an obvious inference to an Astronomer acquainted solely with the elliptic theory of the Moon? In the first case, the computed place being *before* the observed, it would *seem* that the equation of the centre, to be subducted from the mean anomaly, had not been taken of sufficient magnitude; in the latter case, it would *seem* that the equation of the centre had been taken too large.

Let us take another case: suppose, instead of comparing the computed with the observed place, that it was intended to deduce

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* The anomlay is here supposed to be reckoned from apogee.

the quantity of the equation of the centre from an observation of the Moon in syzygy. In that case, the equation of the centre, reckoned as the difference of the true and mean longitudes, would result *too small* a quantity. And this circumstance has really happened. For, the antient Astronomers who determined the elements of the lunar orbit by means of eclipses, when the Moon is in syzygy, have assigned too small a quantity to the equation of the centre.

In the preceding instance, when the Moon is in syzygy and the apsides in quadrature, the determination of the equation of the centre would be too small by the maximum value of the *Evection* ($1^{0}\ 20'\ 29''.5$). But, in other positions of the apsides, the effect of the evection is to lessen, though not by its whole quantity, the equation of the centre.

Astronomers, having found that the augmentation and diminution of the equation of the centre arose from an inequality, soon ascertained the inequality to be periodical; in other words, that, after passing through all its degrees of magnitude, from 0 to its maximum value, it would recur. Now, of such recurring quantities the cosines and sines of angles are most convenient representations; for instance, $\pm K . \sin. E$ is competent to represent the *Evection*: its maximum value is K, when $E = 90^{0}$: and it is nothing, when E is. If then, the value of K could be assigned and the form for E, the numerical quantity of the Evection could be always exhibited. After the comparison of numerous observations, and after many trials, it was found that

$$K = 1^{0}\ 20'\ 29''.5, \text{ and } E = 2\,(☽ - ☉) - A,$$

A representing the mean anomaly of the Moon, and $☽ - ☉$ signifying the angular distance of the Sun and Moon, or, the difference of their mean longitudes viewed from the Earth.

In the *equation*

$$1^{0}\ 20'\ 29''.5 . \sin. [2\,(☽ - ☉) - A],$$

$1^{\circ}\ 20'\ 29''.5$ is called the *coefficient*, and $2\,(☽ - ☉) - A$ the *argument*.

If we represent the equation of the centre by

$$(6^{0}\ 17'\ 54''.49) \sin. A,$$

in which, the coefficient $6^{\circ}\ 17'\ 54''.49$, is the greatest equation, and A (the mean anomaly) the argument, the Moon's longitude expressed by means of the two equations, that of the centre *, and the evection, would stand thus:

$$☽\text{'s longitude} =$$

$$☽\text{'s mean long.} - (6^{\circ}\ 17'\ 54''.49) \sin. A$$
$$- (1^{\circ}\ 20'\ 29''.5) \sin. [2(☽ - ☉) - A];$$

now in syzygies $☽ - ☉ = 0$; $\therefore \sin. [2(☽ - ☉) - A] = - \sin. A$; consequently, in this case, the former expression becomes

$$☽\text{'s longitude} =$$

$$☽\text{'s mean long.} - (6^{\circ}\ 17'\ 54''.49) \sin. A + (1^{\circ}\ 20'\ 29''.5) \sin. A,$$

in which, the argument for the Evection assumes that form, which is the general one of the *equation of the centre;* and on this account, the former is sometimes said to *confound* itself with the latter, in syzygies. It also seems to lessen it, since the preceding expression may be put under this form,

$$☽\text{'s longitude} =$$

$$☽\text{'s mean long.} - (6^{\circ}\ 17'\ 54''.49 - 1^{\circ}\ 20'\ 29''.5) \sin. A,$$

in which, the coefficient of $\sin. A$ would be the difference of the two coefficients $6^{\circ}\ 17'\ 54''.49$, and $1^{\circ}\ 20'\ 29''.5$; and, accordingly, A being the argument of the *Equation of the Centre,* that equation would appear to be lessened.

The Evection itself, and, very nearly, its exact quantity, were discovered by Ptolemy in the first century after Christ, but the cause of it remained unknown till the time of Newton. That great Philosopher shewed that it arose from *one kind of alteration* which the Moon's centripetal force towards the Earth receives from the Sun's perturbation. Let us see how it may be explained:

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* If A be the mean anomaly, the equation of the centre cannot be represented by a single term such as $a \sin. A$, but by a series of terms, such as $a \sin. A + b \sin. 2A + c \sin. 3A + \&c.$ in which, however, the coefficients b, c, &c. decrease very fast.

When the line of the apsides is in syzygies, the Equation of the centre (p. 677,) is increased. The Equation of the centre depends on the eccentricity; (see pp. 662,) an increase therefore in the former would indicate an increase in the latter. Hence, if it can be shewn that the Moon's orbit must, when the line of the apsides is in syzygies, be made more eccentric by the action of the Sun's disturbing force, an adequate explanation will be afforded of the increase of the equation of the centre above its mean value; which increase is styled the *Evection.*

Again, when the line of the apsides is in quadratures, the Equation of the centre is lessened: the eccentricity therefore (see expression, p. 473,) is lessened: and now, in order to afford an explanation, it is necessary to shew that, in this position of the line of the apsides, the Sun's disturbing force necessarily renders the orbit less eccentric.

The Sun's disturbing force admits of two resolutions, one in the direction of the radius vector of the Moon's orbit: the other in the direction of a tangent to the orbit. The former sometimes augments, at other times, diminishes the gravity of the Moon towards the Earth, and always (see Newton, Sect XI, Prop. 66,) proportionally to the Moon's distance from the Earth. When the Moon is in syzygy, it diminishes; consequently, in the first case, when the line of the apsides is also in syzygy, the perigean gravity, which is the greatest, (since it varies inversely as the square of the distance) is diminished, and by the least quantity; the apogean gravity, the least, is also diminished, but by the greatest quantity: the disproportion therefore between the two gravities is augmented; the ratio between them becomes greater than that of the inverse square of the distance: the Moon, therefore, if moving towards perigee, is brought to the line of the apsides in a point between its former and mean place and the Earth: or, if moving towards apogee, reaches the line of the apsides in a point more remote from the Earth than its former and mean place. The orbit then becomes more eccentric; the equation of the centre is increased; and, the increase is the *Evection.*

Thus is the first case accounted for. In the second, the Sun's resolved force *increases* the gravity of the Moon towards the Earth, and, as it has been said, proportionally to the distance. The perigean gravity, therefore, which is the greatest, is increased by the least quantity; the apogean, the least, is also increased, and by the greatest quantity. The disproportion, therefore, between these two gravities is lessened; the ratio between them is less than that of the inverse square of the distance. The Moon, therefore, if moving towards perigee, meets the line of the apsides, in a point more remote from the Earth than the mean place of the perigee: if moving towards the apogee, in a point between the Earth and the mean place of the apogee. The orbit, by these means, becomes less eccentric; the Equation of the centre is diminished, and, the diminution is the *Evection.*

We will now proceed to consider the third inequality called

The Variation. (See *Physical Astronomy,* pp. 217, &c.)

By comparing the Moon's place computed, from her mean motion, the equation of the centre, and the Evection, with her observed place, Tycho Brahe, in the sixteenth century, discovered that the two places did not always agree. They agreed only in opposition and conjunction, and varied most, when the Moon was half way between quadratures and syzygies, that is, in *Octants.* At those points the new inequality seemed to be at its maximum value ($35' \; 41''.6$).

It appeared clearly from the observations, that this new inequality was connected with the angular distance of the Sun and Moon: and that its *argument* must involve, or, be some function of, that distance. At length, it was found, that the equation due to the inequality, was

$$(35' \; 41''.6) \,.\, \sin. 2\,(☽ - ☉)$$

$35' \; 41''.6$ being the *coefficient,* and $2\,(☽ - ☉)$ the *argument.*

According to the above form, the variation is 0 in syzygies and in quadratures, and at its maximum ($35' \; 41''.6$) in octants.

If now, by means of this new equation, we farther correct the expression (p. 679,) for the Moon's place, we shall have

$$☽\text{'s longitude} =$$

$$☽\text{'s mean longitude} - (6^\circ\ 17'\ 54''.49)\sin. A$$
$$- (1\ \ 20\ \ 29.5)\,.\sin.\,[2\,(☽ - ⊙) - A]$$
$$+ (35'\ 41''.6)\,.\sin.\,2\,(☽ - ⊙).$$

We will now proceed to Newton's explanation of the cause of this inequality.

One effect, from a resolved part of the Sun's disturbing force, we have already perceived in the Evection. The *Variation* is occasioned by the other resolved part, that which acts in the direction of a tangent to the Moon's orbit. This latter force will accèlerate the Moon's velocity in every point of the quadrant which the Moon describes, in moving from quadrature to conjunction. The force will be greatest in octants and nothing in conjunction; and, when the Moon is past conjunction, the tangential force will change its direction, and retard the Moon's motion. The greatest acceleration, therefore, of the Moon's velocity must happen in syzygy: exactly at the termination or cessation of the accelerating force. At that point, therefore, the Moon's velocity must differ most from her mean, or, rather, from that velocity which she would have, if the effect of the accelerating tangential force were abstracted. When the Moon moves from that point, her place at the end of any portion of time, a day, for instance, will be beyond her mean place, or beyond the place of an imaginary Moon endowed with a motion from which the effect of *Variation* is abstracted. At the end of the second portion of time, the real Moon will have described a space less, by reason of the retarding force (see l. 15,) than the space described in the first, but, still, greater than the space described by the imaginary Moon; so that, at the end of the second portion of time, the two Moons will be distant from each other, by the effect of two separations; and, for succeeding portions of time, the real Moon will still continue describing greater angular spaces than the imaginary Moon, and the separation of the two Moons, which is the accumulation of the individual excesses, will continue, till the retarding force, by the continuance of its action, and the increase of its quantity, shall have reduced the Moon's velocity to its mean state: at that term which is the octant, the separation will cease to increase, and will be at its greatest. And this greatest separation, $35' \, 41''.6$, is the maximum effect of the *Variation:* and the separation, previously described, in any point between conjunction and octants, is its common effect.

The preceding reasoning is precisely similar to that which was used in p. 469, on the subject of the greatest equation of the centre. At the apogee, the mean velocity differs most from the true, and then the two Suns are together; and, they are most separated, when the real Sun moves with its mean angular velocity.

We will now proceed to a fourth inequality called,

The Annual Equation. (See *Physical Astronomy*, pp. 237, &c.)

The two former inequalities, of which the periods are short, may be ascertained by observing the Moon during one revolution. But, in order to detect this fourth inequality, it is necessary to compare similar positions of the Moon, computed according to the theory of the three preceding inequalities, in different months of the year. If the computed place agreed with the observed place in January, it would not in March, and it would most differ in July. The inequality was soon found to have a connexion with the Earth's distance from the Sun, and its *equation* was at length found to be

$$11' \, 11''.97 \times \sin. \odot\text{'s mean anomaly},$$

$11' \, 11''.97$ being the coefficient, and $\odot$'s mean anomaly the argument.

According to the preceding form, the maximum ($11' \, 11''.97$) of the annual equation happens when the Sun's mean anomaly is $= 90^0$, or 270^0. The equation is nothing, either when the Earth is in the aphelion or perihelion of its orbit.

If now, by means of this new equation, we farther correct the expression for the Moon's longitude, we shall have

$$☽\text{'s longitude} =$$

$$\begin{aligned} ☽\text{'s mean longitude} &- (6^0 \; 17' \; 54''.49) \sin. A \\ &- (1^0 \; 20' \; 29''.5) \sin. [2(☽ - \odot) - A] \\ &+ \quad (35' \; 41''.6) \sin. 2(☽ - \odot) \\ &+ \quad (11' \; 11''.97) \sin. \odot\text{'s mean anomaly}, \end{aligned}$$

(see *Physical Astronomy*, p. 239.)

We will now proceed to an explanation of the cause of this inequality.

The *Variation* has been explained from the effect of that resolved part of the Sun's disturbing force which acts in the direction of the tangent; the *Evection,* from the effect of the resolved part in the direction of the radius vector, and which effect alters the ratio of the perigean and apogean gravities from that of the inverse square of the distance. The present inequality depends not, on any immediate effect, either of the one, or of the other resolved part; but on *an alteration in the mean effect* of the disturbing force in the direction of radius; and, which mean effect lessens the gravity of the Moon towards the Earth.

By the mean effect, that is meant to be understood, which is the result of the disturbing forces in the direction of the radius in one revolution. The disturbing force does not always diminish the Moon's gravity to the Earth; it does in opposition and conjunction, but it augments the gravity in quadratures (see Newton Sect. XI; Prop. 66). The augmentation however, is only half the diminution (Newton, Prop. 66, Cor. 7). In the course therefore of a synodic revolution, there results, what may be called a mean force tending to diminish the Moon's gravity to the Earth, the measure of the mean force being equal to (see Newton, Prop. 66.)

$$\frac{\odot\text{'s mass} \times \text{rad. ☽'s orbit}}{\text{cube } \oplus\text{'s dist. from } \odot}.$$

By reason of this diminution, the Moon is enabled to preserve a greater distance from the Earth than it could do, by the influence of gravity alone. But, since the disturbing force acts in the direction of the radius, the equal description of areas is not altered (see Newton, Prop. 66). The area however varying as the product of the radius vector and the arc (the measure of the real velocity) and the former (see l. 26.) being increased, the real velocity must be diminished: so also must the angular, which varies inversely as the square of the distance.

These results are derived from that effect of the disturbing force of the Sun, which is a mean effect diminishing the Moon's gravity. If this mean effect of diminution be increased, similar results will follow, but in an enlarged degree; the Moon's angular velocity will be still more diminished and her distance from the Earth increased: now the measure of the mean effect is

$$\frac{\odot\text{'s mass} \times \text{rad. }☽\text{'s orbit}}{(\oplus\text{'s distance from }\odot)^3},$$

which will be increased, by diminishing the denominator: and is, therefore, in nature, increased when the Earth approaches the Sun. That circumstance happens in winter. In winter, therefore, the Moon's gravity to the Earth is more diminished, by the Sun's disturbing force, than in summer. Her angular velocity therefore is more diminished. A greater time is requisite to the description of a complete revolution round the Earth: in other words, a periodic month is longer in winter than in summer. Now, as the Earth approaches the Sun, its velocity increases. An acceleration therefore of the Earth's motion is attended, by reason of this new inequality, with a retardation of the Moon's, and reversely. On this account it is that, the *Annual Equation* is said to resemble the equation of the Sun's centre. For, supposing the Sun to be approaching his perigee, then his place (reckoning from apogee and neglecting the perturbations of the planets) is equal to the mean anomaly − the equation of the centre (E), E *decreasing* as the Sun approaches the perigee; if m be the Moon's place independently of the *annual equation* (e), then her place, corrected by that, is $m + e$, e increasing (since it varies as sin. $\odot$'s mean anomaly,) and affected with a contrary sign.

When the annual equation is $\pm$ (11′ 11″.976) sin. $\odot$'s mean anomaly, the corresponding Equation of the centre for the Sun is (1° 55′ 26″.3748) sin. $\odot$'s mean anomaly.

We have now gone through the explanation of the three principal lunar inequalities, which were discovered before the time of Newton and the rise of Physical Astronomy. These inequalities were, by reason of their magnitude, *fished out*, (as a late writer has significantly expressed it) from the rest. The discovery of the rest, in number 28 *, is entirely due to Physical

* Strictly speaking there are more than 28. But Astronomers have *confined* themselves to this number, since other equations, that analytically present themselves, never rise to a numerical value worth considering.

Astronomy. Without the aid of this latter science, it would have been, perhaps, impossible, from mere observation and conjecture, to have assigned the forms of the arguments. These latter being ascertained, it is the proper business of observation to assign the numerical value of their coefficients.

The three equations that have been explained are, with regard to their magnitudes, eminent above the rest; but, it must not be forgotten, the other equations, on the footing of theory, are of equal importance, and in practice, considering the use that is now made of the Lunar Tables, of very essential importance.

The three equations, with all the others, are derived from theory by the same process. And, as we have seen, the causes of the former may, independently of any formal calculation, be discerned in certain modifications of the Sun's disturbing force. The causes of the other equations are not so easily discernible: yet, the sources of some of them may be pointed out in certain changes, which the conditions or circumstances belonging to the three principal equations must necessarily undergo.

For instance, suppose the Moon and the line of the nodes to be in syzygies; then, the Sun's disturbing force, represented by part of a line joining the Sun and Moon, lies entirely in the plane of the Moon's orbit; and two resolutions of it, one in the direction of the radius, the other of the tangent, are sufficient. But, the nodes are regressive; in a subsequent position of them, then, the line representing the Sun's disturbing force, will be inclined to the plane of the Moon's orbit: consequently, a threefold resolution of the force is requisite, the third being in a direction perpendicular to the plane of the Moon's orbit; consequently, if the line representing the absolute quantity of the disturbing force be supposed to be the same, the resolved parts in the directions of the radius and of the tangent must be less than they were before. The inequalities caused by them must therefore be less, and less, according to the position of the nodes. Hence, if the equation of the evection

$$1^{\circ}\ 20'\ 29''.5 \times \sin. 2\left[(☽ - ☉) - A\right]$$

were adapted to the first position of the nodes, it could not suit the second, since the longitude of the nodes forms no part of the argument [2 (☽ — ☉) — *A*]. For this reason, therefore, a correction would be wanting for the Evection, that is a *new equation*, the argument of which should depend on the position of the nodes *. The same cause, the change in the Sun's disturbing force from its direction being more or less inclined to the Moon's orbit, must introduce new corrections, that is, *new equations*, belonging to the variation and annual equation.

Again, the annual equation arises from the change in that mean effect of the Sun's disturbing force by which the Moon's gravity is diminished. In adjusting therefore the value of the coefficient of the annual equation, the Moon's gravity must be supposed to be of a certain value: consequently, the Moon must be assumed to be at a certain distance from the Earth. When therefore the Moon is at a different distance, the *Equation*, if adjusted for the previous distance, cannot suit this: a small correction, therefore, or a *new Equation* will be necessary, the argument of which must involve or contain, in its expression, the Moon's distance, or her mean anomaly, or some term connected with these quantities †.

Again, the argument for the variation involves simply the angular distance of the Sun and Moon; and its coefficient must be supposed to be settled for certain values of the Moon's gravity and the Sun's disturbing force; and, consequently, when the Sun and Moon are at certain distances from the Earth. The changes therefore in those distances, which are continually happening, must render necessary two corrections, or two *new equations:* one for the approach of the Sun to the Earth, the other for the elongation of the Moon from the Earth. Generally, any equation

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* The equation in Lalande, p. 180, is

$$60''.4 \times \text{sin. 2 dist. ☽'s ☊ from ☉.}$$

† The supplementary equation, according to Mayer, is

$$42'' \text{ sin. (☽'s mean anom. — ☉'s mean anom.)}$$

which however is not the sole correcting equation due to this cause. See Lalande, *Astron.* tom. II, p. 178.

furnished with its numerical coefficient on the supposition of the Sun and Moon revolving round the Earth in circular orbits, will require new supplemental or subsidiary equations due to the real and elliptical forms of the orbits*.

Again, the inclination of the Moon's orbit is variable; therefore any equations adjusted to a mean state of inclination will require subsidiary equations, to correct the errors consequent on changes in that state.

From considerations like the preceding, the existence of the *smaller inequalities* is established: and, by an attentive consideration of the circumstances that occasion them, the forms of their arguments may be detected; with much less certainty however, than by the direct investigation of the disturbed place of the Moon.

It is one thing to prove the existence of an *inequality*, and another to establish the necessity of its corresponding *equation*. Whether it is expedient to introduce the latter, is a matter of mere numerical consideration. The correction of a correction, the subsidiary equation to a principal equation, is, in the lunar theory, very minute: and some equations, arising from the causes that have been enumerated, are so minute, as to be disregarded by the practical Astronomer.

We have at present considered only the inequalities that affect the Moon's longitude: but the Sun's disturbing force causes also *inequalities* in the Moon's *latitude* and in her *parallax*.

The inequalities of the latitude and of the parallax have nothing peculiar in them, nor distinct, (whether we regard their physical cause or the mode of ascertaining the laws of their variation,) from the inequalities of longitude. It is not necessary therefore to dwell on them, since the latter have been explained.

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* The evection, for instance, is variable from the variation of the distances of the Sun from the Moon and Earth: and for the purpose of correcting the evection, there are 4 subsidiary, or, as Lalande calls them, *accessary* equations, which in his Tables are the 5th, 6th, 7th, and 9th. See *Astron.* tom. II, p. 177.

We will only mention, that the principal inequality in latitude, and its law, were discovered by Tycho Brahé, and by the comparison of observations of the greatest latitudes of the Moon, at different epochs, and when that planet was differently situated, relatively to the nodes of its orbit. The equation is

$$(8' \, 47''.15) \, . \, \cos. \, 2 \odot\text{'s distance from } \rightmoon\text{'s } \ascnode .$$

(See Lalande, tom. II, p. 193. Mayer, *Theoria Lunæ*, p. 57. Laplace, *Mec. Cel.* Liv. VII, p. 283, &c. French Tables, Introduction.)

If the Moon's orbit coincided with the plane of the ecliptic, the Sun's disturbing force, resolved into the directions of a tangent to the Moon's orbit and of a radius vector, could only, by the first resolution, alter the law of elliptical angular motion, and, by the second, the length of the radius vector (such as it would be in an ellipse); in other words, it could only produce inequalities in longitude and in parallax, for the parallax varies inversely as the radius vector. But, the Moon's orbit being inclined to the ecliptic, the Sun's disturbing force (represented by a line drawn from the Moon towards the Sun) cannot be entirely resolved into the two former directions: a third resolved part will remain perpendicular to the plane of the Moon's orbit, which will cause the Moon to deviate from that plane; in other words, will cause inequalities in the Moon's latitude.

In order to correct these inequalities in the Moon's latitude, eleven equations are necessary, according to Lalande, (see *Astron.* tom. II. p. 193.) In the New French Tables an additional one is inserted.

The Lunar Tables we now possess, and which present us, under a commodious form, the results of the several preceding *Equations*, and from which in fact the Moon's place is computed in the Nautical Almanack, are of great extent and accuracy. It is almost unnecessary to observe, that they are the fruit of long and laborious research: of some conjectures, many revisions, and new helps from theory. The computers of the Nautical Almanack, have, within the space of forty years, used four different sets of Tables: 1. Mayer's Tables corrected by Mason:

2. Mason's Tables of 1780: 3. Mason's Tables, corrected by Lalande from Laplace's Equations of the Acceleration of the Moon's Motion, &c: 4. Burg's Tables edited by Delambre, and published by Mr. Vince in the third Volume of his *Astronomy*. The computers of the *Connoisance des Tems*, since 1817, have used Burckhardt's Tables.

The Moon's place, at any given time, is found by the addition of a great number of terms technically called *Equations*. An equation consists of its *coefficient* and its *argument*. The latter, although it may be found out by a species of orderly and regulated conjecture, is yet most surely obtained from theory, (see *Physical Astronomy*, Chap. XIV, p. 240.) The numerical value of its coefficient is best determined from observations. Now the Tables being once formed, a question arises concerning the means of examining and correcting them: in the first place then we must find their errors, and, in the second, from those errors find the corrections. As this is a subject of some complication, and as its development will afford an illustration of several of the preceding principles and processes, we will consider it fully in the ensuing Chapter.

CHAP. XXXIV.

On the Methods of finding the Errors and Corrections of the Lunar Tables.

THE Moon's places, that is, its longitudes, latitudes, &c. are computed from the Lunar Tables, and then inserted in the Nautical Almanack. To examine then the accuracy of the longitudes and latitudes so inserted, is, in fact, to examine the truth of the Tables from which they were computed.

The means of examining the truths of the results in the Nautical Almanack, are, amongst other means, the observations made at Greenwich. Those observations are of north polar distances and right ascensions: but the immediate results of computations, made from the Lunar Tables, are lunar latitudes and longitudes: we must then, from the latter, derive the corresponding north polar distances and right ascensions, and compare them with the observed, or, we must institute a comparison between the latitudes and longitudes, computed from the observations, and the latitudes and longitudes computed from the Tables. We shall adopt the latter plan.

In the Greenwich Observations for 1812, p. 190, we find the following results obtained by means of the mural circle:

North Polar Distances.

1812.	Bar.	Therm. In.		N. P. D.
Nov. 18,	29.38	40	☽ L. L.	75° 34′ 9″.7
&c.	&c.			
	29.58	38	Arcturus	69 49 25.6

Transits over the Meridian.

* $3^h\ 57^m\ 0^s.66$	☽ 2 L. $12^h\ 5^m\ 19^s.8$ mean time.
4 25 24.3	Aldebaran.
13 15 31.98	Spica Virginis.
14 7 18.43	Arcturus.

The above observations are, if we may use such an expression, in their rough state. In order to fit them for the computations of the Moon's longitude and latitude, they require several reductions.

(1.) In the first place the north polar distance must be corrected on account of the *index error* (see pp. 112, &c.)

(2.) According to the zenith distance of the lower limb, and the states of the barometer and thermometer, the north polar distance must be corrected for refraction, (see pp. 213, &c.)

(3.) The north polar distance, corrected as above, must be farther corrected, on account of parallax, (see pp. 311, &c.)

(4.) The north polar distance of the Moon's centre must be found by subtracting, from the distance of the lower limb, the Moon's semi-diameter.

(5.) If the computation be made for the time of the transit of the *Moon's second limb*, the above north polar distance, which is a *meridional* north polar distance, must be corrected for its change, during the Moon's passing over a space equal to its semi-diameter.

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* These transits were made with the mural circle: the old transit instrument being thought defective. They are called, in the Observations, *Corrected Transits*, being corrected on account of some small inequalities found to obtain in the intervals of the wires.

The mural circle not being a good transit instrument, it would be hardly fair, if the question were one of great accuracy, to examine the results of Lunar Tables by such an instrument. The observations, however, made with it, are sufficiently accurate for the purpose of illustration.

With regard to the reduction of the *transit* observations;

In the first place the observed transit is to be corrected on account of the error of the clock, (see pp. 104, &c.)

(6.) Secondly, the right ascension of the Moon's centre is to be found by subtracting, from the above right ascension of the second limb, the angle subtended at the pole of the equator by the Moon's semi-diameter.

Moon's North Polar Distance found.

☽'s L. L. N. P. D.	75° 34′ 9″.7
Index Error *	+ 6.6
	75 34 16.3
Co-latitude	38 31 20
	37 2 56.3
Refraction	0 0 43.75
	37 3 40.05
Parallax	0 36 36.3
	36 27 3.75
☽'s semi-diameter	0 16 40
	36 10 23.75
Co-latitude	38 31 20
N. P. D. ☽'s centre	74 41 43.75

* The index error is derived by taking the mean of a great number of differences between the tabulated or computed north polar distances, and the instrumental distances, (see pp. 112, &c.). We will subjoin instances of results afforded by two stars; the process is precisely the same for any other.

Nov. 18,

Bar.	Therm. In.	Star.	Instl. N. P. D.
29.59	39	β Ursæ Minoris	15° 5′ 0″.5
29.58	38	Arcturus	69 49 25.6

Jan. 1,

Refraction.

(See pp. 245, &c.): also Tables of Refraction, in Vol. I. of Greenwich Observations, 1812,

Log. to 37° 3′ 40″	1.63327
Corr. barometer and thermometer.........	10.00774
(Log. 43.75)	1.64101

Jan. 1, 1812, N. P. D. β Ursæ Minoris				15° 4′ 34″.25
Corrections.				
Refraction	(p. 243.)......	+ 25″.419		
Propor[l]. Annual Variation	(p. 407.)......	+ 12.98		
Aberration	(p. 286.)......	+ 4.24	....	33.869
Lunar Nutation	..........(Chap. XIV.)	− 8.33		
Solar Nutation		− 0.44		
				15 5 8.119
	Instrumental N. P. D.			15 5 0.5
	Index Error			+ 7.62

Again,

Jan. 1, 1812, N. P. D. Arcturus			69° 50′ 0″.11
Corrections.			
Refraction	+ 35″.68		
Aberration	+ 0.74		
Lunar Nutation	+ 7.64	....	− 27.78
Solar Nutation	+ 0.46		
Refraction	− 35.68		
Variation	− 16.74		69 49 32.33
Instrumental N. P. D.			69 49 25.6
Index Error			+ 6.73
From β Ursæ Minoris			+ 7.62
Mean....................................			7.17

This is the index error from two observations, one of each star: but the mean index error (6″.6) which has been used (see p. 693, l. 10,) in reducing the observations, was obtained from 149 observations, made, during 44 days, with 21 stars. Of such observations, 7 were made of β Ursæ Minoris, 10 of Arcturus. The mean of the 7 was 7″.16: of the 10, 6″.3.

Parallax

Horizontal equatoreal parallax $= 61' \, 7''.7$ * $= 3667''.7$,

Log. 3667.7	3.5643938
Correction	8841
	3.5635097
Log. sin. $36^\circ \, 52' \, 28''.4$	9.7781972
(Log. 2196.3)	3.3417069

In order to make the correction (5), we must find the time the Moon takes in describing its semi-diameter: now the angle at the pole subtended by the semi-diameter is (see p. 90,)

$$16' \, 40'' \times \text{sec. } 15^\circ \, 18' \, 26'' = 1036''.7 = 17' \, 16''.7,$$

but whilst the meridian, by reason of the Earth's rotation, is describing this angular space ($17' \, 16''.7$) the Moon moves to the eastward. We must find then the Moon's *retardation*. If we assume $13^0 \, 30'$ for the mean angular retardation, we have

$$346^0 \, 30' : 17' \, 16''.7 :: 24^h : 71^s.811 \dagger.$$

Therefore the Moon is $1^m \, 11^s.8$ in describing its semi-diameter; but it appears from the Nautical Almanack of 1812, (p. 126,) that the Moon's change of declination in 12^h was about $1^0 \, 29'$, and consequently in $1^m \, 11^s.8$, about $8''.7$. Deducting, therefore, this quantity from the above meridional north polar distance, we have

$$\text{N. P. D. } ☽\text{'s centre} = 74^\circ \, 41' \, 35''.05.$$

* There are two corrections in deducing the parallax from the horizontal equatoreal parallax: one, on account of the diminution of the radius of the Earth in an oblate spheroid: this in the latitude of Greenwich is effected by subtracting the logarithm .0008841 from the logarithm of the horizontal parallax. The second correction is on account of the angle, which a line drawn from the centre of the Earth to the place of observation, makes with the direction of the plumb-line at the same place. This correction is effected by subtracting $11' \, 11''.6$ from the zenith distance when its sine is to be multiplied into the parallax, in order to deduce the parallax of altitude.

† See a Table for this and like computations in Wollaston's *Fasciculus*, p. 79.

Moon's Right Ascension found.

First, to find the error of the clock, (see pp. 101, &c.) On Nov. 18, 1812, at the time of the Moon's transit.

Computed Right Ascension.		Observed R. A.	Clock too fast.
Aldebaran Æ, 1812, . .	$4^h\ 25^m\ 8^s.576$		
Aberr. prec^n. $4^s.30$ } Nutation -0.65 } . .	0 0 3.65		
See Chaps. XI, XII, &c.	4 25 12.226	$4^h\ 25^m\ 24^s.3$	$12^s.07$
Spica Virginis	$13^h\ 15^m\ 18^s.1$		
Aberr. prec^n. $1^s.71$ } Nutation . . . $-.56$ } . .	0 0 1.15		
	13 15 19.25	13 15 31.98	12.73
Arcturus	$14^h\ 7^m\ 5^s.28$		
Aberr. prec^n. $1^s.12$ } Nutation . . . $-.74$ } . .	0 0 0.38		
	14 7 5.66	14 7 18.43	12.77
Sum of times and errors		31 48 14.71	37.57
Mean time and error		10 36 4.9	12.52

gain of clock * in $10^h = 0^s.7$, nearly.

Hence, at $3^h\ 57^m\ 0^s.66$ the time of the Moon's passage, the clock was $12^s.04$ too fast, and, accordingly,

Æ ☽'s 2 L. . . . . $3^h\ 56^m\ 48^s.63 = 59^\circ\ 12^m\ 9''.45$
(see p. 695,) angle subtended by ☽'s radius 0 17 16.7

Æ ☽'s centre 58 54 52.75

Hence, the elements and process for computing the longitude and latitude of the Moon, at the time of the transit of its second limb over the meridian of Greenwich, are as follow (see pp. 158, &c.)

* There is no rate of the clock given in the Greenwich Observations, the clock having been taken down and adjusted to sidereal time, on the 18th.

Latitude.

☽'s Æ $58^{\circ}\ 54'\ 52''.75$ .... sin. $\frac{1}{2}$ (90 — Æ).... 9.4280638
2

18.8561276

N. P. D.	$74^{\circ}\ 41'\ 35''.05$	sin.	9.9843137
I.	23 27 35.1	sin.	9.5999970
S.	98 9 10.15	2)	18.4404383
$\frac{1}{2}$ *S*.	49 4 35.07		9.2202196
	9 33 28.1	∴	$M = 9^{\circ}\ 33'\ 28''.1$
$\frac{1}{2}S + M$, nearly,	58 38 3	sin.	9.9313873
$\frac{1}{2}S - M$, nearly,	39 31 7	sin.	9.8036816
		2)	19.7350689
	(sin. $47^{\circ}\ 29'\ 10''$)		9.8675345

∴ the distance from the north pole of the ecliptic is $94^{\circ}\ 58'\ 20''$
and the latitude (south) 4 58 20

Longitude.

Δ	$= 94^{\circ}\ 58'\ 20''$	sin.	9.9983626
I	= 23 27 35.1	sin.	9.5999970
δ	= 74 41 35.05		19.5983596
	2) 193 7 30.15		
(nearly) $\frac{1}{2}$ sum	96 33 45.	sin.	9.9971450
$\frac{1}{2}$ sum — δ..	21 52 10	sin.	9.5711180
		(20 added)	39.5682630
			19.5983596
		2)	19.9699034
		(sin. $75^{\circ}\ 0'\ 14''$)	9.9849517

∴ 90° + longitude = $150^{\circ}\ 0'\ 28''$,
and longitude = 60 0 28.

Such are the values of the latitude and longitude of the Moon, computed from immediate observations. In order to compare such values, with the values of the latitude and longitude inserted in the Nautical Almanack, we must reduce the latter, which are computed for Greenwich at the apparent times of its noon and midnight, to the observed time of the transit of the Moon's limb. In the record of the observations (see p. 692,) the mean time of such transit is expressed. As we wish, however, to explain every part of the present investigation, we will now deduce the mean and apparent times of the transit.

On the 18th the Sun's transit was not observed at Greenwich: we will, therefore, compute it after the manner of pp. 527, &c.

Sun's mean longitude, 1812, ...	$9^s\ 9^\circ\ 59'\ 50''.9$
Motion to Nov. 18, ...	10 17 22 42 .2
Mean longitude Nov. 18, ...	19 27 22 33 . 1
In time (rejecting 24^h)	$15^h\ 49^m\ 30^s.2$
Equation of equinoxes..............	— .64
	15 49 29.56
Right ascension Moon's second limb....	3 56 48.63
Apparent time of transit	12 7 19.07
Acceleration	0 1 59.15
Mean time of transit	12 5 19.9

Value of the Moon's Latitude and Longitude, at $12^h\ 5^m\ 19^s.9$ *computed from the Nautical Almanack.* See the *Nautical Almanack* for Nov. 18, 1812, &c.

Moon's Latitude.	First Diff. d'.	Second Diff. d''.	Third Diff. d'''.
18th Noon $4^\circ\ 59'\ 58''$			
	$-\ 1'\ 18''$		
Midnight 4 58 40		$-\ 5'\ 13''$	
	$-\ 6\ 31$		$+\ 16''$
19th Noon 4 52 9		$-\ 4\ 57$	
	$-\ 11\ 28$		$+\ 20$
Midnight 4 40 41		$-\ 4\ 37$	
	$-\ 16\ 5$		
20th Noon 4 24 36			

Now the intervals between these latitudes are 12 hours of apparent time and, therefore, in applying the differential theorem, we must find the value of x in such time. If, therefore, we assume the latitude of the Moon, on the midnight of Nov. 18th as the first term, we have

$$x = \frac{5^m\ 19^s.9 + 14^m\ 27^s.1}{12} = .027476;$$

$\therefore$ since $d' = -\ 6'\ 31''$; $\quad xd' = -\ 10''.74$,

and since $d'' = -\ 4'\ 57''$, $\quad x \cdot \frac{x-1}{2}\, d'' = +\ 3.97$,

$$d''' = +\ 20'', \quad x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3}\, d''' = +\,0.17;$$

$\therefore$ latitude $= 4^\circ\ 58'\ 40'' - 6''.6 \ldots\ldots = 4^\circ\ 58'\ 33''.4$,

nearly, but the latitude computed from immediate observations was, see p. 697, ... 4 58 20

the error of the Tables 0 0 13.4

Longitude.			
Moon's Longitude.	d'.	d''.	d'''.
18th Noon $1^s\ 22^\circ\ 12'\ 25''$			
	$+7^\circ\ 35'\ 25''$		
Midnight 1 29 47 50		$-2'\ 54''$	
	$+7\ \ 32\ \ 31$		$-1'\ 17''$
19th Noon 2 7 20 21		$-4\ \ 11$	
	$+7\ \ 28\ \ 20$		$-1\ \ \ 5$
Midnight 2 14 48 41		$-5\ \ 16$	
	$+7\ \ 23\ \ \ 4$		
20th Noon 2 22 11 45			

Here, the first term being $1^s\ 29^\circ\ 47'\ 50''$

$$\begin{aligned} d' &= 0\ \ 7\ \ 32\ \ 31 = 27151'', \\ d'' &\quad\ -\ \ 4\ \ 11 = -\ 251, \\ d''' &\quad\ -\ \ 1\ \ \ 5 = -\ \ 65, \end{aligned}$$

and $x = .027476$,

therefore we have, by substitution, from the differential theorem

$$\left.\begin{array}{r} ☽\text{'s long., on 18th at } 12^h\ 19^m\ 47^s, = 1^s\ 29^0\ 47'\ 50'' \\ +\ 12\ \ 26 \\ +\ \ 0\ \ \ 3.35 \\ -\ \ 0\ \ \ 0.57 \end{array}\right\} = 2^s\ 0^0\ 0'\ 18''.78$$

but (see p. 697,) the longitude computed from immediate observations was. . . . } . . . . $2^s\ 0^0\ 0'\ 28''$

∴ error of Lunar Tables . . . . $-\ 9''.22$

We subjoin two other instances, in which the zenith distances of the Moon were observed by the brass mural quadrant, and the transits by the old transit instrument, (see pp. 33, 65, of Greenwich Observations.)

1811.	Transits reduced.	Rate of Clock.	Stars, &c.
	$19^h\ 37^m\ 41^s.50$	+ 0.46	γ Aquilæ
Sept. 27,	19 41 58.78	+ 0.48	α Aquilæ
	19 46 26.80		β Aquilæ
	20 11 47.08		☽ 1 L. $7^h\ 48^m\ 37^s.4$ mean time
	21 56 39.96		α Aquarii.
	19 37 42.26	+ 0.76	γ Aquilæ
Sept. 28,	19 41 59.56	+ 0.78	α Aquilæ
	19 46 27.70	+ 0.90	β Aquilæ
	20 35 25.36		α Cygni
	21 12 55.32		☽ 1 L. $8^h\ 45^m\ 38^s.8$ mean time

Sept.	Bar.	Therm. In.	Refraction.		*Zenith Distances.* Ext^r. Division.
27,	29.22	51	2′ 23″.3	☽ 's L. L.	68° 44′ 18″.2
28	29.36	53	2 4.7	☽ 's L. L.	65 53 23.8

Moon's North Polar Distance found.

27th, instrumental zenith dist. ☽ 's L. L. . .	68° 44′ 18″.2
Error of Collim.	−5
	68 44 13.2
Refraction	0 2 23.2
	68 46 36.4
Parallax	0 55 26.2
	67 51 10.2
Moon's semi-diameter	0 16 15.7
	67 34 54.5
Co-latitude	38 31 20
North polar distance of Moon's centre on the meridian	106 6 14.5
Change of north polar distance	+ 6.23
North polar distance of Moon's centre when 1 L. is on the meridian . . .	106 6 20.73

The values of the parallax and change of north polar distance, used in lines 5 and 9, are thus computed:

1st *Parallax.* Equatoreal horizontal parallax 59′ 40″,

Log. 3580.	3.5538830
(See p. 50, Vince, vol. III.)	8841
	3.5529989
Log. sin (68° 46′ 36″.4 − 11′ 11″.6).	9.9689466
(Log. 3326.18).	3.5219455

2nd. *Change of the Moon's North Polar Distance during the time of the describing its Semi-diameter.*

Time of describing Moon's radius (p. 695,) . .		1^m $10^s.5$
Change of decl^n. S. (*Naut. Alm*^{k}.) in 12 hours	−	1° 4′
in 12^m	−	1′ 4″
in 1^m $10^s.5$ . . .	−	6.23
Or, decrease of north polar distance.	−	6.23.

Again on 28th, zenith distance L. L.	65°	53′	23″.8
Collim.			− 5
	65	53	18.8
Refraction	0	2	4.7
	65	55	23.5
Parallax, (see below l. 11,)	0	54	58
	65	0	25.5
Moon's semi-diameter	0	16	27
Zenith dist. of Moon's centre on meridian	64	43	58.5
Change of north polar distance (l. 17,) . .			+ 9.2
	64	44	7.7
Co-latitude	38	31	20.
North polar distance of the Moon.	103	15	27.7

Parallax.

Horizontal equatoreal parallax 60′ 25″ = ·3625″

Log. 3625	3.5593080
	8841
	3.5584239
Log. sin. 65° 44′ 11″.9	9.9598359
(Log. 3298)	3.5182598

Change in North Polar Distance.

Time of describing Moon's radius	1^m $10^s.9$
By Nautical Almanack, change in 12^h	− 1° 34′
in 12^m	− 1′ 34″
In 1^m $10^s.9$	9.2

Moon's Right Ascension found.

First, error of clock found on the 27th,

	R. A. *from Theory and Tables.*			R. A. by Clock (p. 700,)	Clock too fast.
Aquilæ γ	19^h	37^m	$18^s.60$	. . . . . . $41^s.5$	$22^s.9$
Aquilæ α	19	41	35.88	. . . . . . 58.78	22.9
Aquilæ β	19	46	4.02	. . . . . . 26.8	22.78
α Aquarii	21	56	7.2	. . . . . . 29.96	22.76
	4) 81	1	5.7		4) 91.34
therefore at	20	15	16.4	mean error of clock	22.83

Moon's transit by clock, p. 700, ...	20^h	11^m	$47^s.08$
True Æ Moon's 1 L. on the 27th.	20	11	24.25

Next, gain of clock in 24^h from three stars of the Eagle (see p. 700,) = $\frac{1}{3}(.76+.78+.9)=.82$; $\therefore$ in 25^h....	0^h	0^m	$0^s.83$, nearly,
Clock too fast on 27th..............	0	0	22.83
too fast on 28th	0	0	23.66
Moon's transit by clock	21	12	55.32
True Æ Moon's 1 L. on the 28th	21	12	31.66
Hence, expressed in space, on 27th, right ascension Moon's 1 L. ..	302^0	$51'$	$3''.75$
Angle subtended by Moon's radius ($975''.58 \times$ co-sec. $106^0\ 6'$)	0	16	55.4
Right ascension of Moon's centre	303	7	59.15
On 28th, Right ascension Moon's 1 L.	318	7	54.9
Angle of Moon's radius ($987''. \times$ co-sec. $103^0\ 15'$)	0	16	54.03
	318	24	48.9, nearly.

Computation of the Moon's latitude and longitude, (see pp. 159, &c.)

Latitude. Sept. 27th.

Moon's Æ	303°	7′	59″.15		
	90				
2)	213	7	59.15		
	106	33	59.57	.... sin.	9.9815873
					2
					19.9631746
North polar distance	106°	6′	20″.73	.... sin.	9.9826106
I	23	27	42.5	.... sin.	9.6000333
2)	129	34	3.23	2)	19.5458185
$\frac{1}{2}S$	64	47	1.61		9.7729092
	36	21	21.8		$M = 36^0\ 21'\ 21''.8$
$\frac{1}{2}S+M$	101	8	23.4	.... sin.	9.9917392
$\frac{1}{2}S-M$	28	25	39.8	....	9.6776523
				2)	19.6693915
			(43° 6′ 45″.8)......		9.8346957

Hence, complement of the latitude = 86° 13′ 31″.6

and latitude = 3 46 28.4.

Longitude. Sept. 27th.

$\Delta =$	86°	$13'$	$31''.6$	... sin.	9.9990567
$I =$	23	27	42 . 5	... sin.	9.6000333
$\delta =$	106	6	20.73		19.5990900
2)	215	47	34.84		
$\frac{1}{2}$ sum	107	53	47.42	... sin.	9.9784604
$\frac{1}{2}$ sum $-\delta$	1	47	26.69	... sin.	8.4948395
				(20 added)	38.4732999
					19.5990900
				2)	18.8742099
					9.4371049

which is the log. sine of $15^\circ\ 52'\ 41''.4$, and of $375^\circ\ 52'\ 41''.4$.

Hence, taking the last value, (which the value of the Moon's right ascension points out as the right one),

$$90^\circ + \text{longitude} = 0^s\ 751^\circ\ 45'\ 22''.8$$
$$\text{and longitude} = 0\ 661\ 45\ 22.8$$
$$(\text{rejecting } 360^\circ) = 0\ 301\ 45\ 22.8$$
$$= 10\ \ 1\ \ 45\ 22.8.$$

Latitude. Sept. 28th.

Moon's Æ	318°	$24'$	$48''.9$		
	90				
2)	228	24	48 . 9		
	114	12	24 . 4	...... sin.	9.9600290
					2
					19.9200580
North polar distance	103°	$15'$	$27''.7$	.... sin.	9.9882684
I	23	27	42 . 5	.... sin.	9.6000333
2)	126	43	10 . 2	2)	19.5083597
$\frac{1}{2}S$....	63	21	35 . 1	....	9.7541798
	34	35	44	.... ($M = 34^\circ\ 35'\ 44''$)	
$\frac{1}{2}S+M$....	97	57	19 . 1	.... sin.	9.9958003
$\frac{1}{2}S-M$	28	45	51 . 1	.... sin.	9.6823306
				2)	19.6781309
				(sin. $43^\circ\ 39'\ 26''.3$)	9.8390654

Hence, the complement of latitude is . . $87^\circ\ 18'\ 52''.6$
and the latitude, nearly .... $2\ \ 41\ \ 7.3$.

Longitude. Sept. 28th.

Δ	87°	18′	52″.6	..... sin.	9.9995168
I........	23	27	42 . 5	..... sin.	9.6000333
δ........	103	15	27 . 7		19.5995501
2)	214	2	2 . 8		
$\frac{1}{2}$ sum	107	1	1 . 4	....	9.9805574
$\frac{1}{2}$ sum $-\delta$.......	3	45	33 . 7	....	8.8166798
				(20 added)	38.7972372
					19.5995501
				2)	19.1976871
					9.5988435

which is the logarithmic sine of 383° 23′ 38″.6; therefore
longitude + 90° .... = 766 47 17.2
and (reject^g. 12 signs) the long. = 316 47 17 . 2 = 10^s 16°47′17″.2.

Latitudes and Longitudes deduced from the Nautical Almanack.

Since these latitudes and longitudes are expressed in the Nautical Almanack, for apparent noon and midnight, it is necessary to know the time of the passage of the Moon,

Sun's epoch for 1811,	9^s	10°	14′	10″.5				
Mean motion to Sept. 27,	8	25	8	20 . 7				
Mean longitude on 27,	18	5	22	31 . 2	in time	12^h	21^m	30^s.08
Mean motion for 1 day	0	0	59	8 . 333				
Mean longitude on 28,	18	6	21	39 . 5	in time	12	25	26.63

but equation of the equinoxes in right ascension is —.26.

Hence, on 27th sidereal time (see p. 702,)	20^h	11^m	24^s.25
Sun's mean longitude from true equinox	12	21	29.82
Approximate time	7	49	54.43
Acceleration, (see p. 526,)...................	0	1	16.98
Mean time of transit of Moon's first limb	7	48	37.45

On the 28th, Sidereal time, (see p. 703,)	21^h	12^m	$31^s.66$
Sun's mean longitude reckoned from true equinox	12	25	26.37
Approximate time, nearly	8	47	5.3
Acceleration	0	1	26.35
Mean time of transit of Moon's first limb	8	45	38.95

But these are the mean times: the apparent times may be obtained by adding to them the equations of time. Now the equation of time proportional to $7^h\ 48^m\ 37^s$, on Sept. 27th, is $8'\ 54''$ subtractive of apparent time, and Sept. 28th, $9'\ 14''.6$. Hence, the times are

on the 27th, $7^h\ 57^m\ 31^s.45$; $\therefore$ x (see p. 699,) $= .66322$
on the 28th, 8 54 53.55, and x .... $= .74298$.

Moon's Latitudes.	d'.	d''.	d'''.
27th, Noon $4^\circ\ 3'\ 36''$			
	$-\ 27'\ 0''$		
Midnight. . 3 36 36		$-\ 3'\ 46''$	
	$-\ 30\ 46$		$+\ 24''$
28th, Noon 3 5 50		$-\ 3\ 22$	
	$-\ 34\ 8$		$+\ 29$
Midnight. . 2 31 42		$-\ 2\ 53$	
	$-\ 37\ 0$		$+\ 37$
29th, Noon 1 54 41		$-\ 2\ 16$	
	$-\ 39\ 17$		
Midnight. . 1 15 24			

Hence, for the

	Twenty-seventh.	Twenty-eighth.
a	$= 4^\circ\ 3'\ 36''$	$3^\circ\ 5'\ 50''$
d'	$= -\ 27\ \ 0$	$-\ 34\ \ 8$
d''	$= -\ \ 3\ 46$	$-\ \ 2\ 53$
d'''	$= +\ \ 0\ 24$	$+\ 0\ 37$
x	$= .66322$	$.7429$
$\frac{x-1}{2}$	$= -.16839$	$-.128547$
$\frac{x-2}{3}$	$= -.4456$	$-.419031$..

Hence, the latitudes =, respectively,

$$\left.\begin{array}{l} 4^\circ\ 3'\ 36'' \\ -17\ \ 54 \\ +\ 0\ \ 25.24 \\ +\ 0\ \ \ 1.19 \end{array}\right\} = 3^\circ\ 46'\ 8'' \qquad \left.\begin{array}{l} 3^\circ\ \ 5'\ 50'' \\ -25\ \ 21.47 \\ +16\ \ \ 0.52 \\ +\ 1\ \ \ 0.48 \end{array}\right\} = 2^\circ\ 40'\ 46''.5$$

Moon's Longitudes.	d′.	d″.	d‴.
27th, Noon 9ˢ 27° 2′ 32″			
	7° 7′ 14″		
Midnight... 10 4 9 46		5′ 30″	
	7 12 44		− 23″
28th, Noon 10 11 22 30		5 7	
	7 17 51		− 28
Midnight... 10 18 40 21		4 39	
	7 22 30		− 40
29th, Noon 10 26 2 51		3 59	
	7 26 29		
Midnight... 11 3 29 20			

Hence, for the

	Twenty-seventh.	Twenty-eighth.
a =	9ˢ 27° 2′ 32″	10ˢ 11° 22′ 30″
d' =	7 7 14	7 17 51
d'' =	5 30	4 39
d''' =	− 23	− 40

*and Moon's longitudes =

$$\left.\begin{array}{r} 9^s\ 27^\circ\ 2'\ 32'' \\ +\ 4\ 43\ 21.2 \\ -\ 36.855 \\ -\ 1.144 \end{array}\right\} = 10^s\ 1^\circ\ 45'\ 15''.2 \text{ on 27th,}$$

$$\left.\begin{array}{r} 10^s\ 11^\circ\ 22'\ 30'' \\ +\ 5\ 25\ 16.9 \\ -\ 26.64 \\ -\ 1.6 \end{array}\right\} = 10^s\ 16^\circ\ 47'\ 18''.6 \text{ on 28th,}$$

* In order to place the whole of the detail under the eye of the student, we subjoin the arithmetical computation. What is here effected by

If we now exhibit, under one point of view, the results obtained from observations, and those results that are computed from the Nautical Almanack, we shall have

	Transit of Moon's Limb, Mean Time.	Moon's Latitude from Observation.	Moon's Latitude from Tables.	Error of Table.
1811,				
Sept. 27,	$7^h\ 48^m\ 37^s.45$	$3^\circ\ 46'\ 28''.4$	$3^\circ\ 46'\ 8''$	$-20''.4$
28,	8 45 38 .95	2 41 7 .3	2 40 46.5	-20.8
1812,				
Nov. 18,	12 5 19 .9	4 58 20	4 58 33.2	$+13.2$

		Moon's Longitude from Observation.	Moon's Longitude from Tables.	Error of Table.
1811,				
Sept. 27,		$10^s\ 1^\circ\ 45'\ 22''.8$	$10^s\ 1^\circ\ 45'\ 15''.2$	$-7''.6$
28,		10 16 47 17 .2	10 16 47 18.6	$+1.4$
1812,				
Nov. 18,		2 0 0 28	2 0 0 18.8	-9.2*

---

by the *differential theorem*, might have been, and in practice is, effected, but less accurately, by Tables of second differences,

L. x ....9.8216628 ....9.8216628 } 19.04798

L. 7° 7′ 14″...4.4088164   L. $\frac{x-1}{2}$ 9.2263163 }   L. $\frac{x-2}{3}$...9.64894

4.2304792   L. 5′ 30″ 2.5185139   L. 23″......1.36173

1.5664930   0.05865

No. = 4° 43′ 21″.2   No. = −36″.855   No. − 1″.1446

L. x ....9.8709339 .... 9.8709339 } 18.97999

L. 7° 17′ 51″... 4.4194766   L. $\frac{x-1}{2}$ 9.1090629 }   L. $\frac{x-2}{3}$ 9.62225

4.2904105   L. 4′ 39″ 2.4456042   L. 40″ 1.60206

1.4256010   0.20430

No. = 5° 25′ 16″.9   No. = 26″.64   No. = 1″.6

* See Note in opposite page.

Results like those that have been just obtained serve, as we have before observed, a double purpose: they become tests of the accuracy of the Lunar Tables, and means of correcting them. It is obvious how they perform the first office. The mode of performing the second has also been already explained in Chapter XXI. The Moon's place, previously to its insertion in the *Ephemerides of England*, &c. is computed from the Lunar Tables on certain conditions, as they may be called: that is, the mean epoch, the mean motion, the equation of the centre, the longitude of the apogee, and the equations expounding the modifications of the Sun's disturbing force, &c. are all assumed of certain magnitudes: which magnitudes may be erroneous: all, perhaps, in slight degrees, some certainly erroneous: since, otherwise, the Moon's computed place ought to agree with the observed, the observations being supposed to be exact. Although, in correcting the Tables, we may be more assured of the exactness of some of the *elements* than of others, yet it is the safer and the more scientific plan to suppose them all erroneous: and to form equations such as

$$a.dL + b.dm + c.dE + f.dp + \&c. = C,$$

in which dL, dm, &c. shall represent the variations or errors of the longitude, equation of the centre, &c. and C shall be such a quantity as we have just deduced in p. 708, and there represented,

---

* The results do not exactly agree with the results obtained by the computers of the Nautical Almanack, who, by order of the Board of Longitude, and for the purpose of ascertaining the relative accuracy of the several Lunar Tables, have compared the Greenwich Observations, from 1783 to 1819, with the Moon's longitudes and latitudes set down in the Nautical Almanack, and in the *Connoisance des Tems*. The disagreements are found amongst the latitudes: which may arise from the Moon's parallaxes being computed from different Tables, or from Tables constructed on different *oblatenesses* of the Earth. *Some* differences must occur, since in the comparisons, the Moon's places, at the times of the transits of its limbs, were deduced by means of the Tables of second differences, which cannot give results so exact, (we are speaking of arithmetical exactness) as the differential theorem is able to give.

according to the case by $-7''.6$, $+1''.4$, $-9''.2$, &c. In order to deduce the values of the errors of the elements we must form, at least, as many equations as there are supposed errors: but in practice, for reasons already assigned in Chapter XXI, a great number of equations are selected and combined together to form one equation. If the *variations of the elements* are in number 10, 10 sets of equations must be formed, and then the values of the variations or errors, or, under a different name, the *corrections* of the elements of the Tables, must be deduced by the ordinary but laborious process of elimination. By such means the present Lunar Tables have been advanced to their present state of perfection.

We must now pass on to other matters: and those will next claim our attention, which are connected with, and depend on, the lunar theory. Of such sort are eclipses and the methods of computing, at assigned times, the distances of the Moon from the Sun and certain fixed stars. Both subjects are of considerable extent, intricacy, and practical utility, since both, with different degrees however of accuracy, may be made subservient to the determination of the longitudes of places.

By the latter term we mean, in the most general sense, any points on the Earth's surface, whether such are permanent land-stations, or the temporary places of vessels at sea. For the determination of the longitudes of places of the latter description, lunar eclipses are of no use: and indeed, of but small use in fixing the longitudes of land-stations: not, however, from any defect in the lunar theory, but from the practical uncertainty of marking the times when the *phases* of an eclipse commence and terminate. Lunar eclipses might be excluded from a work, the scope of which should be strictly limited to subjects of merely practical utility. A wider range, however, has already been taken in the present Treatise; and, acting on a like plan, we will, in the next Chapter, treat of Lunar Eclipses: which are certainly phenomena of great interest, of celebrity in the History of

Astronomy, and of importance in settling certain of the lunar elements *.

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* The uncertainty of the *time* of an eclipse, to the amount of a minute of time, vitiates the determination of the longitudes of places. But an error of that magnitude would be but of little consequence, when the happenings of eclipses, distant from each other by several centuries, are employed in fixing such an element of the lunar theory, as the Moon's mean motion.

CHAP: XXXV.

On Eclipses of the Moon.

In Chapter IV, a lunar eclipse was shewn to arise from such an interposition of the Earth between the Moon and Sun, as to cause the shadow of the Earth to fall on part, or on the whole, of the Moon's disk.

This prescription of circumstance is necessary: since an opaque body, interposed at a certain distance between the Sun and Moon, does not necessarily cause an eclipse: for instance, if the diameter of the interposed body should be below a certain magnitude, its shadow would not reach the Moon.

The existence, therefore, of eclipses depends on the relative magnitudes of the Sun and Earth, supposing the mutual distances of the Sun, Earth, and Moon, to be assigned.

The Moon being in opposition, and at her mean distance, the apparent diameters of the Sun and Earth, seen from the Moon's centre, are $31' 59''.08$, and $1^{\circ} 55' 8''$. Now, at the extremity, or conical point of the Earth's shadow, the apparent diameters of the Sun and Moon are the same. The Moon, therefore, must be considerably nearer to the Earth than the extremity of the Earth's shadow: or, what amounts to the same, the length of that shadow must be greater than the Moon's distance from the Earth. By computation, it is found to be four times as great.

The eccentricity of the Moon's orbit being very small, equal only to 0.0548553, it would follow, if the above result, relative to the length of the shadow, were established for any distance of the Moon from the Earth, that in all distances the shadow would extend far beyond the Moon. In fact by an easy computation we have the following results:

	Length of Axes of Shadow.
☉ in perigee	212.896 rad. ⊕
at mean distance	216.531
in apogee	220.238.

Hence, the least length of the shadow is more than 212 radii of the Earth, whereas the Moon's distance from the Earth never exceeds 64 radii.

Hence it appears a lunar eclipse must always happen whenever the Earth is *interposed* between the Sun and Moon; understanding, by such expression, the Earth's centre to lie in a line joining the centres of the Sun and Moon. In this latter situation of the three bodies, the Moon is in opposition. In such *kind* of opposition, an eclipse must always happen, and there would be only that kind, if the plane of the Moon's orbit coincided with that of the ecliptic.

The Moon's orbit being inclined to the ecliptic, and, opposition meaning nothing more, than the difference, in longitude, of a semi-circle, or of 180^0, the Moon may be in opposition, and still either directly above or below the right line joining the centres of the Sun and Earth; and, consequently, may either be above or below the conical shadow, the axis of which lies in the direction of the above-mentioned line.

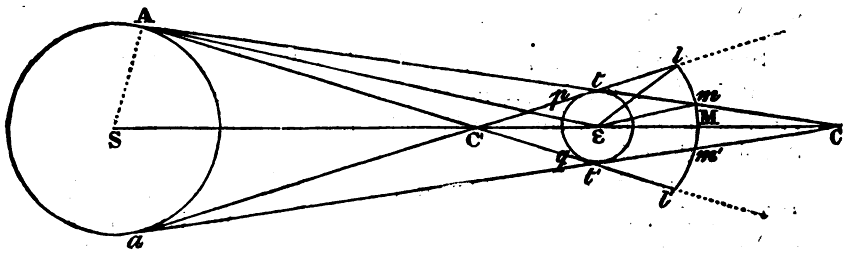
Since the inclination of the Moon's orbit, (see p. 661,) is about $5^\circ\ 9'$, if the Moon in opposition should be either in its greatest northern or southern latitude, that is, either $5^\circ\ 9'$ above or below the ecliptic, no eclipse can take place, since the greatest section of the Earth's shadow at the Moon never exceeds $64'$. But, in the next succeeding opposition, after the lapse of a synodic period, the Moon cannot be again in her greatest latitude, since, the synodic period being greater than the sidereal, the Moon would, on that account, have approached the ecliptic, even supposing the nodes to have been stationary. But the nodes, instead of being stationary, are, during a synodic period, regressive to the amount of $1^{\circ}\ 35'$. For this reason, then, as well as for the one just stated, the Moon approaches the ecliptic. In succeeding oppositions, the Moon, by the operation of both causes, would approach nearer and nearer to the ecliptic, till at length an opposition would occur, in which the Moon would be either, exactly, or very nearly, in its node: and if in its node, then it would be in the ecliptic, and in such case, an eclipse must happen.

An eclipse may happen, if the Moon be *near* to the node of her orbit; the least degrees of proximity are called the *Lunar Ecliptic Limits*.

These limits are easily determined from the inclination of the Moon's orbit, the Moon's apparent diameter, and the apparent diameter of a section of the Earth's shadow at the Moon. The two former conditions may be supposed to be known by previous methods, (see pp. 661, &c.) and it is the latter only that now requires to be investigated.

Apparent Diameter of a Section of the Earth's Shadow at the Moon.

Let S represent the Sun's centre, E the Earth's, and let the circles described round the centres S, E represent sections of those bodies. Draw AtC, $at'C$, tangents to the circular sections



of the Sun and Earth, and the triangular space included within tC, $t'C$, will represent the section of the conical shadow of the Earth. Let mMm' be part of the Moon's orbit, then the section of the Earth's shadow at the Moon is mMm', and its apparent semi-diameter at the Earth, which we have to estimate, is the angle mEM*.

$$\angle mEM = \angle Emt - \angle ECm,$$
$$= \angle Emt - (\angle AES - \angle EAt).$$

Let $\angle\ Emt$, the angle subtended at the Moon by the Earth's radius, or the Moon's horizontal parallax, be denoted by....P,

$\angle\ AES$, the Sun's apparent semi-diameter, by..........$\frac{D}{2}$,

$\angle\ EAt$, the angle subtended by the Earth's radius at the Sun, or the Sun's horizontal parallax, by..................p.

Hence,

The apparent semi-diameter of ⊕'s shadow $= p + P - \frac{D}{2}$.

Hence, the distance of the centres of the Moon and of the Earth's shadow, when the Moon's disk just touches the shadow, will be the preceding expression plus the Moon's apparent semi-diameter $\left(\frac{d}{2}\right)$, that is,

$$p + P - \frac{D}{2} + \frac{d}{2}.$$

If we take $P = 57'\ 1''$, $p = 8''.8$, and $\frac{D}{2} = 16'\ 1''.3$, we shall have

The mean apparent semi-diameter of ⊕'s shadow $= 41'\ 8''.5$, which is nearly three apparent semi-diameters of the Moon.

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* We have, more than once, adverted to the necessary defect which diagrams in Astronomy are subject to, in representing distances and magnitudes according to their true proportion in nature. The Figure in the preceding page is an instance of it. The Earth's radius is there made not less than one-third of the Sun's, whereas it is about $\frac{1}{110}$th part. But, if it had been so drawn, we should have had a most inconvenient diagram, in which it would have been difficult to discern the lines and angles, which are the subjects of investigation.

Hence, since the Moon in the space of an hour moves over a space nearly equal to its diameter, the Moon may be entirely within the shadow, about two hours, or a total eclipse may endure that time.

In order to find the greatest value of the preceding expression, we must take the greatest parallax of the Moon, and the least of the Sun: for, since there is a constant ratio between the Sun's horizontal parallax and his apparent semi-diameter, the latter will be the least when the former is: and although in the expression the parallax is additive, yet its diminution below its mean or even its greatest quantity is trifling, relatively to that of its apparent diameter.

Hence, since the ☽ 's greatest horizontal parallax is $1^{\circ}\ 1'\ 29''$
and the ☉ 's least semi-diameter 15 45.48
the corresponding parallax of the ☉ 0 8.6

We have, nearly,
the greatest semi-diameter of the ⊕'s shadow .... $= 45'\ 52''$,
and the diameter .... $= 1^{\circ}\ 31'\ 44''$.

Precisely after this manner, and by the same formula, namely, $\left(p + P - \frac{D}{2}\right)$ may the apparent diameters of the Earth's shadow be computed, for other distances of the Sun and the Moon. Thus,

		Apparent Diameter of ⊕'s Shadow.
☉ in perigee.	☽ in apogee	1° 15′ 24″.3036
	at mean distance	1 23 2.31
	in perigee	1 30 40.3164
☉ at mean distance.	☽ in apogee	1 15 56.8656
	at mean distance	1 23 34.872
	in perigee	1 31 12.8784
☉ in apogee.	☽ in apogee	1 16 28.2936
	at mean distance	1 24 6.3
	in perigee	1 31 44.3064

In p. 714, there is given an expression for the length of the Earth's shadow, in terms of the Earth's radius obtained from the value $\frac{D}{2} - p$, of the angle Ect; thus

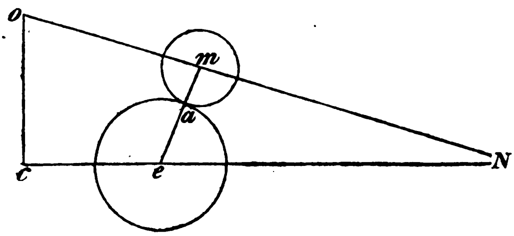
$$Ec = \frac{Et}{\sin. \angle Ect} = \frac{\text{rad.} \oplus}{\sin. \left(\frac{D}{2} - p\right)}.$$

Since there is a constant ratio (see p. 651,) between the Sun's semi-diameter and horizontal parallax, (which ratio is that of the radius of the Sun to the radius of the Earth, and in numbers, as 110 : 1 nearly), the denominator of the preceding fraction may be expressed either, in terms of the semi-diameter, or of the parallax; thus,

$$\text{Length of shadow} = \frac{\text{rad.} \oplus}{\sin. (109\, p)},$$

$$\text{or} = \frac{\text{rad.} \oplus}{\sin. \frac{109\, D}{220}}.$$

But to return to the investigation of the extreme cases in which eclipses can happen. To the greatest apparent semi-diameter of the Earth's shadow (see p. 714,) add the greatest apparent semi-diameter of the Moon, and the result will be the greatest apparent distance of the Moon's centre from the ecliptic, at



which an eclipse can happen. Thus, in the Figure, if Ne be part of the ecliptic, Nm part of the Moon's orbit, e the centre of a section of the Earth's shadow; if we take (see p. 716,) εa in its greatest value, equal to $45' \; 52''$, and ma, the greatest apparent semi-diameter of the Moon, $= 16' \; 45''.5$, then me, $= 62' \; 37''.5$, is the greatest distance of the Moon at which an eclipse can happen. If the distance be greater, there can be no eclipse, if less, and less within certain limits, there may or may not be an eclipse; its happening depending on the relative proximities of the Earth to the Sun and Moon.

The *ecliptic limit* Ne, corresponding to the greatest value of me, may be thus computed:

By Naper's Rules,

$$\text{rad.} \times \sin. me = \sin. Ne \times \sin. \angle eNm;$$

$\therefore$ taking $me = 62' \; 38''$, and the inclination of the Moon's orbit, (what it generally is, in these circumstances,) equal to $5^\circ \; 17'$, we have

$$\begin{array}{ll} 10 + \log. \sin. 62' \; 38'' \ldots\ldots\ldots\ldots & 18.2605076 \\ \quad \log. \sin. \; 5^\circ \; 17' \ldots\ldots\ldots\ldots & 8.9641697 \\ \hline \therefore \log. \sin. Ne \ldots\ldots\ldots\ldots & 9.2963379 \\ \hline \end{array}$$

$$\therefore Ne = \ldots\ldots\ldots\ldots \; 11^\circ \; 25' \; 40'', \text{ nearly.}$$

The species of eclipse represented in the above Figure, where the two circular sections of the Moon and shadow are in contact, is called an *Appulse.*

The opposition of the Moon must have happened soon before this appulse, if the direction of the Moon's motion be supposed from m towards N. For, the Moon moving more quickly* than the Sun, and consequently, than the centre (e) of the shadow, cannot long have quitted a point o, such that the corresponding position of the centre of the shadow would be at c. And in these positions of the Moon and shadow, the former is in opposition.

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* The diurnal motions of the Moon and Sun are respectively $13^\circ \; 10' \; 35''.027$, and $59' \; 8''.33$.

In the computation of eclipses there are several expedients employed for abridging its labour. Eclipses are to be expected when the Moon is near her node, and in opposition. But the labour of a direct and formal computation may frequently be spared, by roughly ascertaining certain limits, beyond which, it is useless to expect an eclipse. Thus, as we have seen in the preceding page, if *Ne* be greater than $11^0\ 26'$, no eclipse can happen. But *Ne* is the difference of the true longitudes of the centre of the ⊕'s shadow and of the ☽ 's ☊ at the time of the appulse; the time of appulse differs a little from the time of true opposition, and therefore, for two causes, from the time of mean opposition. The mean longitude of the centre of the Earth's shadow differs from the true longitude, by reason of the equation of the centre, and other small equations. If therefore, we compute the *mean* longitude of the Earth's shadow at the time of *mean* opposition, it will differ from the longitude of *e*, (see Fig. p. 717,) at the time of appulse for three causes; the difference, of the times of appulse and of true opposition, of the times of mean and true opposition, and of the mean and true longitudes. But, notwithstanding these sources of inequality, the consequent error in the value of *Ne* computed, from the mean longitude of the Earth, and for the time of mean opposition, is within certain limits; and accordingly M. Delambre states that, if *Ne* be $> 12^0\ 36'$, there cannot be an eclipse, if $< 9^0$, there must be one. Between 9^0, and $12^0\ 36'$, the happening of the eclipse is doubtful, and the doubt must be removed by a more exact calculation. The time of mean opposition may be computed from the Tables of the Sun and Moon. But, the computation is facilitated by means of a Table of *Epacts*. The *Epact for a year*, meaning the Moon's age at the beginning of the year, the age commencing from the last *mean* conjunction; and the *Epact for any month*, meaning the Moon's age at the beginning of the month, supposing the age to have begun from the beginning of the year. Delambre in his Astronomical Tables has given a new method of computing the probable times of the happening of eclipses. (See Vince, vol. III. Introduction, p. 56.)

In the preceding explanations we have supposed an eclipse to begin when the Moon enters the Earth's shadow at *m'*. A spectator at the Moon in any point within m' and m, (see Fig. p. 714,) would, by reason of the intervention of the Earth, be unable to see any part of the Sun's disk. But, before and after this *eclipse*, properly so called, the Moon's light would be obscured; or, what amounts to the same thing, the spectator, on the Moon's surface, previously to being entirely deprived of the Sun's light, would lose sight of portions of his disk. In order to determine, when this obscuration first begins, and when it ends, draw two tangents $AC'\,q\,l'$, $a\,C'p\,l$, to the Sun and Moon; then, the moment the Moon enters $l'\,l$, part of the Sun's light is stopped; or, a spectator at the Moon situated any where between $l'm'$ sees part only of the Sun's disk. Entering $m'm$, the spectator loses sight of the Sun entirely; emerging from $m'm$, he regains, in his progress through $m\,l$, the sight of successively greater portions of the disk, and finally, emerging from $m\,l$, he again sees the full orb of the Sun.

The space included within the lines $p\,l$, $q\,l'$, is the section of what is, properly enough, denominated the *Penumbra;* and its angle is $l\,C'\,l'$.

Angle of the Penumbra.

$$\angle AC'S = \angle AES + \angle EAC',$$
$$= \odot\text{'s apparent semi-diameter} + \odot\text{'s hor. parallax},$$
$$= \frac{D}{2} + p.$$

Hence, may be deduced,

The Apparent Semi-diameter of a Section of the Penumbra at the Moon's Orbit.

$$\text{For,}\quad \angle lEC = \angle ElC' + \angle EC'l$$
$$= ☽\text{'s hor. par}^{x}. + \frac{D}{2} + p$$
$$= P + p + \frac{D}{2}.$$

From this formula, as in the case of the umbra (p. 716,) the several values of the apparent semi-diameter of the penumbra, corresponding to certain positions of the Sun and Moon, may be computed.

Since the apparent semi-diameter of the Moon's penumbra is

$$P + p + \frac{D}{2},$$

the distance of the centres of the Moon and shadow, when the Moon first enters the penumbra, is

$$P + p + \frac{D}{2} + \frac{d}{2};$$

d representing the Moon's apparent diameter.

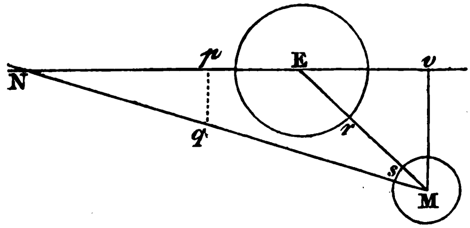
In the preceding investigations we have supposed the cones of the umbra and penumbra to be formed by lines drawn from the Sun and touching the Earth's surface. This, probably, is not the exact case in nature; for, the apparent diameter of the Earth's shadow is found, by observation, to be somewhat greater than what would result from the preceding formula. This circumstance is, with great appearance of probability, accounted for, by supposing those solar rays, that, from their direction, would glance by and rase the Earth's surface, to be stopped and absorbed by the lower strata of the atmosphere. In such a case, the conical boundary of the Earth's shadow would be formed by certain rays exterior to the former and would be larger.

This is not the sole effect of the atmosphere in eclipses; but, another, totally of a different nature, results from it. Certain of the Sun's rays, instead of being stopped and absorbed, are bent from their rectilinear course, by the refracting power of the atmosphere; so as to form a cone of faint light interior to that cone which has been mathematically described as the Earth's shadow. The effect of this, or the phenomenon of which the preceding statement is presumed to be the explanation, is a reddish light visible on the Moon's disk, during an eclipse.

We will now proceed to shew how the time, duration and magnitude, of a lunar eclipse, may be computed.

Let NqM represent part of the Moon's orbit, vEN the ecliptic, N the node.

Suppose the Moon's place of opposition to be q, p being the corresponding place of the centre of the Earth's shadow, and



the latter to describe Ep, whilst the Moon's centre describes Mq. Let also

$m =$ ☽ 's horary motion in longitude,
$n =$ ☽ 's motion in latitude,
$s =$ ☉ 's (or, the shadow's centre's) motion in longitude,
$\lambda =$ ☽ 's latitude when in opposition at q,
$t =$ time from q to M,
$c =$ distance of M from E (ME);

then, in the time t, the ☽ 's motion in longitude $= mt\ (vp)$,
in latitude $= nt\ (Mv - pq)$
the ☉ 's motion in longitude $= st\ (Ep)$;

consequently, $Mv = pq + nt = \lambda + nt$, and $Ev = pv - Ep = mt - st$;

$$\therefore c^2\ (ME^2) = Mv^2 + Ev^2 = (\lambda + nt)^2 + (mt - st)^2,$$

which expression expanded produces a quadratic equation, of which t is the quantity to be determined, and the value of which will depend on that of c; or, if we assign to c such values as belong to the different *phases* of an eclipse, the results will be intervals of time between the happening of such phases, and the time of opposition, which latter time may be computed from the Tables of the Sun and Moon.

If in the preceding expression for t^2, we substitute, after expansion tan. θ instead of $\frac{n}{m - s}$, there will result

$$n^2 t^2 + 2\lambda n \sin.^2 \theta . t = (c^2 - \lambda^2) \sin.^2 \theta,$$

and if from this, by the Rule for the solution of a quadratic equation, we deduce the value of t, we shall have

$$t = \frac{1}{n}[-\lambda \sin.^2 \theta \pm \sin. \theta \sqrt{(c^2 - \lambda^2 \cos.^2 \theta)}],$$

from which expression, as it has been stated, may be deduced values of the time corresponding to any assigned values of c.

For instance, if we wish to determine the time from opposition, at which the Moon first enters the Earth's penumbra, we must assume (see p. 721,)

$$c = P + p + \frac{D}{2} + \frac{d}{2}.$$

t has two values corresponding to the same value of c, the second of which will denote the time at which the Moon quits the penumbra. If we wish to determine the time at which the Moon enters the umbra, we must assume, (see p. 721,)

$$c = P + p + \frac{d}{2} - \frac{D}{2}.$$

If we wish to determine the time when the whole disk has just entered the shadow, we must subduct d from the preceding value, and make

$$c = P + p - \frac{d}{2} - \frac{D}{2},$$

and similarly for other phases.

The two values (t', t'') of t are

$$t' = \frac{1}{n}[-\lambda \sin.^2 \theta + \sin. \theta \sqrt{(c^2 - \lambda^2 \cos.^2 \theta)}],$$

$$t'' = \frac{1}{n}[-\lambda \sin.^2 \theta - \sin. \theta \sqrt{(c^2 - \lambda^2 \cos.^2 \theta)}],$$

which values can never equal each other, except the quantity under the radical sign, that is, $c^2 - \lambda^2 . \cos.^2 \theta = 0$;

in which case the value of t, namely $-\frac{\lambda \sin.^2 \theta}{n}$, represents the *middle* of the eclipse, the distance (c) of the centres being $\lambda \cos. \theta$.

This value ($\lambda \cos. \theta$) of c corresponding to the middle of the eclipse, is the least distance, or, the nearest approach of the centres of the Moon and shadow. For, if by the rules for finding the maxima and minima of quantities, we deduce from the expression, p. 723, l. 3, the value of t, it will be found equal to $-\dfrac{\lambda \sin.^2 \theta}{n}$.

The nearest approach of the centres being known, the magnitude of the eclipse is easily ascertained. Thus, on the supposition that $\lambda \cos. \theta$ is less than the distance $\left(P+p+\frac{d}{2}-\frac{D}{2}\right)$ at which the Moon's limb just touches the shadow, some part of the Moon's disk is eclipsed; and the portion of the diameter of the eclipsed part is

$$P + p + \frac{d}{2} - \frac{D}{2} - \lambda \cos. \theta.$$

The portion of the diameter of the non-eclipsed part, is the Moon's apparent diameter (d) minus the preceding expression, and, therefore, is

$$\lambda \cos. \theta + \frac{d}{2} + \frac{D}{2} - P - p.$$

If this expression should be equal nothing, the eclipse would be *just* a total one. If the expression should be negative, the eclipse may be said to be *more than* a total one, since the upper boundary of the Moon's disk would be below the upper boundary of the section of the shadow: and the distance of the two boundaries would be the preceding expression.

The preceding formulæ for the parts eclipsed, which are parts of the Moon's diameter, are usually expressed in twelfths of that diameter; which twelfths are, with no great propriety of language, called *Digits*. Thus, if the part eclipsed should be $24' \ 52''$, the Moon's diameter being $33' \ 18''$; then, the part eclipsed $= \dfrac{24' \ 52''}{33' \ 18''} \times \overset{\text{Digits.}}{12} = \overset{\text{Digits.}}{8.96}$.

By p. 723, the second root of the quadratic, or

$$t'' = -\frac{1}{n}[\lambda \sin.^2 \theta + \sin. \theta \sqrt{(c^2 - \lambda^2 \cos.^2 \theta)}],$$

which is negative with respect to the other value t'; that is, if the first be previous to opposition, the latter is subsequent to it: hence the whole duration of that part of the eclipse which takes place between equal values of the distance of the centres is the sum of the two times, and therefore =

$$t' + (-t'') = \frac{2}{n} \sin. \theta \sqrt{(c^2 - \lambda^2 \cos.^2 \theta)}.$$

If in this expression we substitute that value of c, which is $P + p + \frac{d}{2} - \frac{D}{2}$, (see p. 723,) the quantity

$$\frac{2}{n} \sin. \theta \sqrt{(c^2 - \lambda^2 \cos.^2 \theta)},$$

denotes the time from the Moon's first entering, to her finally quitting the shadow or *umbra*. And, if we substitute for c, $P + p + \frac{d}{2} + \frac{D}{2}$, (see p. 723,) the resulting expression will denote the whole time of an eclipse, from the Moon's first entering till her finally quitting the *penumbra*.

Example.

Of the Eclipse, which happened on March 17, 1764, *it is required to calculate the beginning, middle, and the end; also the number of Digits eclipsed.*

By the Lunar and Solar Tables it appears that the epoch, or the time of true opposition, happened on the 18th of March 1764, at $0^h\ 6^m\ 12^s$, mean solar time at Paris (reckoned from midnight).

By the above-mentioned Tables the following numerical results were obtained.

☽ 's lat. at the time of opposition λ =	38′	42″	N.
☽ 's horary motion in latitude ... n =	−3	26	(lat. decreasing)
☽ 's horary motion in longitude .. m =	37	23	
☉ 's horary motion in longitude .. s =	2	29	
☽ 's apparent diameter .... d =	33	18	
☽ 's corresponding hor[l]. parallax P =	61	0	
☉ 's apparent diameter .... D =	32	10	
☉ 's corresponding hor[l]. parallax p =	0	9.	

Hence, (see p. 722,)

$$\tan.\ \theta = \frac{n}{m - s} = -\frac{3'\ 26''}{34'\ 54''} = -\frac{206}{2094};$$

$$\therefore \theta = -5^0\ 37'\ 6''.5.$$

Hence, (see p. 724,) the middle of the eclipse, or,

$$-\frac{\lambda \sin.^2 \theta}{n} = \frac{2322}{206} \times \sin.^2 (5^0\ 37'\ 6''.5) = 6^m\ 29^s.$$

This is the time reckoned from the epoch of opposition, which is March 18, $0^h\ 6^m\ 12^s$, consequently, the middle of the eclipse was March 18, $0^h\ 12^m\ 41^s$. Now, in order to find the times when the Moon first entered and when it finally quitted the shadow, we must first compute (see p. 723,) the corresponding values of c, and accordingly we have

$$c = \frac{d}{2} - \frac{D}{2} + p + P = 61'\ 43'',$$

or, adding (see p. 721,) $1'\ 40''$ for the effect of the Earth's atmosphere,

$$c = 63'\ 23'',$$

which value being substituted in

$$-\frac{\lambda \sin.^2 \theta \pm \sin.\ \theta \sqrt{(c^2 - \lambda^2 \cos.^2 \theta)}}{n},$$

the two resulting values (t'', t') of t are

$$\begin{array}{lllll} \text{(end of eclipse)} & t'' = 6^m\ 29^s & + & 1^h\ 26^m\ 8^s = & 1^h\ 32^m\ 37^s \\ \text{(beginning)} & t' = 6\ \ 29 & - & 1\ \ 26\ \ 8 = & -1\ \ 19\ \ 39 \end{array}$$

and consequently, the duration of the eclipse . . . $2^h\ 52^m\ 16^s$.

Since $t' = -1^h\ 19^m\ 39^s$ is negative, the commencement of the eclipse happened before the time of opposition, therefore, at Paris, it happened $1^h\ 19^m\ 39^s$ before March 18, $0^h\ 6^m\ 12^s$, that is, on March 17, $22^h\ 46^m\ 33^s$, and the eclipse terminated $1^h\ 32^m\ 37^s$ after the time of opposition March 18, $0^h\ 6^m\ 12^s$, that is, on March 18, $1^h\ 38^m\ 49^s$.

Since the preceding times are computed, according to the usage of French Astronomers, from midnight, and since, at the time of opposition, the Moon was nearly on the meridian, it is plain that the whole of this eclipse must have been seen at Paris, and could not have been seen on the hemisphere opposite to that, on which Paris is situated.

The distance of the centres corresponding to the middle of the eclipse, and to the greatest *phase*, that is, to the greatest quantity of eclipsed disk, or

$$\lambda \cos.\theta = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 38'\ 31''.$$

The eclipsed part, or

$$\frac{d}{2} - \frac{D}{2} + p + P - \lambda \cos.\theta = ..23'\ 12'',$$

or (see p. 721,), accounting for the effect of atmosphere, 24′ 52″,

$$\text{and expressed in digits} = \overset{\text{Digits.}}{12} \times \frac{24'\ 52''}{33'\ 18''} = \overset{\text{Digits.}}{8.96}.$$

In deducing the equation that involves the time (t) we supposed the Moon to describe the space Mq, whilst the centre of the shadow described Ep: and, expressed by means of the horary motions, the line pv was $= mt$*, and the line, which is the difference of Mv and pq, was $= nt$. According to this notation, therefore, the tangent of the inclination of the Moon's orbit $\left(\text{which} = \frac{Mv}{Nv}\right) = \frac{nt}{mt} = \frac{n}{m}$. Now the Moon approaches the shadow for two reasons, one of which is its motion in latitude, (nt), the other the *excess* ($mt - st$) of its motion in longitude above that of the shadow. Hence, its approach to the shadow would evidently be the same, if we suppose the centre of the shadow to be *quiescent*, the Moon to move with its proper motion in latitude (nt), and besides with an imaginary proper motion, in longitude, equal to the relative one, $mt - st$; with such an hypothesis the equation (see p. 722,)

$$c^2 = (\lambda + nt)^2 + (m - s)^2\, t^2,$$

would equally result, and the same conclusions relative to t, &c.

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* The Reader must observe that mt, nt, &c. are not lines like pq, &c. but the products of two algebraical symbols, m, t and n, t.

would also equally result. In this case, since we suppose the shadow to be at rest and the two motions of the Moon to be nt, and $(m - s)\,t$, the Moon must move towards the shadow along an imaginary orbit, the tangent of whose inclination would be $\frac{nt}{(m - s)\,t}$, or $\frac{n}{m - s}$, an inclination greater therefore than that of the real orbit.

This imaginary orbit, (which originates by a species of translation of the equation involving t,) has, for the purpose of graphically representing the phases of an eclipse, been invented by Astronomers, and been termed the Moon's *relative* Orbit. If we prolong the line pq below q, by a quantity equal to $n \times t$, so that the whole line, beginning from p, may be equal to $\lambda + nt$ ($\lambda = pq$) and then, from the extremity of the prolonged line, draw a line parallel to pv, towards M, and equal to $(m - s)\,t$, and lastly, join p and the extremity of the line parallel to pv; the joining line will represent a portion of the relative orbit, and be equal to ME (c).

The *relative* orbit is a mere mathematical fiction, convenient enough for representing the phases of an eclipse, but not essential to their computation, as the very fact of the preceding computations, made without reference to it, sufficiently proves. If, however, by independent reasonings, it be established and laid down as the basis of investigation, then may all the preceding results relative to the duration and quantity of an eclipse be obtained. It may not be improper to note, that the artifice of computation which substitutes tan. θ instead of $\frac{n}{m - s}$, when geometrically exhibited, introduces the *relative* orbit.

In the preceding computations of the duration, &c. of a lunar eclipse, we have supposed the motion of the Sun in longitude, and the motions of the Moon in longitude and latitude to be uniform. This, during the short continuance of an eclipse, is nearly, but not exactly, true. The error of the supposition, however, may be corrected by means of the Lunar and Solar Tables, which give the true motions of the Sun and Moon for every instant of time, and then the eclipse may be computed to the greatest exactness.

Since the computation of eclipses, (especially, of solar,) is attended with considerable difficulties, it is natural to search for expedients that may lessen them. Now, an eclipse depends on two circumstances, the syzygy of the Moon, and the proximity to the node of its orbit. The first circumstance, whether it be an opposition or a conjunction, recurs after a synodic period, or, 29 days. But, at the end of this period, the proximity of the Moon to the node of its orbit cannot be the same, in degree, as it was at the beginning. It must, according as the Moon is approaching or receding from the node, be less or greater. This arises from the *regression* of the nodes. But, the nodes still regressing, before they have performed a circuit of the heavens, an opposition or conjunction must happen, in which the Moon would be either exactly, or very nearly, at the same distance from the node, as it was at the beginning of the period. If, for the sake of illustration, we suppose the synodic period to be 30 days, and the Sun after quitting the node of the Moon's orbit, to return to the same after 330 days, then at the end of this latter period, and after eleven lunations, if the Sun and Moon should have been in conjunction, or opposition, at the beginning, they would be again so, and besides the Moon would be in the same degree of proximity to the node. If, however, the return of the Sun to the node should not be performed exactly in 330 days, but in 330 days 12 hours, then at the end of 661 days, after two revolutions with respect to the node and 60 lunations, the Moon would be in syzygy with the Sun, and at the same distance from the node, as it was at the beginning. Now, if the Moon, at different periods, be in syzygy with the Sun, and at the same distance from the node, the same phases of an eclipse must be always seen at those periods (supposing the mutual distances of the Moon, Sun, and Earth, not to alter). Hence, an eclipse computed for one period would serve for other periods, and, eclipses could be predicted; since, after the lapse of a certain number of days, they would recur.

A lunation, and the Sun's period with regard to the node of the Moon's orbit, are not of the values, which, in the preceding illustration, we have supposed them to be. The former is $29^d\ 12^h\ 44^m\ 2^s.8$, (29.530588) the latter $346^d\ 14^h\ 52^m\ 16^s.032$ (346.61963). But, with these true values, the period of the recurrence of the Moon to the same position, relatively to the Sun and the node of its orbit, is to be determined on the same principles, which, indeed, are those which have been previously used on the occasion of the transits of *Venus* and *Mercury* over the Sun's disk, (see p. 613.). We must find two numbers in the proportion of 29.530588 to 346.61963: if not exactly, nearly so, employing the method of continued fractions. Now two numbers, nearly so, are 19 and 223; the Moon's node, therefore, after 223 lunations has, relatively to the Sun, returned 19 times to the same position. And accordingly at the end of 223 lunations, that is, of 18 years 11 days *, there are the same conditions requisite for an eclipse, as at the beginning; after such interval, then eclipses, solar as well as lunar, will recur, and in the same order. If we know, therefore, previous, we can predict subsequent, eclipses.

This simple method of predicting eclipses was known to the antient Astronomers. It, however, is not exact, since 19 to 223, is only an approximate ratio: even were it exact, still the lunar inequalities, the periodical and secular, would prevent the Moon from being at the end of $18^y\ 11^d$, or of $36^y\ 22^d$, &c. precisely at the same distance from the node, as at the beginning.

The method, however, may, with advantage, be used for ascertaining, very nearly, the happening of eclipses; after which, the exact times may be calculated by means of the Astronomical Tables.

By means of the period of 223 lunations, called by the Chaldean Astronomers, the *Saros*, eclipses may be predicted; but, independently of this, there is, for finding directly those syzygies at which eclipses may happen, the method of *Astronomical Epacts*, (see p. 719).

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* More exactly, $18^y\ 10^d\ 7^h\ 43^m$, or $18^y\ 11^d\ 7^h\ 43^m$, accordingly as four or five leap years happen in the interval of 223 lunations.

CHAP. XXXVI.

On Solar Eclipses.

An eclipse of the Sun, is caused by the interposition of the Moon between the Sun and Earth; in consequence thereof, the whole, or part of the Sun's light is prevented from falling on certain parts of the Earth's surface.

A spectator, deprived of the whole of the Sun's light, is involved in the Moon's shadow; deprived of part, in the penumbra.

A material circumstance of distinction exists between lunar and solar eclipses: the former are seen, at the same time, by every spectator who sees the Moon above his horizon. The latter may be seen by different spectators at different times; or may be seen by one spectator and not by another. The passage of the Moon's shadow across the Earth's surface, during a solar eclipse, has been properly likened to that of the shadow of a cloud.

In the case of the Moon, it was shewn, that, if that body were within certain limits of distance from the node of her orbit, an eclipse must happen in opposition; because, (see p. 712,) the shadow of the Earth, in all distances of the Moon and Sun, extends far beyond the lunar orbit. The length of the Moon's shadow must be determined as that of the Earth's has been, on the same principles and by similar formulæ. But, the result, in certain respects, will be different. The Moon's shadow will never extend far beyond the Earth, and sometimes will fall short of it. Hence, the happening of a solar eclipse will depend not solely on the ecliptic limits, but also on the relative distances of the Sun, Moon, and Earth.

In order to determine the length of the Moon's shadow, we may use the Figure of page 714.

$$\text{Now, by p. 717, } CE = \frac{Et}{\sin.\ \angle\ ECt},$$

$$= \frac{Et}{\sin.\ (\angle\ AES - \angle\ EAt)}.$$

In this case E must represent the Moon, and accordingly $\angle\ AES$, which is the apparent semi-diameter of the Sun seen from *the Moon*, is equal to

$$\text{apparent semi-diameter} \odot \text{ seen from } \oplus \times \frac{\text{dist. } \odot \text{ from } \oplus}{\text{dist. } \odot \text{ from } ☽},$$

and the angle EAt is the Sun's horizontal parallax belonging to the *Moon*, and equal, therefore, to

$$\odot\text{'s horizontal parallax for } \oplus \times \frac{☽\text{'s rad.}}{\oplus\text{'s rad.}} \times \frac{\text{dist. } \odot \text{ from } \oplus}{\text{dist. } \odot \text{ from } ☽}.$$

Hence, calling the radii of the Moon and Earth, r, R, and the distances of the Sun from the Moon, and Earth, k, K respectively, there results

$$\text{length of Moon's shadow} = \frac{r}{\sin.\left(\frac{D}{2} \times \frac{K}{k} - p\,\frac{rK}{Rk}\right)}$$

$$= \frac{r}{\sin.\left\{\left(\frac{D}{2} - p\,\frac{r}{R}\right)\frac{K}{k}\right\}}$$

$$= \frac{r}{\sin.\left\{\left(\frac{D}{2} - p\,\frac{r}{R}\right)\frac{P}{P-p}\right\}}$$

For, since $p = \frac{R}{K}$, and $P = \frac{R}{K-k}$, $\frac{K}{k} = \frac{P}{P-p}$.

By means of this formula, we have

	Length of Shadow.	☽'s Dist.
$\odot$ in apogee, ☽ in perigee	59.730	55.902
$\odot$ in perigee, ☽ in apogee	57.760	63.862

And this latter case is one of those mentioned in p. 731, and in which the Moon's shadow never reaches the Earth.

The formula for the length of the Earth's shadow has been adapted so as to express the length of the Moon's shadow. Similar alterations may be applied to the other formulæ. For instance, (see p. 715,)

the appa^t. semi-diam. of ⊕'s shadow $= \angle Emt - (\angle AES - \angle EAt)$.

Now we have already shewn (p. 732,) that

$$\angle AES - \angle EAt = \left(\frac{D}{2} - p\frac{r}{R}\right)\frac{P}{P-p},$$

and $\angle Emt$, (the Moon being at E, and the Earth at M,) equals the ☽'s apparent semi-diameter $\left(\frac{d}{2}\right)$.

Hence,

the appa^t. semi-diam^r. of ☽'s shadow $= \frac{d}{2} - \left(\frac{D}{2} - p\frac{r}{R}\right)\frac{P}{P-p}$

$$\left(\text{since see p. 651, } \frac{d}{2P} = \frac{r}{R}\right) = \frac{d-D}{2} \times \frac{P}{P-p}.$$

Hence, when the Moon's apparent diameter (d) equals the Sun's (D), the apparent semi-diameter of the Moon's shadow is equal nothing; or, the vertex of the conical shadow just reaches the Earth.

When the Moon's apparent diameter (d) is less than the Sun's (D), the expression for the apparent diameter of a section of the Moon's shadow is negative; in other words, the shadow never reaches the Earth.

In a similar manner may the formulæ for the penumbra of the Earth be transformed, and adapted to the case of the Moon.

In order to find the distance of the centres of the Moon's shadow and of the Earth, when the Earth's disk just touches the section of the Moon's shadow, we must add to the expression, l. 13, the apparent semi-diameter of the Earth, seen from the Moon, which, in other words, is the Moon's horizontal parallax (P). Hence

$$\text{distance} = P + \frac{D - d}{2} \times \frac{P}{P - p}.$$

From this expression the solar ecliptic limits may be computed, precisely as the lunar were (see p. 718,) and they will be found equal to $17^{\circ}\ 21'\ 27''$.

The same diagram and formulæ, as we have seen, apply equally to solar as to lunar eclipses; and, to a spectator placed in the Moon, our solar eclipses must appear, precisely, as lunar eclipses appear to us; the fictitious spectator might also compute the duration, and magnitude, of an eclipse caused by the shadow of the globe on which he is placed, by processes like those which have already been used, (p. 722,) in the case of lunar eclipses. The forms of the resulting equations, and the steps of the process, would be the same in each case. It would be only necessary to make such slight alterations as we have already made. And, under this point of view, there is no difference between lunar and solar eclipses. The computation of the one is as easy as that of the other. But, still the fact is, the subject of solar is much more difficult than that of lunar eclipses. There is then some material circumstance of difference between them, which it is now necessary to point out.

In the preceding computations relative to lunar eclipses, no consideration was had of any particular parts of the Moon's disk which might either be covered by, or approach within a certain distance of, the Earth's shadow. In the ingress, for instance, merely the time of contact was determined, and nothing said concerning the position of the point of contact relatively to any fixed point in the Moon's equator. The lunar latitude and longitude of the point of contact is a matter of indifference to the observer on the Earth's surface. But, to an observer at the Moon, the case is quite different: to such an one, the eclipse does not begin when the Earth's shadow comes in contact with the Moon's disk, but when it begins to obscure his station. Now, in the predicament of this fictitious observer at the Moon, during what to us is a lunar eclipse, is an observer at the Earth during a solar eclipse. It is necessary for him to know when, and how long, the shadow of the Moon will obscure a station of an assigned longitude and latitude.

Solar eclipses then are more difficult of computation because more is required to be done in them, than in lunar eclipses. If in the investigation of the latter, there had been solved a problem, in which it was required to determine the time when a particular point on the Moon's surface was eclipsed, then from such solution we should possess the means of determining, what it is essential to determine, in solar eclipses.

The method, however, of computing lunar eclipses (given in pp. 722, &c.) may be adapted to solar; and in such a manner as to determine the times of the happening of the latter at an assigned place. This we will endeavour to explain.

First, that method may (making such substitutions as have already been made in pp. 722, &c.) be employed in computing the time and duration of a solar eclipse with reference to the *whole disk* of the Earth; that is, the eclipse being supposed to begin at the first contact between the Moon's shadow and any part of the Earth, and to end at the last contact.

At any time (t) included within the duration (T) of such an eclipse, we are able to compute the apparent distance of the centres of the Sun and Moon, supposing the spectator to be placed in the centre of the Earth. The problem is precisely the same as the one in p. 722, relative to a lunar eclipse. Corresponding to the time t, the Solar and Lunar Tables, will furnish us with the longitude of the Sun, the longitude and latitude of the Moon, &c.; such quantities in fact, as λ, m, ρ, &c.; and, involving these quantities precisely as they were in pp. 722, &c., an equation exactly similar to the one of p. 722, would result: and from its solution, since t is supposed to be given, c would result; but if c be assigned, then is t the resulting quantity.

If, instead of a spectator in the Earth's centre, we suppose one on the surface, in what respects and degree ought the conditions of the preceding problem to be changed? The latitudes and longitudes (l, λ), computed for the former spectator, cannot belong to the latter, because angular distances (and such are latitudes and longitudes) seen from the centre are not the same as when seen from the surface. They differ however solely by *parallax*. If therefore the true longitudes and latitudes at any time be diminished by parallax, the resulting longitudes and latitudes (l', λ') will belong to a spectator on the Earth's surface, for the same time. These latter being substituted as in page 722, the equation

$$n^2 t^2 + 2\lambda' n \sin.^2 \theta \times t = (c^2 - \lambda'^2) \sin.^2 \theta,$$

will express the relation between t and c.

In finding therefore the time, at which, the apparent distance of the centres of the Sun and Moon should be of an assigned magnitude, or in finding the magnitude for an assigned time, the chief thing required to be done, is to diminish the angular distances, which the Astronomical Tables furnish us with, by the effects of parallax in the directions of those angular distances.

The angular distances, as we have seen (p. 735,) are measured along the circles of latitude and longitude. What we require then, are formulæ for computing the *parallaxes in longitude and latitude*. The investigation of such formulæ is the chief object of the ensuing Chapter.

That Chapter is on the *Occultation of fixed Stars by the Moon*. A subject which, equally with solar eclipses, requires the aid of formulæ for computing the parallax in longitude and latitude. The investigation of those formulæ might have been introduced into the present Chapter, but it was judged right to defer it to the next, because its subject may *mathematically* be viewed in the light of the simplest case of a solar eclipse. For, if from this last we make abstraction of all the ordinary phenomena, the two cases are similar. In the one, we have to find the apparent distance of the centres of the Sun and Moon; in the other, the apparent distance of the centre of the Moon and a fixed star. In each we must take the latitudes and longitudes from the Tables, and then correct such for parallax; but the latter case is somewhat the more simple, because it is necessary to compute the parallax in latitude and longitude for one body only, namely, the Moon; the other, the fixed star, having no parallax.

There is a third phenomenon, *The Transit of an inferior Planet over the Sun's Disk*, which is nearly similar to an occultation and a solar eclipse in its general circumstances, and is exactly so in its mathematical conditions. In the two latter phenomena, the Moon by its interposition obscures the light of the Sun, or suddenly extinguishes that of the star: in the former, the planet successively darkens parts of the Sun's disk; this effect then, like an occultation, is a species of eclipse. But, without any forced analogies or violation of the proprieties of language, it is a sufficient reason for classing these phenomena together, that it is *mathematically convenient* so to do. To each, the same equations and formulæ apply; and, as we shall hereafter perceive, they may all be employed in attaining the same object, the determination of the longitudes of places.

The next Chapter will put us in possession of the means of computing the apparent distance of the centres of the Sun and Moon. If that distance be the sum of the semi-diameters of those bodies, their disks will be just in contact, and the corresponding time will be that of the beginning or the end of an eclipse. Such, considering the practical use of solar eclipses in determining the longitudes of places, is the essential problem; and to that we shall restrict ourselves: still, it must not be forgotten, it is only one out of many that may be proposed on the same subject.

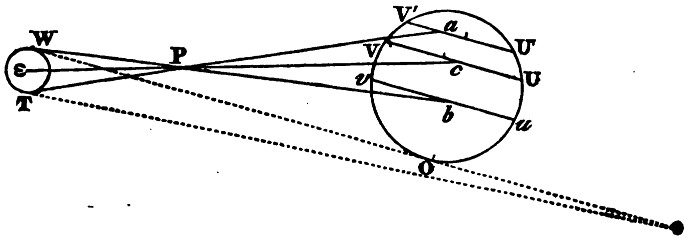
The times of the beginnings of solar eclipses can be exactly noted: which is the circumstance which gives them utility and distinguishes them from lunar. In order therefore that the observer may be prepared to note the times of the phases of an eclipse, he ought to know them approximately at least, by previous computation. This he may do by computing, for the several times included within the whole duration of the eclipse, the apparent distances of the centres of the Sun and Moon: and, then, from such results he may determine nearly (which is all he wants) the time when the distance shall be equal the sum of the semi-diameters of those bodies.

CHAP. XXXVII.

On the Occultation of fixed Stars by the Moon.

PARALLAX enters as a condition into almost all Astronomical calculations; because we agree to reckon, from the centre of the Earth, observations which we must make on its surface. The parallax in its greatest value (the horizontal,) being the greatest angle under which the Earth's radius can be seen from an heavenly body, is less, the more distant the body. Fixed stars are so distant that they have no parallax, or, at the most, a very small one. Were the Moon equally distant, her centre, or any point of her disk, would be seen at the same angular distance from a fixed star, whether the Earth's centre or its surface were the spectator's place. If her disk therefore were in contact with a fixed star, the contact would be seen, at the same instant of time, by an imaginary spectator in the Earth's centre, and by all spectators (to whom the Moon should be visible) on its surface. The same instant of time, however, would be differently reckoned by different spectators, according to the situation of their meridians. If 3^h were the time of observation at Greenwich, the time might be 7^h at a place to its east, or might be noon at a place to its west. And, in this case, the mere differences of the *reckoned* times of the happening of the phenomenon would be the angular distances of the several meridians, or the differences of *the longitudes* of the stations of the several observers.

The Moon, by reason of its great relative proximity, is more affected by parallax than any other heavenly body. Suppose in the Figure (which is intended subsequently to illustrate the transit of *Venus*) $V'VOU$, &c. to be the Moon's disk, $W\epsilon T$ the Earth *, then a spectator at W would see a star ✻ in apparent contact with the point O in the Moon's disk, and (if the Moon's centre be supposed moving towards WO) in the instant of time immediately



previous to an occultation. A spectator at T would see the star ✻ separated from the Moon's disk; a spectator in ϵ, the Earth's centre would also see it separated but by a less angle. To these latter spectators the instant of contact, immediately preceding an occultation, would not have arrived. Hence, it is plain, that the *absolute time* of an occultation would be different to different observers; and, accordingly, the mere difference of the *reckoned* times of the happening of the phenomenon, would not, in all cases, give the difference of the longitudes of the places of observation. Account must also be made of that difference in the absolute time, which would be nothing, were it not for the effects of parallax.

The effects of parallax in longitude and latitude are usually computed by a process of considerable length, involving several subordinate ones. These latter, being distinct steps in the investigation, may be proposed as independent problems. And, on such occasions, authors have been accustomed so to treat a complicated process. They resolve it into its parts, and propose such for solution under the form of problems, and towards the beginnings of their treatises. The object in view, in this arrangement, is the accommodation of the student, who, it is intended, should thus separately subdue the parts of a formidable calcula-

* P and the lines VU, $V'U'$, &c. are of no use in the present illustration.

tion. But, in this case, he must be content to learn the solutions of problems, without discerning the objects of their application. He must take them on trust, and consider that, although not of independent and immediate, they may be of subsidiary and future, use.

In the present instance it is intended to resolve the process for computing the parallax in longitude and latitude into its several parts; previously to propose such parts as problems for solution; and then to proceed immediately to their use and application. On this plan, therefore, we are required to find

The right ascension of the mid-heaven, *or of the Medium Cœli.*

The altitude of the *Nonagesimal.*

The longitude of the Nonagesimal.

1st. *The Right Ascension of the Mid-Heaven.*

The right ascension of the *mid-heaven* has been already explained (see p. 527.). It is, at any assigned time, the right ascension of a point of the equator on the meridian at that time, or, should a star be then on the meridian, it is the right ascension of such star. In like manner should the Sun, either the true, or the imaginary mean, Sun, then the true right ascension of the former, or the mean longitude of the latter, would be the right ascension of the mid-heaven. Suppose, the star, or the Sun, to have passed the meridian and to be to the west of it, then the right ascension of the *Mid-heaven* must be the right ascension of the star, or of the Sun, plus the angular distance of the star or Sun from the meridian, that is, plus the *hour* or *horary angle* (see p. 10,) of the star or Sun. If the true Sun be used in the computation, the right ascension of the mid-heaven will be the

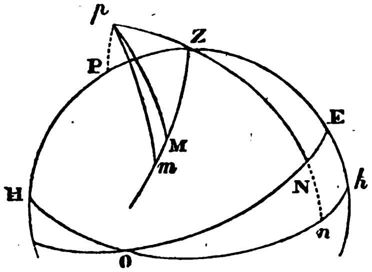
$\odot$'s true right ascension + true time from meridian . . . . (A).

If the mean Sun, then the right ascension required is

$\odot$'s mean longitude + mean time.

The Altitude of the Nonagesimal.

The *Nonagesimal* is that point of the ecliptic, which, at any assigned time, is the highest above the horizon. If Hh be the horizon, ONE a portion of the ecliptic, and if ON be taken $= 90^0$, the point N is the nonagesimal, and its height is Nn;



Nn being the continuation of a vertical circle passing through N and the zenith Z.

Nn the height of the nonagesimal is (see *Trig*. p. 129,) the measure of the spherical angle EOH, the inclination of the ecliptic to the horizon.

$$pN\,(= \text{a quadrant}) = pZ + ZN,$$
$$\text{also } Zn\,(= \text{a quadrant}) = Nn + ZN;$$
$$\therefore pZ = Nn,$$

or, pZ is equal to the height of the nonagesimal and measures the inclination of the ecliptic to the horizon.

In order to find pZ, take P the pole of the equator, then, in the triangle PpZ, we have

PZ the co-latitude of the place,

Pp the obliquity of the ecliptic,

$\angle\, pPZ = 270^0 -$ right ascension of the Mid-heaven.

Since the right ascension of E is the same as the right ascension of the Mid-heaven.

This then is that case of oblique spherical triangles, in which, from two sides and an included angle, it is required to find the third side; a problem of the same kind as that of the latitude of a star to be determined from its right ascension and north polar distance (see p. 159,) and which we shall similarly solve by the aid of a subsidiary angle (θ), (see *Trig*. p. 170).

Assume then θ such, that

$$\tan.^2 \theta = \frac{\text{sin. obl}^y. \times \text{cos. lat.} \times \text{ver. sin.} (90^\circ + Æ \text{ of mid-heaven})}{\text{ver. sin. (co-latitude} - \text{obliquity)}},$$

then, ver. sin. pZ = ver. sin. (co-lat. − obliquity) × sec.$^2 \theta$*

$$\text{or, sin.} \frac{pZ}{2} = \text{sin.} \frac{1}{2} (\text{co-lat.} - \text{obliquity}) \times \text{sec.}\ \theta,$$

and in logarithms,

$$\text{log. sin.} \frac{pZ}{2} = 10 + \text{log. sin.} \frac{1}{2} (\text{co-lat.} - \text{obliquity}) + \text{log. sec.}\ \theta.$$

The complement of the altitude (pZ) of the nonagesimal is ZN, and is sometimes called the *Latitude of the Zenith.*

Longitude of the Nonagesimal.

p, P being the poles of the ecliptic and the equator, the arc pP, if continued, must pass through the solstitial point; therefore, the longitude of P is 90°; and the longitude of N (the longitude of the nonagesimal) is

the longitude of P plus the angle $PpN (= PpZ)$.

Now,

sin. PpZ = cosec. height of nonagesimal × sin. pPZ × cos. lat.

or, (see *Trig.* p. 159,)

$$\text{cos.}^2 \tfrac{1}{2} pPZ . \text{sin.}\ pZ . \text{sin.}\ pP$$
$$= \text{sin.} \tfrac{1}{2} (pP + pZ + PZ) . \text{sin.} \tfrac{1}{2} (pP + pZ - PZ),$$

from either of which expressions PpZ may be computed.

From the right ascension of the mid-heaven have been found the height and longitude of the nonagesimal; from these latter we may proceed to, what indeed are the chief objects of search, the parallaxes in longitude and latitude.

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* Examples to these several methods will be given under that belonging to the general problem of 'the distance of two bodies.'

Parallax in Longitude.

Let M be the true place of an heavenly body, m its apparent place depressed, in a vertical circle ZMm, by the effect of parallax, (see Chap. XII,) then the parallax in longitude is the angle Mpm, the measure of which, since Mm is small, is very nearly the fluxion, or the differential of the angle ZpM: and such we shall assume it to be. Now, let

L, l, be the latitudes of M, m, $(= 90^\circ - pM,\ 90^\circ - pm,)$
K, k the angles ZpM, Zpm,
h, (pZ) the height of the Nonagesimal,
p, the common parallax, P $(= p \,.\, \text{sec. alt.})$ the horizontal,
a, the parallax in longitude; δ the parallax in latitude,
Z, z, the zenith distances ZM, Zm.

Then, by *Trigonometry*, p. 157, we have

$$\text{cot.}\, z \,.\, \text{sin.}\, h = \text{cot.}\, k \,.\, \text{sin.}\, \angle pZm + \text{cos.}\, h \,.\, \text{cos.}\, \angle pZm.$$

Of this equation take the *differential* or fluxion, and, since $\angle pZm$ is constant, and dk or $\dot{k} = a$, and dz, or $\dot{z} = p$, there results

$$p \,.\, \frac{\text{sin.}\, h}{\text{sin.}^2\, z} = a \,.\, \frac{\text{sin.}\ \angle pZm}{\text{sin.}^2\, k}.$$

But,

$$\text{sin.}\, pZm = \text{sin.}\, k \times \frac{\text{sin.}\ pm}{\text{sin.}\ Zm} = \text{sin.}\, k \times \frac{\text{cos.}\ l}{\text{sin.}\ z};$$

$$\therefore a, \text{ the parallax in longitude}, = \frac{p}{\text{sin.}\ z} \times \frac{\text{sin.}\, h \,.\, \text{sin.}\, k}{\text{cos.}\ l},$$

$$= P\, \frac{\text{sin.}\, h \,.\, \text{sin.}\, k}{\text{cos.}\ l} \text{ (very nearly).}$$

In this expression $k = K + dk = K + a$; $\therefore a$, the quantity sought, is contained in the formula that is meant to express its value. This is a frequent case in which there is an appearance of arguing in a circle. In order to evade such arguing we must approximate to the value of a, by supposing, in the first case, k to equal K: thus, first find a value (e) of a from this expression

$$a\ (e) = P\, \frac{\text{sin.}\, h \,.\, \text{sin.}\ K}{\text{cos.}\ L},$$

then investigate a nearer value of a, from

$$a = P \cdot \frac{\sin. h \cdot \sin. (K + e)}{\cos. L},$$

and, if this last value be not sufficiently accurate, the above process must be repeated.

Parallax in Latitude.

By a formula similar to that which we have just used, and which differs from it only, in the circumstance of the angle k being used for $p Z m$, l for z, &c., we have

in $\triangle\ Zpm$, $\tan. l \sin. h = \cot. pZm \cdot \sin. k + \cos. h \cdot \cos. k$,

in $\triangle\ ZpM$, $\tan. L \sin. h = \cot. pZm \sin. K + \cos. h \cdot \cos. K$,

eliminate, from these two equations, $\cot. pZm$, and there results

$$\sin. h(\tan. L. \sin. k - \tan. l. \sin. K) = \cos. h(\sin. k \cos. K - \cos. k. \sin. K)$$
$$= \cos. h \times \sin. (k - K).$$

Now, $k - K = a$, and $\sin. (k - K) = \sin. a = a$ (nearly) $= P \dfrac{\sin. h \cdot \sin. k}{\cos. L}$: substituting $\therefore$ and dividing by $\sin. h \times \sin. k$,

$$\tan. L - \tan. l \frac{\sin. K}{\sin. k} = P \frac{\cos. h}{\cos. L};$$

$$\therefore \tan. L - \tan. l = P \frac{\cos. h}{\cos. L} - \tan. l \left(1 - \frac{\sin. K}{\sin. k}\right)$$

$$= P \frac{\cos. h}{\cos. L} - \frac{\tan. l}{\sin. k} \cdot (\sin. k - \sin. K).$$

$$\text{Now, } \tan. L - \tan. l = \frac{\sin. (L - l)}{\cos. L \cdot \cos. l},$$

$$\text{and } \sin. k - \sin. K = 2 \cdot \cos. \left(\frac{k + K}{2}\right) \sin. \left(\frac{k - K}{2}\right),$$

and since, $k - K = a$, $\dfrac{k + K}{2} = K + \dfrac{a}{2}$: substitute, and

$$\frac{\sin. (L - l)}{\cos. L. \cos. l} = P \frac{\cos. h}{\cos. L} - \frac{2 \tan. l}{\sin. k} \cdot \left\{\cos. \left(K + \frac{a}{2}\right) \sin. \frac{a}{2}\right\}.$$

But sin. $(L - l) = $ sin. $dl = $ sin. $\delta = \delta$, nearly, and sin. $\frac{a}{2} = \frac{a}{2}$

$$= \frac{P \text{ sin. } h \text{ sin. } k}{2 \text{ cos. } L};$$

$\therefore \delta$, the par. in lat., $= P$ cos. h.cos. $l - P$.sin. $h$sin. $l \times$ cos. $\left(K + \frac{a}{2}\right)$*

This expression, since $l = L - \delta$, is under the same predicament as the former one, (p. 743,) and must be treated in the same manner; that is, we must find a value of δ by supposing $l = L$, and then a nearer value. Since the Moon's latitude is never very large, and at the time of an eclipse (for computing which the above expressions are useful) is always very small, (and consequently sin. l is very small) we may assume, as a first step in the approximation,

$$\delta = P \text{ cos. } h \text{ cos. } L \; (= f \text{ suppose,})$$

and then the second step may be made by computing δ, from

$$\delta = P \text{ cos. } h \text{ cos. } (L - f) - P \text{ sin. } h \text{ sin. } (L - f) \,.\, \text{cos. } \left(K + \frac{a}{2}\right)$$

and the investigation continued will give more exact values of δ, the parallax in latitude †.

The formulæ for computing the parallaxes in longitude and latitude have been deduced by, what has technically been called, *the Method of the Nonagesimal.* This method, of no recent invention, naturally suggested itself, as Lalande observes, to the mind of Kepler. For, parallax takes place in a vertical circle, therefore, if the heavenly body were situated in a vertical circle, such as $pZNn$ passing through N the nonagesimal point, the effect of parallax, in such a circle, would be nothing in longitude but would take place, altogether, in latitude; since ON, the

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* See *Mem.* Gottingen, tom. II, p. 168; where Mayer has given, very nearly, the same expressions; also Lalande, tom. II, p. 305. Edit. 3.

† The expressions for the parallaxes in right ascension and declination may easily be deduced from the preceding processes. We must then consider p to be the pole of the equator.

ecliptic, is perpendicular to pZN. Again, if the Moon, always near to the ecliptic at the time of an eclipse, should also be near to the nonagesimal, then the greater its altitude the less would be the parallax in latitude, (see Lalande, tom. II, p. 291.)

Distance of the Moon and a Star at the time of an Occultation.

Computing by the preceding formulæ the parallaxes, we must apply them, with their proper signs, to the true longitudes and latitudes furnished by the Tables, or by observation, and the results will be the apparent longitudes and latitudes of the centre of the Moon and of the star. Suppose these to be l, l', k, k', respectively; then, in order to find the distance (D), we have (in a triangle such as Mpm, Fig. p. 741), the two sides $90^0 - l$, $90^0 - l'$ (analogous to Mp, mp), and the included angle, $k - k'$ (analogous to Mpm); and D is the side opposite to the angle $k - k'$: therefore, (*Trig.* pp. 139, 172, &c.),

$$\cos. D = \cos. l . \cos. l' \cos. (k - k') + \sin. l . \sin. l',$$

and substituting for $\cos. D$, &c. $1 - 2 \sin.^2 \frac{D}{2}$, &c. there results

$$\sin.^2 \frac{D}{2} = \sin.^2 \left(\frac{l - l'}{2}\right) + \cos. l . \cos. l' . \sin.^2 \left(\frac{k - k'}{2}\right),$$

whence D may be deduced, and most conveniently, by means of a subsidiary angle, (see the page just referred to).

The preceding method is not confined to the case of an occultation, but is equally applicable to the finding of the distances of the Sun and Moon during a solar eclipse, and of the Sun and an inferior planet during a transit. And, in all the cases, since the distances are small, a more simple formula for computing D may be introduced. For, D may be considered as the hypothenuse of a right-angled triangle, the sides of which are $l - l'$, and $(k - k)$ cos. l *, in which case

* For $k - k'$ is the arc on the great circle, $(k - k')$. cos. l, on the parallel; for instance, in Fig. p. 9, if $a\,b = \angle\ a\,P\,b\ (k - k')$ $s\,s' = a\,b\ .\cos.\ s\,b = (k - k') \cos.\ s\,b$.

$$D^2 = (l - l')^2 + (k - k')^2 . \cos.^2 l$$

$$= (l - l')^2 \left\{1 + \left(\frac{k - k'}{l - l'}\right)^2 \cos.^2 l\right\};$$

$$\therefore\ D = (l - l') . \sec. \theta,$$

$$\text{making } \tan. \theta = \frac{k - k'}{l - l'} . \cos. l.$$

The latter expression for the value of D is easily deducible from the former, by substituting in the former $\frac{D}{2}$, $\frac{l - l'}{2}$, &c. instead of their sines, which may be done with inconsiderable error, by reason of the smallness of those angles, during the contiguity of the Moon and star, &c.

The first term of the expression for $\sin.^2 D$, (see p. 746,) is $\sin.^2 \left(\frac{l - l'}{2}\right)$. In which expression l, l', are the apparent latitudes, therefore if δ, δ', were the parallaxes, and Δ the difference of the true latitudes, we should have

$$l - l' = \Delta + \delta - \delta'.$$

Suppose now one of the bodies (that to which the latitude l' belongs) to have no parallax in latitude, but the other to have a parallax equal to $\delta - \delta'$, then, still as before,

$$l - l' = \Delta + (\delta - \delta'),$$

and a similar result will hold good with regard to $\sin.^2 \frac{k - k'}{2}$; therefore, if the coefficient of this latter term, instead of being $\cos. l . \cos. l'$, were a constant quantity a, for instance, (or involved merely the *difference* of the parallaxes), the distance D would result precisely of the same value $\sin.^2 D$ from the expression

$$\sin.^2 \frac{D}{2} = \sin.^2 \frac{l - l'}{2} + a . \sin.^2 \frac{k - k'}{2},$$

if, instead of assigning to each body its proper parallax, we suppose one to be entirely without, and *attributed* to the other an imaginary parallax in latitude and longitude, equal to the difference of the real parallaxes. And in this case, the rule given by Astronomers, (see Lalande, 434, tom. II, and *Cagnoli*, p. 463,) would be proved to be true. Since, however, the coefficient cos. l. cos. l', is not a constant quantity such as a, but [since it equals $\frac{1}{2}$ cos. $(l - l') +$ cos. $(l + l')$], involves, besides the difference, the *sum* of the parallaxes, the rule is not perfectly exact. It, however, is nearly so, since $\sin.^2 \frac{k - k'}{2}$, which is multiplied into cos. l. cos. l', is a very small quantity.

We have spoken of the general case of the Problem, when the distance of the centres of two heavenly bodies is to be found. But, if we speak of each particular case, then we must say, the rule is slightly inaccurate in a solar eclipse and in a transit, but exact in an occultation, since one of the bodies, the fixed star, is devoid of parallax.

The *Distance of the Centres* is the last step in the mathematical process belonging to the subject of the occultation of a fixed star by the Moon; and, since the process is somewhat complicated, we will endeavour to illustrate it, and its subordinate methods, by an Example.

Required the apparent Distance of Antares *from the Centre of the Moon at the instant of Immersion, which was observed at Paris in April* 6, 1749, $13^h\ 1^m\ 20^s$, *Apparent Time**.

(1.) *Right Ascension of the Mid-Heaven.*

Convert the time into degrees and take from the Tables the Sun's longitude, and we have (see p. 740,)

$$\text{Æ of Mid-heaven } (A) = 15^0\ 58' + 195^0\ 29'$$
$$= 211^0\ 18'$$

Since, $15^0\ 58' = \odot$'s Æ,
and $195\ \ 20 = 13^h\ 1^m\ 20^s$.

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* Lalande, tom. II, pp. 437, &c.

(2.) *Altitude of the Nonagesimal*, (*see* 1[st] *Form*, p. 742,)

log. sin.	23°	28′	22″	(obliquity)	9.60022 *
cos.	48	38	50	(lat. cor. see p. 329,)	9.82000
ver. sin.	301	18	0	$(90^{\circ} + A)$	9.68167
					29.10189
ver. sin.	17	52	48	(co-lat. – obliquity)	8.68395 . . . . (*a*)
					20.41794 = 2 log. tan. θ
2 sec.	58	16	54	(θ)	20.55845
				(*a*)	8.68395
				20 + log. ver. sin. pZ =	29.24240

$\therefore pZ$ (h), the altitude of the nonagesimal, is $34^{\circ}\ 23'\ 9''$.

(3) *Longitude of the Nonagesimal*, (*see Form*, p. 742.) †.

$pZ\,(h)$	34°	23′	9″	. . . . log. sin. =	9.75186
Pp	23	28	22	. . . . sin.	9.60022
PZ	41	21	10	. . . . (*b*)	1935208
sum . . =	99	12	41		
$\frac{1}{2}$ sum	49	36	20.5	. . . . log. sin.	9.88172
$\frac{1}{2}$ sum – PZ	8	15	10.5	. . . . sin.	9.15697
				(20 added)	39.03869
				(*b*)	19.35208
				2 log. cos. PpZ =	19.68661

$\therefore PpZ = 91^{\circ}\ 36'\ 30''$, and consequently, (see p. 742,) the longitude of the nonagesimal $= 181^{\circ}\ 36'\ 30''$.

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* Five decimals are sufficient: more, such is the nature of the process, would not add to the accuracy of the result.

† The angle PpZ being nearly 90°, is the reason, why it is expedient to use the second, (see p. 742,) of the formulæ, which, in the first instance, gives only half the angle PpZ. For a more full explanation of this point, consult *Trig*. Chap. V.

Hence, since by the Lunar Tables the longitude of the Moon was $245^\circ\ 31'\ 42''.4$, K, or the Moon's distance from the nonagesimal, (see Fig. p. 741,)

$$\text{is } 245^\circ\ 31'\ 42''.4 - 181^\circ\ 36'\ 30'' = 63^\circ\ 55'\ 12''.$$

(4.) *Parallax in Longitude,* (see p. 743,)

log. $0^\circ\ 57'\ 16''.2$ (P, from Tables)	3.53608	sum =
log. sin. 34 23 9 (h)	9.75186	13.28890
Ar. com. cos. 3 47 58 .7 (L ☽ 's true lat.)	0.00096	
sin. 64 10 * ($K + a$)	9.95427	
(rejecting 10)	3.24317 = log. $29'\ 10''$	

$\therefore\ \epsilon$, or the first approximate value of a, is $29'\ 10''$, and

$$K' + \epsilon = 64^\circ\ 24'\ 22'',$$

log. sin. $64^\circ\ 24'\ 22''$ ($K + \epsilon$)	9.95515
Sum (see p. 744,) rejecting 10	3.28890
(rejecting 10)	3.24405 = log. $29'\ 14''.1$;

$\therefore\ a$, the parallax in longitude, is $29'\ 14''.1$.

(5.) *Parallax in Latitude,* (see p. 744.)

Computation of the first part of the expression,

log. P	3.53608	sum = 13.45267
log. cos. $34^\circ\ 23'\ 9''$ (h)......	9.91659	
cos. 3 47 58.7 (L)	9.99903	
(rejecting 20)	3.45170 = log. $47'\ 9''$; $\therefore\ 47'\ 9''$	

is the first approximate value of δ.

Again,

log. cos. $4^\circ\ 35'\ 7''.7$ ($L + \delta$)	9.99861
log. P + log. cos. h....	3.45267
(rejecting 10)	3.45128 = log. $47'\ 6''.7$, 2d value of δ.

* K (see l. 4,) $= 63^\circ\ 55'\ 12''$, and, since a is some small quantity, it is *conjecturally* taken, in the first trial, equal to $14'\ 48''$, which added to K, makes $K + a = 64^\circ\ 10'$.

Computation of the second part of the expression,

log. $P \times$ sin. h (see p. 745, l. 3.) 3.28794

log. cos. $64^\circ\ 9'\ 49''\ \left(K + \frac{\alpha}{2}\right)$ 9.63929

sin. 4 35 8.7 (☽ 's latitude) 8.90283

(rejecting 20) 1.83006 = log. $1'\ 9''$

Since the Moon's latitude was south, this last part ($1'\ 9''$) of the parallax in latitude must be added; consequently, the whole parallax in latitude (δ) = $47'\ 6''.7 + 1'\ 9'' = 48'\ 15''$, nearly. Hence, applying the parallaxes thus found to the true longitude and latitude,

☽ 's apparent long. = $245^\circ\ 31'\ 42''.4 + 29'\ 14''.1 = 246^\circ\ 0'\ 56''.5$

☽ 's apparent lat. = 3 47 58.7 + 48 15 = 4 36 13.7.

(6.) *Apparent Distance of the Moon and Antares,* (see p. 747.)

Long. of *Antares* (k') . . $246^\circ\ 16'\ 19''.2$ . . lat. (l') $4^\circ\ 32'\ 10''.5$

☽ 's longitude (k) . . . . 246 0 56.5 . . lat. (l) 4 36 13.7

$k' - k$ . . . . 0 15 22.7 . . $l - l'$ . . 0 4 3.2

$\therefore$ log. cos. $4^\circ\ 34'\ 12''\ \left(\frac{l + l'}{2}\right)$* 9.9986171

log. . . . . 0 15 22.7 . . . . 2.9650605

Ar. comp. log. 0 4 3.2 . . . . 7.6140364

10.5777140 = log. tan. θ

log sec. 75 11 21 (θ) 10.5923906

Ar. comp. log. 0 4 3.2 . . . . 7.6140364

log. $951''.38$ = 2.9783542

therefore the distance required is $15'\ 51''.38$.

By the preceding process the *apparent* distance of a fixed star and of the Moon's centre has been found at the instant of *occul-*

* $\frac{l + l'}{2}$ used instead of l.

tation. A process, almost entirely the same, will give the distance of the Sun and Moon in a solar eclipse, and the distance of the Sun, and of an inferior planet, during the *transit* of the latter across the Sun's disk. The *difference* in the processes is pointed out in the Rule of p. 748: which Rule directs us to *suppose* one body to be devoid of parallax, and the other to be invested with a parallax, equal to the difference of the parallaxes of the two bodies.

The above process, as it stands, is rather long and would have been much more so, had we deduced from Tables, the Moon's real longitude and latitude. But we, in fact, know the latter quantities from the Nautical Almanack, or may deduce them by interpolation. The computers of *occultations*, are so enabled to abridge their labours. The utility of such labours will be more fully explained in a subsequent Chapter: but we will not dismiss the present without giving to the students a slight idea of the principle and manner of using the result of the preceding computations.

The Moon's latitude and longitude (see p. 746,) are computed for the instant of time, at which the star is on the Moon's disk. When the *time is given* we can, from the Lunar Tables, or from the results from those Tables registered in the Nautical Almanack, compute directly, or by interpolation, the Moon's latitude, longitude, and semi-diameter. But, since the Nautical Almanack, (confining our views to its results) is computed for Greenwich, we cannot, should the occultation be observed at Cambridge, determine the time at the former place, except we know how much it is to the *west* of the latter place. For instance, an occultation is observed at Cambridge, at 11^h: the Moon's latitudes are expressed in the Nautical Almanack for *Greenwich*, noon and midnight: we must not, therefore, by interpolation, compute the latitude corresponding to 11^h, but the latitude to 11^h minus corresponding the time due to the difference of the longitudes of Greenwich and Cambridge. The determination, however, of such *difference* is one of the special uses of the problem. The thing, therefore, requisite to be known in the process of solution, is the result of such process. We must, therefore, assume some quantity as the difference, and compute, agreeably to such assumption, the Moon's latitude and longitude: thence, as it is pointed out in the preceding pages, we compute the distance of the Moon's centre, and of the star on its disk: such distance is the Moon's semi-diameter. But we can also determine the Moon's semi-diameter, by interpolating between the values expressed in the Nautical Almanack, for noon and midnight, its value corresponding to 11^h minus the assumed time of the difference of the longitudes of Greenwich and Cambridge. Should that difference be assumed, as it probably will be, erroneously, the two values of the semi-diameter compared together will not agree. The quantity of their disagreement will become an index of the error of the original assumption, and the means of amending it: and, by repetition of process, of completely correcting it.

By computing the parallaxes in longitude and latitude, we have, in the preceding pages, deduced the Moon's apparent longitudes and latitudes from her true, and thence the apparent distance of the Moon from the star. If we reverse the process, we may deduce the true distance of the Moon and star: and some authors make the same use of the true, as, according to the above explanation, may be made of the apparent, (see Vince, vol, I. pp. 334, &c.)

CHAP. XXXVIII.

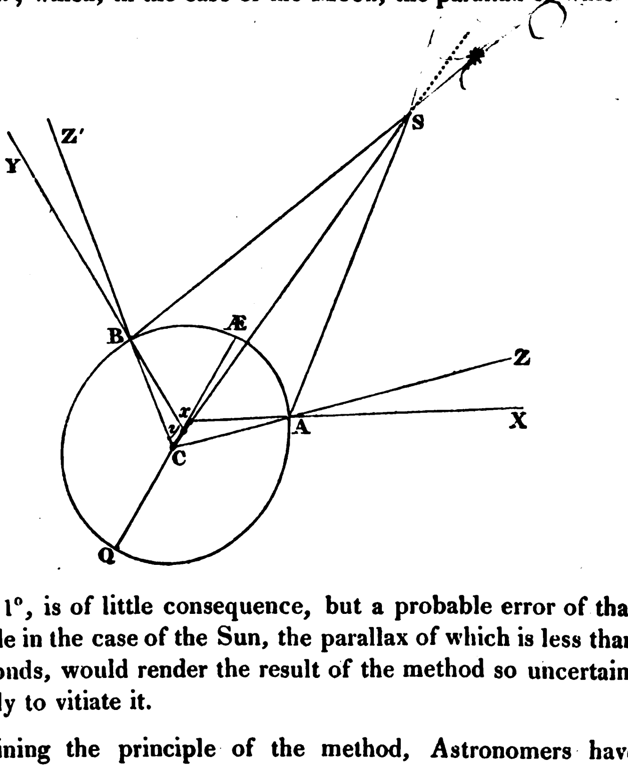
On the Transits of Venus and Mercury over the Sun's Disk.

WE have already stated in p. 736, that the phenomena of eclipses, occultations, and transits are very nearly alike in their general circumstances, and exactly alike in their mathematical theories. In those theories, the essential problem to be solved is the apparent angular distance of two heavenly bodies, in apparent proximity to each other, when viewed by a spectator on an assigned station on the Earth's surface.

In an eclipse and occultation, the Sun's parallax is supposed to be known: were it supposed to be known in a transit, there would be an additional circumstance of similarity between its theory and the theories of the former phenomena: for, they would have the same object, and would equally serve to the determination of the longitudes of places. And, in point of fact, this is the present state of the case. One transit of *Venus* has already answered the special purpose of determining the parallax of the Sun, and future transits may be used, either to confirm the accuracy of that determination, or for the general purposes which *eclipses*, in their extended signification, (see p. 736,) are made subservient to.

It is the object of the present Chapter to explain the use that *has been made* of the transit of *Venus*; or, to shew the special use of that phenomenon in determining the important element of the Sun's parallax.

The Sun's parallax is the angle subtended at the Sun by the Earth's radius; which angle can be found, if another subtended by a chord, lying between two known places, can. And to find this latter angle is the object of the method given in Chap. XII, pp. 325, &c. If we refer to that Chapter we shall find the angle ASB to be the object of investigation. Now, in its instrumental measurement, an error of three or four seconds may be committed; which, in the case of the Moon, the parallax of which



is about 1°, is of little consequence, but a probable error of that magnitude in the case of the Sun, the parallax of which is less than nine seconds, would render the result of the method so uncertain, as entirely to vitiate it.

Retaining the principle of the method, Astronomers have sought to correct its error, by *computing*, instead of instrumentally measuring, an angle such as ASB, or an angle from which it may be immediately deduced.

Suppose, for the sake of illustration, S to be a point in Venus's disk, and BS continued to be a tangent to the Sun's disk: then the direction of a line AS would be to the left of the Sun's disk. In other words, the moment of contact or ingress would have arrived to a spectator at B, but not to a spectator at A. It would, however, arrive some minutes after, when by the retrograde motion (see p. 556,) of *Venus*, the line AS, always a tangent to the disk of *Venus*, should become one to that of the Sun. Suppose AS, in this latter direction (to the right of its present position) to intersect BS produced in some point S' situated in the Sun's disk: then, the angle SAS' is proportional to the time elasped between the contacts at B and A: which time is known from observation and the ascertained difference of longitudes of the places B and A: suppose it t, and let h be the horary approach of *Venus* to the Sun (about 240″); then,

$1 : t :: h : ht$, which is equal to the angle SAS',

which is by these means *computed*.

SAS' being known, $SS'A$, or $AS'B$, may be determined from the known ratio between SA and SS'.

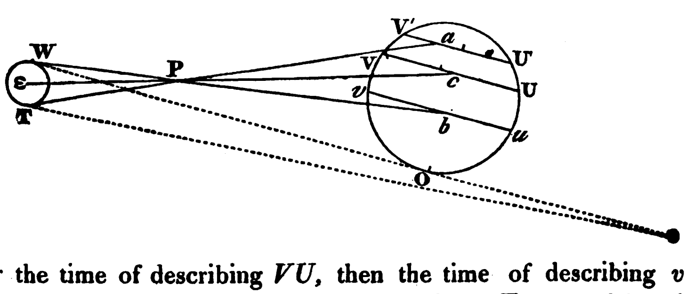
The preceding is a very imperfect description of the method that was actually used in the problem of the transit of *Venus*. But it shews the principle of the method and the reason of its superior accuracy: for, since the time of contact can be observed to be within three or four seconds, or since the limit of the error in time is about three seconds, and since the excess of the horary motion of *Venus* above the Sun's is 240″, that is, 4″ in 1^m, or $\frac{1''}{15}$ in 1^s, an error of 6^s (3^s at each place of observation) would only cause an error of $\frac{6''}{15}$ in the estimation of the angle SAS', and an error in the estimation of $SS'A$, (on which the parallax depends) less in the proportion of SA to SS', that is, in the case of *Venus*, of one to two and a half nearly.

The imperfection of the method, as it has been described, consists in this; that it requires to be known, what it is very difficult to determine, the difference of the longitudes of the places A and B. For, t is the difference of *actual* or *absolute time*, which depends on the reckoned time at each place of observation, and the difference of the longitudes of those places. If the contact was observed at Greenwich at 3^h 40^m, and at a place 15° east of Greenwich, at $4^h\ 41^m$, the difference in absolute time would be only 1^m; since 1^h, in the reckoned time, is entirely due to the difference of the meridians. We shall, however, in the subsequent pages, see a method of getting rid of the imperfection which we have just noted.

The longitude of the Cape of Good Hope, which had been long the station of an European Colony, and where the transit of 1761 was observed, was known to a considerable degree of accuracy. That of Otaheite, where it was expedient to observe the transit of 1769, was not known. And, from the difficulty of ascertaining with sufficient precision this nice condition of the longitude, Astronomers, by modifying their process of calculation, have got rid of it entirely. Instead of observing the ingress, they observe the duration of the transit, and from the difference of durations, at different places, deduce the difference of the parallaxes of *Venus* and the Sun, and then the Sun's parallax.

The difference in the durations of transits does not amount to many minutes. To make it as large as possible, it is expedient so to select the places of observation, that, at one, the duration should be accelerated, at another, retarded beyond the *true time* of duration; which true time is supposed to be that which would be observed at the Earth's centre.

If P were *Venus*, ϵ the Earth, W a place towards the north pole (Wardhus for instance) and T (Otaheite) towards the south, and $V'V$, &c. the Sun's disk, then the *true* line of transit, seen from the centre ϵ would be VU: from W, vu would be the line; from T, $V'U'$. If T should be the *true duration* of the transit,

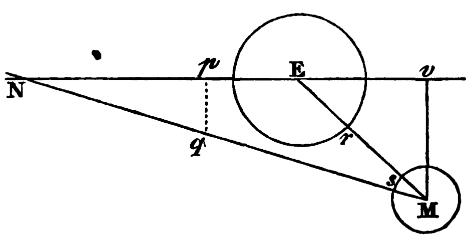


or the time of describing VU, then the time of describing vu nearer to the Sun's centre than VU, would be $T + t$: of describing $V'U'$ more remote than VU from the Sun's centre, $T-t'$: and, accordingly, the difference of the durations of the transits seen from T and W, would be $T+t-(T-t')=t+t'$. This, as it is plain, is entirely the effect of parallax, and, as it is also plain, the effect is compounded of the parallaxes of Venus and the Sun: since changes in the distances of P and of the Sun will produce changes in the dimensions of the lines $V'U'$, vu.

We will now proceed to treat the subject mathematically, and to deduce, by means of a simple equation, the *difference* of the parallaxes of Venus and the Sun. That difference being determined, the values of both the parallaxes may be deduced by means of Kepler's law relative to the periods of planets, and their distances from the Sun.

In the subsequent mathematical process we shall have a proof of what we have more than once asserted, namely, the similarity of the mathematical theories of eclipses, occultations, and transits. For, T, $T+t$, $T-t'$ will be computed by means of the formula employed in Chap. XXXV. The only difference in the computation of T and of $T+t$ consists in assuming in the former, the angular distances seen from the Earth's centre and given by the Astronomical Tables, and in the latter, those angular distances *corrected* for the effects of parallax in longitude and latitude.

In the above-mentioned formula, the time and the apparent angular distance of two heavenly bodies were involved. And the diagram employed on that occasion will suit the present*. Instead of E and M representing the centres of the Earth's shadow and



* The same diagram will serve for an occultation, M being the Moon, and E the star.

the Moon, let them represent the centres of the Sun and Venus; then, EM will represent the distance of their centres previous to a transit, or after one: and, the Tables of the Sun and of the planets, will, as in an eclipse (see p. 725,) furnish us with quantities analogous to λ, m, n, &c. Suppose then, at the time of conjunction,

♀ 's lat. . . . . λ. . . . . horary motion in lat. . . . . . n
♀ 's long. . . . . . . l. . . . . . horary motion in long. . . . . m
☉ 's horary motion in long. . . . . . s.

If we form an equation, precisely as the one in p. 722, was formed, we shall have

$$n^2 t^2 + 2\lambda n t \, . \, \sin.^2\theta = (c^2 - \lambda^2) \, . \, \sin.^2\theta,$$

$$\text{whence, } t = \frac{1}{n}\left[-\lambda \sin.^2\theta \pm \sin.\theta \sqrt{(c^2 - \lambda^2 \cos.^2\theta)}\right],$$

t being the time from conjunction, and c the distance of the centres.

In this equation substitute, instead of c, the sum of the apparent semi-diameters of the Sun and *Venus*, and the resulting time will be that of the first or last *exterior contact:* substitute the difference, and the resulting time will be that of the first or last *interior contact*. The duration of a transit is the difference between the times of the last and first exterior contacts, and is to be found exactly as the duration of an eclipse was in pp. 726, &c.

The times which we have mentioned, as resulting from the preceding equation, would be noted by a spectator in the Earth's centre: they belong to the points V, U, and the line VU. But to a spectator at T, for instance, the contact instead of at V would appear to take place at V'; and, it would appear to happen at a time, different from (T') the computed time of its happening at V, at $T' + t'$, for instance, t' being a small quantity and entirely the effect of parallax.

The latitudes and longitudes of *Venus* and the Sun continually altering, those quantities at the time $T' + t'$ from conjunction would be different from what they were at the time T': their change would be proportional to t'. The time T' being computed from the preceding equation, the corresponding latitudes and longitudes may be taken from the Tables, or may be easily computed from their values at the time of conjunction. At this latter time, we have supposed the latitude of *Venus* to be λ. It is convenient for us to use that symbol (λ) to denote the latitude at the time T' of contact; let also the corresponding longitudes of *Venus* and the Sun be l, l'; and the horary motions m, n, s: then (see p. 722,) at the time t' from contact,

♀'s long. ...... $l + mt'$ .... ♀'s lat. ...... $\lambda + nt'$,
☉'s long. ...... $l' + st'$.

And accordingly, the distance of the centres (such as EM) would be the hypothenuse of a right-angled triangle, of which the sides, are, respectively, $(l + mt') - (l' + st')$, and $\lambda + nt'$.

These angular distances belong to the centre of the Earth; but when they are diminished, as in the case of an occultation, (see p. 746,) by the parallaxes in longitude and latitude, they are made to belong to a spectator on the Earth's surface. Let the parallaxes in longitude be a, a'; in latitude δ, δ'; then, the sides of the right-angled triangle are

$$(l + mt' - a) - (l' + st' - a'), \text{ and } \lambda + nt' - \delta + \delta',$$
$$\text{or } l - l' + (m - s)t' - (a - a'), \text{ and } \lambda + nt' - (\delta - \delta').$$

The hypothenuse is the distance of the centres. But, the time is that at which a contact of the limbs of the Sun and *Venus* is seen; if the contact therefore be an *internal* one, (when the whole of *Venus's* disk is just within the Sun's), the distance will be the difference of the semi-diameters of *Venus* and the Sun: let it equal Δ, then,

$$\Delta^2 = [l - l' + (m - s)t' + a' - a]^2 + (\lambda + nt' + \delta - \delta')^2.$$

In which expression, $a - a'$, $\delta - \delta'$, and t' are very small quantities; rejecting therefore their squares and products in the expression expanded;

$$\Delta^2 = (l - l')^2 + 2(l - l') \times (m - s)t' - 2(l - l') \times (a - a')$$
$$(a) \qquad + \lambda^2 + 2\lambda nt' - 2\lambda(\delta - \delta').$$

But, since by hypothesis, (see l. 6,) l, l', &c. are the longitudes, &c. at the time of contact seen from the centre, we have

$$\Delta^2 = (l - l')^2 + \lambda^2,$$

thence deducing t' from (a),

$$t' = \frac{(l - l')(a - a') + \lambda(\delta - \delta')}{(l - l')(m - s) + \lambda n}.$$

In this expression, l, l', λ, m, s, n, are to be computed from the Tables, and the parallaxes in longitude and latitude (a, a', δ, δ') are to be computed from the expressions in pages 743, &c. that is, if P, P' represent the horizontal parallaxes of *Venus* and the Sun,

$$a = \frac{P\,.\,\text{sin.}\,h\,.\,\text{sin.}\,k}{\text{cos. lat. } ♀}, \quad a' = \frac{P'\,.\,\text{sin.}\,h\,.\,\text{sin.}\,k'}{1},$$

$$\delta = P\,\text{cos.}\,h\,.\,\text{cos. } ♀\text{'s app. lat.}$$

$$-P\,\text{sin.}\,h\,.\,\text{sin. } ♀\text{'s app. lat.} \times \text{cos.}\left(\frac{K+\delta}{2}\right),$$

$$\delta' = P'\,\text{cos.}\,h \text{ (since } ⊙\text{'s apparent latitude is nearly} = 0.)$$

At the time of a transit, *Venus's* latitude is very small, and her longitude is nearly equal to that of the Sun, the coefficients of P, P', therefore, in the expressions for a, a', and for δ, δ', must be nearly equal. Let these coefficients be a, a', b, b' respectively, then

$$t' = \frac{(l - l')(aP - a'P') + \lambda(bP - b'P')}{(l - l')(m - s) + \lambda n};$$

or, since $aP - a'P' = a'(P - P') + (a - a')P$, and $(a - a')P$, as well as $(b - b')P$, are very small quantities and may be neglected, we have

$$t' = \frac{a'l - a'l' + \lambda b'}{(l - l')(m - s) + \lambda n} \times (P - P').$$

From this equation, if t' should be known from observation, $P - P'$, the excess of the parallax of *Venus* above that of the Sun, (which is the object of investigation,) could be determined. We must consider, therefore, by what means t' may be ascertained.

The Astronomical Tables, from which the quantities, l, l', &c. are supposed to be taken, are computed for Greenwich. At such a place, let the time of the conjunction of *Venus* and the Sun be T; then, at any place to the west of Greenwich and distant by a longitude $= M$ (expressed in time), the reckoned time, at which the conjunction would be seen from the centre of the Earth, would be $T - M$; the time of internal contact, seen also from the centre, would be $T - M + T'$; and the time, at which the contact would be seen from the place of observation (whose longitude is M) would be

$$T - M + T' + t'.$$

Now, the observer, by means of his regulated clock, is able to note this time; suppose it H', then

$$t' = H' - T + M - T', \text{ and consequently,}$$

$$H' - T + M - T' = \frac{a'l - a'l' + \lambda b'}{(l - l')(m - s) + \lambda n} \times (P - P')$$

$= f(P - P')$, f representing the coefficient of $P - P'$.

From this equation $P - P'$ could be determined, if M, the longitude of the place, were known. We must, however, for the reasons alledged in p. 757, seek to dispense with that condition. This is simply effected by observing the *last* interior contact, that is, the one immediately preceding the egress of *Venus's* disk from the Sun. Let the quantities analogous to T', H', and belonging to this last contact be T'', H'', and the coefficient of $P - P'$ (analogous to f) be f'; then,

$$H' - T + M - T' = f(P - P')$$
$$H'' - T + M - T'' = f'(P - P'),$$

consequently,

$$H' - H'' - (T' - T'') = (f - f')(P - P') \ldots (h)$$

$$\text{and } P - P' = \frac{H' - H'' - (T' - T'')}{f - f'}.$$

This expression is deduced by observing at the same place the times of ingress and egress. If we take a second place of observation, then there will result an equation similar to (h), such as

$$H_{,} - H_{,,} - (T' - T'') = (f_{,} - f_{,,})(P - P'),$$

and subtracting this from the former (h),

$$(H'-H'')-(H_{\prime}-H_{\prime\prime}) = [(f-f')-(f_{\prime}-f_{\prime\prime})]\times(P-P')(h')$$

whence, we have the value of $P - P'$, obtained from the difference of the *durations* of the transit*.

The parallax is inversely as the distance; but, by observation and the Planetary Theory, (see Chap. XVII,) the ratio of the distances of the Earth from *Venus* and the Sun, is known, and therefore the ratio of P *to* P'; let it be as $g : 1$, and let the coefficient of $P-P'$ in (h') be q, the left hand side being $= A$; then

$$(g-1)\,q\,P' = A,$$

$$\text{and } P' = \frac{A}{q\,(g-1)}.$$

This is the value of P' when the Sun is at some distance ρ from the Earth. At the mean distance (1)

$\odot$'s horizontal parallax (nearly his mean) $= \rho P'$.

The preceding formula, applied to the transit of *Venus* which happened in 1769, would give

$$P - P' = \frac{1416}{65.72962} \times 1'' = 21''.5428.$$

And the Astronomical Tables, at the epoch of the observations, gave

$\oplus$'s distance from $\odot$ (ρ) 1.01515

♀'s distance from $\odot$ 72619

$$\text{and therefore } g - 1 = \frac{72619}{28896}, \text{ and}$$

$$P' \text{ the Sun's parallax} = 21''.5428 \times \frac{28896}{72619} = 8''.5721$$

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* This last operation, although unnecessary in the preceding simple statement, is not so in practice: since, by means of it, the errors of the Tables introduced into the calculation as arbitrary quantities are got rid of.

and (see p. 763, l. 14,) the ☉'s hor. par. $= 8''.5721 \times 1.01513 = 8''.7017$*

In the fraction $\frac{1416}{65.72962}$ $(= P - P')$ the numerator is obtained from observations on the times of contact. If that numerator had been $1416 - 65.72962$, the quotient, instead of being $21''.5428$, would have been $20''.5428$. In other words, a difference of $65^s.72962$, made in noting the times of the transit,

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* The equation (see p. 763,) for determining the difference of the parallaxes of Venus and the Sun, was obtained by observing, at *different* places, the *differences* of the durations of the transits. The transit of 1767, was observed at several places, and an exact result was endeavoured to be obtained, by taking the *mean* of several results. The following are the results and their mean according to M. Delambre:

	Sun's Parallax.	Difference of Parallaxes.
Taïti, (Otaheité) Wardhus	8.7094	21''.561
Taïti, Kola	8.5503	21 . 166
Taïti, Cajanebourg	8.3863	20 . 762
Taïti, Hudson's Bay	8.5036	21 . 066
Taïti, Paris and Petersburgh	8.7780	21 . 730
California, Wardhus	8.6160	21 . 330
California, Kola	8.3880	20 . 765
California, Cajanebourg	8.1636	20 . 208
California, Hudson's Bay	8.1521	20 . 284
California, Paris and Petersburgh	8.7155	21 . 576
Hudson's Bay, Wardhus	9.1266	22 . 592
Hudson's Bay, Kola	8.4589	20 . 941
Hudson's Bay, Cajanebourg	8.1730	20 . 233
Hudson's Bay, Paris and Petersburgh	9.2491	22 . 897

Here the mean of the first 5 results is, nearly, ... 8''.59
of the next 5 .... 8 . 41
of the next 4 .... 8 . 75
of all .... 8 . 57.

would have produced an error of one second only in the difference of the parallaxes, and consequently, an error in the Sun's parallax less in the ratio of 28896 to 72619, or (of 2 to 5 nearly). Or, what amounts to the same thing, it would have required an error in time equal to $164^s \left(= 65.7 \times \frac{5}{2}\right)$ to have produced an error of $1''$ in the value of the Sun's parallax.

The special Astronomical use of the transit of *Venus* is, as it has been observed, the determination of the Sun's horizontal parallax. But, that important element being once determined, the transit of an inferior planet, even with regard to its use and object, may be made to enter the class of eclipses and occultations, and, like them, be made subservient to the determination of the longitudes of places.

That a transit may be adapted to this latter purpose, is evident from the equation of p. 762, namely,

$$H' - T + M - T' = f.(P - P'),$$

for in that, if $P - P'$ be supposed to be known, M, the longitude of the place of observation, is the only unknown quantity.

Transits, however, are phenomena of such rare occurrence, that their use, in this latter respect, is very inconsiderable *.

.The fixed stars, the Sun, the planets, and the Moon, with their peculiar and connected theories, have already been treated of. There is another class of heavenly bodies, called *Comets*,

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* The transit of Mercury was used by M. Kohler to determine the longitude of Dresden, see *Phil. Trans.* 1787, p. 47: and by Short to determine the difference of longitudes of Paris and Greenwich, (see *Phil. Trans.* 1763, vol. LIII, p. 158.). M. Delambre, however, and properly, says 'Le mouvement relatif est si lent et les observations de l' entreé et de la soireè sont en consequence si peu susceptibles de precision qu'on ne doit recourir a ce moyen que faute d' autres' (*Mem. Inst.* tom. II, p. 442,) see also *Phil. Trans.* vol. LIII, pp. 30, and 300: also vol. LII: *Mem. Acad. Paris,* 1761: *Phil. Trans.* No. 348, p. 454, (Halley's account) and Horrox's *Venus in Sole visa.*

which ought not to be passed over. Yet their strictly mathematical theory is so difficult, that, instead of attempting to put the Student in possession of it, we shall content ourselves with acquainting him with some of its general circumstances, and with referring him to ampler sources of information.

CHAP. XXXIX.

On Comets.

COMETS are bodies occasionally seen in the heavens, with ill-defined and faint disks, and usually accompanied with a *coma* or stream of faint light in the direction of a line drawn from the Sun through the Comet.

Comets resemble the Moon and planets in their changes of place amongst the fixed stars: but, they differ from them in never having been observed to perform an entire circuit of the heavens. There are also other points of difference; the inclinations of the planes of their orbits observe not the limits of the Zodiac, as the planes of the orbits of the Moon and planets do; and, the motions of some of them are not according to the order of the signs.

Comets, like planets, move in ellipses, but, of such great eccentricity, that thence has arisen a ground of distinction, and Comets are said to *differ* from planets, because they move in orbits so eccentric. The eccentricities of those that have been observed have been found so great, that parabolas would nearly represent them.

What are called the elements of a Comet's orbit are less in number than those of a planet's, being only five. It is impossible from the observations made, during one appearance of a Comet, to compute the major axis of its orbit and its period, and consequently the area described by it in a given time: what Astronomers seek to compute, and what they with difficulty compute, are the perihelion distance; its place, or longitude; the epoch of that longitude; the longitude of the ascending node, and the inclination of the orbit.

The elements of the orbits of planets are capable of being determined by observations made on the meridian: by longitudes and latitudes computed from right ascensions and declinations. Comets, however, require observations of a different kind: by the rotation of the Earth they are brought on the meridian, but, (from their proximity to the Sun whilst they are visible,) not during the night, when alone the faintness of their light does not prevent them from being discerned. They must therefore be observed *out* of the meridian; and, in that position, the differences between their right ascensions and declinations and those of a known contiguous star must be determined.

It is difficult to make these latter observations with accuracy by reason of the doubtful and ill-defined disk of the Comet; and a small error in the observations will materially affect the elements of the orbit.

The rigorous solution of the problem of the elements of a Comet's orbit requires three observations only. But then the solution is attended with so many difficulties, that in this, as in other like cases, Astronomers have sought, by the indirect methods of trial and conjecture, to avoid them. If, (and this case always happens) more than three observations are obtained, the redundant ones are employed in correcting and confirming previous results.

The periodic time, as we have observed, cannot be determined from observations during one appearance of a Comet. If known, it can only be so, by recognising the Comet during its second appearance. And the only mode of recognising a Comet, is by the identity of the elements of its orbit with those of the orbit of a Comet already observed. If the perihelion distance, the positions of the perihelion and of the nodes, the inclination of the orbit, are the same or nearly so, we may presume, with considerable probability, that the Comet we are observing, has been previously in the vicinity of the Sun; and that, after moving round by the aphelion of its oval orbit, it has again returned towards its perihelion distances.

Comets not having been formerly observed with great accuracy, it so happens, that the period of one alone, that of the Comet observed in 1682, 1607, and 1531, is known to any degree of certainty. Its period is presumed to be about 76 years. Assuming the Earth's mean distance to be unity, the perihelion distance of the Comet was 0.58, and the major axis of the orbit 35.9. The inequalities which are noted in its period are supposed to arise from the influence of some disturbing forces *.

The chief business of the present Treatise, hitherto, has been with calculations founded on observations made on the meridian. But, there are many important processes dependent on angular distances observed *out* of the meridian: such, for instance, as those for ascertaining the latitude and longitude of a ship at sea. The nature of the observations, in these cases, requires a peculiar instrument; which, besides being adapted to the measuring of angular distances out of the meridian, may be held in the hand of the observer, and used by him, even when he becomes unsteady by the motion of the vessel. The description and use of such an instrument will be explained in the ensuing Chapter.

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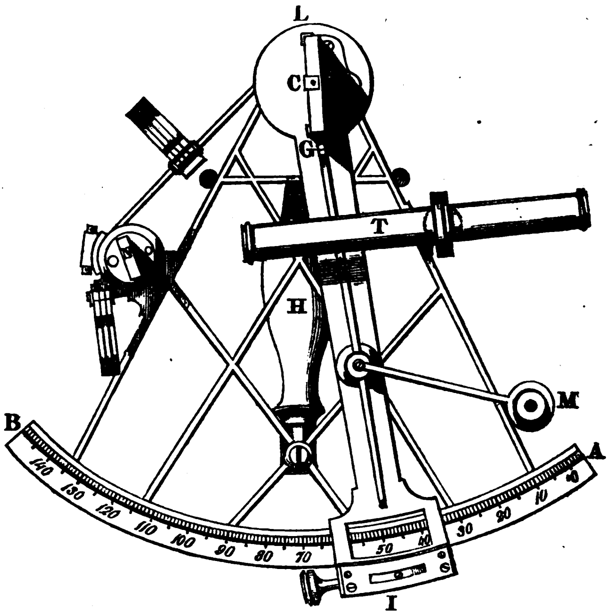
* On the subject of Comets, see Laplace, *Mec. Celeste*, Liv. II, p. 20, &c. Biot, tom. III, Add. p. 186, Englefield: Cagnoli, p. 429, Newton, *Arith. Univ.* Sect. 4, Chap. II, Prob. 30.

CHAP. XL.

ON THE APPLICATION OF ASTRONOMICAL ELEMENTS AND RESULTS, DEDUCED FROM MERIDIONAL OBSERVATIONS, TO OBSERVATIONS MADE OUT OF THE MERIDIAN.

On Hadley's Quadrant and the Sextant.

THE larger figure is intended to represent a *Sextant*, as it is usually fitted up, with its handle *H*, the telescope *T*, the micro-



scope *M* moveable about a centre, and capable of being adjusted so as to *read off* the divisions on the graduated limb *AB*. The less Figure is intended as a sketch of the larger and for the purpose of explaining its properties.

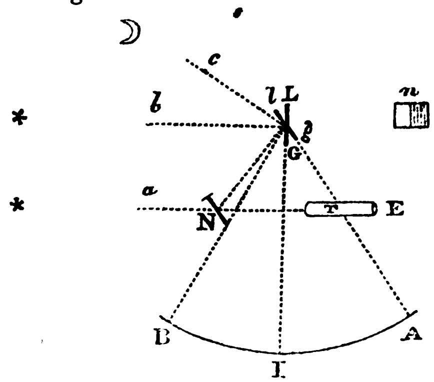
LCG and *N* (in the large Figure) must be supposed to represent the edges of two plane reflectors; the planes of which are perpendicular to the plane of the instrument in which the graduated limb and the connecting bars lie. The upper part of the reflector *N*, which is fixed, and called *the Horizon glass*, is transparent and free from quicksilver, as in *n* (in the small Figure) which is represented as *N* appears when viewed through the tube of the telescope *T*. The other reflector *LCG* (*the index glass*) is attached to the limb and index *I*, and with them moveable round a centre placed near *C*. Now, the instrument is so constructed that, when the index *I* is moved up to *A* and points to *o* on the graduated arc, the planes of the two reflectors *LCG* and *N* are exactly parallel to each other. In the small Figure, *lg* represents this position of *LG*.

In this position of the index *I* and the reflector *LG*, if the eye at *E* (small Figure) look through the upper part of the horizon glass at *N*, and perceive a distant object such as a star (✱), it will also perceive the image of the same star reflected from the under and silvered part of *N*. For, by hypothesis, the reflectors are parallel: and since the star is extremely distant, two rays from it (*a N*, *b g*) falling on *N* and *LG* must be parallel; therefore the latter ray, after two reflections, the first at *LG*, the second at *N*, must proceed towards the eye in the direction of *a N* produced.

Suppose now, the eye still looking through the telescope at the same object (✱), the index *I*, the limb *GI*, and with them the reflector *LCG*, to be moved from *A* towards *B* (*LGI* is their position in the small Figure); in this case the star ✱ can no longer be seen by two reflexions, but some other object such as the ☽ may: and if so, two objects, the ✱ and ☽, would be seen nearly in contact; the former in the upper part of the horizon glass *N*, the latter on the lower silvered part.

In consequence then of this translation of the index *I* from *A*, where it was opposite *o*, to another position between *A* and *B*;

two objects (✱ and ☽) inclined to each other at a certain angle (*bgc* in small Figure) are brought into contact. If, therefore, the arc moved through (*AI* in the small Figure) bore any relation to the angular distance of the two objects, and we could ascertain such relation, we should, in such case, be able by measuring *AI*, or by reading off its graduations, to determine the angular distances of the two observed objects. This relation we will proceed to investigate.



In the first position (*LG*) both the direct and the reflected rays from ✱ are seen in the direction of the telescope (*T*); the direct ray from ✱ is always seen in the same direction. But, in the new position, the reflected ray (in order that ☽ may be seen) must also be seen in that direction; therefore, the ray must come from the under part of *N* in the same direction: and therefore, since *N* is fixed, the ray must always be incident on *N* in the same direction, and consequently be *always reflected* from *LCG* in the same direction. What we have then, to determine is reduced to this. *To find the inclination of two incident rays, such, that the position of the reflector being changed (from* LG *to* lg *for instance,) each shall be reflected into the same direction.*

Let the first incident ray (and consequently the reflected ray) be inclined to the reflector at an angle $= A$: let the reflector be moved through an angle $= \theta$, and towards the reflected ray : (for instance, from the position *gl* to *GL* in the small Fig.), then the angle between the reflected ray and the plane in its new position $= A - \theta$ between the first incident ray and the plane . . . . . . $= A + \theta$.

But, by the laws of reflection, the second incident ray must form with the reflector, an angle equal to that which the reflected ray does; an angle, therefore, $= A - \theta$. Now, the difference between the angles which the incident rays form with the same position of the plane, is no other than the inclination of the incident rays, equal, therefore, to

$$(A + \theta) - (A - \theta), \text{ or, } 2\theta.$$

This is the important principle in the construction of the instrument. For, suppose the arc AB to be one-sixth part of a circle, and the index I, when the two objects are seen in contact, be one-third of the way between AB; then, the inclination of the two reflectors (for the reflector N is always parallel to the first position lg) would be one-third of one-sixth of 360^0 or 20^0: and, accordingly, the angular distances of the two objects would be 40^0. Instead of dividing AB into a number of degrees proportional to its magnitude (60^0 for instance, if $AB = \frac{1}{6}$th circumference), it is usual to divide it into *twice* that number. In such a graduation the number of degrees, minutes, &c. intercepted between *o* and the index will at once determine the angular distance of the two objects.

The objects must be brought into contact: in the case of a star and the Moon, the former must be made just to touch the limb of the latter: in the case of the Sun and Moon, their two limbs must be made to touch.

For the sake of illustration, we have supposed the two objects to be a star and the Moon: and, in practice, those are frequently the observed bodies. But, the instrument is capable of measuring the angular distance of any two objects lying in any plane: the Sun and Moon, for instance, and in such cases there are certain darkened glasses, near to N, and between N and L (see Fig.) contrived for the purpose of lowering the Sun's light to that of the Moon's, or the Moon's to that of a star's.

The uppermost and lowest points in the disks of the Sun, or of the Moon, may be considered as two objects; therefore, their distances, which are the diameters of the Sun and the Moon, may be measured by the described instrument. Instead of the points in the direction of a vertical circle, we may observe two opposite points in an horizontal direction : and, accordingly, we can measure the horizontal diameters of the Sun and Moon.

If we make a star, or the upper or the lower limb of the Sun or Moon, to be one object, and the point in the horizon directly beneath to be the other, we can measure their angular distance, which, in these cases, is either the *altitude* of the star, or the altitude either of the upper or the lower limb of the Sun and Moon. In this observation, the horizon is viewed through the upper part of the reflector *N*, which is the reason why that is called the *horizon*-glass. At sea, where the horizon is usually defined with sufficient accuracy, the altitude of the Sun or of a star may be taken, by the above method; but at land the inequalities of the Earth's surface oblige us to have recourse to a new expedient, in the contrivance of what is called an *Artificial*, sometimes a *False Horizon*. This, in its simplest state, is a basin either of water, or of quicksilver: to the image of the Sun or other object seen therein we must direct the telescope *T*, and view it directly through the upper part of *N*, and then move, backwards, or forwards the limb and index, till by the double reflexion, the upper or the under limb of the reflected Sun is brought into contact, or exactly made to touch the under or the upper limb of the image of the Sun seen in the *Artificial Horizon*. The angle shewn by the instrument is double either of the altitude of the Sun's upper or under limb: subtract or add the Sun's diameter, divide by two, and the result is the altitude of the Sun's centre: all other proper corrections, instrumental as well as theoretical, being supposed to be made.

It is evident from the preceding description, that the plane of the instrument must be held in the plane of the two bodies, the angular distance of which is required: in a vertical plane, therefore, when altitudes are measured; in an horizontal, when, for instance, the horizontal diameters of the Sun and Moon are to be taken. In the management of the instrument, this adjustment of its plane, or the holding it in the plane of the two bodies, is the most difficult part.

The instrument is to be held by the handle *H*, and generally is, in the left hand of the observer: his right being employed in moving and adjusting the index, its connected limb, and the reflector *LCG*. Its great and eminent advantage is, that it does not require to be fixed, nor that the observer using it should himself be steady. It is the chief instrument in Nautical Astronomy: since by its means alone, the position of a vessel at sea may be determined.

The instrument represented and described in this Chapter is, the *sextant:* which is an improvement on the *quadrant,* called, from its inventor, *Hadley's Quadrant* *. Besides these, on the same principle, but of better contrivance, is the *reflecting circle* †: also, Borda's *reflecting repeating* circle, on the principle of Mayer's. (See *Mem. Gottingen,* tom. II, also *Tabulæ Motuum,* &c. 1770).

We subjoin two instances of the uses of the sextant.

Angular Distance of the Sun's Centre, and of the Horizon (*at Sea,*) *or* (see p. 774,) *Altitude of the Sun's Centre.*

Alt. ⊙'s lower limb	49^0	$10'$	$0''$
(*a*) ⊙'s semidiameter	0	15	51
	49	25	51
‡ Refrac. (Chap. X.)	0	0	43
true alt. ⊙'s centre	49	25	8

Distance of eastern and western limbs, or ⊙'s horizontal diameter	$31'$ $42''$
(*a*) ⊙'s semi-diameter	15 51

Altitude of the ⊙*'s Centre, by means of the Artificial Horizon,* (see p. 774,)

By inst. ⊙'s upper limb	100^0	$2'$	$47''$
Apparent altitude	50	1	23.5
(*b*) ⊙'s semi-diameter	0	15	50
	49	45	33.5
Refraction	0	0	43
True alt. ⊙'s centre	49	44	50.5
⊙'s horizontal diameter		$31'$	$40''$
(*b*) ⊙'s semi-diameter		15	50

* Described in the *Phil. Trans.* Year 1738, No. 420, p. 147.

† Invented by Mr. Troughton: for a description of it, see Rees' *Encyclopedia,* new edit. Art. *Circle.*

‡ The Nautical Tables of Refraction include within their results the correction for the Sun's parallax.

The sextant (using that as the generic name of like instruments) is, as it has appeared, a secondary instrument, but capable of performing, in an imperfect degree indeed, several astronomical operations. It measures, and generally, angular distance. It affords us, therefore, the means of determining the latitude of a place, from the meridional altitude of the Sun or a star, since such *meridional* altitude is the angular distance of the horizon and star when on the meridian. From two observed altitudes, one of which is meridional, and the declination of the observed body, we are able, by computation, to determine the *time* of the other observed altitude. From the same data the azimuth of the observed body may be determined. By means of the observed distance, between a star and the Moon, we derive a method (a thing hereafter to be explained) of determining the *longitude* of a place. So that, as it has been said, the sextant is itself and alone a sort of portable Observatory, capable of performing many astronomical operations, but all imperfectly. This would naturally be expected on the ground, that an instrument of general uses cannot be excellent when employed in special ones.

The succeeding Chapter will contain several methods adapted to the uses of the sextant, and to the uses of instruments performing like operations.

CHAP. XLI.

On the Mode of computing Time and the Hour of the Day; by the Sun; by the Transit of Stars; by equal Altitudes; by the Altitude of the Sun or of a Star.

We will preface the methods that ought to be considered, perhaps, as the special objects of this Chapter, with some that are adapted to observations made on the meridian.

Transit of the Sun over the Meridian.

When the Sun's centre is on the meridian, it is *true* or *apparent* noon. It can be determined to be there, by means of a transit instrument. With this, observing the contacts of the Sun's western and eastern limbs with the middle vertical wire, note, by means of the clock, the interval of time, and half that interval added to the time of the contact of the western, or subtracted from that of the eastern, will give the time at which the Sun's centre is on the meridian. For greater accuracy, the times of contact of the Sun's limbs with the vertical wires to the right and left of the middle one may be noted, (see pages 96, &c.)

The time thus determined is *apparent* noon; in order to deduce the *mean* time, apply the *Equation of time*, (see Chap. XXII.). For instance, the *equation* on Nov. 8, 1808, is stated in the Nautical Almanack to be $-$ $16^m\ 3^s.7$, therefore, when the Sun's centre on that day was on the meridian, the mean solar time was $12^h - 16^m\ 3^s.7$, or $11^h\ 43^m\ 56^s.3$; 12^h being supposed to denote the time when the centre of the *mean* Sun is on the meridian.

Transit of a fixed Star; of the Moon; of a Planet over the Meridian.

The mean Sun leaves a meridian and returns to the same in 24^h, describing $360° 59' 8''.3$; $59' 8''.3$ being the increase of its mean right ascension in that time. Since the mean Sun, by its definition, moves equably, the time from mean noon must be always proportional to the Sun's distance from the meridian. If a star, then, were on the meridian, the time would be proportional to the Sun's angular distance from the star; it would be proportional, therefore, to the difference of the right ascensions of the star and the Sun, at the time when the star is on the meridian.

The Sun's right ascension in the Nautical Almanack is expressed solely for noon, that is, when the Sun's centre shall be on the meridian of Greenwich; and such right ascension continually increasing, will be greater when the star comes on the meridian, and the Sun is more to the west, than it was at noon. In the interval between the transits of the Sun and star, the former will have moved to the east, and towards the latter, by an increase of right ascension proportional to the interval. The angular distance therefore of the star and Sun, or the difference of their right ascensions, when the former is on the meridian, is

$\ast$'s Æ − ⊙'s Æ (*at preceding noon*) − increase of ⊙'s Æ,

and to this angular distance is the time proportional.

The time from noon is nearly proportional to the $\ast$'s right ascension − ⊙'s right ascension at noon; therefore, the increase of ⊙'s right ascension is nearly proportional to that angle. If a therefore denote the increase of the Sun's right ascension in 24^h, we have the time =

$$\ast\text{'s Æ} - \odot\text{'s Æ} - \frac{D}{24} \times a,$$

(making $D = \ast$'s Æ − ⊙'s Æ.)

EXAMPLE.

A Star in Capricorn *whose* Æ $= 20^h\ 30^m\ 7^s$ *was on the Meridian at Greenwich, Nov.* 8, 1808. *Required the time.*

✱'s Æ	20^h	30^m	7^s
By Naut. Alm. ☉'s Æ (noon of Nov. 8.) ..	14	53	52 *
✱'s Æ — ☉'s Æ (D)...............	5	36	15
☉'s Æ Nov. 9............	14	57	53.5
8.............	14	53	52
a=...........	0	4	1.5

* The Sun's right ascension is expressed in time, the Moon's in degrees, and to be expressed in the hours, minutes, &c. of *sidereal time*, must be converted into such at the rate of 15° for 1^h; for $\frac{24}{360} = \frac{1}{15}$.

For facilitating this operation and its reverse, appropriate Tables are provided; but, it may be, nearly with as much ease, effected by dividing and multiplying by 4. Thus, to convert $7^h\ 21^m\ 56^s.21 = 7^h\ 21^m\ 56^s\ 12'''$ into degrees, &c. begin with the minutes, and take the fourth of them, then, of the seconds, &c. reckoning the minutes of the quotient as degrees, the seconds as minutes, &c. thus:

$$\begin{array}{rccc} & 4)21^m & 56^s & 12''' \\ \hline & 5^\circ & 29' & 3'' \\ \text{But } 7^h = & 105 & & \\ \hline & 110 & 29 & 3 \end{array}$$

For the reverse operation, multiply by 4, reckoning the seconds of the product as thirds, the minutes as seconds, &c.

$$\begin{array}{lcl} \text{Thus} \ldots\ldots\ldots\ldots & 36^\circ\ 8'\ 34''\ 30''' & \ldots\ldots\ldots\ldots (36^\circ = 30 + 6 = 2^h + 6^\circ) \\ & 4 & \\ \hline & 2^h\ 24^m\ 34^s\ 18'''\ 0 & \end{array}$$

or dividing 18''' by 6 to reduce it to a decimal, the product is $2^h\ 24^m\ 34^s.3$.

The reasons of the two operations are these; in the first we ought to multiply by 15, or, which is the same thing, by $\frac{60}{4}$; therefore we may divide by 4 and dispense with the multiplication by 60, by merely *raising the denomination* of the quotient; for $60 \times 1'' = 1'$. In the second case, we ought to *divide* by 15, or which is the same thing

$\therefore$ apparent time $=$

$$5^h\ 36^m\ 15^s - \frac{5^h\ 36^m\ 15^s}{24^h} \times 4^m\ 1^s.5 = 5^h\ 35^m\ 19^s.3,$$

and the mean time $=$

$$5^h\ 35^m\ 19^s.3 - 16^m\ 2^s \text{ (the equation of time)} = 5^h\ 19^m\ 17^s.3.$$

Since the increase of the Sun's mean Æ is $59'\ 8''.3$ in 24 hours, a meridian of the Earth describes, in that time, $360^0\,59'\,8''.3$; therefore, it describes 360^0 in $24^h \times \dfrac{360^0}{360^0\,59'\,8''.3}$, or in $23^h\ 56^m\ 4^s.09$. This is the time of the Earth's rotation, or the length of a sidereal day, expressed in mean solar time. If the Sun, therefore, and a Star were together on the meridian on a certain day, on the succeeding one, the Star would return sooner, or more quickly, to the meridian by $3^m\ 55^s.9$ of mean solar time. On this account, stars are said to be *accelerated.* The *acceleration* on mean solar time, therefore, when the Star and Sun are distant by 360^0, or by 24 *of sidereal time,* is $3^m\ 55^s.909$; when distant by 180^o, or by 12^h *of sidereal time,* it is $1^m\ 57^s.955$; when distant by 60^o, or 4^h, it is $39^s.388$, and generally the *acceleration* is

$$\frac{\ast\text{'s Æ} - \odot\text{'s Æ}}{24^h} \times 3^m\ 55^s.909\,*.$$

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thing, we may *multiply* by $\frac{1}{15}$ or $\frac{4}{60}$; therefore, we may multiply solely by 4, and dispense with the division by 60 by merely *lowering the denomination* of the product; for $\frac{1'}{60} = 1''$.

* Twenty-four sidereal hours $= 23^h\ 56^m\ 4^s.092$ of mean solar time, and, $23^h\ 56^m\ 4^s.092$ $(= 23^h.93447) : 24 : 24 : 24^h.065709$, in other words,

24 mean solar hours $= 24^h\ 3^m\ 56^s.55$ of sidereal time.

Now *acceleration* for 24^h is	....	3^m	$55^s.909$
3^m	....	0	0.491
56^s	....	0	0.153
.55	....	0	0.15
		3	56.558

and $3^m\ 56^s.55$ deduced from $24^h\ 3^m\ 56.55$ leaves 24^h of solar time, as it ought to do.

This is only another mode of expressing the rule given in p. 778; instead of the increase of the Sun's mean right ascension, in 24 hours of mean solar time, we there took the real increase between two apparent noons.

There are* Tables constructed for the *Acceleration of stars on mean solar time,* which render the computation of the hour, by means of the transit of a fixed star, very easy; the rule is,

the time = ✱'s Æ — ☉'s Æ — acceleration.

Thus, in the former instance,

✱'s Æ	20^h	30^m	7^s
Nov. 8. ☉'s mean Æ..........	15	9	57.3
	5	20	9.7
Acceleration....................	0	0	52.3
Mean time	5	19	17.4

The right ascensions of the Sun and of the stars, are always expressed in sidereal time; and care must be taken to distinguish the hours, minutes, &c. of that time, from the hours, minutes, &c. of mean solar time. If we subtract, from an angle expressed in the symbols of sidereal time, the *acceleration,* the remainder is expressed in mean solar time. Thus,

A star is to the east of the meridian $30^0\ 30'$, or	2^h	$2'$	$0''$
The acceleration, or the Sun's motion in $2^h\ 2'$ ..	0	0	19.99
	2	1	40.01

therefore in $2^h\ 1^m\ 40^s.01$ of mean solar time, the star will be on the meridian.

The time is proportional to a *less* angle than the difference of the right ascension of the star and the Sun; or, stars are *accelerated,* because the Sun, in the interval between his transit and that of the star, moves towards the latter. In the case of the Moon then, the time must be proportional to a *greater* angle than the difference of the Sun's right ascension on the preceding noon, and the Moon's; or, the Moon must be *retarded;* because, in

* Zach's Tables *d'Aberration,* &c. Tab. XXVI.

the interval between the transit of the Sun and that of the Moon, the latter, by its greater motion in right ascension, has increased its angular distance from the former. It would be easy, as in the former case, to compute the hour from the Moon's transit over the meridian, (or what is the same thing, to find the hour of the Moon's transit), but instead of it, we will give a formula applicable to all cases:

Let the increment of ☉'s Æ in 24^h be . . . . a
of a ✱, or of the ☽, or of a planet . . . . A.

Let also the difference between the right ascension of the heavenly body and that of the Sun at *the preceding noon,* expressed in sidereal time, be . . . . . t

then, if $a = A$, the hour of transit will be proportional to t
if $a > A$, ... to some less angle . . $t - \tau$
if $a <$ . . . . to some greater . . . $t + \tau$.

Hence in the first case, which can only happen with a planet, the time of transit is proportional to t; that is, if the Sun's right ascension when on the meridian be $30^0\ 30'$, or $2^h\ 2^m$, less than that of the planet, the latter will be on the meridian at $2^h\ 2^m$ of solar time.

In the second case, $a > A$

$$24 : a - A :: t - \tau : \tau; \ \therefore \tau = t \times \frac{a - A}{24 + a - A}.$$

In the third case $a < A$

$$24 : A - a :: t + \tau : \tau; \ \therefore \tau = t \times \frac{A - a}{24 + a - A}.$$

Hence, in the second case, the time of transit $= t - t \times \dfrac{a - A}{24 + a - A}$

in the third, $= t + t \times \dfrac{A - a}{24 + a - A}$, or, $t - t \times \dfrac{a - A}{24 + a - A}$

therefore, in both cases,

$$\text{the time of transit} = t\left(1 - \frac{a - A}{24 + a - A}\right)$$

$$\text{(expanding)} = t\left\{1 - \frac{a - A}{24} + \left(\frac{a - A}{24}\right)^2 - \left(\frac{a - A}{24}\right)^3 + \&c.\right\}$$

Hence in the case of a fixed star, when $A = 0$, the time of

$$*\text{'s transit} = t - \frac{at}{24} + \left(\frac{a}{24}\right)^2 t - \&c.$$

in which the two first terms (which are sufficient) give the rule of computation used in p. 778, l. 28.

In the case of the Moon, $a = A$; therefore all the terms are additive, and

$$\text{the time of ☽'s transit} = t + \frac{A-a}{24} t + \left(\frac{A-a}{24}\right)^2 t + \&c.$$

In the case of a planet, a may be less or greater than A; if equal, then the time of transit $= t$, as before, p. 782, l. 13.

There is one case which has not been mentioned, that in which a planet is *retrograde* (see Chap. XXIII.). In this case, the approach of the Sun and planet is greater than that of the Sun and a star, and the same approach, as if, instead of the Sun having a motion in right ascension equal to a, we suppose it endowed with a motion equal to $a + A$; substituting therefore in the form, p. 783, l. 29, $a + A$ instead of a time of the planet's

$$\text{transit} = t - \frac{a+A}{24} . t + \left(\frac{a+A}{24}\right)^2 . t - \&c.$$

When the planet is *stationary*, its hour of passage is evidently the same as that of a fixed star which has the same right ascension.

Example.

Let it be required to find the time of the Moon's passing the Meridian of Greenwich, June 13, 1791.

June 14, ☽'s Æ . .	15^h	43^m	32^s	☉'s Æ . .	5^h	30^m	38^s
13, ditto	14	42	32	ditto	5	26	29.1
	1	1	$0 = A$		0	4	$8.9 = a$

June 13, ☽'s Æ . .	14	42	32	A	1	1	0
☉'s Æ . .	5	26	29.1	a	0	4	8.9
	9	16	$2.9 = t$		0	56	$51.1 = A - a$

$$\therefore t \ldots\ldots 9^h\ 16^m\ 2^s.9 = \begin{cases}\text{approx. time}\\ \text{of ☽'s transit.}\end{cases}$$

$$t\,.\frac{A-a}{24},\ \text{or}\ \frac{9^h\ 16^m\ 2^s.9}{24} \times 56^m\ 51^s.1 \quad 0\ \ 21\ \ 57$$

$$9\ \ 37\ \ 59.9 = \begin{cases}\text{more cor-}\\ \text{rect time.}\end{cases}$$

$$t\left(\frac{A-a}{24}\right)^2 \ldots\ldots\ldots\ldots \quad 0\ \ \ 0\ 49.8$$

$$9\ \ 38\ \ 49.7 = \begin{cases}\text{still more}\\ \text{corr}^{\text{t}}\text{.time.}\end{cases}$$

This last result (in apparent time) is sufficiently exact for Astronomical purposes *.

The second additional term $21^m\ 54^s.7 = \dfrac{9^h\ 16^m\ 29^s}{24^h} \times 56^m\ 51^s.1$, is evidently the *proportional part* † of $56^m\ 51^s.1$, corresponding to $9^h\ 16^m\ 29^s$; the third additional term, $49^s.8$, =

$$\left(\frac{A-a}{24}\right)^2.t = \frac{A-a}{24} \times \frac{A-a}{24}.t = \frac{21^m\ 54^s.7}{24^h} \times (A-a)$$

$= \dfrac{21^m\ 54^s.7}{24^h} \times 56^m\ 51^s.1$ is evidently the *proportional part* of $56^m\ 51^s.1$, corresponding to the time $21^m\ 54^s.7$. This is the explanation of the rule, as it is sometimes given by Astronomers, which directs us to find a first, and a second proportional, and to add them to the approximate time of the Moon's transit, in order to find a more correct time. (See Nautical Almanack, 1811, pp. 154, 155. Also Wollaston's *Fasciculus*, Appendix, p. 76.)

The hour, or the mean solar time, may be determined or computed from the transit of a fixed star; and, much more exactly, than from the transit of the Moon or of a planet. With regard therefore to these two latter, the object of the preceding methods

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* See in pp. 702, 705, &c. the time of the Moon's transit, found from the observed sidereal time of the transit of its limb.

† Tables are computed for facilitating these operations.

is to determine from Astronomical Tables, the times of their transits, or passages over the meridian, rather than the hour of the day from the transits.

Time determined by the Sidereal Clock.

If we can determine the time from the transit of a fixed star, it is an immediate inference that we can determine it from the sidereal clock. For, the clock is regulated by the observed transits of stars, and when regulated, we may suppose it always to indicate the right ascension of some imaginary star: Thus,

July 1, 1790, time by sidereal clock	13^h	20^m	15^s
☉'s mean longitude (by Tables)	6	54	35.86
	6	25	39.14
* *Acceleration* (Maskelyne, Tab. XXI.)......	0	1	3.1
Mean solar time	6	24	36.04

The preceding computations of transits †, &c. have been made for Greenwich, for which place our Astronomical Tables, and the Nautical Almanack are constructed. For any other place, we must account for the difference of longitude. Thus, to find, on July 9, 1808, the Sun's right ascension at noon, at a place 35^0 (2^h 20^m) east of Greenwich, we have only to find the Sun's right ascension 2^h 20^m previous to noon time at Greenwich: which is easily done by subtracting from the right ascension at noon the proportional increase of right ascension in 2^h 20^m: thus,

July 10, ... ☉'s Æ	7^h	17^m	$48^s.5$
9, ... ditto	7	13	43.2
Increase of Æ in 24^h		4	5.3
Proportional increase in 2^h 20^m = ...		0	33

∴ Sun's Æ, at noon, at the required place, = 7^h 17^m $15^s.5$.

* The *Acceleration* is the Sun's mean motion in right ascension, and by this latter title it is called by Maskelyne in the Table referred to. See Wollaston's *Fasciculus*, Appendix, p 69.

† See another Example in pp. 705, 706, &c.

A similar method must be used to find the Moon's right ascension, or longitude, &c. at noon, at any given place, with this difference, however, that the change of right ascension will not be simply proportional to the time, but must be computed more exactly by the differential method and series $\left(a + x\,d' + x \cdot \frac{x-1}{2} d'' + \&c.\right)$ See *Trigonometry*, p. 259, also pp. 706, &c. of this Work.

We now proceed to the methods of determining the time, by observations made out of the meridian.

The Method of equal, or of corresponding, Altitudes.

The principle of the method is this: before noon, if the Sun be the body to be observed, note its altitude and the time, and wait till the Sun, in the afternoon, descends to an equal altitude; half the time elapsed between the two observations is, nearly, the distance of each observation from noon.

The same process is to be used with a star or planet: half the sum of the times between two equal altitudes observed, respectively, in the east and west, is, in time, the star's passage of the meridian; exactly the passage of the star, very nearly that of the planet.

The sole condition respecting altitudes mentioned in the preceding description is their equality. The corresponding altitudes, therefore, may be taken at any distance from the meridian. Hence, if we had ten altitudes in the east, and ten corresponding ones in the west, half the sum of the times for each pair would be the star's passage over the meridian: and, accordingly, one-twentieth of the sum of the times would be the *mean* time of it.

In this operation, as before when only one pair of altitudes is employed, the result is only *nearly* true, if the observed body be the Sun or a planet: since, in either case, the declination is changed during the interval of the observations.

With regard to the instruments necessary to the above operations, a sextant may be used, in default of better instruments, or when, as would be the case at sea, fixed instruments cannot be used. But the *better* instruments are astronomical quadrants, (see pp. 58, &c.) declination circles, repeating circles, or any of that class which are furnished with movements in azimuth, and will serve as *equal altitude* instruments. With any instrument of such sort, properly adjusted, *clamp* the telescope at a certain graduation of the limb of the instrument, and a little above what, probably, may then be the star's altitude, (the star being supposed to be in the east). Turn the instrument towards the star, and note the time when it passes through the middle point of the horizontal wire, in the field of the telescope (the point *a* in the figure of p. 58.). Note also the time when the star, after having passed the meridian, descends to (*a*), the middle point of the horizontal wire. Half the interval, as it has been already said, is the sidereal time of the star's passing the meridian. But in order to procure a *mean* result (see p. 786,) repeat the first operation (l. 6, &c.) after the telescope shall have been elevated through a certain number of graduations, 20′ for instance. The second observation being made, make a third, fourth, &c. the telescope, at each, being raised through 20′. When the star shall have passed the meridian, go through the same operations, but in an inverse order. For instance, Lacaille who constantly deduced his time from *corresponding altitudes,* made the following observations of the star Arcturus.

Altitudes.	Times East and West.	Sum of Times.	Times of Transit.
43° 10′	$10^h\ 55^m\ 47^s$ 17 11 55.5	$28^h\ 7^m\ 42^s.5$	$14^h\ 3^m\ 51^s.25$
43 30	10 57 57 17 9 45.5	28 7 42.5	51.25
43 50	11 0 7.5 17 7 35	28 7 42.5	51.25
44 10	11 2 18.5 17 5 24.5	28 7 43	51.5
		112 30 50.5	
		14 3 51.31	

Here the least hour-angle from one pair of observations is $14^h\ 3^m\ 51^s.25$, the greatest $14^h\ 3^m\ 51^s.5$, and the mean of 4 pairs of observations is $14^h\ 3^m\ 51^s.31$.

If the telescope of the instrument be furnished with a micrometer, having a wire moveable but always preserving its parallelism to the horizontal wire (to *hf* in the figure of p. 58,) two observations may be made at each position of the telescope, one when the star is bisected by the moveable wire, the other when it is bisected by the horizontal. The object of this is to procure a greater number of results, in order to deduce a truer *mean* result.

The following Table, from Lacaille, contains the observations made with the horizontal wire, and the subsidiary observations made with the moveable one.

Altitudes.	Star's Time in the East.	Star's Time in the West.	Sums of Times.
$43^0\ 10'$	$10^h\ 55^m\ 47^s$	$17^h\ 11^m\ 55^s.5$	$28^h\ 7^m\ 42^s.5$
	51.5	50.5	42
30	57 57	9 45.5	42.5
	58 2	40.5	42.5
50	11 0 7.5.	7 35	42.5
	12	30	42
44 30	2 18.5	5 24.5	43
	23	20	43

mean . . . . $28^h\ 7^m\ 42^s.5$

sidereal time of star's passing the meridian 14 3 51.25.

Here the mean time of the star's passage over the meridian, is $14^h\ 3^m\ 51^s.25$, instead of $14^h\ 3^m\ 51^s.31$ as it was in p. 787.

If we examine the preceding Table, the greatest time of transit from a single pair of observations is, (regarding only the seconds,) $51^s.5$, the least $51^s.0$. Lacaille, therefore, could rely on determining, by his method and with his instrument, the time of the star's transit to within a quarter of a second.

In the preceding illustration the star Arcturus was the body observed. Should the Sun or a planet be the object, then instead of noting the time of bisection, as it is called, we must note the time of contact of the upper or lower limb with the horizontal wire. But this is not the only circumstance of difference. The Rule itself (see p. 786.) must be altered, since, from the change of declination during the observations of two corresponding altitudes, half the sum of times cannot be exactly the sidereal time of the Sun's, or planet's passage of the meridian.

This point is easily explained. Suppose the Sun's north declination to be increasing. In such a case the Sun, after passing the meridian, will be *longer* in descending to the *corresponding* altitude in the west, than it was in ascending from the eastern altitude to the meridian. Half the interval, therefore, would have the effect of throwing the meridian too much to the west, or, of retarding the time of transit. What remains then is to investigate a correction of the time dependent on the change of declination.

In a triangle ZPS, where Z is the zenith, P the pole, S the Sun, the angle ZPS measures the time $\left(\frac{t}{2}\right)$ from noon, and by *Trigonometry*, p. 139,

$$\cos.\frac{t}{2}\times\sin.ZP\times\sin.PS=\cos.ZS-\cos.ZP\times\cos.PS.$$

Now, $\frac{t}{2}$ being the exact time from noon, if PS remain constant, we have to ascertain the variation in $\frac{t}{2}$, from the variation in PS: for that purpose, it will be sufficient to deduce the proportion between the *differentials* or *fluxions* of these quantities; accordingly, taking the differential of the above equation,

$$-\frac{dt}{2}.\sin.\frac{t}{2}.\sin.ZP\sin.PS+d\,(PS)\cos.PS\cos.\frac{t}{2}.\sin.ZP=$$
$$d.(PS).\sin.PS\cos.ZP,$$

or putting $\frac{dt}{2}=\epsilon,\ d\,(PS)=\delta$, and reducing,

$$\epsilon=\delta\left(\tan.\text{decl}^{n}.\times\cot.\frac{t}{2}-\tan.\text{lat.}\times\text{cosec.}\frac{t}{2}\right).$$

$$\text{or} = \frac{\delta}{\sin. \frac{t}{2}} \left(\text{tan. decl}^{\text{n}}. \times \cos. \frac{t}{2} - \text{tan. lat.}\right)$$

$$= \frac{\delta}{\sin. \frac{t}{2}} \left(\text{tan. lat.} - \text{tan. decl}^{\text{n}}. \times \cos. \frac{t}{2}\right),$$

if the declination, during the observations, should decrease.

As this operation of corresponding altitudes is an useful one, and of frequent occurrence, M. Zach has enabled us (see Nouvelles *Tables d' Aberration*, &c. pp. 29, &c.) to compute the correction e by means of two Tables. The two Tables are constructed from the above formula thus modified. Let H be the latitude, D the Sun's declination, and let δ, instead of denoting the change of declination during half the interval of the observations, denote the daily change: instead of δ, therefore, we must write $\frac{\delta}{24} \times \frac{t}{2}$. If also $\frac{t}{2}$ is to be expressed in hours and parts of an hour, we must write $\sin 15^{\circ} \times \frac{t}{2}$, instead of $\sin. \frac{t}{2}$, &c. So that e, expressed in time, becomes

$$e = \frac{\delta}{360^{\circ}} \times \frac{\frac{t}{2}}{\sin. 15^{\circ} \times \frac{t}{2}} \left(\text{tan. } H - \text{tan. } D \,.\, \cos. 15^{\circ} \,.\, \frac{t}{2}\right)$$

$$= \frac{\delta}{360^{\circ} \,.\, \sin. 15^{\circ}} \cdot \frac{\sin. 15^{\circ}}{\sin. 15^{\circ} \,.\, \frac{t}{2}} \cdot \frac{t}{2} \,.\, \text{tan. } H$$

$$- \frac{\delta \text{ tan. } D}{36 \,.\, \text{tan. } 150^{\circ}} \cdot \frac{\text{tan. } 150^{\circ}}{10 \,.\, \text{tan. } 15^{\circ} \,.\, \frac{t}{2}} \cdot \frac{t}{2},$$

$$\text{make } a = \frac{\delta}{360^{\circ} \,.\, \sin. 15^{\circ}},$$

$$\tan. a = \frac{\sin. 15^0}{\sin. 15^0 . \frac{t}{2}} . \frac{t}{2},$$

$$b = -\frac{\delta \tan. D}{36 . \tan. 150^0},$$

$$\tan. \beta = \frac{\tan. 150^0}{10 . \tan. 15^0 . \frac{t}{2}} . \frac{t}{2},$$

and

$$\epsilon = a \tan. a \tan. H + b . \tan. \beta.$$

Here a, β depending on $\frac{t}{2}$ (half the interval of the observations) are taken from the same Table (Tab. XVIII.) the argument of which Table is $\frac{t}{2}$, and a and b depending on the Sun's declination (and, therefore, on the Sun's longitude) are taken from a second Table (Tab. XIX.) the argument of which is the Sun's true longitude.

Thus, suppose with a sextant we took a double altitude

$(76^0\ 50')$ at $9^h\ 47^m\ 50^s$ A. M.

and 3 0 14.5 P. M.

then since 2 12 10

is the distance of the first observation from noon,

$\frac{1}{2}(5^h\ 12^m\ 14^s.5)$

or 2 36 7.25

is half the interval $\left(\frac{t}{2}\right)$ of the observations; entering then Tab. XVIII. with the argument $2^h\ 36^m\ 7^s.25$, we obtain

$$a = 46^0\ 55'\ 16'',$$
$$\beta = 10\ \ 30\ \ 5,$$

and entering Tab. XIX. with $5^s\ 4^0\ 33'\ 55''$, which, nearly, is the Sun's longitude for August 28th, 1822, we have

$$a = 13''.726,$$
$$b = 10 . 295.$$

Hence, Falmouth being the place of observation (the latitude of which is $50^\circ\ 8'$), we have

log. tan. $46^\circ\ 55'\ 16''$	10.0292440		
log. tan. 50 8	10.0782398		
log. $13''.726$	1.1375440		
	1.2450278	.. No. +	$17''.58$
log. tan. $10^\circ\ 30'\ 5''$	9.2679669		
log. $10''.295$	1.0126264		
	1.2805933	.. No. —	1 .908
			15 .67

This $(+\ 15''.67)$ then is the correction to be added to $\frac{1}{2}(9^h\ 47^m\ 50^s + 15^h\ 0^m\ 4^s.5)$, or $12^h\ 23^m\ 57^s.25$, in order to have the time of apparent noon, which accordingly is

$$12^h\ 24^m\ 12^s.92.$$

This is the result from one pair of corresponding altitudes: but, as soon as one observation is made, preparation is made for another by advancing (see p. 787,) the limb of the telescope on the limb of the instrument, 10 or 20 minutes: for instance, in the example from which the above times were taken, the second double altitude was 77°, and the times before and after noon were, respectively,

(see p. 791,)................	9^h	48^m	$31^s.5$
and (adding 12^h).............	14	59	24 .5
the half, or time from noon	12	23	58
the correction computed as above ..		+	15 .67
∴ the time from noon...........	12	24	13 .67.

As in the case of the observed times of the corresponding altitudes of a star, the *mean* of all the results is to be taken as the true result. All the observations are subjoined.

Place, Falmouth: Time, August 28, 1822.				
Double Altitudes.	Times A. M.	Times P. M.	Corrections.	Times of Apparent Noon.
76^0 $56'$	9^h 47^m 50^s	3^h 0^m $4^s.5$	$15''.67$	12^h 24^m $12^s.93$
77 0	48 31.5	2 59 24.5	.67	13.67
10	49 10.5	58 45.5	.65	13.65
20	49 49.4	58 5.5	.64	13.09
30	50 30.6	57 26	.62	13.92
40	51 10.5	56 47	.60	14.50
50	51 50.2	56 6.5	.58	13.93
78 0	52 31	55 25	.56	13.56
5	52 50	55 6	.55	13.55
10	53 11	54 46.5	.54	14.29
80 0	10 0 41.2	47 16	.40	14.0
10	1 23.2	46 34	.39	13.99
20	2 4.4	45 51	.38	13.08
40	3 28.8	44 30	.33	14.73
50	4 10.5	43 47	.31	14.06
81 0	4 52.8	43 4.6	.32	13.72
		Mean		12^h 24^m $13^s.792$

We have given instances of a star and the Sun: the method will also apply, with equal facility, to a planet. The second Table (XIX.) of M. Zach cannot indeed be used because its argument is the Sun's longitude, but it is easy to dispense with it by computing the change of the planet's declination in 24 hours.

Thus,

$$\epsilon = \frac{\delta}{360^0 . \sin. 15^\circ} \tan. a . \tan. H - \frac{\delta}{36 . \tan. 150^0} \tan. D . \tan. \beta,$$

in which ϵ can be computed, if δ be known.

Example.

April 8, 1809, Mars was observed at Florence, and the following were the conditions:

latitude of Florence, or	H =	43° 46′ 40″
south declination of Mars, or	D =	5 9 40
diurnal change of declination, or	δ	+ 6 38
half the interval of observation, or	$\frac{t}{2}$	$4^h\ 10^m$.

Hence,

log. 398″ (= 6′ 38″)........ =	2.5998831	
log. $\frac{1}{360 \,.\, \sin.\ 15^0}$	8.0307013	
log. tan. 50° 33′ 40″ (α)	0.0848395	
log. tan. 43 46 40 (H)........	9.9814658	
	0.6968897	No. 4″.97.

Again,

log. 398″	2.5998831	
log. $\frac{1}{36 \,.\, \tan.\ 30^0}$	8.6822581	
log. tan. 7° 8′ 16″ (β)...........	9.0976954	
log. tan. 5 9 40 (D)...........	8.9557974	
	9.3356340	No. − 0″.22
	the correction	4.75.

Since the change of the Sun's declination may be had from the Nautical Almanack, a calculation, exactly similar to the preceding, will apply to the corresponding altitudes of the Sun, and be equally simple with the one of p. 791, from which, indeed, it does not much differ.

The above method of determining the time from *corresponding altitudes* is the best of all methods, when we are not provided with a fixed and adjusted transit instrument. It is, as M. Zach observes, capable of great exactness, and is independent of the rectification of the instrument. It requires the aid solely of a chronometer, sufficiently good to mark the times during an interval of 5 or 6 hours. Those astronomical elements, such as the latitude of the place, the altitude of a star, its right ascension, &c. which are requisite to be known in the following methods, need not be known in this.

Time determined from an observed Altitude of the Sun.

The altitude of the Sun is to be observed and corrected as it was in page 775; then, we have to find the angle ZPS (h), from ZS ($90^0 - A$) thus determined, from the Sun's north polar distance (p) given by the Tables, and from the latitude (L) of the place, known or previously determined by observation. Then by *Trig.* pp. 139, &c. making $h = ZPS$, we have cos. h

$$= \frac{\cos. ZS - \cos. ZP \times \cos. PS}{\sin. ZP \,.\, \sin. PS} = \frac{\sin. A - \sin. L \,.\, \cos. p}{\cos. L \,.\, \sin. p};$$

$$\therefore 2 \,.\, \sin.^2 \frac{h}{2} = 1 - \cos. h = \frac{\cos. L \,.\, \sin. p + \sin. L \,.\, \cos. p - \sin. A}{\cos. L \,.\, \sin. p}$$

$$= \frac{\sin. (p + L) - \sin. A}{\cos. L \,.\, \sin. p}$$

$$= \frac{2}{\cos. L \,.\, \sin. p} [\cos. \tfrac{1}{2}(p + L + A) \sin. \tfrac{1}{2}(p + L - A)],$$

and, in logarithms, $\quad 2 \log. \sin. \frac{h}{2} = 20 +$

$$\log. \cos. \tfrac{1}{2}(p + L + A) + \log. \sin. \tfrac{1}{2}(p + L - A) - \log. \cos. L - \log. \sin. p.$$

EXAMPLE.

The Sun's Altitude being 39° 5′ 28″; *his North Polar Distance, from Nautical Almanack,* 74° 51′ 50″, *and the Latitude of Place,* 52° 12′ 42″; *it is required to deduce the Time.*

$$
\begin{array}{llll}
 & L = 52^\circ\ 12'\ 42'' \ldots\ldots \cos. & = 9.7872806 & \\
 & p = 74\ \ 41\ \ 50\ \ldots\ldots \sin. & = 9.9846660 & \\
 & A = 39\ \ \ 5\ \ 28 & \quad 19.7719466 & \ldots (a) \\
\text{sum} & 166\ \ 10\ \ \ \ \ \ \ \ \ \ 20 & & \\
\frac{1}{2}\,\text{sum} & 83\ \ \ 5 \ldots\ldots \cos. = 9.0807189 & & \\
\frac{1}{2}\,\text{sum} - A & 43\ \ 59\ \ 32 \ldots \sin. = 9.8417102 & & \\
 & \qquad\qquad\qquad\quad 38.9224291 & & \\
 & \qquad\qquad\qquad (a)\ 19.7719466 & & \\
 & \qquad\qquad\qquad 2)\ 19.1504825 & &
\end{array}
$$

$$\log.\ \sin.\frac{h}{2} = 9.5752412 = \log.\ \sin.\ 22^\circ\ 5'\ 20''\tfrac{1}{3}$$

$$\therefore h = 44^\circ\ 10'\ 40''\tfrac{2}{3} = \text{(in time) } 2^h\ 56^m\ 43^s, \text{ nearly.}$$

This is the time for Greenwich; for any other place, we must correct p, taken from the Nautical Almanack, by adding to it, or subtracting from it, the change in the Sun's north polar distance, proportional to the difference of longitude between Greenwich, and the place of the observed altitude.

Time determined from an observed Altitude of a fixed Star.

The altitude is to be observed as in the former instance: the latitude is supposed to be known from previous observation, and, the star's north polar distance from his *mean* north polar distance (contained in Tables) corrected for the several inequalities of precession, aberration, and nutation; (see Chapters XI, &c.) Then, the computation of the angle ZPS, or of h, will be exactly the same as in the preceding case. That angle will be the star's angular distance from the meridian; therefore, since the star's right ascension is known, the right ascension of a point of an imaginary star, at that time supposed to be on the meridian, is known. But, the right ascension of a star on the meridian being known, the hour of the day is (see pp. 779, &c.)

All stars on the meridian at the same time have the same right ascension; therefore, we may place the imaginary star on the equator, and then (see p. 748,) its right ascension will be that of the *Mid-Heaven*; consequently we may give the rule for finding the time under the following form:

$$* \text{'s } Æ \pm h = Æ \text{ of mid-heaven,}$$

$$Æ \text{ of mid-heaven} - \odot\text{'s } Æ - \textit{acceleration} = \text{time (see p. 780.)}$$

EXAMPLE.

April 14, 1780. *In Latitude* 48° 56′, *Longitude* W = 66° ($4^h\ 24^m$) *the Altitude of* Aldebaran *in the West was observed* = 22° 20′ 8″. *Required the Time.*

L = 48° 56′ 0″	....	cos. =	9.8175235
p = 73 56 59	....	sin.	9.9827322
A = 22 17 50	(refrac. = 2′ 18″)		19.8002557
2) 145 10 49			20
$\frac{1}{2}$ sum = 72 35 24	....	cos.	9.4759722
$\frac{1}{2}$ sum − A = 50 17 34	....	sin.	9.8861065
			39.3620787
			19.8002557
			2) 19.5618230
		log. sin. $\frac{h}{2}$ =	9.7809115

[= l sin. 37° 8′ 39″.75;

$\therefore h$ = 74° 17′ 19″.5
*'s Æ = 65 49 49 .5 (by Tables)

*'s Æ + h = 140 7 9 = Æ of mid-heaven.

But, April 14, ☉'s Æ	=	1^h	31^m	1^s
April 15, ...	=	1	34	42
Increase in 24^h	=	0	3	41 ∴ prop^l. inc^e. in $4^h\ 24^m$ = 40^s.

Hence, Æ of mid-heaven (140° 7′ 9″)	=	9^h	20^m	28 .6
☉'s Æ (= $1^h\ 31^m\ 1^s + 40^s$)	=	1	31	41
		7	48	47.6
Acceleration (see p. 780,)		0	1	16.8
∴ apparent time	=	7	47	30.8

This method, as a practical one, is inferior to the former, partly from the greater length of its computations, and partly from the difficulty of exactly noting the altitude of a star with a sextant. The errors of the Solar Tables affect both methods. In order to lessen the errors of observation, several successive altitudes, distant from each other by nearly equal intervals of time, are noted, and the *mean* altitude deduced corresponding to a *mean* time.

In the sextant there is always some difficulty (and consequently some chance of error) in *reading* off the graduations at the end of each observation. This kind of error is avoided, at least much lessened, in *repeating circles*. Since, with such instruments the *reading off* is not made till after all the observations. The reading off then is the *sum* of all the several altitudes (if they are altitudes which are observed), and the mean altitude is to be had by dividing the above *sum* by the number of observations.

In an Observatory, that has its instruments fixed in the plane of the meridian, the time of apparent noon is easily determined. It may be also ascertained by a sextant, which (see p. 774,) is adapted to measure altitudes: by means of it we can determine when the Sun is at its greatest altitude, or in the meridian. But the altitude of the Sun, when near to the meridian, varying very little, it is difficult to ascertain by a sextant the precise time of the greatest altitude, and consequently, that of apparent noon. Out of the meridian, the variations of altitude are quicker: where they are most quick, then, an error in the altitude (and such there will always be in an observation with a sextant) must be of the least consequence, since it least affects the time; which time would be truly computed by the preceding method, if the altitude were rightly observed.

Since the altitude changes most slowly, when the star is near the meridian, either towards the south or the north, it seems probable, that it would change most rapidly, half way between the north and south; and this is the case, as we shall prove in the solution of a problem, which is usually thus announced.

Given the Error in Altitude; it is required to find where the corresponding Error in Time will be the least.

By p. 795,

$$\cos. h = \frac{\sin. A - \sin. L . \cos. p}{\cos. L . \sin. p},$$

take the differential or fluxion of this equation, and put $d h = \epsilon$, $d A = \alpha$, then

$$-\epsilon \sin. h = \alpha \frac{\cos. A}{\cos. L . \sin. p},$$

but by *Trigonometry*, $\sin. h \times \sin. p = \sin. PZS \times \cos. A$;

$$\therefore \epsilon = \frac{\alpha}{\sin. PZS \times \cos. L};$$

consequently, if L and α, the error in altitude, be given, ϵ is least, when sin. PZS is the greatest, that is, when $PZS = 90^{\circ}$, or the azimuth, is 90^{0}, or the body is on the *Prime Vertical:* which is that vertical circle which passes through the east and west points.

The above is the reason of the precept given by Dr. Maskelyne at p. 152, Nautical Almanack, in which he directs the altitude to be observed near the west and east points. To this precept may be added another; that those stars should be selected for observation, which move most quickly; those, therefore, which are situated near the equator.

Besides the error of altitude, there may be an error in the assumed latitude. For, between the observation which determines the latter from the Sun's meridian altitude, and the observation of the altitude, the observer, if on board a ship, may have changed his place, and, if so, has probably changed his latitude. The relation between its error and that of the time may be determined exactly as the relation between ϵ and δ was in p. 789. Instead of making PS to vary, we must make ZS, $(90 - L)$; let λ be the variation of L, then,

$$\epsilon = \lambda \left(\tan. \text{dec.} \times \text{cosec.} \frac{t}{2} - \tan. \text{lat.} \times \cot. \frac{t}{2}\right).$$

There are several methods and instruments used to ascertain, in the interval between observations, the situation of the ship. Dating from a latitude and longitude astronomically determined, navigators *carry* on a latitude and longitude by *account*. This they are enabled to do, by the chronometer, by the *Log* (by which instrument they ascertain the ship's velocity,) and by an instrument of which we shall now give a short account, and called

The Magnetic Compass.

The *Needle* of the Magnetic compass, is a thin bar of steel, made to move about a centre, in a plane nearly horizontal; which needle in different parts of the Earth points to different parts of the horizon. In scarcely any place, is its direction true north and south. The *Magnetic North,* almost always, differs from the true. And the difference is, technically, called the *Variation* of the compass, differing in degree at different places, and not remaining the same at the same place. Navigators are provided with charts of this *Variation*. Therefore, by observing the *variation* they are to form some probable conjecture of the situation of the ship: and if, by independent means, they know the latter condition, they will be able to examine and to correct the charts.

We must now then consider by what astronomical methods the deviation of the *Magnetical* from the *true* north may be ascertained.

The Magnetic north is always known from the direction of the Magnetic needle. The true north may be computed from the Sun's azimuth, at the time of his rising, or from his observed altitude at any other time. The azimuth is the angle PZS; the computation of which is exactly similar to that of the hour angle ZPS (h) in p. 795.

Let the declination and zenith distance of the Sun be d, z, then,

$$\cos. PZS = \frac{\cos. PS - \cos. ZP \,.\, \cos. ZS}{\sin. ZP \,.\, \sin. ZS} = \frac{\sin. d - \sin. L \cos. z}{\cos. L \,.\, \sin. z}$$

when the Sun rises, or is on the horizon, $z = 90^0$;

$$\therefore \cos. z = 0, \text{ and } \sin. z = 1,$$

and cos. *PZS*, or sin. *amplitude* * $= \dfrac{\sin. d}{\cos. L}$.

In other situations, deducing 2 log. sin. $\dfrac{PZS}{2}$, exactly as 2 log. sin. $\dfrac{h}{2}$ was, in p. 795, we have

$$2 \log. \sin. \text{azimuth} = 20 + \log. \cos. \tfrac{1}{2}(L + z + d) +$$
$$\log. \sin. \tfrac{1}{2}(L + z - d) - \log. \cos. L - \log. \sin. z.$$

Example to the First Method.

In Lat. 51° 52′ *N. the Sun's Declination being* 23° 28′ *N. Required the amplitude, in the Morning.*

$d = 23° 28'$	sin. 9.6001181	
$L = 51 \cdot 52$	cos. 9.7906325	
	9.8094856	= log. sin. 40° 9′ 26′

$\therefore$ the Sun's distance from the east point $= 40° 9' 26''$.

Or the computed true amplitude is	40°	9′	26″	N. E.
$\therefore$ if the amplitude by the compass be....	52	12	28	N. E.
the variation of the compass........	12	3	2	

This operation cannot be a very exact one, since the *computed* amplitude is the amplitude of the Sun when its centre is on the *true* horizon. The observation with the compass can only be made when the Sun is on the *visible* horizon.

Some precautions, therefore, must be taken: and the writers on Nautical Astronomy direct us to take, with the compass, the amplitude of the Sun's centre when the lower limb appears elevated above the horizon by a space somewhat greater than the Sun's semi-diameter. This, however, must needs be an imperfect and rude operation.

* The amplitude is frequently appropriated to signify the complement of the azimuth, when the star *rises* or *sets*.

Example to the Second Method.

In Lat. 51° 32′, *the Sun's Declination being* 23° 28′, *and his Altitude corrected for Refraction* 46° 20′. *Required the Azimuth.*

$$
\begin{array}{rll}
L = 51°32' \ldots\ldots & \cos. = 9.7938317 & \\
z = 43\ 40 \ \ldots\ldots & \sin. = 9.8391396 & \\
d = 23\ 28 & \quad 19.6329713 & (a) \\
\text{sum} = 118\ 40 & \quad 20 & \\
\tfrac{1}{2}\ \text{sum} = 59\ 20 \ldots\ldots & \cos. = 9.7076064 & \\
\tfrac{1}{2}\ \text{sum} - d = 35\ 52 \ldots\ldots & \sin. \quad 9.7678242 & \\
 & \quad 39.4754306 & \\
 & (a)\ 19.6329713 & \\
 & 2)19.8424593 & \\
 & \quad 9.9212296 & = \log. \sin. 56°31'28''
\end{array}
$$

∴ the Sun's azimuth = 56° 31′ 28″.

We will now briefly explain the

Methods of regulating Chronometers.

We have already in pp. 100, &c. explained the method of regulating an Astronomical Clock by means of a fixed transit instrument. But it is necessary, in geodesical operations, for instance, to employ portable instruments and chronometers, and we have now to explain by what means the latter may be regulated, or, rather, their irregularities detected and valued.

The *error* of a chronometer at any time is the difference between the time deduced from astronomical phenomena, and the time its index denotes. The *rate* of a chronometer is the difference between two successive *errors*: it is called the *daily* rate when it is the difference between two errors that happen at the interval of twenty-four hours; or, the *daily* rate may be made to mean the quotient arising from dividing the difference of two more distant errors by the number of intervening days. In order to know, from astronomical phenomena, the time when we are not possessed of a transit instrument, there is no better method than that of *corresponding altitudes* taken by means of an equal altitude instrument, or sextant. In the Example of p. 793, the mean of sixteen observations gave

$$12^h\ 24^m\ 13^s.792,$$

as the apparent time by the chronometer of the Sun's transit over the meridian. Now on the day of observation (August 28, 1822.) the equation of time was $1^m\ 9^s.3$ additive of apparent time; consequently, the chronometer, if it had been properly adjusted to mean solar time, ought to have denoted

$$12^h\ 1^m\ 9^s.3,$$

as the time of the Sun's transit.

The *error*, therefore, of the chronometer on that day (the difference between $12^h\ 24^m\ 13^s.792$, and $12^h\ 1^m\ 9^s.3$) was $23^m\ 4^s.492$, and hence, as a general rule, *correct the chronometer's time of the Sun's transit (determined as above, or by like methods) by the equation of time with a contrary sign, and the result is the time of mean noon by the chronometer.*

We have been speaking of portable chronometers to be examined or regulated at different stations. Now the *equation of time*, of which we have just spoken, is the *equation* when the Sun is on the meridian of the place of observation, and, consequently, not (except in particular cases,) the *equation* inserted in the Nautical Almanack; which latter *equation* is the correction of the apparent time of the Sun's transit over the meridian of Greenwich. In practice, therefore, it will be, almost always, necessary to compute the equation of time for the noon of the place of observation. This is easily done: for instance, if the place of observation were Cadiz, the longitude of which is $25^m\ 8^s$ west of Greenwich, it would be necessary to compute the *equation of time*, for a time $25^m\ 8^s$ *after* the noon of Greenwich. Suppose the observation made on September 8, 1808: in the Nautical Almanack, p. 98, we have

equation of time subtractive $2^m\ 29^s.4$, difference $20^s.4$, and, therefore, the difference, corresponding to $25^{m}\ 8^{s}$,

$$= 20^{s}.4 \times \frac{25^{m}\ 8^{s}}{24^{h}} = 0^{s}.36 \text{ nearly};$$

consequently, the equation of time when the Sun was on the meridian at Cadiz, is equal to

$$2^{m}\ 29^{s}.76,$$

$$\text{or nearly, } 2^{m}\ 29^{s}.8.$$

This, and the previous explanation are sufficient for the following example, and the mode of solving it.

Example.

In September 1808, at Cadiz (longitude $25^{m}\ 8^{s}$, latitude $36^{\circ}\ 31'$ N.) by means of corresponding altitudes (see p. 786,) the following times of noon were obtained *:

Times of Noon.	Equation of Time for Cadiz.	Times of Mean Noon.	Chronometer too slow.	Differences.
Sept. 8, $11^{h}\ 51^{m}\ 48^{s}.38$	$2^{m}\ 29^{s}.8$	$11^{h}\ 54^{m}\ 18^{s}.18$	$5^{m}\ 41^{s}.82$	$-\ 12^{s}.64$
11, 50 59.22	3 31.6	54 30.82	5 29.18	$-\ 16.11$
15, 49 51.83	4 55.1	54 46.93	5 13.07	$-\ 12.53$
18, 49 1.46	5 58	54 59.46	5 0.54	$-\ 12.51$
21, 48 11.27	7 0.7	55 11.97	4 48.03	$-\ 11.85$
24, 47 21.22	8 2.6	55 23.82	4 36.18	65.64

Here the sum of differences in 16 days is $65^{s}.64$, and, accordingly, the mean daily rate, estimated by dividing the sum by the number of days, is $-\ 4^{s}.1025$.

* The column of *equations of time* for Cadiz is formed by adding .4 (nearly the proportional difference, see above) to the equations of time expressed in the Nautical Almanack.

If we estimate the daily rates, by dividing the numbers in the last column, by the numbers of intervening days (3, 4, &c.) we shall have the mean daily rates

from Sept. 8 to 11	$-4^s.21$
11 to 15	-4.03
15 to 18	-4.18
18 to 21	-4.17
21 to 24	-3.95

which differ slightly from the preceding mean daily rate of p. 804.

This is, in effect, the method of determining the *errors and daily rates* of chronometers, by whatever operation or process the time of apparent noon be determined: whether such time be determined by a transit instrument* or be computed (see pp. 795, &c.) from the observed altitude of the Sun or a star, and the latitude of the place of observation.

The present Chapter, unlike the preceding ones, is not confined to the same subject. It contains several methods unconnected as to their nature, and capable of being classed together only because they are useful, or subsidiary to the same astronomical instrument, such as the sextant. We shall soon speak of other uses of that instrument, and of its principal one in determining the longitude of a vessel at sea. That subject, however, claims

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* The *rate* of a chronometer may be determined by a telescope even if it should not be fixed in the plane of the meridian. It is only necessary to take care that the wires of the telescope be at right angles to the star's motion. The interval between two successive returns of the same star to one of these wires is a sidereal day, which differs from a mean solar day by the *acceleration:* so that a chronometer, exactly adjusted to mean solar time, ought to note 24^h — *acceleration* during two successive transits of the star over the same wire of the telescope. Thus, May 3, α Libræ passed the vertical wire of a fixed telescope at .... $10^h\ 44^m\ 41^s$

acceleration	0	3	55.9
	10	40	45.1
but chronometer at the ✳'s transit on May 3, noted.........	10	40	47
∴ rate	+ 0		2.1

a separate Chapter: the present we will conclude with the solution of a few astronomical problems, as they may be called, flowing easily from the Trigonometrical formula, of which, such frequent use has already been made.

If h be the hour angle, z the zenith distance, L the latitude of the place, p the polar distance of the star or Sun, then

$$\cos. h = \frac{\cos. z - \sin. L . \cos. p}{\cos. L . \sin. p}.$$

When the Sun rises or sets, $z = 90^0$, $\cos. z = 0$;

$$\therefore \cos. h = -\frac{\sin. L . \cos. p}{\cos. L . \sin. p} = -\tan. L . \cot. p,$$

the negative sign indicating that, if p be $< 90°$, h is $> 90°$, in other words, that, if the Sun have north declination, h will be greater than 6 hours, or that the length of the day will exceed 12 hours.

Again, if $h = 0$,

$$\begin{aligned} \cos. z &= \cos. L \sin. p + \sin. L . \cos. p \\ &= \sin. (p + L) \\ &= \cos. [p - (90^0 - L)]. \end{aligned}$$

If P, Z, S, be the places of the pole, zenith, Sun (or star),

$$\cos. ZS = \cos. (PS - ZP),$$

and $ZS = PS - ZP$, the body being on the meridian. In this case, then, ZS the meridional zenith distance, is the least zenith distance, since in every other position of S, there is formed a triangle ZPS, in which $PS - ZP$ is $< ZS$.

Twilight is the light of the Sun, when below the horizon, faintly reflected by the atmosphere; and, by computation, it is found to be just sensible when the Sun is within 18^0 of the horizon; or, when $z = 118^0$. We may find the time, therefore, of twilight's beginning or ending, by substituting in the preceding expression, or in that which is immediately deduced from it, (see p. 795,) instead of A $(= 90^0 - z)$, $-$ 18^0.

The *duration* of twilight is the interval of time due to the Sun's ascending or descending through 18^0, it is, therefore, equal to the difference of the last, and that expression (p. 806, l. 9,) which expresses the time of the Sun's rising or setting.

The boundary of twilight, a small circle, parallel to the horizon and 18^0 from it, is called the *Almacanter*.

The length of a day, in its common acceptation, is the interval of time between the rising and setting of the Sun; it is, therefore, equal to twice the angle h, estimated from that expression of cos, h, in which $A = 0$, that is, it is equal to $2 \,.\, \tan.\, L \,.\, \text{co-tan.}\, p$.

$$\text{At the equinoxes, } p \text{ the } \odot\text{'s N.P.D.} = 90^0;$$

$$\therefore\ \cot.\, p = 0;\ \therefore\ \cos.\, h = 0;\ \therefore\ h = 90^0 = \text{(in time) } 6^h;$$

$$\therefore\ \text{the length of the day} = 12^h.$$

At the solstices, p, either, $= 90^0 - 23^0\ 28'$, or $90^0 + 23^0\ 28'$; therefore, the lengths of the longest and shortest day at Greenwich are to be computed from this expression,

$$\cos.\, h = \mp\ 2 \tan.\ 51^0\ 28'\ 39''.5 \times \tan.\ 23^0\ 28',$$

the upper sign $-$, for the longest day, denoting h to be $> 90^0$, and the lower sign $+$, for the shortest, denoting h to be $< 90^0$, and equal to the supplement of the former.

If we wish to investigate the latitude in which the Sun's centre, in its greatest depression, just reaches, but does not descend below, the horizon, we must make $h = 180^0$,

$$\text{then } \cos.\ 180^0 = -1 = -\tan.\ L \,.\, \cot.\, p = -\frac{\tan.\ L}{\tan.\ p};$$

$$\therefore\ \tan.\ L = \tan.\, p, \text{ and } L = p = 90^0 - \text{declination},$$

or, the co-latitude of the place equals the Sun's declination.

In a similar way, and still using the expression for cos. h, we may express the relation between the latitude and the Sun's declination, when there is *just* twilight all night; thus, z being the zenith distance, since

$$\cos.\ h = \frac{\cos.\ z - \sin.\ L \,.\, \cos.\, p}{\cos.\ L \,.\, \sin.\, p},$$

$$\cos.\ 180^0 = \frac{\cos.\ 118^0 - \sin.\ L \,.\, \cos.\, p}{\cos.\ L \,.\, \sin.\, p};$$

$$\therefore \sin. L \cos. p - \cos. L \sin. p, \text{ or } \sin.(L-p), = \cos. 118^\circ = -\sin. 18^\circ;$$

$$\therefore L - p = -18^0, \text{ or } L - (90^0 - \odot\text{'s dec.}) = -18^\circ;$$

$$\therefore \odot\text{'s declination} = (90^0 - L) - 18^0.$$

If L therefore be given, search in the Nautical Almanack for that declination, which equals the difference of the co-latitude and 18^0.

Since, $L = p - 18^0$, and the least value of p, is $66^0\ 32'$; therefore the *least* value of L is $48^0\ 32'$; or in latitudes less than $48^0\ 32'$, there never can be twilight all night.

CHAP. XLII.

On Geographical Latitude.

LATITUDE of places at land, (see p. 11, &c.)

1st. *Method by the Altitudes of Circumpolar Stars.*

This method has been already described in pp. 129, &c. Another instance of it is subjoined, in which, the circumpolar star is that particular one, which, for distinction, is called the *Pole Star,* (the *a Polaris* of Astronomical Catalogues.)

By means of an Astronomical Circle, (see Chap. V,) the following zenith distances (*Z. D.*) were observed at Dublin Observatory on August 23, 1808 :

Greatest Z. D.	38° 18′ 59″.1
Refraction (barom. 29.97, thermom. 67,) ...	0 0 44 .01
Corrected Z. D.	38 19 43 . 11
Least Z. D. of *a Polaris*	34° 53′ 10″.1
Refraction, (barom. 29, 99, thermom. 58,) ..	0 0 39 . 45
Corrected Z. D	34 53 49 . 55
	38 19 43 . 11
	2) 73 13 32 . 66
∴ co-latitude of Observatory, ...	36 36 46 . 33

and latitude is 53° 23′ 13″.67.

2dly, *Method by the Zenith distances of Stars near the Zenith.*

This method determines merely the difference of latitude by means of an instrument, (the zenith sector) capable of measuring small zenith distances with great exactness. We have had already (pp. 12, &c.) specimens of it, and we here subjoin another.

EXAMPLE.

By observation, at the College of Mazarin, *Mem. Acad.* 1755.)

Z. D. of γ *Draconis reduced* (see p. 380,) to Jan. 1750,	2^0	$40'$	$15''$
At Greenwich Z. D. reduced to the same epoch	0	3	4.5
(The star is to the north of both zeniths) diff. lat. ...	2	37	10.5
Hence, if the latitude of Greenwich be	51	28	39.5
Latitude of Observatory, at College of Mazarin....	48	51	29

It is essential, as it has been fully explained in pp. 306, &c. that, for finding the difference of latitudes, by this operation, the zenith distances of the star observed at different epochs, should be reduced to the same. If, however, we should be possessed of two observations of the same star, made on the same day, of the same year, then, since the corrections of aberration, precession, and nutation, (see Chap. XI, XIII, XIV,) would be the same in each observation, it would be necessary merely to apply the corrections for refraction, before we subtracted or added the zenith distances.

This method of determining the latitude, and capable of great accuracy, was employed in the Trigonometrical Survey of England. See *Phil. Trans.* 1803, pp. 483, &c.

Method of determining the Latitude, by reducing to the Meridian the observed Zenith Distances of the Sun, or a Star when near to the Meridian.

The principle and peculiar processes of this method have already, in substance, been explained in pp. 417, 418, &c. The illustrations there given, were, with observations, made with the *circle* of the Dublin Observatory. It now remains to adapt the method to observations made with small portable instruments: for with such observations and instruments is our present concern.

By observing the zenith distance of a star out of the meridian, and by *reducing it* to the meridian, we obtain a result which is equal to the star's meridional zenith distance. When, therefore, as in the instance of the star Arcturus, (see pp. 422, &c.) we observe four zenith distances, two before, two after, the star's transit over the meridian, we obtain four meridional zenith distances: one-fourth of the sum of which, the *mean* meridional zenith distance, is to be held, according to astronomical usage, and as it probably is, a more true value than any individual zenith distance.

It follows from this, that if we could multiply our observations near to the meridian, we should obtain a truer value of the star's meridional zenith distance. But, with an instrument, such as that of the Dublin *Circle*, there are limits to such multiplication. From the size of the instrument, the *readings off* at the three verniers cannot be very quickly effected: add to this, the instrument must be *adjusted* at each observation: so that, at the distance of ten or twelve minutes of time from the meridian, more than two observations cannot be *conveniently* made; and if we begin to observe the star at greater distances from the meridian, the computations of the *corrections* (see pp. 420, &c.) become more operose and less exact.

With instruments, however, of less magnitude which the observer can adjust and *read off*, without hardly shifting his position, a greater number of observations may be made; and no instrument is so fitted to the rapid multiplication of observations as the *repeating circle*, because, in that, the *readings off* are not made till the termination of the observations.

We shall soon give an instance of a meridional zenith distance, deduced from twenty-six observations made out of the meridian. But the advantage of so many observations, is not solely that of giving, by their number, a more exact *mean* result. It is easy to see, by referring to pp. 420, &c. that the corrections c, c', c'' become less, the nearer the star is to the meridian: it will, therefore, frequently happen (it will always so happen with those stars which are selected for the use of *repeating circles*) that, in computing the reduction, we may confine our computation to that of the first correction: since the second and third corrections, which must be inconsiderable, except in the extreme observations (those which are most to the east and west) will have scarcely any effect on the mean result.

Thus, if there should be twenty-six results, and the values of the second and third corrections should amount to one-fourth of a second, the mean result could only be affected by them to the amount of $\frac{1}{104}$th of a second.

Let us suppose, however, that we are able, either by computing the three corrections or only one, to determine the star's meridional zenith distance: such distance, if corrected solely on account of refraction, and not on account of the inequalities of precession, aberration and nutation, is an *apparent zenith distance*. If, therefore, the star be to the south of the pole and zenith, the co-latitude (ZP) is to be obtained by subtracting the above *apparent zenith distance*, from the star's *apparent north polar distance*. If the star be south of the pole, but between the pole and zenith, the co-latitude is the sum of the above two apparent zenith distances. If, however, we choose to correct the observed zenith distance by the *equations* due to precession, &c. we must then instead of the above-mentioned *apparent* polar distance, use the *mean* polar distance. The result in each case, as it has been abundantly explained in the preceding pages, must be the same. The formulæ of reduction which we shall use in the succeeding Examples, are those which are given at p. 420, in which A depends on the latitude of the place, and C on A and the star that is observed. In two of the Examples that follow, the places of observation are Dunkirk and Leith: at the former the pole star was observed, at the latter the Sun.

Hence, for these two places, the latitudes of which are respectively 51° 2′ 5″ and 55° 58′ 4″, we have (see pp. 420, 421,) the following computations of log. A,

Dunkirk.		Leith.	
log. sin. 1″......	4.68557	sum	16.73672
2 log...15	2.35218	log. cos. 55° 58′ 41″....	9.74781
ar. com. 2	9.69896		26.48453
log. cos. 51° 2′ 5″	9.79854		
	26.53526		

Hence, for Dunkirk, (see p. 421,)

$$\log. C = 6.53526 + \log. \sin. D + \log. \text{cosec}. z - 20 + 2 \log. h'.$$

For Leith,

$$\log. C = 6.48453 + \log. \sin. D + \log. \text{cosec}. z - 20 + 2 \log. h'.$$

We will now, in the case of Dunkirk, farther reduce the value of log. C; for which end it is necessary to take account of the other conditions of the observations.

The observed star was Polaris: the time Dec. 19, 1808;

therefore, since co-lat. .... =	38°	57′	55″		
and (from Tables) ∗'s N.P.D. =	1	42	18.5	.... sin.	8.47357
$(ZP - PS)$	37	15	36.5	.. cosec.	10.21793
			(from l. 2,)	....	6.53526
					25.22676

Accordingly,

$$\log. C = 5.22676 + 2 \log. h',$$

which is the formula of computation, from which the correction C is to be computed, when h' the horary angle is given.

Suppose, for instance, a value of h' to be $27^{m}\ 42^{s}$,

$$
\begin{array}{rr}
{}^{*}\log. 27^{m}\ 42^{s} = & 3.22063 \\
 & 2 \\
\hline
 & 6.44126 \\
 & 5.22676 \\
\hline
 & 1.66803 = \log. 46''.561,
\end{array}
$$

and so for other values. The following Table contains the values of h', according to the observations (made in the instance we are quoting) and the corresponding values of the corrections.

* These logarithms may be had very conveniently from Mendoza's Tables, (Tab. XV.)

Values of h'.	Logarithms of C.	Values of C.
27^m 42^s	1.66802	46″.561
26 26	1.62736	42 .400
25 38	1.60068	39 .875
24 57	1.57720	37 .775
24 17	1.55368	35 . 783
23 39	1.53072	33 . 940
22 58	1.50526	32 . 008
15 18	1.15244	14 . 205
14 34	1.10978	12 . 877
5 47	0.30742	2 . 030
2 21	9.52520	0 . 335
1 45	9.26914	0 . 185
1 1	8.79742	0 . 063
4 35	0.10542	1 . 274
17 15	1.25664	18 . 057
21 10	1.43436	27 . 187
21 52	1.46262	29 . 015
22 28	1.48614	30 . 630
23 8	1.51154	32 . 475
23 47	1.53560	32 . 324
24 19	1.55488	35 . 883
25 37	1.60010	39 . 820
26 20	1.62408	42 . 080
28 3	1.67892	47 . 744
29 56	1.73538	54 . 373
35 34	1.88514	76 . 761
	26)	767 . 66
	Mean value of C	29 . 52

The values of h' are thus to be obtained. Note by the chronometer the hour of the passage of *Polaris* over the meridian, using a transit instrument, or, in default thereof, a sextant or *repeating circle*, or any instrument that enables us to take (see pp. 786, &c.) *corresponding altitudes*. Note, also, by the same chronometer the times of the several observed zenith distances: the differences of the hours of transit, and of the hours of observation are, the chronometer going sidereal time, the hour angles. Thus, in the instance we are considering, the hour of the transit of Polaris was $0^h\ 24^m\ 44^s$, and the times of the first and second observations were, respectively, $23^h\ 57^m\ 2^s$, $23^h\ 58^m\ 18^s$, consequently the two corresponding values of h' are

$$2^m\ 58^s + 24^m\ 44^s$$

and
$$1\ \ 42 + 24\ \ 44,$$

or, respectively, $27^m\ 42^s$, $26^m\ 26^s$, (see the Table of p. 814).

The values of the preceding hour angles depend on the chronometer or clock going exactly sidereal time. This may not be the case. The pendulum may be retarded. The consequence of which would be that the number of beats between each observation, and the star's passage over the meridian, would be too small. The *corrections,* or *reductions,* therefore, which depend on such hour angles would be all too small, and, by consequence, the whole reduction. It will be necessary, therefore, should the retardation be considerable, to apply a corresponding correction. But should the clock be nearly adjusted to sidereal time, the last-mentioned correction will be inconsiderable, since the observations are seldom made at a greater distance of time from the meridian, than 20 minutes.

It may happen that the chronometer of the observer is adjusted to mean solar time. Such chronometer, therefore, may be immediately used in obtaining the values of the horary angles, or the times from noon, when the Sun is the body observed: but should, which usually happens, a star be the observed body, the hour angles, for the reasons just stated, will be all too small. They must, therefore, be all increased in the proportion (see p. 780,) of $24^h\ 3^m\ 56^s.55$ to 24^h, or be corrected for retardation. Since we may consider a clock adjusted to mean solar time as a retarded sidereal clock.

We will now deduce a formula of correction for the retardation (or should it so happen the acceleration) of a pendulum, applicable to any small degrees of retardation.

Formula of Correction for the Retardation of the Pendulum.

If a seconds' pendulum loses, in 24 hours, r seconds, it must beat $86400 - r$ timès, instead of 86400.

The true value, therefore, of an hour angle h' noted by such a pendulum is

$$h' \cdot \frac{86400}{86400 - r}, \text{ or } h' + \frac{r h'}{86400 - r};$$

if, therefore, we substitute this true value instead of h', in $\sin.^2 \frac{h'}{2}$, we have

$$\sin.^2 \frac{h'}{2} \text{ equal to } \left(\sin. \frac{h'}{2} + \cos. \frac{h'}{2} \cdot \frac{h'}{2} \cdot \frac{r}{86400 - r}\right)^2,$$

nearly, since h' is a small quantity; but $\frac{h'}{2} = \sin. \frac{h'}{2}$, nearly, and $2 \cos. \frac{h'}{2} \cdot \sin. \frac{h'}{2} = \sin. h' = 2 \sin. \frac{h'}{2}$, nearly.

Hence, the above formula becomes

$$\sin.^2 \frac{h'}{2} \cdot \left(1 + \frac{2 r}{86400 - r}\right).$$

If we refer to p. 419, the first term of the expression for δ is

$$2 \sin.^2 \frac{h'}{2} \cdot \frac{\cos. L \cdot \cos. D}{\sin. 1''. \sin. z} \ldots\ldots (C);$$

which, by increasing h' on account of the retardation of the pendulum, will be increased by

$$2 \sin.^2 \frac{h'}{2} \cdot \frac{\cos. L \cdot \cos. D}{\sin. 1''. \sin. z} \cdot \frac{2 r}{86400 - r},$$

so that C representing the first correction on the supposition that the values of h' are exact, or that the pendulum is accurately adjusted to sidereal time, (supposing a star to be observed) the additional correction for the retardation of the pendulum will be

$$\frac{2 C r}{86400 - r}.$$

What now remains to be done with regard to the instance before us, is the deduction of the numerical value of the latitude, according to the actual circumstances (the zenith distances, barometer, thermometer, &c.) of the observation

Mean of 26 zenith distances	37°	15′	20″.89
Refraction	0	0	46.41
Apparent mean instrumental zenith distance	37	16	7.3
Reduction, (see p. 814,)...................		−	29.52
Retardation [the daily rate (r) of clock being 69^s.5]			− .05
	37	15	37.73
North Polar Distance, (p. 813,)	1	42	18.5
Co-latitude............................	38	57	56.23
Latitude of Dunkirk	51	2	3.77

This is the value of the latitude of Dunkirk from 26 observations, or, from one series of that number, made with a repeating circle. It differs, however, considerably (by several seconds) from the mean value deduced by Mechain and Delambre, from several hundreds of observations, and which are detailed in the second Volume of the *Base du Systême Metrique*, p. 273 to p. 293. The latitude of Dunkirk from the mean of these observations is concluded to be about 51° 2′ 8″.7, *using a certain formula of refractions*: for, as we have shewn in pp. 220, &c., the latitude of a place is no absolute value (we speak of our means of determining values) but depends on the assumed law of refraction, (see also on this subject, tom. II, *du Systême Metrique*, pp. 640, &c.)

We subjoin as a second Example, one taken from the abovementioned Work (*Base du Systême Metrique*).

Example II.

Paris, Rue de Paradis, 17 Dec^r. 1798.

Approximate latitude	48° 51′ 38″ .. cos. =	9.81815
N. P. D. of Polaris (the star observed) ..	1 45 40.16 .. sin. =	8.48760
z $(ZP - PS)$	. cosec.	10.19767
const. log. or sum of log. sin. 1″, 2 log. 15, arith. comp. 2. .		6.73671
		5.24007

therefore, see p. 421, the formula of computation for Paris with the pole star, at the time of observation, is

$$\log. C = 5.24007 + 2 \log. h'.$$

52^m 4^s sidereal time of pole star on the meridian.
42 clock too slow,

51 22 hour of ✱'s passage by the clock,

	Values of h'.	Values of C.
$24^m\ 37^s$	$26^m\ 45^s$	$44''.77$
26 51	24 31	37 . 62
28 3	23 19	34 . 10
29 20	22 2	31 . 21
30 58	20 24	26 . 04
32 0	19 22	23 . 47
33 3	18 19	20 . 99
33 55	17 27	19 . 05
35 12	16 10	16 . 35
36 24	14 58	14 . 02
37 55	13 27	11 . 32
39 39	11 43	8 . 59
		12) 287 . 53
		23 . 96

Mean of 12 zenith distances..........	$39^\circ\ 22'\ 18''.93$
Meridional Z. D..................	39 21 54 . 97
Refraction......................	0 0 46 . 42
True Z. D......................	39 22 41 . 39
✱'s N. P. D.....................	1 45 40 . 16
Height of equator.................	41 8 21 . 55
Latitude........................	58 51 38 . 45

The numbers in the first column are the times of observation by the clock; the numbers in the second are formed by deducting the former numbers from $51^m\ 22^s$, the star's time of transit. The numbers representing the values of C in the third column, do not exactly agree with those in the *Base du Systême*, &c. p. 311, &c. which latter were taken from a Table (p. 250,) constructed for the latitude of Dunkirk and the pole star. The sum of the corrections instead of being, as we obtained it, $287''.53$, is stated to be $288''.14$.

The corrections of third column, p. 818, are merely the first corrections computed, as we have shewn, from

$$\log. C = 5.24007 + 2 \log. h',$$

the formulæ for computing the other two corrections are (see pp. 421, &c.)

$$\log. C' = 7.19899 + 4 \log. h',$$
$$\log. C'' = 4.95046 + 4 \log. h',$$

the greatest value of log. C', therefore, in the preceding instance, when $h' = 26^m\ 45^s$, is

$$4 \log. 26^m\ 45^s + 7.19899 = 0.02091,$$

and, accordingly, $C' = 1''.05$.

In the following observations which were made at Barcelona, and for the purpose of determining its latitude, the clock was adjusted to mean Solar time, and consequently, according to what was said in p. 815, in computing the reduction it is necessary either to increase the hour angles marked by the clock, or to correct the reduction computed on the supposition of the hour angles expressing sidereal time.

Example III.

Barcelona the place of observation, Capella the Star observed, the Time, March 16, 1794.

Approximate latitude....41° 22′ 43″......cos. =	9.87527
∗'s N. P. D..........44 13 50 ....sin.	9.84344
z.................. 4 23 27 .....cosec.	11.11600
Sum of log. sin. 1″, 2 log. 15, arith. comp. 2....	6.73671
(See p. 421,) constant logarithm in log. C........	7.57142
(See p. 421,) sum of 2 log. sin. 1″ 2 log. 15, arith. comp. 12.....	0.64413
Constant logarithm in log. C'....................	8.21555
Again, (see p. 421,)............................	4.38454
2 constant logarithm in log. C..................	5.14284
Log. cot. z................................	11.11472
Constant logarithm in log. C''..................	0.64210

Hence, the three formulæ of computation are

$$\log. C = 7.57142 + 2 \log. h',$$
$$\log. C' = 8.21555 + 4 \log. h',$$
$$\log. C'' = 0.64210 + 4 \log. h'.$$

The three formulæ are given, since Capella passing near to the zenith of Barcelona, renders the third correction of some moment, when the star is observed at more than five minutes of time from the meridian.

✱'s Æ........	5^h	23^m	$32^s.1$
clock too slow ..	0	12	36.1
time of ✱'s transit	5	10	56

	Values of h'.	Values of C.
5^h 7^m 45^s	3^m 11^s	$135''.98$
8 55	2 1	54.57
10 27	0 29	3.13
11 20	0 24	2.15
12 42	1 46	41.88
13 51	2 55	114.15
		6) 351.86
	mean reduction........	58.64

Now log.5 8.64........... 1.76823

$r = 3^m\ 55^s.9$, and log. $\dfrac{471.8}{86164.1}$........... 7.73843

(log. 321)............ 9.50666

Hence, allowing for the *retardation* of the clock on sidereal time, (see p. 815,) the value of C, the first of the corrections, is

$$58''.64 + 0''.32, \quad \text{that is,} \quad 58''.96.$$

If we compute C', C'', from the formulæ of p. 819, we have

Horary angle.	Values of C'.	Values of C''.
3^m 11^s	.002	.584
2 1		.094
0 29		
0 24		
1 46		.055
2 55		.410
		1.143
		.002
		6) 1.145
		.19

The values corresponding to the horary angles 29^s, 24^s, &c. are too inconsiderable to be made account of. But, as it appears, the reduction obtained solely from *C*, is affected by the values of *C*′, *C*″, only to the amount of 0″.19.

We have now given examples of different stars, and different rates of the chronometer. In the fourth Example, which is subjoined, the zenith distances of the Sun's upper limb are observed, and the times of observation noted by a chronometer adjusted to mean solar time.

Example IV.

(*From the Philosophical Transactions,* 1819.) *Leith Fort. Approximate Latitude* 55° 58′ 41″. *Longitude* 12^m 46^s.7 *West. Sept.* 17, 1818. *Barometer* 30.05 *Inches. Thermometer* 66°.

Time of Apparent Noon by the Chronometer.
*0^h 3^m 15^s

	Times from Apparent Noon.	Values of C.
23^h 52^m 28^s	10^m 47^s	2′ 38″.6
23 54 21	8 54	1 48.05
0 10 6	6 51	1 4
0 11 26	8 11	1 31.34
0 13 6	9 51	2 12.34
0 14 19	11 4	2 47.02
		6) 12 1.35
		2 0.22

* Chronometer 8^m 42^s.18 too fast.

For, equation of time at Greenwich (subtractive)......	0^h	5^m	27^s	diff. 2″.2
Proportional difference for 12^m 46^s (longitude).....	0	0	0.2	
∴ equation at Leith..........................	0	5	27.2	
Or, time of apparent noon.....................	23	54	32.8	
Add..	0	8	42.2	nearly.
Time by chronometer..........................	24	3	15	

From preceding page		2	0.22
$\frac{1}{6}$ Sum (319° 57′ 38″.4) of the corrected angles read off on the repeating circle	53h	19m	36s.4
	53	17	36.18
Refraction 1′ 15″.85, parallax 7″.03, difference....	0	1	8.82
☉'s semi-diameter	0	15	57.26
	53	34	42.26
Change of declination, (see pp. 442, &c.)...........	0	0	2.62
Z. D. ☉'s centre	53	34	39.64
* ☉'s apparent declination on the meridian at Leith..	2	24	1.6
Latitude at Leith	55	58	41.24

In the above case, the chronometer was not *exactly* regulated to sidereal time. Its retardation, however, was too small to affect the preceding results.

For a like cause, that of minuteness, the corrections C', C'', are not taken account of in the above computation †.

---

* September 17, ☉'s declination by N. A... 2° 24′ 14″
18 .... 2 1 0

	0	23	14	in 24^h
∴	0	0	11.62	in 12^m
	0	0	0.74	in 46^s
	0	0	12.36	in $12^m 46^s$

∴ ☉'s declination on the meridian at Leith.... 2 24 1.64.

† C', C'', computed from their formulæ, are as follows

$10^m\ 47^s$	$8^m\ 54^s$	$6^m\ 51^s$	$8^m\ 11^s$	$9^m\ 51^s$	$11^m\ 4^s$
C' =.045	.0136	.00476	.0079	.0203	.0324
C''=.029	.0209	.00732	.0149	.0312	.0498

the whole value, therefore, of the corrections, or their sum computed from the formula, (see pp. 420, &c.)

$$C - C' - C'',$$

will be 12′ 0″.972, one-sixth of which is 2′ 0″.162, instead of 2′ 0″.22, as was deduced in page 821. The difference, then, in the two results is only 0″.06.

A great part of the Second Volume of the *Base du Système Metrique*, is occupied with computations, like the preceding, for determining the latitudes of Dunkirk, Barcelona, Paris, &c. The Observer's instruments were, as it has been already mentioned, small repeating circles, their chief star of observation, Polaris; but, besides, other stars, Capella, β Ursæ Minoris, ζ Ursæ Majoris, β Pollucis, β Tauri, &c. were observed, and as, with each of these stars, a vast number of observations were made, it was found to be most commodious to construct separate Tables of *reduction*, (see pp. 302, &c. *Base Metrique*,) for each star and place: for, it is evident from the formulæ of computation given in pp. 421, 819, that the *reduction* depends on the star, the time of its observation, and the latitude of the place.

The preceding methods cannot be practised at sea, where the motion of the vessel renders the level and plumb line useless. In order, then, to determine the latitude of a ship at sea, recourse must be had to the sextant. By means of that the necessary observations are to be made. The results obtained from them, with the aid of Solar and other Tables, give (under skilful management) the latitude to within half a mile: an accuracy sufficient for the navigator, but quite inferior to that which may be obtained from the repeating circle, and its appropriate methods.

LATITUDE OF A VESSEL AT SEA.

Method by the Meridional Altitude of the Sun.

If the latitude and the declination be of the same denomination, that is, either both north, or both south, then, the latitude $= \text{Z. D.} \odot + \text{decl.} \odot$

$$\text{or} = \text{decl.} \odot - \text{Z. D.} \odot, \text{ if decl.} > \text{lat.}$$

If the latitude and declination be of different denominations then, the latitude $= \text{Z. D.} \odot - \text{decl.} \odot$.

EXAMPLE.

July 24, 1783. *Longitude* 54° (3^h 36^m) *West of Greenwich, the Altitude of the Sun's Lower Limb was observed by the Sextant to be* 59° 16′. *Required the Latitude.*

Altitude of the Sun's lower limb	59°	16′	0″
Refraction (Chap. X.)	0	−	34
Parallax (Chap. XII.)	0	+	4
Sun's semi-diameter	0	15	48
True altitude of Sun's centre	59	31	18
∴ Z. D.	30	28	42
Sun's decl. (found as in p. 822,)	19	51	0
∴ latitude (*N*)	50	19	42

By the Meridional Altitude of a fixed Star.

March 29, 1783. *South Latitude, the Meridional Altitude of* Procyon *was* 77° 27′ 15″: *the Height of the Observer's Eye,* 22 *Feet above the Surface of the Sea. Required the Latitude.*

Meridional alt. of *Procyon*	77°	27′	15″
Refraction	0	0	− 13
Dip of the horizon	0	− 4	28
True alt. of ✳	77	22	34
∴ true zen. dist.	12	37	26 S.
Decl. of *Procyon* (from Tables)	5	46	17 N.
∴ latitude	6	51	9 S.

In this Example, a correction called the *Dip*, and not before mentioned, is made. That correction arises from the increase of the apparent altitude occasioned by the elevation of the observer above the surface of the sea *.

* See Tables for computing the Dip: Mendoza's Tables I, II. Lax's Tables VIII, IX.

By the Meridional Altitude of the Moon.

March 26, 1810. *Longitude* 40° 47′ *West of Greenwich, the Altitude of the Moon's Upper Limb was observed to be* 46° 14′ 19″. *Required the Latitude.*

Alt. of Moon's upper limb	46°	14′	19″
Horiz. $\frac{1}{2}$ diam. .... 16′ 6″; Augmentation (see p. 656,) 0 12 } semi-diameter	0	16	18
Apparent alt. of Moon's centre	45	58	1
(Horiz. Parallax 59′ 7″) Parallax (p. 323,)	0	41	0
Refraction	0	—	55
True alt. Moon's centre	46	38	6
Declination S.	17	42	0
Alt. of equator, or co-latitude	64	20	6
∴ latitude	25	39	54 N.

The difference of the parallax and refraction is given as *one result* in Astronomical Tables, (See Tab. VIII. of the Requisite Tables: also Tab. VIII. of Mr. Mendoza's.)

Of these three methods, the first, in which the altitude of the Sun is observed, is most commonly used: the second, very rarely, by reason of the difficulty of observing the star's altitude with a sextant: the third, as it is plain, can only be used in certain parts of the month; and, since in all the observed body must be on the meridian, clouds may prevent any of the three from being used. A subsidiary method, therefore, is provided, in which the latitude may be computed from two observed altitudes of the Sun, and the interval of time between the observations.

Method of finding the Latitude by two Altitudes of the Sun and the Time between.

We have already used a triangle ZPS, and we will now introduce another, ZPs, exactly similar to it: in which s is a position of the Sun, separated from that of S, by the angle SPs, and, in time, by the interval t. Conceive the places S, s (S being nearest to the meridian) to be joined by the arc Ss of a great circle; then we have given

ZS, Zs ($90 - a$, $90 - a'$) the observed zenith distances,

PS, Ps (p, p,) equal N. P. D. of the Sun,

and $\angle SPs$ (t) measuring the interval between the observations. Now the investigation will consist of several steps, which all tend to the finding of the angle ZsP; for, that being found, we have given Zs, Ps, and the included angle ZsP, to find ZP the co-latitude. The steps for finding ZsP are according to the following order. First,

Ss is found; then $\angle PsS$; next $\angle ZsS$, and last,

$$\angle ZsP = \angle PsS - \angle ZsS.$$

Ss *found.*

Cos. Ss = cos. SPs . sin. SP . sin. sP + cos. SP . cos. sP (*Trigonometry*, p. 139.)

$$\therefore\ 1 - \cos. Ss, \text{ or, } 2 \sin.^2 \frac{Ss}{2} = 1 - \cos.^2 p - \cos. t \sin.^2 p$$

$$= \sin.^2 p \,.\, 2 \,.\, \sin.^2 \frac{t}{2}; \text{ and in log}^s.$$

$$\log. \sin. \frac{Ss}{2} = \log. \sin. p + \log. \sin. \frac{t}{2} - 10.$$

Angle SsP *found.*

$$\text{Sin. } SsP = \frac{\sin. p \,.\, \sin. t}{\sin. Ss}*,$$

$$\cos. SsP = \frac{\cos. p\, (1 - \cos. Ss)}{\sin. p \,.\, \sin. Ss};$$

---

* The angle might be deduced from this expression; but the last in practice, is more convenient, since, by taking out the log. sin. $\frac{t}{2}$, we can, without turning over the leaves, take out the log. cot. $\frac{t}{2}$.

$$\therefore \tan. SsP = \frac{\sin. t . \sin.^2 p}{\cos. p (1 - \cos. Ss)} = \frac{\sin. t . \sin.^2 p}{2 . \cos. p . \sin.^2 p . \sin.^2 \frac{t}{2}},$$

$$= \frac{\cot. \frac{t}{2}}{\cos. p}.$$

In logarithms,

$$\log. \tan. SsP = 10 + \log. \cot. \frac{t}{2} - \log. \cos. p.$$

Angle ZsS *found.*

$$\cos. ZsS = \frac{\cos. ZS - \cos. Ss . \cos. Zs}{\sin. Ss . \sin. Zs} = \frac{\sin. a - \sin. a' . \cos. Ss}{\cos. a' . \sin. Ss};$$

$$\therefore 1 + \cos. ZsS, \text{ or } 2 \cos.^2 \frac{ZsS}{2}$$

$$= 2 . \sin. \left(\frac{Ss + a + a'}{2} - a'\right) \cos. \left(\frac{Ss + a + a'}{2} - Ss\right)$$

$$\times \operatorname{cosec}. Ss . \sec. a'.$$

In logarithms,

$$2 \log. \cos. \frac{ZsS}{2} = \log. \sin. \left(\frac{Ss + a + a'}{2} - a'\right)$$

$$+ \log. \cos. \left(\frac{Ss + a + a'}{2} - Ss\right)$$

$$+ \log. \operatorname{cosec}. Ss + \log. \sec. a' - 20.$$

Now $\angle ZsS$ being found, $\angle ZsP = \angle SsP - \angle ZsS$ is known.

ZP *the Co-latitude found.*

In the triangle ZsP we have Zs, Ps and the angle ZsP given, and the side ZP is required. This side will be found by the formula of p. 171, *Trigonometry*.

Thus,

$$\log. \sin. M = \tfrac{1}{2}\left(2 \log. \cos. \frac{ZsP}{2} + \log. \sin. p + \log. \cos. a' - 20\right)$$

$$\text{and log. sin. } \frac{ZP}{2} = \tfrac{1}{2} \left\{ \begin{array}{l} \text{log. sin. } \left(\frac{p}{2} + \frac{90 - a'}{2} + M\right) \\ + \text{ log. sin. } \left(\frac{p}{2} + \frac{90 - a'}{2} - M\right) \end{array} \right\}$$

This method, although it may be called a *direct* one, cannot give an exact result, because, in the first operation (see p. 826,) the Sun's declination is supposed not to alter during the observations. It will be necessary, therefore, to introduce a correction dependent on the change of declination.

EXAMPLE.

$a = 42^{\circ}\ 14'\ 0''$, $\quad p = 81^{\circ}\ 43'\ 30''$

$a' = 16\ \ 5\ \ 47 \quad p' = 81\ \ 45\ \ 0$

$\frac{p + p'}{2}$ (mean N. P. D.) 81 44 15,

t, the interval between the observations, 3^{h}, or in space 45°.

Ss		$\angle SsP$	
	10 10		10 10
sin. 81° 44′ 15″ . .	9.9954800	cot. 22° 30′ 0″. .	10.3827757
sin. 22 30	9.5828397	cos. 81 44 15. . .	9.1574825
(sin. 22° 15′ 16″)	9.5783197	(tan. 86° 35′ 36″.3)	11.2252932
∴ $Ss = 44^{\circ}\ 30'\ 32''$		∴ $SsP = 86^{\circ}\ 35'\ 36''.3$	

ZsS.

				−20 = −20
a =	42° 14′ 0″			
a' =	16 5 47	. . . . sec. =		10.0173684
Ss =	44 30 32	. . . . . . cosec. =		10.1542695
sum . .	102 50 19			
½ sum . .	51 25 9.5			
½ sum − Ss	6 54 37.5	. . . . cos. =		9.9968337
½ sum − a'	35 19 22.5	. . . . sin. =		9.7620664
			2)	19.9305380
		(cos. 22° 36′ 36″)		9.9652690

$$\therefore\ ZsS = 45^\circ\ 13'\ 12''$$
$$\text{but } SsP = 86\ \ 35\ \ 36.3$$
$$\therefore\ ZsP = 41\ \ 22\ \ 24.3$$

ZP.

2 log. cos. 20° 41′ 12″ 19.9421120
log. sin. 81 44 15 .... 9.9954800
log. cos. 16 5 47 .... 9.9826315

2) 19.9202235

9.9601117 ; $\therefore\ M = 65^\circ\ 49'\ 3''$.

Again,

$$p = 81^\circ\ 44'\ 15''$$
$$90 - a' = 73\ \ 54\ \ 13$$
$$2)\ 155\ \ 38\ \ 28$$
$$\tfrac{1}{2}\text{ sum } 77\ \ 49\ \ 14$$
$$M\ 65\ \ 49\ \ \ 3$$

$\frac{1}{2}$ sum $+ M$ 143 38 17 .... sin. = 9.7729698
$\frac{1}{2}$ sum $- M$ 12 0 11 .... sin. 9.3179879

2) 19.0909577

sin. (20° 33′ 25″) 9.5454788

$$\therefore\ ZP = 41^\circ\ \ 6'\ 50''$$
$$\text{latitude} = 48\ \ 53\ \ 10.$$

The formula of correction, for a change in the Sun's declination, which happens between the two observations, is

$$\pm (D - d)\, \frac{\cos.\dfrac{a + a'}{2} \, . \, \sin.\dfrac{a - a'}{2}}{\cos.\, D \, . \cos.\, L \, . \sin.^2 \dfrac{t}{2}}.$$

D being the Sun's declination, at the mean time between the observations, and *d* being the less declination.

Now if the whole change of declination be 1′ 30″,

$$D - d = \tfrac{1}{2}(1'\ 30'') = 0''.75 \ldots \text{log.} = \quad 9.8751$$

$$\frac{a + a'}{2} = 29\quad 9\quad 53 \ldots\ldots\ldots \text{cos.} \quad 9.9411$$

$$\frac{a - a'}{2} = 13\quad 4\quad 7 \ldots\ldots\ldots \text{sin.} \quad 9.3548$$

$$D = 8\quad 15\quad 45 \ldots\ldots\ldots \text{sec.} \quad 10.0045$$

$$L = 48\quad 53\quad 10 \ldots\ldots\ldots \text{sec.} \quad 10.1820$$

$$\frac{t}{2} = 22\quad 30\quad 0 \ldots\ldots 2\ \text{cosec.} \quad 20.8343$$

(60 taken away) .1918 (log. 1′.55.)

Hence, the correction is + 1′.55, or + 1′ 33″,

and since the value of L is 48° 53′ 10″,

the corrected latitude is 48 54 43.

This method founded on the false supposition of the constancy of the Sun's declination during the observations, with the subsequent correction for the change of declination, form a process as long as that would have been in which no change should have been supposed. It is scarcely worth the while to set down all the logarithmic operations in the latter method, but we subjoin the formulæ and their several arithmetical results.

In the triangle PSs, S belongs to the greater altitude, and Ps is the greater N. P. D, and we have to determine, from the two sides and the included angle, the third side and the other angles.

Given Quantities.

$$Ps = 81^\circ\ 45'\ 0''$$

$$PS = 81\quad 43\quad 30$$

$$\angle\ SPs = 45\quad 0\quad 0.$$

Formulæ, (Trig. p. 167.)

$$\tan.\frac{PSs+PsS}{2}=\cot.\frac{SPs}{2}.\cos.\frac{Ps-PS}{2}.\sec.\frac{Ps+PS}{2},$$

$$\tan.\frac{PSs-PsS}{2}=\cot.\frac{SPs}{2}.\sin.\frac{Ps-PS}{2}.\operatorname{cosec}.\frac{Ps+PS}{2},$$

$$\sin.Ss=\sin.Ps.\frac{\sin.SPs}{\sin.PSs}.$$

Results.

$$PSs=86^\circ\ 37'\ 26''$$
$$PsS=86\ \ 33\ \ 46.5$$
$$Ss=44\ \ 30\ \ 28.$$

ZsS is to be determined from the formula of 828, by substituting the present values of Ss, instead of the value therein used: if this be done,

$$ZsS=45^\circ\ 13'\ 10'',\quad ZSs=112^\circ\ 54'\ 54'',$$
$$\text{but, } PsS=86\ \ 33\ \ 46.5$$
$$\therefore\ ZsP=41\ \ 20\ \ 36.5$$

In order to determine ZP, we must also use the same formulæ as were used in p. 829. The results of those formulæ (substituting instead of their former values, the new values of ZsP and Ps, namely, $41^\circ\ 20'\ 36''.5$, and $81^\circ\ 45'$,) will be

$$M=65^\circ\ 49'\ 49''.7$$
$$\tfrac{1}{2}ZP=20\ \ 32\ \ 46.25$$
$$\text{and therefore latitude}=48\ \ 54\ \ 27.5$$

differing from the former result by 15.5 seconds.

We may derive from this method the following mode of correcting the approximate latitude obtained by the first process of pp. 826, &c., and dispense with the correction of page 829. Thus, the value of PsS, deduced in this page, is an exact value: so is Ss; therefore, ZSs, deduced from ZS, Zs (given quantities,) and Ss, is also an exact value. Compute, then, the angle ZSP, from ZS, PS, and that value of ZP which results from the approximate method of pp. 826, &c. If such be a true value of ZP, $ZSs - ZSP$ ought to equal PSs: or, not being equal, their difference will indicate how much, and which way, the value of ZP ought to be changed, in order to procure a more exact agreement. For instance, from

$L = 48^\circ\ 53'\ 10''$ first approximate value, p. 829.
$p = 81\ \ 43\ \ 30$ least N. P. D. corresponding to greatest altitude,
$a = 42\ \ 14\ \ \ 0$ greatest altitude,

and this formula, to wit

$$\cos.^2 \frac{ZSP}{2} = \sin. \left(\frac{L+p-a}{2}\right) \cos. \left(\frac{L+a-p}{2}\right) \sec. a . \operatorname{cosec}. p,$$

may be derived

$$\frac{ZSP}{2} = 13^\circ\ 10'\ 2'', \text{ and } ZSP = 26^\circ\ 20'\ 4''$$

but (see p. 831,)	$ZSs =$	112	54	54
	$\therefore\ PSs =$	86	34	50
but the true value (see p. 831,) of	$PSs =$	86	37	26
	difference	0	2	36

consequently, since ZSs is an exact value, this difference can only arise from ZSP being too large. In order to discover how much we must either augment or diminish the latitude, for the purpose of properly diminishing ZSP, we have this equation,

$$\cos. ZSP = \frac{\sin. L - \sin. a . \cos. p}{\cos. a . \sin. p},$$

$$\text{whence } - d\,(ZSP) . \sin. ZSP = dL . \frac{\cos. L}{\cos. a . \sin. p},$$

we must, therefore, in order to diminish ZSP, augment the latitude, and by the result from the preceding differential formula: thus,

log. $2'\ 36''$, or log. $2''.6$	=	0.41497
log. cos. a	=	9.86947
log. sin. p	=	9.99550
log. cos. ZSP	=	9.64700
log. sec. L	=	10.18190
		0.10884 = log. $1''.284$.

$$dL \therefore = 0^{\circ}\ 1'\ 17''$$

$$\text{and, since } L = 48\ 53\ 10$$

$$\text{corrected latitude} = 48\ 54\ 27.$$

These latter observations and processes have been introduced because they fully explain the method which Dr. Brinkley has given in the Nautical Almanack of 1825, for finding the latitude from the observed altitudes of two known stars. Instead of S, s being two different positions of the Sun, suppose those points to denote two different stars: then the angle SPs will be the difference of the right ascensions of the two stars, and since Ps, PS, the north polar distances of the two stars, and SPs the difference of their right ascensions is known, their distance Ss, and the angle PsS can be computed: which latter quantities, for certain pairs of stars are, in the Nautical Almanack, already computed for the use of the Observer. For instance, the first pair of stars in Table I. (see *Nautical Almanack* 1825, p. 5, of Appendix,) are Capella and Sirius. Now, for 1822, taking

N. P. D. of Capella =	$44^{\circ}\ 11'\ 42''$	Æ	$5^{h}\ 3^{m}\ 33^{s}$
of Sirius	106 28 40	Æ	6 37 18
		difference	1 33 45

we may, as in page 831, and by the same formula, find Ss (D) and the angles PSs, PsS, one of which like PSs is the *angle of comparison* (C) and answers the same end. Their values will be, according to the above data,

$$Ss\ (D =)\ 65^{\circ}\ 47'\ 48''$$

$$PSs\ (C) = 17\ 41\ 50$$

$$PsS = 155\ 16\ 51,$$

and these values (very nearly the same) are expressed in Tab. I. to save the Observer, as we have said, the trouble and difficulty of computation. The parts of the Rule for finding the latitude are, in substance, precisely the same as those we have already used in pages 831, 832, for finding the latitude from two altitudes of the Sun, and the time between. Dr. Brinkley, indeed, instead of a process wholly logarithmic, uses one partly so, and partly constructed by the aid of natural cosines.

The latitude in the first method (see p. 827,) before correction, was supposed to be approximately found, on the supposition of the Sun's declination remaining constant. But we may suppose it approximately known by *account*, as Dr. Brinkley supposes it in his method of two stars, and correct as before.

These methods, whether the Sun be twice observed after a short interval, or two stars be observed at the same time, have been invented for the use of the mariner; and when they are practised whilst the vessel is in motion, the latter has, in one respect, a considerable advantage over the former: which is, that it is not necessary to make in it any allowance for a change of latitude, which it is almost always necessary to do in the other method *.

Instead of the direct method (if such it may be called) of finding the latitude from two altitudes, and the intervening time, several indirect and approximate methods, and made easy by proper Tables, have been invented (see *Nautical Almanack* 1797, 1798, 1799, 1800, 1822: Mendoza's and Lax's Tables on *Nautical Astronomy*. Delambre, tom. III. pp. 641, &c. *Phil. Mag.* 1821, pp. 81, &c.)

It is evident, the preceding methods (pp. 823, &c.) which are the only ones that can be practised at sea, may be practised at land, when the sextant is used with an artificial horizon, (see p. 774.). But then, they are to be used only when no great accuracy is required, and in default of better instruments. The errors of observation with the sextant, and those of the Solar Tables, must always be presumed to be of some magnitude; and, of both of these errors, the above-mentioned methods necessarily partake.

* The inconvenience of the *latter* method is the difficulty of observing, with accuracy, the altitudes of stars.

CHAP. XLIII.

On Geographical Longitude.

THE Earth revolves round its axis in $23^h\ 56^m\ 4^s.091$ of mean solar time; but, a meridian passing through the Sun returns to it after the lapse of a greater time, viz. 24^h, and consequently, after describing a greater angle than 360^0. This arises from the increase of the Sun's right ascension in the time of the Earth's rotation; the mean value of which increase is $59'\ 8''.3$: consequently, the angle, through which a meridian revolves in a mean solar day of 24 hours, is $360^0\ 59'\ 8''.3$.

If we suppose a number of meridians to be drawn at equal intervals, that is, to form successively with each other, equal angles at the poles, then, in the course of 24 hours, each of these meridians (supposing their planes produced) will pass through the Sun and, since both the Earth's rotation, and the Sun's mean motion in right ascension, are supposed to be uniform, at equal intervals of time. If the meridian of a given place passed through the Sun at the beginning of the 24 hours, it would again pass through it at the end; 24 hours then of mean solar time would correspond to 360 degrees of longitude; for, the whole scale of longitude must be comprehended between the eastern and western sides of the meridian of the same place. At places situated on the meridian opposite that on which the Sun was at 0^h, or, in civil reckoning, at 12 at noon, the time would be 12^h, or 12 at night; and 12^h would correspond to 180 degrees of longitude. At places situated on the meridian, at right angles to the former, the time would be 6^h or 18^h; or, in civil reckoning, 6 in the morning, or 6 in the evening; and accordingly, 6 and 18 hours of mean solar time, would correspond to 90^0, or 270^0 of longitude; and similarly for intermediate meridians.

The selection of a meridian, from which the longitudes of all other places are to be reckoned, is entirely arbitrary. The English have selected that which passes through the Royal Observatory at Greenwich: it is called the *First Meridian*, and its longitude is called 0^h. The French use a different one: their *Premier Meridien* passes through the Observatory at Paris, and is $9^m\ 21^s$ east of the former.

If then at Greenwich, (and consequently at all places through which its meridian passes) the Sun were $7°\ 30'$ to the west of the meridian, or the time were $0^h\ 30^m$, at other places, the meridians of which should be $15°$, $30°$, $45°$, &c. distant from that of Greenwich and to the east, or which should have, respectively, $15°$, $30°$, $45°$, &c. of *east longitude*, the times, or the reckoned hours of the day, would be, respectively, $1^h\ 30^m$, $2^h\ 30^m$, $3^h\ 30^m$, &c. At places, $10°$, $20°$, $30°$, &c. of *west longitude*, the times would be respectively, $23^h\ 50^m$, $23^h\ 10^m$, $22^h\ 30^m$, &c. or in civil reckoning, $11^h\ 50^m$, $11^h\ 10^m$, $10^h\ 30^m$, &c. in the morning. Now, some of the methods of determining the longitude, depend solely on the reverse of this; that is, they find the differences between the reckoned time at a given place and at Greenwich, and thence deduce the difference of longitude, or, (since that of Greenwich is 0), the real longitude, converting the time into degrees at the rate of 15 for each hour.

The methods that depend solely on the difference of the reckoned times, are those which are connected with phenomena that happen and are observed at the same point of *absolute time*. Such phenomena are the eclipses of the Moon and of the satellites of *Jupiter*. There are other methods, however, which depend partly on the difference of the reckoned, and partly on that of the absolute times. Such are founded on the phenomena of solar eclipses, of occultations, and of transits, which are not observed, at the same point of absolute time, at all parts of the Earth's surface. (See p. 738.)

This may be illustrated by an instance. Berlin is $44^m\ 10^s$ east of Paris; therefore, if an eclipse of one of *Jupiter's* satellites were observed to happen at the latter place at $13^h\ 1^m\ 20^s$, it would be reckoned to happen at the former at $13^h\ 45^m\ 30^s$: for, since the phenomenon takes place by the actual falling of the shadow on the satellite, the observer at Berlin must see it at the same point of absolute time, as the observer at Paris. But, the occultation of *Antares* by the Moon, (see p. 748,) was observed at Paris at $13^h\ 1^m\ 20^s$, and at Berlin at $14^h\ 6^m\ 19^s$. The difference ($1^h\ 4^m\ 59^s$) of the *reckoned* times, then, is not entirely due to the *difference of meridians* ($44^m\ 10^s$), but partly to that, and partly to the difference in the absolute times of the observations of the phenomena: which latter difference, equal to $20^m\ 49^s$, is entirely the effect of parallax. In the former case, the satellite was *obscured* by the shadow of *Jupiter*, in this latter, the star is *concealed* by the interposition of the Moon.

The methods of finding the longitude, then, naturally arrange themselves into two classes: one belonging to phenomena of the first description, the other, to phenomena of the second. The methods of the former being very simple in their application, but not very accurate in their results; the latter requiring tedious computations, but capable of great exactness. We will, however, first shew how to determine

The Longitude by a Chronometer or Time-keeper.

From the *error* of a chronometer at the beginning of a period and its daily *rate*, we can, supposing the latter constant, determine the error at the end of the period. If the chronometer on June 1, be $2^m\ 13^s$ too slow, and its daily rate be $-0^s.5$, on June 10, its error will be $2^m\ 18^s$. This is an arithmetical operation: but we can also determine the *error* from astronomical phenomena: by means of the Sun's transit observed by a transit instrument, by equal altitudes, or by calculations from absolute altitudes, (see pp. 104, 786, 796.) Should the two errors, thus differently found, not agree, the inference would be that the *rate* of the chronometer had, during the interval, varied.

In this we suppose the observer to have remained at the same station, at Greenwich, for instance. But should he, in the interval of the two observations, have journeyed to a station west of Greenwich, to Edinburgh, for instance, he would have to account for the difference of the longitudes of the two stations, before he could rightly estimate the equability of the chronometer's rate.

We may illustrate this point by an instance taken from the *Philosophical Transactions*, 1819. Part III, p. 384.

Thus, June 15, 1818, the *equation of time* at Greenwich being $-5^s.6$, the Sun's centre was on the meridian at $11^h\ 59^m\ 54^s.4$ of mean time, but the chronometer noted $11^h\ 58^m\ 38^s.6$, it was, therefore, slow by $1^m\ 15^s.8$, and its daily rate being $-0^s.2$, on Sept. 17, it ought, on the supposed constancy of the daily rate, to have been slow by $1^m\ 34^s.6$: in other words, it ought to have noted the time of noon by $11^h\ 52^m\ 58^s.4$, since $-5^m\ 27^s$ being the equation of time at Greenwich, the mean time of apparent noon was $11^h\ 54^m\ 33^s$. Now the chronometer was carried to Edinburgh, and there examined on Sept. 17, by one of the methods mentioned in pp. 786, 802. The longitude of Edinburgh, known by previous methods, is $12^m\ 46^s.7$ west, and the equation of time for that place on the noon of September 17, being $-5^m\ 27^s.2$, the time of apparent noon was $11^h\ 54^m\ 32^s.8$, but the chronometer denoted $12^h\ 3^m\ 14^s.4$; it was, therefore, too fast by $8^m\ 41^s.6$, but if $-0^s.2$ had been its rate, it ought to have been fast by $12^m\ 46^s.7 - 1^m\ 34^s.6$, or $11^m\ 12^s.1$: instead then of having lost in 94 days $18^s.8$, the chronometer had really lost $11^m\ 12^s.1 - 8^m\ 41^s.6$, or $2^m\ 50^s.5$, and its daily rate instead of $-0^s.2$, appeared to be $-1^s.8$.

By methods, then, like this it is ascertained that chronometers by being transported from one place to another change their daily rate, or, widely depart from that mean rate, which, if their construction be good, they preserve at a fixed station. A chronometer, therefore, cannot be relied on for determining the longitudes of places, especially if it be conveyed over land. Their rates are less subject to variation at sea, from the less jolting mode of transport. But the uncertainty attendant on one chronometer is almost entirely got rid of, by the use of several. In the present year, the longitude of Funchal in the island of Madeira has been so determined. Ten or twelve chronometers were taken from Greenwich to Falmouth, and their errors and rates examined at that latter place, by the method of corresponding altitudes. They were then taken to Madeira, and subjected to a like examination, and the longitude determined by a mean of results.

Longitude by an Eclipse of the Moon.

By means of a perfect chronometer we could always, and in all places, determine the longitude. By lunar eclipses which are rare, we can determine the longitude, only occasionally and at particular conjunctures; but, when such occur, by the following method. The times at which eclipses happen, at the place of observation, are to be computed, by one of the methods given in pp. 396, &c., or, which is commonly the case, may be known by a chronometer previously regulated by observation. The times at Greenwich, previously computed, are inserted in the Nautical Almanack, or may be computed by the observer from the Lunar Tables. The difference of these times is the longitude.

Since the Lunar Tables are not exact, the comparison of the same eclipse, actually observed at two different places, will give the difference of their longitudes much more accurately than the comparison of the eclipse observed at one place, and *computed for another.*

EXAMPLE.

1729, *Aug.* 28. *By observations of Cassini at Paris* (Mem. Acad. 1779.) *and of Mr. Stevenson at Barbados* (Phil. Trans. N°. 416. p. 441.)

At Paris, Imm. ☽	12^h	19^m	13^s	Emer. ☽	13^h	59^m
At Barbados, Imm.....	8	11	0	Emer.........	9	51
	4	8	13		4	8

By the mean of the two, the difference of longitude is, $4^h\ 8^m\ 6^s.5$ or $62^0\ 1'\ 30''$: that is, Barbados is $62^{\circ}\ 1'\ 30''$ west of Paris.

This method of determining the longitude is rarely used, since, by reason of a penumbra, it is difficult to ascertain the exact time of contact of the Earth's shadow with the Moon's limb. The time is uncertain, to the extent of 2^m, or 30′. It has been proposed to amend the method, by observing the contact of the Earth's shadow with some remarkable spots in the Moon's disk. (See *Phil. Trans.* 1786, pp. 415, &c.)

Longitude by the Eclipses of Jupiter's Satellites.

This method, although an inexact one, is yet better than the preceding, and for two reasons; the first is, the more frequent recurrence of the eclipses of Jupiter's satellites than of lunar eclipses. The first satellite, for instance, is regularly eclipsed at intervals of forty-two hours. The second reason is, that the times of the immersion and emersion of the satellites, can be more exactly noted than the times of the contacts of the Earth's shadow with the Moon's limb.

This is, however, only a relative excellence. In noting the eclipses of the first satellite, the time must be considered as uncertain to the amount of 20 or 30 seconds. Two observers, in the same room, observing with different telescopes, the same eclipse, will frequently disagree in noting its time, to the amount of 15 or 20 seconds; and the difference will not be always the same way: that is, the telescope by which an emersion is the soonest seen on one occasion, will not always maintain its superiority. As a general fact, however, the telescope of the greatest power will cause immersions to appear later, and emersions sooner: and this is the reason why observers are directed in the Nautical Almanack, (p. 151,) to use telescopes of a certain power.

The eclipses of the first satellite cannot, as it has been remarked, be observed very exactly. But there is much greater uncertainty in noting the times of the eclipses of the other satellites. M. Delambre thinks that the time of an eclipse of the fourth satellite, may be doubtful to the amount of 4′. Still the method of determining the longitude by the eclipses is much practised, because it can be frequently and conveniently practised. A good telescope, an adjusted chronometer, and the Nautical Almanack, are all the apparatus wanted. We subjoin an Example,

Example.

At the Cape of Good Hope, May 9, 1769,

Emer. 1st Satellite.	$10^h\ 46^m\ 45^s$
At Greenwich, by computation (Naut. Alm.).	9 33 12
Difference of meridians	1 13 33

or the Cape is $18^\circ\ 23'\ 15''$ to the east of Greenwich. The remark which was applied to the former case, applies to this. If we use the emersion *observed* at Greenwich, instead of the emersion *computed* for Greenwich, we shall avoid the errors of the Tables of Jupiter's satellites, and obtain a more exact value of the longitude.

We now proceed to the methods of determining the longitude by means of phenomena of the second class; those, which are not seen by all spectators at the same point of absolute time.

The Longitude determined by an occultation of a fixed Star by the Moon.

In pp. 748, &c. the apparent distance of *Antares*, from the Moon was computed, for the instant previous to its occultation, and found equal to $15'\ 51''.38$. The place of observation was Paris: the hour or apparent time $13^h\ 1^m\ 20^s$ (the mean time $13^h\ 3^m\ 32^s.8$): and the formula for the computation of the distance, was

$$D^2 = (l-l')^2 + (k-k')^2 . \cos.^2 l \quad (a).$$

In this formula, l, k, are the *apparent* latitude and longitude of the Moon, obtained, by adding to the true, (see p. 744,) the computed parallaxes in longitude and latitude.

The true longitude and latitude of the Moon were taken, from Lunar Tables *computed for the meridian of Paris*, and for $13^h\ 3^m\ 32^s.8$ *mean solar time at Paris:* and were found, respectively, equal to $9^s\ 5^\circ\ 31'\ 42''.4$ and $3^\circ\ 47'\ 58''.7$. (See p. 749.)

If then the Lunar Tables be correct, D would result from the preceding formula (a) exactly of its proper value, such as the

Tables would assign, or (since D is, in this case, the Moon's semi-diameter) such as might easily be ascertained by observation. But, if D computed from the formula (a) should differ from the value of the Moon's semi-diameter assigned by the Tables, such circumstance would be a proof of, the existence of errors in the Tables. And, the difference between the two values of D, would enable us to deduce an equation between the corresponding errors in the Moon's latitude and longitude. In this case, an occultation would serve to *correct the errors of the Lunar Tables.*

But, as it has been already explained in Chap. XXXIV, there is another method of correcting the Lunar Tables. On the day of observation, the Moon's declination and right ascension are observed, and thence, her latitude and longitude are computed. The respective differences between these, and her latitude and longitude computed from the Lunar Tables, will give, for that day, their errors.

Since we have the means then of ascertaining the errors, we will *suppose* the Lunar Tables to be perfectly correct. Let us now see, by what means, D is to be computed, in a place of observation, for the *Meridian of which, there are no Tables constructed.*

In such a place, the observer must use Tables computed for another meridian: either, for the meridian of Greenwich, or for that of Paris: either the Nautical Almanack, or the *Connoissance des Tems**. By these, he must compute l, and k, and accordingly, previously must compute the Moon's true latitude and longitude, that is, the latitude and longitude that belong to the centre of the Earth. The values of these latter depend on the time for which they are computed, and, *on the time as it is reckoned either at Greenwich or Paris.* Now, although (see

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* These Ephemerides may be considered a species of lunar and solar Tables, in which certain results, most commonly wanted in practice, and computed from the *general* Tables, are inserted. Such results are the Moon's right ascension, declination, longitude, latitude, parallax, and semi-diameter, for noon and midnight.

Chap. XLI.) the time, at the place of observation, can be exactly known, that, *at the place for which the Tables are computed*, cannot, except by a knowledge of the *longitude* of the former place.

This is easily illustrated: the occultation of *Antares* was observed at Berlin at $14^h\ 7^m\ 31^s$, mean solar time. The Observer at that place in order to compute, by the *French Tables*, the Moon's true longitude, must know the corresponding time at Paris. If he *assume* Berlin to be 44^m east of Paris, the corresponding mean time, at the latter place would be, $13^h\ 23^m\ 31^s$: and the Moon's true longitude computed for $13^h\ 23^m\ 31^s$, would be $8^s\ 5^0\ 43'\ 16''$. But, if he *assume* the difference of longitude to be $39^m\ 49^s$, the corresponding time at Paris will be $14^h\ 27^m\ 42^s$: and the Moon's true longitude computed for $14^h\ 27^m\ 42^s$, will be $8^s\ 5^0\ 45'\ 35''$. The computations for the Moon's true latitude will be similarly affected by a change in the hypothesis of the longitude of Berlin.

A small error in that hypothesis will very little affect the computation * of the parallaxes in longitude and latitude, which depend chiefly on the hour angle; consequently, since the apparent differ from the true longitudes and latitudes, solely by the parallaxes, the change, or error in the hypothesis of the difference of meridians, will produce the same difference in the apparent, as in the true longitudes and latitudes of the Moon.

Hence it follows, that an error in the assumed longitude of Berlin (that being still the place used for illustration) will produce errors in the computation of l, k; and consequently, in the computation of D from,

$$D^2 = (l - l') + (k - k')^2 \cos.^2 l \quad (a),$$

there must be an error in the resulting value of D.

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* If we examine the formulæ of computation, (1), (2), (3), &c. in p. 748, &c. we shall perceive that the parallaxes depend principally on the hour angle which is not changed by altering the hypothesis of the longitude.

Now, the principle of finding the longitude of Berlin, consists in *correcting* the assumed longitude, by means of the error in D. The correction is thus made.

The Moon's latitude and longitude (l, k,) being supposed to be erroneous, let their true value be $l + nt$, $k + mt$, n, m being the Moon's horary motions in latitude and longitude, and t, as an unknown quantity, representing the time, or the error of the hypothesis of the *difference of the meridians;* then, if Δ be the Moon's true semi-diameter, we have

$$\Delta^2 = (l + nt - l')^2 + (k + mt - k')^2 . \cos.^2 l \quad (b),$$

and from this and the preceding equation (a), t is to be determined.

If we suppose, what will always be the case in practice, the longitude of the place of observation to be *nearly* known, and, consequently, the hypothesis of its value to differ but little from the true value, t will be a small quantity; and, if we neglect its square in the expansion of (b), we shall have

$$\Delta^2 = (l - l')^2 + 2nt.(l - l') + [(k - k')^2 + 2mt(k - k')] \cos.^2 l.$$

Subtracting (a) from this,

$$\Delta^2 - D^2 = 2t[n(l - l') + m.(k - k')\cos.^2 l]$$

and, consequently,

$$t = \frac{\Delta^2 - D^2}{2[n(l - l') + m(k - k').\cos.^2 l]} \ldots.(c).$$

This value of t, (an approximate one) is the correction to the assumed longitude: suppose, the longitude $= T$, then its corrected value is $T' \pm t$; and, if a still more correct value be required, compute again by means of this *corrected hypothesis* of the difference of the meridians ($T \pm t$), the true latitudes and longitudes of the Moon; thence deduce correcter values of l, k, and find a new approximation (t') from the expression (c). The longitude, after this second correction, will be $T \pm t \pm t'$.

This method, from an assumed approximate value, is capable of determining the true value of the longitude, to the greatest exactness. And, we need not be solicitous concerning the *nearness* of the first approximation to the truth. An eclipse of one of

Jupiter's satellites, which is easily observed, will afford us a first value of the longitude, we might almost say, more than sufficiently near. For, we may even take as a first value, the difference of the *reckoned times* of the occultation at the two places which in the preceding illustration was $1^h\ 5^m$, and which (see pp. 837, &c.) is considerably different from the true value.

We have already illustrated the method, by supposing the occultation to have been observed at Berlin, and the Moon's longitude and latitude to have been computed by Paris Tables. We will now attempt to exemplify the mode of computing the *correction* (t), by supposing the occultation to have been *observed at Paris,* and the Moon's longitude and latitude to be computed by Tables adapted to the *Meridian of Greenwich.*

The immersion (see p. 748,) was observed at Paris at $13^h\ 1^m\ 20^s$. In order to find the corresponding time at Greenwich, suppose the latter place to be 9^m west of the former; then, the reckoned time would be $13^h\ 1^m\ 20^s - 9^m$, or $12^h\ 52^m\ 20^s$; for this time, compute the Moon's longitude; the simplest mode of effecting which, now, would be, to take from the Nautical Almanack the Moon's longitudes on April 6th at midnight, and April 7th at noon; to find their difference, and then to add to the former that part of the difference which is proportional to $52^m\ 20^s$. The result would be the Moon's true longitude at $12^h\ 52^m\ 20^s$. (See pp. 784, &c.) Compute in the same way the Moon's latitude: suppose the above quantities to be exactly of those values which are assigned to them in the Example of pp. 748, &c.; then, the parallaxes, &c., being computed exactly as in that Example, the Moon's semi-diameter will be found (see p. 751,) equal to $15'\ 51''.3$. If the Tables be perfectly correct, and the longitude be rightly assumed, such computed value of the semi-diameter ought to be equal to the semi-diameter assigned by the same Tables. But, the latter is found to be $15'\ 37''.7$. The difference or error $13''.6$, *assuming* the Tables to be correct, must arise then solely from an error in the hypothesis of the longitude: computing that error from

$$t = \frac{\Delta^2 - D^2}{2\,[n(l-l') + m\,(k-k')\cos.^2 l]},$$

in which $\Delta = 15' \ 37''.7, \ldots\ldots\ldots \ l - l' = \ 4' \ 3''.2$
$D = 15 \ 51 .3, \ldots\ldots\ldots \ k - k' = 15 \ 22.7$

$l = 4^{0} \ 36'$, and n and m are the hourly motions *; t will be found nearly $= 25^{s}$. The corrected longitude of Paris then is $9^{m} \ 25^{s}$, and a repetition of the process will give a value still more correct.

Since the illustration of the method of correcting the assumed longitude was our chief object, we have supposed the Lunar Tables to be correct. But, in practice, their errors, which are frequently considerable, must be always attended to.

If the occultation be observed under a known meridian, such as that of Greenwich or of Paris, then, it may be made subservient to the correction of the Lunar Tables. For such an end, Mayer has employed the immersion and emersion of *Aldebaran* †. And, it is easy to see, since the errors in the computation of the Moon's distance from the star, can be only three ‡ (those of the lunar longitude and latitude and of the assumed longitude of the place of observation,) that three observations, to wit, of an immersion, at a place of an ascertained longitude, and of an immersion and an emersion at a place whose longitude is required, will furnish three equations sufficient to correct the three errors abovementioned. (See Cagnoli, *Trig*. pp. 470, &c.)

In page 753, allusion was made to a method, of deducing the longitude from an occultation, in some respects the reverse of the preceding. In the method alluded to, the true latitude and longitude of the *point* of occultation are deduced by correcting the apparent latitude and longitude of the star on account

* To obtain n, m, the hourly motions, compute the Moon's apparent latitudes and longitudes, for $12^{h} \ 51^{m} \ 40^{s}$, and for $13^{h} \ 51^{m} \ 40^{s}$: and the respective differences of these quantities will be the hourly motions in latitude and longitude. In the computation they were assumed to be $1' \ 54''$ and $36' \ 31''$; which are not, however, their exact values.

† Mayer's Lunar Tables, 1770, pp. 39, 40.

‡ The Moon's semi-diameter, on the day of the occultation, may be measured or computed by means of an observation, and accordingly, any error, in it's value assigned by the Tables, corrected.

of parallax. The true latitude of the Moon is taken from the Nautical Almanack. The true distance D, or the semi-diameter of the Moon may be taken from the same source, or may be determined by observation : and thence may the Moon's longitude be determined: for, supposing in the equation (p. 747,)

$$D^2 = (l - l')^2 + (k - k')^2 . \cos.^2 l,$$

that, l, k, &c. represent the true latitudes and longitudes: if D, l, l', are known, $k - k'$ may be determined; and, since k', or the true longitude of the point of occultation is known, k the longitude of the Moon's centre is.

Suppose, then, that by these means, and separate calculations, we obtained, from an occultation, at two different places, the following results :

Greenwich, long. ☽'s centre	$67^{\circ}\ 22'\ 26''.1$	hour $= 8^h\ 37^m\ 36^s.8$
Dublin	67 18 43.3	8 4 51.5
	0 3 42.8	0 32 45.3

then, $3'\ 42''.8$, is the difference between the Moon's true longitudes at the *absolute* times of the observed occultation: and if the Moon's horary motion be $30'\ 9''.2$, the difference would correspond to $7^m\ 23^s.3$, in time. The occultation therefore at Greenwich *really* happened later than the occultation at Dublin by $7^m\ 23^s.3$: but, it is *reckoned* to happen later at the former by $32^m\ 45^s.3$: consequently part of this, or that part which remains after $7^m\ 23^s.3$ is subducted, is solely due to the difference of the longitudes of the two places: Dublin therefore is *east* of Greenwich, $25^m\ 22^s$.

The Longitude determined by means of a Solar Eclipse.

This method, in all its parts, is like the preceding. The distance (D) which is to be computed, instead of being the Moon's semi-diameter, will be the sum of the semi-diameters of the Sun and Moon. The immersion of the star will correspond to the first exterior contact of the limbs of the Sun and Moon, the emersion to the last. Thence will result, two equations for correcting, if the Lunar and Solar Tables be correct, the hypothesis (see p. 846,) of the assumed longitude. But, since we can also observe other *Phases* of the eclipse, that, for instance, of the nearest approach of the centres (see pp. 724, 732,) we may deduce equations sufficient to correct both the errors of the Tables and the error of the assumed longitude of the place of observation.

We will now proceed to the description of an excellent method of finding the longitude, which cannot be ranged under either of the two preceding classes.

Method of determining the Longitude by means of the Passage of the Moon over the Meridian.

Let us suppose the meridian of a given place, produced to the heavens, to pass through the Moon, the Sun, and a fixed star. In the next instant, the Sun by its motion in right ascension will separate itself from the star; the Moon, by her greater motion in right ascension, both from the star and Sun, and the meridian, by the rotation of the Earth, from the star, Sun and Moon. In other words, in the instant of time (whatever be its magnitude) after that on which the three bodies were on the meridian, the star will be most to the west of the meridian, the Moon least, and the Sun will be in an intermediate position.

The meridian after quitting these bodies, will approach towards them with different degrees of velocity, and will reach them after different intervals of time. It will again pass through the star, after describing 360^0, in $23^h\ 56^m\ 4^s.09$; through the Sun, after describing $360^0\ 59'\ 8''.3$, in 24^h; and, through the Moon, after describing an angle equal to the sum of 360^0 and the increase of the Moon's right ascension in 24^h, and in a time equal to the sum of 24 hours, and of the Moon's *retardation* (see p. 783,) in 24 hours.

This takes place in the interval between two successive transits of the Moon over the same meridian. A spectator on a *different* meridian must note similar effects; but less in degree, and less, proportionally to the distance of his, from the first, meridian. He will note an increase in the Sun's right ascension, (or a separation of the Sun from the fixed star) but less than $59' \, 8''.3$: an increase in the Moon's right ascension (or a separation of the Moon from the star), but less than its increase between two successive transits: and, consequently, an excess of the increase of the Moon's right ascension above that of the Sun's, but less than the excess that takes place between two successive transits of the Moon over the first meridian.

Hence, if the spectator, on this second meridian, knows, or is able to compute, the respective increases in right ascension of the Moon and Sun that take place between two successive passages of the Moon over the first meridian, then, since he is able, by actual observation, to ascertain, at the times of their passages, the right ascension of the Sun and Moon, he may, by simple proportion, determine his longitude; and in fact, he has three ways of effecting it: either with the Sun and star; or with the Moon and star; or with the Moon and Sun. Since, however, the first method by reason of the slow motion of the Sun, is not convenient and *practically* useful, we shall not notice it, but consider only the two latter.

Let E be the increase of the Moon's right ascension during two successive transits over the first meridian, e the difference between the Moon's right ascension at the Moon's first passage over the first meridian, and her right ascension at the passage over the second meridian, then,

$$E : e :: 360^{\circ} : 360 \times \frac{e}{E} = \text{difference of the meridians.}$$

This is the case with the Moon and star: and, with the Moon and Sun, there is this only difference, that E (E') must denote the *excess* of the increase of the Moon's right ascension above that of the Sun between two successive transits of the Moon; and e (e') the difference between the hours of Moon's passages over the second and first meridian: for, the hour of the Moon's passage is proportional to the angular distance which then exists between the Sun and Moon.

We must now endeavour to render the above formula more convenient for computation, so that (which ought in practical

Astronomy to be our constant aim) we may avail ourselves of the facilities of the Nautical Almanack.

E is the increase of right ascension between two successive transits of the Moon over the first meridian; it is, therefore, equal to the increase of right ascension in twenty-four hours, plus the increase of right ascension due or proportional to, the Moon's *retardation* (see p. 783,) in twenty-four hours. We have therefore this rule in the case of the Moon and star:

Find from the Nautical Almanack, (see p. 786,) the increase of the Moon's right ascension in twenty-four hours.

Compute also by the rule in p. 155 of the Nautical Almanack, (or from the expression in this Treatise, p. 782,) the Moon's retardation in twenty-four hours.

To the increase (A) of the Moon's right ascension in 24^h add the increase proportional to the retardation : call the sum E.

Then, substituting in p. 849, l. 25, 24^h instead of 360^0, we have

$$\text{log. longitude} = \text{log. } 24 + \text{log. } e - \text{log. } E.$$

In the case of the Moon and Sun, the rule is somewhat more simple: for E' converted into time in the case of the Moon, is the Moon's retardation, and e' is the proportional retardation between the transits at the first and second meridian. The third step, therefore, in the preceding rule, in this case, need not be made.

The above rule is adapted to the Nautical Almanack. But, it is easy to substitute, instead of it, a general formula of computation expressed in symbols. Thus, let A, a, be the respective increases of the right ascensions of the Moon and Sun in twenty-four hours; then, since the interval between two successive passages of the Moon over the meridian is

$$24^h + 24 \times \frac{A - a}{24} + 24 \left(\frac{A - a}{24}\right)^2 + 24 \left(\frac{A - a}{24}\right)^3 + \&c.$$

(since in this case $t = 24^h$, see p. 782,) the *retardation* in 24^h must equal

$$A - a + \frac{(A - a)^2}{24} + \frac{(A - a)^3}{(24)^2} + \&c.$$

and the increase of A, due to the retardation, must equal

$$\frac{A}{24}\left(A - a + \frac{(A - a)^2}{24} + \frac{(A - a)^3}{(24)^2} + \&c.\right)$$

and consequently, (see p. 783,)

$$E = A + A\left\{\frac{A - a}{24} + \left(\frac{A - a}{24}\right)^2 + \left(\frac{A - a}{24}\right)^3 + \&c.\right\}$$

and the longitude =

$$\frac{24 \times e}{A\left\{1 + \frac{A - a}{24} + \left(\frac{A - a}{24}\right)^2 + \left(\frac{A - a}{24}\right)^3 + \&c.\right\}} \ldots (1).$$

In the case of the Sun,

$$E' = A - a + \frac{(A - a)^2}{24} + \frac{(A - a)^3}{24^2} + \&c.$$

and $e' = e - \epsilon$, where ϵ expresses the star's *acceleration,* (see p. 780,) proportional to the time corresponding to the difference of meridians. Hence, the longitude =

$$\frac{24 \times (e - \epsilon)}{(A - a)\left\{1 + \frac{A - a}{24} + \left(\frac{A - a}{24}\right)^2 + \left(\frac{A - a}{24}\right)^3 + \&c.\right\}} \ldots (2).$$

Since $e - \epsilon : A - a :: e : A$, it is plain, the two expressions are, as they ought to be, equal.

The Moon's right ascension is expressed in the Nautical Almanack for every 12^h. Instead therefore of the difference of the increases of right ascension $(A - a)$ in 24 hours, we may employ the difference $\left(\frac{A - a}{2}\right)$ in 12 hours: and accordingly in the Rule, (p. 850, l. 9, &c.) and in the two expressions (1), (2), we must use 12^h instead of 24^h.

The denominators of the expressions, (1), (2), are, strictly speaking, infinite series; but, in practice it will be sufficiently accurate to take the sums of three of their terms.

The application of the Rule of p. 850, to Examples will now be much more easy than it was some years ago: since the Nautical Almanack, will, in future, express the Moon's right ascensions for noon and midnight in degrees, minutes, and *seconds*. We may, therefore, either compute the retardation by the formula of p. 783, or by the Rule given by Dr. Maskelyne in the Nautical Almanack, his explanation of its use, &c.: or by computing by the method given in pp. 698, &c. the time of the Moon's passage over the meridian: since the difference of two successive passages will immediately give us the Moon's retardation in 24 hours. If the passages of the Moon over the meridian of Greenwich were expressed as far as seconds of sidereal, or other, time, the application of the Rule would be still more simple.

EXAMPLE.

April 8, 1800.

Æ of Moon's centre observed at Greenwich....	12^h	36^m	$26^s.6$
On a meridian to the west, ...	12	47	56.7
$e =$	0	11	30.1

By computation from Nautical Almanack

Increase of ☽'s right ascension in 24^h, or A	52^m	6^s
of ☉'s, ... or a	3	39.3
$A - a =$	48	26.7
Moon's retardation in 24^h, or time proportional to the description of $A - a$ (see p. 782,) also Nautical Almanack, Explanation of Rules }	50	7.8
Proportional increase of $52^m\ 6^s$, in $50^m\ 7^s.8$.	1	48.8
$\therefore E\ (= 52^m\ 6^s + 1^m\ 48^s.8)$.	53	54.8

Hence, by the Rule, p. 850,

log. 24	1.3802112
log. $11^m\ 30^s.1$	2.8389120
	4.2191232
log. 53 54.8	3.5098474
	0.7092758 = log. 5.12007;

therefore the longitude $= 5^h.12007 = 5^h\ 7^m\ 12^s.25$.

We will now solve the same Example, by the second method, which is founded on the difference between the hours of the Moon's passages over the meridian, instead of the difference of her right ascensions at those passages. We will also use 12 instead of 24 hours (see p. 852.)

EXAMPLE.

Moon's passage at Greenwich	11^h	26^m	$47^s.82$
—— at the place of observation	11	37	29 . 5
e' or, $e - \epsilon =$	0	10	41.68
Moon's retardation, or E		25	3 . 9

Hence, log. 12	1.0791812
log. 10^m $41^s.68$	2.8073185
	3.8864997
log. 25 3 . 9	3.1772190
	.7092807 = log. 5.1201

$$\therefore \text{ longitude } = 5^h.1201 = 5^h\ 7^m\ 12^s.36.$$

The results are expressed as far as decimals of a second, merely for *arithmetical* exactness, and with no view of signifying that, in practice, any such exactness is attainable. The method is an excellent one, if it will determine the longitude within 10 seconds: and its original author Mr. Pigott, does not think it capable of a greater degree of accuracy. (See *Phil. Trans.* 1786, p. 419.)

The method, indeed, in a point of view strictly theoretical, cannot be minutely accurate. For the Moon's motion is continually variable, and the increase of its right ascension in 24 hours, will not be 24 times the increase in one hour. But if, from the strict laws of the lunar motions, we corrected the method, we should probably obtain an exactness of no practical value; since, we might only get rid of errors much less than the almost unavoidable errors of observation.

Any means, however, of rendering the method more accurate and simple, are not to be neglected. And, on the ground of accuracy, we shall probably gain something, by employing, instead of the sidereal clock, one of the stars that regulate it: and, *that star*, which shall happen to be nearest the Moon in right ascension and declination. Let both Observers note the right ascensions of this star and of the Moon, at the times of their transits over their meridians; then since, in a short interval, the clocks will not err much, the *difference of the differences in right ascension*, on which the method depends, will be given with sufficient accuracy for its successful application.

Again, the method will be rendered more simple, if instead of computing the transit of the Moon's centre, we are content to note merely the *transit of one of her limbs*. This we may do, with little error, if the required longitude be not great. For, the error, if there be any, can arise, solely from a change in the Moon's semi-diameter during the interval between the transits over the two meridians.

EXAMPLE. (See Vince's *Astronomy*, p. 533.)

June 13, 1791. At Greenwich, difference of Æ of ☽'s first limb, and of *α Serpentis* } . . . $28^m\ 31^s.18$

Difference, at Dublin 27 24.74

$1\quad 6.44 = e$

By Nautical Almanack, $\frac{A}{2}$ 30 30

$\frac{a}{2}$ 2 4.4

$\frac{A-a}{2}$ 28 25.6

Retardation, (see p. 782.) 29 35.2

Increase of $\frac{A}{2}$ proportional to retardation 1 15.2

$\therefore E\ (= 30^m\ 30^s + 1^m\ 15^s.2)$ 31 45.2

$$
\begin{array}{ll}
\text{Hence, log. } 12 \ldots\ldots\ldots\ldots & 1.0791812 \\
\text{log. } 1^m\ 6^s.44 \ldots\ldots\ldots & 1.8224296 \\
\hline
 & 2.9016108 \\
\text{log. } 31^m\ 45^s.2 \ldots\ldots\ldots & 2.2799406 \\
\hline
 & 0.6216702 = \text{log. } .418475
\end{array}
$$

$\therefore$ the longitude $= 25^m\ 6^s.5$*.

The method of finding the longitude, by an occultation and the eclipses of the Sun and Moon, would, even if they could be practised, be of no use at sea, by reason of the rare occurrence of the phenomena on which they depend. A voyage might be completed before any eclipse happened. The mariner, who continually changes his place, requires a constant method of determining the change of longitude; a method, accordingly, depending on phenomena, continually occurring. Now, the passages of the Moon over the meridian, and the eclipses of *Jupiter's* Satellites, are phenomena of such character. But, of neither of these can he avail himself: for, the method founded on the former requires a nice observation with a telescope adjusted to move in the plane of the meridian: which is an operation evidently impracticable on board a ship. And the other method, on trial, has been found to be equally impracticable. Yet all that is wanted, for its success is, a contrivance that shall enable the Observer to direct, with steadiness, a telescope of sufficient power, towards *Jupiter*. (See Naut. Alm. p. 151.)

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* The principle of the preceding method is to be found, in a letter from Mr. Pigott, to Dr. Maskelyne, inserted in the *Philosophical Transactions* for 1786, pp. 417, &c.; and the method was used by the former in determining the longitude of York. The rule, however, p. 417, given by its author, is inaccurate: immaterially so, with regard to a place of so small a longitude as York, but to the extent, nearly, of 3 degrees, if we should seek to determine, by it, the longitude of a place that exceeds 5^h. This inaccuracy, as well as those of other authors, (see Vince's *Practical Astronomy*, p. 91. Wollaston's *Fasciculus*, Appendix, p. 76) who have adopted Mr. Pigott's method, was first pointed out in the *Phil. Mag.* vol. XV.

From the defect, however, of the preceding methods, has arisen one of singular simplicity and ingenuity, in which the sole instrument employed is the *Sextant*. This we shall now proceed to describe and illustrate.

Method of determining the Longitude by the Distance of the Moon from a fixed Star, or from the Sun.

1. By means of the sextant (see Chap. XL.) observe the distance between a star and one of the limbs of the Moon; or between the limbs of the Sun and Moon; then, by adding or subtracting, in the former case, the Moon's semi-diameter, and in the latter, the sum of the semi-diameters of the Sun and Moon, there will result either the distance between the Moon's centre and the star, or between the centres of the Sun and Moon.

2. If there be two Observers besides the one, who takes the above distance, let them, at the instant that distance is taken, observe the altitudes of the Moon and Star, or of the Moon and Sun. If there be only one Observer, he must take the altitudes immediately before and after the observation of the distance, and endeavour to allow for the changes of altitude, that may have taken place in the intervals between their observations and that of the distance.

3. These observations being made, the true altitudes must be deduced from the apparent and observed, by correcting the latter for parallax and refraction, (see Chap. XI, XII.). Which correction, in practice, is effected by means of Tables.

4. The observed distance being an apparent one, must be reduced to a true distance, or, (as it is technically expressed,) must be *cleared* of the effects of parallax and refraction. This must be effected in every case, by a distinct computation from a proper formula.

5. The true distance being obtained, find the hour, minute, &c. of *Greenwich time* corresponding to it. This is effected by appropriate Tables, previously computed and inserted in the Nautical Almanack. In these Tables the Moon's distances from

certain stars are inserted for every 3^h: and thence, by an easy calculation, the time corresponding to an intermediate distance may be found.

6. Compute the time at the place of observation from the corrected altitude of the Sun or star, the Sun's or star's north polar distance (furnished by Tables), and the latitude.

7. The difference between this latter time and the time at Greenwich, is the *longitude*.

The first thing in the preceding statement that requires our attention, is the

Formula for deducing the True from the observed Distance.

Conceive S, M to be the true places of the star and Moon in two vertical circles SZ, MZ, forming at the zenith Z, the angle MZS; then, since (see Chap. XI, XII.) both parallax and refraction take place entirely in the directions of vertical circles, some point s above S, in the circle ZS, will be the apparent place of the star, and m below M, (since, in the case of the Moon, the *depression* by parallax is greater than the *elevation* by refraction) will be the apparent place of the Moon: let

D (SM) be the true, d (sm) the apparent distance,

A, a $(90^0 - ZM,\ 90^0 - ZS)$ the true altitudes,

H, h $(90^0 - Zm,\ 90^0 - Zs)$ the apparent altitudes;

then,

$$\text{in } \triangle\ SZM,\ \cos. SZM = \frac{\cos. D - \sin. A \,.\, \sin. a}{\cos. A \,.\, \cos. a},$$

$$\text{in } \triangle\ sZm,\ \cos. sZm\,(=SZM) = \frac{\cos. d - \sin. H \,.\, \sin. h}{\cos. H \,.\, \cos. h},$$

and D is to be deduced by equating these two expressions.

Hence,

$$\cos. D = (\cos. d - \sin. H \,.\, \sin. h)\frac{\cos. A \,.\, \cos. a}{\cos. H \,.\, \cos. h} + \sin. A \,.\sin. a,$$

$$= [\cos. d + \cos.(H+h) - \cos. H . \cos. h] \frac{\cos. A . \cos. a}{\cos. H . \cos. h} + \sin. A . \sin. a$$

$$= 2 . \cos. \tfrac{1}{2} (H + h + d) . \cos. \tfrac{1}{2} (H + h - d)^* \frac{\cos. A . \cos. a}{\cos. H . \cos. h},$$

$$- (\cos. A \cos. a - \sin. A \sin. a.)$$

But the last term $= \cos. (A + a)$; subtract both sides of the equation from 1; then, since

$$1 - \cos. D = 2 . \sin.^2 \frac{D}{2}, \text{ and } 1 + \cos. (A + a) = 2 . \cos.^2 \frac{A+a}{2},$$

we have, dividing by 2, and making F to represent $\dfrac{\cos. A . \cos. a}{\cos. H . \cos. h}$,

$$\sin.^2 \frac{D}{2} = \cos.^2 \tfrac{1}{2} (A + a) - \cos. \tfrac{1}{2} (H+h+d) \cos. \tfrac{1}{2} (H+h-d) \times F$$

$$= \cos.^2 \tfrac{1}{2} (A + a) \left(1 - \frac{\cos. \frac{1}{2} (H+h+d) . \cos. \frac{1}{2} (H+h-d)}{\cos.^2 \frac{1}{2} (A+a)} \times F\right)$$

and, if we make the fraction, on the right-hand side of the equation, $= \sin.^2 \theta$, we shall have

$$\sin.^2 \frac{D}{2} = \cos.^2 \tfrac{1}{2} (A + a) . \cos.^2 \theta,$$

$$\text{and } \sin. \frac{D}{2} = \cos. \tfrac{1}{2} (A + a) . \cos. \theta.$$

Hence, by logarithms, the rule of computation is

1st, $2 . \log. \sin. \theta = \log. \cos. \tfrac{1}{2} (H+h+d) + \log. \cos. \tfrac{1}{2} \pm (H+h-d)$
$+ \log. \cos. A + \log. \cos. a + \text{ar. com. } \log. \cos. H$
$+ \text{ar. com. } \log. \cos. h - 2 \log. \cos. \tfrac{1}{2} (A + a)$,

and 2ndly, $\log. \sin. \dfrac{D}{2} = \log. \cos. \tfrac{1}{2} (A + a) + \log. \cos. \theta - 10$†.

---

* Cos. $\tfrac{1}{2} (d - H - h)$ if d be $> H + h$.

† This formula of computation is Borda's. If in p. 857, bottom line, instead of substituting for sin. H sin. h, cos. H . cos. h, — cos. $(H + h)$, we substitute cos. $(H - h)$ — cos. H . cos. h, we may deduce the formula, which is the basis of Dr. Maskelyne's Rule inserted in the Introduction to Taylor's *Logarithms*, pp. 60, &c.

The other parts (1), (2), &c. p. 856, of the statement* have either already received explanation, in the preceding pages of this Treatise, or are so plain as to need none. We proceed therefore to an Example.

EXAMPLE.

June 5, 1793, about an hour and an half after noon, in $10^\circ\ 46'\ 40''$ south latitude, and 149° longitude, *by account* (see p. 800), by means of several observations, it appeared, that

Distance of nearest limbs of ☉ and ☽	83°	26′	46″
Altitude of lowest limb of ☉	48	16	10
Altitude of upper limb of ☽	27	53	30

Here, see (1) p. 856, we must add to the distance, the semi-diameters of the Sun and Moon, taking them from the Nautical Almanack.

The apparent distance of limbs of ☽ and ☉	83°	26′	46″
semi-diameter of ☉	0	15	46
of ☽	0	14	54
Augmentation propor[l]. to altitude, (see p. 657,)	0	0	7
Apparent distance (d) of centres...........	83	57	33

* The distance (see p. 856, bottom line,) between the Moon and a fixed star is easily computed from their latitudes and the difference of their longitudes, the proper formula is

$$\sin.^2 \frac{D}{2} = \sin.^2 \left(\frac{l-l'}{2}\right) + \cos. l . \cos. l' . \sin.^2 \frac{k-k'}{2},$$

(see p. 746: also *Trig.* pp. 170, &c.) l, l', k, k', representing, in this case, the true latitudes and longitudes.

The Moon's latitude and longitude being computed and inserted in the Nautical Almanack, for noon and midnight, the Moon's distances from certain stars are computed, by the above formula, for those times; and, the distances for the intermediate times, at 3^h, 6^h, &c. are determined by *interpolation*, or by the aid of the differential formula. The latitudes and longitudes of the stars, are either to be computed, (see pp. 158, &c.) from their right ascensions and declinations, or to be immediately taken from certain Tables. (See Lalande's Tables, Nautical Almanack 1773, *Connois. des Tems*, an. 12.)

Reduction of the Apparent to the True Altitude. (See [3] p. 856.)

Altitude of Sun's lower limb	48° 16′ 10″
Dip (see p. 824.)	−0 4 24
	48 11 46
Semi-diameter	0 15 46
Apparent altitude of Sun's centre (h)	48 27 32
Refr. − Par. − correct. for Therm	−0 0 43
True alt. of Sun's centre (a)	48 26 49

Altitude of Moon's upper limb	27° 53′ 30″
Dip	−0 4 24
	27 49 6
Semi-diameter	0 15 1
Apparent altitude of Moon's centre (H)	27 34 5
Par. − Refr. + corr. for Therm	0 46 43
True altitude of Moon's centre (A)	28 20 48

Reduction of the Apparent to the True Distance.
(See [5] p. 856, and *Formula*, p. 858.)

d	83° 57′ 33″		
h	48 27 32	ar. co. cos. =	.1783835
H	27 34 5	ar. co. cos. =	.0523390
sum	159 59 10		
$\frac{1}{2}$ sum	79 59 35	cos. =	9.2399686
$d - \frac{1}{2}$ sum	3 57 58	cos. =	9.9989587
a	48 26 49	cos. =	9.8217187
A	28 20 48	cos. =	9.9445275
$A + a =$	76 47 37		39.2358960
$\frac{1}{2}(A + a) =$	38 23 48	2 log. cos.	19.7883324
		2)	19.4475636
		log. sin. $\theta =$	9.7237818 = log. sin. 31° 57′ 33″

Hence,	log. cos. 31 57 53	9.9285875
	log. cos. 38 23 48	9.8941662
	(10 taken away)	9.8227537 = log. sin. 41° 40′ 27″ ½

$$\therefore \frac{D}{2} = 41^\circ\ 40'\ 27''\tfrac{2}{3},$$

$$\text{and, } D = 83\ \ 20\ \ 55, \text{ nearly.}$$

Time at Greenwich computed. [See (5) p. 856.]

By Nautical Almanack, (p. 70.)

Dist. ☽ from ⊙ $\begin{cases} \text{at } 15^h \ldots .83^\circ\ 6'\ 1'' \ldots . D = 83^\circ\ 20'\ 55'' \\ \text{at } 18\ \ldots .84\ \ 28\ \ 26 \ldots \text{ at } 15^h\ 83\ \ 6\ \ 1 \end{cases}$

Increase of dist. in 3^h = 1 22 25   0 14 54

Hence,

$1^\circ\ 22'\ 25'' : 14'\ 54'' :: 3^h$: time corres^g. to the increase of $14'\ 54''$

* Hence, log. 3 = .4771213
log. 894″ = 2.9513375
3.4284588
log. 4945″ = 3.6941663
$\bar{1}.7342925$ = log. $0^h.5425$ = log. $32^m\ 33^s$.

Hence, the time at Greenwich = $15^h\ 32^m\ 33^s$.

Time at the Place of Observation computed. [See (6) p. 857, *also*, pp. 795, &c.]

L (Lat.)	10°	$16'$	$40''$	. . cos.	9.9929749
p	113	22	48	. . sin.	9.9627922
a	48	26	49		19.9557671
sum . .	172	6	17		
$\frac{1}{2}$ sum . .	86	3	8.5	cos.	8.8378712
$\frac{1}{2}$ sum $- a$	37	36	19.5	sin.	9.7854864
				(20 added)	38.6233576
					19.9557671
				2)	18.6675905
					9.3337902 = log. sin. $12^\circ\ 27'\ 17''\frac{1}{2}$

* As this is a frequent operation in Nautical Astronomy, it is facilitated by means of Tables of *Proportional Logarithms*, in which log. $3^h = 1$. See Requisite Tables, Tab. XV. also Mendoza's Tables, Tab. XIV.

$\therefore$ hour angle (see p. 795, &c.) $= 24^{\circ}\ 54'\ 35''$
(and in time, by Rule, p. 779,) $= 1^{h}\ 39^{m}\ 38^{s}.3$.

Hence, see (7) p. 857,

Time at Greenwich, ...	15^{h}	32^{m}	33^{s}
at place of observation	1	39	38.3
Long. from Greenwich *reckoning by the west*	13	52	54.7

$\therefore$ longitude *east* of Greenwich $10^{h}\ 7^{m}\ 4^{s}.3$.

We will give a second Example, in which the *lunar distance* is the Moon's distance from a known star.

Example II.

Dec. 14, 1818, at $12^{h}\ 10^{m}$, nearly: latitude $36^{\circ}\ 7'$ N., longitude by account $11^{h}\ 52^{m}$, the following observations were made; the eye of the Observer being about 19 feet above the surface of the sea,

Observed Alt. of Regulus.	Observed Alt. of Moon's L. L.	Observed Dist. of Moon's nearest Limb and Star.	
$28^{\circ}\ 29'\ 17''$	$61^{\circ}\ 26'\ 12''$	$33^{\circ}\ 15'\ 25''$	
$-\ 4\ 18$	$-\ 4\ 18$		.... dip of the horizon
	$+\ 14\ 56$	$+\ 14\ 56$	☽'s semi-diameter.
$(h)\ 28^{\circ}\ 24'\ 59''$	$(H)\ 61^{\circ}\ 36'\ 50''$	$(d)\ 33^{\circ}\ 30'\ 21''$	
Ref^{n}. $-\ 1\ 45$	$-\ 0\ 31.1$		
Parallax [see below (p)]	$+\ 25\ 40.5$		
$(a)\ 28\ 23\ 14$	$(A)\ 62\ \ 2\ \ 0$, nearly.		

(p) Horizontal Parallax .... $53'\ 59''$,
log. $53'\ 59''$ .... $= 3.51041$
log. $61^{\circ}\ 36'$ 9.67726
3.18767

$\therefore$ parallax $= 1540''.5$
$= 25'\ 40''.5$.

Hence, see p. 858,

$$\begin{array}{rl} d = & 33^{0}\ 30'\ 21'' \\ h = & 28\ \ 24\ \ 59 \ldots \text{sec. } 0.0557579 \\ H = & 61\ \ 36\ \ 50 \ldots \text{sec. } 0.3229307 \\ \hline \tfrac{1}{2}\text{sum} = & 61\ \ 46\ \ \ 5 \ldots \text{cos. } 9.6748997 \\ \tfrac{1}{2}\text{sum} - d = & 28\ \ 15\ \ 44 \ldots \text{cos. } 9.9448723 \\ a = & 28\ \ 23\ \ 14 \ldots \text{cos. } 9.9443616 \\ A & 62\ \ \ 2\ \ \ 0 \ldots \text{cos. } 9.6711338 \\ \hline & \qquad\qquad\qquad 39.6139560 \\ \tfrac{1}{2}(A+a) & 45\ \ 12\ \ 37\ \ \ 2\text{ cos. } 19.6957706 \\ \hline & \qquad\qquad\qquad 2)\ 19.9181854 \\ \hline & \qquad\qquad\qquad 9.9590927 \text{ (log. sin. } 65^{0}\ 31'\ 13'') \end{array}$$

$$\begin{array}{l} \text{again, cos. } 65^{\circ}\ 31'\ 18'' = 9.6173895 \\ \qquad\quad \text{cos. } 45\ \ 12\ \ 37 \qquad 9.8478853 \\ \hline \qquad\qquad\qquad 9.4652748 = \text{log. sin. } 16^{0}\ 58'\ 24''.2; \\ \qquad\qquad \therefore D = 33^{\circ}\ 56'\ 48''.4. \end{array}$$

Time at Greenwich (see Nautical Almanack for 1818, p. 140.)

$$\text{Dist. ☽ from } * \begin{cases} 0^{h} \ldots 33^{\circ}\ 58'\ 7'' \\ 3 \ \ldots 32\ \ 30\ \ 3 \end{cases} \qquad \begin{array}{l} 33^{\circ}\ 58'\ \ 7'' \\ 33\ \ 56\ \ 48.4 \ldots\ldots (D) \end{array}$$

$$\qquad\qquad\qquad 1\ \ 28\ \ 4 \qquad\qquad\qquad 0\ \ \ 1\ \ 18.6.$$

$$\text{Hence, the time at Greenwich} = 0^{h} + \frac{1'\ 18''.6}{1^{0}\ 28'\ 4''} \times 3^{h} = 2^{m}\,40^{s}.6.$$

Time of Observation, at the Place of Observation.

$$\begin{array}{rl} a = & 28^{\circ}\ 23'\ 14'' \\ L = & 36\ \ \ \ 7\ \ \ \ 0 \ldots \text{sec. } 0.0926862 \\ p = & 77\ \ \ \ 9\ \ 66 \ldots \text{cosec. } 0.0110020 \\ \hline \tfrac{1}{2}\text{sum} = & 70\ \ 49\ \ 40.3 \ldots \text{cos. } 9.5164147 \\ \tfrac{1}{2}\text{sum} - a = & 42\ \ 26\ \ 26.3 \ldots \text{sin. } 9.8291911 \\ \hline & \qquad\qquad\qquad 2)\ 19.4492940 \\ \hline & \qquad\qquad\qquad 9.7246470 \text{ log. sin. } 32^{0}\ 2'\ 10''; \end{array}$$

∴ the horary angle .... $= 64^{\circ}\ 4'\ 20'' =$	4^{h}	16^{m}	$17^{s}.3$	
But star's right ascension .... =	9	58	43.3	
Right Ascension of mid-heaven ...... =	5	42	26	
From N. A., the Sun's Æ on the meridian of the place of observation =	17	27	12	
Approximate time	12	15	14	
Acceleration	0	2	0.4	
Time at ship	12	13	13.6	
	11	46	46.4	
Time at Greenwich	0	2	40.6	
Longitude	11	49	27	W.

Instead of computing the time from the altitude of the star, we might have computed it from the Moon's altitude, which can be more exactly observed. The computation will be as follows:

☽ 's true alt. (A)	62°	2′	0″		
☽ 's N. P. D. p	63	37	20	cosec.	0.0477480
L	36	7	0	.. sec.	0.0926862
½ sum	80	53	10	.. cos.	9.1997481
½ sum − A	18	51	10	.. sin.	9.5093874
				2)	18.8495697
					9.4247848 sin. 15° 25′ 22″.25.

∴ ☽ 's horary angle $= 30^{0}\ 50'\ 44''.5 =$	2^{h}	3^{m}	23^{s},	nearly,
☽ 's right ascension..............	7	45	51	
Right Ascension of mid-heaven	5	42	28	
Sun's right ascension	17	27	12	
	12	15	16	
Acceleration	0	2	0.5	
	12	13	15.5	
	11	46	44.5	
Time at Greenwich	0	2	40.5,	nearly,
Longitude	11	49	25	

The process for finding the longitude from the distance of the Moon from a *star*, similar to the preceding, is, in deducing the true from the observed altitude, somewhat more simple; but, more tedious in the computation of the time from the altitude.

The computation of deriving the time from the star's altitude, it is desirable to supersede, by reason, of the probable errors that will be made in observing the star's altitude*. And it may be superseded, by finding the time and regulating the chronometer by a previous or a subsequent observation of the Sun's altitude: by allowing for the change in longitude (see p. 802, &c.) during the two observations; and then by *computing* the star's altitude, from its north polar distance, the latitude, and the *estimated* time.

The proper formula of computation for this occasion is one that has repeatedly occurred, (see pp. 795, &c.) If L be the latitude, p the north polar distance, h the estimated hour angle and a the altitude, then,

$$\sin. a = \sin. L . \cos. p + \cos. L \sin. p . \cos. h,$$

whence, a may be computed by means of a subsidiary angle. (See *Trig.* pp. 169, &c.)

Hence, the process for finding the longitude, although it does not essentially require the chronometer, is rendered more easy and accurate by its aid.

This is not the sole use of the chronometer. It enables the Observer to use the *mean* of several observed distances of the Moon from a star, or the Sun, instead of a single one. For, he cannot, without error, take the *mean*, except he know the several intervals of time that separate the successive observations. The chronometer enables him to ascertain these intervals.

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* The practical inconvenience of this method, is of the same kind as that which occurs in Dr. Brinkley's method of finding the latitude from the observed altitudes of two known stars: except in the twilight, or by Moonlight, it is very difficult to see the horizon distinctly, when you can see the star. Lacaille was accustomed to use precautions in order to be able to see the horizon.

Thus, in the following observations:

	Time by Watch.	Star's Altitude.	Altitude Moon's Upper Limb.	Dist. Moon's L. from Star.
	13^h 1^m 50^s	43° $0'$ $30''$	67° $28'$ $0''$	45° $19'$ $45''$
	2 25	7 0	67 11 0	19 15
	3 21	14 0	66 59 0	18 45
	4 14	20 30	66 51 0	18 30
	5 11	29 0	66 36 0	18 15
	6 8	38 0	66 32 0	18 0
Sums	23 9	109	401 37 0	112 30
$\frac{1}{6}$th or means	13 3 52	43 18 10	66 56 10	45 18 45

And, generally, the *elements* of the computation in the *lunar method* are the *means* of several observations, not the results of individual ones.

Since, in Nautical Astronomy, the finding of the longitude is the most important and most difficult operation, several expedients have been devised for facilitating it. *The distance has been cleared**, (see p. 857,) by a formula different, from that which has been given in p. 857, although derived from the same funda-

* M. Delambre has given in Chap. XXXVI. (and there is no great difficulty in the deduction) about 20 different formulæ. The leisure of scientific men cannot be more innocently employed. It is profitably employed when, after comparison, it selects that formula which, sufficiently exact, is the least liable, in its application, to the mistakes of merely practical men: such, as in general, mariners are. But a proper formula once adopted, and invested with its Rules and Tables, ought not hastily to be got rid of. It is no sufficient reason to get rid of it, to be able to supply a method a little more simple, and a little less long. There is no great harm, indeed, in perplexing a mere mathematician. But it is a very mischievous innovation to disturb the technical memory of an old seaman, and to unsettle his familiar rules of computation. Every one, man of science or not, knows, from his own experience, the great value of *fixed* rules, in conducting arithmetical operations.

mental expression. Instead of a logarithmic computation, one proceeding solely by addition, and furnished with appropriate Tables, has been substituted. But, for a satisfactory explanation of the means and artifices, by which, on this occasion, the labour of computation is abridged and expedited, we must refer to the treatises that contain them. (See *Requisite Tables:* their explanation and use. Mendoza's *Treatise on Nautical Astronomy:* Brinkley, *Irish Transactions*, 1808: *Connoissance des Tems* for 1808, and for years 12 and 14: Mackay, *On the Longitude*, Lax's Tables.)

If we wish to reduce, to one of the classes (see p. 837,) the preceding method of finding the longitude, we shall find that it belongs to the second. The principle on which it rests, is, indeed, precisely the same as that which forms the basis of the second method (see p. 841,) of finding the longitude from an occultation; for,

Analogous to the distance D	83°	20′	55″,	at 1^h	39^m	38^s	
is the ☽'s longitude at Dublin,	67	18	43.3,	at 8	4	51.5	
Analogous to the distance	84	28	26,	at 18			(*Greenwich*)
is the Moon's longitude.....	67	22	26.1,	at 8	37	36.8	

(for the Moon's longitude is a species of distance, being the distance of her place referred to the ecliptic from ♈). And the reduction of $84^0\ 28'\ 26''$ to $83^0\ 20'\ 55''$ by taking away $1^0\ 7'\ 31''$, corresponding to $2^h\ 27^m\ 27^s$, is analogous to the reduction of $67^0\ 22'\ 26''.1$ to $67^0\ 18'\ 43''.3$, by taking away $3'\ 42''.8$, corresponding to $7^m\ 23^s.3$; $1^0\ 22'\ 25''$, being, in the former case, the change of the Moon's distance in 3^h and $30'\ 9''.2$, in the latter, the change of the Moon's longitude in 1^h: that is, in other words, the Moon's *horary motion in longitude.*

The problems then of deducing the longitude from an occultation, and from the distance of the Moon from a star, are the same in principle; but the former is more difficult in its process, because, in *clearing* the observation of parallax, it is necessary to compute its resolved parts in the directions of longitude and latitude; whereas, in the latter, the entire effects of parallax, which take place in altitude, are alone considered.

The former, as a practical method of determining the longitude, is exceedingly more accurate than the latter*; because, we are enabled to mark the distance, which is the Moon's semi-diameter, and the corresponding time, which is that either of the immersion or emersion, with much greater precision, than we can measure the distance by means of a sextant, and compute the time from an observed altitude. But, as it has been observed in p. 855, the degree of accuracy does not alone determine the adoption of a method; we are obliged, in finding the longitude at sea, by the exigencies of the case, to rely solely on, what is called technically, the *Lunar Method.*

In finding the longitudes of places at land, circumstances also must determine which of the preceding methods must be adopted. Several have been proposed, not as if they might be indifferently used, but that Observers may select from them, what are suited to their several wants, means, and opportunities. If the Observer, furnished with a telescope and chronometer, wishes readily and speedily to determine the longitude of the place where he is, he may use the method of the eclipses of *Jupiter's* satellites, (see p. 840,) and obtain a result probably within 30 or 40 seconds of the truth. If he has the means of adjusting a telescope to move nearly in the plane of the meridian, the method of the transits of the Moon and of a fixed star, (see p. 856,) will afford a more accurate result, and with an error, perhaps, not exceeding ten seconds. But, if great accuracy be required, and expedition be not, then the Observer must wait for the opportunity of a solar

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* "For the present, I infer, we may take the difference of meridians (Greenwich and Paris) 9^m 20^s, as being within a few seconds of the truth, till some *occultations* of fixed stars by the Moon, already observed, or hereafter to be observed, in favourable circumstances, and carefully calculated, shall enable us to establish it with the *last exactness.*" Maskelyne, *On the Latitude and Longitude of Greenwich*, &c. *Phil. Trans.* 1787, p. 186. See also *Phil. Trans.* 1790, p. 230.

eclipse, or, what is better, of an occultation *, and thence compute the longitude †.

The several methods have their peculiar advantages and disadvantages: the last, which is the most accurate, requires computations of considerable length and nicety; the first, probably inaccurate to the extent of $\frac{1}{8}$th of a degree, requires scarcely any. The second is more accurate, and may constantly be used, and therefore, on the whole, it is perhaps the readiest and best practical method.

The *Lunar method,* which is the least exact, is yet founded on the most refined theory, and the most complicated calculations. It depends, for its accuracy, entirely on previous computations. We cannot, in applying it, compare, as in the case of an occultation, (pp. 841, &c.) *actual* observations of the same phenomenon, or give accuracy to the result, by *correcting* (see p. 842,) the errors of the Tables. But, the mariner must be guided by the result, such as it comes out at the time of the observation, and which, a few hours after, will have lost all its utility.

In page 849, it was mentioned, that, in a merely theoretical point of view, the longitude ought to be afforded as a result, from

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* An occultation affords a more exact practical result than a solar eclipse, because, in the former, the instant of immersion can be marked with greater precision, than the instant of contact in the latter.

The *recurrence* of occultations may be found as those of eclipses were, p. 730. We must find two numbers in the proportion, or nearly so, of $27^d.321661$ (the Moon's sidereal period) to $6793^d.42118$ (the sidereal revolution of the nodes): which numbers are 17, and 4227: and the period of recurrence is $316^y\ 72^d.1$ ($=4227 \times 27^d.321661$).

† In speaking of the errors in the determination of the longitude, we have supposed the *mean*, of several observations accurately made with excellent instruments, to be taken. The errors of *single* observations will be much greater than what have been assigned to them. With the first satellite of *Jupiter* it may amount to $3^m\ 44^s$ according to Mr. Short. (See his Paper in the *Phil. Trans.* 1763, p. 167, for determining the difference of longitude between Greenwich and Paris, from the transits of *Mercury* over the Sun's disk).

the separation, during a given interval, of the Sun from a star; but that the slow motion of the former, deprived the method of all practical utility. Now, the material circumstance that coufers, what accuracy it possesses, on the *Lunar method,* is the Moon's quick change of place. Were the change greater, the method would be more accurate. For instance, the Moon now moves through 1° in about 2 hours, and therefore, an error of $1'$, in observing and computing her distance, causes an error of 2 minutes of time, or of $30'$ of longitude. But, if she moved through the same space (1°) in $\frac{1}{2}$ hour, then the error of $1'$ would cause only an error of 30^{s} of time, and of $7'\frac{1}{2}$ of longitude.

Hence it follows, that the first satellite which moves round *Jupiter* in less than two days, (see p. 629,) must enable an Observer on that planet to determine, very exactly, the longitude of his station: as exactly, as we can determine the latitude of a place.

CHAP. XLIV.

On the Calendar.

THE Sun naturally regulates the beginnings, ends, and durations of the seasons; and, the calendar is constructed to distribute and arrange the smaller portions of the year.

The calendar divides the year into 12 months, containing, in all, 365 days; now, it is desirable that it should always denote the same parts of the same season by the same days of the same months, that, for instance, the summer and winter solstices, if once happening on the 21st of June and 21st of December, should, ever after, be reckoned to happen on the same days; that, the date of the Sun's entering the equinox, the natural commencement of spring, should, if once, be always on the 20th of March. For thus, the labours of agriculture, which really depend on the situation of the Sun in the heavens, would be simply and truly regulated by the calendar.

This would happen, if the civil year of 365 days were equal to the astronomical; but, (see p. 529, &c.) the latter is greater; therefore, if the calendar should invariably distribute the year into 365 days, it would fall into this kind of confusion; that, in progress of time, and successively, the vernal equinox would happen on every day of the civil year. Let us examine this more nearly.

Suppose the excess of the astronomical year above the civil to be exactly 6 hours, and, on the noon of March 20th of a certain year, the Sun to be in the equinoctial point; then, after the lapse of a civil year of 365 days, the Sun would be on the meridian, but not in the equinoctial point; it would be to the west of that point; and would have to move 6 hours in order to reach it, and to complete (see pp. 197, &c.) the astronomical or tropical year.

At the completions of a second, and a third civil year, the Sun would be still more and more remote from the equinoctial point: and would be obliged to move, respectively, for 12 and 18 hours, before he could rejoin it, and complete the astronomical year.

At the completion of a fourth civil year, the Sun would be more distant, than on the two preceding ones, from the equinoctial point. In order to rejoin it, and to complete the astronomical year, he must move for 24 hours, that is, for *one whole day*. In other words, the astronomical year would not be completed till the beginning of the next astronomical day; till, in civil reckoning, the *noon of March* 21st.

At the end of four more common civil years, the Sun would be in the equinox on the noon of March 22. At the ends of 8 and 64 years, on March 23, and April 6, respectively; at the end of 736 years, the Sun would be in the vernal equinox on September 20. And, in a period of about 1508 years, the Sun would have been in every sign of the Zodiac on the same day of the calendar, and in the same sign on every day.

If the excess of the astronomical above the civil year, were really, what we have supposed it to be, 6 hours, this confusion of the calendar might be, most easily, avoided. It would be necessary merely to make every fourth civil year to consist of 366 days; and, for that purpose, to interpose, or to *intercalate* a day in a month previous to March. By this *intercalation* what would have been March 21st is called March 20th; and, accordingly, the Sun would be still in the equinox on the same day of the month.

This mode of correcting the calendar was adopted by Julius Cæsar. The fourth year into which the intercalary day is introduced was called *Bissextile* *: it is now frequently called the *Leap*

* The *Bissextus* dies ante Calendas, being the intercalated day in the Julian Calendar.

year. The correction is called the *Julian correction,* and the length of a mean Julian year is equal to $365^{d}.25$.

If the astronomical year (see p. 529,) be equal to $365^{d}.242264$, it is less than the mean Julian by $0^{d}.007736$. The *Julian* correction. therefore, itself needs a correction. The calendar, regulated by it, would, in progress of time, become erroneous, and would require *reformation.*

The intercalation of the Julian correction being too great, its effect would be to *antedate* the happening of the equinox. Thus, (to return to the old illustration) the Sun, at the completion of the fourth civil year, now the Bissextile, would have passed the equinoctial point, by a time equal to four times $0^{d}.007736$: at the end of the next Bissextile, by eight times $0^{d}.007736$: at the end of 129 years, nearly by one day. In other words, the Sun would have been in the equinoctial point 24 *hours previously,* or on the *noon of March* 19th.

In the lapse of ages, this error would continue and be increased. Its accumulation in 1292 years would amount, nearly, to 10 days, and then, the vernal equinox would be reckoned to happen on March 10th.

The error into which the calendar had fallen, and would continue to fall, was noticed by Pope Gregory in 1582. At his time, the length of the year was known to greater precision, than at the time of Julius Cæsar. It was supposed equal to $365^{d}\ 5^{h}\ 49^{m}\ 16^{s}.23$. Gregory, desirous that the vernal equinox should be reckoned on or near March 21st, (on which day it happened in the year 325, when the Council of Nice was held,) ordered that the day succeeding the 4th of October 1582, instead of being called the 5th, should be called the 15th; thus, suppressing 10 days, which, in the interval between the years 325 and 1582, represented, nearly, the accumulation of error arising from the *excessive intercalation of the Julian correction.*

This act *reformed* the calendar: in order to *correct* it in future ages, it was prescribed that, at certain convenient periods, the intercalary day of the Julian correction should be omitted. Thus, the centenary years, 1700, 1800, 1900, are (as every year divisible by 4 is) according to the Julian correction, Bissextiles, but on these it was ordered that the intercalary day *should not be inserted:* inserted again in 2000, but not inserted in 2100, 2200, 2300; and so on for succeeding centuries *.

This is a most simple mode of regulating the calendar. It corrects the insufficiency of the Julian correction by omitting, in the space of 400 years, 3 intercalary days. And, it is easy to estimate the degree of its accuracy. For, the real error of the Julian correction is $0^d.007736$ in 1 year, consequently, $4 \times 0^d.7736$, or $3^d.0944$ in 400 years. Consequently, 0^d0944, or, $2^h\ 15^m\ 56^s.16$ in 400 years, or 1 day in 4237 years is the measure of the degree of inaccuracy in the Gregorian correction. Against such, it perhaps, is not worth the while to make any formal provision in the mode of regulating the calendar.

The calendar may be thus examined and regulated, without the aid of mathematical processes and formulæ. Yet, on this subject, the method of *continued Fractions* † is frequently

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* M. Delambre proposed to keep the calendar correct on this principle. Assuming the length of the year to be equal to $365^d.24\frac{2}{9}$, in 9 years the excess above the common civil year would be $24 \times 9 + 2$, or $2^d.18$

in 450 years	109
in 900	218
in 3600	872

According to the Julian correction there would be in 3600 years (3600 divided by 4 gives 900,) 900 intercalations, or 900 Bissextiles, too many by 28.

The Gregorian calendar casts out 27; in order, then, to cast out the 28th, and to keep the calendar right, it is merely necessary to make the year 3600 and its multiples common years.

† Since the excess of the tropical year above the civil is $0^d.242264$, the exact intercalation is that of 242264 days, in 1000000 years. But, since this intercalation would be of no practical use, we must find numbers nearly in the ratio of 242264 to 1000000: which may be effected by the method of continued fractions, as in pages 279, 280, &c. See on this subject, Euler's *Algebra*. Addition, pp. 426, &c. edit. 1774.

employed. This, however, is to use an instrument too *fine* for the occasion. The results have a degree of exactness, beyond what we require, or can practically avail ourselves of. The only thing, in the correction of the calendar, that requires a high degree of mathematical science, is the determination of the length of the astronomical year. Had this been known, to a greater exactness, by the Astronomers of the time of Julius Cæsar, the Julian correction would, probably, have superseded the necessity of the Gregorian.

ERRATA ET ADDENDA.

P. 7. l. 13. for 'greater,' read 'greatest.'
P. 9. l. 15. for 'more,' read 'move.'
P. 10. l. 3. for $b\,P\,b$, read $b\,P\,a$.
P. 16. last line, read $k\,WE'$ and $ES\,k$.
P. 17. l. 8. from bottom, for 'notions,' read 'nations.'
P. 18. l. 4. for 'is first,' read 'it is first.'
P. 30. l. 2. for 'night,' read 'day.'
P. 40. l. 14. for $\frac{360^\circ}{10}$, read $\frac{360^\circ}{1^\circ}$.
P. 194. l. 6. for 'exacted,' read 'exact.'
P. 344. l. 19. instead of $\frac{1}{2}(50''.1)^2$, &c., read $\frac{1}{2}$ sin. $1''(50''.1)^2$.

P. 697. the value of the obliquity $I,=23^\circ\ 27'\ 35''.1$, was taken from the N. A. of 1812, but all the values of I therein expressed are wrong to the amount of 8 seconds and upwards. The value of I on Nov. 12, 1812, ought to have been $23^\circ\ 27'\ 43''.6$: in which case, the resulting latitude would have been $4^\circ\ 58'\ 27''.6$: the value of the longitude will be very slightly affected by the change in the value of the obliquity.

P. 703. the two last figures in the logarithmic value of $106^\circ\ 6'\ 20''.73$, instead of 06 ought to have been 10. If these and the following figures be corrected, the complement of the latitude will be $86^\circ\ 13'\ 29''.2$. But the longitude in p. 704, is derived from the latitude: and if, in the calculation of the longitude, the above altered value of the computed latitude be substituted, the resulting value of the longitude will be $10^s\ 1^\circ\ 45'\ 13''.3$. The observation of Nov. 12, 1812, was made with the new mural circle: but those of Sept. 27, and 28, 1811, with the brass quadrant, which, it is now known, has, since it was first put up, changed its figure. The *changes* have not been accounted for in pp. 701, &c. these amount to $+7''.3$, $+6''.6$, corrections additive to the north polar distance of the 27th and 28th, and, if the calculations be made with the north polar distances so corrected, the resulting latitudes and longitudes on the 27th and 28th, will be respectively,

$$3^\circ\ 46'\ 23''.4, \qquad 2^\circ\ 41'\ 1'',$$
$$10^s\ 1^\circ\ 45'\ 11''.7, \qquad 10^s\ 16^\circ\ 47'\ 12'',$$

and the errors of the Tables in latitude $-15''.4$, $-14''.5$,
in longitude $+\ 3.5$, $+\ 6$.

ERRATA ET ADDENDA.

The results now agree much more nearly with those printed by order of the Board of Longitude. The *elements* of the latter results are now, by the kindness of the Astronomer Royal, in the Author's possession: they differ, however, in some small respects, from what he has used.

The above Errata are few in number, and not of much moment: others, no doubt, will be detected: the Author, however, does not anticipate the detection of many, relying on the careful and intelligent superintendence which the Work, during its progress, has received from the Rev. Dr. FRENCH, Master of Jesus College.