

## INTRODUCTION

### GENERAL CONSIDERATIONS

Toward the end of the 19th century, many physicists believed that all of the principles of physics had been discovered. The accepted principles, now called *classical physics*, included laws relating to Newton's mechanics, Gibbs' thermodynamics, LaGrange's and Hamilton's elasticity and hydrodynamics, Maxwell-Boltzmann molecular statistics, and Maxwell's Equations. However, the discovery that the intensity of blackbody radiation goes to zero, rather than infinity as predicted by the prevailing laws, provided an opportunity for new principles to be discovered. In 1900, Planck made the revolutionary assumption that energy levels were quantized, and that atoms of the blackbody could emit light energy only in amounts given by  $h\nu$ , where  $\nu$  is the radiation's frequency and  $h$  is a proportionality constant (now called Planck's constant). This assumption also led to our understanding of the photoelectric effect and ultimately to the concept of light as a particle called a photon. A similar course arose in the development of the model of the electron. In 1923, de Broglie suggested that the motion of an electron has a wave aspect where the wavelength,  $\lambda$ , is inversely proportional to the electron's momentum,  $p$ , as  $\lambda = \frac{h}{p}$ . This concept seemed unlikely according to the familiar properties of electrons such as charge, mass and adherence to the laws of particle mechanics. But, the wave nature of the electron was confirmed by Davisson and Germer in 1927 by observing diffraction effects when electrons were reflected from metals.

Experiments by the early part of the 20th century had revealed that both light and electrons behave as waves in certain instances and as particles in others. This was unanticipated from preconceptions about the nature of light and the electron. Early 20th century theoreticians proclaimed that light and atomic particles have a "wave-particle duality" that was unlike anything in our common-day experience. The wave-particle duality is the central mystery of the presently accepted atomic model, *quantum mechanics*, the one to which all other mysteries could ultimately be reduced.

Three atomic theories have been developed to explain the seemingly mysterious physics of the atomic scale. The earlier theories of Bohr and Schrödinger assume that the laws of physics that are valid in the macroworld do not hold true in the microworld of the atom. In contrast, the Mills theory is based on the foundation that laws of physics

valid in the macroworld do hold true in the microworld of the atom. In the present case, the predictions which arise from the equations of light and atomic particles are completely consistent with observation, including the wave-particle duality of light and atomic particles.

### Three Atomic Theories

The theories of Bohr, Schrödinger, and presently Mills all give the identical equation, Eq. (I.1), for the principal energy levels of the hydrogen atom.

$$E = -\frac{Z^2 e^2}{8\pi\epsilon_0 n^2 a_H} = -\frac{Z^2}{n^2} \times 2.1786 \times 10^{-18} \text{ J} = -\frac{Z^2}{n^2} \times 13.598 \text{ eV} \quad (\text{I.1})$$

However, only Mills theory is derived from first principles. The theories of Bohr and Schrödinger depend on specific postulates to yield Eq. (I.1). A relationship exists between the theories of Bohr and Schrödinger with respect to that of Mills which involves these postulates.

### Mills Theory-*a classical quantum theory*

One-electron atoms include the hydrogen atom, He II, Li III, Be IV, and so on. The mass-energy and angular momentum of the electron are constant; this requires that the equation of motion of the electron be temporally and spatially harmonic. Thus, the classical wave equation applies and

$$\nabla^2 - \frac{1}{v^2} \frac{\delta^2}{\delta t^2} \rho(r, \theta, \phi, t) = 0 \quad (\text{I.2})$$

where  $\rho(r, \theta, \phi, t)$  is the charge density function of the electron in time and space. In general, the wave equation has an infinite number of solutions. To arrive at the solution which represents the electron, a suitable boundary condition must be imposed. It is well known from experiments that each single atomic electron of a given isotope radiates to the same stable state. Thus, Mills chose the physical boundary condition of nonradiation of the bound electron to be imposed on the solution of the wave equation for the charge density function of the electron. The condition for radiation by a moving charge is derived from Maxwell's equations. To radiate, the spacetime Fourier transform of the current-density function must possess components synchronous with waves traveling at the speed of light [1]. Alternatively,

*For non-radiative states, the current-density function must not possess spacetime Fourier components that are synchronous with waves traveling at the speed of light.*

Proof that the condition for nonradiation by a moving point charge

is that its spacetime Fourier transform does not possess components that are synchronous with waves traveling at the speed of light is given by Haus [1]. The Haus derivation applies to a moving charge-density function as well because charge obeys superposition. The Haus derivation is summarized below.

The Fourier components of the current produced by the moving charge are derived. The electric field is found from the vector equation in Fourier space ( $\mathbf{k}$ ,  $\omega$ -space). The inverse Fourier transform is carried over the magnitude of  $\mathbf{k}$ . The resulting expression demonstrates that the radiation field is proportional to  $\mathbf{J} \left( \frac{\omega}{c} \mathbf{n}, \omega \right)$ , where  $\mathbf{J}(\mathbf{k}, \omega)$  is the spacetime Fourier transform of the current perpendicular to  $\mathbf{k}$  and  $\mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|}$ . Specifically,

$$\mathbf{E}(\mathbf{r}, \omega) \frac{d\omega}{2\pi} = \frac{c}{2\pi} \rho(\omega, \mathbf{r}) d\omega d\mathbf{r} \sqrt{\frac{\mu_0}{\epsilon_0}} \mathbf{n} \times \mathbf{n} \times \mathbf{J} \left( \frac{\omega}{c} \mathbf{n}, \omega \right) e^{i \frac{\omega}{c} \mathbf{n} \cdot \mathbf{r}} \quad (\text{I.3})$$

The field  $\mathbf{E}(\mathbf{r}, \omega) \frac{d\omega}{2\pi}$  is proportional to  $\mathbf{J} \left( \frac{\omega}{c} \mathbf{n}, \omega \right)$ , namely, the Fourier component for which  $\mathbf{k} = \frac{\omega}{c} \mathbf{n}$ . Factors of  $\omega$  that multiply the Fourier component of the current are due to the density of modes per unit volume and unit solid angle. An unaccelerated charge does not radiate in free space, not because it experiences no acceleration, but because it has no Fourier component  $\mathbf{J} \left( \frac{\omega}{c} \mathbf{n}, \omega \right)$ .

The time, radial, and angular solutions of the wave equation are separable. The motion is time harmonic with frequency  $\omega_n$ . To be a harmonic solution of the wave equation in spherical coordinates, the angular functions must be spherical harmonic functions. A zero of the spacetime Fourier transform of the product function of two spherical harmonic angular functions, a time harmonic function, and an unknown radial function is sought. The solution for the radial function which satisfies the boundary condition is a delta function

$$f(r) = \frac{1}{r^2} \delta(r - r_n) \quad (\text{I.4})$$

where  $r_n$  is an allowed radius. Thus, bound electrons are described by a charge-density (mass-density) function which is the product of a radial delta function ( $f(r) = \frac{1}{r^2} \delta(r - r_n)$ ), two angular functions (spherical harmonic functions), and a time harmonic function. Thus, an electron is a spinning, two-dimensional spherical surface, called an *electron*

**orbitsphere**, that can exist in a bound state at only specified distances from the nucleus. More explicitly, the orbitsphere comprises a two-dimensional spherical shell of moving charge. The corresponding current pattern of the orbitsphere comprises an infinite series of correlated orthogonal great circle current loops. The current pattern (shown in Figure 1.4) is generated over the surface by two orthogonal sets of an infinite series of nested rotations of two orthogonal great circle current loops where the coordinate axes rotate with the two orthogonal great circles. Each infinitesimal rotation of the infinite series is about the new x-axis and new y-axis which results from the preceding such rotation. For each of the two sets of nested rotations, the angular sum of the rotations about each rotating x-axis and y-axis totals  $\sqrt{2}\pi$  radians. The current pattern gives rise to the phenomenon corresponding to the spin quantum number.

Mills has built on this result by deriving a complete atomic theory based on first principles. The novel theory unifies Maxwell's Equations, Newton's Laws, and General and Special Relativity. The central feature of this theory is that all particles (atomic-size particles and macroscopic particles) obey the same physical laws.

The Mills theory solves the wave equation for the charge density function of the electron. The radial function for the electron indicates that the electron is two-dimensional. Therefore, the angular mass-density function of the electron,  $A(\theta, \phi, t)$ , must be a solution of the wave equation in two dimensions (plus time),

$$\nabla^2 - \frac{1}{v^2} \frac{\delta^2}{\delta t^2} A(\theta, \phi, t) = 0 \quad (I.5)$$

where  $\rho(r, \theta, \phi, t) = f(r)A(\theta, \phi, t) = \frac{1}{r^2} \delta(r - r_n)A(\theta, \phi, t)$  and  $A(\theta, \phi, t) = Y(\theta, \phi)k(t)$

$$\frac{1}{r^2 \sin \theta} \frac{\delta}{\delta \theta} \left( \sin \theta \frac{\delta}{\delta \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\delta^2}{\delta \phi^2} - \frac{1}{v^2} \frac{\delta^2}{\delta t^2} A(\theta, \phi, t) = 0 \quad (I.6)$$

where  $v$  is the linear velocity of the electron. The charge-density functions including the time-function factor are

$$l = 0$$

$$\rho(r, \theta, \phi, t) = \frac{e}{8\pi r^2} [\delta(r - r_n)] [Y_\ell^m(\theta, \phi) + Y_0^0(\theta, \phi)] \quad (I.7)$$

$$l \neq 0$$

$$\rho(r, \theta, \phi, t) = \frac{e}{4\pi r^2} [\delta(r - r_n)] [Y_0^0(\theta, \phi) + \text{Re}\{Y_\ell^m(\theta, \phi)[1 + e^{i\omega_n t}]\}] \quad (I.8)$$

where

$\text{Re}\{Y_\ell^m(\theta, \phi)[1 + e^{i\omega_n t}]\} = \text{Re}[Y_\ell^m(\theta, \phi) + Y_\ell^m(\theta, \phi)e^{i\omega_n t}] = P_\ell^m(\cos\theta)\cos m\phi + P_\ell^m(\cos\theta)\cos(m\phi + \omega_n t)$   
 and  $\omega_n = 0$  for  $m = 0$ . And, the Fourier transform of the charge density function which is derived in the Spacetime Fourier Transform of the Electron Function Section is a solution of the four-dimensional wave equation in frequency space ( $\mathbf{k}$ ,  $\omega$ -space).

$$M(s, \theta, \phi, \omega) = 4\pi \frac{\sin(2s_n r_n)}{2s_n r_n} \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} (\pi \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\frac{1}{2} \nu + \frac{1}{2}}{(\pi \cos \theta)^{2\nu+1} 2^{\nu+1}} \frac{2\nu!}{(\nu-1)!} s^{-2\nu} \quad (\text{I.9})$$

$$2\pi \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} (\pi \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\frac{1}{2} \nu + \frac{1}{2}}{(\pi \cos \theta)^{2\nu+1} 2^{\nu+1}} \frac{2\nu!}{(\nu-1)!} s^{-2\nu} \frac{1}{4\pi} [\delta(\omega - \omega_n) + \delta(\omega + \omega_n)]$$

The motion on the orbitsphere is angular; however, a radial component exists due to Special Relativistic effects. Consider the radial wave vector of the sinc function. When the radial projection of the velocity is  $c$

$$\mathbf{s}_n \cdot \mathbf{v}_n = \mathbf{s}_n \cdot \mathbf{c} = \omega_n \quad (\text{I.10})$$

the relativistically corrected wavelength is

$$r_n = \lambda_n \quad (\text{I.11})$$

(i.e. the lab frame motion in the angular direction goes to zero as the velocity approaches the speed of light). Substitution of Eq. (I.11) into the sinc function results in the vanishing of the entire Fourier transform of the current-density function. Thus, spacetime harmonics of  $\frac{\omega_n}{c} = k$  or

$$\frac{\omega_n}{c} \sqrt{\frac{\epsilon}{\epsilon_0}} = k \text{ for which the Fourier transform of the current-density}$$

function is nonzero do not exist. Radiation due to charge motion does not occur in any medium when this boundary condition is met.

### Relationship Between the Theories of Bohr and Schrödinger with Respect to that of Mills

J. J. Balmer showed, in 1885, that the frequencies for some of the lines observed in the emission spectrum of atomic hydrogen could be expressed with a completely empirical relationship. This approach was later extended by J. R. Rydberg, who showed that all of the spectral lines of atomic hydrogen were given by the equation:

$$\bar{\nu} = R \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \quad (\text{I.12})$$

where  $R = 109,677 \text{ cm}^{-1}$ ,  $n_f = 1, 2, 3, \dots$ ,  $n_i = 2, 3, 4, \dots$ , and  $n_i > n_f$ . In 1911,

Rutherford proposed a planetary model for the atom where the electrons revolved about the nucleus (which contained the protons) in various orbits. There was, however, a fundamental conflict with this

model and the prevailing classical physics. According to classical electromagnetic theory, an accelerated particle radiates energy (as electromagnetic waves). Thus, an electron in a Rutherford orbit, circulating at constant speed but with a continually changing direction of its velocity vector is being accelerated; thus, the electron should constantly lose energy by radiating and spiral into the nucleus.

An explanation was provided by Bohr in 1913, when he assumed that the energy levels were quantized and the electron was constrained to move in only one of a number of allowed states. Niels Bohr's theory for atomic hydrogen was based on an unprecedented postulate of stable circular orbits that do not radiate. Although no explanation was offered for the existence of stability for these orbits, the results gave energy levels in agreement with Rydberg's equation. Bohr's theory was a straightforward application of Newton's laws of motion and Coulomb's law of electric force. According to Bohr's model, the point particle electron was held to a circular orbit about the relatively massive point particle nucleus by the balance between the coulombic force of attraction between the proton and the electron and centrifugal force of the electron.

$$\frac{e^2}{4\pi\epsilon_0 r^2} = \frac{m_e v^2}{r} \quad (\text{I.13})$$

Bohr postulated the existence of stable orbits in defiance of classical physics (Maxwell's Equations), but he applied classical physics according to Eq. (I.13). Then Bohr realized that the energy formula Eq. (I.1) was given by postulating nonradiative states with angular momentum

$$L_z = m_e v r = n\hbar \quad n = 1, 2, 3, \dots \quad (\text{I.14})$$

and by solving the energy equation classically. The Bohr radius is given by substituting the solution of Eq. (I.14) for  $v$  into Eq. (I.13).

$$r = \frac{4\pi\epsilon_0 \hbar^2 n^2}{m_e e^2} = n^2 a_0 \quad n = 1, 2, 3, \dots \quad (\text{I.15})$$

The total energy is the sum of the potential energy and the kinetic energy. In the present case of an inverse squared central field, the total energy (which is the negative of the binding energy) is one half the potential energy [2]. The potential energy,  $\phi(\mathbf{r})$ , is given by Poisson's Equation

$$\phi(\mathbf{r}) = - \frac{\rho(\mathbf{r}') dV'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} \quad (\text{I.16})$$

For a point charge at a distance  $r$  from the nucleus the potential is

$$\phi(r) = - \frac{e^2}{4\pi\epsilon_0 r} \quad (\text{I.17})$$

Thus, the total energy is given by

$$E = -\frac{Z^2 e^2}{8\pi\epsilon_0 r} \quad (\text{I.18})$$

Substitution of Eq. (I.15) into Eq.(I.18) with the replacement of the electron mass by the reduced electron mass gives Eq. (I.1).

Bohr's model was in agreement with the observed hydrogen spectrum, but it failed with the helium spectrum, and it could not account for chemical bonds in molecules. The prevailing wisdom was that the Bohr model failed because it was based on the application of Newtonian mechanics for discrete particles. And, its limited applicability was attributed to the unwarranted assumption that the energy levels are quantized.

In 1923, de Broglie suggested that the motion of an electron has a wave aspect—  $\lambda = \frac{h}{p}$ . This was confirmed by Davisson and Germer in 1927 by observing diffraction effects when electrons were reflected from metals. Schrödinger reasoned that if electrons have wave properties, there must be a wave equation that governs their motion. And, in 1926, he proposed the Schrödinger equation

$$H \psi = E \psi \quad (\text{I.19})$$

where  $\psi$  is the wave function,  $H$  is the wave operator, and  $E$  is the energy of the wave. The Schrödinger equation solutions are three dimensional in space and four dimensional in spacetime

$$\nabla^2 \psi - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} (r, \theta, \phi, t) = 0 \quad (\text{I.20})$$

where  $\psi(r, \theta, \phi, t)$  according to quantum theory is the probability density function of the electron as described below. When the time harmonic function is eliminated [3],

$$-\frac{\hbar^2}{2\mu} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] + U(r) \psi(r, \theta, \phi) = E \psi(r, \theta, \phi) \quad (\text{I.21})$$

In general, the Schrödinger equation has an infinite number of solutions. To arrive at the solution which represents the electron, a suitable boundary condition must be imposed. Schrödinger postulated a boundary condition:  $\psi = 0$  as  $r \rightarrow \infty$ , which leads to a purely mathematical model of the electron. The general form of the solutions for  $\psi(r, \theta, \phi)$  are

$$\psi(r, \theta, \phi) = \sum_{l,m} f_{lm}(r) Y_{lm}(\theta, \phi) \quad (\text{I.22})$$

where the solutions for the angular part of Eq. (I.21),  $Y_{lm}(\theta, \phi)$ , are the spherical harmonics

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) e^{im\phi} \quad (\text{I.23})$$

The angular part of Eq. (I.21) is the generalized Legendre equation which is derived from the Laplace equation by Jackson (Eq. (3.9) of Jackson [4]). For the case that the potential energy is a constant times the wavenumber of the electron,  $k$  (a constant times the inverse of the de Broglie wavelength of the electron-- $k = \frac{2\pi}{\lambda}$ ;  $\lambda = \frac{h}{p}$ ), the radial part of Eq.

(I.21) is just the Bessel equation, Eq. (3.75) of Jackson [4] with  $\nu = l + \frac{1}{2}$ .

(In the present case of an inverse squared central field, the magnitude of each of the binding energy and the kinetic energy is one half the potential energy [2], and the de Broglie wavelength requires that the kinetic energy,  $\frac{p^2}{2m_e}$ , is a constant times the wavenumber squared.)

Thus, the solutions for  $f_{lm}(r)$  are

$$f_{lm}(r) = \frac{A_{lm}}{r^{1/2}} J_{l+1/2}(kr) + \frac{B_{lm}}{r^{1/2}} N_{l+1/2}(kr) \quad (\text{I.24})$$

It is customary to define the spherical Bessel and Hankel functions, denoted by  $j_l(x)$ ,  $n_l(x)$ ,  $h_l^{(1,2)}(x)$ , as follows:

$$\begin{aligned} j_l(x) &= \frac{\pi}{2x}^{1/2} J_{l+1/2}(x) \\ n_l(x) &= \frac{\pi}{2x}^{1/2} N_{l+1/2}(x) \\ h_l^{(1,2)}(x) &= \frac{\pi}{2x}^{1/2} [J_{l+1/2}(x) \pm iN_{l+1/2}(x)] \end{aligned} \quad (\text{I.25})$$

For  $l=0$ , the explicit forms are:

$$\begin{aligned} j_0(x) &= \frac{\sin x}{x} \\ n_0(x) &= -\frac{\cos x}{x} \\ h_0^{(1)}(x) &= \frac{e^{ix}}{ix} \end{aligned} \quad (\text{I.26})$$

Eq. (I.21) has the general form

$$H\psi = E\psi \quad (\text{I.27})$$

The energy is given by

$$\psi H\psi dv = E \int \psi^2 dv; \quad (\text{I.28})$$

Typically, the solutions are normalized.

$$\int \psi^* \psi dv = 1 \tag{I.29}$$

Thus,

$$\int \psi H \psi dv = E \tag{I.30}$$

A physical interpretation of Eq. (I.30) is sought. Schrödinger interpreted  $\psi^*(x) \psi(x)$  as the charge-density or the amount of charge between  $x$  and  $x + dx$  ( $\psi^*$  is the complex conjugate of  $\psi$ ). Presumably, then, he pictured the electron to be spread over large regions of space. Three years after Schrödinger's interpretation, Max Born, who was working with scattering theory, found that this interpretation led to logical difficulties, and he replaced the Schrödinger interpretation with the probability of finding the electron between  $x$  and  $x + dx$  as

$$|\psi(x)|^2 dx \tag{I.31}$$

Born's interpretation is generally accepted. Nonetheless, interpretation of the wave function is a never-ending source of confusion and conflict. Many scientists have solved this problem by conveniently adopting the Schrödinger interpretation for some problems and the Born interpretation for others. This duality allows the electron to be everywhere at one time—yet have no volume. Alternatively, the electron can be viewed as a discrete particle that moves here and there (from  $r = 0$  to  $r = \infty$ ), and  $\psi^* \psi$  gives the time average of this motion. According to the Copenhagen interpretation, every observable exists in a state of superposition of possible states and observation or the potential for knowledge causes the wavefunction corresponding to the possibilities to collapse into a definite state. The postulate of quantum measurement asserts that the process of measuring an observable forces the state vector of the system into an eigenvector of that observable, and the value measured will be the eigenvalue of that eigenvector. Thus, Eq.(I.30) corresponds to collapsing the wave function, and  $E$  is the eigenvalue of the eigenvector.

However, an alternative interpretation of Eq. (I.30) and the corresponding solutions for  $\psi$  exist. In the case that  $\psi$  is a function given by Eqs. (I.24-I.26), Eq. (I.30) is equivalent to an inverse Fourier transform. The spacetime inverse Fourier transform in three dimensions in spherical coordinates is given [5,6] as follows:

$$M(s, \theta, \phi) = \int_0^\infty \int_0^\pi \int_0^{2\pi} \rho(r, \theta, \phi) \exp(-i2\pi sr[\cos \theta \cos \theta + \sin \theta \sin \theta \cos(\phi - \phi)]) r^2 \sin \theta dr d\theta d\phi \tag{I.32}$$

With circular symmetry [5]

$$M(s, ) = 2\pi \int_0^\pi \rho(r, \theta) J_0(2\pi sr \sin \theta) \exp(-i2\pi sr \cos \theta) r^2 \sin \theta dr d\theta \quad (I.33)$$

With spherical symmetry [5],

$$M(s) = 4\pi \int_0^\infty \rho(r) \text{sinc}(2sr) r^2 dr = 4\pi \int_0^\infty \rho(r) \frac{\sin sr}{sr} r^2 dr \quad (I.34)$$

The Schrödinger equation (Eq. (I.21)) can be transformed into a sum comprising a part that depends only on the radius and a part that is a function of angle only. By separation of variables, and substitution of the eigenvalues corresponding to the angular part [7], the Schrödinger equation becomes the radial equation,  $R(r)$ , given by

$$-\frac{\hbar^2}{2\mu r^2} \frac{d}{dr} r^2 \frac{dR}{dr} + \frac{\hbar^2 l(l+1)}{2\mu r^2} + U(r) R(r) = ER(r) \quad (I.35)$$

Consider the case that  $l = 0$ , that the potential energy is a constant times the wavenumber, and that the radial function is a spherical Bessel function as given by Eqs. (I.24-I.26). In this case, Eq. (I.35) is given by

$$4\pi \int_0^\infty \frac{\sin sr}{sr} - \frac{\hbar^2}{2\mu r^2} \frac{d}{dr} r^2 \frac{d}{dr} + U(r) \frac{\sin sr}{sr} r^2 dr = E4\pi \int_0^\infty \frac{\sin sr}{sr} \frac{\sin sr}{sr} r^2 dr \quad (I.36)$$

Eq. (I.34) is the Fourier transform integral in spherical coordinates with spherical symmetry. The left hand side (LHS) of Eq. (I.36) is equivalent to the LHS of Eq. (I.30) wherein  $\psi$  is given by Eq. (I.26). Then the LHS of Eq. (I.36) is the Fourier transform integral of  $H\psi$  wherein the kernel is

$r^2 \frac{\sin sr}{sr}$ . The integral of Eq. (I.30) gives  $E$  which is a constant. The energy  $E$  of Eq. (I.27) is a constant such as  $b$ . Thus,  $H\psi$  according to Eq. (I.27) is a constant times  $\psi$ .

$$H\psi = b\psi \quad (I.37)$$

where  $b$  is a constant. Since  $b$  is an arbitrary constant, consider the following case wherein  $b$  is the Rydberg formula:

$$b = -\frac{Z^2 e^2}{8\pi\epsilon_0 n^2 a_H} \quad (I.38)$$

Then the energy of Eq. (I.30) is that given by Eq. (I.1). But, the Schrödinger equation can be solved to give the energy corresponding to the radial function given by Eq. (I.4) of the Mills theory. The radial function used to calculate the energy is a delta function which corresponds to an inverse Fourier transform of the solution for  $\psi$ .

$$(s) = \delta(s - s_n) \quad (I.39)$$

With a change of variable, Eq. (I.39) becomes Eq. (I.4).

Eq. (I.36) can be expressed as follows

$$4\pi \int_0^\infty \frac{\sin sr}{sr} - \frac{\hbar^2}{2\mu r^2} \frac{d}{dr} r^2 \frac{d}{dr} + U(r) \frac{\sin s_n r}{s_n r} r^2 dr = E4\pi \int_0^\infty \frac{\sin sr}{sr} \frac{\sin s_n r}{s_n r} r^2 dr \quad (I.40)$$

It follows from Eq. (I.34) that the right side integral is the Fourier transform of a radial Dirac delta function.

$$4\pi E \int_0^{\infty} \frac{\sin s_n r}{s_n r} \frac{\sin sr}{sr} r^2 dr = E \frac{\delta(s - s_n)}{4\pi s_n^2} \quad (\text{I.41})$$

Substitution of Eq. (I.37) into Eq. (I.40) gives

$$4\pi b \int_0^{\infty} \frac{\sin s_n r}{s_n r} \frac{\sin sr}{sr} r^2 dr = b \frac{\delta(s - s_n)}{4\pi s_n^2} \quad (\text{I.42})$$

Substitution of Eq. (I.41) and Eq. (I.42) into Eq. (I.40) gives

$$b\delta(s - s_n) = E\delta(s - s_n) \quad (\text{I.43})$$

Consider the case where  $b$  is given by

$$b = -\frac{\hbar^2}{2m_e n \frac{a_0}{Z^2} s} = -\frac{\frac{1}{2} Z^2 e^2}{8\pi\epsilon_0 s} \quad (\text{I.44})$$

and  $s_n$  is given by

$$s_n = na_H \quad (\text{I.45})$$

where  $r_n = na_H$ . According to the duality and change of scale properties of Fourier transforms [8], ***the energy equation of the Mills theory and that of quantum mechanics are identical***, the energy of a radial Dirac delta function of radius equal to an integer multiple of the radius of the hydrogen atom. The total energy of the electron is given by Poisson's Equation (Eq. (I.17)) and the relationship that the total energy is one half the potential energy in the case of an inverse squared central force [2].

$$E = \int_0^{\infty} E\delta(r - r_n) dr = - \int_0^{\infty} \delta(r - r_n) \frac{\frac{1}{2} Z^2 e^2}{8\pi\epsilon_0 r} dr = -\frac{\frac{1}{2} Z^2 e^2}{8\pi\epsilon_0 r_n} = -\frac{Z^2 e^2}{8\pi\epsilon_0 n^2 a_H} \quad (\text{I.46})$$

As was the case with the Bohr theory, quantum mechanics which is based on the Schrödinger equation and modifications of the Schrödinger equation has encountered several obstacles that have proved insurmountable. For examples:

1.) The Schrödinger equation failed to predict the electron spin, and it provides no rational basis for the phenomenon of spin, the Pauli exclusion principle, or Hund's rules. Instantaneous exchange of information between particles is required, which violates Special Relativity. According to this model, the electron must spin in one dimension and give rise to a Bohr magneton; yet, classically the energy of a magnetic moment is  $\frac{\mu^2}{r^3}$  which in the present case is infinity (by substitution of  $r = 0$  for the model that the electron is a point particle), not the required  $mc^2$ . This interpretation is in violation of Special Relativity [9]. A modification of the Schrödinger

equation was developed by Dirac to explain spin which relies on the unfounded notions of negative energy states of the vacuum, virtual particles, and gamma factors.

2.) Quantum mechanics assumes that atomic-size particles obey different physical laws than macroscopic objects. For example, according to Maxwell's equations the electron described by a Schrödinger one-electron wave function would radiate. Quantum Electrodynamics (QED) based on vacuum energy fluctuations (zero point fluctuation (ZPF) energy) was invented to address some of these issues, but rigorous solutions of QED result in no solutions other than infinities. (Radiated photons make an infinite contribution to the perturbation series).

The failure of quantum mechanics is attributed to the unwarranted assumption that first principles such as Maxwell's Equations do not apply to the electron and the incorrect notion that the electron is described by a probability distribution function of a point particle. The success of the Schrödinger equation can be understood in terms of the nature of the solutions to the wave equation. In general, the solutions are separable, provide three quantum numbers, and yield eigenvalues. By adjusting the arbitrary constants of the separable solutions, the desired eigenvalues can be obtained.

The fourth quantum number arises naturally in the Mills theory as derived in The Electron g Factor Section. The Stern-Gerlach experiment implies a magnetic moment of one Bohr magneton and an associated angular momentum quantum number of 1/2. Historically, this quantum number is called the spin quantum number,  $s$  ( $s = \frac{1}{2}$ ;  $m_s = \pm \frac{1}{2}$ ).

Conservation of angular momentum of the orbitsphere permits a discrete change of its "kinetic angular momentum" ( $\mathbf{r} \times m\mathbf{v}$ ) by the field of  $\frac{\hbar}{2}$ , and concomitantly the "potential angular momentum" ( $\mathbf{r} \times e\mathbf{A}$ ) must change by  $-\frac{\hbar}{2}$ . The flux change,  $\phi$ , of the orbitsphere for  $r < r_n$  is determined as follows:

$$\mathbf{L} = \frac{\hbar}{2} - \mathbf{r} \times e\mathbf{A} \quad (\text{I.47})$$

$$= \frac{\hbar}{2} - \frac{e2\pi rA}{2\pi} \quad (\text{I.48})$$

$$= \frac{\hbar}{2} - \frac{e\phi}{2\pi} \quad (\text{I.49})$$

In order that the change of angular momentum,  $\mathbf{L}$ , equals zero,  $\phi$  must be  $\phi_0 = \frac{h}{2e}$ , the magnetic flux quantum. Thus, to conserve angular

momentum in the presence of an applied magnetic field, the orbitsphere magnetic moment can be parallel or antiparallel to an applied field as observed with the Stern-Gerlach experiment, and the flip between orientations ( a rotation of  $\frac{\pi}{2}$ ) is accompanied by the "capture" of the magnetic flux quantum by the orbitsphere. And, the total energy of the flip transition is the sum of Eq. (1.148), the energy of a fluxon treading the orbitsphere and Eq. (1.136), the energy of reorientation of the magnetic moment.

$$E_{mag}^{spin} = 2 \mu_B \mathbf{B} + \frac{\alpha}{2\pi} \mu_B \mathbf{B} \quad (I.50)$$

$$E_{mag}^{spin} = 2(1 + \frac{\alpha}{2\pi}) \mu_B \mathbf{B} \quad (I.51)$$

$$E_{mag}^{spin} = 2g\mu_B \mathbf{B} \quad (I.52)$$

The spin-flip transition can be considered as involving a magnetic moment of g times that of a Bohr magneton. The factor g is redesignated the fluxon g factor as opposed to the anomalous g factor and its value is 1.00116. The experimental value is 1.00116.

The orbitsphere is a resonator cavity which traps photons of discrete frequencies. The radius of an orbitsphere increases with the absorption of electromagnetic energy. The solutions to Maxwell's equations for modes that can be excited in the orbitsphere resonator cavity give rise to four quantum numbers, and the energies of the modes are the experimentally known hydrogen spectrum.

*The subscript n is used in Eq. (I.46); the quantization condition appears in the Excited States of the One Electron Atom (Quantization) Section. Quantization arises as "allowed" solutions of the wave equation corresponding to a resonance between the electron and a photon.*

More explicitly, it is well known that resonator cavities can trap electromagnetic radiation of discrete resonant frequencies. The orbitsphere is a resonator cavity which traps photons of discrete frequencies. Thus, photon absorption occurs as an excitation of a resonator mode. The "trapped photon" is a "standing electromagnetic wave" which actually is a circulating wave that propagates along each great circle current loop of the orbitsphere. The time-function factor,  $k(t)$ , for the "standing wave" is identical to the time-function factor of the orbitsphere in order to satisfy the boundary (phase) condition at the orbitsphere surface. Thus, the angular frequency of the "trapped photon" has to be identical to the angular frequency of the electron

orbitsphere,  $\omega_n$ . Furthermore, the phase condition requires that the angular functions of the "trapped photon" have to be identical to the spherical harmonic angular functions of the electron orbitsphere. Combining  $k(t)$  with the  $\phi$ -function factor of the spherical harmonic gives  $e^{i(m\phi - \omega_n t)}$  for both the electron and the "trapped photon" function. The photon is "glued" to the inner orbitsphere surface and the outer nuclear surface as photon source charge-density with a radial electric field.

From the application of the nonradiative boundary condition, the instability of excited states as well as the stability of the "ground" state arise naturally in the Mills theory as derived in Stability of Atoms and Hydrinos Section. In addition to the above known states of hydrogen (Eq. (I.1)), the theory predicts the existence of a previously unknown form of matter: hydrogen atoms and molecules having electrons of lower energy than the conventional "ground" state, called *hydrinos* and *dihydrinos*, respectively, where each energy level corresponds to a fractional quantum number.

The central field of the proton corresponds to integer one charge. Excited states comprise an electron with a trapped photon. In all energy states of hydrogen, the photon has an electric field which superposes with the field of the proton. In the  $n = 1$  state, the sum is one, and the sum is zero in the ionized state. In an excited state, the sum is a fraction of one (i.e. between zero and one). Derivations from first principles given by Mills demonstrate that each "allowed" fraction corresponding to an excited state is  $\frac{1}{\text{integer}}$ . The relationship between the electric field equation and the "trapped photon" source charge-density function is given by Maxwell's equation in two dimensions.

$$\mathbf{n} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = \frac{\sigma}{\epsilon_0} \quad (\text{I.53})$$

where  $\mathbf{n}$  is the radial normal unit vector,  $\mathbf{E}_1 = 0$  ( $\mathbf{E}_1$  is the electric field outside of the orbitsphere),  $\mathbf{E}_2$  is given by the total electric field at  $r_n = na_H$ , and  $\sigma$  is the surface charge-density. The electric field of an excited state is fractional; therefore, the source charge function is fractional. It is well known that fractional charge is not "allowed". The reason is that fractional charge typically corresponds to a radiative current density function. The excited states of the hydrogen atom are examples. They are radiative; consequently, they are not stable. Thus, an excited electron decays to the first nonradiative state corresponding to an integer field,  $n = 1$ . Equally valid from first principles are electronic states where the sum of the photon field and the central field are an integer. These states are nonradiative. A catalyst can effect a transition

between these states.

### Instability of Excited States

For the excited (integer quantum) energy states of the hydrogen atom,  $\sigma_{photon}$ , the two dimensional surface charge due to the "trapped photons" at the orbitsphere, is given by Eqs. (2.6) and (2.11).

$$\sigma_{photon} = \frac{e}{4\pi(r_n)^2} Y_0^0(\theta, \phi) - \frac{1}{n} \left[ Y_0^0(\theta, \phi) + \text{Re} \left\{ Y_\ell^m(\theta, \phi) \left[ 1 + e^{i\omega_n t} \right] \right\} \right] \delta(r - r_n) \quad n = 2, 3, 4, \dots, \quad (\text{I.54})$$

Whereas,  $\sigma_{electron}$ , the two dimensional surface charge of the electron orbitsphere is

$$\sigma_{electron} = \frac{-e}{4\pi(r_n)^2} \left[ Y_0^0(\theta, \phi) + \text{Re} \left\{ Y_\ell^m(\theta, \phi) \left[ 1 + e^{i\omega_n t} \right] \right\} \right] \delta(r - r_n) \quad (\text{I.55})$$

The superposition of  $\sigma_{photon}$  (Eq. (I.54)) and  $\sigma_{electron}$  (Eq. (I.55)), where the spherical harmonic functions satisfy the conditions given in the Angular Function Section, is equivalent to the sum of a radial electric dipole represented by a doublet function and a radial electric monopole represented by a delta function.

$$\sigma_{photon} + \sigma_{electron} = \frac{e}{4\pi(r_n)^2} Y_0^0(\theta, \phi) \dot{\delta}(r - r_n) - \frac{1}{n} Y_0^0(\theta, \phi) \delta(r - r_n) - 1 + \frac{1}{n} \left[ \text{Re} \left\{ Y_\ell^m(\theta, \phi) \left[ 1 + e^{i\omega_n t} \right] \right\} \right] \delta(r - r_n) \quad n = 2, 3, 4, \dots, \quad (\text{I.56})$$

where

$$\left[ +\delta(r - r_n) - \delta(r - r_n) \right] = \dot{\delta}(r - r_n) \quad (\text{I.57})$$

The spacetime Fourier transform of Eq. (I.56), the superposition of  $\sigma_{photon}$  (Eq. (I.54)) and  $\sigma_{electron}$  (Eq. (I.55)) is

$$M(s, \theta, \phi, \omega) = 4\pi s_n \frac{\cos(2s_n r_n)}{2s_n r_n} \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} (\pi \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\frac{1}{2} \nu + \frac{1}{2}}{(\pi \cos \theta)^{2\nu+1} 2^{\nu+1}} \frac{2\nu!}{(\nu-1)!} s^{-2\nu} \\ + 2\pi \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} (\pi \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\frac{1}{2} \nu + \frac{1}{2}}{(\pi \cos \theta)^{2\nu+1} 2^{\nu+1}} \frac{2\nu!}{(\nu-1)!} s^{-2\nu} \frac{1}{4\pi} [\delta(\omega - \omega_n) + \delta(\omega + \omega_n)] \quad (\text{I.58})$$

Consider the radial wave vector of the cosine function of Eq. (I.58). When the radial projection of the velocity is  $c$

$$\mathbf{s}_n \cdot \mathbf{v}_n = \mathbf{s}_n \cdot \mathbf{c} = \omega_n \quad (\text{I.59})$$

the relativistically corrected wavelength is

$$r_n = \lambda_n \quad (\text{I.60})$$

Substitution of Eq. (I.60) into the cosine function does not result in the vanishing of the Fourier Transform of the current-density function.

Thus, spacetime harmonics of  $\frac{\omega_n}{c} = k$  or  $\frac{\omega_n}{c} \sqrt{\frac{\epsilon}{\epsilon_0}} = k$  do exist for which the

Fourier Transform of the current-density function is nonzero. An excited state is metastable because it is the sum of nonradiative (stable) and radiative (unstable) components and de-excites with a transition probability given by the ratio of the power to the energy of the transition [10].

### Stability of "Ground" and Hydrino States

For the below "ground" (fractional quantum) energy states of the hydrogen atom,  $\sigma_{photon}$ , the two dimensional surface charge due to the "trapped photon" at the electron orbitsphere, is given by Eqs. (5.13) and (2.11).

$$\sigma_{photon} = \frac{e}{4\pi(r_n)^2} Y_0^0(\theta, \phi) - \frac{1}{n} \left[ Y_0^0(\theta, \phi) + \text{Re} \left\{ Y_\ell^m(\theta, \phi) [1 + e^{i\omega_n t}] \right\} \right] \delta(r - r_n) \quad n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \quad (\text{I.61})$$

And,  $\sigma_{electron}$ , the two dimensional surface charge of the electron orbitsphere is

$$\sigma_{electron} = \frac{-e}{4\pi(r_n)^2} \left[ Y_0^0(\theta, \phi) + \text{Re} \left\{ Y_\ell^m(\theta, \phi) [1 + e^{i\omega_n t}] \right\} \right] \delta(r - r_n) \quad (\text{I.62})$$

The superposition of  $\sigma_{photon}$  (Eq. (I.61)) and  $\sigma_{electron}$ , (Eq. (I.62)) where the spherical harmonic functions satisfy the conditions given in the Angular Function Section is a radial electric monopole represented by a delta function.

$$\sigma_{photon} + \sigma_{electron} = \frac{-e}{4\pi(r_n)^2} \frac{1}{n} Y_0^0(\theta, \phi) + 1 + \frac{1}{n} \text{Re} Y_\ell^m(\theta, \phi) [1 + e^{i\omega_n t}] \delta(r - r_n) \quad n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \quad (\text{I.63})$$

As given in the Spacetime Fourier Transform of the Electron Function Section, the radial delta function does not possess spacetime Fourier components synchronous with waves traveling at the speed of light (Eqs. (I.9-I.11)). Thus, the below "ground" (fractional quantum) energy states of the hydrogen atom are stable. The "ground" ( $n = 1$  quantum) energy state is just the first of the nonradiative states of the hydrogen atom; thus, it is the state to which excited states decay.

### Catalytic Lower-Energy Hydrogen Electronic Transitions

Comparing transitions between below "ground" (fractional quantum) energy states as opposed to transitions between excited (integer quantum) energy states, it can be appreciated that the former are not effected by photons; whereas, the latter are. Transitions are symmetric with respect to time. Current density functions which give rise to photons according to the boundary condition are created by photons in the reverse process. Excited (integer quantum) energy states correspond to this case. And, current density functions which do not give rise to photons according to the nonradiative boundary condition

are not created by photons in the reverse process. Below "ground" (fractional quantum) energy states correspond to this case. But, atomic collisions and nonradiative energy transfer can cause a stable state to undergo a transition to the next stable state. The transition between two stable nonradiative states effected by a collision with an resonant energy sink is analogous to the reaction of two atoms to form a diatomic molecule which requires a third-body collision to remove the bond energy [11].

### Energy Hole Concept

The nonradiative boundary condition and the relationship between the electron and the photon give the "allowed" hydrogen energy states which are quantized as a function of the parameter  $n$ . Each value of  $n$  corresponds to an allowed transition effected by a resonant photon which excites the electronic transition. In addition to the traditional integer values (1, 2, 3,...) of  $n$ , values of fractions are allowed which correspond to transitions with an increase in the central field (charge) and decrease in the size of the hydrogen atom. This occurs, for example, when the electron couples to another electronic transition or electron transfer reaction which can absorb energy, an energy sink. This transition reaction of the electron of hydrogen to a lower energy state occurs by the *absorption of an energy hole by the hydrogen atom*. The absorption of an energy hole destroys the balance between the centrifugal force and the resulting increased central electric force. Consequently, the electron undergoes a transition to a lower energy nonradiative state. Thus, the corresponding reaction from an initial energy state to a lower energy state effected by an energy hole is called a *transition reaction*.

From energy conservation, the energy hole of a hydrogen atom which excites resonator modes of radial dimensions  $\frac{a_H}{m+1}$  is

$$m \times 27.2 \text{ eV} \quad (I.64)$$

where  $m = 1, 2, 3, 4, \dots$

After resonant absorption of the energy hole, the radius of the orbitsphere,  $a_H$ , decreases to  $\frac{a_H}{m+1}$  and after  $p$  cycles of transition

reaction, the radius is  $\frac{a_H}{mp+1}$ . In other words, the radial ground state

field can be considered as the superposition of Fourier components. The removal of negative Fourier components of energy  $m \times 27.2 \text{ eV}$ , where  $m$  is an integer increases the positive electric field inside the spherical shell by  $m$  times the charge of a proton. The resultant electric field is a time-harmonic solution of Laplace's Equations in spherical coordinates. In

this case, the radius at which force balance and nonradiation are achieved is  $\frac{a_H}{m+1}$  where  $m$  is an integer. In decaying to this radius from the "ground" state, a total energy of  $[(m+1)^2 - 1^2] \times 13.6 \text{ eV}$  is released. The transition reaction is hereafter referred to as the **BlackLight Process**. The source of energy holes may not be consumed in the transition reaction; therefore it is a hydrogen catalyst.

An efficient catalytic system that hinges on the coupling of three resonator cavities involves potassium. For example, the second ionization energy of potassium is  $31.63 \text{ eV}$ . This energy hole is obviously too high for resonant absorption. However,  $K^+$  releases  $4.34 \text{ eV}$  when it is reduced to  $K$ . The combination of  $K^+$  to  $K^{2+}$  and  $K^+$  to  $K$ , then, has a net energy change of  $27.28 \text{ eV}$ .

$$27.28 \text{ eV} + K^+ + K^+ + H \frac{a_H}{p} \rightarrow K + K^{2+} + H \frac{a_H}{(p+1)} + [(p+1)^2 - p^2] \times 13.6 \text{ eV} \quad (\text{I.65})$$

$$K + K^{2+} \rightarrow K^+ + K^+ + 27.28 \text{ eV} \quad (\text{I.66})$$

And, the overall reaction is

$$H \frac{a_H}{p} \rightarrow H \frac{a_H}{(p+1)} + [(p+1)^2 - p^2] \times 13.6 \text{ eV} \quad (\text{I.67})$$

Note that the energy given off as the atom undergoes a transition to a lower energy level is much greater than the energy lost to the energy hole. Also, the energy released is large compared to conventional chemical reactions.

### Disproportionation of Energy States

Lower-energy hydrogen atoms, *hydrinos*, can act as a source of energy holes that can cause transition reactions because the excitation and/or ionization energies are  $m \times 27.2 \text{ eV}$  (Eq. (I.64)). The general equation for the absorption of an energy hole of  $27.21 \text{ eV}$ ,  $m = 1$  in Eq. (I.64), during the transition cascade for the  $p$ th cycle of the hydrogen-type atom,  $H \frac{a_H}{p}$ , with the hydrogen-type atom,  $H \frac{a_H}{m'}$ , that is ionized as the source of energy holes that causes the transition reaction is represented by

$$27.21 \text{ eV} + H \frac{a_H}{m'} + H \frac{a_H}{p} \rightarrow H^+ + e^- + H \frac{a_H}{(p+1)} + [(p+1)^2 - p^2] \times 13.6 \text{ eV} - (m'^2 - 2) \times 13.6 \text{ eV}$$

$$(I.68)$$

$$H^+ + e^- \rightarrow H \frac{a_H}{1} + 13.6 \text{ eV} \quad (I.69)$$

And, the overall reaction is

$$H \frac{a_H}{m'} + H \frac{a_H}{p} \rightarrow H \frac{a_H}{1} + H \frac{a_H}{(p+1)} + [2p + 1 - m'^2] \times 13.6 \text{ eV} + 13.6 \text{ eV} \quad (I.70)$$

For example, the equation for the absorption of an energy hole of 27.21 eV,  $m = 1$  in Eq. (I.64), during the transition cascade for the third cycle of the hydrogen-type atom,  $H \frac{a_H}{3}$ , with the hydrogen-type atom,  $H \frac{a_H}{2}$ , that is ionized as the source of energy holes that causes the transition reaction is represented by

$$27.21 \text{ eV} + H \frac{a_H}{2} + H \frac{a_H}{3} \rightarrow H^+ + e^- + H \frac{a_H}{4} + [4^2 - 3^2] \times 13.6 \text{ eV} - 27.21 \text{ eV} \quad (I.71)$$

$$H^+ + e^- \rightarrow H \frac{a_H}{1} + 13.6 \text{ eV} \quad (I.72)$$

And, the overall reaction is

$$H \frac{a_H}{2} + H \frac{a_H}{3} \rightarrow H \frac{a_H}{1} + H \frac{a_H}{4} + [4^2 - 3^2 - 4] \times 13.6 \text{ eV} + 13.6 \text{ eV} \quad (I.73)$$

Disproportionation may be the predominant mechanism of hydrogen electronic transitions to lower energy levels of interstellar and solar hydrogen and hydrinos. Hydrogen transitions to electronic energy levels below the "ground" state corresponding to fractional quantum numbers match the spectral lines of the extreme ultraviolet background of interstellar space and from the sun. This assignment given in the Spectral Data of Hydrinos from the Dark Interstellar Medium and Spectral Data of Hydrinos, Dihydrinos, and Hydrino Hydride Ions from the Sun Section resolves the paradox of the identity of dark matter, accounts for many celestial observations such as: diffuse H emission is ubiquitous throughout the Galaxy whereby widespread sources of flux shortward of 912 Å are required [12], and resolves many solar problems. The energy of the emission line for the transition given by Eqs. (I.71-

I.73) whereby  $H \frac{a_H}{2}$  is ionized as the source of the energy hole of 27.2 eV,  $m = 1$  in Eq. (I.64), that causes transition reaction is 40.8 eV (See Table 1 of the Foreword Section).

$$H \frac{a_H}{3} \quad H \frac{a_H}{2} \quad H \frac{a_H}{4} \quad (I.74)$$

In summary, the mathematics of the theories of Bohr, Schrödinger, and presently Mills converge to Eq. (I.1) as the principal energy levels of the hydrogen atom.

$$E_n = -\frac{e^2}{n^2 8\pi\epsilon_0 a_H} = -\frac{13.598 \text{ eV}}{n^2} \quad (I.75a)$$

$$n = 1, 2, 3, \dots \quad (I.75b)$$

where  $a_H$  is the Bohr radius for the hydrogen atom (52.947 pm),  $e$  is the magnitude of the charge of the electron, and  $\epsilon_0$  is the vacuum permittivity. However, the physics is quite different. Only the Mills theory is derived from first principles and holds over a scale of spacetime of 45 orders of magnitude-it correctly predicts the nature of the universe from the scale of the quarks to that of the cosmos. And, only the Mills theory predicts fractions as "allowed" states. Explicitly, Mills theory gives Eq. (I.75a) as the energy-level equation for atomic hydrogen, but the restriction on " $n$ ", Eq. (I.75b), should be replaced by Eq. (I.75c).

$$n = 1, 2, 3, \dots, \text{ and } n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \quad (I.75c)$$

A number of experimental observations lead to the conclusion that atomic hydrogen can exist in fractional quantum states that are at lower energies than the traditional "ground" ( $n = 1$ ) state. The corresponding process, the catalytic release of thermal energy as electrons are induced undergo transitions to lower energy levels corresponding to fractional quantum numbers, represents a virtually limitless source of clean, inexpensive energy.

## References

1. Haus, H. A., "On the radiation from point charges", American Journal of Physics, 54, (1986), pp. 1126-1129.
2. Fowles, G. R., Analytical Mechanics, Third Edition, Holt, Rinehart, and Winston, New York, (1977), pp. 154-156.
3. McQuarrie, D. A., Quantum Chemistry, University Science Books, Mill Valley, CA, (1983), pp. 78-79.
4. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley &

- Sons, New York, (1962), pp. 84-108.
5. Bracewell, R. N., The Fourier Transform and Its Applications, McGraw-Hill Book Company, New York, (1978), pp. 252-253.
  6. Siebert, W. McC., Circuits, Signals, and Systems, The MIT Press, Cambridge, Massachusetts, (1986), p. 415.
  7. McQuarrie, D. A., Quantum Chemistry, University Science Books, Mill Valley, CA, (1983), pp. 221-224.
  8. Siebert, W. McC., Circuits, Signals, and Systems, The MIT Press, Cambridge, Massachusetts, (1986), p. 416.
  9. Pais, A., "George Uhlenbeck and the discovery of electron spin", *Physics Today*, 42, Dec., (1989), pp. 34-40.
  10. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, New York, (1962), pp. 758-763.
  11. N. V. Sidgwick, The Chemical Elements and Their Compounds, Volume I, Oxford, Clarendon Press, (1950), p.17.
  12. Labov, S., Bowyer, S., "Spectral observations of the extreme ultraviolet background", *The Astrophysical Journal*, 371, (1991), pp. 810-819.

## A NEW ATOMIC THEORY DERIVED FROM FIRST PRINCIPLES

To overcome the limitations of quantum mechanics, physical laws which are exact on all scales are sought. Rather than engendering the electron with a wave nature as suggested by the Davisson-Germer experiment and fabricating a set of associated postulates and mathematical rules for wave operators, a new theory is derived from first principles.

The novel theory unifies Maxwell's Equations, Newton's Laws, and General and Special Relativity. Theoretical predictions conform with experimental observations. The closed form calculations of a broad spectrum of fundamental phenomena contain fundamental constants only. Equations of the one electron atom are derived which give four quantum numbers, the spin/nuclear hyperfine structure, the Rydberg constant, the stability of atoms, the ionization energies, the equation of the photon, the equation of the electron in free space, the results of the Stern-Gerlach experiment, the electron g factor, the spin angular momentum energies, the excited states, the results of the Davisson-Germer experiment, the parameters of pair production, and the hyperfine structure interval of positronium. Ionization energies of two and three electron atoms are given as well as the bond energies, vibrational energies, rotational energies, and bond distances of hydrogen-type molecules and molecular ions.

From the closed form solution of the helium atom, the predicted electron scattering intensity is derived. The closed form scattering equation matches the experimental data; whereas, calculations based on the Born model of the atom utterly fail at small scattering angles. The implications for the invalidity of the Schrödinger and Born model of the atom and the dependent Heisenberg Uncertainty Principle are discussed. The theory of collective phenomena including statistical mechanics, superconductivity, Quantum Hall effects, and the Aharonov-Bohm effect is given. Atomic equations of gravitation are derived which provide the basis of the atomic, thermodynamic, and cosmological arrows of time, and the equation of the expansion of the universe. The gravitational equations with the equivalence of the particle production energies permit the equivalence of mass/energy and the spacetime metric from which the gravitational constant and the masses of the leptons, the quarks, and nucleons are derived. The basis of the antigravitational force is presented with supporting experimental evidence. The magnetic moments of the nucleons are derived. The beta decay energy of the neutron, and the binding energy of deuterium are calculated. The theory of alpha decay is derived.

In addition to the above known phenomena and characteristics of

fundamental particles and forces, the theory predicts the existence of a previously unknown form of matter: hydrogen atoms and molecules having electrons of lower energy than the conventional "ground" state called *hydrinos* and *dihydrinos*, respectively, where each energy level corresponds to a fractional quantum number. The existence of hydrinos explains the spectral observations of the extreme ultraviolet background emission from interstellar space, which characterizes dark matter, and it provides an explanation of the solar neutrino paradox. The experimental confirmation of the existence of fractional quantum energy levels of hydrogen atoms and molecules is presented. The data shows the process of hydrino production to be an exothermic reaction that represents a limitless clean energy source. The principles are as follows:

#### **Foundations:**

- Conservation of mass-energy;
- Conservation of linear and angular momentum;
- Maxwell's Equations;
- Newton's Laws;
- Special Relativity.

Next, the condition that a bound electron cannot radiate energy is imposed. The mathematical formulation for zero radiation is that the function that describes the motion of the electron must not possess spacetime Fourier components that are synchronous with waves traveling at the speed of light. The permissible solutions of the electron function are derived as a boundary value problem with the application of the nonradiative boundary condition.

#### **Solution to the Electron Functions**

From these laws and the non-radiative condition the following are a summary of some of the salient features of the theory derived in subsequent sections:

- Bound electrons are described by a charge-density (mass-density) function which is the product of a radial delta function ( $f(r) = \delta(r - r_n)$ ), two angular functions (spherical harmonic functions), and a time harmonic function. Thus, an electron is a spinning, two-dimensional spherical surface, called an electron

orbisphere, that can exist in a bound state at only specified distances from the nucleus. More explicitly, the orbisphere comprises a two dimensional spherical shell of moving charge. The corresponding current pattern of the orbisphere comprises an infinite series of correlated orthogonal great circle current loops. The current pattern (shown in Figure 1.4) is generated over the surface by two orthogonal sets of an infinite series of nested rotations of two orthogonal great circle current loops where the coordinate axes rotate with the two orthogonal great circles. Each infinitesimal rotation of the infinite series is about the new x-axis and new y-axis which results from the preceding such rotation. For each of the two sets of nested rotations, the angular sum of the rotations about each rotating x-axis and y-axis totals  $\sqrt{2}\pi$  radians. The current pattern gives rise to the phenomenon corresponding to the spin quantum number.

- The total function that describes the spinning motion of each electron orbisphere is composed of two functions. One function, the spin function, is spatially uniform over the orbisphere, spins with a quantized angular velocity, and gives rise to spin angular momentum. The other function, the modulation function, can be spatially uniform—in which case there is no orbital angular momentum and the magnetic moment of the electron orbisphere is one Bohr magneton—or not spatially uniform—in which case there is orbital angular momentum. The modulation function also rotates with a quantized angular velocity. Numerical values for the angular velocity, radii of allowed orbispheres, energies, and associated quantities are calculated.
- Orbisphere radii are calculated by setting the centripetal force equal to the electric and magnetic forces.
- The orbisphere is a resonator cavity which traps photons of discrete frequencies. The radius of an orbisphere increases with the absorption of electromagnetic energy. The solutions to Maxwell's equations for modes that can be excited in the orbisphere resonator cavity give rise to four quantum numbers, and the energies of the modes are the experimentally known hydrogen spectrum. The spectrum of helium is the solution of Maxwell's equations for the energies of modes of this resonator cavity with a contribution from electron-electron spin and orbital interactions.

- Excited states are unstable because the charge-density function of the electron plus photon have a radial doublet function component which corresponds to an electric dipole. The doublet possesses spacetime Fourier components synchronous with waves traveling at the speed of light; thus it is radiative. The charge-density function of the electron plus photon for the  $n = 1$  principal quantum state of the hydrogen atom as well as for each of the  $n = \frac{1}{\text{integer}}$  states mathematically is purely a radial delta function. The delta function does not possess spacetime Fourier components synchronous with waves traveling at the speed of light; thus, each is nonradiative.
- The spectroscopic linewidth arises from the classical rise-time band-width relationship, and the Lamb Shift is due to conservation of energy and linear momentum and arises from the radiation reaction force between the electron and the photon.
- The photon is an orbitsphere with electric and magnetic field lines along orthogonal great circles.
- Upon ionization, the orbitsphere radius goes to infinity and the electron becomes a plane wave (consistent with double-slit experiments) with the de Broglie wave length,  $\lambda = h / p$ .
- The energy of atoms is stored in their electric and magnetic fields. Chemical bonding occurs when the total energy of the participant atoms can be lowered with the formation of two dimensional equipotential energy surfaces (molecular orbitals) where the motion is along geodesics, and a general form of the nonradiative boundary condition is met. Zero order vibration occurs because it gives rise to a nonradiative lower energy state.
- Certain atoms and ions serve as catalysts to release energy from hydrogen to produce an increased binding energy hydrogen atom having a binding energy of  $\frac{13.6 \text{ eV}}{\frac{1}{p^2}}$  where  $p$  is an integer greater than 1, designated as  $H \frac{a_H}{p}$  where  $a_H$  is the radius of the hydrogen atom. Increased binding energy hydrogen atoms called

hydrinos are predicted to form by reacting an ordinary hydrogen atom with a catalyst having a net enthalpy of reaction of about the potential energy of hydrogen in its first nonradiative state,  $m \cdot 27.2 \text{ eV}$ , where  $m$  is an integer. This catalysis releases energy from the hydrogen atom with a commensurate decrease in size of the hydrogen atom,  $r_n = na_H$ . For example, the catalysis of  $H(n=1)$  to  $H(n=1/2)$  releases  $40.8 \text{ eV}$ , and the hydrogen radius decreases from  $a_H$  to  $\frac{1}{2}a_H$ . For example, potassium ions can provide a net enthalpy of a multiple of that of the potential energy of the hydrogen atom. The second ionization energy of potassium is  $31.63 \text{ eV}$ ; and  $K^+$  releases  $4.34 \text{ eV}$  when it is reduced to  $K$ . The combination of reactions  $K^+$  to  $K^{2+}$  and  $K^+$  to  $K$ , then, has a net enthalpy of reaction of  $27.28 \text{ eV}$ . The process is hereafter referred to as the Atomic BlackLight Process.

- The existence of fractional quantum energy levels of hydrogen atoms, molecules, and hydride ions as the product of the BlackLight Process—a new energy source has been confirmed experimentally.
- The Schwarzschild metric gives the relationship whereby matter causes relativistic corrections to spacetime that determines the curvature of spacetime and is the origin of gravity. The correction is based on the boundary conditions that no signal can travel faster than the speed of light including the gravitational field that propagates following particle production from a photon wherein the particle has a finite gravitational velocity given by Newton's Law of Gravitation.
- It is possible to give the electron a spatial velocity function having negative curvature and, therefore, cause antigravity. An engineered spacecraft may be feasible.
- Fundamental particle production occurs when the energy of the particle given by the Planck equation, Maxwell's Equations, and Special Relativity is equal to  $mc^2$ , and the proper time is equal to the coordinate time according to General Relativity. The gravitational equations with the equivalence of the particle production energies permit the equivalence of mass/energy and the spacetime metric from which the gravitational constant and the masses of the leptons, the quarks, and nucleons are derived.

- The gravitational equations with the equivalence of the particle production energies permit the equivalence of mass/energy ( $E = mc^2$ ) and spacetime ( $\frac{c^3}{4\pi G} = 3.22 \times 10^{34} \frac{kg}{sec}$ ). Spacetime expands as mass is released as energy which provides the basis of the atomic, thermodynamic, and cosmological arrows of time. Entropy and the expansion of the universe are large scale consequences. The universe is closed independently of the total mass of the universe, and different regions of space are isothermal even though they are separated by greater distances than that over which light could travel during the time of the expansion of the universe. The universe is oscillatory in matter/energy and spacetime with a finite minimum radius, the gravitational radius; thus, the gravitational force causes celestial structures to evolve on a time scale that is greater than the period of oscillation. The equation of the radius of the universe,

$$, is = \frac{2Gm_U}{c^2} + \frac{cm_U}{c^3} - \frac{cm_U}{c^3} \cos \frac{2\pi t}{\frac{2\pi Gm_U}{c^3} sec} m . \text{ The calculated}$$

Hubble constant is  $H_0 = 78.6 \frac{km}{sec Mpc}$ . Presently, stars exist which

are older than the elapsed time of the present expansion as stellar evolution occurred during the contraction phase. The maximum energy release of the universe which occurs at the beginning of the expansion phase is

$$P_U = \frac{\frac{m_e c^2}{\sqrt{\frac{2GM}{c^2 \lambda_c}}}}{\tau \sqrt{\frac{2GM}{c^2 \lambda_c}}} = \frac{c^5}{4\pi G} = 2.89 \times 10^{51} W .$$

- Superconductivity arises when the lattice is a band-pass for the magnetic field of an array of magnetic dipoles; so, no energy is dissipated with current flow.
- The Quantum Hall Effect arises when the forces of crossed electric and magnetic fields balance and the lattice is a band-pass for the magnetic field of an array of magnetic dipoles.
- The vector potential component of the electron's angular momentum gives rise to the Aharonov-Bohm Effect.

- Alpha decay occurs as a transmission of a plane wave through a potential barrier.
- The proton and neutron functions each comprise a linear combination of a constant function and three orthogonal spherical harmonic functions resulting in three quark/gluon functions per nucleon. The nucleons are locally two dimensional.

## SECTION I

### Atoms and Molecules

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## THE ONE ELECTRON ATOM

One-electron atoms include the hydrogen atom, He(II), Li(III), Be(IV), and so on. The mass-energy and angular momentum of the electron are constant; this requires that the equation of motion of the electron be temporally and spatially harmonic. Thus, the classical wave equation (4-dimensional Laplace equation) applies and

$$\nabla^2 - \frac{1}{v^2} \frac{\delta^2}{\delta t^2} \rho(r, \theta, \phi, t) = 0 \quad (1.1)$$

where  $\rho(r, \theta, \phi, t)$  is the function of the electron in time and space. In each case, the nucleus contains  $Z$  protons and the atom has a net positive charge of  $(Z - 1)e$ . All forces are central and Special Relativity applies. Thus, the coordinates must be three dimensional spherically harmonic coordinates plus time. The time, radial, and angular solutions of Laplace's Equation are separable. The motion is time harmonic with frequency  $\omega_n$ . To be a harmonic solution of Laplace's equation in spherical coordinates, the angular functions must be spherical harmonic functions.

### THE BOUNDARY CONDITION OF NONRADIATION AND THE RADIAL FUNCTION - THE CONCEPT OF THE "ORBITSPHERE"

A zero of the spacetime Fourier transform of the product function of two spherical harmonic angular functions, a time harmonic function, and an unknown radial function is sought.

#### The Boundary Condition

The condition for radiation by a moving charge is derived from Maxwell's equations. To radiate, the spacetime Fourier transform of the current-density function must possess components synchronous with waves traveling at the speed of light [1]. Alternatively,

*For non-radiative states, the current-density function must not possess spacetime Fourier components that are synchronous with waves traveling at the speed of light.*

#### Derivation of the Condition for Nonradiation

Proof that the condition for nonradiation by a moving point charge is that its spacetime Fourier transform does not possess components that are synchronous with waves traveling at the speed of light is given by Haus [1]. The Haus derivation applies to a moving charge-density function as well because charge obeys superposition. The Haus

derivation is summarized below.

The Fourier components of the current produced by the moving charge are derived. The electric field is found from the vector equation in Fourier space ( $\mathbf{k}$ ,  $\omega$ -space). The inverse Fourier transform is carried over the magnitude of  $\mathbf{k}$ . The resulting expression demonstrates that the radiation field is proportional to  $\mathbf{J} \left( \frac{\omega}{c} \mathbf{n}, \omega \right)$ , where  $\mathbf{J}(\mathbf{k}, \omega)$  is the spacetime Fourier transform of the current perpendicular to  $\mathbf{k}$  and  $\mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|}$ . Specifically,

$$\mathbf{E}(\mathbf{r}, \omega) \frac{d\omega}{2\pi} = \frac{c}{2\pi} \rho(\omega, \mathbf{r}) d\omega d\mathbf{r} \sqrt{\frac{\mu_0}{\epsilon_0}} \mathbf{n} \times \mathbf{n} \times \mathbf{J} \left( \frac{\omega}{c} \mathbf{n}, \omega \right) e^{i \frac{\omega}{c} \mathbf{n} \cdot \mathbf{r}} \quad (1.2)$$

The field  $\mathbf{E}(\mathbf{r}, \omega) \frac{d\omega}{2\pi}$  is proportional to  $\mathbf{J} \left( \frac{\omega}{c} \mathbf{n}, \omega \right)$ , namely, the Fourier component for which  $\mathbf{k} = \frac{\omega}{c} \mathbf{n}$ . Factors of  $\omega$  that multiply the Fourier component of the current are due to the density of modes per unit volume and unit solid angle. An unaccelerated charge does not radiate in free space, not because it experiences no acceleration, but because it has no Fourier component  $\mathbf{J} \left( \frac{\omega}{c} \mathbf{n}, \omega \right)$ .

### Derivation of the Boundary Condition

In general, radial solutions of the Helmholtz wave equation are spherical Bessel functions, Neumann functions, Hankel functions, associated Laguerre functions, and the radial Dirac delta function. The Dirac delta function eliminates the radial dependence and reduces the number of dimensions of the Helmholtz wave equation from four to three. The solution for the radial function which satisfies the boundary condition is three dimensional delta function in spherical coordinates--a spherical shell [2]

$$f(r) = \frac{1}{r^2} \delta(r - r_n) \quad (1.3)$$

where  $r_n$  is an allowed radius. The Fourier Transform of the radial Dirac delta function is a sinc function. For time harmonic motion, with angular velocity,  $\omega$ , the relationship between the radius and the wavelength is

$$2\pi r = \lambda \quad (1.4)$$

Consider the radial wave vector of the sinc function, when the radial projection of the velocity is  $c$ , the relativistically corrected wavelength is

$$r = \lambda \quad (1.5)$$

Substitution of Eq. (1.5) into the sinc function results in the vanishing of the entire Fourier Transform of the current-density function.

### SPACETIME FOURIER TRANSFORM OF THE ELECTRON FUNCTION

The electron charge-density (mass-density) function is the product of a radial delta function ( $f(r) = \frac{1}{r^2} \delta(r - r_n)$ ), two angular functions (spherical harmonic functions), and a time harmonic function. The spacetime Fourier transform in three dimensions in spherical coordinates plus time is given [3,4] as follows:

$$M(s, \theta, \phi, \omega) = \int_0^{2\pi} \int_0^\pi \int_0^\infty \rho(r, \theta, \phi, t) \exp(-i2\pi sr[\cos \theta \cos \phi + \sin \theta \sin \phi \cos(\phi - \theta)]) \exp(-i\omega t) r^2 \sin \theta dr d\theta d\phi dt \quad (1.6)$$

With circular symmetry [3]

$$M(s, \theta, \omega) = 2\pi \int_0^\pi \rho(r, \theta, t) J_0(2\pi sr \sin \theta) \exp(-i2\pi sr \cos \theta) r^2 \sin \theta \exp(-i\omega t) dr d\theta dt \quad (1.7)$$

With spherical symmetry [3],

$$M(s, \omega) = 4\pi \int_0^\infty \rho(r, t) \text{sinc}(2sr) r^2 \exp(-i\omega t) dr dt \quad (1.8)$$

The solutions of the classical wave equation are separable.

$$\rho(r, \theta, \phi, t) = f(r)g(\theta)h(\phi)k(t) \quad (1.9)$$

The orbitsphere function is separable into a product of functions of independent variables,  $r, \theta, \phi$ , and  $t$ . The radial function which satisfies the boundary condition is a delta function. The time functions are of the form  $e^{i\omega t}$ , the angular functions are spherical harmonics, sin or cosine trigonometric functions or sums of these functions, each raised to various powers. The spacetime Fourier transform is derived of the separable variables for the angular space function of  $\sin \phi$  and  $\sin \theta$ . It follows from the spacetime Fourier transform given below that other possible spherical harmonics angular functions give the same form of result as the transform of  $\sin \theta$  and  $\sin \phi$ . Using Eq. (1.8),  $F(s)$ , the space Fourier transform of ( $f(r) = \delta(r - r_n)$ ) is given as follows:

$$F(s) = 4\pi \int_0^\infty \frac{1}{r^2} \delta(r - r_n) \text{sinc}(2sr) r^2 dr \quad (1.10)$$

$$F(s) = 4\pi \text{sinc}(2sr_n) \quad (1.11)$$

***The subscript n is used hereafter; however, the quantization condition appears in the Excited States of the One Electron Atom (Quantization) Section. Quantization arises as***

**"allowed" solutions of the wave equation corresponding to a resonance between the electron and a photon.**

Using Eq. (1.7),  $G(s, \theta)$ , the space Fourier transform of  $g(\theta) = \sin \theta$  is given

as follows where there is no dependence on  $\phi$ :

$$G(s, \theta) = 2\pi \int_0^\pi \sin \theta J_0(2\pi sr \sin \theta) \exp(-i2\pi sr \cos \theta) \sin \theta r^2 d\theta dr \quad (1.12)$$

$$G(s, \theta) = 2\pi \int_0^\pi r^2 \sin^2 \theta J_0(2\pi sr \sin \theta) \cos(2\pi sr \cos \theta) d\theta dr \quad (1.13)$$

From Luke [5] and [6]:

$$J_\nu(z) = \frac{1}{2} z^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n \frac{z}{2}^{2n}}{n! (\nu + n + 1)!} = \frac{1}{2} z^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n \frac{z}{2}^{2n}}{n! (\nu + n)!} \quad (1.14)$$

Let

$$Z = 2\pi sr \sin \theta \quad (1.15)$$

With substitution of Eqs. (1.15) and (1.14) into Eq. (1.13),

$$G(s, \theta) = 2\pi \int_0^\pi r^2 \sin^2 \theta \sum_{n=0}^{\infty} \frac{(-1)^n (\pi sr \sin \theta)^{2n}}{n! n!} \cos(2\pi sr \cos \theta) d\theta dr \quad (1.16)$$

$$G(s, \theta) = 2\pi \int_0^\pi r^2 \sum_{n=0}^{\infty} \frac{(-1)^n (\pi sr \sin \theta)^{2n}}{n! n!} \sin^{2(n+1)} \theta \cos(2\pi sr \cos \theta) d\theta dr \quad (1.17)$$

$$G(s, \theta) = 2\pi \int_0^\pi r^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (\pi sr \sin \theta)^{2(n-1)}}{(n-1)! (n-1)!} \sin^{2n} \theta \cos(2\pi sr \cos \theta) d\theta dr \quad (1.18)$$

From Luke [7], with  $\text{Re}(\nu) > -\frac{1}{2}$ :

$$J_\nu(z) = \frac{\frac{1}{2} z^{-\nu}}{\frac{1}{2} \nu + \frac{1}{2}} \int_0^\pi \cos(z \cos \theta) \sin^{2\nu} \theta d\theta \quad (1.19)$$

Let

$$z = 2\pi sr \cos \theta, \text{ and } n = \nu \quad (1.20)$$

Applying the relationship, the integral of a sum is equal to the sum of the integral to Eq. (1.18), and transforming Eq. (1.18) into the form of Eq. (1.19) by multiplication by

$$1 = \frac{\left(\frac{1}{2}\right)^{\nu + \frac{1}{2}} (\pi sr \cos \theta)^{\nu}}{(\pi sr \cos \theta)^{\nu} \left(\frac{1}{2}\right)^{\nu + \frac{1}{2}}} \quad (1.21)$$

and by moving the constant outside of the integral gives:

$$G(s, \theta) = 2\pi \int_0^{\pi} r^2 \frac{(-1)^{\nu-1} (\pi r \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\left(\frac{1}{2}\right)^{\nu + \frac{1}{2}} (\pi sr \cos \theta)^{\nu}}{(\pi sr \cos \theta)^{\nu} \left(\frac{1}{2}\right)^{\nu + \frac{1}{2}}} \sin^{2\nu} \theta \cos(2\pi sr \cos \theta) d\theta dr \quad (1.22)$$

$$G(s, \theta) = 2\pi \int_0^{\pi} r^2 \frac{(-1)^{\nu-1} (\pi r \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\left(\frac{1}{2}\right)^{\nu + \frac{1}{2}} (\pi sr \cos \theta)^{\nu}}{(\pi sr \cos \theta)^{\nu} \left(\frac{1}{2}\right)^{\nu + \frac{1}{2}}} \sin^{2\nu} \theta \cos(2\pi sr \cos \theta) d\theta dr \quad (1.23)$$

Applying Eq. (1.19),

$$G(s, \theta) = 2\pi \int_0^{\pi} r^2 \frac{(-1)^{\nu-1} (\pi r \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\left(\frac{1}{2}\right)^{\nu + \frac{1}{2}}}{(\pi sr \cos \theta)^{\nu}} J_{\nu}(2\pi sr \cos \theta) dr \quad (1.24)$$

Using the Hankel transform formula from Bateman [8]:

$$\int_0^{\infty} r^{-\frac{1}{2}} (rs)^{\frac{1}{2}} J_{\nu}(rs) dr = s^{\frac{1}{2}} \quad (1.25)$$

and the Hankel transform relationship from Bateman [9], the general Eq. (1.31) is derived as follows:

$$f(x) \text{ Hankel Transform } g(y; \nu) = \int_0^{\infty} f(x) (xy)^{\frac{1}{2}} J_{\nu}(xy) dx \quad (1.26)$$

$$x^m f(x), m = 0, 1, 2, \dots \text{ Hankel Transform } y^{\frac{1}{2}-\nu} \frac{d^m}{y dy} y^{m+\nu-\frac{1}{2}} g(y; m + \nu) \quad (1.27)$$

$$\int_0^{\infty} r^{\nu} r^{-\frac{1}{2}} (rs)^{\frac{1}{2}} J_{\nu}(rs) dr = s^{\frac{1}{2}-\nu} \frac{d^{\nu}}{s ds} s^{\nu+\frac{1}{2}} s^{\frac{1}{2}} \quad (1.28)$$

$$\int_0^{\infty} r^{\nu} s^{\frac{1}{2}} J_{\nu}(rs) dr = \frac{s^{\frac{1}{2}-\nu}}{s^{\nu}} \frac{d^{\nu}}{ds^{\nu}} [s^{(2\nu)}] \quad (1.29)$$

$$\int_0^{\infty} r^{\nu} s^{\frac{1}{2}} J_{\nu}(rs) dr = s^{\frac{1}{2}-2\nu} \frac{2\nu!}{(\nu-1)!} s^{\nu} = \frac{2\nu!}{(\nu-1)!} s^{\frac{1}{2}-\nu} \quad (1.30)$$

$$\int_0^{\infty} r^{\nu} s^{-\frac{1}{2}} s^{\frac{1}{2}} J_{\nu}(rs) dr = \frac{2\nu!}{(\nu-1)!} s^{-\nu} \quad (1.31)$$

Collecting the  $r$  raised to a power terms, Eq. (1.24) becomes,

$$G(s, \nu) = 2\pi \int_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} (\pi \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\frac{1}{2}^{\nu+\frac{1}{2}}}{(\pi s \cos \theta)^{\nu}} r^{\nu} J_{\nu}(2\pi s r \cos \theta) dr \quad (1.32)$$

Let  $r = \frac{r'}{2\pi \cos \theta}$ ;  $dr = \frac{dr'}{2\pi \cos \theta}$ ,

$$G(s, \nu) = 2\pi \int_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} (\pi \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\frac{1}{2}^{\nu+\frac{1}{2}}}{(\pi s \cos \theta)^{\nu}} \frac{r^{\nu}}{(2\pi \cos \theta)^{\nu+1}} J_{\nu}(sr') dr' \quad (1.33)$$

By applying Eq. (1.31), Eq. (1.33) becomes,

$$G(s, \nu) = 2\pi \int_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} (\pi \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\frac{1}{2}^{\nu+\frac{1}{2}}}{(\pi s \cos \theta)^{\nu} (2\pi \cos \theta)^{\nu+1}} \frac{2\nu!}{(\nu-1)!} s^{-\nu} \quad (1.34)$$

By collecting power terms of  $s$ , Eq. (1.34) becomes,

$$G(s, \nu) = 2\pi \int_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} (\pi \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\frac{1}{2}^{\nu+\frac{1}{2}}}{(\pi \cos \theta)^{2\nu+1} 2^{\nu+1}} \frac{2\nu!}{(\nu-1)!} s^{-2\nu} \quad (1.35)$$

$H(s, \nu)$ , the space Fourier transform of  $h(\phi) = \sin \phi$  is given as follows where there is no dependence on  $\theta$ :

The spectrum of  $\sin \phi$  and  $\sin \theta$  are equivalent. Applying a change of variable to the Fourier transform of  $g(\theta) = \sin \theta$ .

$$\theta = \phi \implies$$

Therefore,  $\theta$  replaces  $\phi$  in Eq. (1.35),

$$H(s, \nu) = 2\pi \int_{\nu=1}^{\infty} \frac{(-1)^{\nu-1} (\pi \sin \theta)^{2(\nu-1)}}{(\nu-1)!(\nu-1)!} \frac{\frac{1}{2}^{\nu+\frac{1}{2}}}{(\pi \cos \theta)^{2\nu+1} 2^{\nu+1}} \frac{2\nu!}{(\nu-1)!} s^{-2\nu} \quad (1.36)$$

The time Fourier transform of  $K(t) = \text{Re}\{\exp(i\omega_n t)\}$  where  $\omega_n$  is the

angular frequency is given [4] as follows:

$$\int_0^{\infty} \cos \omega_n t \exp(-i\omega t) dt = \frac{1}{2\pi} \frac{1}{2} [\delta(\omega - \omega_n) + \delta(\omega + \omega_n)] \quad (1.37)$$

A very important theorem of Fourier analysis states that the Fourier transform of a product is the convolution of the individual Fourier transforms [10]. By applying this theorem, the spacetime Fourier transform of an orbitsphere,  $M(s, r, \omega)$  is of the following form:

$$M(s, r, \omega) = F(s) G(s, r) H(s, r) K(\omega) \quad (1.38)$$

Therefore, the spacetime Fourier transform,  $M(s, r, \omega)$ , is the convolution of Eqs. (1.11), (1.35), (1.36), and (1.37).

$$M(s, r, \omega) = 4\pi \text{sinc}(2sr_n) \int_{v=1}^{\infty} 2\pi \frac{(-1)^{v-1} (\pi \sin \theta)^{2(v-1)}}{(v-1)!(v-1)!} \frac{\frac{1}{2} \theta^{v+\frac{1}{2}}}{(\pi \cos \theta)^{2v+1} 2^{v+1}} \frac{2v!}{(v-1)!} s^{-2v} \\ 2\pi \int_{v=1}^{\infty} \frac{(-1)^{v-1} (\pi \sin \theta)^{2(v-1)}}{(v-1)!(v-1)!} \frac{\frac{1}{2} \theta^{v+\frac{1}{2}}}{(\pi \cos \theta)^{2v+1} 2^{v+1}} \frac{2v!}{(v-1)!} s^{-2v} \frac{1}{4\pi} [\delta(\omega - \omega_n) + \delta(\omega + \omega_n)] \quad (1.39)$$

The condition for nonradiation of a moving charge-density function is that the spacetime Fourier transform of the current-density function must not have waves synchronous with waves traveling at the speed of

light, that is synchronous with  $\frac{\omega_n}{c}$  or synchronous with  $\frac{\omega_n}{c} \sqrt{\frac{\epsilon}{\epsilon_0}}$  where

is the dielectric constant of the medium. The Fourier transform of the charge-density function of the orbitsphere (bubble of radius  $r$ ) is given by Eq. (1.39). In the case of time harmonic motion, the current-density function is given by the time derivative of the charge-density function. Thus, the current-density function is given by the product of the constant angular velocity and the charge-density function. The Fourier transform of current-density function of the orbitsphere is given by the product of the constant angular velocity and Eq. (1.39). Consider the radial and time parts of,  $J$ , the Fourier transform of the current-density function where the angular transforms are not zero:

$$J = \omega_n \text{sinc} 2sr_n \frac{1}{4\pi} [\delta(\omega - \omega_n) + \delta(\omega + \omega_n)] = \omega_n \frac{\sin 2\pi sr_n}{2\pi sr_n} \frac{1}{4\pi} [\delta(\omega - \omega_n) + \delta(\omega + \omega_n)] \quad (1.40)$$

For the case that the current-density function is constant, the delta function of Eq. (1.40) is replaced by a constant. For time harmonic motion, with angular velocity,  $\omega_n$ , Eq. (1.40) is nonzero only for  $\omega = \omega_n$ ; thus,  $-\infty < s < \infty$  becomes finite only for the corresponding wavenumber,

$s_n$ . The relationship between the radius and the wavelength is

$$v_n = \lambda_n f_n \quad (1.41)$$

$$v_n = 2\pi r_n f_n = \lambda_n f_n \quad (1.42)$$

$$2\pi r_n = \lambda_n \quad (1.43)$$

The motion on the orbitsphere is angular; however, a radial component exists due to Special Relativistic effects. Consider the radial wave vector of the sinc function. When the radial projection of the velocity is  $c$

$$\mathbf{s}_n \cdot \mathbf{v}_n = \mathbf{s}_n \cdot \mathbf{c} = \omega_n \quad (1.44)$$

the relativistically corrected wavelength is<sup>1</sup>

$$r_n = \lambda_n \quad (1.45)$$

(i.e. the lab frame motion in the angular direction goes to zero as the velocity approaches the speed of light as given by Eq. (24.15)).

Substitution of Eq. (1.45) into the sinc function results in the vanishing of the entire Fourier Transform of the current-density function. Thus,

spacetime harmonics of  $\frac{\omega_n}{c} = k$  or  $\frac{\omega_n}{c} \sqrt{\frac{\epsilon}{\epsilon_o}} = k$  do not exist for which the

Fourier Transform of the current-density function is nonzero. Radiation due to charge motion does not occur in any medium when this boundary condition is met. [Note that the boundary condition for the solution of the radial function of the hydrogen atom with the Schrödinger equation is that  $\psi = 0$  as  $r \rightarrow \infty$ . Here, however, the boundary condition is derived from Maxwell's equations: For non-radiative states, the current-density function must not possess spacetime Fourier components that are synchronous with waves traveling at the speed of light. An alternative derivation which provides acceleration without radiation is given by Abbott [11]] Bound electrons are described by a charge-density (mass-density) function which is the product of a radial delta function, Eq. (1.3), two angular functions (spherical harmonic functions), and a time harmonic function. This is a solution of Laplace's Equation. Thus,

---

<sup>1</sup> The special relativistic length contraction relationship observed for a laboratory frame relative to an inertial frame moving at constant velocity  $v$  in the direction of velocity  $v$  is

$$l = l_o \sqrt{1 - \frac{v^2}{c^2}}$$

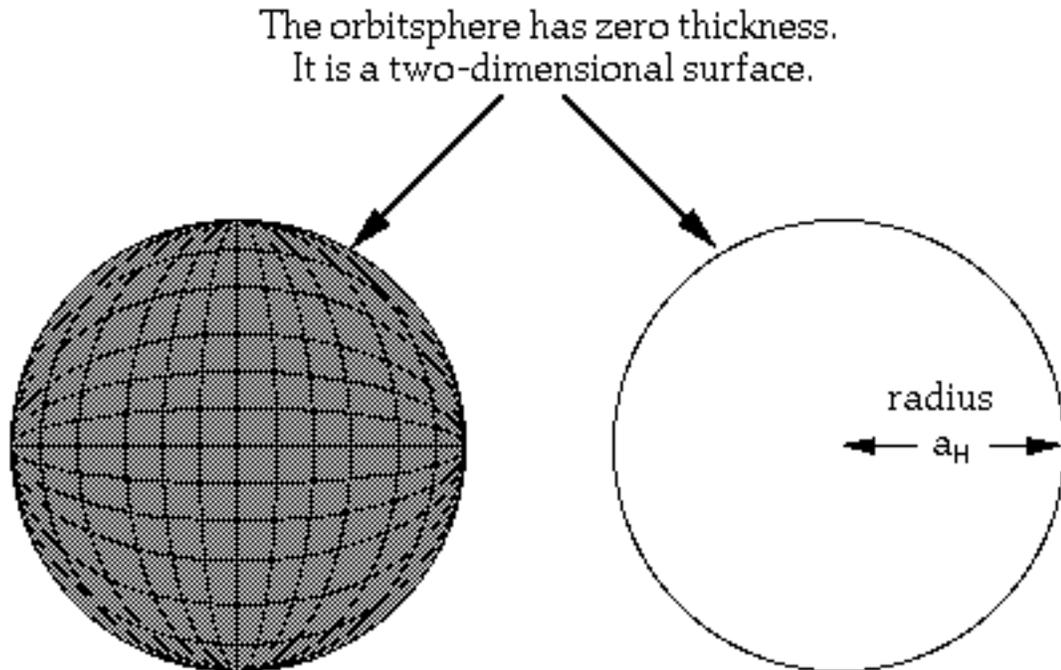
Consider the distance on a great circle given by

$$\int_0^{2\pi} r d\theta = r \int_0^{2\pi} d\theta = 2\pi r$$

The distance undergoes length contraction only in the  $\theta$  direction as  $v \rightarrow c$ . Thus, as  $v \rightarrow c$  the distance on a great circle approaches its radius which is the relativistically contracted electron wavelength.

this radial function implies that allowed states are two-dimensional spherical shells (zero thickness) of charge-density (and mass density) at specific radii  $r_n$ . These shells are referred to as electron orbitspheres. See Figure 1.1 for a pictorial representation of an orbitsphere.

Figure 1.1. The orbitsphere is a two dimensional spherical shell with the Bohr radius of the hydrogen atom.



Given time harmonic motion and a radial delta function, the relationship between an allowed radius and the electron wavelength is given by Eq. (1.43). Using the de Broglie relationship for the electron mass where the coordinates are spherical,

$$\lambda_n = \frac{h}{p_n} = \frac{h}{m_e v_n} \quad (1.46)$$

and the magnitude of the velocity for every point on the orbitsphere is

$$v_n = \frac{\hbar}{m_e r_n} \quad (1.47)$$

### THE ANGULAR FUNCTION

The radial function for the electron indicates that the electron is two-dimensional. Therefore, the angular mass-density function of the electron,  $A(\theta, \phi, t)$ , must be a solution of the Laplace equation in two dimensions (plus time),

$$-\frac{1}{v^2} \frac{\delta^2}{\delta t^2} A(\theta, \phi, t) = 0 \quad (1.48)$$

where  $\rho(r, \theta, \phi, t) = f(r)A(\theta, \phi, t) = \frac{1}{r^2} \delta(r - r_n)A(\theta, \phi, t)$  and  $A(\theta, \phi, t) = Y(\theta, \phi)k(t)$

$$\frac{1}{r^2 \sin \theta} \frac{\delta}{\delta \theta} \sin \theta \frac{\delta}{\delta \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\delta^2}{\delta \phi^2} - \frac{1}{v^2} \frac{\delta^2}{\delta t^2} A(\theta, \phi, t) = 0 \quad (1.49)$$

where  $v$  is the linear velocity of the electron. Conservation of momentum and energy allows the angular functions and time functions to be separated.

$$A(\theta, \phi, t) = Y(\theta, \phi)k(t) \quad (1.50)$$

Charge is conserved as well, and the charge of an electron is superimposable with its mass. That is, the angular mass-density function,  $A(\theta, \phi, t)$ , is also the angular charge-density function.

The electron orbitsphere experiences a constant potential energy because it is fixed at  $r = r_n$ . In general, the kinetic energy for an inverse squared electric force is half the potential energy. It is the rotation of the orbitsphere that causes spin angular momentum. The rotational energy of a rotating body,  $E_{rot}$ , is

$$E_{rot} = \frac{1}{2} I \omega^2 \quad (1.51)$$

where  $I$  is the moment of inertia and  $\omega$  is the angular velocity. The angular velocity must be constant (at a given  $n$ ) because  $r$  is constant and the energy and angular momentum are constant. The allowed angular velocities are related to the allowed frequencies by

$$\omega_n = 2\pi\nu_n \quad (1.52)$$

The allowed frequencies are related to allowed velocities by

$$\nu_n = v_n \lambda_n \quad (1.53)$$

The allowed velocities and angular frequencies are related to  $r_n$  by

$$v_n = r_n \omega_n \quad (1.54)$$

$$\omega_n = \frac{\hbar}{m_e r_n^2} \quad (1.55)$$

$$v_n = \frac{\hbar}{m_e r_n} \quad (1.56)$$

The sum of the  $L_i$ , the magnitude of the angular momentum of each infinitesimal point of the orbitsphere of mass  $m_i$ , must be constant. The constant is  $\hbar$ .

$$|\mathbf{L}_i| = |\mathbf{r} \times m_i \mathbf{v}| = m_e r_n \frac{\hbar}{m_e r_n} = \hbar \quad (1.57)$$

where the velocity is given by Eq. (1.47). The vector projections of the

orbisphere spin angular momentum relative to the Cartesian coordinates are given in the Spin Angular Momentum of the Orbisphere with  $\ell = 0$  Section.

In the case of an excited state, the charge density function of the electron orbisphere can be modulated by the corresponding "trapped" photon to give rise to orbital angular momentum about the z-axis. The "trapped photon" is a "standing electromagnetic wave" which actually is a circulating wave that propagates around the z-axis. Its source current superimposes with each great circle current loop of the orbisphere. In order to satisfy the boundary (phase) condition at the orbisphere surface, the angular and time functions of the photon must match those of its source current which modulates the orbisphere charge density function as given in the Equation of the Electric Field Inside the Orbisphere Section. The time-function factor,  $k(t)$ , for the photon "standing wave" is identical to the time-function factor of the orbisphere. Thus, the angular frequency of the "trapped photon" has to be identical to the angular frequency of the electron orbisphere,  $\omega_n$  given by Eq. (1.55). However, the linear velocity of the modulation component is not given by Eq. (1.54)--the orbital angular frequency is with respect to the z-axis; thus, the distance from the z-axis must be substituted for the orbisphere radius of Eq. (1.54). The vector projections of the orbital angular momentum and the spin angular momentum of the orbisphere are given in the Rotational Parameters of the Electron (Angular Momentum, Rotational Energy, and Moment of Inertia) Section. Eq. (1.49) becomes

$$-\frac{\hbar^2}{2I} \frac{1}{\sin \theta} \frac{\delta}{\delta \theta} \sin \theta \frac{\delta}{\delta \theta} + \frac{1}{\sin^2 \theta} \frac{\delta^2}{\delta \phi^2} A(\theta, \phi, t) = E_{rot} A(\theta, \phi, t) \quad (1.58)$$

The spacetime angular function,  $A(\theta, \phi, t)$ , is separated into an angular and a time function,  $Y(\theta, \phi)k(t)$ . The solution of the time harmonic function is  $k(t) = e^{i\omega_n t}$ . When the time harmonic function is eliminated,

$$-\frac{\hbar^2}{2I} \frac{1}{\sin \theta} \frac{\delta}{\delta \theta} \sin \theta \frac{\delta}{\delta \theta} + \frac{1}{\sin^2 \theta} \frac{\delta^2}{\delta \phi^2} Y(\theta, \phi) = E_{rot} Y(\theta, \phi) \quad (1.59)$$

Eq. (1.59) is the equation for the rigid rotor. The angular function can be separated into a function of  $\theta$  and a function of  $\phi$  and the solutions are well known [11]. The energies are given by

$$E_{rot} = \frac{\hbar^2 \ell(\ell + 1)}{2I} \quad \ell = 0, 1, 2, 3, \dots, \quad (1.60)$$

where the moment of inertia,  $I$ , is derived in the Rotational Parameters of the Electron (Angular Momentum, Rotational Energy, and Moment of Inertia) Section. The angular functions are the spherical harmonics,

$Y_\ell^m(\phi, \theta) = P_\ell^m(\cos \theta)e^{im\phi}$ . The spherical harmonic  $Y_0^0(\phi, \theta) = 1$  is also a solution. The real part of the spherical harmonics vary between  $-1$  and  $1$ . But the mass of the electron cannot be negative; and the charge cannot be positive. Thus, to insure that the function is positive definite, the form of the angular solution must be a superposition:

$$Y_0^0(\theta, \phi) + Y_\ell^m(\theta, \phi) \quad (1.61)$$

(Note that  $Y_\ell^m(\phi, \theta) = P_\ell^m(\cos \theta)e^{im\phi}$  are not normalized here as given by Eq. (3.53) of Jackson [12]; however, it is implicit that magnitude is made to satisfy the boundary condition that the function is positive definite and Eq. (1.63) is satisfied.)  $Y_0^0(\theta, \phi)$  is called the angular spin function

corresponding to the quantum numbers  $s = \frac{1}{2}$ ;  $m_s = \pm \frac{1}{2}$  as given in the Spin

Angular Momentum of the Orbitsphere with  $\mathfrak{l} = 0$  Section and the Stern-Gerlach Experiment Section.  $Y_\ell^m(\theta, \phi)$  is called the angular orbital

function corresponding to the quantum numbers

$\ell = 0, 1, 2, 3, 4, \dots$ ;  $m_\ell = -\ell, -\ell + 1, \dots, 0, \dots, +\ell$ .  $Y_\ell^m(\theta, \phi)$  can be thought of as a modulation function. The charge-density of the entire orbitsphere is

the total charge divided by the total area,  $\frac{-e}{4\pi r_n^2}$ . The fraction of the

charge of an electron in any area element is given by

$$N[Y_0^0(\theta, \phi) + Y_\ell^m(\theta, \phi)]r_n^2 \sin \theta d\theta d\phi, \quad (1.62)$$

where  $N$  is the normalization constant. Therefore, the normalization constant is given by

$$-e = Nr_n^2 \int_0^{2\pi} \int_0^\pi [Y_0^0(\theta, \phi) + Y_\ell^m(\theta, \phi)] \sin \theta d\theta d\phi \quad (1.63)$$

For  $\mathfrak{l} = 0$ ,  $N = \frac{-e}{8\pi r_n^2}$ . For  $\mathfrak{l} \neq 0$ ,  $N = \frac{-e}{4\pi r_n^2}$ . The charge-density functions

including the time-function factor are

$$\mathfrak{l} = 0$$

$$\rho(r, \theta, \phi, t) = \frac{e}{8\pi r^2} [\delta(r - r_n)] [Y_\ell^m(\theta, \phi) + Y_0^0(\theta, \phi)] \quad (1.64)$$

$$\mathfrak{l} \neq 0$$

$$\rho(r, \theta, \phi, t) = \frac{e}{4\pi r^2} [\delta(r - r_n)] \left[ Y_0^0(\theta, \phi) + \text{Re} \left\{ Y_\ell^m(\theta, \phi) [1 + e^{i\omega_n t}] \right\} \right] \quad (1.65)$$

where

$$\text{Re} \left\{ Y_\ell^m(\theta, \phi) [1 + e^{i\omega_n t}] \right\} = \text{Re} \left[ Y_\ell^m(\theta, \phi) + Y_\ell^m(\theta, \phi) e^{i\omega_n t} \right] = P_\ell^m(\cos \theta) \cos m\phi + P_\ell^m(\cos \theta) \cos(m\phi + \omega_n t)$$

and  $\omega_n = 0$  for  $m = 0$ . The photon equations which correspond to the

orbitsphere states, Eqs. (1.64) and (1.65) are given in the Excited States

of the One Electron Atom (Quantization) Section. For  $n = 1$ , and  $\ell = 0$ ,  $m = 0$ , and  $s = 1/2$ , the charge (and mass) distribution is spherically symmetric and  $M_{1,0,0,1/2} = -14.41 \text{ Cm}^{-2}$  everywhere on the orbitsphere.

Similarly, for  $n = 2$ ,  $\ell = 0$ ,  $m = 0$ , and  $s = 1/2$ , the charge distribution everywhere on the sphere is  $M_{2,0,0,1/2} = -3.602 \text{ Cm}^{-2}$ . For  $n = 2$ ,  $\ell = 1$ ,  $m = 0$ , and  $s = 1/2$ , the charge distribution varies with  $\theta$ .  $Y_1^0(\phi, \theta)$  is a maximum at  $\theta = 0^\circ$  and the charge-density is also a maximum at this point,  $M_{2,1,0,1/2}(\theta = 0^\circ) = -7.203 \text{ Cm}^{-2}$ . The charge-density decreases as  $\theta$  increases; a minimum in the charge-density is reached at  $\theta = 180^\circ$ ,  $M_{2,1,0,1/2}(\theta = 180^\circ) = 0 \text{ Cm}^{-2}$ .

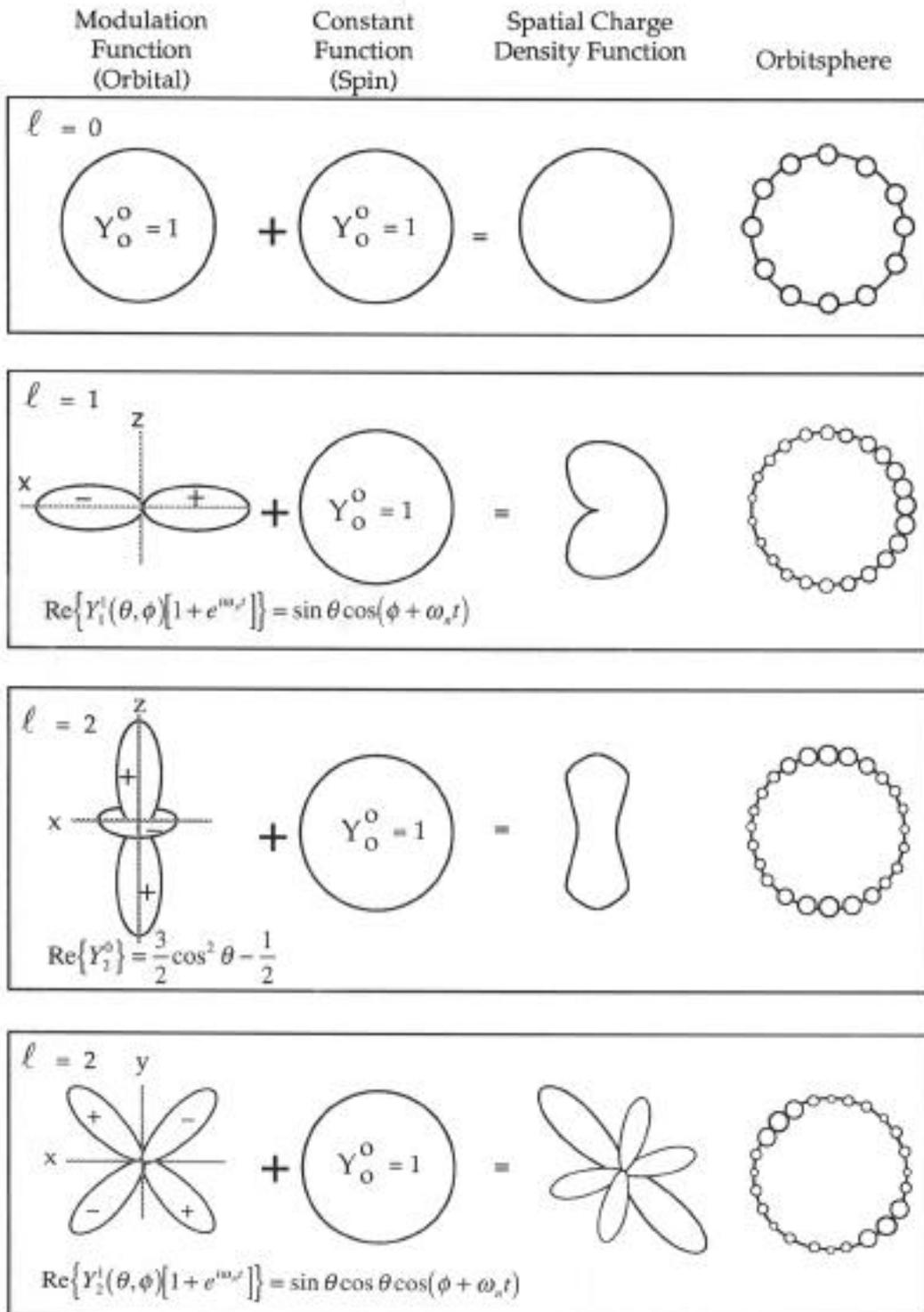
For  $\ell = 1$  and  $m = \pm 1$ , the spherical harmonics are complex, and the angular functions comprise linear combinations of

$$Y_{1,x} = \sin \theta \cos \phi \quad (1.66)$$

$$Y_{1,y} = \sin \theta \sin \phi \quad (1.67)$$

Each of  $Y_{1,x}$  and  $Y_{1,y}$  is the component factor part of a phasor. They are not components of a vector; however, the  $x$  and  $y$  designation corresponds, respectively, to the historical  $p_x$  and  $p_y$  probability density functions of quantum mechanics.  $Y_{1,x}$  is a maximum at  $\theta = 90^\circ$  and  $\phi = 0^\circ$ ;  $M_{2,1,x,1/2}(90^\circ, 0^\circ) = -3.602 \text{ Cm}^{-2}$ . Figure 1.2 gives pictorial representation of how the modulation function changes the electron density on the orbitsphere for several  $\ell$  values. (When the electron charge appears throughout this text in a function involving a linear combination of the spin and orbital functions, it is implicit that the charge is normalized.)

Figure 1.2 The orbital function modulates the constant (spin) function. (shown for  $t = 0$ ; cross-sectional view)



### THE ORBITSPIHERE EQUATION OF MOTION FOR $\mathfrak{l} = 0$

The orbitsphere equation of motion for  $\mathfrak{l} = 0$  is solved as a boundary value problem. The boundary conditions are: 1.) each infinitesimal point of the orbitsphere must move along a great circle; 2.) every such infinitesimal point must have the same angular and linear velocity given by Eqs. (1.55) and (1.56), respectively; 3.) the current of the orbitsphere must give rise a magnetic moment of a Bohr magneton and the corresponding magnetic field; 4.) the magnetic moment must align completely parallel or antiparallel with an applied magnetic field in agreement with the Stern-Gerlach experiment; 5.) the energy of the transition of the alignment of the magnetic moment with an applied magnetic field must be given by Eq. (1.151); 6.) the projection of the angular momentum of the orbitsphere onto the z-axis must be  $\pm \frac{\hbar}{2}$ , and 7.) the projection of the angular momentum of the orbitsphere onto an axis which precesses about the z-axis must be  $\pm \sqrt{\frac{3}{4}}\hbar$ .

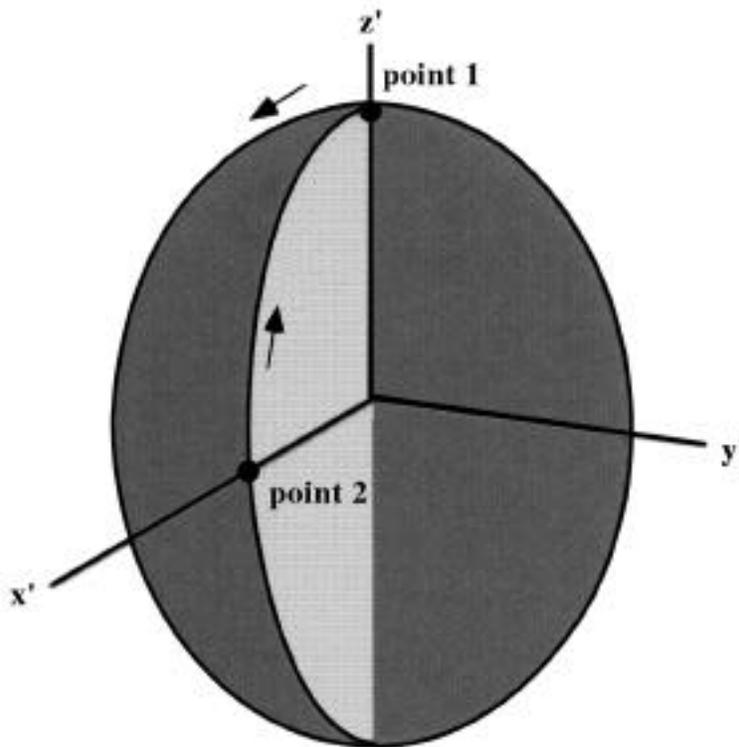
In the derivation of Eqs. (1.58) and (1.59), the moment of inertia, typically caused by a point particle or a reduced mass, is  $mr^2$ . Here, however, the mass is in the form of a two-dimensional, spherical shell. Assume that  $\mathfrak{l} = 0$  and that the electron mass and charge is uniformly distributed over the orbitsphere. Each point on the sphere with mass  $m_i$  has the same angular velocity ( $\omega_n$ ), the same magnitude of linear velocity ( $v_n$ ), and the same moment of inertia ( $m_i r_n^2$ ). The motion of each point of the orbitsphere is along a great circle, and the motion of each great circle is correlated with the motion on all other great circles. The orbitsphere is *not* analogous to a globe, where  $I = \frac{2}{3}mr^2$ , spinning about some axis. The velocity of a point mass on a spinning globe is a function of  $\theta$ . On the orbitsphere, however, each point mass has the same velocity (magnitude); the velocity is not a function of  $\theta$ . Each point must travel on a great circle such that all points have the same velocity (magnitude) and angular frequency. The uniform charge-density function of the orbitsphere is constant in time due to the motion of the current along great circles. The current flowing into any given point of the orbitsphere equals the current flowing out, but the current pattern of the orbitsphere is not uniform. The equation of motion for each point mass which gives the current pattern of the orbitsphere is generated as follows:

(Here a procedure is used to generate the current pattern of the orbitsphere from which the physical properties are derived in the Spin

Angular Momentum of the Orbitsphere with  $l = 0$  Section and are shown to match the boundary conditions.)

Consider the electron to be evenly distributed within two orthogonal great circle current loops. Then consider two infinitesimal point masses (charges), one and two, of two orthogonal great circle current loops. The Cartesian coordinate system wherein the first current loop lies in the yz-plane, and the second current loop lies in the xz-plane is designated the orbitsphere reference frame.

Figure 1.3 Two infinitesimal point masses (charges) of two orthogonal great circle current loops in the orbitsphere frame.



The current pattern of the orbitsphere comprises an infinite series of correlated orthogonal great circle current loops. It is generated by an infinite series of nested rotations of two orthogonal great circle current loops each about the new x-axis and new y-axis which results from the preceding such rotation. Each such two orthogonal great circle current loops wherein the first current loop lies in the yz-plane, and the second current loop lies in the xz-plane of the orbitsphere reference frame is an element of the infinite series. The first such orthogonal great circle

current loops is shown in Figure 1.3. The second element of the series is generated by rotation of the first element by an infinitesimal angle  $\alpha$  about the first x-axis followed by a rotation by the same infinitesimal angle  $\alpha$  about the new (second) y-axis to form a second x-axis. The third element of the series is generated by the rotation of the second element by the infinitesimal angle  $\alpha$  about the second x-axis followed by the rotation by the same infinitesimal angle  $\alpha$  about the new (third) y-axis. In general, the  $(n + 1)$ th element of the series is generated by the rotation of the  $n$ th orbitsphere coordinate system by the infinitesimal angle  $\alpha$  about the  $n$ th x-axis followed by the rotation of the  $n$ th orbitsphere coordinate system by the infinitesimal angle  $\alpha$  about the  $(n + 1)$ th new y-axis. The orbitsphere is given by reiterations of the successive rotations where the summation of the rotation about each of

the x-axis and the y-axis is  $\sum_{n=1}^{\sqrt{2}\pi/\alpha} \alpha = \sqrt{2}\pi$  which rotates the final z-axis to

the first negative z-axis, the final x-axis to the first -y-axis, and the final y-axis to the first -x-axis. (The total angle,  $\sqrt{2}\pi$ , is the hypotenuse of the triangle having the sides of  $\pi$  radians corresponding to x-axis rotations and  $\pi$  radians corresponding to y-axis rotations.) Then the reiterations of the successive rotations is continued about the  $n$ th x-axis followed by the rotation of the  $n$ th orbitsphere coordinate system by the infinitesimal angle  $\alpha' = -\alpha$  about the  $(n + 1)$ th new y-axis where the magnitude of the summation of the rotation about each of the x-axis and

the y-axis is  $\sum_{n=1}^{\sqrt{2}\pi/|\alpha'|} |\alpha'| = \sqrt{2}\pi$ . The final step rotates the final z-axis to the

first z-axis, the final x-axis to the first x-axis, and the final y-axis to the first y-axis. Thus, the orbitsphere is generated from two orthogonal great circle current loops which are rotated about the  $n$ th x-axis then about the  $(n + 1)$ th y-axis in two steps. The first step comprises all rotations by  $\alpha$ , and the second step comprises all rotations by  $\alpha'$ . In the case of the  $n$ th element of the first step, the intersection of the two orthogonal great circle current loops occurs at the  $n$ th z-axis which is along a great circle in a plane rotated  $\frac{\pi}{4}$  with respect to the 1st xz-plane

and 1st yz-plane of Figure 1.3. In the case of the  $n$ th element of the second step, the intersection of the two orthogonal great circle current loops occurs at the  $n$ th z-axis which is along a great circle in a plane rotated  $\frac{\pi}{4}$  with respect to the 1st yz-plane and the 1st negative xz-plane

(the plane containing the negative x-axis and the positive z-axis) of Figure 1.3.

Consider two point masses, one and two, in the reference frame of the orbitsphere at time zero. Point one is at  $x' = 0$ ,  $y' = 0$ , and  $z' = r_n$  and point two is at  $x' = r_n$ ,  $y' = 0$ , and  $z' = 0$ . Let point one move on a great circle toward the negative  $y'$ -axis, as shown in Figure 1.3, and let point two move on a great circle toward the positive  $z'$ -axis, as shown in Figure 1.3. The equations of motion, in the reference frame of the orbitsphere are given by

point one:

$$\dot{x}_1 = 0 \quad \dot{y}_1 = -r_n \sin(\omega_n t) \quad \dot{z}_1 = r_n \cos(\omega_n t) \quad (1.68)$$

point two:

$$\dot{x}_2 = r_n \cos(\omega_n t) \quad \dot{y}_2 = 0 \quad \dot{z}_2 = r_n \sin(\omega_n t) \quad (1.69)$$

The great circles are rotated by an infinitesimal angle  $\alpha$  (a rotation around the x-axis) and then by  $\alpha$  (a rotation around the new y-axis). The coordinates of each point on the rotated great circle is expressed in terms of the first (x,y,z) coordinates by the following transforms:

point one:

$$\begin{matrix} x_1 & \cos(\alpha) & -\sin^2(\alpha) & -\sin(\alpha)\cos(\alpha) & \dot{x}_1 \\ y_1 & 0 & \cos(\alpha) & -\sin(\alpha) & \dot{y}_1 \\ z_1 & \sin(\alpha) & \cos(\alpha)\sin(\alpha) & \cos^2(\alpha) & \dot{z}_1 \end{matrix} \quad (1.70)$$

and  $\alpha' = -\alpha$  replaces  $\alpha$  for  $\frac{\sqrt{2}\pi}{\alpha}$   $\alpha = \sqrt{2}\pi$ ;  $\frac{\sqrt{2}\pi}{|\alpha'|}$   $|\alpha'| = \sqrt{2}\pi$

point two:

$$\begin{matrix} x_2 & \cos(\alpha) & -\sin^2(\alpha) & -\sin(\alpha)\cos(\alpha) & \dot{x}_2 \\ y_2 & = & 0 & \cos(\alpha) & -\sin(\alpha) & \dot{y}_2 \\ z_2 & \sin(\alpha) & \cos(\alpha)\sin(\alpha) & \cos^2(\alpha) & & \dot{z}_2 \end{matrix} \quad (1.71)$$

and  $\alpha' = -\alpha$  replaces  $\alpha$  for  $\sum_{n=1}^{\frac{\sqrt{2}\pi}{\alpha}} \alpha = \sqrt{2}\pi$ ;  $\sum_{n=1}^{\frac{\sqrt{2}\pi}{|\alpha'|}} |\alpha'| = \sqrt{2}\pi$

The total orbitsphere is given by reiterations of Eqs. (1.70) and (1.71). The output given by the non primed coordinates is the input of the next iteration corresponding to each successive nested rotation by the infinitesimal angle where the summation of the rotation about each of

the x-axis and the y-axis is  $\sum_{n=1}^{\frac{\sqrt{2}\pi}{\alpha}} \alpha = \sqrt{2}\pi$  and  $\sum_{n=1}^{\frac{\sqrt{2}\pi}{|\alpha'|}} |\alpha'| = \sqrt{2}\pi$ .

The current pattern corresponding to point one and point two shown with 8.49 degree increments of the infinitesimal angular variable  $\alpha$  ( $\alpha'$ ) of Eqs. (1.70) and (1.71) is shown from three perspectives in Figures 1.4 A, 1.4 B, and 1.4 C. The complete orbitsphere current pattern corresponds to all such correlated points, point one and point two, of the orthogonal great circles shown in Figure 1.3 which are rotated according to Eqs. (1.70) and (1.71) where  $\alpha$  ( $\alpha'$ ) approaches zero and the summation of the infinitesimal angular rotations of  $\alpha$  ( $\alpha'$ ) about the successive x-axes and y-axes is  $\sqrt{2}\pi$ .

Figure 1.4 A. The current pattern of the orbitsphere shown with 8.49 degree increments of the infinitesimal angular variable  $\alpha$  ( $\alpha'$ ) from the perspective of looking along the z-axis.

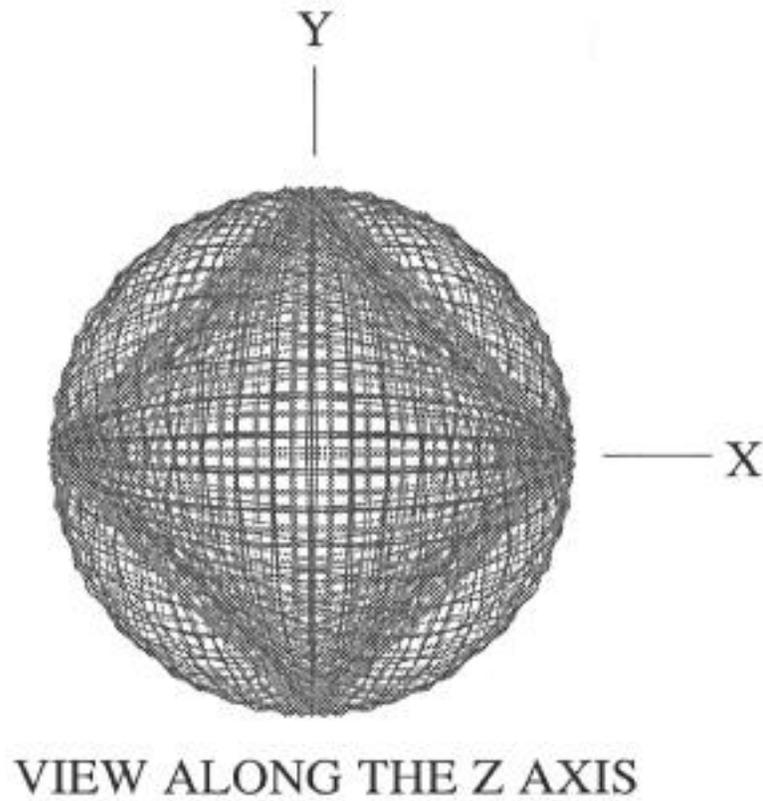
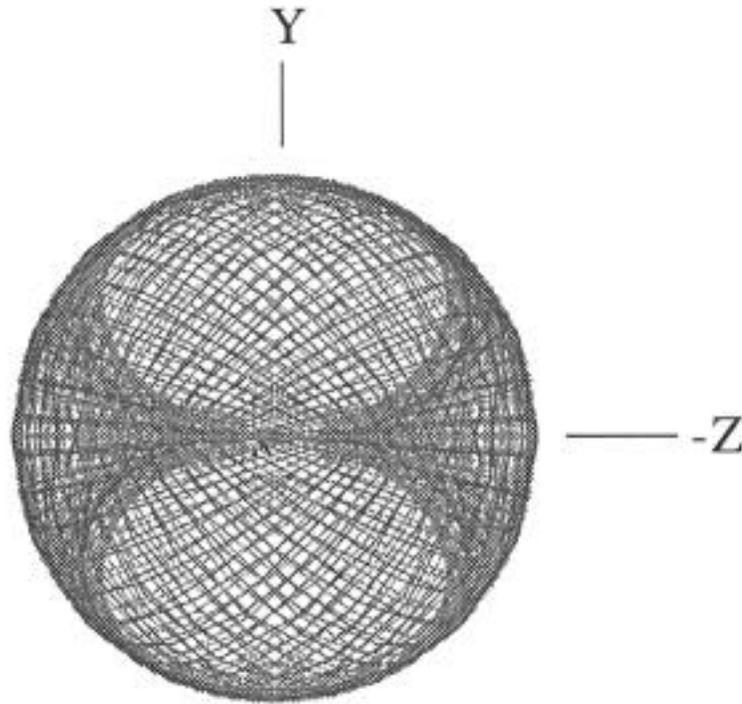
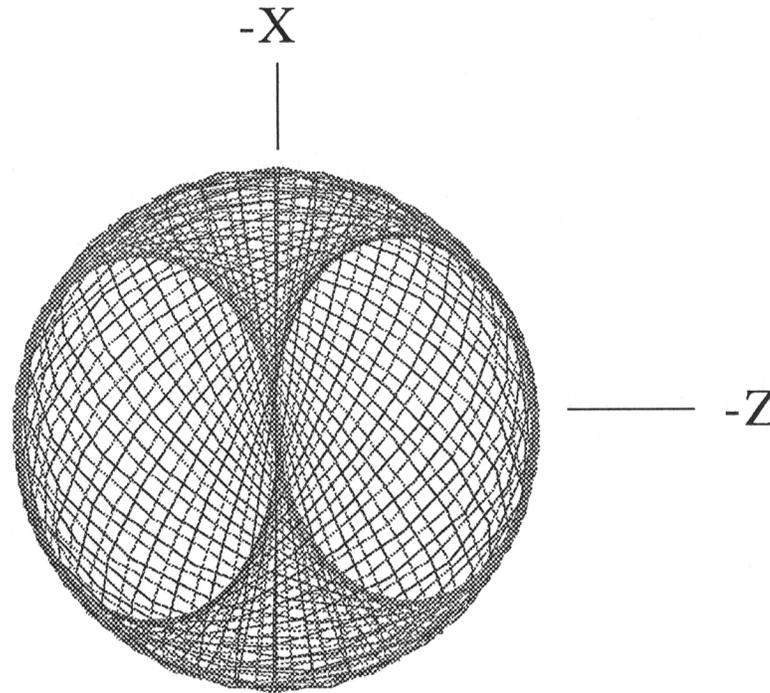


Figure 1.4 B. The current pattern of the orbitsphere shown with 8.49 degree increments of the infinitesimal angular variable  $\alpha$  ( $\alpha'$ ) from the perspective of looking along the x-axis.



VIEW ALONG THE +X AXIS

Figure 1.4 C. The current pattern of the orbitsphere shown with 8.49 degree increments of the infinitesimal angular variable  $\alpha$  ( $\alpha'$ ) from the perspective of looking along the y-axis.



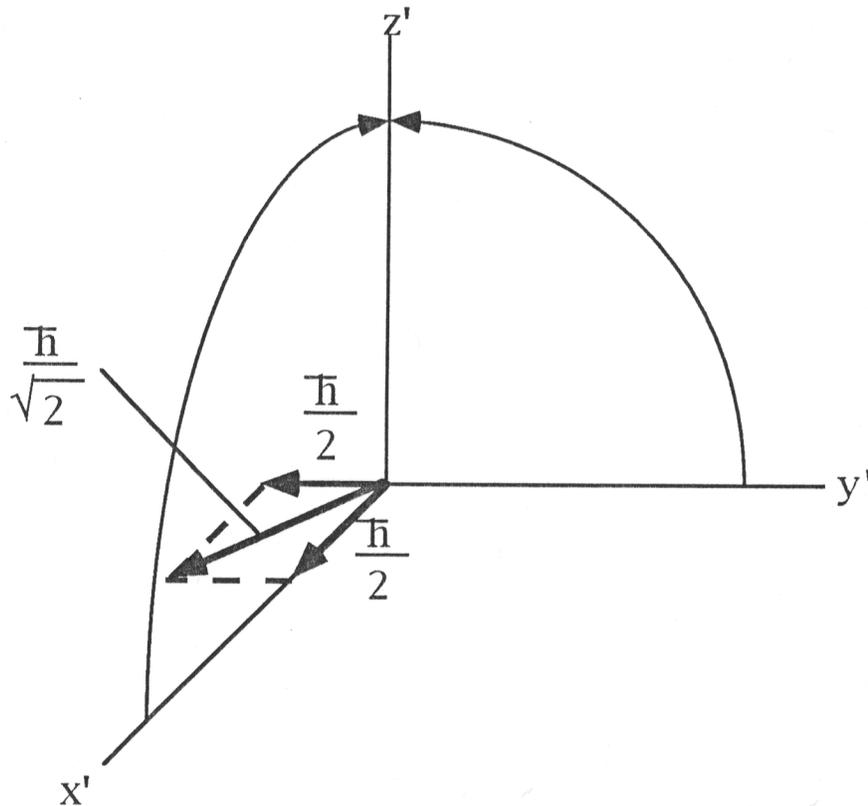
VIEW ALONG THE Y AXIS

**SPIN ANGULAR MOMENTUM OF THE ORBITSPHERE WITH  $\ell = 0$**

As demonstrated in Figures 1.3 and 1.4, the orbitsphere is generated from two orthogonal great circle current loops which are rotated about the  $n$ th x-axis then about the  $(n + 1)$ th y-axis in two steps. The first step comprises all rotations by  $\alpha$ , and the second step comprises all rotations by  $\alpha'$ . In the case of the  $n$ th element of the first step, the intersection of the two orthogonal great circle current loops occurs at the  $n$ th z-axis which is along a great circle in a plane rotated  $\frac{\pi}{4}$  with respect to the 1st xz-plane and the 1st yz-plane of Figure 1.3. In the case of the  $n$ th element of the second step, the intersection of the two orthogonal great circle current loops occurs at the  $n$ th z-axis which is along a great circle in a plane rotated  $\frac{\pi}{4}$  with respect to the 1st yz-plane and the 1st negative xz-plane (the plane containing the negative x-

axis and the positive z-axis) of Figure 1.3. The mass density,  $\frac{m_e}{4\pi r_1^2}$ , of the orbitsphere of radius  $r_1$  is uniform. However, the projections of the angular momentum of the great circle current loops of the orbitsphere onto the z-axis and onto the xy-plane can be derived by considering two orthogonal great circle current loops of Figure 1.5 each of mass  $\frac{m_e}{2}$  which generate the current pattern of the orbitsphere in two steps. (Here the physical properties of the orbitsphere are derived following the procedure used to generate the current pattern of the orbitsphere given in the Orbitsphere Equation of Motion for  $\ell = 0$  Section and are shown to match the boundary conditions.)

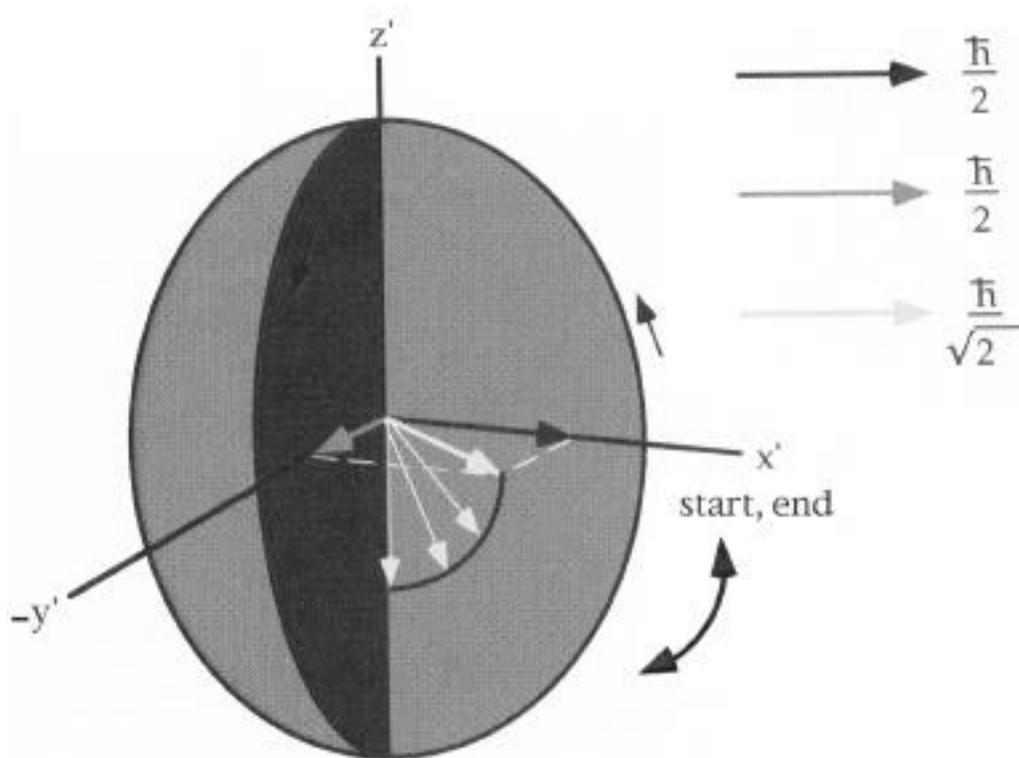
Figure 1.5 A. The angular momentum of the orthogonal great circle current loops in the xy-plane is  $\frac{\hbar}{\sqrt{2}}$ .



For step one, the resultant angular momentum vector of the orthogonal great circle current loops of magnitude  $\frac{\hbar}{\sqrt{2}}$  moves along a

great circle oriented at an angle of  $\frac{\pi}{4}$  to the 1st xz-plane and the 1st yz-plane. For the vector current directions shown in Figure 1.5 A, as the  $Y_0^0(\phi, \theta)$  orbitsphere function is partially generated in step one, the resultant angular momentum vector moves along the great circle from the 1st xy-plane to the 1st negative z-axis and back to the xy-plane. The trajectory of the resultant angular momentum vector is shown in Figure 1.5 B.

Figure 1.5 B. The trajectory of the resultant angular momentum vector of the orthogonal great circle current loops of magnitude  $\frac{\hbar}{\sqrt{2}}$  during step one.



The total sum of the magnitude of the angular momentum of each infinitesimal point of the orbitsphere is  $\hbar$  (Eq. (1.57)). Thus, the angular momentum of each great circle is  $\frac{\hbar}{2}$ . The planes of the great circles are oriented at an angle of  $\frac{\pi}{2}$  with respect to each other, and the resultant angular momentum is  $\frac{\hbar}{\sqrt{2}}$  in the xy-plane. Now, allow the

summation of the rotations by  $\alpha$  to go from zero to  $\sqrt{2}\pi$ . For step one, the vector projection of the angular momentum onto the xy-plane goes as the magnitude of  $\frac{\hbar}{\sqrt{2}} \cos\theta$  ( $\left|\frac{\hbar}{\sqrt{2}} \cos\theta\right|$ ) for  $0 \leq \theta \leq \frac{\pi}{2}$  where  $\theta$  is defined as the angle of the resultant angular momentum vector of the orthogonal great circle current loops that moves along a great circle oriented at an angle of  $\frac{\pi}{4}$  to the 1st xz-plane and the 1st yz-plane as shown in Figure 1.5 B. The trajectory of the resultant angular momentum vector is from  $\theta = 0$  to  $\theta = \frac{\pi}{2}$ , and then from  $\theta = \frac{\pi}{2}$  to  $\theta = 0$ . The vector projection of the angular momentum onto the negative z-axis goes as  $\left|\frac{\hbar}{\sqrt{2}} \sin\theta\right|$  as shown in Figure 1.5 B. In each case, the projection of the angular momentum is periodic over the range of  $\theta$  corresponding to  $\alpha$  which generates the angular momentum distribution. The projection in the xy-plane varies in magnitude from a maximum of  $\frac{\hbar}{\sqrt{2}}$  to zero to  $\frac{\hbar}{\sqrt{2}}$  again. The projection onto the negative z-axis varies in magnitude from zero to a maximum of  $\frac{\hbar}{\sqrt{2}}$  to zero again. The total of each projection,  $\langle L_{xy} \rangle_{\alpha}$  and  $\langle L_z \rangle_{\alpha}$ , is the integral of the magnitude of the vector as a function of  $\theta$ . The result is the root mean square value (rms) of the maximum magnitude which is multiplied by one half corresponding to two steps (i.e. the electron angular momentum is distributed over 1/2 of the surface of a sphere in the first step, and the mirror image of the angular momentum distribution is generated in the second step given *infra*).

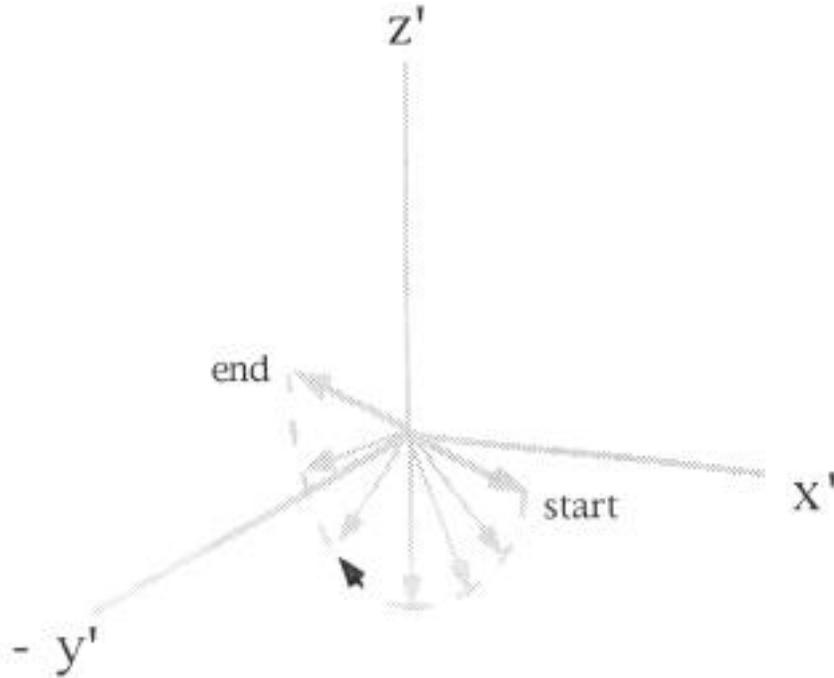
$$\langle L_{xy} \rangle_{\alpha} = \frac{1}{2} \frac{\hbar}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{\hbar}{4} \quad (1.72)$$

$$\langle L_z \rangle_{\alpha} = \frac{1}{2} \frac{\hbar}{\sqrt{2}} \frac{1}{\sqrt{2}} = \frac{\hbar}{4} \quad (1.73)$$

For step two, the resultant angular momentum vector of the orthogonal great circle current loops of magnitude  $\frac{\hbar}{\sqrt{2}}$  moves along a great circle oriented at an angle of  $\frac{\pi}{4}$  to the 1st yz-plane and the 1st negative xz-plane (the plane containing the negative x-axis and the positive z-axis). For the vector current directions shown in Figure 1.5 A, as the  $Y_0^0(\phi, \theta)$  orbitosphere equation of motion is completely generated in step two, the resultant angular momentum vector moves along the great circle from the xy-plane to the negative z-axis and back to the xy-plane such that the vector projections on to the z-axis all add positively and

the vector projections into the xy-plane sum to zero. The trajectory of the resultant angular momentum vector is shown in Figure 1.5 C.

Figure 1.5 C. The trajectory of the resultant angular momentum vector of the orthogonal great circle current loops of magnitude  $\frac{\hbar}{\sqrt{2}}$  during step two.



For step two, the trajectory of the resultant angular momentum vector is from  $\theta = 0$  to  $\theta = \pi$ . The vector projection of the angular momentum onto the xy-plane goes as  $\left| \frac{\hbar}{\sqrt{2}} \cos\theta \right|$  for  $0 \leq \theta \leq \frac{\pi}{2}$  and  $-\left| \frac{\hbar}{\sqrt{2}} \cos\theta \right|$  for  $\frac{\pi}{2} \leq \theta \leq \pi$  as shown in Figure 1.5 C. The projection of the angular momentum is a periodic function of  $\theta$  corresponding to  $\alpha'$  which generates the angular momentum distribution. The projection in the xy-plane varies in magnitude from a maximum of  $\frac{\hbar}{\sqrt{2}}$  to zero to  $-\frac{\hbar}{\sqrt{2}}$ . For step two, the vector projection into the xy-plane,  $\langle L_{xy} \rangle_{\alpha'}$  is zero, but the scalar sum of the angular momentum in the xy-plane is the absolute value of the integral of the magnitude of the vector as a function of  $\theta$ . The result is the root mean square value (rms) of the maximum magnitude which is multiplied by one half corresponding to two steps

(i.e. the electron angular momentum is distributed over 1/2 of the surface of a sphere in the first step, and the mirror image of the angular momentum distribution is generated in the second step). The scalar sum is given by the magnitude of Eq. (1.72).

The vector projection of the angular momentum onto the negative z-axis goes as  $\left| \frac{\hbar}{\sqrt{2}} \sin\theta \right|$  for  $0 < \theta < \pi$  as shown in Figure 1.5 C. The vector projection onto the z-axis is periodic over the range of  $\theta$  corresponding to  $\alpha$  which generates the angular momentum distribution and varies in magnitude from zero to a maximum of  $\frac{\hbar}{\sqrt{2}}$  to zero again. The total of each projection,  $\langle L_z \rangle_\alpha$ , is the integral of the magnitude of the vector as a function of  $\theta$ . The result is the root mean square value (rms) of the maximum magnitude which is multiplied by one half corresponding to two steps (i.e. the electron angular momentum is distributed over 1/2 of the surface of a sphere in the first step, and the mirror image of the angular momentum distribution is generated in the second step). The vector sum is given by Eq. (1.73).

The total angular momentum of the orbitsphere is  $\hbar$  (Eq. (1.57)). The vector projection of the angular momentum into the xy-plane is given by Eq. (1.72), and the scalar sum of the projection of the angular momentum into the xy-plane is  $\frac{\hbar}{2}$ . Consider steps one and two. As demonstrated by Figures 1.3, 1.4, and 1.5, each contribution to vector sum of the z component of the orbitsphere angular momentum is positive. Thus, the z-projection of the angular momentum is  $\frac{\hbar}{2}$ .

Consider the case of a magnetic field applied to the orbitsphere. The magnetic moment corresponding to the angular momentum along the z-axis results in the alignment of the z-axis of the orbitsphere with the magnetic field. The angular momentum in the xy-plane precesses about the applied field; thus, the time average angular momentum in the xy-plane is zero. The angular momentum of the precessing orbitsphere can be given as an equivalent vector which precesses about the z-axis which possesses a scalar projection of the angular momentum into the xy-plane of  $\frac{\hbar}{2}$  and a vector projection of the angular momentum onto the z-axis of  $\frac{\hbar}{2}$ . S the projection of the orbitsphere angular momentum that precesses about the z-axis called the spin axis at an angle of  $\theta = \frac{\pi}{3}$  and an angle of  $\phi = \pi$  with respect to  $\langle L_{xy} \rangle_\alpha$  given by Eq. (1.72) is

$$\mathbf{S} = \pm \sqrt{\frac{3}{4}} \hbar \quad (1.74)$$

$\mathbf{S}$  rotates about the z-axis at the Larmor frequency; thus,  $\langle S_z \rangle$ , the time averaged projection of the orbit's angular momentum onto the axis of the applied magnetic field is  $\pm \frac{\hbar}{2}$ . To verify the validity of Eq. (1.74), consider the components of the angular momentum along the z-axis and in the xy-plane from the perspective of a frame that rotates with  $\mathbf{S}$  and from the perspective of the stationary or laboratory frame. In the rotating frame  $\phi = 0$  is defined in the direction of the resultant angular momentum vector shown in Figure 1.5A. From Eq. (1.72), the angular momentum in this direction is  $\frac{\hbar}{4}$ . The angular momentum in the direction  $\phi = \pi$  with respect to this direction is  $\sqrt{\frac{3}{4}} \hbar \sin \frac{\pi}{3} = \frac{3}{4} \hbar$ . Thus, in the rotating frame, the resultant scalar angular momentum in the xy plane is  $\frac{\hbar}{2}$ .  $\mathbf{S}$  forms a cone in the nonrotating laboratory frame with a total angular momentum of  $\hbar$ . The projection of this angular momentum onto the z-axis is  $\pm \hbar \cos \frac{\pi}{3} = \pm \frac{\hbar}{2}$ . (The same result is obtained from the approach given by Eq. (3.35).) The plus or minus sign corresponds to the two possible vector orientations which are observed with the Stern-Gerlach experiment described below.

### ROTATIONAL PARAMETERS OF THE ELECTRON (ANGULAR MOMENTUM, ROTATIONAL ENERGY, AND MOMENT OF INERTIA)

One result of the correlated motion along great circles is that some of the kinetic energy is not counted in the rotational energy. That is, for any spin axis there will be an infinite number of great circles with planes passing through that axis with  $\theta$  angles other than  $90^\circ$ . All points on any one of these great circles will be moving, but not all of that motion will be part of the rotational energy; only that motion perpendicular to the spin axis will be part of the rotational energy. Thus, the rotational kinetic energy will always be less than the total kinetic energy.

Furthermore, the following relationships must hold.

$$E_{rotational} = \frac{1}{2} I \omega^2 = \frac{1}{2} m_e v^2 \quad (1.75)$$

$$I \omega = \hbar \quad (1.76)$$

$$I = m_e r^2 \quad (1.77)$$

Furthermore, it is known from the Stern-Gerlach experiment that a beam of silver atom splits into two components when passed through an

inhomogeneous magnetic field. This experiment implies a magnetic moment of one Bohr magneton and an associated angular momentum quantum number of  $1/2$ . Historically, this quantum number is called the spin quantum number, and that designation will be retained. The angular momentum can be thought of arising from a spin component or equivalently an orbital component of the spin. The z-axis projection of the spin angular momentum was derived in the Spin Angular Momentum of the Orbitosphere with  $\ell = 0$  Section.

$$L_z = I\omega \mathbf{i}_z = \pm \frac{\hbar}{2} \quad (1.78)$$

where  $\omega$  is given by Eq. (1.55); so,

$$\ell = 0$$

$$|L_z| = I \frac{\hbar}{m_e r^2} = \frac{\hbar}{2} \quad (1.79)$$

Thus,

$$I_z = I_{spin} = \frac{m_e r_n^2}{2} \quad (1.80)$$

From Eq. (1.51),

$$E_{rotational\ spin} = \frac{1}{2} [I_{spin} \omega^2] \quad (1.81)$$

From Eqs. (1.55) and (1.80),

$$E_{rotational} = E_{rotational\ spin} = \frac{1}{2} I_{spin} \frac{\hbar^2}{m_e r_n^2} = \frac{1}{2} \frac{m_e r_n^2}{2} \frac{\hbar^2}{m_e r_n^2} = \frac{1}{4} \frac{\hbar^2}{2I_{spin}} \quad (1.82)$$

When  $\ell = 0$ , the spherical harmonic is not a constant and the charge-density function is not uniform over the orbitosphere. Thus, the angular momentum can be thought of arising from a spin component and an orbital component.

### Derivation of the Rotational Parameters of the Electron

In the derivation of Eq. (1.59) and its solution for  $E_{rotational}$  (Eq. (1.60)), the moment of inertia,  $I$ , was assumed by McQuarrie [11] to be the moment of inertia of a point particle,  $mr_n^2$ . However, the correct equation of the electron is a two dimensional shell with constant or a constant plus a spherical harmonic angular dependence. In that case, the relationships given by Eqs. (1.75) to (1.77) must hold.

The substitution of  $NI$  for  $I$  in the rigid rotor problem [11] where  $N$  is a constant does not change the form of the previous solution given by Eq. (1.60). However, the result that

$$N = \frac{\ell(\ell+1)}{\ell^2 + 2\ell + 1}^{\frac{1}{2}} < 1 \quad (1.83)$$

derived below gives

$$E_{rotational} = \frac{\hbar^2 \ell(\ell+1)}{2I(\ell^2 + 2\ell + 1)} \quad (1.84)$$

and gives the moment of inertia of the orbitsphere,  $I_{orbital}$ , where  $\ell \geq 0$  as

$$NI = I_{orbital} = m_e r_n^2 \frac{\ell(\ell+1)}{(\ell^2 + 2\ell + 1)^{\frac{1}{2}}} \quad (1.85)$$

The solution of Eq. (1.59) for  $|L|$ , the magnitude of the orbital angular momentum is [11]

$$|L| = \hbar \sqrt{\ell(\ell+1)} \quad (1.86)$$

where  $I$  of Eq. (1.59) is the moment of inertia of a point charge. It is demonstrated by Eq. (1.57) that the total sum of the magnitudes of the angular momenta of the infinitesimal points of the electron orbitsphere is  $\hbar$ ; therefore, the magnitude of the angular momentum of an electron orbitsphere must be less than  $\hbar$ , and the moment of inertia must be less than that given by  $m_e r_n^2$ . For example, the moment of inertia of the uniform spherical shell,  $I_{RS}$ , is [13]

$$I_{RS} = \frac{2}{3} m r_n^2 \quad (1.87)$$

Thus, Eq. (1.86) must be multiplied by a fraction,  $\frac{1}{K}$ , to give the correct angular momentum. Given that generally  $L$  is

$$L = I\omega \mathbf{i}_z \quad (1.88)$$

then

$$I_{orbital} \omega \mathbf{i}_z = \hbar \frac{1}{K} \sqrt{\ell(\ell+1)}, \quad (1.89)$$

where  $\omega$  is given by Eq. (1.55). The orbital moment of inertia,  $I_{orbital}$ , is

$$I_{orbital} = m_e r_n^2 \frac{1}{K} \sqrt{\ell(\ell+1)} \quad (1.90)$$

The total kinetic energy,  $T$ , of the orbitsphere is

$$T = \frac{1}{2} m_e v_n^2 \quad (1.91)$$

Substitution of Eq. (1.56) gives

$$T = \frac{\hbar^2}{2m_e r_n^2} \quad (1.92)$$

$E_{rotational}$  of the rigid shell is given by Eq. (1.51) with  $I$  given by Eq. (1.87).

$E_{rotationalorbital}$  of the orbitsphere is given by Eq. (1.60) multiplied by the

fraction  $\frac{1}{K^2}$  so that Eqs. (1.75) to (1.77) hold with  $I = m_e r_n^2$ .

$$E_{rotationalorbital} = \frac{\hbar^2}{2I} \frac{\ell(\ell+1)}{K^2} \quad (1.93)$$

Eq. (1.59) can be expressed in terms of the variable  $x$  which is

substituted for  $\cos\theta$ . The resulting function  $P(x)$  is called Legendre's equation and is a well-known equation in classical physics. It occurs in a variety of problems that are formulated in spherical coordinates. When the power series method of solution is applied to  $P(x)$ , the series must be truncated in order that the solutions be finite at  $x = \pm 1$ . The solution to Legendre's equation given by Eq. (1.60) is the maximum term of a series of solutions corresponding to the  $m$  and  $\ell$  values [11,14]. The rotational energy must be normalized by the total number of states—each corresponding to a set of quantum numbers of the power series solution. As demonstrated in the Excited States of the One Electron Atom (Quantization) Section, the quantum numbers of the excited states are

$$\begin{aligned} n &= 2, 3, 4, \dots \\ \ell &= 1, 2, \dots, n-1 \\ m &= -\ell, -\ell+1, \dots, 0, \dots, +\ell \end{aligned}$$

In the case of an orbitosphere excited state, each rotational state solution of Eq. (1.59) (Legendre's equation) corresponds to a multipole moment of the charge-density function (Eq. (1.65)).  $E_{\text{rotationalorbital}}$  is normalized by  $N$ , the total number of multipole moments.  $N$ , the total number of multipole moments where each corresponds to an  $\ell$  and  $m_\ell$  quantum number of an energy level corresponding to a principal quantum number of  $n$  is

$$N = \sum_{\ell=0}^{n-1} \sum_{m_\ell=-\ell}^{+\ell} 2\ell+1 = n^2 = \ell^2 + 2\ell + 1 \quad (1.94)$$

Thus,  $K^2$  is equal to  $N$  given by Eq. (1.94). Substitution of Eq. (1.94) into Eq. (1.93) gives

$$E_{\text{rotationalorbital}} = \frac{\hbar^2}{2I} \frac{\ell(\ell+1)}{\ell^2 + 2\ell + 1} \quad (1.95)$$

Substitution of Eq. (1.94) into Eq. (1.90) gives the orbital moment of inertia.

$$I_{\text{orbital}} = m_e r_n^2 \frac{\ell(\ell+1)}{\ell^2 + 2\ell + 1}^{\frac{1}{2}} \quad (1.96)$$

In the case of the excited states, the orbitosphere charge-density function for  $\ell \neq 0$ , Eq. (1.65), is the sum of two functions of equal magnitude.  $L_z$ , total is given by the sum of the spin and orbital angular momenta. The principal energy levels of the excited states are split when a magnetic field is applied. The energy shift due to spin and orbital angular momenta are given in the Spin and Orbital Splitting Section.

$\ell \neq 0$

$$L_{z\text{total}} = L_{z\text{spin}} + L_{z\text{orbital}} \quad (1.97)$$

Similarly, the orbital rotational energy arises from a spin function (spin angular momentum) modulated by a spherical harmonic angular function (orbital angular momentum). The time-averaged orbital rotational energy is zero; the magnitude is given by Eq. (1.95); the rotational energy due to spin is given by Eq. (1.82); the total kinetic energy is given by Eq. (1.92).

$$\langle E_{rotational\,orbital} \rangle = 0 \quad (1.98)$$

The demonstration that the modulated orbitsphere solutions are solutions of the wave equation appears in Box 1.1.

**BOX 1.1. DERIVATION OF THE ROTATIONAL PARAMETERS OF THE ELECTRON FROM A SPECIAL CASE OF THE WAVE EQUATION--THE RIGID ROTOR EQUATION**

For a time harmonic charge density function, Eq. (1.49) becomes

$$\frac{1}{r^2 \sin \theta} \frac{\delta}{\delta \theta} \sin \theta \frac{\delta}{\delta \theta}_{r,\phi} + \frac{1}{r^2 \sin^2 \theta} \frac{\delta^2}{\delta \phi^2}_{r,\theta} + \frac{\omega^2}{v^2} A(\theta, \phi) = 0 \quad (1)$$

Substitution of the velocity about a Cartesian coordinate axis,  $v = \rho \omega$ , into Eq. (1) gives

$$\frac{1}{r^2 \sin \theta} \frac{\delta}{\delta \theta} \sin \theta \frac{\delta}{\delta \theta}_{r,\phi} + \frac{1}{r^2 \sin^2 \theta} \frac{\delta^2}{\delta \phi^2}_{r,\theta} + \frac{\omega^2}{(\rho \omega)^2} A(\theta, \phi) = 0 \quad (2)$$

Substitution of Eq. (1.55) into Eq. (1.2) gives

$$\frac{1}{r^2 \sin \theta} \frac{\delta}{\delta \theta} \sin \theta \frac{\delta}{\delta \theta}_{r,\phi} + \frac{1}{r^2 \sin^2 \theta} \frac{\delta^2}{\delta \phi^2}_{r,\theta} + \frac{\omega_n^2}{\rho \frac{\hbar}{m_e r_n^2}} A(\theta, \phi) = 0 \quad (4)$$

Multiplication by the denominator of the second term in Eq. (3) gives

$$\rho \frac{\hbar}{m_e r_n^2} \frac{1}{r^2 \sin \theta} \frac{\delta}{\delta \theta} \sin \theta \frac{\delta}{\delta \theta}_{r,\phi} + \frac{1}{r^2 \sin^2 \theta} \frac{\delta^2}{\delta \phi^2}_{r,\theta} + \omega_n^2 A(\theta, \phi) = 0 \quad (4)$$

Substitution of Eq. (1.51) gives

$$\rho \frac{\hbar}{m_e r_n^2} \frac{1}{r^2 \sin \theta} \frac{\delta}{\delta \theta} \sin \theta \frac{\delta}{\delta \theta}_{r,\phi} + \frac{1}{r^2 \sin^2 \theta} \frac{\delta^2}{\delta \phi^2}_{r,\theta} + \frac{2E_{rot}}{I} A(\theta, \phi) = 0 \quad (5)$$

The total rotational energy is given by the superposition of  $\ell$  quantum states corresponding to a multipole expansion of total rotational energy of the orbitsphere. The total number,  $N$ , of multipole moments where

each corresponds to an  $\ell$  and  $m_\ell$  quantum number of an energy level corresponding to a principal quantum number of  $n$  is

$$N = \sum_{\ell=0}^{n-1} \sum_{m_\ell=-\ell}^{+\ell} = \sum_{\ell=0}^{n-1} 2\ell + 1 = \ell^2 + 2\ell + 1 = n^2 \quad (6)$$

Summing over all quantum states gives

$$\sum_{\ell=0}^{n-1} \sum_{m_\ell=-\ell}^{+\ell} \rho \frac{\hbar^2}{m_e r_n^2} \frac{1}{r^2 \sin\theta} \frac{\delta}{\delta\theta} \sin\theta \frac{\delta}{\delta\theta} \Big|_{r,\phi} + \frac{1}{r^2 \sin^2\theta} \frac{\delta^2}{\delta\phi^2} \Big|_{r,\theta} + \sum_{\ell=0}^{n-1} \sum_{m_\ell=-\ell}^{+\ell} \frac{2E_{rot}}{I} A(\theta, \phi) = 0 \quad (7)$$

Each of the orbital energy, orbital moment of inertia, and orbital angular momentum is a modulation of the orbitsphere function. Thus, the sum of  $\rho^2$  over all  $\ell$  quantum numbers is  $r_n$ . Substitution of

$\rho_z = r_n \cos\theta$ ;  $\rho_x = r_n \sin\theta \cos\phi$ ;  $\rho_y = r_n \sin\theta \sin\phi$  into Eq. (7) gives

$$r_n \frac{\hbar^2}{m_e r_n^2} \frac{1}{r_n^2 \sin\theta} \frac{\delta}{\delta\theta} \sin\theta \frac{\delta}{\delta\theta} \Big|_{r,\phi} + \frac{1}{r_n^2 \sin^2\theta} \frac{\delta^2}{\delta\phi^2} \Big|_{r,\theta} + (\ell^2 + 2\ell + 1) \frac{2E_{rot}}{I} A(\theta, \phi) = 0 \quad (8)$$

where  $\frac{2E_{rot}}{I}$  is the constant,  $\omega_n$  given by Eq. (1.55), and  $r = r_n$ . Eq. (8) can be expressed in terms of the rotational energy of any given mode by dividing the denominator of the first term by,  $K^2$ , the factor corresponding to the vector projection of the rotational energy onto the z-axis.

$$\frac{I\hbar^2}{2m_e^2 r_n^4 (\ell^2 + 2\ell + 1)} \frac{1}{\sin\theta} \frac{\delta}{\delta\theta} \sin\theta \frac{\delta}{\delta\theta} \Big|_{r,\phi} + \frac{1}{\sin^2\theta} \frac{\delta^2}{\delta\phi^2} \Big|_{r,\theta} + E_{rot} A(\theta, \phi) = 0 \quad (9)$$

In the case that  $E_{rot}$  is the total rotational energy which is equal to the kinetic energy of the orbitsphere given by Eq. (1.92) and that the moment of inertia is given by

$$I = m_e r_n^2 \quad (10)$$

Eq. (9) becomes equivalent to Eq. (1.59).

$$\frac{1}{N} \frac{\hbar^2}{2I} \frac{1}{\sin\theta} \frac{\delta}{\delta\theta} \sin\theta \frac{\delta}{\delta\theta} \Big|_{r,\phi} + \frac{1}{\sin^2\theta} \frac{\delta^2}{\delta\phi^2} \Big|_{r,\theta} + E_{rot \ total} A(\theta, \phi) = 0 \quad (11)$$

where  $N$  is one. Eq. (11) applies to all of the multipole modes of the rotational energy with the appropriate moment of inertia,  $I$ , and factor  $N$ ; thus, the rotational energy of each mode is given by Eq. (1.58) with these conditions. Eq. (9) can be expressed in terms of the rotational energy of any given mode by dividing the first term by,  $K^2$ , the factor corresponding to the vector projection of the rotational energy and the

moment of inertia onto the z-axis.

$$\frac{\hbar^2}{2m_e^2 r_n^4 K^2 (\ell^2 + 2\ell + 1)} \frac{1}{\sin\theta} \frac{\delta}{\delta\theta} \sin\theta \frac{\delta}{\delta\theta} + \frac{1}{\sin^2\theta} \frac{\delta^2}{\delta\phi^2} + E_{rot} A(\theta, \phi) = 0 \quad (12)$$

where in the case of the spherical harmonics,  $N = \ell^2 + 2\ell + 1$ . From Eq. (1.51) and Eq. (1.88), Eq. (12) can be expressed as

$$\frac{\hbar^2}{m_e^2 r_n^4 K^2 (\ell^2 + 2\ell + 1)} \frac{1}{\sin\theta} \frac{\delta}{\delta\theta} \sin\theta \frac{\delta}{\delta\theta} + \frac{1}{\sin^2\theta} \frac{\delta^2}{\delta\phi^2} + \frac{L^2}{I^2} A(\theta, \phi) = 0 \quad (13)$$

In the case of the spherical harmonic functions with Eq. (1.88) and Eq. (1.55), Eq. (12) gives

$$\sqrt{\frac{\hbar^2 (\ell(\ell + 1))}{m_e^2 r_n^4 K^2 (\ell^2 + 2\ell + 1)}} = \frac{L}{I} = \frac{\hbar}{m_e r_n^2} \quad (14)$$

Thus,

$$\sqrt{\frac{(\ell(\ell + 1))}{(\ell^2 + 2\ell + 1)}} = K \quad (15)$$

Eq. (12) becomes Eq. (11) where the rotational energy is given by Eq. (1.95).

$$E_{rotational\ orbital} = \frac{\hbar^2}{2I} \frac{\ell(\ell + 1)}{\ell^2 + 2\ell + 1} \quad (16)$$

and the orbital moment of inertia is given by Eq. (1.96).

$$I_{orbital} = m_e r_n^2 \frac{\ell(\ell + 1)}{\ell^2 + 2\ell + 1} \quad (17)$$

The Substitution of Eqs. (1.65), (6), and (16) into Eq. (11) gives

$$-\frac{\hbar^2}{2I} \frac{\ell(\ell + 1)}{\ell^2 + 2\ell + 1} + \frac{\hbar^2}{2m_e r_n^2} \sqrt{\frac{\ell(\ell + 1)}{\ell^2 + 2\ell + 1}} = 0 \quad (18)$$

Substitution of Eq. (17) into Eq. (18) gives

$$-\frac{\hbar^2}{2m_e r_n^2 \sqrt{\frac{\ell(\ell + 1)}{\ell^2 + 2\ell + 1}}} \frac{\ell(\ell + 1)}{\ell^2 + 2\ell + 1} + \frac{\hbar^2}{2m_e r_n^2} \sqrt{\frac{\ell(\ell + 1)}{\ell^2 + 2\ell + 1}} = 0 \quad (19)$$

$$0 = 0 \quad (20)$$

Thus, the modulated orbitsphere solutions are shown to be solutions of the wave equation by their substitution into the wave equation (Eqs. (18-20)). The present derivation of the rigid rotor equation given by the substitution of

$$E_{rot} = \frac{1}{2} I \omega_n^2$$

$$\omega_n = \frac{\hbar}{m_e r_n^2} \quad (21)$$

$$v = \rho \omega_n$$

is consistent with the wave equation relationship:

$$v = \lambda \frac{\omega}{2\pi} \quad (22)$$

Whereas, Schrodinger's derivation from the Helmholtz equation [1] with the substitution of

$$\lambda = \frac{h}{m_e v} \quad (23)$$

gives the rigid rotor equation with the paradox that

$$v^2 = \frac{h}{m_e} \frac{\omega}{2\pi} \quad (24)$$

which is not the wave relationship,

$$v = \lambda \frac{\omega}{2\pi} \quad (25)$$

## References

1. McQuarrie, D. A., Quantum Chemistry, University Science Books, Mill Valley, CA, (1983), pp. 78-79.

## MAGNETIC PARAMETERS OF THE ELECTRON (BOHR MAGNETON)

### The Magnetic Field of an Orbitsphere from Spin

The orbitsphere is a shell of negative charge current comprising correlated charge motion along great circles. For  $\ell = 0$ , the orbitsphere gives rise to a magnetic moment of 1 Bohr magneton [16] as shown in the Derivation of the Magnetic Field Section,

$$\mu_B = \frac{e\hbar}{2m_e} = 9.274 \times 10^{-24} \text{ JT}^{-1}, \quad (1.99)$$

and a magnetic field derived below.

$$\mathbf{H} = \frac{e\hbar}{m_e r_n^3} (\mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta) \quad \text{for } r < r_n \quad (1.100)$$

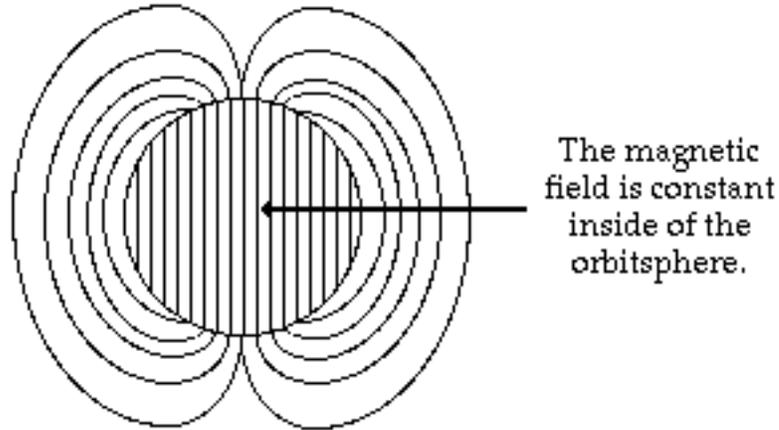
$$\mathbf{H} = \frac{e\hbar}{2m_e r^3} (\mathbf{i}_r 2\cos\theta - \mathbf{i}_\theta \sin\theta) \quad \text{for } r > r_n \quad (1.101)$$

It follows from Eq. (1.99), the relationship for the Bohr magneton, and relationship between the magnetic dipole field and the magnetic moment  $\mathbf{m}$  [17] that Eqs. (1.100) and (1.101) are the equations for the magnetic field due to a magnetic moment of a Bohr magneton,  $\mathbf{m} = \mu_B \mathbf{i}_z$  where  $\mathbf{i}_z = \mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta$ . Note that the magnetic field is a constant for  $r < r_n$ . See Figure 1.6. It is shown in the Magnetic Parameters of the Electron

(Bohr Magneton) Section that the energy stored in the magnetic field of the electron orbitsphere is

$$E_{mag,total} = \frac{\pi\mu_o e^2 \hbar^2}{m_e^2 r_1^3} \quad (1.102)$$

Figure 1.6. The magnetic field of an electron orbitsphere.



### Derivation of the Magnetic Field

Consider Figure 1.6. The magnetic field must satisfy the following relationships:

$$\mathbf{H} = 0 \text{ in free space} \quad (1.103)$$

$$\mathbf{n} \times (\mathbf{H}_a - \mathbf{H}_b) = \mathbf{K} \quad (1.104)$$

$$\mathbf{n} \cdot (\mathbf{H}_a - \mathbf{H}_b) = 0 \quad (1.105)$$

$$\mathbf{H} = - \nabla \psi \quad (1.106)$$

The z component of the current,  $|i|$ , for a current loop of total charge,  $e$ , oriented at an angle  $\theta$  with respect to the z-axis is given as the product of the charge, the angular velocity (The orbitsphere angular velocity is given by Eq. (1.55).), and  $\sin\theta$ .

$$|i| = \frac{e\hbar}{m_e r_n^2} \sin\theta \quad (1.107)$$

Consider the orbitsphere depicted in Figures 1.3, 1.4, and 1.5. The surface current-density function,  $\mathbf{K}_{i_\phi}$ , is perpendicular to the angular momentum. As shown in the Spin Angular Momentum Section, the

vector projection of the orbitsphere angular momentum onto the xy-plane goes as  $|\cos \alpha|$  as shown in Figure 1.5 B. It is periodic over the range of  $\alpha$  and  $\alpha'$  and varies in magnitude from a maximum of  $\frac{\hbar}{\sqrt{2}}$  to zero to  $\frac{\hbar}{\sqrt{2}}$  again. The projection of the charge-density of the orbitsphere onto the xy-plane (perpendicular to the z-axis) which carries the incremental current,  $\mathbf{i}_\phi$ , is a function of  $\sin \theta$ . The angular function of the current-density of the orbitsphere is normalized to that of one electron.

$$N = \frac{1}{\pi \int_0^\pi \sin^2 \theta \sin \theta d\theta} = \frac{3}{4} \quad (1.108)$$

Due to the precession of the S-axis about the z axis, the time averaged projection of the angular momentum of the electron orbitsphere onto the xy-plane is zero. Therefore, the current corresponding to the total charge of the electron is about the z-axis, and the angular velocity of the spinning orbitsphere is twice that of a stationary orbitsphere. As shown in Figure 1.5 B, the projection of the angular momentum is only onto the negative z-axis of length  $r_n$ . Thus, the incremental current-density  $d\mathbf{K}i_\phi$  along the z-axis is given by dividing  $\mathbf{i}_\phi$  by the length,  $r_n$ . The current-density of the orbitsphere in the incremental length  $dz$  is

$$\mathbf{K}(\rho, \phi, z) = \mathbf{i}_\phi 2N \frac{e\hbar}{m_e r_n^3} = \mathbf{i}_\phi \frac{3}{2} \frac{e\hbar}{m_e r_n^3} \quad (1.109)$$

Because

$$z = r \cos \theta \quad (1.110)$$

a differential length

$$dz = -\sin \theta r_n d\theta \quad (1.111)$$

and so the current-density in the differential length  $r_n d\theta$  as measured along the periphery of the orbitsphere is a function of  $\sin \theta$ . Thus, the surface current-density function is given by

$$\mathbf{K}(r, \theta, \varphi) = \mathbf{i}_\phi \frac{3}{2} \frac{e\hbar}{m_e r_n^3} \sin \theta \quad (1.112)$$

Substitution of Eq. (1.112) into Eq. (1.104) gives

$$H_\theta^a - H_\theta^b = \frac{3}{2} \frac{e\hbar}{m_e r_n^3} \sin \theta \quad (1.113)$$

To obtain  $H_\theta$ , the derivative of with respect to  $\theta$  must be taken, and this suggests that the  $\theta$  dependence of be taken as  $\cos \theta$ . The field is finite at the origin and is zero at infinity; so, solutions of Laplace's equation in spherical coordinates are selected because they are consistent with these conditions [18].

$$= C \frac{r}{r_n} \cos \theta ; \quad r < r_n \quad (1.114)$$

$$= A \frac{r_n^2}{r} \cos \theta ; \quad r > r_n \quad (1.115)$$

The negative gradient of these potentials is

$$\mathbf{H} = \frac{-C}{r_n} (\mathbf{i}_r \cos \theta - \mathbf{i}_\theta \sin \theta) \quad \text{for } r < r_n \quad (1.116)$$

$$\mathbf{H} = \frac{A}{r_n} \frac{r_n^3}{r} (\mathbf{i}_r 2 \cos \theta + \mathbf{i}_\theta \sin \theta) \quad \text{for } r > r_n \quad (1.117)$$

The continuity conditions of Eqs. (1.104), (1.105), (1.112), and (1.113) and are applied to obtain the following relationships among the variables

$$\frac{-C}{r_n} = \frac{2A}{r_n} \quad (1.118)$$

$$\frac{A}{r_n} - \frac{C}{r_n} = \frac{3}{2} \frac{e\hbar}{m_e r_n^3} \quad (1.119)$$

Solving the variables algebraically gives the magnetic fields of an electron:

$$\mathbf{H} = \frac{e\hbar}{m_e r_n^3} (\mathbf{i}_r \cos \theta - \mathbf{i}_\theta \sin \theta) \quad \text{for } r < r_n \quad (1.120)$$

$$\mathbf{H} = \frac{e\hbar}{2m_e r^3} (\mathbf{i}_r 2 \cos \theta - \mathbf{i}_\theta \sin \theta) \quad \text{for } r > r_n \quad (1.121)$$

### Derivation of the Energy

The energy stored in the magnetic field of the electron is

$$E_{mag} = \frac{1}{2} \mu_o \int_{\text{O O O}}^{2\pi\pi} H^2 r^2 \sin \theta dr d\theta d \quad (1.122)$$

$$E_{mag \text{ total}} = E_{mag \text{ external}} + E_{mag \text{ internal}} \quad (1.123)$$

$$E_{mag \text{ internal}} = \frac{1}{2} \mu_o \int_{\text{O O O}}^{2\pi\pi r_1} \frac{e\hbar}{m_e r_1^3} \cos^2 \theta + \sin^2 \theta r^2 \sin \theta dr d\theta d \quad (1.124)$$

$$= \frac{2\pi\mu_o e^2 \hbar^2}{3m_e^2 r_1^3} \quad (1.125)$$

$$E_{mag \text{ external}} = \frac{1}{2} \mu_o \int_{\text{O O}_1}^{2\pi\pi} \frac{e\hbar}{2m_e r_1^3} 4\cos^2 \theta + \sin^2 \theta r^2 \sin \theta dr d\theta d\phi \quad (1.126)$$

$$= \frac{\pi \mu_o e^2 \hbar^2}{3m_e^2 r_1^3} \quad (1.127)$$

$$E_{mag \text{ total}} = \frac{2\pi \mu_o e^2 \hbar^2}{3m_e^2 r_1^3} + \frac{\pi \mu_o e^2 \hbar^2}{3m_e^2 r_1^3} \quad (1.128)$$

$$E_{mag \text{ total}} = \frac{\pi \mu_o e^2 \hbar^2}{m_e^2 r_1^3} \quad (1.129)$$

### STERN-GERLACH EXPERIMENT

The sum of the  $\mathbf{L}_i$ , the magnitude of the angular momentum of each infinitesimal point of the orbitsphere of mass  $m_i$ , must be constant. The constant is  $\hbar$ .

$$|\mathbf{L}_i| = |\mathbf{r} \times m_i \mathbf{v}| = m_e r_n \frac{\hbar}{m_e r_n} = \hbar \quad (1.130)$$

where the velocity is given by Eq. (1.47). Furthermore, it is known from the Stern-Gerlach experiment that a beam of silver atoms is split into two components when passed through an inhomogeneous magnetic field. The measured angular momentum in the direction of the applied field (spin axis) is  $\pm \frac{\hbar}{2}$ , and the magnitude of the angular momentum

vector which precesses about the spin axis is  $\sqrt{\frac{3}{4}}\hbar$ . As demonstrated in the Orbitsphere Equation of Motion Section, the projection of the total orbitsphere angular momentum onto the spin axis is  $\pm \frac{\hbar}{2}$ , and the

projection onto  $\mathbf{S}$ , the axis which precesses about the spin axis, is  $\sqrt{\frac{3}{4}}\hbar$ .

The Stern-Gerlach experiment implies a magnetic moment of one Bohr magneton and an associated angular momentum quantum number of  $1/2$ . Historically, this quantum number is called the spin quantum number,  $s$  ( $s = \frac{1}{2}$ ;  $m_s = \pm \frac{1}{2}$ ), and that designation is maintained.

### ELECTRON g FACTOR

As demonstrated by Purcell [19], when a magnetic field is applied to an electron in a central field which comprises a current loop, the

orbital radius does not change, but the velocity changes as follows:

$$\mathbf{v} = \frac{er\mathbf{B}}{2m_e} \quad (1.131)$$

The angular momentum of the electron orbitsphere is  $\hbar$  as given by Eq. (1.57), and as demonstrated in Figure 1.5,  $\frac{\hbar}{2}$  of the orbitsphere angular momentum is in the plane perpendicular to any applied magnetic field. The angular momentum in the presence of an applied magnetic field is

$$\mathbf{L} = \mathbf{r} \times (m_e \mathbf{v} + e\mathbf{A}) \quad (1.132)$$

where  $\mathbf{A}$  is the vector potential of the external field evaluated at the location of the orbitsphere. Conservation of angular momentum of the orbitsphere permits a discrete change of its "kinetic angular momentum" ( $\mathbf{r} \times m\mathbf{v}$ ) by the field of  $\frac{\hbar}{2}$ , and concomitantly the "potential angular momentum" ( $\mathbf{r} \times e\mathbf{A}$ ) must change by  $-\frac{\hbar}{2}$ . The flux change,  $\phi$ , of the orbitsphere for  $r < r_n$  is determined as follows:

$$\mathbf{L} = \frac{\hbar}{2} - \mathbf{r} \times e\mathbf{A} \quad (1.133)$$

$$= \frac{\hbar}{2} - \frac{e2\pi rA}{2\pi} \quad (1.134)$$

$$= \frac{\hbar}{2} - \frac{e\phi}{2\pi} \quad (1.135)$$

In order that the change of angular momentum,  $\mathbf{L}$ , equals zero,  $\phi$  must be  $\phi_0 = \frac{h}{2e}$ , the magnetic flux quantum. Thus, to conserve angular momentum in the presence of an applied magnetic field, the orbitsphere magnetic moment can be parallel or antiparallel to an applied field as observed with the Stern-Gerlach experiment, and the flip between orientations ( a rotation of  $\frac{\pi}{2}$ ) is accompanied by the "capture" of the magnetic flux quantum by the orbitsphere "coils" comprising infinitesimal loops of charge moving along geodesics (great circles).

The energy to flip the orientation of the orbitsphere due to its magnetic moment of a Bohr magneton,  $\mu_B$ , is

$$E_{mag}^{spin} = 2\mu_B \mathbf{B} \quad (1.136)$$

where

$$\mu_B = \frac{e\hbar}{2m_e} \quad (1.137)$$

The energy change corresponding to the "capture" of the magnetic flux quantum is derived below. From Eq. (1.129) for one electron,

$$E_{mag} = \frac{\pi\mu_0 e^2 \hbar^2}{(m_e)^2 r_n^3} \quad (1.138)$$

is the energy stored in the magnetic field of the electron. The orbitsphere is equivalent to a Josephson junction which can trap integer numbers of fluxons where the quantum of magnetic flux is  $\phi_0 = \frac{h}{2e}$ .

Consider Eq. (1.138). During the flip transition a fluxon treads the orbitsphere at the speed of light; therefore, the radius of the orbitsphere in the lab frame is  $2\pi$  times the relativistic radius in the fluxon frame. Thus, the energy of the transition corresponding to the "capture" of a fluxon by the orbitsphere,  $E_{mag}^{fluxon}$ , is

$$E_{mag}^{fluxon} = \frac{\pi\mu_0 e^2 \hbar^2}{(m_e)^2 (2\pi r_n)^3} \quad (1.139)$$

$$= \frac{\mu_0 e^2}{4\pi^2 m_e r_n} \frac{e\hbar}{2m_e} \frac{h}{2e\pi r_n^2} \quad (1.140)$$

$$= \frac{\mu_0 e^2}{4\pi^2 m_e r_n} \mu_B \frac{\phi_0}{A} \quad (1.141)$$

where  $A$  is the area and  $\phi_0$  is the magnetic flux quantum.

$$E_{mag}^{fluxon} = 2 \frac{e^2 \mu_0}{2m_e r_n} \frac{1}{4\pi^2} \mu_B B \quad (1.142)$$

where the  $n$ th fluxon treading through the area of the orbitsphere is equivalent to the applied magnetic flux. Furthermore, the term in brackets can be expressed in terms of the fine structure constant,  $\alpha$ , as follows:

$$\frac{e^2 \mu_0}{2m_e r_n} = \frac{e^2 \mu_0 c v}{2m_e v r_n c} \quad (1.143)$$

Substitution of Eq. (1.47) gives

$$\frac{e^2 \mu_0 c v}{2\hbar c} \quad (1.144)$$

Substitution of

$$c = \sqrt{\frac{1}{\epsilon_0 \mu_0}} \quad (1.145)$$

and

$$\alpha = \frac{\mu_0 e^2 c}{2\hbar} \quad (1.146)$$

gives

$$\frac{e^2 \mu_0 c v}{2\hbar c} = 2\pi\alpha \frac{v}{c} \quad (1.147)$$

The fluxon treads the orbitsphere at  $v = c$ . Thus,

$$E_{mag}^{fluxon} = 2 \frac{\alpha}{2\pi} \mu_B B \quad (1.148)$$

The principal energy of the transition of reorientation of the orbitsphere is given by Eq. (1.136). And, the total energy of the flip transition is the sum of Eq. (1.148), the energy of a fluxon treading the orbitsphere and Eq. (1.136), the energy of reorientation of the magnetic moment.

$$E_{mag}^{spin} = 2 \mu_B \mathbf{B} + \frac{\alpha}{2\pi} \mu_B \mathbf{B} \quad (1.149)$$

$$E_{mag}^{spin} = 2(1 + \frac{\alpha}{2\pi}) \mu_B \mathbf{B} \quad (1.150)$$

$$E_{mag}^{spin} = 2g\mu_B \mathbf{B} \quad (1.151)$$

The magnetic moment of Eq. (1.136) is twice that from the gyromagnetic ratio as given by Eq. (2.36) of the Orbital and Spin Splitting Section. The magnetic moment of the electron is the sum of the component

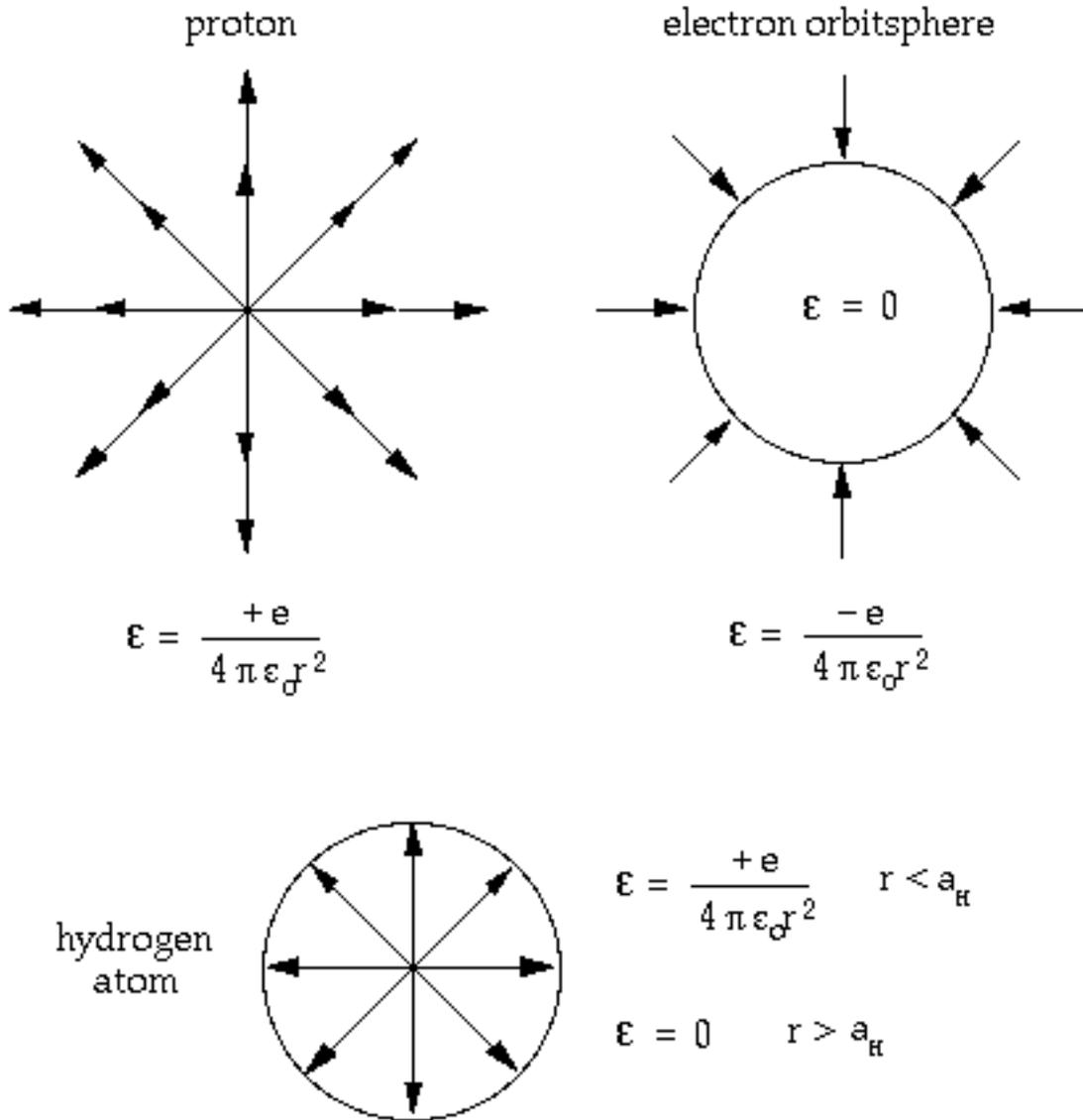
corresponding to the kinetic angular momentum,  $\frac{\hbar}{2}$ , and the component corresponding to the vector potential angular momentum,  $\frac{\hbar}{2}$ , (Eq.

(1.132). The spin-flip transition can be considered as involving a magnetic moment of  $g$  times that of a Bohr magneton. The factor  $g$  is redesignated the fluxon  $g$  factor as opposed to the anomalous  $g$  factor and its value is 1.00116. The experimental value is 1.00116. Additional small corrective terms to the  $g$  factor arise as a result of the radiative reaction force [20].

#### DETERMINATION OF ORBITSPHERE RADII, $r_n$

The one-electron orbitsphere is a spherical shell of negative charge (total charge =  $-e$ ) of zero thickness at a distance  $r_n$  from the nucleus (charge =  $+Ze$ ). It is well known that the field of a spherical shell of charge is zero inside the shell and that of a point charge at the origin outside the shell [21]. See Figure 1.7.

Figure 1.7 The electric fields of a proton, an electron, and a hydrogen atom.



Thus, for a nucleus of charge  $Z$ , the force balance equation for the electron orbitsphere is obtained by equating the forces on the mass and charge densities. For the ground state,  $n = 1$ , the centrifugal force of the electron is given by

$$\mathbf{F}_{centrifugal} = \frac{m_e}{4\pi r_1^2} \frac{\mathbf{v}_1^2}{r_1} \tag{1.152}$$

where  $\frac{m_e}{4\pi r_1^2}$  is the mass density of the orbitsphere. The centripetal force

is the electric force,  $F_{ele}$ , between the electron and the nucleus.

$$F_{ele} = \frac{e}{4\pi r_1^2} \frac{Ze}{4\pi \epsilon_o r_1^2} \quad (1.153)$$

where  $\epsilon_o$  is the permittivity of free-space.

The second centripetal force is an electrodynamic force, a force dependent on the second derivative of charge position which respect to time, which arises between the electron and the nucleus. The motion of each point in the magnetic field of the nucleus will cause a relativistic central force,  $F_{mag}$ , which acts on each point mass. The magnetic central force is derived as follows from the Lorentzian force which is relativistically corrected. Each infinitesimal point of the orbitsphere moves on a great circle, and each point charge has the charge-density

$\frac{e}{4\pi r_n^2}$ . As given in the Proton and Neutron Section, the proton is

comprised of a linear combination of three constant functions and three orthogonal spherical harmonic quark/gluon functions. From the photon inertial reference frame at the radius of each infinitesimal point of the electron orbitsphere, the proton charge distribution is given as the product of the quark and gluon functions which gives rise to a uniform distribution. The magnetic flux of the proton in the  $v = c$  inertial frame at the electron radius follows from McQuarrie [16]:

$$\mathbf{B} = \frac{\mu_o e \hbar}{2m_p r_n^3} \quad (1.154)$$

And, the magnetic flux due to a nucleus of charge  $Z$  and mass  $m$  is

$$\mathbf{B} = \frac{\mu_o Z_1 e \hbar}{2m r_n^3} \quad (1.155)$$

The motion of each point will cause a relativistic central force,  $F_{i mag}$ , which acts on each point mass. The magnetic central force is derived as follows from the Lorentzian force which is relativistically corrected. The Lorentzian force density on each point moving at velocity  $v$  given by Eq. (1.47) is

$$\mathbf{F}_{mag} = \frac{e}{4\pi r_n^2} \mathbf{v} \times \mathbf{B} \quad (1.156)$$

Substitution of Eq. (1.47) for  $\mathbf{v}$  and Eq. (1.155) for  $\mathbf{B}$  gives

$$\mathbf{F}_{mag} = \frac{1}{4\pi r_1^2} \frac{Z_1 e^2 \mu_o}{2m_e r_n} \frac{\hbar^2}{m r_n^3} \quad (1.157)$$

The term in brackets can be expressed in terms  $\alpha$ . From Eqs. (1.143-1.147)

$$\frac{Z_1 e^2 \mu_o}{2m_e r_n} = 2\pi \alpha Z_1 \frac{v}{c} \quad (1.158)$$

It can be shown that the relativistic correction to Eq. (1.157) is the

reciprocal of Eq. (1.158). Consider an inertial frame following a great circle of radius  $r_n$  with  $v = c$ . The motion is tangential to the radius; thus,  $r_n$  is Lorentzian invariant. But, the tangential distance along a great circle is  $2\pi r_n$  in the laboratory frame and  $r_n$  in the  $v = c$  frame. The charge is relativistically invariant, whereas, the mass is not. The relativistic correction to the laboratory frame mass relative to the  $v = c$  frame is  $2\pi$ . The correction follows from the Lorentz transformation of the electron's invariant angular momentum of  $\hbar$ . It is shown by Purcell [22] that the force on a moving charge due to a moving line of charge is a relativistic electric force due to Lorentzian contraction of the line charge density. The force is proportional to  $\frac{v}{c}$  where  $v$  is the electron's velocity. Thus, it follows that the electron mass in the laboratory frame relative to the  $v = c$  inertial frame is which is also proportional to  $\frac{v}{c}$ .

Following the derivation of Purcell with the substitution of the relativistic mass density for the charge density gives the electron mass correction to the electrodynamic force as

$$m_e = 2\pi \frac{v}{c} m_{e\text{Rest}} \quad (1.159)$$

Furthermore, due to invariance of charge under Gauss's Integral Law, the radius term in the brackets of Eq. (1.157) is relativistically corrected. The radius of the electron relative to the  $v = c$  frame,  $r_\alpha^*$ , is relativistically corrected as follows. From Eq. (1.43) the relationship between the radius and the electron wavelength is

$$2\pi r = \lambda \quad (1.160)$$

Using the de Broglie Eq. (1.46) with  $v = c$

$$\lambda = \frac{h}{mv} = \frac{h}{mc} \quad (1.161)$$

With substitution of Eq. (1.160) into Eq. (1.161)

$$r_\alpha^* = \frac{\hbar}{mc} = \tilde{\lambda}_c = \alpha a_o \quad (1.162)$$

where  $\tilde{\lambda}_c$  is the Compton wavelength bar, and where  $a_o$  is the Bohr radius. The radius of the electron orbitsphere in the  $v = c$  frame is  $\tilde{\lambda}_c$ , and the relativistic correction due to length contraction can be determined as a boundary value problem. Eq. (1.162) can be expressed in terms of a relativistic correction,  $n$ , which multiplies,  $r_1$ , the radius of the electron orbitsphere in the lab frame. The lab frame electron radius is taken as  $\frac{a_o}{Z_2}$  which is consistent with Eq. (1.169); thus, it is the solution of our boundary value problem as shown as follows.

$$r_{\alpha}^* = \frac{\hbar}{mc} = \tilde{\lambda}_c = \alpha a_o = \frac{na_o}{Z_2} = nr_1 \quad (1.163)$$

It follows from Eq. (1.163) that the radius,  $r_n$ , of Eq. (1.157) must be corrected by the factor  $\alpha Z_2$ . By correcting the radius and the mass, the relativistic correction is  $\frac{1}{2\pi\alpha Z_2 \frac{v}{c}}$ . In this case,  $Z_1 = Z_2$ ; thus, 1 is

substituted for the term in brackets in Eq. (1.157); therefore,

$$\mathbf{F}_{mag} = -\frac{1}{4\pi r_1^2} \frac{\hbar^2}{mr_n^3} \quad (1.164)$$

The force balance equation is given by equating the centrifugal and centripetal force densities:

$$\frac{m_e}{4\pi r_1^2} \frac{v_1^2}{r_1} = \frac{e}{4\pi r_1^2} \frac{Ze}{4\pi\epsilon_o r_1^2} - \frac{1}{4\pi r_1^2} \frac{\hbar^2}{mr_n^3} \quad (1.165)$$

Using Eq. (1.47),

$$r_1 = \frac{4\pi\epsilon_o \hbar^2}{Ze^2 \mu_e} \quad (1.166)$$

where the reduced electron mass,  $\mu_e$ , is

$$\mu_e = \frac{m_e m}{m_e + m} \quad (1.167)$$

The Bohr radius is

$$a_o = \frac{4\pi\epsilon_o \hbar^2}{e^2 m_e} \quad (1.168)$$

And, the radius given by force balance between the centrifugal force and central electrostatic force alone is

$$r_1 = \frac{4\pi\epsilon_o \hbar^2}{Ze^2 m_e} = \frac{a_o}{Z} \quad (1.169)$$

And, for hydrogen,  $m$  of Eq. (1.167) is

$$m = m_p \quad (1.170)$$

Substitution of the reduced electron mass for the electron mass gives,  $a_H$ , the Bohr radius of the hydrogen atom.

$$a_H = \frac{4\pi\epsilon_o \hbar^2}{e^2 \mu_e} \quad (1.171)$$

Thus, Eq. (1.166) becomes

$$r_1 = \frac{a_H}{Z} \quad (1.172)$$

## ENERGY CALCULATIONS

The potential energy  $V$  between the electron and the nucleus separated by the radial distance radius  $r_1$  is,

$$V = \frac{-Ze^2}{4\pi\epsilon_0 r_1} = \frac{-Z^2 e^2}{4\pi\epsilon_0 a_H} = -Z^2 \times 4.3675 \times 10^{-18} \text{ J} = -Z^2 \times 27.2 \text{ eV} \quad (1.173)$$

Because this is a central force problem, the kinetic energy,  $T$ , is  $-\frac{1}{2}V$ .

$$T = \frac{Z^2 e^2}{8\pi\epsilon_0 a_H} = Z^2 \times 13.59 \text{ eV} \quad (1.174)$$

The same result can be obtained from  $T = \frac{1}{2}m_e v_1^2$  and Eq. (1.47).

Alternatively, the kinetic energy, which is equal to the stored electric energy,  $E_{ele}$ , can be calculated from

$$T = E_{ele} = -\frac{1}{2}\epsilon_0 \int_0^{r_1} \mathbf{E}^2 dv \quad (1.175)$$

where  $\mathbf{E} = -\frac{Ze}{4\pi\epsilon_0 r^2}$ . Thus, as the orbitsphere shrinks from  $r_1$  to  $r_1$ ,

$$E_{ele} = -\frac{Z^2 e^2}{8\pi\epsilon_0 a_H} = -Z^2 \times 2.1786 \times 10^{-18} \text{ J} = -Z^2 \times 13.598 \text{ eV} \quad (1.176)$$

The calculated Rydberg constant is  $109,677.58 \text{ cm}^{-1}$ ; the experimental Rydberg constant is  $109,677.58 \text{ cm}^{-1}$ . Furthermore, a host of parameters can be calculated for the hydrogen atom, as shown in Table 1.1.

Table 1.1. Some calculated parameters for the hydrogen atom ( $n = 1$ ).

|  |  |  |
|--|--|--|
| radius   | $r_1 = a_H$                              | $5.2918 \times 10^{-11} \text{ m}$       |
| potential energy                                   | $V = \frac{-e^2}{4\pi\epsilon_0 a_H}$    | $-27.196 \text{ eV}$                     |
| kinetic energy                                     | $T = \frac{e^2}{8\pi\epsilon_0 a_H}$     | $13.598 \text{ eV}$                      |
| angular velocity (spin)                            | $\omega_1 = \frac{\hbar}{m_e r_1^2}$     | $4.13 \times 10^{16} \text{ rads}^{-1}$  |
| linear velocity                                    | $v_1 = r_1 \omega_1$                     | $2.19 \times 10^6 \text{ ms}^{-1}$       |
| wavelength   | $\lambda_1 = 2\pi r_1$                   | $3.325 \times 10^{-10} \text{ m}$        |
| spin quantum number                                | $s = \frac{1}{2}$                        | $\frac{1}{2}$                            |
| moment of Inertia                                  | $I = m_e r_1^2 \sqrt{s(s+1)}$            | $2.209 \times 10^{-51} \text{ kgm}^2$    |
| angular kinetic energy                             | $E_{angular} = \frac{1}{2} I \omega_1^2$ | $11.78 \text{ eV}$                       |
| magnitude of the angular momentum                  | $\hbar$                                  | $1.0545 \times 10^{-34} \text{ Js}$      |
| projection of the angular momentum onto the S-axis | $S = \hbar \sqrt{s(s+1)}$                | $9.133 \times 10^{-35} \text{ Js}$       |
| projection of the angular momentum onto the z-axis | $S_z = \frac{\hbar}{2}$                  | $5.273 \times 10^{-35} \text{ Js}$       |
| mass density                                       | $\frac{m_e}{4\pi r_1^2}$                 | $2.589 \times 10^{-11} \text{ kgm}^{-2}$ |
| charge-density                                     | $\frac{e}{4\pi r_1^2}$                   | $14.41 \text{ Cm}^{-2}$                  |

Table 1.2 gives the radii and energies for some one-electron atoms.

In addition to the energies, the wavelength, angular frequency, and the linear velocity can be calculated for any one-electron atom from Eqs. (1.46), (1.55), and (1.56). Values are given in Table 1.3.

Table 1.2. Calculated energies (non-relativistic) and calculated ionization energies for some one-electron atoms.

| Atom                    | Calculated $r_1^a$<br>( $a_0$ ) | Calculated Kinetic Energy <sup>b</sup><br>(eV) | Calculated Potential Energy <sup>c</sup><br>(eV) | Calculated Ionization Energy <sup>d</sup><br>(eV) | Experimental Ionization Energy <sup>e</sup><br>(eV) |
|-------------------------|---------------------------------|--|--|---|---|
| <i>H</i>                | 1.000                           | 13.59  | -27.18   | 13.59   | 13.59   |
| <i>He</i> <sup>+</sup>  | 0.500                           | 54.35  | -108.70  | 54.35   | 54.58   |
| <i>Li</i> <sup>2+</sup> | 0.333                           | 122.28   | -244.56  | 122.28  | 122.45  |
| <i>Be</i> <sup>3+</sup> | 0.250                           | 217.40   | -438.80  | 217.40  | 217.71  |
| <i>B</i> <sup>4+</sup>  | 0.200                           | 339.68   | -679.36  | 339.68  | 340.22  |
| <i>C</i> <sup>5+</sup>  | 0.167                           | 489.14   | -978.28  | 489.14  | 489.98  |
| <i>N</i> <sup>6+</sup>  | 0.143                           | 665.77   | -1331.54   | 665.77  | 667.03  |
| <i>O</i> <sup>7+</sup>  | 0.125                           | 869.58   | -1739.16   | 869.58  | 871.39  |

a from Equation (1.169)

b from Equation (1.174)

c from Equation (1.173)

d from Equation (1.176)

e experimental

It is noteworthy that the potential energy is a constant (at a given  $n$ ) because the electron is at a fixed distance,  $r_n$ , from the nucleus. And, the kinetic energy and velocity squared are constant because the atom does not radiate at  $r_n$  and the potential energy is constant.

Table 1.2. Calculated radii, angular frequencies, linear velocities, and wavelengths for the  $n = 1$  state of some one-electron atoms (non-relativistic).

| Atom                    | $r_1^a$<br>( $a_0$ ) | angular <sup>b</sup><br>velocity<br>( $10^{17} \text{ rad s}^{-1}$ ) | linear <sup>c</sup><br>velocity<br>( $10^6 \text{ ms}^{-1}$ ) | wavelength <sup>d</sup><br>( $10^{-10} \text{ m}$ ) |
|-------------------------|----------------------|--|---|---|
| <i>H</i>                | 1.000                | 0.413  | 2.19  | 3.325   |
| <i>He</i> <sup>+</sup>  | 0.500                | 1.65   | 4.38  | 1.663   |
| <i>Li</i> <sup>2+</sup> | 0.333                | 3.72   | 6.56  | 1.108   |
| <i>Be</i> <sup>3+</sup> | 0.250                | 6.61   | 8.75  | 0.831   |
| <i>B</i> <sup>4+</sup>  | 0.200                | 10.3   | 10.9  | 0.665   |
| <i>C</i> <sup>5+</sup>  | 0.167                | 14.9   | 13.1  | 0.554   |
| <i>N</i> <sup>6+</sup>  | 0.143                | 20.3   | 15.3  | 0.475   |
| <i>O</i> <sup>7+</sup>  | 0.125                | 26.5   | 17.5  | 0.416   |

a from Equation (1.169)

b from Equation (1.55)

c from Equation (1.56)

d from Equation (1.46)

It should be noted that the linear velocity is an appreciable percent of the velocity of light for some of the atoms in Table 1.2—5.9% for  $O^{7+}$  for example. Relativistic corrections must be applied before a comparison between the total energy and ionization energy (Table 1.2) is made.

### References

1. Haus, H. A., "On the radiation from point charges", *American Journal of Physics*, 54, (1986), pp. 1126-1129.
2. Jackson, J. D., *Classical Electrodynamics*, Second Edition, John Wiley & Sons, New York, (1962), p. 111.
3. Bracewell, R. N., *The Fourier Transform and Its Applications*, McGraw-Hill Book Company, New York, (1978), pp. 252-253.
4. Siebert, W. McC., *Circuits, Signals, and Systems*, The MIT Press, Cambridge, Massachusetts, (1986), p. 415.

5. Luke, Y. L., Integrals of Bessel Functions, McGrall-Hill, New York, p.22.
6. Abramovitz and Stegagun (3rd Printing 1965), p. 366, eq. 9.1.10, and p. 255, eq. 6.1.6.
7. Luke, Y. L., Integrals of Bessel Functions, McGrall-Hill, New York, p.30.
8. Bateman, H., Tables of Integral Transforms, Vol. III, McGraw-Hill, New York, (1954), p. 33.
9. Bateman, H., Tables of Integral Transforms, Vol. III, McGraw-Hill, New York, (1954), p. 5.
10. Reynolds, G. O., DeVelis, J. B., Parrent, G. B., Thompson, B. J., The New Physical Optics Notebook, SPIE Optical Engineering Press, (1990).
11. Abbott, T. A., Griffiths, D. J., Am. J. Phys., Vol. 153, No. 12, (1985), pp. 1203-1211.
12. McQuarrie, D. A., Quantum Chemistry, University Science Books, Mill Valley, CA, (1983), pp. 206-221.
13. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, New York, (1962), p. 99.
14. Fowles, G. R., Analytical Mechanics, Third Edition, Holt, Rinehart, and Winston, New York, (1977), p. 196.
15. Pauling, Linus, Wilson, E., Bright, Introduction to Quantum Mechanics with Applications to Chemistry, McGraw-Hill Book Company, New York, (1935), pp. 118-121.
16. McQuarrie, D. A., Quantum Chemistry, University Science Books, Mill Valley, CA, (1983), pp. 238-241.
17. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, New York, (1962), p. 178.
18. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, New York, (1962), pp. 194-197.
19. Purcell, E. M., Electricity and Magnetism, McGraw-Hill, New York, (1965), pp. 370-375.
20. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, New York, (1962), Chapter 17.
21. Bueche, F., Introduction to Physics for Scientists and Engineers, McGraw-Hill, (1975), pp. 352-353.
22. Purcell, E. M., Electricity and Magnetism, McGraw-Hill, New York, (1965), pp. 170-199.

## EXCITED STATES OF THE ONE ELECTRON ATOM (QUANTIZATION)

### EQUATION OF THE ELECTRIC FIELD INSIDE THE ORBITSPHERE

It is well known that resonator cavities can trap electromagnetic radiation of discrete resonant frequencies. The orbitsphere is a resonator cavity which traps photons of discrete frequencies. Thus, photon absorption occurs as an excitation of a resonator mode. The "trapped photon" is a "standing electromagnetic wave" which actually is a circulating wave that propagates around the z-axis, and its source current superimposes with each great circle current loop of the orbitsphere. The time-function factor,  $k(t)$ , for the "standing wave" is identical to the time-function factor of the orbitsphere in order to satisfy the boundary (phase) condition at the orbitsphere surface. Thus, the angular frequency of the "trapped photon" has to be identical to the angular frequency of the electron orbitsphere,  $\omega_n$  given by Eq. (1,55). Furthermore, the phase condition requires that the angular functions of the "trapped photon" have to be identical to the spherical harmonic angular functions of the electron orbitsphere. Combining  $k(t)$  with the  $\phi$ -function factor of the spherical harmonic gives  $e^{i(m\phi - \omega_n t)}$  for both the electron and the "trapped photon" function. The photon is "glued" to the inner orbitsphere surface and the outer nuclear surface as photon source charge-density with a radial electric field. Thus, the "trapped photon" is analogous to a gluon described in the Proton and Neutron Section and is different from a photon in free space as described in the Equation of the Photon Section.

For a spherical resonator cavity, the relationship between an allowed radius and the "photon standing wave" wavelength is

$$2\pi r = n\lambda \quad (2.1)$$

where  $n$  is an integer. Now, the question arises: given that this is a resonator cavity, which nonradiative states are possible where the transition is effected by a "trapped photon"? For the electron orbitsphere, a spherical resonator cavity, the relationship between an allowed radius and the electron wavelength is

$$2\pi(nr_1) = 2\pi r_n = n\lambda_1 = \lambda_n \quad (2.2)$$

where

$$n = 1, 2, 3, 4, \dots$$

$$n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

$\lambda_1$  is the allowed wavelength for  $n = 1$

$r_1$  is the allowed radius for  $n = 1$

An electron in the ground state,  $n = 1$ , is in force balance including the electrodynamic force which is included by using the reduced electron

mass as given by Eqs. (1.166), (1.171), and (1.172).

$$\frac{m_e v_1^2}{r_1} = \frac{Ze^2}{4\pi\epsilon_o r_1^2} \quad (2.3)$$

When an electron in the ground state absorbs a photon of sufficient energy to take it to a new non-radiative state,  $n = 2,3,4,\dots$ , force balance must be maintained. This is possible only if the central field is equivalent to that of a central charge of  $\frac{Ze}{n}$ , and the excited state force balance equation is

$$\frac{m_e v_n^2}{r_n} = \frac{1}{n} \frac{Ze^2}{4\pi\epsilon_o r_n^2} \quad (2.4)$$

where  $r_1$  is the "ground" state radius of the electron, and  $r_n$  is the  $n$ th excited state radius of the electron. The radius of the  $n$ th excited state follows from Eq. (1.172) and Eq. (2.4).

$$r_n = na_H \quad (2.5)$$

The reduction of the charge from  $Ze$  to  $\frac{Ze}{n}$  is caused by trapping a photon in the orbitsphere, a spherical resonator cavity. The photon's electric field creates a "standing wave" in the cavity with an effective charge of  $-1 + \frac{1}{n} Ze$  (at  $r_n$ ). The total charge experienced by the electron is the sum of the proton and "trapped photon" charge components. The equation for these "trapped photons" can be solved as a boundary value problem of Laplace's equation. For the hydrogen atom, the boundary conditions are that the electric field is in phase with the orbitsphere and that the radial function for the electric field of the "trapped photon" at  $r_n$  is

$$\mathbf{E}_{r_{photon}} = -1 + \frac{1}{n} \frac{e}{4\pi\epsilon_o (r_n)^2} \quad n = 2,3,4,\dots, \quad (2.6)$$

The general form of the solution to Laplace's equation in spherical coordinates is

$$(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [A_{\ell,m} r^{\ell} + B_{\ell,m} r^{-(\ell+1)}] [Y_0^0(\theta, \phi) + Y_{\ell}^m(\theta, \phi)] \quad (2.7)$$

All  $A_{\ell,m}$  are zero because the electric field given by the potential must be inversely proportional to the radius to obtain force balance. The electric field is the gradient of the potential

$$\mathbf{E} = - \quad (2.8)$$

$$\begin{aligned}\mathbf{E}_r &= -\frac{\delta}{\delta r} \hat{i}_r \\ \mathbf{E}_\theta &= -\frac{1}{r} \frac{\delta}{\delta \theta} \hat{i}_\theta \\ \mathbf{E}_\phi &= -\frac{1}{r \sin \theta} \frac{\delta}{\delta \phi} \hat{i}_\phi\end{aligned}\tag{2.9}$$

Thus,

$$\mathbf{E}_r = \sum_{\ell=0}^{\ell} \sum_{m=-\ell}^{m=\ell} B_{\ell,m} (\ell+1) r^{-(\ell+2)} [Y_0^0(\theta, \phi) + Y_\ell^m(\theta, \phi)]\tag{2.10}$$

Given that  $\mathbf{E}(\text{proton}) = \frac{+e}{4\pi\epsilon_o r_n^2}$ , and that the electric fields of the proton and "trapped photon" must superimpose to yield a field equivalent to a central point charge of  $\frac{+Ze}{n}$ , the "trapped photon" electric field for each mode is determined as follows. The time-function factor and the angular-function factor of the charge-density function of the orbitsphere (Eqs. (1.64) and (1.65)) at force balance must be in phase with the electric field of the "trapped photon". The relationship between the electric field equation and the "trapped photon" source charge-density function is given by Maxwell's equation in two dimensions.

$$\mathbf{n} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = \frac{\sigma}{\epsilon_o}\tag{2.11}$$

where  $\mathbf{n}$  is the radial normal unit vector,  $\mathbf{E}_1 = 0$  ( $\mathbf{E}_1$  is the electric field outside of the orbitsphere),  $\mathbf{E}_2$  is given by the total electric field at  $r_n = na_H$ , and  $\sigma$  is the surface charge-density. Thus,

$$\begin{aligned}\mathbf{E}_{r_{\text{photon}n,l,m}}|_{r_n=na_H} &= \frac{e}{4\pi\epsilon_o (na_H)^2} -1 + \frac{1}{n} [Y_0^0(\theta, \phi) + \text{Re}\{Y_\ell^m(\theta, \phi)[1 + e^{i\omega_n t}]\}] \\ \omega_n &= 0 \text{ for } m = 0\end{aligned}\tag{2.12}$$

$$\begin{aligned}&= \sum_{\ell=0}^{\ell} \sum_{m=-\ell}^{m=\ell} -B_{\ell,m} (\ell+1) (na_H)^{-(\ell+2)} [Y_0^0(\theta, \phi) + \text{Re}\{Y_\ell^m(\theta, \phi)[1 + e^{i\omega_n t}]\}] \\ \omega_n &= 0 \text{ for } m = 0\end{aligned}\tag{2.13}$$

Therefore,

$$\sum_{\ell=0}^{\ell} \sum_{m=-\ell}^{m=\ell} -B_{\ell,m} = \frac{e (na_H)^\ell}{4\pi\epsilon_o (\ell+1)} -1 + \frac{1}{n}, \text{ and}\tag{2.14}$$

$$\begin{aligned}\mathbf{E}_{r_{\text{photon}n,l,m}} &= \frac{e (na_H)^\ell}{4\pi\epsilon_o} \frac{1}{r^{(\ell+2)}} -Y_0^0(\theta, \phi) + \frac{1}{n} [Y_0^0(\theta, \phi) + \text{Re}\{Y_\ell^m(\theta, \phi)[1 + e^{i\omega_n t}]\}] \\ \omega_n &= 0 \text{ for } m = 0 \\ n &= 1, 2, 3, 4, \dots \\ \ell &= 1, 2, \dots, n-1\end{aligned}\tag{2.15}$$

$$m = -\ell, -\ell + 1, \dots, 0, \dots, +\ell$$

$\mathbf{E}_{r_{total}}$  is the sum of the "trapped photon" and proton electric fields,

$$\mathbf{E}_{r_{total}} = \frac{e}{4\pi\epsilon_0 r^2} + \frac{e(na_H)^\ell}{4\pi\epsilon_0 r^{(\ell+2)}} - Y_0^0(\theta, \phi) + \frac{1}{n} \left[ Y_0^0(\theta, \phi) + \text{Re} \left\{ Y_\ell^m(\theta, \phi) [1 + e^{i\omega_n t}] \right\} \right]$$

$$\omega_n = 0 \text{ for } m = 0 \quad (2.16)$$

For  $r = na_H$  and  $m = 0$ , the total radial electric field is

$$\mathbf{E}_{r_{total}} = \frac{1}{n} \frac{e}{4\pi\epsilon_0 (na_H)^2} \quad (2.17)$$

All boundary conditions are met for the electric fields and the wavelengths of the "trapped photon" and the electron. Thus, Eq. (2.16) is the solution for the excited modes of the orbitsphere, a spherical resonator cavity. And, the quantum numbers of the electron are  $n$ ,  $\ell$ ,  $m$  ( $m_\ell$ ), and  $m_s$  (Described in the Stern-Gerlach Experiment Section).

### PHOTON ABSORPTION

The energy of the photon which excites a mode in a stationary spherical resonator cavity from radius  $a_H$  to radius  $na_H$  is

$$E_{photon} = \frac{e^2}{4\pi\epsilon_0 a_H} \left( 1 - \frac{1}{n^2} \right) = h\nu = \hbar\omega \quad (2.18)$$

After multiplying Eq. (2.18) by  $\frac{a_H}{a_H} = \frac{4\pi\epsilon_0 \hbar^2}{e^2 m_e a_H}$ , where  $a_H$  is given by Eq.

(1.171),  $\omega_{photon}$  is

$$\omega_{photon} = \frac{\hbar}{m_e a_H^2} \left( 1 - \frac{1}{n^2} \right) \quad (2.19)$$

In the case of an electron orbitsphere, the resonator possesses kinetic energy before and after the excitation. The kinetic energy is always one-half of the potential energy because the centripetal force is an inverse squared central force. As a result, the energy and angular frequency to excite an electron orbitsphere is only one-half of the values above, Eqs. (2.18) and (2.19). From Eq. (1.55), the angular velocity of an electron orbitsphere of radius  $na_H$  is

$$\omega_n = \frac{\hbar}{m_e (na_H)^2} \quad (2.20)$$

The change in angular velocity of the orbitsphere for an excitation from  $n = 1$  to  $n = n$  is

$$\omega = \frac{\hbar}{m_e (a_H)^2} - \frac{\hbar}{m_e (na_H)^2} = \frac{\hbar}{m_e (a_H)^2} \left( 1 - \frac{1}{n^2} \right) \quad (2.21)$$

The kinetic energy change of the transition is

$$\frac{1}{2} m_e (v)^2 = \frac{1}{2} \frac{e^2}{4\pi\epsilon_0 a_H} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2} \hbar\omega \quad (2.22)$$

The change in angular velocity of the electron orbitsphere, Eq. (2.21), is identical to the angular velocity of the photon necessary for the excitation,  $\omega_{\text{photon}}$  (Eq. (2.21)). The energy of the photon necessary to excite the equivalent transition in an electron orbitsphere is one-half of the excitation energy of the stationary cavity because the change in kinetic energy of the electron orbitsphere supplies one-half of the necessary energy. The change in the angular frequency of the orbitsphere during a transition and the angular frequency of the photon corresponding to the superposition of the free space photon and the photon corresponding to the kinetic energy change of the orbitsphere during a transition are equivalent. The correspondence principle holds. It can be demonstrated that the resonance condition between these frequencies is to be satisfied in order to have a net change of the energy field [1].

The excited states of hydrogen are given in Table 2.1.

Table 2.1. Calculated energies (non-relativistic; no spin-orbit interaction; no electronic spin/nuclear spin interaction) and ionization energies for the hydrogen atom in the ground state and some excited states.

| $n$ | $Z$            | Calculated<br>$r_n^a$<br>( $a_H$ ) | Calculated<br>Kinetic<br>Energy <sup>b</sup><br>(eV) | Calculated<br>Potential<br>Energy <sup>c</sup><br>(eV) | Calculated<br>Ionization<br>Energy <sup>d</sup><br>(eV) | Experimental<br>Ionization<br>Energy <sup>e</sup><br>(eV) |
|-----|----------------|------------------------------------|--|--|---|---|
| 1   | 1              | 1.000                              | 13.589   | -27.21   | 13.598  | 13.595  |
| 2   | $\frac{1}{2}$  | 2.000                              | 3.397  | -6.803   | 3.400   | 3.393   |
| 3   | $\frac{1}{3}$  | 3.000                              | 1.510  | -3.023   | 1.511   | 1.511   |
| 5   | $\frac{1}{5}$  | 5.000                              | 0.544  | -1.088   | 0.544   | 0.544   |
| 10  | $\frac{1}{10}$ | 10.000                             | 0.136  | -0.272   | 0.136   | 0.136   |

<sup>a</sup> from Equation (2.5)

<sup>b</sup> from  $T = -\frac{1}{2}V$

<sup>c</sup> from Equation (1.173)

<sup>d</sup> from Equation (2.22)

<sup>e</sup> experimental

## SELECTION RULES

The multipole fields of a radiating source can be used to calculate the energy and angular momentum carried off by the radiation [2]. For definiteness we consider a linear superposition of electric ( $l, m$ ) multipoles with different  $m$  values, but all having the same  $l$ , and following Eq. (16.46) of Jackson [2], write the fields as

$$\begin{aligned} \mathbf{B}_l &= a_E(l, m) \mathbf{X}_{lm} h_l^{(0)}(kr) e^{i\omega t} \\ \mathbf{E}_l &= \frac{i}{k} \times \mathbf{B}_l \end{aligned} \quad (2.23)$$

For harmonically varying fields, the time-averaged energy density is

$$u = \frac{1}{16\pi} (\mathbf{E} \cdot \mathbf{E}^* + \mathbf{B} \cdot \mathbf{B}^*) \quad (2.24)$$

In the radiation zone, the two terms are equal. Consequently, the energy in a spherical shell between  $r$  and  $(r + dr)$  (for  $kr \gg 1$ ) is

$$dU = \frac{dr}{8\pi k^2} \sum_{m,m'} a_E^*(l, m') a_E(l, m) \mathbf{X}_{lm'}^* \cdot \mathbf{X}_{lm} d \quad (2.25)$$

where the asymptotic form (Eq. (16.13) of Jackson [2]) of the spherical Hankel function has been used. With the orthogonality integral (Eq. (16.44) of Jackson [2]) this becomes

$$\frac{dU}{dr} = \frac{1}{8\pi k^2} \sum_m |a_E(l, m)|^2 \quad (2.26)$$

independent of the radius. For a general superposition of electric and magnetic multipoles, the sum over  $m$  becomes a sum over  $l$  and  $m$  and  $|a_E|^2$  becomes  $|a_E|^2 + |a_M|^2$ . The total energy in a spherical shell in the radiation zone is thus an *incoherent sum* over all multipoles.

The time-averaged angular-momentum density is

$$\mathbf{m} = \frac{1}{8\pi c} \text{Re}[\mathbf{r} \times (\mathbf{E} \times \mathbf{B}^*)] \quad (2.27)$$

The triple cross product can be expanded, and the electric field substituted to yield, for a superposition of electric multipoles,

$$\mathbf{m} = \frac{1}{8\pi\omega} \text{Re}[\mathbf{B}^* (\mathbf{L} \cdot \mathbf{B})] \quad (2.28)$$

Then the angular momentum in a spherical shell between  $r$  and  $(r + dr)$  in the radiation zone is

$$d\mathbf{M} = \frac{dr}{8\pi\omega k^2} \text{Re} \sum_{m,m'} a_E^*(l, m') a_E(l, m) (\mathbf{L} \cdot \mathbf{X}_{lm'})^* \mathbf{X}_{lm} d \quad (2.29)$$

With the explicit form (Eq. (16.43) of Jackson [2]) for  $X_{lm}$ , Eq. (2.29) can be written

$$\frac{d\mathbf{M}}{dr} = \frac{1}{8\pi\omega k^2} \text{Re} \sum_{m,m'} a_E^*(l, m') a_E(l, m) Y_{lm'}^* \mathbf{L} Y_{lm} d \quad (2.30)$$

From the properties of  $LY_{lm}$  listed in Eq. (16.28) of Jackson [2] and the orthogonality of the spherical harmonics, we obtain the following expressions for the Cartesian components of  $\frac{d\mathbf{M}}{dr}$

$$\frac{dM_x}{dr} = \frac{1}{16\pi\omega k^2} \text{Re} \sum_m \left[ \sqrt{(l-m)(l+m+1)} a_E^*(l, m+1) + \sqrt{(l+m)(l-m+1)} a_E^*(l, m-1) \right] a_E(l, m) \quad (2.31)$$

$$\frac{dM_y}{dr} = \frac{1}{16\pi\omega k^2} \text{Im} \sum_m \left[ \sqrt{(l-m)(l+m+1)} a_E^*(l, m+1) - \sqrt{(l+m)(l-m+1)} a_E^*(l, m-1) \right] a_E(l, m) \quad (2.32)$$

$$\frac{dM_z}{dr} = \frac{1}{8\pi\omega k^2} \sum_m m |a_E(l, m)|^2 \quad (2.33)$$

These equations show that for a general  $l$ th order electric multipole that

consists of a superposition of different  $m$  values, only the  $z$  component of the angular momentum is relatively simple.

For a multipole with a single  $m$  value,  $M_x$  and  $M_y$  vanish, while a comparison of Eq. (2.33) and Eq. (2.25) shows that

$$\frac{dM_z}{dr} = \frac{m}{\omega} \frac{dU}{dr} \quad (2.34)$$

independent of  $r$ . Experimentally, the photon can carry  $\pm \hbar$  units of angular momentum. Thus, during excitation the spin, orbital, or total angular momentum of the orbitsphere can change by zero or  $\pm \hbar$ . The electron transition rules arise from conservation of angular momentum. The selection rules for multipole transitions between quantum states arise from conservation of total angular momentum and component angular momentum where the photon carries  $\hbar$  of angular momentum. The radiation of a multipole of order  $(l, m)$  carries  $m\hbar$  units of the  $z$  component of angular momentum per photon of energy  $\hbar\omega$ .

### ORBITAL AND SPIN SPLITTING

The ratio of the square of the angular momentum,  $M^2$ , to the square of the energy,  $U^2$ , for a pure  $(l, m)$  multipole follows from Eq. (2.25) and Eqs. (2.31-2.33)

$$\frac{M^2}{U^2} = \frac{m^2}{\omega^2} \quad (2.35)$$

The magnetic moment is defined [3] as

$$\mu = \frac{\text{charge} \times \text{angular momentum}}{2 \times \text{mass}} \quad (2.36)$$

The radiation of a multipole of order  $(l, m)$  carries  $m\hbar$  units of the  $z$  component of angular momentum per photon of energy  $\hbar\omega$ . Thus, the  $z$  component of the angular momentum of the corresponding excited state electron orbitsphere is

$$L_z = m\hbar \quad (2.37)$$

Therefore,

$$\mu_z = \frac{em\hbar}{2m_e} = m\mu_B \quad (2.38)$$

where  $\mu_B$  is the Bohr magneton. The presence of a magnetic field causes the principal excited state energy levels of the hydrogen atom (Eq. (2.22)) to split by the energy  $E_{mag}^{orb}$  corresponding to the interaction of the magnetic flux with the magnetic moment given by Eq. (2.38). This energy is called orbital splitting.

$$E_{mag}^{orb} = m\mu_B \mathbf{B} \quad (2.39)$$

As is the case with spin splitting given by one half the energy of Eq. (1.151) which corresponds to the transition between spin states, the energy of the electron is increased in the case that the magnetic flux is

antiparallel to the magnetic moment, or the energy of the electron is decreased in the case that the magnetic flux is parallel to the magnetic moment. The spin and orbital splitting energies superimpose; thus, the principal excited state energy levels of the hydrogen atom (Eq. (2.22)) are split by the energy  $E_{mag}^{spin/orb}$

$$E_{mag}^{spin/orb} = m \frac{e\hbar}{2m_e} \mathbf{B} + m_s g \frac{e\hbar}{m_e} \mathbf{B} \quad (2.40)$$

where it follows from Eq.(2.15) that

$$n = 2,3,4,\dots$$

$$\ell = 1,2,\dots, n-1$$

$$m = -\ell, -\ell+1,\dots,0,\dots, +\ell$$

$$m_s = \pm \frac{1}{2}$$

For the electric dipole transition, the selection rules are

$$\begin{aligned} m &= 0, \pm 1 \\ m_s &= 0 \end{aligned} \quad (2.41)$$

Splitting of the energy levels in addition to that given by Eq. (2.40) occurs due to a relativistic effect described in the Spin-Orbital Coupling Section. Also, a very small shift which is observable by radio-frequency spectroscopy is due to conservation of energy and linear momentum and arises from the radiation reaction force between the electron and the photon. This so-called Lamb Shift is described in the Resonant Line Shape and Lamb Shift Section.

Decaying spherical harmonic currents on the surface of the orbitsphere give rise to spherical harmonic radiation fields during emission; conversely, absorbed spherical harmonic radiation fields produce spherical harmonic currents on the surface of the orbitsphere to effect a transition. Transition intensities,  $I$ , are given by the integral of the product of the multipole of the photon,  ${}^p X_{l,m}(\theta, \phi)$ , and the initial,  ${}^i X_{l,m}(\theta, \phi)$ , and final,  ${}^f X_{l,m}(\theta, \phi)$ , states as is the case with classical electro-dynamics calculations involving antennas.

$$I = I_0 \left| \int_0^\pi \int_0^{2\pi} {}^i X_{l,m}(\theta, \phi) {}^p X_{l,m}(\theta, \phi) {}^f X_{l,m}(\theta, \phi) \sin \theta d\phi d\theta \right|^2 \quad (2.42)$$

The distribution of multipole radiation and the multipole moments of the orbitsphere for absorption and emission are derived in Jackson [4]. Some of the simpler angular distributions are listed in Table 2.2.

Table 2.2. Some of the simpler angular distributions of multipole radiation

and the multipole moments of the orbitsphere for absorption and emission.

---

| l               | m   |   |                                       |
|-----------------|---|---|---------------------------------------|
|                 | 0   | ±1  | ±2                                    |
| 1<br>Dipole     | $\frac{3}{8\pi} \sin^2 \theta$                | $\frac{3}{16\pi} (1 + \cos^2 \theta)$                   |                                       |
| 2<br>Quadrupole | $\frac{15}{8\pi} \sin^2 \theta \cos^2 \theta$ | $\frac{5}{16\pi} (1 - 3\cos^2 \theta + 4\cos^4 \theta)$ | $\frac{5}{16\pi} (1 - \cos^4 \theta)$ |

---

### RESONANT LINE SHAPE AND LAMB SHIFT

The spectroscopic linewidth arises from the classical rise-time band-width relationship, and the Lamb Shift is due to conservation of energy and linear momentum and arises from the radiation reaction force between the electron and the photon. It follows from the Poynting Power Theorem (Eq. (7.27)) with spherical radiation that the transition probabilities are given by the ratio of power and the energy of the transition [5]. The transition probability in the case of the electric multipole moment given by Jackson [5] as

$$Q_{lm} = \frac{3}{l+3} e (na_0)^l \tag{2.43}$$

is [5]

$$\frac{1}{\tau} = \frac{\text{power}}{\text{energy}}$$

$$\frac{1}{\tau} = \frac{\frac{2\pi c}{[(2l+1)!!]^2} \frac{l+1}{l} k^{2l+1} |Q_{lm} + Q'_{lm}|^2}{[\hbar\omega]} = 2\pi \frac{e^2}{h} \frac{\sqrt{\epsilon_0}}{\sqrt{\mu_0}} \frac{2\pi}{[(2l+1)!!]^2} \frac{l+1}{l} \frac{3}{l+3} (kna_0)^{2l} \omega \tag{2.44}$$

This rise-time gives rise to,  $\tau$ , the spectroscopic line-width. The relationship between the rise-time and the band-width is given by Siebert [6].

$$\tau^2 = 4 \frac{\int_0^\infty t^2 h^2(t) dt}{\int_0^\infty h^2(t) dt} - \frac{\left( \int_0^\infty t h^2(t) dt \right)^2}{\int_0^\infty h^2(t) dt} \quad (2.45)$$

$$\tau^2 = 4 \frac{\int_0^\infty f^2 |H(f)|^2 df}{\int_0^\infty |H(f)|^2 df} \quad (2.46)$$

By application of the Schwartz inequality, the relationship between the rise-time and the band-width is

$$\tau \geq \frac{1}{\pi} \quad (2.47)$$

From Eq. (2.44), the line-width is proportional to the ratio of the Quantum Hall resistance,  $\frac{h}{e^2}$ , and,  $\eta$ , the radiation resistance of free space.

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} \quad (2.48)$$

And, the Quantum Hall resistance given in the Quantum Hall Effect Section was derived using the Poynting Power Theorem. Also, from Eq. (2.44), the line-width is proportional to the fine structure constant,  $\alpha$ ,

$$\alpha = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{e^2}{\hbar} \quad (2.49)$$

During a transition, the total energy of the system decays exponentially. Applying Eqs. (2.45) and (2.46) to the case of exponential decay,

$$h(t) = e^{-\frac{t}{T}} u(t) \quad (2.50)$$

$$|H(f)| = \frac{1}{\sqrt{\frac{1}{T^2} + (2\pi f)^2}} \quad (2.51)$$

where the rise-time,  $\tau$ , is the time required for  $h(t)$  of Eq. (2.50) to decay to  $1/e$  of its initial value and where the band-width,  $\Delta f$ , is the half-power bandwidth, the distance between points at which

$$|H(f)| = \frac{|H(0)|}{\sqrt{2}} \quad (2.52)$$

From Eq. (2.45),

$$\tau = T \quad (2.53)$$

From Eq. (2.46),

$$\Delta f = \frac{1}{\pi T} \quad (2.54)$$

From Eq. (2.53) and Eq. (2.54), the relationship between the rise-time and the band-width for exponential decay is

$$\tau = \frac{1}{\pi} \quad (2.55)$$

Photons obey Maxwell-Boltzmann statistics as given in the Statistical Mechanics Section. The emitted radiation, the summation of an assemble of emitted photons each of an exact frequency and energy given by Eq. (4.8), appears as a wave train with effective length  $c/\nu$ . Such a finite pulse of radiation is not exactly monochromatic but has a frequency spectrum covering an interval of the order  $\nu$ . The exact shape of the frequency spectrum is given by the square of the Fourier Transform of the electric field. Thus, the amplitude spectrum is proportional to

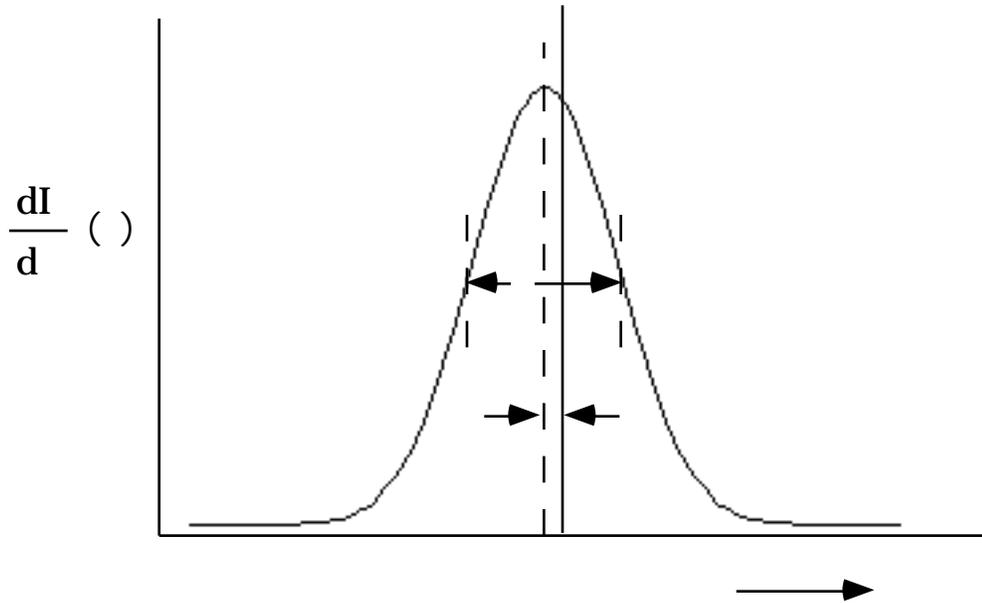
$$\mathbf{E}(\omega) = \int_0^{\infty} e^{-\alpha t} e^{-i\omega t} dt = \frac{1}{\alpha - i\omega} \quad (2.56)$$

The coefficient  $\alpha$  corresponds to the spectroscopic linewidth and also to a shift in frequency that arises from the radiation reaction force between the electron and the photon. The energy radiated per unit frequency interval is therefore

$$\frac{dI(\omega)}{d\omega} = I_0 \frac{1}{2\pi \left[ (\omega - \omega_0 - \alpha)^2 + (\alpha/2)^2 \right]} \quad (2.57)$$

where  $I_0$  is the total energy radiated. The spectral distribution is called a resonant line shape. The width of the distribution at half-maximum intensity is called the half-width or line-breadth and is equal to  $\alpha$ . Shown in Figure 2.1 is such a spectral line. Because of the reactive effects of radiation the line is shifted in frequency. The small radiative shift of the energy levels of atoms was first observed by Lamb in 1947 [7] and is called the Lamb Shift in his honor.

Figure 2.1. Broadening of the spectral line due to the rise-time and shifting of the spectral line due to the radiative reaction. The resonant line shape has width  $\Delta\omega$ . The level shift is  $\omega_0$ .



The Lamb Shift of the  $^2P_{1/2}$  state of the hydrogen atom having the quantum number  $\ell = 1$  is calculated by applying conservation of energy and linear momentum to the emitted photon, electron, and atom. The photon emitted by an excited state atom carries away energy, linear momentum, and angular momentum. The initial and final values of the energies and momenta must be conserved between the atom, the electron, and the photon. (Conservation of angular momentum is used to derive the photon's equation in the Equation of the Photon Section). Consider an isolated atom of mass  $M$  having an electron of mass  $m_e$  in an excited state level at an energy  $E$  and moving with velocity  $\mathbf{V}$  along the direction in which the photon is to be emitted (the components of motion perpendicular to this direction remain unaffected by the emission and may be ignored). The energy above the "ground" state at rest is

$$E + \frac{1}{2} M\mathbf{V}^2 \quad (2.58)$$

When a photon of energy  $E_\nu$  is emitted, the atom and/or electron recoils and has a new velocity

$$\mathbf{V} + \mathbf{v} \quad (2.59)$$

(which is a vector sum in that  $\mathbf{V}$  and  $\mathbf{v}$  may be opposed), and a total energy of

$$\frac{1}{2} M(\mathbf{V} + \mathbf{v})^2 \quad (2.60)$$

By conservation of energy,

$$E + \frac{1}{2} M\mathbf{V}^2 = E_{hv} + \frac{1}{2} M(\mathbf{V} + \mathbf{v})^2 \quad (2.61)$$

so, that the actual energy of the photon emitted is given by

$$E_{hv} = E - \frac{1}{2} M\mathbf{v}^2 - M\mathbf{v}\mathbf{V} \quad (2.62)$$

$$E_{hv} = E - E_R - E_D$$

The photon is thus deficient in energy by a recoil kinetic energy

$$E_R = \frac{1}{2} M\mathbf{v}^2 \quad (2.63)$$

which is independent of the initial velocity  $\mathbf{V}$ , and by a thermal or Doppler energy

$$E_D = M\mathbf{v}\mathbf{V} \quad (2.64)$$

which depends on  $\mathbf{V}$ ; therefore, it can be positive or negative.

Momentum must also be conserved in the emission process. The energy,  $E$ , of the photon is given by Eq. (4.8)

$$E = \hbar\omega = h \frac{\omega}{2\pi} = h\nu = hf = h \frac{c}{\lambda} \quad (2.65)$$

From Special Relativity,

$$E = \hbar\omega = mc^2 \quad (2.66)$$

Thus,  $\mathbf{p}$ , the momentum of the photon is

$$\mathbf{p} = mc = \frac{E_{hv}}{c} \quad (2.67)$$

where  $c$  is the velocity of light, so that

$$M\mathbf{V} = M(\mathbf{V} + \mathbf{v}) + \frac{E_{hv}}{c} \quad (2.68)$$

And, the recoil momentum is

$$M\mathbf{v} = -\frac{E_{hv}}{c} \quad (2.69)$$

Thus, the recoil energy is given by

$$E_R = \frac{E_{hv}^2}{2Mc^2} \quad (2.70)$$

and depends on the mass of the electron and/or atom and the energy of the photon. The Doppler energy,  $E_D$ , is dependent on the thermal motion of the atom, and will have a distribution of values which is temperature dependent. A mean value,  $\bar{E}_D$ , can be defined which is related to the mean kinetic energy per translational degree of freedom

$$\bar{E}_D = \frac{1}{2} kT \quad (2.71)$$

by

$$\bar{E}_D = 2\sqrt{E_k E_R} = E_{hv} \sqrt{\frac{2\bar{E}_k}{Mc^2}} \quad (2.72)$$

where  $k$  is Boltzmann's constant and  $T$  is the absolute temperature. As a result, the statistical distribution in energy of the emitted photons is displaced from the true excited-state energy by  $-E_R$  and broadened by  $E_D$  into a Gaussian distribution of width  $2\bar{E}_D$ . The distribution for absorption has the same shape but is displaced by  $+E_R$ .

For the transition of the hydrogen atom with  $n = 2$  and  $\ell = 0$  in the initial and final states, the emitted angular radiation power pattern is uniform. The linear momentum of the photon is balanced by the recoil momentum of the entire atom of mass  $m_H$ . The recoil frequency of the hydrogen atom,  $f$ , is given by the combining Eqs. (2.65) and (2.70).

$$f = \frac{\omega}{2\pi} = \frac{E_{hv}}{h} = \frac{(E_{hv})^2}{2m_H c^2} = 13 \text{ MHz} \quad (2.73)$$

where  $E_{hv}$  is

$$E_{hv} = 13.6 \left(1 - \frac{1}{n^2}\right) - h f \quad ; \quad h f \ll \ll 1 \quad E_{hv} = 13.6 \left(1 - \frac{1}{n^2}\right) \quad (2.74)$$

However, during the emission of a photon by an excited state atom, with  $\ell \neq 0$ , the angular radiation power pattern is not uniform, and the electron receives the recoil momentum as the charge-density of the electron changes from uniform to uniform plus a spherical harmonic function (angular modulation) as given in the One Electron Atom Section. In the case of  $\ell = 1$ ;  $m_\ell = 0$ , the angular charge-density function is

$$\rho(r, \theta, \phi, t) = \frac{e}{4\pi r_n^2} [\delta(r - r_n)] [Y_0^0(\theta, \phi) + Y_\ell^m(\theta, \phi) \text{Re}[1 + e^{i\omega_n t}]] \quad (2.75)$$

where

$$Y_\ell^m(\theta, \phi) \text{Re}[1 + e^{i\omega_n t}] = \text{Re}[Y_\ell^m(\theta, \phi) + Y_\ell^m(\theta, \phi) e^{i\omega_n t}] = P_\ell^m(\cos \theta) \cos m\phi + P_\ell^m(\cos \theta) \cos(m\phi + \omega_n t)$$

and  $\omega_n = 0$  for  $m = 0$ .

$$Y_{1,z} = \cos \theta \quad (2.76)$$

Figure 1.2 gives pictorial representation of how the modulation function changes the electron density on the orbitsphere for several  $\ell$  values.

The angular function,  $|X_{lm}|^2$ , of the radiation power pattern of the electron in the  $^2P_{1/2}$  ( $\ell = 1$ ;  $m_\ell = 0$ ) state is equivalent to that of a Hertzian dipole.

$$|X_{lm}|^2 = \sin^2 \theta \quad (2.77)$$

The integral of Eq. (2.77) over the surface of a spherical shell is

$$|X_{lm}|_{\ell=1}^2 = \frac{8\pi}{3} \quad (2.78)$$

Thus, the inverse of Eq. (2.78) is the weighting factor of momentum

transfer due to the radiation power pattern. Photons obey Maxwell-Boltzmann statistics as given in the Statistical Mechanics Section. The distribution of the linear momentum transferred from the emitted photons to the electrons is given by the projection of the photon momentum distribution onto the x, y, or z-axis which corresponds to 3 degrees of freedom. The Lamb Shift of the  $^2P_{1/2}$  state of the hydrogen atom is given by the combining Eqs. (2.65), (2.78), and (2.70)

$$f = \frac{\omega}{2\pi} = \frac{E_{hw}}{h} = 3 \frac{(E_{hw})^2}{h^2 m_e c^2} = 1052 \text{ MHz} \quad (2.79)$$

where  $E_{hw}$  is

$$E_{hw} = 13.6 \left[ 1 - \frac{1}{n^2} \frac{1}{|X_{lm}|_{\ell=1}^2} \right] - h f$$

$$E_{hw} = 13.6 \left[ 1 - \frac{1}{n^2} \frac{3}{8\pi} \right] - h f; \quad (2.80)$$

$$h f \ll \Delta$$

$$E_{hw} = 13.6 \left[ 1 - \frac{1}{n^2} \frac{3}{8\pi} \right]$$

Furthermore, it follows from Eq. (2.75), that the recoil energy of the photon corresponding to momentum transfer to the atom for the case of  $\ell = 1$  is one half that of the case where  $\ell = 0$  (Eq. (2.73)).

$$f = \frac{\omega}{2\pi} = \frac{E_{hw}}{h} = \frac{1}{2} \frac{(E_{hw})^2}{2 m_H c^2} = 6.5 \text{ MHz} \quad (2.81)$$

The recoiling electron transfers momentum to the nucleus which binds the electron, and some linear momentum is transferred to the atom as angular momentum. Linear momentum of the electron, atom, and photon are conserved where the propagation vector of the photon does not go through the nucleus; thus, it possesses an equal and opposite component of angular momentum with respect to the atom. The total recoil energy is the sum of the electron component (Eq. (2.79)) and the atom component (Eq. (2.81)). Thus, the calculated Lamb Shift due to both components of linear momentum transfer is

$$f = 1052 \text{ MHz} + 6.5 \text{ MHz} = 1058.5 \text{ MHz}$$

$$f = 1052 \text{ MHz} + 6.5 \text{ MHz} = 1058.5 \text{ MHz} \quad (2.82)$$

The experimental Lamb Shift is 1058 MHz.

The present calculations used the electron rest mass; however, the relativistic mass is required in order to be exact. It is given by Eq. (7.31). In addition to the Lamb Shift, the spectral lines of hydrogen are Zeeman split by spin-orbital coupling and electron-nuclear magnetic interactions.

(As a further example, conservation of linear momentum of the

photon is central to the Mössbauer phenomenon. See Mills patent [8]).

### SPIN-ORBITAL COUPLING

The spin-orbital coupling split is given by the Dirac equation [9] which applies Special Relativity to a spherically symmetric charge distribution in a central field as is the case with the orbitsphere. And, Dirac's spin-orbital interaction operator follows from Eq. (1.164) and Eq. (1.173).

$$E = m_e c^2 \left[ 1 + \frac{\alpha^2}{n^2 - j(j + \frac{1}{2}) + \sqrt{j(j + \frac{1}{2})^2 - \alpha^2}} \right]^{-\frac{1}{2}} ; n = 1, 2, 3, \dots \quad (2.83)$$

The predicted energy difference between the  $2P_{1/2}$  and  $2P_{3/2}$ ,  $2S_{1/2}$  levels of the hydrogen atom,  $E_{s/o}$ , given by Eq. (2.83) is

$$E_{s/o} = \frac{\alpha^4 m_e c^2}{32} \quad (2.84)$$

which corresponds to a frequency of about 11,000 MHz or a wavelength of about 2.7 cm. The experimental value is 10,950 MHz.

### KNIGHT SHIFT

The unpaired electron of the hydrogen atom gives rise to a uniform magnetic field at the nucleus which is given by Eq. (1.120).

$$\mathbf{H} = \frac{e\hbar}{m_e r_n^3} (\mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta) \quad r < r_n \quad (2.85)$$

Multiplication of Eq. (2.85) by the permeability of free space,  $\mu_0$ , and substitution of the Bohr radius of the hydrogen atom,  $a_H$ , given by Eq. (1.171) for  $r_n$  of Eq. (2.85) gives the magnetic flux,  $\mathbf{B}_s$ , at the nucleus due to electron spin.

$$\mathbf{B}_s = \frac{\mu_0 e\hbar}{m_e a_H^3} \mathbf{i}_z = 157.29 \text{ T} \quad (2.86)$$

The proton possesses a magnetic moment which is derived in the Proton and Neutron Section and is given by

$$\mu_p = \frac{\frac{2}{3} e\hbar}{2 \frac{m_p}{2\pi}} \quad (2.87)$$

$E_{mag}^{\text{proton spir}}$ , the energy to flip the orientation of the proton's magnetic moment,  $\mu_p$ , from parallel to antiparallel to the direction of the magnetic flux  $\mathbf{B}_s$  is

$$E_{mag}^{\text{proton spin}} = -2\mu_p \mathbf{B}_s \quad (2.88)$$

As given in the Spin Angular Momentum of the Orbitsphere with  $\ell = 0$  Section, the z directed magnetic field of the nucleus corresponding to the proton magnetic moment given by Eq. (2.87) gives rise to a projection of the angular momentum of the electron onto an axis which precesses about the z-axis of  $\sqrt{\frac{3}{4}}\hbar$ . The projection of the magnetic energy between the electron orbitsphere and the proton is equivalent to that of the angular momentum onto the axis which precesses about the z-axis,  $\sqrt{\frac{3}{4}}$  times that of a Bohr magneton. In the case of the hydrogen atom, the energy to flip the orientation of the proton's magnetic moment,  $\mu_p$ , from parallel to antiparallel to the direction of the magnetic flux  $\mathbf{B}_s$  of the electron is given by the substitution of the magnetic flux  $\mathbf{B}_s$  of Eq. (2.86) multiplied by  $\sqrt{\frac{3}{4}}$  into Eq. (2.88).

$$E_{mag}^{\text{proton spin}} = -2\mu_p \frac{\mu_0 e \hbar}{m_e a_H^3} \sqrt{\frac{3}{4}} = -3.837 \times 10^{-24} \text{ J} = -2.395 \times 10^{-5} \text{ eV} \quad (2.89)$$

The frequency,  $f$ , can be determined from the energy using the Planck relationship, Eq. (2.18).

$$f = \frac{3.837 \times 10^{-24} \text{ J}}{h} = 5.790 \text{ GHz} \quad (2.90)$$

The shift of the NMR frequency of a nucleus by an unpaired electron is called the Knight Shift. The Knight Shift of the hydrogen atom is given by Eq. (2.90) which corresponds to the magnetic flux given by Eq. (2.86). The experimental value is unknown; however, magnetic hyperfine structure shifts of Mossbauer spectra corresponding to magnetic fluxes of 100 T or more due unpaired electrons are common.

### SPIN-NUCLEAR COUPLING

The radius of the hydrogen atom is increased or decreased very slightly due to the force between the magnetic moment of the electron and the magnetic field of the nucleus. The magnetic moment of the electron is a Bohr magneton,  $\mu_B$ , given by Eq. (1.99). The magnetic moment  $\mathbf{m}$  of the proton is given by Eq. (2.87), and the magnetic field of the proton follows from the relationship between the magnetic dipole field and the magnetic moment  $\mathbf{m}$  as given by Jackson [10] where  $\mathbf{m} = \mu_p \mathbf{i}_z$ .

$$\mathbf{H} = \frac{\mu_p}{r^3} (\mathbf{i}_r 2\cos\theta - \mathbf{i}_\theta \sin\theta) \quad (2.91)$$

Multiplication of Eq. (2.91) by the permeability of free space,  $\mu_0$ , gives

the magnetic flux,  $\mathbf{B}_p$ , due to the nucleus.

$$\mathbf{B}_p = \frac{\mu_0 \mu_p}{r^3} (\mathbf{i}_r 2 \cos \theta - \mathbf{i}_\theta \sin \theta) \quad (2.92)$$

The force between the magnetic moment of the electron and the magnetic flux of the proton,  $\mathbf{F}_{S/N}$ , is

$$\mathbf{F}_{S/N} = \frac{1}{4\pi r_1^2} \mu_B \mathbf{i}_z \cdot \mathbf{B}_p \quad (2.93)$$

Substitution of Eq. (2.92) into Eq. (2.93) gives

$$\mathbf{F}_{S/N} = \pm \frac{1}{4\pi r_1^2} \mu_B 3 \frac{\mu_0}{r^4} \mu_p \int_0^\pi (\mathbf{i}_r \cos \theta - \mathbf{i}_\theta \sin \theta) \cdot (\mathbf{i}_r 2 \cos \theta - \mathbf{i}_\theta \sin \theta) d\theta \quad (2.94)$$

$$\mathbf{F}_{S/N} = \pm \frac{1}{4\pi r_1^2} \mu_B 3 \frac{\mu_0}{r^4} \mu_p \mathbf{i}_r \quad (2.95)$$

where the plus corresponds to parallel alignment of the magnetic moments of the electron and proton, and the minus corresponds to antiparallel alignment of the magnetic moments of the electron and proton. The force must be corrected for the vector projection of the angular momentum onto the z-axis. As given in the Spin Angular Momentum of the Orbitsphere with  $\ell = 0$  Section, the z directed magnetic field of the nucleus corresponding to the proton magnetic moment given by Eq. (1.120) gives rise to a projection of the angular momentum of the electron onto an axis which precesses about the z-axis of  $\sqrt{\frac{3}{4}}\hbar$ . The projection of the magnetic force between the electron orbitsphere and the proton is equivalent to that of the angular momentum onto the axis which precesses about the z-axis,  $\sqrt{\frac{3}{4}}$  times that of a Bohr magneton. The force balance equation of the hydrogen atom including the spin/nuclear force is given by substituting Eq. (2.95) multiplied by  $\sqrt{\frac{3}{4}}$  into Eq. (1.165).

$$\frac{m_e}{4\pi r_1^2} \frac{v_1^2}{r_1} = \frac{e}{4\pi r_1^2} \frac{e}{4\pi \epsilon_0 r_1^2} - \frac{1}{4\pi r_1^2} \frac{\hbar^2}{m r_n^3} \pm \frac{1}{4\pi r_1^2} \mu_B 3 \frac{\mu_0}{r^4} \mu_p \sqrt{\frac{3}{4}} \quad (2.96)$$

Using Eq. (1.47),

$$r_{1\pm} = \frac{a_H + \sqrt{a_H^2 \pm \frac{6\mu_0 e \mu_p a_0 \sqrt{\frac{3}{4}}}{\hbar}}}{2} \quad (2.97)$$

where  $r_{1+}$  corresponds to parallel alignment of the magnetic moments of the electron and proton,  $r_{1-}$  corresponds to antiparallel alignment of the magnetic moments of the electron and proton,  $a_H$  is the Bohr radius of

the hydrogen atom given by Eq. (1.171), and  $a_o$  is the Bohr radius given by Eq. (1.168).

### Energy Calculations

The change in the electric energy of the electron due to the slight shift of the radius of the electron is given by the difference between the electric energies associated with the two possible orientations of the magnetic moment of the electron with respect to the magnetic moment of the proton, parallel versus antiparallel. Each electric energy is given by the substitution of the corresponding radius given by Eq. (2.97) into Eq. (1.176). The change in electric energy for the flip from parallel to antiparallel alignment,  $E_{ele}^{S/N}$ , is

$$E_{ele}^{S/N} = \frac{e^2}{8\pi\epsilon_o} \left( \frac{1}{r_{1-}} - \frac{1}{r_{1+}} \right) = 2.878 \times 10^{-24} \text{ J} \quad (2.98)$$

The magnetic energy to flip the orientation of the proton's magnetic moment,  $\mu_p$ , from parallel to antiparallel to the direction of the magnetic flux  $B_s$  of the electron is given by Eq. (2.89).

$$E_{mag}^{proton \text{ spin}} = -2\mu_p \frac{\mu_o e\hbar}{m_e a_H^3} \sqrt{\frac{3}{4}} = -3.837 \times 10^{-24} \text{ J} \quad (2.99)$$

The total energy of the transition from parallel to antiparallel alignment,  $E_{total}^{S/N}$ , is given as the sum of Eqs. (2.98) and (2.99).

$$E_{total}^{S/N} = \frac{e^2}{8\pi\epsilon_o} \left( \frac{1}{r_{1-}} - \frac{1}{r_{1+}} \right) - 2\mu_p \frac{\mu_o e\hbar}{m_e a_H^3} \sqrt{\frac{3}{4}} \quad (2.100)$$

$$E_{total}^{S/N} = 2.878 \times 10^{-24} \text{ J} - 3.837 \times 10^{-24} \text{ J} = -9.592 \times 10^{-25} \text{ J} \quad (2.101)$$

The energy is expressed in terms of wavelength using the Planck relationship, Eq. (2.65).

$$\lambda = \frac{hc}{E_{total}^{S/N}} = 21 \text{ cm} \quad (2.102)$$

The experimental value from astrophysical studies and from electron spin resonance measurements is 21 cm.

### SPIN-NUCLEAR AND ORBITAL-NUCLEAR COUPLING OF HYDRINOS

The theory of a previously unknown form of matter: hydrogen atoms having electrons of lower energy than the conventional "ground" state called *hydrinos*, where each energy level corresponds to a fractional quantum number is given in the Atomic Coulomb Field Collapse--Hydrino Theory--BlackLight Process Section. The radius of the hydrino atom corresponding to the fractional quantum number  $\frac{1}{n}$  where

$n$  is an integer is the Bohr radius of the hydrogen atom divided by  $n$ ,  $\frac{a_H}{n}$ ,

and the central field is  $n$  times that of the proton. The quantum numbers of the electron for below "ground" states are  $n$ ,  $\ell$ ,  $m$  ( $m_\ell$ ), and  $m_s$  as described in the Atomic Coulomb Field Collapse--Hydrino Theory--BlackLight Process Section. The relationship between the quantum numbers is given by Eq. (5.13).

$$\begin{aligned} n &= 2,3,4,\dots \\ \ell &= 1,2,\dots, n-1 \\ m &= -\ell, -\ell+1,\dots,0,\dots, +\ell \end{aligned} \quad (2.103)$$

Photons obey Maxwell's Equations. At the two dimensional surface of the orbitsphere containing a "trapped photon", the relationship between the photon's electric field and its two dimensional charge-density at the orbitsphere is

$$\mathbf{n} \cdot (\mathbf{E}_1 - \mathbf{E}_2) = \frac{\sigma}{\epsilon_0} \quad (2.104)$$

Thus, the photon's electric field acts as surface charge. According to Eq. (2.104), the "photon standing wave" in the electron orbitsphere resonator cavity gives rise to a two dimensional surface charge at the orbitsphere two dimensional surface. The surface charge is given by Eq. (2.104) for a central field strength equal in magnitude to  $ne$ . This surface charge possesses the same angular velocity as the orbitsphere; thus, it is a current with a corresponding magnetic field. As demonstrated in the Orbital and Spin Splitting Section, the z component of the angular momentum of an excited state electron orbitsphere corresponding to a "trapped photon" multipole of order  $(\ell, m)$  is

$$L_z = m\hbar \quad (2.105)$$

Eq. (2.105) also applies in the case of *hydrinos*, hydrogen atoms with below "ground" state electronic energy levels.

The "trapped photon" is a "standing electromagnetic wave" which actually is a circulating wave that propagates along each great circle current loop of the orbitsphere. The time-function factor,  $k(t)$ , for the "standing wave" is identical to the time-function factor of the orbitsphere in order to satisfy the boundary (phase) condition at the orbitsphere surface. Thus, the angular frequency of the "trapped photon" has to be identical to the angular frequency of the electron orbitsphere,  $\omega_n$ . Furthermore, the phase condition requires that the angular functions of the "trapped photon" have to be identical to the spherical harmonic angular functions of the electron orbitsphere. The rotational parameters of the surface current of the "photon standing wave" are given in the Derivation of the Rotational Parameters of the Electron Section. The solution to Legendre's equation given by Eq. (1.60) is the maximum term of a series of solutions corresponding to the  $m$  and

$\ell$  values [11,12]. From Eq. (1.86),  $|\mathbf{L}_o^{photon}|$ , the magnitude of the orbital angular momentum along the axis which precesses about the z-axis is

$$|\mathbf{L}_o^{photon}| = \hbar\sqrt{\ell(\ell+1)} \quad (2.106)$$

Therefore, from Eq. (2.36),

$$\mu = \frac{e\hbar}{2m_e}\sqrt{\ell(\ell+1)} = \mu_B\sqrt{\ell(\ell+1)} \quad (2.107)$$

where  $\mu_B$  is the Bohr magneton. The magnetic moment gives rise to a magnetic field at the nucleus. The magnetic field follows from the relationship between the magnetic dipole field and the magnetic moment  $\mathbf{m}$  as given by Jackson [10] where  $\mathbf{m} = \mu_B\sqrt{\ell(\ell+1)}$  where the z-axis is redesignated as the precessing axis.

$$\mathbf{H} = \frac{2\mu_B}{r_n^3}\sqrt{\ell(\ell+1)}(\mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta) \quad r < r_n \quad (2.108)$$

Multiplication of Eq. (2.108) by the permeability of free space,  $\mu_0$ , and substitution of the Bohr radius of the hydrogen atom,  $a_H$ , given by Eq. (1.171) for  $r_n$  of Eq. (2.108) gives the magnetic flux,  $\mathbf{B}_o$ , at the nucleus due to the orbital angular momentum of the electron.

$$\mathbf{B}_o = \frac{\mu_0 e\hbar}{m_e a_H^3}\sqrt{\ell(\ell+1)}\mathbf{i}_z \quad (2.109)$$

The orbital-nuclear coupling energy,  $E_{\text{mag}}^{\text{proton orb}}$ , the energy to flip the orientation of the proton's magnetic moment,  $\mu_p$ , from parallel to antiparallel to the direction of the magnetic flux  $\mathbf{B}_o$  due to the orbital angular momentum of the electron given by Eq. (2.109), is

$$E_{\text{mag}}^{\text{proton orb}} = -2\mu_p\mathbf{B}_o \quad (2.110)$$

The spin-nuclear and orbital-nuclear coupling energies superimpose. Thus,  $E_{\text{mag}}^{\text{proton spin orb}}$ , the energy to flip the orientation of the proton's magnetic moment,  $\mu_p$ , from parallel to antiparallel to the direction of the magnetic flux  $\mathbf{B}_s$  due to electron spin and the magnetic flux  $\mathbf{B}_o$  due to the orbital angular momentum of the electron is the sum of Eqs. (2.89) and (2.110).

$$E_{\text{mag}}^{\text{proton spin orb}} = -2\mu_p\mathbf{B}_o - 2\mu_p\mathbf{B}_s\sqrt{\frac{3}{4}} \quad (2.111)$$

The spin and orbital moments of inertia, spin and orbital angular momenta, and spin and orbital kinetic energies of an excited state electron orbitsphere are given in the Derivation of the Rotational Parameters of the Electron Section. Substitution of Eq. (1.55) and Eq. (1.96) into Eq. (1.88) gives the magnitude of the orbital component of the angular momentum of an excited state electron orbitsphere corresponding to a multipole of order  $(\ell, m)$ .  $|\mathbf{L}_o^{\text{electron}}|$ , the magnitude of the orbital angular momentum along the axis which precesses about the

z-axis is

$$|\mathbf{L}_o^{electron}| = \hbar \frac{\ell(\ell+1)}{\ell^2 + \ell + 1}^{\frac{1}{2}} \quad (2.112)$$

Eq. (2.112) also applies in the case of *hydrinos*, hydrogen atoms with below "ground" state electronic energy levels. Therefore, from Eq. (2.36),

$$\mu = \frac{e\hbar}{2m_e} \frac{\ell(\ell+1)}{\ell^2 + 2\ell + 1}^{\frac{1}{2}} = \mu_B \frac{\ell(\ell+1)}{\ell^2 + 2\ell + 1}^{\frac{1}{2}} \quad (2.113)$$

where  $\mu_B$  is the Bohr magneton. The force between the magnetic moment of the electron due to orbital angular momentum and the magnetic flux of the proton,  $\mathbf{F}_{O/N}$ , is

$$\mathbf{F}_{O/N} = \frac{1}{4\pi r_1^2} \mu_B \frac{\ell(\ell+1)}{\ell^2 + 2\ell + 1}^{\frac{1}{2}} \mathbf{i}_z \cdot \mathbf{B}_P \quad (2.114)$$

where the z-axis is redesignated as the precessing axis. Substitution of Eq. (2.92) into Eq. (2.114) gives

$$\mathbf{F}_{O/N} = \pm \frac{1}{4\pi r_1^2} \mu_B \frac{\ell(\ell+1)}{\ell^2 + 2\ell + 1}^{\frac{1}{2}} 3 \frac{\mu_0}{r^4} \mu_P \pi (\mathbf{i}_r \cos \theta - \mathbf{i}_\theta \sin \theta) \cdot (\mathbf{i}_r 2 \cos \theta - \mathbf{i}_\theta \cos \theta) d\theta \quad (2.115)$$

$$\mathbf{F}_{O/N} = \pm \frac{1}{4\pi r_1^2} \mu_B \frac{\ell(\ell+1)}{\ell^2 + 2\ell + 1}^{\frac{1}{2}} 3 \frac{\mu_0}{r^4} \mu_P \mathbf{i}_r \quad (2.116)$$

where the plus corresponds to parallel alignment of the magnetic moments of the electron and proton, and the minus corresponds to antiparallel alignment of the magnetic moments of the electron and proton. The force balance equation of the hydrino atom including the spin-nuclear force and the orbital-nuclear force is given by Eq. (5.14), Eq. (2.116), Eq. (1.94), and Eq. (2.96) where the magnitude of the central field is an integer,  $n$ .

$$\frac{m_e}{4\pi r_1^2} \frac{v_1^2}{r_1} = \frac{e}{4\pi r_1^2} \frac{ne}{4\pi \epsilon_o r_1^2} - \frac{1}{4\pi r_1^2} \frac{\hbar^2}{m_e r_n^3} \pm \frac{1}{4\pi r_1^2} \frac{1}{n} \sqrt{\ell(\ell+1)} + \sqrt{\frac{3}{4}} \mu_B 3 \frac{\mu_0}{r^4} \mu_P \quad (2.117)$$

Using Eq. (1.47),

$$r_{1\pm} = \frac{a_H + \sqrt{a_H^2 \pm \frac{6\mu_o e \sqrt{\ell(\ell+1)} + \sqrt{\frac{3}{4}} \mu_P a_o}{\hbar}}}{2n} \quad (2.118)$$

where  $r_{1+}$  corresponds to parallel alignment of the magnetic moments of the electron and proton,  $r_{1-}$  corresponds to antiparallel alignment of the magnetic moments of the electron and proton,  $a_H$  is the Bohr radius of the hydrogen atom given by Eq. (1.171), and  $a_o$  is the Bohr radius given by Eq. (1.168).

### Energy Calculations

The change in the electric energy of the electron due to the slight shift of the radius of the electron is given by the difference between the electric energies associated with the two possible orientations of the magnetic moment of the electron with respect to the magnetic moment of the proton, parallel versus antiparallel. Each electric energy is given by the substitution of the corresponding radius given by Eq. (2.118) into Eq. (1.176) where the magnitude of the central field of the hydrino atom is  $n$  times that of a proton. The change in electric energy for the flip from parallel to antiparallel alignment,  $E_{ele}^{S/N O/N}$ , is

$$E_{ele}^{S/N O/N} = \frac{ne^2}{8\pi\epsilon_o} \left( \frac{1}{r_{1-}} - \frac{1}{r_{1+}} \right) \quad (2.119)$$

The magnetic energy to flip the orientation of the proton's magnetic moment,  $\mu_p$ , from parallel to antiparallel to the direction of the magnetic flux  $B_s$  due to electron spin and the magnetic flux  $B_o$  due to the orbital angular momentum of the electron follows from Eqs.

(2.109) and (2.111) where the radius of the hydrino atom is  $\frac{a_H}{n}$ .

$$E_{mag}^{proton\ spin\ orb} = -2\mu_p \frac{n^3\mu_0 e\hbar}{m_e a_H^3} \sqrt{\ell(\ell+1)} - 2\mu_p \frac{n^3\mu_0 e\hbar}{m_e a_H^3} \sqrt{\frac{3}{4}} = -\sqrt{\ell(\ell+1)} + \sqrt{\frac{3}{4}} \quad 2\mu_p \frac{n^3\mu_0 e\hbar}{m_e a_H^3} \quad (2.120)$$

The total energy of the transition from parallel to antiparallel alignment,  $E_{total}^{S/N O/N}$ , is given as the sum of Eqs. (2.119) and (2.120).

$$E_{total}^{S/N O/N} = \frac{ne^2}{8\pi\epsilon_o} \left( \frac{1}{r_{1-}} - \frac{1}{r_{1+}} \right) - \sqrt{\ell(\ell+1)} + \sqrt{\frac{3}{4}} \quad 2\mu_p \frac{n^3\mu_0 e\hbar}{m_e a_H^3} \quad (2.121)$$

For the case that  $\ell = 0$ ,

$$E_{total}^{S/N O/N} = n^2 2.878 \times 10^{-24} \text{ J} - n^3 3.837 \times 10^{-24} \text{ J} \quad (2.122)$$

The frequency,  $f$ , can be determined from the energy using the Planck relationship, Eq. (2.65).

$$f = \frac{E_{total}^{S/N O/N}}{h} \quad (2.123)$$

From Eq. (2.122), Eq. (2.102), and the Planck relationship, Eq. (2.123), the energy, the wavelength, and the frequency corresponding to the spin-nuclear coupling energy of the hydrino atom with the lower energy state quantum numbers  $n$  and  $\ell$  and with the radius  $\frac{a_H}{n}$  are given in

Table 2.3.

Table 2.3. The spin-nuclear coupling energy of the hydrino atom with the lower energy state quantum numbers  $n$  and  $\ell$  and with the radius  $\frac{a_H}{n}$ .

| $n$ | $\ell$ | Energy<br>( $J \times 10^{23}$ ) | Lambda<br>( $cm$ ) | Wave Number<br>( $cm^{-1}$ ) | Frequency<br>( $GHz$ ) |
|-----|--------|----------------------------------|--------------------|------------------------------|------------------------|
| 1   | 0      | 0.09592                          | 20.71              | 0.04829                      | 1.447                  |
| 2   | 0      | 1.918                            | 1.0355             | 0.9657                       | 28.95                  |
| 2   | 1      | 5.051                            | 0.3933             | 2.543                        | 76.23                  |
| 3   | 0      | 7.769                            | 0.2557             | 3.911                        | 117.2                  |
| 3   | 1      | 20.46                            | 0.09710            | 10.30                        | 308.7                  |
| 3   | 2      | 29.74                            | 0.06678            | 14.97                        | 448.9                  |
| 4   | 0      | 19.95                            | 0.09957            | 10.04                        | 301.1                  |
| 4   | 1      | 52.53                            | 0.03781            | 26.44                        | 792.8                  |
| 4   | 2      | 76.38                            | 0.02601            | 38.45                        | 1153                   |
| 4   | 3      | 99.76                            | 0.01991            | 50.22                        | 1505                   |
| 5   | 0      | 40.77                            | 0.04873            | 20.52                        | 615.2                  |
| 5   | 1      | 107.3                            | 0.01851            | 54.03                        | 1620                   |
| 5   | 2      | 156.1                            | 0.01273            | 78.57                        | 2355                   |
| 5   | 3      | 203.8                            | 0.009746           | 102.6                        | 3076                   |
| 5   | 4      | 251.3                            | 0.007905           | 126.5                        | 3792                   |

### EINSTEIN A COEFFICIENT

An estimate of the transition probability for magnetic multipoles is given by Eq. (16.105) of Jackson [13]. For a magnetic dipole  $\ell = 1$ , and Eq. (16.105) of Jackson is

$$\frac{1}{\tau_M} = \frac{g^2}{mc^2} \frac{\hbar e^2}{c} \frac{\pi}{16} k^2 \omega \quad (2.124)$$

Substitution of

$$k = \frac{\omega}{c} \quad (2.125)$$

into Eq. (2.124) gives

$$\frac{1}{\tau_M} = \frac{g^2}{mc^2} \frac{\hbar e^2}{c} \frac{\pi}{16} \frac{\omega^3}{c^2} \quad (2.126)$$

From Eq. (2.126), the transition probability is proportional to the frequency cubed. The experimental Einstein A coefficient for hydrogen  $H(n=1)$  [14] is

$$A = 2.87 \times 10^{-15} \text{ sec}^{-1} \quad (2.127)$$

The frequencies for the spin/nuclear hyperfine transition of hydrogen  $H(n=1)$  and hydrino  $H(n=1/2)$  are given in Table 2.3. The Einstein A coefficient for hydrino  $H(n=1/2)$  is given by Eq. (2.126) and Eq. (2.127) and the frequencies of Table 2.3.

$$A_{H(n=1/2)} = A_{H(n=1)} \frac{\omega_{H(n=1/2)}^3}{\omega_{H(n=1)}^3} = 2.87 \times 10^{-15} \frac{28.95^3}{1.447^3} \text{ sec}^{-1} = 2.30 \times 10^{-11} \text{ sec}^{-1} \quad (2.128)$$

### INTENSITY OF SPIN-NUCLEAR AND ORBITAL-NUCLEAR COUPLING TRANSITIONS OF HYDRINOS

The intensity,  $I$ , of spin-nuclear and orbital-nuclear coupling transitions of hydrinos can be calculated from the column density of hydrogen or hydrino atoms,  $N(H)$ , and the Einstein A coefficient,  $A_{ul}$ . The column density is given by the product of the number of hydrogen or hydrino atoms per unit volume,  $n_H$ , and the path length,  $\ell$ , which is calculated in steradians from its integral.

$$I = \frac{1}{4\pi} A_{ul} N(H) = \frac{1}{4\pi} A_{ul} n_H \ell \quad (2.129)$$

The number of hydrogen or hydrino atoms per unit volume,  $n_H$ , can be determined from the experimental results of Labov and Bowyer [15]. The number of electronic transitions per atom per second,  $k_1$  (Eq. (5.69) of the Interstellar Disproportionation Rate Section), is equivalent to the number of photons per atom per second,  $A_{ul}$  (Eq. (2.129)).

Equating intensities of photon flux (Eq. (2.129) and the rate of the disproportionation reaction,  $r_{m,m',p}$  Eq. (5.70), gives

$$I = \frac{1}{4\pi} A_{ul} N(H) = \frac{1}{\sqrt{2}} n_H \frac{a_H}{p} \sqrt{\frac{3kT}{m_H}} N(H) \quad (2.130)$$

where  $N(H) = n_H \ell$  is the column density. The intensity reported by Labov and Bowyer for the 304 Å line which is herein assigned as the 1/3 - 1/4 H transition is  $I = 2080_{\pm 40}^{-720} \text{ photons cm}^{-2} \text{ s}^{-1} \text{ sr}^{-1}$  [15]. In the case that  $m=1$ ,  $m'=2$ , and  $p=3$  in Eqs. (5.50-5.52);  $T=50 \text{ K}$ , and  $g_{m,p}=1$  (the result of Förster's theory for the efficiencies of dipole-dipole resonant energy transfers), the column density of hydrino atoms,  $H \frac{a_H}{3}$ , is calculated along the sight-line at  $b=48 \text{ deg}$  to be

$$N(H) = 2 \times 10^{18} \text{ cm}^{-2} \quad (2.131)$$

The calculated density of hydrino atoms,  $H \frac{a_H}{3}$ , is

$$n_H = 4 \times 10^3 \text{ atom} / \text{m}^3 \quad (2.132)$$

Substitution of Eq. (2.132) and Eq. (2.128) into Eq. (2.129) gives the intensity as a function of the path length,  $\ell$ , which is calculated in steradians from its integral.

$$I = \frac{1}{4\pi} A_{ul} N(H) = \frac{1}{4\pi} (2.30 \times 10^{-11} \text{ sec}^{-1}) \frac{4 \times 10^3 \text{ atom}}{\text{m}^3} \frac{1 \text{ photon}}{\text{atom}} \ell \quad (2.133)$$

## References

1. Mizushima, M., Quantum Mechanics of Atomic Spectra and Atomic Structure, W.A. Benjamin, Inc., New York, (1970), p.17.
2. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, New York, (1962), pp. 739-752.
3. McQuarrie, D. A., Quantum Chemistry, University Science Books, Mill Valley, CA, (1983), pp. 238-241.
4. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, New York, (1962), pp. 752-763.
5. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, New York, (1962), pp. 758-763.
6. Siebert, W. McC., Circuits, Signals, and Systems, The MIT Press, Cambridge, Massachusetts, (1986), pp. 488-502.
7. Lamb, W. E., Retherford, R. C., Phys. Rev., Vol. 72, (1947), pp. 241-243.
8. R. L. Mills, EPO Patent Number 86103694.5/0 198 257, Method and Apparatus for Selective Irradiation of Biological Materials, (1986).
9. Berestetskii, V. B., Lifshitz, E. M., Pitaevskii, Quantum Electrodynamics, Pergamon Press, Oxford, (1982), pp. 118-128.
10. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, New York, (1962), p. 178.
11. McQuarrie, D. A., Quantum Chemistry, University Science Books, Mill Valley, CA, (1983), pp. 206-221.
12. Pauling, Linus, Wilson, E., Bright, Introduction to Quantum Mechanics with Applications to Chemistry, McGraw-Hill Book Company, New York, (1935), pp. 118-121.
13. Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, New York, (1962), pp. 758-760.
14. Allen, C. W., Astrophysical Quantities, 3rd Edition, (1973), University of London, The Athlone Press, p. 79.
15. Labov, S., Bowyer, S., "Spectral observations of the extreme ultraviolet background", The Astrophysical Journal, 371, (1991), pp. 810-819.