Chapter 16

Ocean Waves

Looking out to sea from the shore, we can see waves on the sea surface. Looking carefully, we notice the waves are undulations of the sea surface with a height of around a meter, where height is the vertical distance between the bottom of a trough and the top of a nearby crest. The wavelength, which we might take to be the distance between prominent crests, is around 50-100 meters. Watching the waves for a few minutes, we notice that wave height and wave length are not constant. The heights vary randomly in time and space, and the statistical properties of the waves, such as the mean height averaged for a few hundred waves, change from day to day. These prominent offshore waves are generated by wind. Sometimes the local wind generates the waves, other times distant storms generate waves which ultimately reach the coast. For example, waves breaking on the Southern California coast on a summer day may come from vast storms offshore of Antarctica 10,000 km away.

If we watch closely for a long time, we notice that sea level changes from hour to hour. Over a period of a day, sea level increases and decreases relative to a point on the shore by about a meter. The slow rise and fall of sea level is due to the tides, another type of wave on the sea surface. Tides have wavelengths of thousands of kilometers, and they are generated by the slow, very small changes in gravity due to the motion of the sun and the moon relative to Earth.

In this chapter you will learn how to describe ocean-surface waves quantitatively. In the next chapter we will describe tides and waves along coasts.

16.1 Linear Theory of Ocean Surface Waves

Surface waves are inherently nonlinear: The solution of the equations of motion depends on the surface boundary conditions, but the surface boundary conditions are the waves we wish to calculate. How can we proceed?

We begin by assuming that the amplitude of waves on the water surface is infinitely small so the surface is almost exactly a plane. To simplify the mathematics, we can also assume that the flow is 2-dimensional with waves traveling in the x-direction. We also assume that the Coriolis force and viscosity can be neglected. If we retain rotation, we get Kelvin waves discussed in §14.2.

With these assumptions, the sea-surface elevation ζ is:

$$\zeta = a \sin(k\,x - \omega\,t) \tag{16.1}$$

with

$$\omega = 2\pi f = \frac{2\pi}{T}; \qquad k = \frac{2\pi}{L} \tag{16.2}$$

where ω is wave frequency in radians per second, f is the wave frequency in Hertz (Hz), k is wave number, T is wave period, L is wave length, and where we assume, as stated above, that $a \ll 1$.

The *wave period* T is the time it takes two successive wave crests or troughs to pass a fixwed point. The *wave length* L is the distance between two successive wave crests or troughs at a fixed time.

Dispersion Relation Wave frequency ω is related to wave number k by the *dispersion relation*:

$$\omega^2 = g\,k \tanh(kd) \tag{16.3}$$

where d is the water depth anf g is the acceleration of gravity.

Two approximations are especially useful.

1. *Deep-water approximation* is valid if the water depth d is much greater than the wave length L. In this case, $d \gg L$, $kd \gg 1$, and $\tanh(kd) = 1$.

2. *Shallow-water approximation* is valid if the water depth is much less than a wavelength. In this case, $d \ll L$, $kd \ll 1$, and $\tanh(kd) = kd$.

For these two limits of water depth compared with wavelength the dispersion relation reduces to:

$$\text{Deep-water dispersion relation:} \qquad \begin{aligned} \omega^2 &= g\,k \\ d &> L/4 \end{aligned} \tag{16.4}$$

$$\text{Shallow-water dispersion relation:} \qquad \begin{aligned} \omega^2 &= g\,k^2\,d \\ d &< L/11 \end{aligned} \tag{16.5}$$

The stated limits for d/L give a dispersion relation accurate within 10%. Because many wave properties can be measured with accuracies of 5–10%, the approximations are useful for calculating local wave properties. Later we will see how to calculate wave properties as the waves propagate from deep to shallow water.

Phase Velocity The phase velocity c is the speed at which a particular phase of the wave propagates, for example, the speed of propagation of the wave crest. In one wave period T the crest advances one wave length L and the phase speed is $c = L/T = \omega/k$. Thus, the definition of phase speed is:

$$c \equiv \frac{\omega}{k} \tag{16.6}$$

The direction of propagation is perpendicular to the wave crest in the positive x direction. The deep- and shallow-water approximations for the dispersion relation give:

$$\text{Deep-water phase velocity:} \qquad c = \sqrt{\frac{g}{k}} = \frac{g}{\omega} \tag{16.7}$$

$$\text{Shallow-water phase velocity:} \qquad c = \sqrt{g\,d} \tag{16.8}$$

The approximations are accurate to about 5% for limits stated in (16.5, 16.5).

In deep water, the phase speed depends on wave length or wave frequency. Longer waves travel faster. Thus, deep-water waves are said to be dispersive. In shallow water, the phase speed is independent of the wave; it depends only on the depth of the water. Shallow-water waves are non-dispersive.

Group Velocity The concept of group velocity c_g is fundamental for understanding the propagation of linear and nonlinear waves. First, it is the velocity at which a group of waves travels across the ocean. More importantly, it is also the propagation velocity of wave energy. Whitham (1974, §1.3 and §11.6) gives a clear derivation of the concept and the fundamental equation (16.9).

The definition of group velocity in two dimensions is:

$$c_g \equiv \frac{\partial \omega}{\partial k} \tag{16.9}$$

Using the approximations for the dispersion relation:

$$\text{Deep-water group velocity:} \qquad c_g = \frac{g}{2\omega} = \frac{c}{2} \tag{16.10}$$

$$\text{Shallow-water group velocity:} \qquad c_g = \sqrt{g\,d} = c \tag{16.11}$$

For ocean-surface waves, the direction of propagation is perpendicular to the wave crests in the positive x direction. In the more general case of other types of waves, the group velocity is not necessarily in the direction perpendicular to wave crests.

Notice that a group of deep-water waves moves at half the phase speed of the waves making up the group. How can this happen? If we could watch closely a group of waves crossing the sea, we would see waves crests appear at the back of the wave train, move through the train, and disappear at the leading edge of the group. Each wave crest moves at twice the speed of the group.

Do real ocean waves move in groups governed by the dispersion relation? Yes. Walter Munk and colleagues (1963) in a remarkable series of experiments in the 1960s showed that ocean waves propagating over great distances are dispersive, and that the dispersion could be used to track storms. They recorded waves for many days using an array of three pressure gauges just offshore of San Clemente Island, 60 miles due west of San Diego, California. Wave spectra were

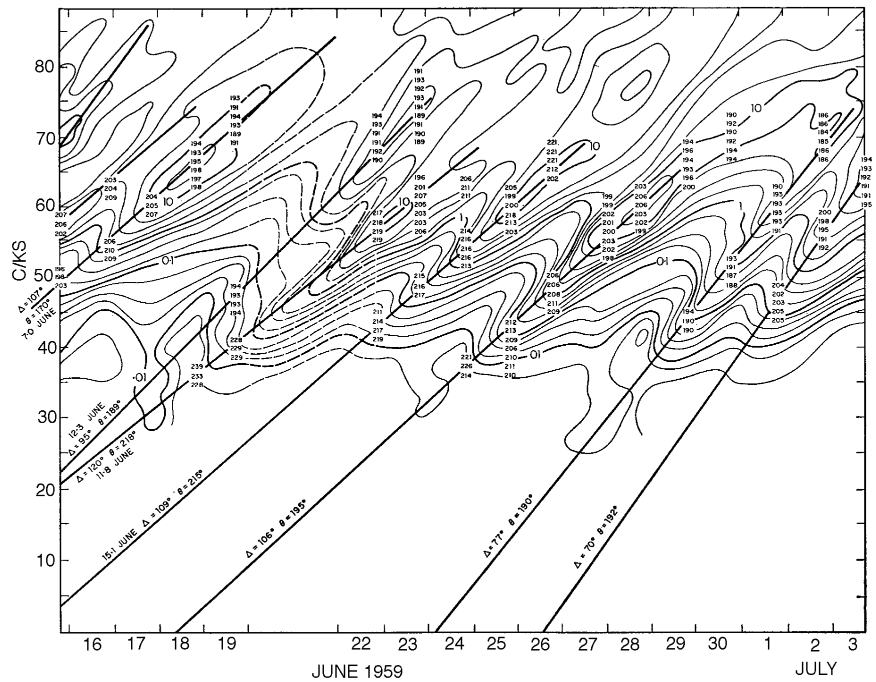

Figure 16.1 Contours of wave energy on a frequency-time plot calculated from spectra of waves measured by pressure gauges offshore of southern California. The ridges of high wave energy show the arrival of dispersed wave trains from distant storms. The slope of the ridge is proportional to distance to the storm. 1 c/ks is 1 cyle per kilosecond = 0.001 Hz; Δ is distance in degrees, θ is direction of arrival of waves at California (from Munk et al. 1963).

calculated for each day's data. (The concept of a spectra is discussed further below.) From the spectra, the amplitudes and frequencies of the low-frequency waves and the propagation direction of the waves were calculated. Finally, they plotted contours of wave energy on a frequency-time diagram (Figure 16.1).

To understand the figure, consider a distant storm that produces waves of many frequencies. The lowest-frequency waves (smallest ω) travel the fastest (16.11), and they arrive before other, higher-frequency waves. The further away the storm, the longer the delay between arrivals of waves of different frequencies. The ridges of high wave energy seen in the figure are produced by individual storms. The slope of the ridge gives the distance to the storm in degrees Δ along a great circle; and the phase information from the array gives the angle to the storm θ. The two angles give the storm's location relative to San Clemente.

The locations of th storms producing the waves in Figure 16.1 were compared with the location of storms plotted on weather maps and in most cases the two agreed well. Thus the ridge labeled 15-1 June was from a storm at a distance $\Delta = 109°$, which corresponded to a storm shown on weather maps at a distance of $\Delta = 111°$, putting the storm south of New Zealand at 55°S and 168°W.

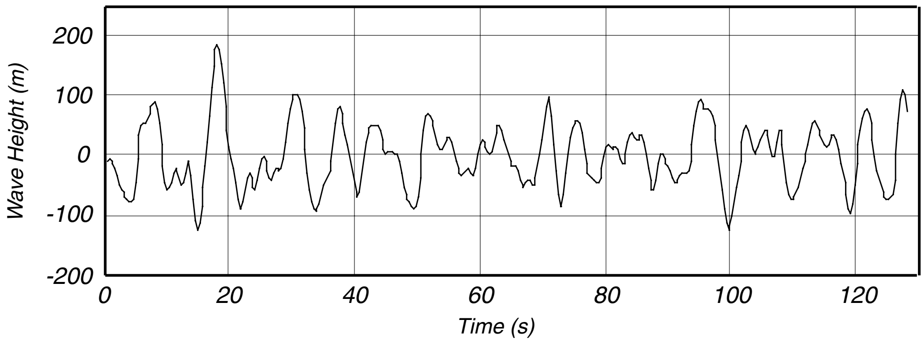

Figure 16.2 A short record of wave height measured by a wave buoy in the North Atlantic.

Wave Energy Wave energy E in Joules per square meter is related to the variance of sea-surface displacement ζ by:

$$E = \rho_w g \left\langle \zeta^2 \right\rangle \tag{16.12}$$

where ρ_w is water density, g is gravity, and the brackets denote a time or space average.

Significant Wave Height What do we mean by wave height? If we look at a wind-driven sea, we see waves of various heights. Some are much larger than most, others are much smaller (Figure 16.2). A practical definition that is often used is the height of the highest 1/3 of the waves, $H_{1/3}$. The height is computed as follows: measure wave height for a few minutes, pick out say 120 wave crests and record their heights. Pick the 40 largest waves and calculate the average height of the 40 values. This is $H_{1/3}$ for the wave record.

The concept of significant wave height was developed during the World War II as part of a project to forecast ocean wave heights and periods. Wiegel (1964: p. 198) reports that work at the Scripps Institution of Oceanography showed

> ... wave height estimated by observers corresponds to the average of the highest 20 to 40 per cent of waves ... Originally, the term significant wave height was attached to the average of these observations, the highest 30 per cent of the waves, but has evolved to become the average of the highest one-third of the waves, (designated H_S or $H_{1/3}$)

More recently, significant wave height is calculated from measured wave displacement. If the sea contains a narrow range of wave frequencies, $H_{1/3}$ is related to the standard deviation of sea-surface displacement (NAS , 1963: 22; Hoffman and Karst, 1975)

$$H_{1/3} = 4 \left\langle \zeta^2 \right\rangle^{1/2} \tag{16.13}$$

where $\left\langle \zeta^2 \right\rangle^{1/2}$ is the standard deviation of surface displacement. This relationship is much more useful, and it is now the accepted way to calculate wave height from wave measurements.

16.2 Nonlinear waves

We derived the properties of an ocean surface wave assuming the wave amplitude was very small. In reality, the amplitude is not small, but the product ka is small, and wave properties can be expanded in a power series of ka (Stokes, 1847). He calculated the properties of a wave of finite amplitude and found:

$$\zeta = a\cos(kx - \omega t) + \frac{1}{2}ka^2\cos 2(kx - \omega t) + \frac{3}{8}k^2a^3\cos 3(kx - \omega t) + \cdots \tag{16.14}$$

The phases of the components for the Fourier series expnasion of ζ in (16.14) are such that non-linear waves have sharpened crests and flattened troughs. The maximum amplitude of the Stokes wave is $a_{max} = 0.07L$. Such steep waves in deep water are called Stokes waves (See also Lamb, 1945, §250).

Knowledge of non-linear waves came slowly until Hasselmann (1961, 1963a, 1963b, 1966), using the tools of high-energy particle physics, worked out to 6th order the interactions of three or more waves on the sea surface. He, Phillips (1960), and Longuet-Higgins and Phillips (1962) showed that n free waves on the sea suface can interact to produce another free wave only if the frequencies and wave numbers of the interacting waves sum to zero:

$$\omega_1 \pm \omega_2 \pm \omega_3 \pm \cdots \omega_n = 0 \tag{16.15a}$$

$$\mathbf{k_1} \pm \mathbf{k_2} \pm \mathbf{k_3} \pm \cdots \mathbf{k_n} = 0 \tag{16.15b}$$

$$\omega_i^2 = g\,k_i \tag{16.15c}$$

where we allow waves to travel in any direction, and $\mathbf{k_i}$ is the vector wave number giving wave length and direction. (16.15a,b) are general requirements for any interacting waves. The fewest number of waves that meet the conditions of (16.15) are three waves which interact to produce a fourth. The interaction is weak; waves must interact for hundreds of wave lengths and periods to produce a fourth wave with amplitude comparable to the incoming waves. The Stokes wave does not meet the criteria of (16.15) and the wave components are not free waves; the higher harmonics are bound to the primary wave.

Wave Momentum The concept of waves momentum has caused considerable confusion (McIntyre, 1981). In general, waves do not have momentum, a mass flux, but they do have a momentum flux. This is true for waves on the sea surface. Ursell (1950) showed that ocean swell on a rotating Earth has no mass transport. His proof seems to contradict the usual textbook discussions of steep, non-linear waves such as Stokes waves. Water particles driven by a Stokes wave move along paths that are nearly circular, but the paths fail to close, and the particles move slowly in the direction of wave propagation. This is a mass transport; and the phenomena is called Stokes drift. On a rotating Earth, the transport is influenced by rotation, and it becomes an inertial current.

Hasselmann (1970) looked closer at the problem and showed how non-linear waves, currents, and inertial oscillations can interact on a rotating Earth. Even over short distances where rotation is not important, packets of steep waves do not have momentum. They do have a forward transport near the surface, and this is balanced by an equal transport in the opposite direction at depth.

Solitary Waves Solitary waves are another class of non-linear waves, and they have very ineresting properties. They propagate without change of shape, and two solitons can cross without interaction. The first soliton was discovered by John Scott Russell (1808–1882), who followed a solitary wave generated by a boat in Edinburgh's Union Canal in 1834.

> Scott witnessed such a wave while watching a boat being drawn along the Union Canal by a pair of horses. When the boat stopped, he noticed that water around the vessel surged ahead in the form of a single wave, whose height and speed remained virtually unchanged. Russell pursued the wave on horseback for more than a mile before returning home to reconstruct the event in an experimental tank in his garden.—Nature 376, 3 August 1995, p. 373.

The properties of a solitary waves result from an exact balance between dispersion which tends to spread the solitary wave into a train of waves, and non-linear effects which tend to shorten and steepen the wave. The type of solitary wave in shallow water seen by Russell, has the form:

$$\zeta = a \ \text{sech}^2 \left[\left(\frac{3a}{4d^3} \right)^{1/2} (x - ct) \right] \tag{16.16}$$

which propagates at a speed:

$$c = c_0 \left(1 + \frac{a}{2d} \right) \tag{16.17}$$

You might think that all shallow-water waves are solitons because they are non-dispersive, and hence they ought to propagate without change in shape. Unfortunately, this is not true if the waves have finite amplitude. The velocity of the wave depends on depth. If the wave consists of a single hump, then the water at the crest travels faster than water in the trough, and the wave steepens as it moves foreward. Eventually, the wave becomes very steep and breaks. At this point it is called a bore. In some river mouths, the incoming tide is so high and the estuary so long and shallow that the tidal wave entering the estuary eventually steepens and breaks producing a bore that runs up the river. This happens in the Amazon in South America, the Severn in Europe, and the Tsientang in China (Pugh, 1987: 249).

16.3 Waves and the Concept of a Wave Spectrum

If we look out to sea, we notice that waves on the sea surface are not simple sinusoids. The surface appears to be composed of random waves of various lengths and periods. How can we describe this surface? The simple answer is, Not very easily. We can however, with some simplifications, come close to describing the surface. The simplifications lead to the concept of the spectrum of ocean waves. The spectrum gives the distribution of wave energy among different wave frequencies of wave lengths on the sea surface.

The concept of a spectrum is based on work by Joseph Fourier (1768–1830), who showed that almost any function $\zeta(t)$ (or $\zeta(x)$ if you like), can be represented over the interval $-T/2 \le t \le T/2$ as the sum of an infinite series of sine and cosine functions with harmonic wave frequencies:

$$\zeta(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nft + b_n \sin 2\pi nft) \tag{16.18}$$

where

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} \zeta(t) \cos 2\pi nft \, dt, \qquad (n = 0, 1, 2, \dots) \tag{16.19a}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} \zeta(t) \sin 2\pi nft \, dt, \qquad (n = 0, 1, 2, \dots) \tag{16.19b}$$

$f = 2\pi/T$ is the fundamental frequency, and nf are harmonics of the fundamental frequency. This form of $\zeta(t)$ is called a *Fourier series* (Bracewell, 1986: 204; Whittaker and Watson, 1963: §9.1). Notice that a_0 is the mean value of $\zeta(t)$ over the interval.

Equations (16.18 and 16.19) can be simplified using

$$\exp(2\pi nft) = \cos(2\pi nft) + i \sin(2\pi nft) \tag{16.20}$$

where $i = \sqrt{-1}$. Equations (16.18 and 16.19) then become:

$$\zeta(t) = \sum_{n=-\infty}^{\infty} Z_n \exp^{i2\pi nft} \tag{16.21}$$

where

$$Z_n = \frac{1}{T} \int_{-T/2}^{T/2} \zeta(t) \exp^{-i2\pi nft} \, dt, \qquad (n = 0, 1, 2, \dots) \tag{16.22}$$

Z_n is called the *Fourier transform* of $\zeta(t)$.

The spectrum $S(f)$ of $\zeta(t)$ is:

$$S(nf) = Z_n Z_n^* \tag{16.23}$$

where Z^* is the complex conjugate of Z. We will use these forms for the Fourier series and spectra when we describing the computation of ocean wave spectra.

We can expand the idea of a Fourier series to include series that represent surfaces $\zeta(x, y)$ using similar techniques. Thus, any surface can be represented as an infinite series of sine and cosine functions oriented in all possible directions.

Now, let's apply these ideas to the sea surface. Suppose for a moment that the sea surface were frozen in time. Using the Fourier expansion, the frozen surface can be represented as an infinite series of sine and cosine functions of different wave numbers oriented in all possible directions. If we unfreeze the surface and let it evolve in time, we can represent the sea surface as an infinite series of sine and cosine functions of different wave lengths moving in all directions. Because wave lengths and wave frequencies are related through the dispersion relation, we can also represent the sea surface as an infinite sum of sine and cosine functions of different frequencies moving in all directions.

Note in our discussion of Fourier series that we assume the coefficients (a_n, b_n, Z_n) are constant. For times of perhaps an hour, and distances of perhaps tens of kilometers, the waves on the sea surface are sufficiently fixed that the assumption is true. Furthermore, non-linear interactions among waves are very weak. Therefore, we can represent a local sea surface by a linear superposition of real, sine waves having many different wave lengths or frequencies and different phases traveling in many different directions. The Fourier series in not just a convenient mathematical expression, it states that the sea surface is really, truely composed of sine waves, each one propagating according to the equations we wrote down in §16.1.

The concept of the sea surface being composed of independent waves can be carried further. Suppose I throw a rock into a calm ocean, and it makes a big splash. According to Fourier, the splash can be represented as a superposition of cosine waves all of nearly zero phase so the waves add up to a big splash at the origin. Furthermore, each individual Fourier wave then begins to travel away from the splash. The longest waves travel fastest, and eventually, far from the splash, the sea consists of a dispersed train of waves with the longest waves furtherest from the splash and the shortest waves closest.

Sampling the Sea Surface Calculating the Fourier series that represents the sea surface is perhaps impossible. It requires that we measure the height of the sea surface $\zeta(x, y, t)$ everywhere in an area perhaps ten kilometers on a side for perhaps an hour. So, let's simplify. Suppose we install a wave staff somewhere in the ocean and record the height of the sea surface as a function of time $\zeta(t)$. We would obtain a record like that in Figure 16.2. All waves on the sea surface will be measured, but we will know nothing about the direction of the waves. This is a much more practical measurement, and it will give the frequency spectrum of the waves on the sea surface.

Working with a trace of wave height on say a piece of paper is difficult, so let's digitize the output of the wave staff to obtain

$$\zeta_j \equiv \zeta(t_j), \qquad t_j \equiv j\Delta \qquad (16.24)$$
$$j = 0, 1, 2, \cdots, N-1$$

where Δ is the time interval between the samples, and N is the total number of samples. The length T of the record is $T = N\,\Delta$. Figure 16.3 shows the first 20 seconds of wave height from Figure 16.2 digitized at intervals of $\Delta = 0.32$ s.

Notice that ζ_j is not the same as $\zeta(t)$. We have absolutely no information about the height of the sea surface between samples. Thus we have converted from an infinite set of numbers which describes $\zeta(t)$ to a finite set of numbers which describe ζ_j. By converting from a continuous function to a digitized function, we have given up an infinite amount of information about the surface.

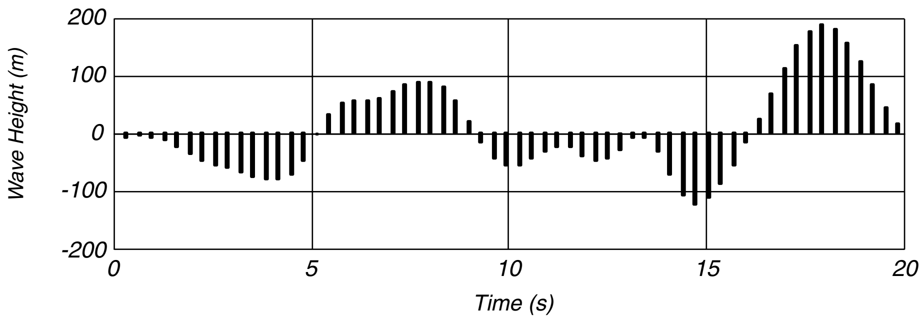

Figure 16.3 The first 20 seconds of digitized data from Figure 16.2.

The sampling interval Δ defines a *Nyquist critical frequency* (Press et al, 1992: 494)

$$Ny \equiv 1/(2\Delta) \tag{16.25}$$

> The Nyquist critical frequency is important for two related, but distinct, reasons. One is good news, the other is bad news. First the good news. It is the remarkable fact known as the *sampling theorem*: If a continuous function $\zeta(t)$, sampled at an interval Δ, happens to be *bandwidth limited* to frequencies smaller in magnitude than Ny, i.e., if $S(nf) = 0$ for all $|nf| \geq Ny$, then the function $\zeta(t)$ is *completely determined* by its samples ζ_j ... This is a remarkable theorem for many reasons, among them that it shows that the "information content" of a bandwidth limited function is, in some sense, infinitely smaller than that of a general continuous function ...
>
> Now the bad news. The bad news concerns the effect of sampling a continuous function that is *not* bandwidth limited to less than the Nyquist critical frequency. In that case, it turns out that all of the power spectral density that lies outside the frequency range $-Ny \leq nf \leq Ny$ is spuriously moved into that range. This phenomenon is called *aliasing*. Any frequency component outside of the range $(-Ny, Ny)$ is *aliased* (falsly translated) into that range by the very act of discrete sampling ... There is little that you can do to remove aliased power once you have discretely sampled a signal. The way to overcome aliasing is to (i) know the natural bandwidth limit of the signal — or else enforce a known limit by analog filtering of the continuous signal, and then (ii) sample at a rate sufficiently rapid to give at least two points per cycle of the highest frequency present.
> —Press et al 1992, but with notation changed to our notation.

Figure 16.4 illustrates the aliasing problem. Notice how a high frequency signal is aliased into a lower frequency if the higher frequency is above the critical frequency. Fortunately, we can can easily avoid the problem: (i) use instruments that do not respond to very short, high frequency waves if we are interested in the bigger waves; and (ii) chose Δt small enough that we lose little useful information. In the example shown in Figure 16.3, there are no waves in the signal to be digitized with frequencies higher than $Ny = 1.5625$ Hz.

Let's summarize. Digitized signals from a wave staff cannot be used to study waves with frequencies above the Nyquist critical frequency. Nor can the signal

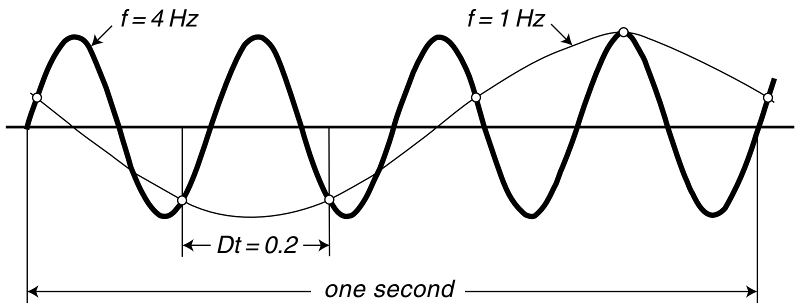

Figure 16.4 Sampling a 4 Hz sine wave (heavy line) every 0.2 s aliases the frequency to 1 Hz (light line).

be used to study waves with frequencies less than the fundamental frequency determined by the duration T of the wave record. The digitized wave record contains information about waves in the frequency range:

$$\frac{1}{T} < f < \frac{1}{2\Delta} \tag{16.26}$$

where $T = N\Delta$ is the length of the time series, and f is the frequency in Hertz.

Calculating The Wave Spectrum The digital Fourier transform Z_n of a wave record ζ_j equivalent to (16.21 and 16.22) is:

$$Z_n = \frac{1}{N}\sum_{j=0}^{N-1} \zeta_j \exp[-i2\pi jn/N] \tag{16.27a}$$

$$\zeta_n = \sum_{n=0}^{N-1} Z_j \exp[i2\pi jn/N] \tag{16.27b}$$

for $j = 0, 1, \cdots, N-1$; $n = 0, 1, \cdots, N-1$. These equations can be summed very quickly using the Fast Fourier Transform, especially if N is a power of 2 (Rabiner and Rader,1972; Press et al. 1992).

The simple spectrum S_n of ζ, which is called the *periodogram*, is:

$$S_n = \frac{1}{N^2}\left[|Z_n|^2 + |Z_{N-n}|^2\right]; \qquad n = 1, 2, \cdots, (N/2-1) \tag{16.28}$$

$$S_0 = \frac{1}{N^2}|Z_0|^2$$

$$S_{N/2} = \frac{1}{N^2}|Z_{N/2}|^2$$

where S_N is normalized such that:

$$\sum_{j=0}^{N-1} |\zeta_j|^2 = \sum_{n=0}^{N/2} S_n \tag{16.29}$$

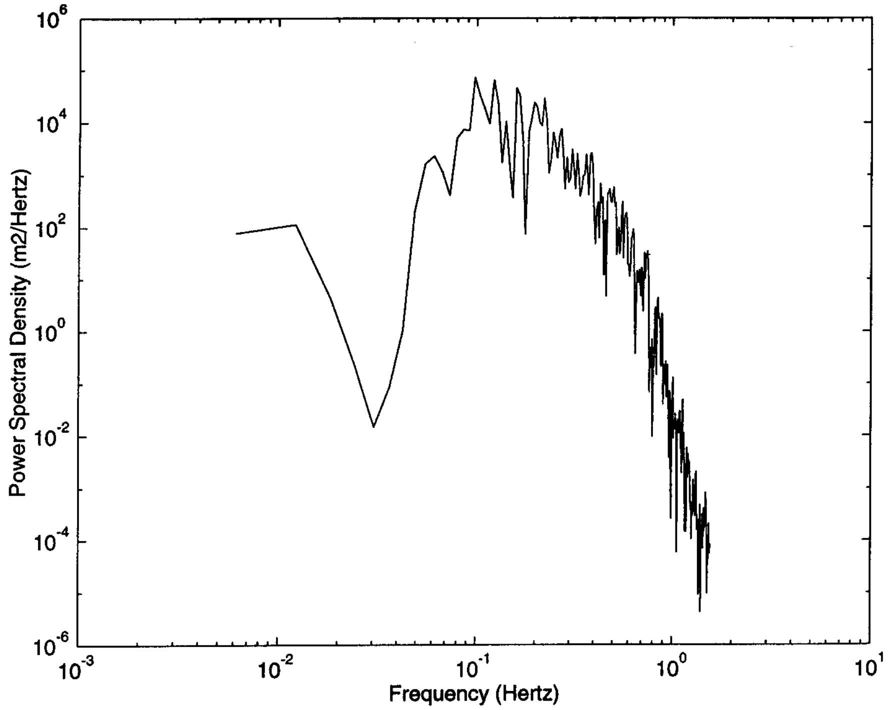

Figure 16.5 The periodogram calculated from the first 164 s of data from Figure 16.2. The Nyquist frequency is 1.5625 Hz.

that is, the variance of ζ_j is the sum of the $(N/2+1)$ terms in the periodogram. Note, the terms of S_n above the frequency $(N/2)$ are symmetric about that frequency. Figure 16.5 shows the periodogram of the time series shown in Figure 16.2.

The periodogram is a very noisy function. The variance of each point is equal to the expected value at the point. By averaging together 10–30 periodograms we can reduce the uncertainty in the value at each frequency. The averaged periodogram is called the spectrum of the wave height (Figure 16.6). It gives the distribution of the variance of sea-surface height at the wave staff as a function of frequency. Because wave energy is proportional to the variance (16.12) the spectrum is called the *energy spectrum* or the *wave-height spectrum*. Typically three hours of wave staff data are used to compute a spectrum of wave height.

Summary We can summarize the calculation of a spectrum into the following steps:

1. Digitize a segment of wave-height data to obtain useful limits according to (16.26). For example, use 1024 samples from 8.53 minutes of data sampled at the rate of 2 samples/second.
2. Calculate the digital, fast Fourier transform Z_n of the time series.
3. Calculate the periodogram S_n from the sum of the squares of the real and imaginary parts of the Fourier transform.

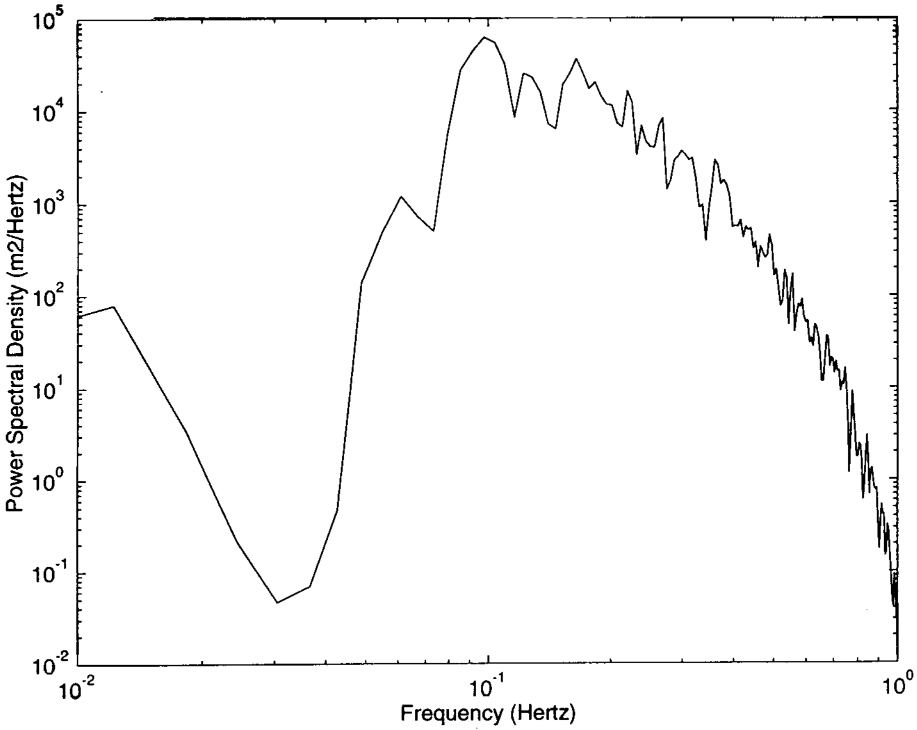

Figure 16.6 The spectrum of waves calculated from 11 minutes of data shown in Figure 7.2 by averaging four periodograms to reduce uncertainty in the spectral values. Spectral values below 0.04 Hz are in error due to noise.

4. Repeat to produce $M = 20$ periodograms.
5. Average the 20 periodograms to produce an averaged spectrum S_M.
6. S_M has values that are χ^2 distributed with $2M$ degrees of freedom.

This outline of the calculation of a spectrum ignores many details. For more complete information see, for example, Percival and Walden (1993), Press et al. (1992: §12), Oppenheim and Schafer (1975), or other texts on digital signal processing.

16.4 Ocean-Wave Spectra

Ocean waves are produced by the wind. The faster the wind, the longer the wind blows, and the bigger the area over which the wind blows, the bigger the waves. In designing ships or offshore structures we wish to know the biggest waves produced by a given wind speed. Suppose the wind blows at 20 m/s for many days over a large area of the North Atlantic. What will be the spectrum of ocean waves at the downwind side of the area?

Pierson-Moskowitz Spectrum Various idealized spectra are used to answer the question in oceanography and ocean engineering. Perhaps the simplest is

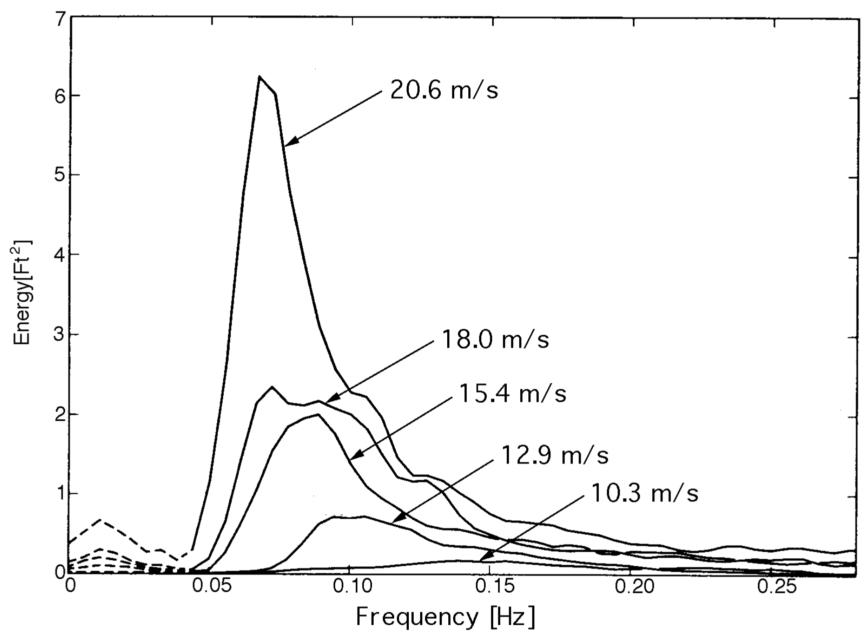

Figure 16.7 Wave spectra of a fully developed sea for different wind speeds according to Moskowitz (1964).

that proposed by Pierson and Moskowitz (1964). They assumed that if the wind blew steadily for a sufficiently long time and over a sufficiently large area, the waves would come into equilibrium with the wind. This is the concept of a *fully developed sea.* Here, a "long time" is roughly ten-thousand wave periods, and a "large area" is roughly ten-thousand wave lengths on a side.

To obtain such the spectrum of a fully developed sea, they used measurements of waves made by accelerometers on British weather ships in the North Atlantic. First, they selected wave data for times when the wind had blown steadily for long times over large areas of the North Atlantic. Then they calculated the wave spectra for various wind speeds, and they found that the spectra were of the form (Figure 16.7):

$$S(\omega) = \frac{\alpha g^2}{\omega^5} \exp\left[-\beta\left(\frac{\omega_0}{\omega}\right)^4\right] \tag{16.30}$$

where $\omega = 2\pi f$, f is the wave frequency in Hertz, $\alpha = 8.1 \times 10^{-3}$, $\beta = 0.74$, $\omega_0 = g/U_{19.5}$ and $U_{19.5}$ is the wind speed at a height of 19.5 m above the sea surface, the height of the anemometers on the weather ships used by Pierson and Moskowitz (1964).

For most airflow over the sea the atmospheric boundary layer has nearly neutral stability, and

$$U_{19.5} \approx 1.026\, U_{10} \tag{16.31}$$

assuming a drag coefficient of 1.3×10^{-3}.

The frequency of the peak of the Pierson-Moskowitz spectrum is calculated by solving $dS/d\omega = 0$ for ω_p, to obtain

$$\omega_p = 0.877\, g/U_{19.5}. \tag{16.32}$$

The speed of waves at the peak is calculated from (16.10), which gives:

$$c_p = \frac{g}{\omega_p} = 1.14\, U_{19.5} \approx 1.17\, U_{10} \tag{16.33}$$

Hence waves with frequency ω_p travel 14% faster than the wind at a height of 19.5 m or 17% faster than the wind at a height of 10 m. This poses a difficult problem: How can the wind produce waves traveling faster than the wind? We will return to the problem after we discuss the JONSWAP spectrum and the influence of nonlinear interactions among wind-generated waves.

The significant wave height is calculated from the integral of $S(\omega)$ to obtain:

$$\left\langle \zeta^2 \right\rangle = 2.74 \times 10^{-3}\, \frac{\left(U_{19.5}\right)^4}{g^2} \tag{16.34}$$

Remembering that $H_{1/3} = 4 \left\langle \zeta^2 \right\rangle^{1/2}$, the significant wave height calculated from the Pierson-Moskowitz spectrum is:

$$H_{1/3} = 0.21\, \frac{\left(U_{19.5}\right)^2}{g} \approx 0.22\, \frac{\left(U_{10}\right)^2}{g} \tag{16.35}$$

Figure 16.8 gives significan wave heights and periods calculated from the Pierson-Moskowitz spectrum.

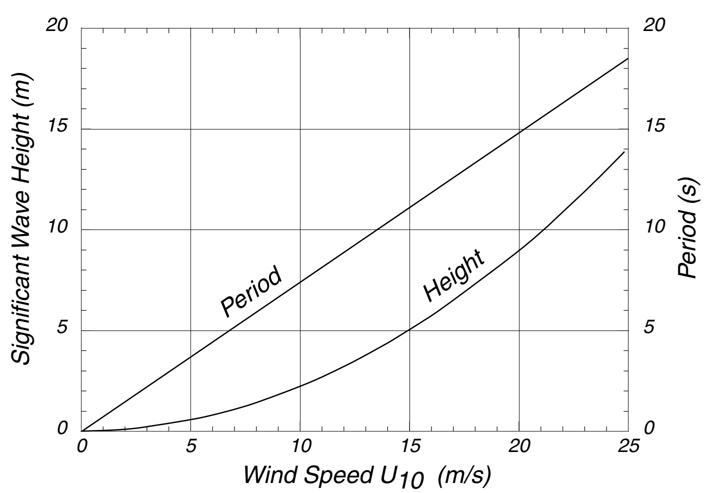

Figure 16.8 Significant wave height and period at the peak of the spectrum of a fully developed sea calculated from the Pierson-Moskowitz spectrum using (16.15 and 16.32).

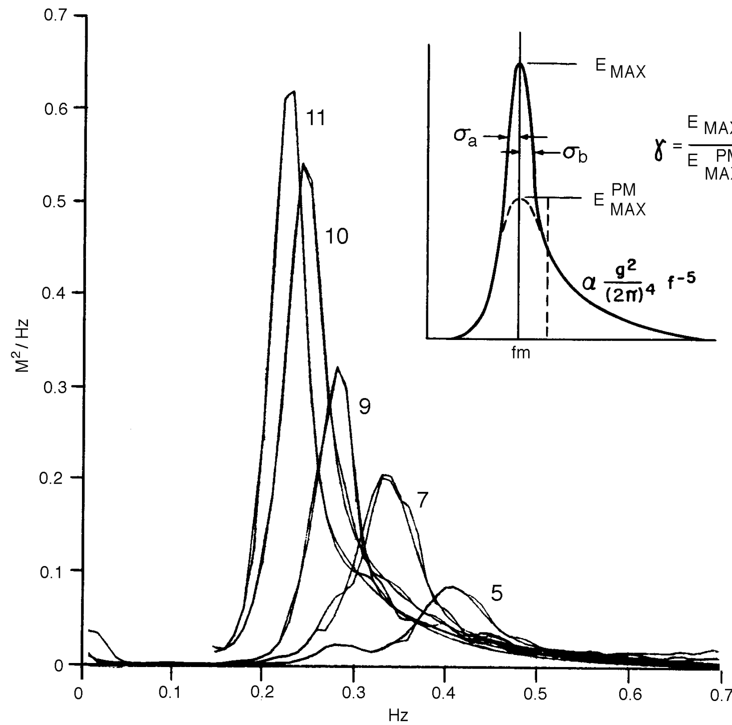

Figure 16.9 Wave spectra of a developing sea for different fetches according to Hasselmann et al. (1973).

JONSWAP Spectrum Hasselmann et al. (1973), after analyzing data collected during the Joint North Sea Wave Observation Project JONSWAP, found that the wave spectrum is never fully developed. It continues to develop through non-linear, wave-wave interactions even for very long times and distances. They therefore proposed a spectrum in the form (Figure 16.9):

$$S(\omega) = \frac{\alpha g^2}{\omega^5} \exp\left[-\frac{5}{4}\left(\frac{\omega_p}{\omega}\right)^4\right]\gamma^r \tag{16.36a}$$

$$r = \exp\left[-\frac{(\omega-\omega_p)^2}{2\,\sigma^2\,\omega_p^2}\right] \tag{16.36b}$$

Wave data collected during the JONSWAP experiment were used to determine the values for the constants in (16.36):

$$\alpha = 0.076 \left(\frac{U_{10}^2}{F\,g} \right)^{0.22} \tag{16.37a}$$

$$\omega_p = 22 \left(\frac{g^2}{U_{10} F} \right)^{1/3} \tag{16.37b}$$

$$\gamma = 3.3 \tag{16.37c}$$

$$\sigma = \begin{cases} 0.07 & \omega \leq \omega_p \\ 0.09 & \omega > \omega_p \end{cases} \tag{16.37d}$$

where F is the distance from a lee shore, called the *fetch*, or the distance over which the wind blows with constant velocity.

Because the spectral values increase with fetch, so too does the energy of the wave field:

$$\langle \zeta^2 \rangle = 1.67 \times 10^{-7} \left(\frac{U_{10}^2}{g} \right) x \tag{16.38}$$

The JONSWAP spectrum is similar to the Pierson-Moskowitz spectrum except that waves continues to grow with distance (or time) as specified by the α term, and the peak in the spectrum is more pronounced, as specified by the γ term. The latter turns out to be particularly important because it leads to enhanced non-linear interactions and a spectrum that changes in time according to the theory of Hasselmann (1966).

Generation of Waves by Wind We have seen in the last few paragraphs that waves are related to the wind. We have, however, put off until now just how they are generated by the wind. Suppose we begin with a mirror-smooth sea (Beaufort Number 0). What happens if the wind suddenly begins to blow steadily at say 8 m/s? Three different physical processes begin:

1. The turbulence in the wind produces random pressure fluctuations at the sea surface, which produces small waves with wavelengths of a few centimeters (Phillips, 1957).

2. Next, the wind acts on the small waves, causing them to become larger. Wind belowing over the wave produces pressure differences along the wave profile causing the wave to grow. The process is unstable because, as the wave gets bigger, the pressure differences get bigger, and the wave grows faster. The instability causes the wave to grow exponentially (Miles, 1957).

3. Finally, the waves begin to interact among themselves to produce longer waves (Hasslemann et al. 1973). The interaction transfers wave energy from short waves generated by Miles' mechanism to waves with frequencies slightly smaller than the frequency of waves at the peak of the spectrum (Figure 16.10). Eventually, this leads to waves going faster than the wind, as noted by Pierson and Moskowitz.

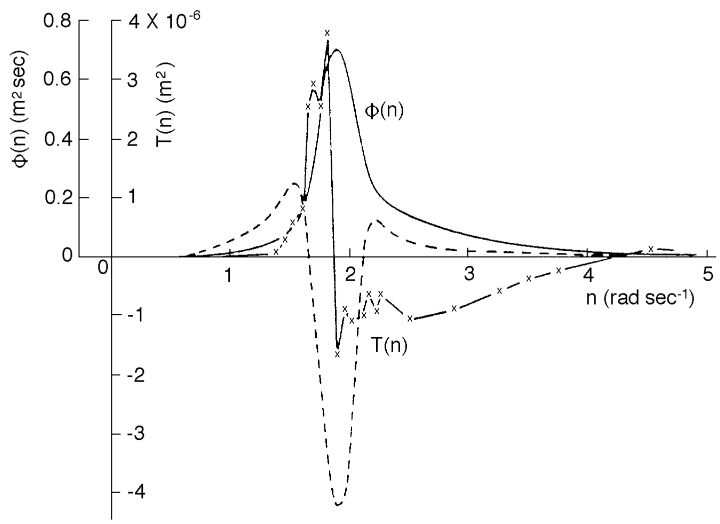

Figure 16.10 The function transfering wave energy from one part of the wave spectrum to another part via nonlinear wave-wave interactions $T(n)$ where n is frequency. Dashed curve is for the theoretical spectrum $\Phi(n)$, $\times - \times$ is for a measured spectrum. The interactions extract energy from high frequency waves and transfer it to low frequency waves, causing the spectrum to change with time and generating waves going faster than the wind, (From Phillips, 1977: 139).

16.5 Wave Forecasting

Our understanding of ocean waves, their spectra, their generation by the wind, and their interactions are now sufficiently well understood that the wave spectrum can be forecast using winds calculated from numerical weather-forecasting models. If we observe some small ocean area, or some area just offshore, we can see waves generated by the local wind, the *wind sea*, plus waves that were generated in other areas at other times and that have propogated into the area we are observing, the *swell*. Forecasts of local wave conditions must include both sea and swell, hence wave forecasting is not a local problem. We saw, for example, that waves off California can be generated by storms more than 10,000 km away.

Various techniques have been used to forecast waves. The earliest attempts were based on empirical relationships between wave height and wave length and wind speed, duration, and fetch. The development of the wave spectrum allowed evolution of individual wave components with frequency f travelling in direction θ of the directional wave spectrum $\psi(f, \theta)$ using

$$\frac{\partial \psi_0}{\partial t} + \mathbf{c_g} \cdot \nabla \psi_0 = S_i + S_{nl} + S_d \tag{16.39}$$

where $\psi_0 = \psi_0(f, \theta; \mathbf{x}, t)$ varies in space ($\mathbf{x}$) and time t, S_i is input from the wind given by the Phillips (1957) and Miles (1957) mechanisms, S_{nl} is the transfer among wave components due to nonlinear interactions (Figure 16.10), and S_d

is dissipation.

The third-generation wave-forecasting models now used by meteorological agencies throughout the world are based on integrations of (16.39) using many individual wave components (The SWAMP Group 1985; The WAMDI Group, 1988; Komen et al, 1994). The forecasts follow individual components of the wave spectrum in space and time, allowing each component to grow or decay depending on local winds, and allowing wave components to interact according to Hasselman's theory. Typically the sea is represented by 300 components: 25 wavelengths going in 12 directions (30°). Each component is allowed to propagate from grid point to grid point, growing with the wind or decaying in time, all the while interacting with other waves in the spectrum. To reduce computing time, the models use a nested grid of points: the grid has a high density of points in storms and near coasts and a low density in other regions. Typically, grid points in the open ocean are 3° apart.

Some recent experimental models take the wave-forecasting process one step further by assimilating altimeter and scatterometer observations of wind speed and wave height into the model. This reduces the errors in the forecasts, but it complicates the calculations. Regional and global forecasts of waves using assimilated satellite data are available from the European Centre for Medium-Range Weather Forecasts. Details of the calculations used for the third-generation models produced by the Wave Analysis Group (WAM) are described in the book by Komen et al. (1994).

NOAA's Ocean Modelling Branch at the National Centers for Environmental Predictions also produces regional and global forecasts of waves. The Branch use a third-generation model based on the Cycle-4 WAM model. It accommodates ever-changing ice edge, and it assimilates buoy and satellite altimeter wave data. The model calculates directional frequency spectra in 12 directions and 25 frequencies at 3-hour intervals up to 72 hours in advance. The lowest frequency is 0.04177 Hz and higher frequencies are spaced logarithmically at increments of 0.1 times the lowest frequency. Wave spectral data are available on a 2.5°× 2.5° grid for deep-water points between 67.5°S and 77.5°N. The model is driven using 10-meter winds calculated from the lowest winds from the Centers weather model adjusted to a height of 10 meter by using a logarithmic profile (8.20). The Branch is testing an improved forecast with 1°× 1.25° resolution (Figure 16.11).

16.6 Measurement of Waves

Because waves influence so many processes and operations at sea, many techniques have been devised for measuring waves. Here are a few of the more commonly used. Stewart (1980) gives a more complete description of wave measurement techniques, including methods for measuring the directional distribution of waves.

Sea State Estimated by Observers at Sea This is perhaps the most common observation included in early tabulations of wave heights. These are the significant wave heights summarized in the U.S. Navy's *Marine Climatological Atlas* and other such reports printed before the age of satellites.

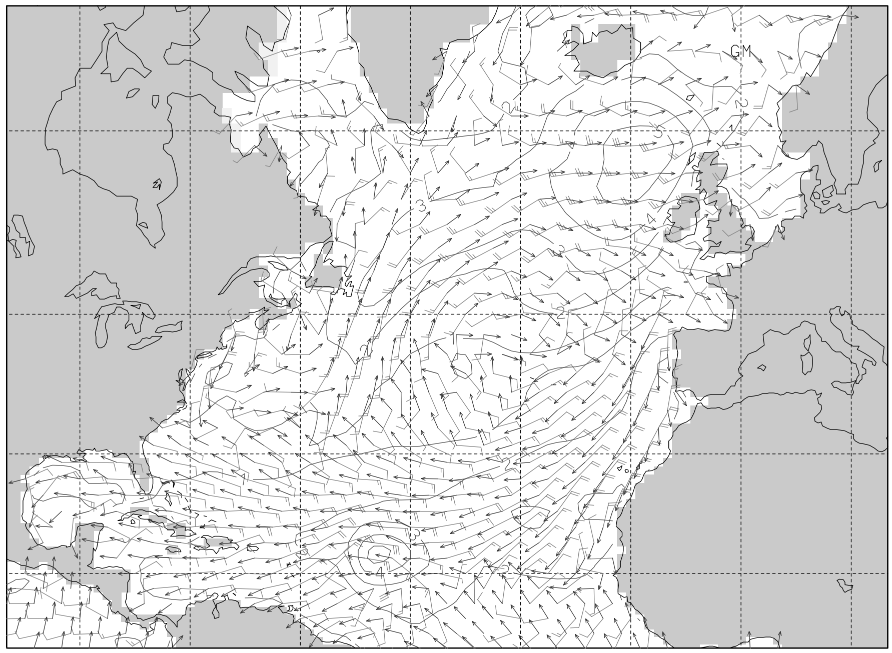

Figure 16.11 Output of a third-generation waveforecast model produced by NOAA's Ocean Modelling Branch for 20 August 1998. Contours are significant wave height in meters, arrows give direction of waves at the peak of the wave spectrum, and barbs give wind speed in m/s and direction. From NOAA Ocean Modelling Branch.

Accelerometer Mounted on Meteorological or Other Buoy This is a less common measurement, although it is often used for measuring waves during short experiments at sea. For example, accelerometers on weather ships measured wave height used by Pierson & Moskowitz and the waves shown in Figure 16.2. The most accurate measurements are made using an accelerometer stabilized by a gyro so the axis of the accelerometer is always vertical.

Double integration of vertical acceleration gives displacement. The double integration, however, amplifies low-frequency noise, leading to the low frequency signals seen in Figures 16.4 and 16.5. In addition, the buoy's heave is not sensitive to wavelengths less than the buoy's diameter, and buoys measure only waves having wavelengths greater than the diameter of the buoy.

Errors are due to failure to measure only vertical acceleration (How to maintain vertical reference?) and drift (low frequency noise) in measurement of acceleration. Overall, careful measurements are accurate to $\pm 10\%$ or better.

Wave Gages Gauges may be mounted on platforms or on the sea floor in shallow water. Many different types of sensors are used to measure the height of the wave or subsurface pressure which is related to wave height. Sound, infrared beams, and radio waves can be used to determine the distance from the sensor to the sea surface provided the sensor can be mounted on a stable platform that does not interfere with the waves. Pressure gauges described in

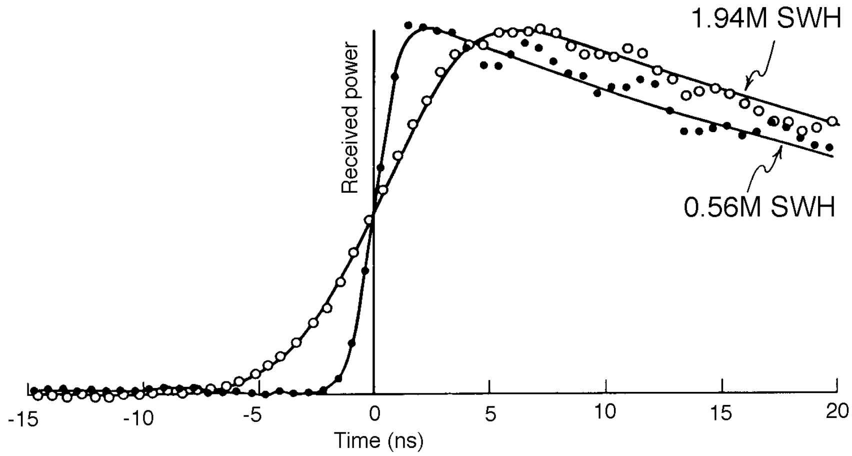

Figure 16.12 Shape of radio pulse received by the Seasat altimeter, showing the influence of ocean waves. The shape of the pulse is used to calculate significant wave height. (From Stewart, 1985).

§6.8 can be used to measure the depth from the sea surface to the gauge. Arrays of bottom-mounted pressure gauges are useful for determining wave directions. Thus arrays are widely used just offshore of the surf zone to determine offshore wave directions.

Pressure gauge must be located within a quarter of a wavelength of the surface because wave-induced pressure fluctuations decrease exponentially with depth. Thus, both gauges and pressure sensors are restricted to shallow water or to large platforms on the continental shelf. Again, accuracy is $\pm\,10\%$ or better.

Satellite Altimeters The satellite altimeters used to measure surface geostrophic currents also measure wave height. Altimeters were flown on Seasat in 1978, Geosat from 1985 to 1988, ERS–1 &2 from 1991, and Topex/Poseidon from 1992. Altimeter data have been used to produce monthly mean maps of wave heights and the variability of wave energy density in time and space. The next step, just begun, is to use altimeter observation with wave forecasting programs, to increase the accuracy of wave forecasts.

The altimeter technique works as follows. Radio pulse from a satellite altimeter reflect first from the wave crests, later from the wave troughs. The reflection stretches the altimeter pulse in time, and the stretching is recorded and used to calculate wave height (Figure 16.12). Accuracy is $\pm 10\%$.

Synthetic Aperture Radars on Satellites These radars map the radar reflectivity of the sea surface with spatial resolution of 6–25 m. Maps of reflectivity often show wave-like features related to the real waves on the sea surface. I say 'wave-like' because there is not an exact one-to-one relationship between wave height and image density. Some waves are clearly mapped, others less so. The maps, however, can be used to obtain additional information about waves, especially the spatial distribution of wave directions in shallow water (Vesecky and

Stewart, 1982). Because the directional information can be calculated directly from the radar data without the need to calculate an image (Hasselmann, 1991), data from radars and altimeters on ERS–1 & 2 are being used to determine if the radar and altimeter observations can be used directly in wave forecast programs.

16.7 Important Concepts

1. Wavelength and frequency of waves are related through the dispersion relation.
2. The velocity of a wave phase can differ from the velocity at which wave energy propagates.
3. Waves in deep water are dispersive, longer wavelengths travel faster than shorter wavelengths. Waves in shallow water are not dispersive.
4. The dispersion of ocean waves has been accurately measured, and observations of dispersed waves can be used to track distant storms.
5. The shape of the sea surface results from a linear superposition of waves of all possible wavelengths or frequencies travelling in all possible directions.
6. The spectrum gives the contributions by wavelength or frequency to the variance of surface displacement.
7. Wave energy is proportional to variance of surface displacement.
8. Digital spectra are band limited, and they contain no information about waves with frequencies higher than the Nyquist frequency.
9. Waves are generated by wind. Strong winds of long duration generate the largest waves.
10. Various idealized forms of the wave spectrum generated by steady, homogeneous winds have been proposed. Two important forms are the Pierson-Moskowitz and JONSWAP spectra.
11. Observations by mariners on ships and by satellite altimeters have been used to make global maps of wave height. Wave gauges are used on platforms in shallow water and on the continental shelf to measure waves. Bottom-mounted pressure gauges are used to measure waves just offshore of beaches. And synthetic-aperture radars are used to obtain information about wave directions.