CHAPTER 11

ONE-DIMENSIONAL COMPRESSIBLE FLOW

11.1 Introduction

Thermodynamics plays an important role in fluid mechanics, particularly when the flow speed is high (comparable to the speed of sound). High-speed flow in gases is called *compressible flow*, since it turns out that only in this case does the density vary much from point to point in the flow; if the velocity is much less than the speed of sound in a gas, then the density is nearly constant everywhere. In this chapter, we introduce some of the basic ideas of compressible flow.

11.2 The Momentum Principle

To analyze fluid flow, we need a principle from mechanics to supplement the laws of thermodynamics. This is the *momentum principle*. When fluid enters or leaves a control volume, it carries momentum with it. Since a packet of fluid of mass δm moving with velocity \vec{V} has momentum $(\delta m)\vec{V}$, the specific momentum (momentum per unit mass) of the fluid packet is simply equal to the velocity \vec{V} . Therefore, if fluid is flowing into a control volume with a mass flow rate of \dot{m} kg/s and each kg brings momentum \vec{V} with it, the momentum inflow rate is $\dot{m}\vec{V}$.

The momentum principle is simply Newton's second law applied to a control volume. It states that the net force on a control volume is equal to the *net* rate of momentum outflow from the control volume, plus the rate of change of momentum stored inside. The momentum stored inside the control volume only



Figure 11.1: Flow in a curved tube.

changes in unsteady flows, which we will not consider here.

Consider the steady flow in a curved tube shown in Fig. 11.1. The flow is assumed to be one-dimensional at points 1 and 2, which means that the velocity is constant over any cross-section and is oriented perpendicular to the crosssection. In this flow, the momentum inflow rate is $\dot{m}\vec{V_1}$, and the momentum outflow rate is $\dot{m}\vec{V_2}$. The *net* momentum outflow rate is $\dot{m}(\vec{V_2} - \vec{V_1})$. Therefore, the momentum principle states that the net force \vec{F} on the control volume is

$$\vec{F} = \dot{m} \left(\vec{V}_2 - \vec{V}_1 \right). \tag{11.1}$$

Note that \vec{F} is the *resultant* of all forces applied to the control volume. The momentum principle doesn't tell us anything about *where* these forces appear, just their vector sum. In general, the forces on the control volume result both from pressure in the fluid where the control surface intersects the fluid, and from stresses in the tube walls where the control surface intersects the walls.

11.3 The Speed of Sound

Sound waves are small pressure disturbances, which travel through a fluid with a characteristic speed. As long as the wavelength of the sound wave λ is long compared to the mean free path in the fluid,¹ the speed of propagation does not depend on the wavelength. (If you listen to a concert from far away the high notes and the low notes both reach you at the same time.)

To calculate the speed of sound, consider the situation shown in Fig. 11.2. A piston at the end of a long tube is suddenly displaced a small distance to the left, locally compressing the gas next to it. This launches a wave down the tube, which travels at the speed of sound. If the piston displacement is very sudden, then the density and pressure will change abruptly across the wavefront, from the undisturbed values ρ and P to the perturbed values $\rho + \Delta \rho$ and $P + \Delta P$ behind the wave. (If the piston were replaced by a loudspeaker, then both the diaphram displacement and the disturbances would instead be periodic.)

Since the piston displacement is rapid, there is little time for heat transfer, and consequently the compression process may be taken as adiabatic. Therefore, the temperature rises to $T + \Delta T$ behind the wave. Since the gas behind the wave is expanding, it is in motion; let its velocity be ΔV . (Note that $\Delta V \ll c$; the wave propagates into the undisturbed gas, much like a water wave moves without transporting water with it.)

We want to determine the speed c the disturbance propagates with. To do

¹The mean free path in air is about 10^{-7} m, so this condition is easily satisfied in air.



Figure 11.2: The piston is rapidly pushed in a small distance, launching a sound wave travelling to the left.



Figure 11.3: The situation as seen by an observer moving with the wave at speed c.

this, it is convenient to switch to a reference frame travelling with the wave (Fig. 11.3). Defining a control volume surrounding the wave, undisturbed fluid enters the control volume from the left at speed c, passes through the wave, and leaves the control volume at speed $c - \Delta V$.

If this wave propagates steadily, then there can be no mass buildup within the control volume. The mass flow rate in the left is

$$\dot{m}_{in} = \rho c A, \tag{11.2}$$

where A is the tube cross-sectional area. The mass flow rate out the right side is

$$\dot{m}_{out} = (\rho + \Delta \rho)(c - \Delta V)A.$$
(11.3)

Equating \dot{m}_{in} and \dot{m}_{out} yields

$$c\Delta\rho - \rho\Delta V - \Delta\rho\Delta V = 0. \tag{11.4}$$

The magnitudes of the pressure, density, and velocity disturbances in real sound waves are very small. In fact, the sound speed is formally defined as the speed *infinitesimal* disturbances propagate with. We will soon let all disturbance quantities become infinitesimal, which means we can ignore the very small $\Delta \rho \Delta V$ term in this equation. With this approximation, we have

$$\Delta V = c \frac{\Delta \rho}{\rho}.$$
 (11.5)

Since $\Delta \rho / \rho \ll 1$, this justifies our claim above that $\Delta V \ll c$.

Equation (11.5) resulted from doing a mass balance on the control volume; let's now see what results from applying the momentum principle. The x momentum leaving the control volume on the right is $\dot{m}(c-\Delta V)$; the x momentum entering on the left is $\dot{m}c$. Therefore, the *net* x momentum leaving the control volume is $-\dot{m}\Delta V$.

The momentum principle says that this equals the net force in the x direction on the control volume. The net force is due to pressure:

$$F_{net} = AP - A(P + \Delta P) = -A\Delta P.$$
(11.6)

Therefore, the momentum principle requires that

$$A\Delta P = \dot{m}\Delta V. \tag{11.7}$$

Using $\dot{m} = \rho Ac$, this becomes

$$\Delta P = \rho c \Delta V. \tag{11.8}$$

Substituting for ΔV from Eq. (11.5), we find

$$\Delta P = c^2 \Delta \rho. \tag{11.9}$$

Finally, an energy balance on the control volume (including the kinetic energy in this reference frame) yields

$$h + \frac{c^2}{2} = (h + \Delta h) + \frac{(c - \Delta V)^2}{2}.$$
 (11.10)

Since the propagation is assumed to be adiabatic, there is no heat transfer term Q in this equation.

To first order in ΔV , the energy balance becomes

$$\Delta h = c \Delta V. \tag{11.11}$$

Using Eq. (11.5), this can be written

$$\Delta h = c^2 \frac{\Delta \rho}{\rho}.\tag{11.12}$$

Now let us take the limit as the disturbances become infinitesimal. Equations (11.9) and (11.12) become

$$dP = c^2 d\rho, \tag{11.13}$$

$$dh = c^2 \frac{d\rho}{\rho}.$$
 (11.14)

(11.15)

At this point, it is interesting to ask what happens to the entropy of the gas as it passes through the wave. We know

$$dh = Tds + vdP = Tds + dP/\rho, \qquad (11.16)$$

so solving for ds yields

$$ds = \frac{1}{T} \left(dh - \frac{dP}{\rho} \right). \tag{11.17}$$

Substituting from above for dh and dP,

$$ds = \frac{1}{T} \left((c^2 d\rho) - \frac{(c^2 d\rho/\rho)}{\rho} \right) = 0.$$
 (11.18)

Therefore,

the propagation of an infinitesimal sound wave is isentropic.

(This would not be true for finite-sized disturbances, which are shock waves.)

With this result, we can use Eq. (11.13) to solve for c. Since s is constant, $dP/d\rho$ for this process is equivalent to $(\partial P/\partial \rho)_s$. Therefore,

$$c = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_s} \tag{11.19}$$

Example 11.1 Estimate the speed of sound in liquid water at 400 K and 1 MPa.

Solution: using the WWW thermodynamic property calculator or TPX, for these conditions $\rho = 937.88 \text{ kg/m}^3$. Now let's perturb ρ by a small amount, say 1.0 kg/m³. Holding *s* constant and setting $\rho = 938.88 \text{ kg/m}^3$, we find P = 3.2676 MPa. Thus,

$$c \approx \left(\frac{2.2676 \times 10^6}{1.0}\right)^{1/2} = 1506 \text{ m/s}.$$

11.3.1 Speed of sound in an ideal gas

In an ideal gas with constant c_p , Pv^k is constant for any isentropic process. Therefore, since $\rho = 1/v$,

$$P = C\rho^k \tag{11.20}$$

for an isentropic process. Differentiating this expression,

$$\left(\frac{\partial P}{\partial \rho}\right)_s = kC\rho^{k-1} = \frac{kP}{\rho}.$$
(11.21)

Using $P = \rho RT$,

$$\left(\frac{\partial P}{\partial \rho}\right)_s = kRT \tag{11.22}$$

and therefore the speed of sound in an ideal gas is

$$c = \sqrt{kRT}.$$
 (11.23)

Recall that R is the gas constant per unit mass: $R = \hat{R}/\hat{M}$.

Example 11.2 Calculate the speed of sound in air at 300 K.



Figure 11.4: The stagnation state on an h-s plot

Solution: For air, k = 1.4 and $\hat{M} = 28.97$. Therefore,

$$c = \sqrt{(1.4)(8314/28.97)(300)} = 347 \text{ m/s.}$$
 (11.24)

Note that the speed of sound in air is much less than in water.

The *Mach Number* M is defined to be the ratio of the fluid velocity to the local speed of sound:

$$M = \frac{V}{c}.\tag{11.25}$$

If M < 1, the flow is *subsonic*, and if M > 1 the flow is *supersonic*.

11.4 Stagnation Properties

When a high-speed flowing gas is decelerated to zero velocity (for example in a diffuser, or when the gas must deflect around an object inserted into it), its temperature, pressure, and density increase, since as a fluid packet decelerates the kinetic energy in the flow is expended by doing compression work on the fluid packet. The state reached by decelerating a fluid moving at speed V reversibly and adiabatically to zero velocity is called the *stagnation state*. The stagnation process is shown in Fig. 11.4.

The stagnation state is defined by the two equations

$$h_0 = h + \frac{V^2}{2} \tag{11.26}$$

and

$$s_0 = s.$$
 (11.27)

Here the subscript 0 denotes the stagnation state, and h_0 is known as the stagnation enthalpy. The pressure attained in the stagnation state is the stagnation pressure $P_0 = P(h_0, s_0)$, and the temperature $T_0 = T(h_0, s_0)$ is the stagnation temperature. Note that given h, s, and V it is always possible to calculate h_0 , s_0 , P_0 , and T_0 , whether or not the problem at hand actually involves decelerating the fluid to a stationary state; the stagnation state is the stagnation properties as properties of the flowing fluid.

Example 11.3 Determine the stagnation temperature and pressure for nitrogen at 10 MPa and 300 K, with speed V = 200 m/s.

Solution: Using the WWW calculator, we find $h = 442.14 \text{ kJ/kg} (4.4214 \times 10^5 \text{ J/kg})$ and s = 2.988 kJ/kg-K. Therefore, $h_0 = 4.4214 \times 10^5 + (300)^2/2 = 4.8714 \times 10^5 \text{ J/kg}$. Using $h_0 = 487.14 \text{ kJ/kg}$, and $s_0 = s = 2.988 \text{ kJ/kg-K}$, the WWW calculator finds $T_0 = 343.44 \text{ K}$, and $P_0 = 15.83 \text{ MPa}$.

For the special case of an ideal gas with constant c_p , Eq. (11.26) becomes

$$c_p T_0 = c_p T + \frac{V^2}{2}.$$
(11.28)

We can re-write this equation in terms of k and Mach number, using $c_p = kR/(k-1)$, $c = \sqrt{kRT}$, and M = V/c:

$$\frac{T_0}{T} = 1 + \frac{k-1}{2}M^2 \tag{11.29}$$

For any isentropic process in an ideal gas with constant c_p ,

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{k/k-1}.$$
(11.30)

Taking state 1 to be the state with velocity V and state 2 to be the stagnation state,

$$\frac{P_0}{P} = \left(\frac{T_0}{T}\right)^{k/k-1} = \left(1 + \frac{k-1}{2}M^2\right)^{k/k-1}$$
(11.31)

If we consider the reverse situation, in which a stationary gas at T_0 and P_0 is accelerated in a nozzle to speed V, Eq. (11.28) still applies, and sets an upper limit on the attainable speed, since T cannot be negative.



Setting T = 0,

$$V_{max} = \sqrt{2c_p T_0}.$$
 (11.32)

This can be rewritten as

$$V_{max} = \sqrt{\frac{2}{k-1}}c_0,$$
 (11.33)

where $c_0 = \sqrt{kRT_0}$ is the sound speed upstream of the nozzle. For k = 1.4, this becomes $V_{max} = 2.2c_0$. Therefore, the maximum velocity air can be accelerated to using an adiabatic nozzle is a little more than twice the upstream sound speed. Of course, since T is dropping as the gas accelerates, the *local* sound speed c is going down, and the *local* Mach number M = V/c diverges to infinity as $V \to V_{max}$.

11.5 Isentropic Flow with Area Change

Consider one-dimensional, steady, flow down a tube with an area A(x) which changes with distance. We will assume the irreversible processes of viscous friction and heat conduction are negligible, in which case the flow is isentropic.

First of all, conservation of mass requires that

$$\rho(x)V(x)A(x) = \text{constant.}$$
(11.34)

Differentiating this and dividing by ρVA yields

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0. \tag{11.35}$$

Conservation of energy requires that

$$h(x) + \frac{V^2(x)}{2} = \text{constant.}$$
 (11.36)

Differentiating yields

$$dh = -VdV. \tag{11.37}$$

We also know from basic thermodynamic principles that

$$dh = Tds + vdP = Tds + (1/\rho)dP.$$
 (11.38)

Since the flow is isentropic, ds = 0, and therefore

$$dh = \frac{dP}{\rho} \tag{11.39}$$

Equating Eq. (11.37) and Eq. (11.39),

$$dP = -\rho V dV. \tag{11.40}$$

This relationship makes sense qualitatively, since if the pressure decreases down the tube (dP/dx < 0), we expect the fluid to accelerate (dV/dx > 0), and visa versa.

Let's now find how the density changes with distance. Since the flow is isentropic,

$$dP = \left(\frac{\partial P}{\partial \rho}\right)_s d\rho. \tag{11.41}$$

But $(\partial P/\partial \rho)_s = c^2$, so

$$dP = c^2 d\rho. \tag{11.42}$$

Putting this into Eq. (11.39), we find

$$\frac{d\rho}{\rho} = -\frac{VdV}{c^2} = -M\frac{dV}{c}.$$
(11.43)

Therefore, the density decreases if the flow accelerates and visa versa $(d\rho < 0 \iff dV > 0)$. Note that the relative change in density for a given dV is greater at high Mach number than at low.

We may eliminate density from Eq. (11.35) by substituting from Eq. (11.43):

$$-\frac{VdV}{c^2} + \frac{dV}{V} + \frac{dA}{A} = 0$$
(11.44)

or

$$\frac{dA}{A} = -\frac{dV}{V} \left[1 - \left(\frac{V}{c}\right)^2 \right].$$
(11.45)

In terms of the Mach number,

$$\frac{dA}{A} = -\frac{dV}{V} \left(1 - M^2\right) \tag{11.46}$$

Note that we have not made any ideal gas assumption, so this equation applies for one-dimensional, isentropic flow of any fluid.



Figure 11.5: Flow in a converging nozzle.

Equation (11.46) tells us how the velocity of the fluid changes as it passes down the tube. An interesting result of this equation is that whether the fluid accelerates (dV > 0) or decelerates (dV < 0) as the tube area changes depends on whether the Mach number is greater than or less than 1.0 (supersonic or subsonic). We may identify 3 cases:

- Subsonic Flow: M < 1. In this case, a converging tube (dA < 0) causes the gas to accelerate (dV > 0), and a diverging tube causes the gas to decelerate. Therefore, a subsonic nozzle is a converging tube, and a subsonic diffuser is a diverging tube.
- Supersonic Flow: M > 1. Now the sign of the right-hand side of Eq. (11.46) is positive, so a converging tube causes the gas to *decelerate*, and a diverging tube causes the gas to accelerate. Therefore, a nozzle to accelerate a supersonic flow must be constructed as a diverging tube, and a supersonic diffuser must be converging.
- Sonic Flow: M = 1. If the Mach number equals one, then the right-hand side becomes zero for any dV. Therefore, sonic flow requires dA = 0: it is not possible to have M = 1 in a portion of the tube where the area is changing with x.

11.6 Flow in a Converging Nozzle

Suppose gas is escaping from a gas tank at pressure P_0 through a small converging nozzle, as shown in Fig. 11.5. The pressure of the surroundings is $P_B < P_0$ (the "back pressure"). The pressure at the exit plane of the nozzle is P_E .

Since the nozzle is converging, dA/dx < 0 and therefore the accelerating flow in the nozzle must be subsonic – it is impossible for supersonic flow to exist in this nozzle. At the end, the flow area expands abruptly. We will assume that dA/dx = 0 at some point very close to the nozzle exit, since the lip will inevitably be at least a little rounded.

Let us consider the flow in the nozzle as a function of P_B for given P_0 . If $P_B = P_0$, then the system is in mechanical equilibrium, and there is no flow, so $\dot{m} = 0$. For P_B slightly less than P_0 , gas will flow, and the pressure through the nozzle will fall continuously from P_0 to $P_E = P_B$. As P_B is lowered further, \dot{m} increases, and still $P_E = P_B$.

But at some P_B the flow becomes sonic at the exit; call this value P^* . What happens if P_B is now reduced below P^* ? The way the flow "communicates" the change in P_B to the flow upstream is through acoustic waves, which travel at the sound speed. As long as the flow is everywhere subsonic, it is possible for sound waves to propagate upstream, and the flow everywhere in the nozzle adjusts to the change in P_B .

But at $P_B = P^*$, the exit flow is coming out at the local speed of sound. Sound waves propagate upstream with this speed *relative to the moving gas*. In the lab frame, the sound waves get nowhere – it is analogous to a fish trying to swim upstream in a river when the river is flowing downstream as fast as the fish can swim upstream.

Therefore, the information that P_B is now below P^* can't be conveyed to the flow in the nozzle; it is unaware of the change. Therefore, everything about the flow in the nozzle – the total mass flow rate, P(x), V(x), etc. – is identical to the results for $P_B = P^*$, even though now $P_B < P^*$.

In particular, $P_E = P^*$, not P_B . Once the gas emerges from the nozzle, it finds itself suddenly at higher pressure than the surroundings. It rapidly adjusts to P_B through a set of complex, three-dimensional expansion waves. The pressure distribution would look something like that shown in Fig. 11.6.

11.7 Choked Flow

Once the flow at the exit of a converging nozzle becomes sonic, the downstream conditions no longer affect the flow upstream of the shock wave. Therefore, the mass flow rate through the nozzle is *unaffected* by downstream pressure once M = 1 at the nozzle exit. When this occurs, we say the flow is *choked*.

We can calculate the pressure ratio P_0/P_B above which the flow becomes choked, and the mass flow rate under choked conditions for the case of an ideal gas with constant c_p . Let the upstream conditions be T_0 and P_0 , and denote the conditions where M = 1 at the exit by T^* and P^* . Then from Eqs. 11.29



Figure 11.6: Pressure vs. distance for flow through a converging nozzle.

and 11.31,

$$\frac{T_0}{T^*} = 1 + \frac{k-1}{2} \tag{11.47}$$

and

$$\frac{P_0}{P^*} = \left(1 + \frac{k-1}{2}\right)^{k/k-1}.$$
(11.48)

For k = 1.4, these expressions become $T_0/T^* = 1.2$ and $P_0/P^* = 1.89$. Thus, a pressure ratio from inlet to outlet of the tube of only 1.9 is sufficient to choke the flow for air.

To solve for the choked mass flow rate, we may use the result that \dot{m} is equal to ρVA at any cross-section of the nozzle. For simplicity let's use the throat:

$$\dot{m} = \rho^* A^* c.$$
 (11.49)

Here A^* is the area of the throat, and we have used the fact that the flow is sonic at the throat. Using the expression for c for an ideal gas, we have

$$\dot{m} = A^* \frac{P^*}{RT^*} \sqrt{kRT^*}$$
 (11.50)

$$= \left[\frac{P^*}{P_0}\sqrt{\frac{k}{R}\frac{T_0}{T^*}}\right]\frac{P_0A^*}{\sqrt{T_0}}$$
(11.51)

The terms in brackets can be evaluated for a given k and \hat{M} (needed to calculate R). For air, this expression reduces to

$$\dot{m} = 0.0404 \frac{A^* P_0}{\sqrt{T_0}} \tag{11.52}$$



Figure 11.7: A converging-diverging nozzle.

in SI units.

Therefore, for a given gas, the maximum (choked) mass flow rate through a nozzle depends only on three quantities: the area of the nozzle where the flow is sonic, and the stagnation temperature and pressure upstream.

Example 11.4 Air enters a converging nozzle with an entrance diameter of 15 cm and an exit diameter of 2 cm. The entrance conditions are 6 MPa and 300 K. Treating air as an ideal gas with constant c_p , determine the maximum mass flow rate possible through the nozzle.

Solution: The maximum flow rate is the choked flow rate. From Eq. (11.52),

$$\dot{m}_{max} = 0.0404 \frac{(\pi)(0.01 \text{ m})^2(6 \times 10^6 \text{ Pa})}{\sqrt{300 \text{ K}}} = 4.39 \text{ kg/s}.$$

11.8 Flow in a Converging-Diverging Nozzle

Consider now attaching a diverging section onto the converging nozzle of the previous section, producing the situation shown in Fig. 11.7.

As before, upstream of the nozzle the flow velocity is negligible, and the pressure and temperature are the stagnation values P_0 and T_0 , respectively. Outside the nozzle, the pressure is P_B .

As P_B is lowered below P_0 , gas begins to flow through the nozzle. In the converging section, the flow accelerates and the pressure and temperature drop. If the flow is still subsonic when it reaches the minimum-area position in the nozzle (the "throat"), then as it passes into the diverging portion it decelerates, with a corresponding rise in P and T. It emerges with $P_e = P_B$, and $T_e = T_0 - V_e^2/2c_p$. Note that this behavior is completely consistent with Eq. (11.46). Since $M \neq 1$ at the throat where dA/dx = 0, it must be that dV/dx = 0 here, as it is.

If P_B is reduced further to P^* , the flow at the throat will become sonic. From Eq. (11.46), in this case it is no longer necessary for dV/dx = 0 at the



Figure 11.8: Flow in a converging-diverging nozzle. If P_B is low enough that the flow at the throat is sonic $(M^* = 1)$, then the flow continues to expand supersonically in the diverging portion (red line). In order to match P_B at the nozzle exit, at some point the flow suddenly transitions from supersonic to subsonic in a shock wave, with a sudden rise in pressure. After the shock, the diverging nozzle acts as a diffuser, and the pressure rises to P_B . The location of the shock depends on P_B , beginning at the throat when P_B is just low enough to produce supersonic flow, and moving out toward the nozzle exit as P_B decreases.

throat, so the flow can continue to accelerate into the diverging portion, where (now that it is supersonic) it continues to be accelerated.

As the supersonic flow accelerates in the diverging portion, the pressure continues to drop. The value of P_B has no effect on the flow in this portion of the nozzle, since any information about P_B would propagate at the speed of sound, and so would be swept downstream and would not reach the upstream flow. Therefore, if the gas continues to accelerate isentropically to the exit of the nozzle the pressure of the gas at the nozzle exit would not generally match P_B .

But somehow the pressure must come to P_B at or after the nozzle exit. What really happens is that if the pressure in the nozzle drops below P_B , then at some point in the diverging portion the gas undergoes a sudden and irreversible transition to subsonic flow. This transition is known as a *shock wave*, and will be discussed in the next section. Typically a shock wave normal to the flow will stand somewhere in the diverging portion of the nozzle, at a position such that $P_e = P_B$. The pressure vs. distance plot for various P_B values now looks something like the plot shown in Fig. 11.8.

In the isentropic portion of the flow (before the shock wave), we can use the condition ρVA =constant to solve for the velocity at any position in the nozzle.



Figure 11.9: Area ratio A/A^* for k = 1.4.

We may write

$$\rho VA = \rho^* cA^* \tag{11.53}$$

which using the ideal gas equation of state and $c = \sqrt{kRT}$ becomes²

$$\frac{A}{A^*} = \frac{1}{M} \frac{P^*}{P} \sqrt{\frac{T}{T^*}}.$$
(11.54)

If we now write this as

$$\frac{A}{A^*} = \frac{1}{M} \frac{P^*/P_0}{P/P_0} \sqrt{\frac{T/T_0}{T^*/T_0}}$$
(11.55)

and use eqs. (11.29) and (11.29), we find

$$\frac{A}{A^*} = \frac{1}{M} \left[\left(\frac{2}{k+1} \right) \left(1 + \frac{k-1}{2} M^2 \right) \right]^{(k+1)/2(k-1)}.$$
(11.56)

This equation is plotted in Fig. 11.9 for k = 1.4. Note that for a given area ratio, there are two possible Mach numbers — one subsonic, and the other supersonic.

11.9 Normal Shock Waves

Shock waves which stand perpendicular to the flow direction are called *normal* shock waves, in contrast to *oblique* shock waves which can occur when two- and

²You should verify this.



Figure 11.10: A steady shock wave. Note that ΔP , $\Delta \rho$, and ΔT are *not* assumed to be infinitesimal.

three-dimensional flow is considered. A shock wave is a discontinuity in the pressure, velocity, density, and temperature of the gas. It may seem suprising to you that these properties can be discontinuous, but in fact the equations describing conservation of mass, momentum, and energy, along with the ideal gas law, do admit discontinuous solutions.³

By carrying out balances of mass, momentum, and energy on a shock wave, we may derive relationships between the gas properties on either side of the shock. The approach is very much like that we did to derive the speed of sound, except that now the perturbations are not infinitesimal.

Consider a situation in which gas with velocity V_1 in the x direction passes through a stationary normal shock and leaves with velocity V_2 . Conservation of mass requires

$$\rho_1 V_1 = \rho_2 V_2. \tag{11.57}$$

We have divided through by the tube area, since it is the same on each side of the shock. The momentum principle states

$$(P_1 - P_2) = (\dot{m}/A)(V_2 - V_1), \tag{11.58}$$

which can also be written

$$P_1 + \rho_1 V_1^2 = P_2 + \rho_2 V_2^2, \qquad (11.59)$$

since $\dot{m}/A = \rho_1 V_1 = \rho_2 V_2$. The energy balance is

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2},\tag{11.60}$$

 $^{^{3}}$ A more detailed treatment considering viscosity and heat conduction would show that a shock wave has a finite thickness, which is of order the mean free path.

which is equivalent to

$$h_{0,1} = h_{0,2}.\tag{11.61}$$

Therefore, the stagnation enthalpy is constant across a shock.

Also, we can write equations of state that have to be satisfied in both states. If the fluid may be treated as an ideal gas in both states, then

$$P_1 = \rho_1 R T_1, \tag{11.62}$$

$$P_2 = \rho_2 R T_2 \tag{11.63}$$

If state 1 is fully specified, the four equations (11.57), (11.59), (11.60), and (11.63) may be used to solve for the four unknowns P_2 , T_2 , ρ_2 , and V_2 .

For the case of an ideal gas with constant c_p , this set of equations may be solved analytically. Equation (11.60) becomes

$$T_{0,1} = T_{0,2},\tag{11.64}$$

which, using Eq. (11.29), may be written

$$\frac{T_2}{T_1} = \frac{1 + \frac{k-1}{2}M_1^2}{1 + \frac{k-1}{2}M_2^2} \tag{11.65}$$

Using $\rho = P/RT$, $c = \sqrt{kRT}$, and M = V/c, Eq. (11.59) reduces to

$$\frac{P_2}{P_1} = \frac{1 + kM_1^2}{1 + kM_2^2} \tag{11.66}$$

Equation (11.57) becomes

$$\frac{P_2}{P_1} = \sqrt{\frac{T_2}{T_1}} \frac{M_2}{M_1}.$$
(11.67)

Substituting eqs. (11.65) and (11.66) into Eq. (11.67), we can solve for M_2 :

$$M_2^2 = \frac{M_1^2 + \frac{2}{k-1}}{\frac{2k}{k-1}M_1^2 - 1}$$
(11.68)

Since there is no heat transfer in the shock, the entropy production rate as the gas flows through the shock is simply

$$\dot{\mathcal{P}}_s = \dot{m}(s_2 - s_1) = \dot{m}(c_p \ln(T_2/T_1) - R \ln(P_2/P_1)).$$
 (11.69)

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It may be shown that this is positive only if $M_1 > 1$. Therefore, the flow entering the shock must be supersonic. Or, in the frame of reference of the gas upstream of the shock, this is equivalent to saying the shock wave must propagate at a supersonic (not subsonic) speed.