Chapter 12

Vorticity in the Ocean

Most of the fluid flows with which we are familiar, from bathtubs to swimming pools, are not rotating, or they are rotating so slowly that rotation is not important except maybe at the drain of a bathtub as water is let out. As a result, we do not have a good intuitive understanding of rotating flows. In the ocean, rotation and the conservation of vorticity strongly influence flow over distances exceeding a few tens of kilometers. The consequences of the rotation leads to results we have not seen before in our day-to-day dealings with fluids. For example, did you ask yourself why the curl of the wind stress leads to a mass transport in the north-south direction and not in the east-west direction? What is special about north-south motion? In this chapter, we will explore some of the consequences of rotation for flow in the ocean.

12.1 Definitions of Vorticity

In simple words, vorticity is the rotation of the fluid. The rate of rotation can be expressed various ways. Consider a bowl of water sitting on a table in a laboratory. The water may be spinning in the bowl. In addition to the spinning of the water, the bowl and the laboratory are rotating because they are on a rotating Earth. The two processes are separate, and we can consider two types of vorticity.

Planetary Vorticity Everything on Earth, including the oceans, the atmosphere, and bowls of water rotates with the Earth. This rotation imparted by Earth is the *planetary vorticity* f. It is twice the local rate of rotation of Earth:

$$f \equiv 2\,\Omega\sin\varphi \tag{12.1}$$

Planetary vorticity is the Coriolis parameter we used earlier to discuss flow in the ocean. It is greatest at the poles where it is twice the rotation rate of Earth. Note that the vorticity vanishes at the equator and that the vorticity in the southern hemisphere is negative because φ is negative.

Relative Vorticity The ocean and atmosphere do not rotate at exactly the same rate as Earth. They have some rotation relative to Earth due to currents

and winds. Relative vorticity ζ is the vorticity due to currents in the ocean. Mathematically it is:

$$\zeta \equiv \operatorname{curl}_{z} \mathbf{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$
(12.2)

where $\mathbf{V} = (u, v)$ is the horizontal velocity vector, and where we have assumed that the flow is two-dimensional. This is true if the flow extends over distances greater than a few tens of kilometers. ζ is the vertical component of the threedimensional vorticity vector ω , and it is sometimes written ω_z . ζ is positive for counter-clockwise rotation viewed from above. This is the same sense as Earth's rotation in the northern hemisphere.

Note on Symbols Symbols commonly used in one part of oceanography often have very different meaning in another part. Here we use ζ for vorticity, but in Chapter 10, we used ζ to mean the height of the sea surface. We could use ω_z for relative vorticity, but ω is also commonly used to mean frequency in radians per second. I have tried to eliminate most confusing useage, but the dual use of ζ is one we will have to live with. Fortunately, it shouldn't cause much confusion.

For a rigid body rotating at rate Ω , curl $\mathbf{V} = 2 \Omega$. Of course, the flow does not need to rotate as a rigid body to have relative vorticity. Vorticity can also result from shear. For example, at a north/south western boundary in the ocean, u = 0, v = v(x) and $\zeta = \partial v(x)/\partial x$.

 ζ is usually much smaller than f, and it is greatest at the edge of fast currents such as the Gulf Stream. To obtain some understanding of the size of ζ , consider the edge of the Gulf Stream off Cape Hatteras where the velocity decreases by 1 m/s in 100 km at the boundary. The curl of the current is approximately (1 m/s)/(100 km) = 0.13 cycles/day = 1 cycle/week. Hence even this large relative vorticity is still almost seven times smaller than f. More typical values of relative vorticity, such as the vorticity of eddies, is a cycle per month.

Absolute Vorticity The sum of the planetary and relative vorticity is called *absolute vorticity*:

Absolute Vorticity
$$\equiv (\zeta + f)$$
 (12.3)

We can obtain an equation for absolute vorticity in the ocean by a simple manipulation of the equations of motion for frictionless flow. We begin with:

$$\frac{Du}{Dt} - f v = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$
(12.4a)

$$\frac{Dv}{Dt} + f u = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$
(12.4b)

If we expand the substantial derivative, and if we subtract $\partial/\partial y$ of (12.4a) from $\partial/\partial x$ of (12.4b), we obtain after some algebraic manipulations:

$$\frac{D}{Dt}\left(\zeta+f\right) + \left(\zeta+f\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$
(12.5)



Figure 12.1 Sketch of fluid flow used for deriving conservation of potential vorticity (From Cushman-Roisin, 1994).

In deriving (12.15) we used:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} = \beta v$$

recalling that f is independent of time t and eastward distance x.

Potential Vorticity The rotation rate of a column of fluid changes as the column is expanded or contracted. This changes the vorticity through changes in ζ . To see how this happens, consider barotropic, geostrophic flow in an ocean with depth H(x, y, t), where H is the distance from the sea surface to the bottom. That is, we allow the surface to have topography (figure 12.1).

Integrating the continuity equation (7.19) from the bottom to the top of the ocean gives (Cushman-Roisin, 1994):

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \int_{b}^{b+H} dz + w \Big|_{b}^{b+H} = 0$$
(12.6)

where b is the topography of the bottom, and H is the depth of the water. The boundary conditions require that flow at the surface and the bottom be along the surface and the bottom. Thus the vertical velocities at the top and the bottom are:

$$w(b+H) = \frac{\partial(b+H)}{\partial t} + u \frac{\partial(b+H)}{\partial x} + v \frac{\partial(b+H)}{\partial y}$$
(12.7)

$$w(b) = u \frac{\partial(b)}{\partial x} + v \frac{\partial(b)}{\partial y}$$
(12.8)

Substituting (12.7) and (12.8) into (12.6) we obtain

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{D}\frac{DH}{Dt} = 0$$

Substituting this into (12.5) gives:

$$\frac{D}{Dt}\left(\zeta+f\right) - \frac{\left(\zeta+f\right)}{H}\frac{DH}{Dt} = 0$$

which can be written:

$$\frac{D}{Dt}\left(\frac{\zeta+f}{H}\right) = 0$$

The quantity within the parentheses must be constant; and it is called *potential* vorticity Π . Potential vorticity is conserved along a fluid trajectory:

Potential Vorticity =
$$\Pi \equiv \frac{\zeta + f}{H}$$
 (12.9)

For baroclinic flow in a continuously stratified fluid, the potential vorticity can be written (Pedlosky, 1987: §2.5):

$$\Pi = \frac{\zeta + f}{\rho} \cdot \nabla \lambda \tag{12.10}$$

where λ is any conserved quantity for each fluid element. In, particular, if $\lambda = \rho$ then:

$$\Pi = \frac{\zeta + f}{\rho} \frac{\partial \rho}{\partial z} \tag{12.11}$$

assuming the horizontal gradients of density are small compared with the vertical gradients, a good assumption in the thermocline. In most of the interior of the ocean, $f \gg \zeta$ and (12.11) is written (Pedlosky, 1996, eq 3.11.2):

$$\Pi = \frac{f}{\rho} \frac{\partial \rho}{\partial z} \tag{12.12}$$

This allows the potential vorticity of various layers of the ocean to be determined directly from hydrographic data without knowledge of the velocity field.

12.2 Conservation of Vorticity

The angular momentum of any isolated spinning body is conserved. The spinning body can be an eddy in the ocean or the Earth in space. If the the spinning body is not isolated, that is, if it is linked to another body, then angular momentum can be transfered between the bodies. The two bodies need not be in physical contact. Gravitational forces can transfer momentum between bodies in space. We will return to this topic in Chapter 17 when we discuss tides in the ocean. Here, let's look at conservation of vorticity in a spinning ocean.

Friction is essential for the transfer of momentum in a fluid. Friction transfers momentum from the atmosphere to the ocean through the thin, frictional, Ekman layer at the sea surface. Friction transfers momentum from the ocean



Figure 12.2 Sketch of the production of relative vorticity by the changes in the height of a fluid column. Left: vertical stretching reduces the moment of inertia of the column, causing it to spin faster; **Right:** vetical shrinking increases the moment of inertia of the column, causing it to spin slower (from von Arx, 1962).

to the solid earth through the Ekman layer at the sea floor. Friction along the sides of subsea mountains leads to pressure differences on either side of the mountain which causes another form of drag called *form drag*. This is the same drag that causes wind force on cars moving at high speed. In the vast interior of the ocean, however, the flow is frictionless, and vorticity is conserved. Such a flow is said to be *conservative*.

Conservation of Potential Vorticity The conservation of potential vorticity couples changes in depth, relative vorticity, and changes in latitude. All three interact.

- 1. Changes in the depth H of the flow causes changes in the relative vorticity. The concept is analogous with the way figure skaters decreases their spin by extending their arms and legs. The action increases their moment of inertia and decreases their rate of spin (Figure 12.2).
- 2. Changes in latitude require a corresponding change in ζ . As a column of water moves equatorward, f decreases, and ζ must increase (Figure 12.3). If this seems somewhat mysterious, von Arx (1962) suggests we consider a barrel of water at rest at the north pole. If the barrel is moved southward, the water in it retains the rotation it had at the pole, and it will appear to rotate counterclockwise at the new latitude where f is smaller.

Consequences of Conservation of Potential Vorticity The concept of conservation of potential vorticity has far reaching consequences, and its application to fluid flow in the ocean gives a deeper understanding of ocean currents.

1. In the ocean f tends to be much larger than ζ and thus f/H = constant. This requires that the flow in an ocean of constant depth be zonal. Of



Figure 12.3 Angular momentum tends to be conserved as columns of water change latitude. This causes changes in relative vorticity of the columns (from von Arx, 1962).

course, depth is not constant, but in general, currents tend to be eastwest rather than north south. Wind makes small changes in ζ , leading to a small meridional component to the flow (see figure 11.3).

- 2. Barotropic flows are diverted by seafloor features. Consider what happens when a flow that extends from the surface to the bottom encounters a sub-sea ridge (Figure 12.4). As the depth decreases, $\zeta + f$ must also decrease, which requires that f decrease, and the flow is deflected toward the equator. This is called *topographic steering*. If the change in depth is sufficiently large, no change in latitude will be sufficient to conserve potential vorticity, and the flow will be unable to cross the ridge. This is called *topographic blocking*.
- 3. The conservation of vorticity provides an alternate explanation for the existance of western boundary currents (Figure 12.5). Consider the gyrescale flow in an ocean basin, say in the North Atlantic from 10°N to 50°N. The wind blowing over the Atlantic adds negative vorticity. As the



Figure 12.4 Barotropic flow over a sub-sea ridge is deflected equatorward to conserve potential vorticity (from Dietrich, *et al.*, 1980).



Figure 12.5 Conservation of potential vorticity can clarify why western boundary currents are necessary. Left: Vorticity input by the wind ζ_{τ} balances the change in potential vorticity Π in the east as the flow moves southward and f decreases; but the two do not balance in the west where Π must decrease as the flow moves northward and f increases. Right: Vorticity in the west is balanced by relative vorticity ζ generated by shear in the western boundary current.

water flows around the gyre, the vorticity of the gyre must remain nearly constant, else the flow would spin up or slow down. The negative vorticity input by the wind must be balanced by a source of positive vorticity.

The source of positive vorticity must be boundary currents: the winddriven flow is baroclinic, which is weak near the bottom, and bottom friction cannot transfer vorticity out of the ocean. Hence, we must decide which boundary contributes. Flow tends to be zonal, and east-west boundaries will not solve the problem. In the east, potential vorticity is conserved: the input of negative relative vorticity is balanced by a decrease in potential vorticity as the flow turns southward. Only in the west is vorticity not conserved, and a strong source of positive vorticity is required. The vorticity is provided by the current shear in the western boundary current as the current rubs against the coast causing the northward velocity to go to zero at the coast (Figure 12.5, right).

In this example, friction transfers angular momentum from the wind to the ocean and eddy viscosity—friction—transfers angular momentum from the ocean to the solid earth.

12.3 Vorticity and Ekman Pumping

Rotation places another very interesting constraint on the geostrophic flow field. To help understand the constraints, let's first consider flow in a fluid with constant rotation. Then we will look into how vorticity constrains the flow of a fluid with rotation that varies with latitude. An understanding of the constraints leads to a deeper understanding of Sverdrup's and Stommel's results descussed in the last chapter.

Fluid dynamics on the f plane: the Taylor-Proudman Theorem The influence of vorticity due to Earth's rotation is most striking for geostrophic flow of a fluid with constant density ρ_0 on a plane with constant rotation $f = f_0$. From Chapter 10, the three components of the geostrophic equations and the

continuity equations are:

$$f v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x}$$
(12.13a)

$$f u = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} \tag{12.13b}$$

$$g = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} \tag{12.13c}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
(12.13d)

Taking the z derivative of (12.13a) and using (12.13c) gives:

$$-f_0 \frac{\partial v}{\partial z} = -\frac{1}{\rho_0} \frac{\partial}{\partial z} \left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x} \left(-\frac{1}{\rho_0} \frac{\partial p}{\partial z}\right) = \frac{\partial g}{\partial x} = 0$$
$$f_0 \frac{\partial v}{\partial z} = 0$$
$$\therefore \quad \frac{\partial v}{\partial z} = 0$$

Similarly, for the u-component of velocity (12.13b). Thus, the vertical derivative of the horizontal velocity field must be zero.

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \tag{12.14}$$

This is the *Taylor-Proudman Theorem*, which applies to slowly varying flows in a homogeneous, rotating, inviscid fluid. The theorem places strong constraints on the flow:

If therefore any small motion be communicated to a rotating fluid the resulting motion of the fluid must be one in which any two particles originally in a line parallel to the axis of rotation must remain so, except for possible small oscillations about that position—Taylor (1921).

Hence, geostrophic flow past a seamount requires that the flow go around the seamount; it cannot go over the seamount. Taylor (1921) explicitly derived (12.14) and (12.16) below. Proudman (1916) independently derived the same theorem but not as explicitly.

Further consequences of the theorem can be obtained by eliminating the pressure terms from (12.13a & 12.13b) to obtain:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial}{\partial x} \left(\frac{1}{f_0 \rho_0} \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{1}{f_0 \rho_0} \frac{\partial p}{\partial x} \right)$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{f_0 \rho_0} \left(-\frac{\partial^2 p}{\partial x \partial y} + \frac{\partial^2 p}{\partial x \partial y} \right)$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
(12.15)

Because the fluid is incompressible, the continuity equation (12.13d) requires

$$\frac{\partial w}{\partial z} = 0 \tag{12.16}$$

Furthermore, because w = 0 at the sea surface and at the sea floor, if the bottom is level, there can be no vertical velocity on an f-plane. Note that the derivation of (12.16) did not require that density be constant. It requires only slow motion in a frictionless, rotating fluid.

Fluid Dynamics on the beta plane: Ekman Pumping If (12.16) is true and there can be no gradient of vertical velocity in an ocean with constant planetary vorticity, how then can the divergence of the Ekman transport at the sea surface lead to vertical velocities at the surface or at the base of the Ekman layer? The answer can only be that one of the constraints used in deriving (12.16) must be violated. One constraint that can be relaxed is the requirement that $f = f_0$.

Consider then flow on a beta plane. If $f = f_0 + \beta y$, then (12.13d) becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{f\rho_0} \frac{\partial^2 p}{\partial x \partial y} + \frac{1}{f\rho_0} \frac{\partial^2 p}{\partial x \partial y} - \frac{\beta}{f} \frac{1}{f\rho_0} \frac{\partial p}{\partial x}$$
(12.17)

$$f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\beta v \tag{12.18}$$

where we have used (12.13) to obtain v in the right-hand side of (12.18).

Using the continuity equation, and recalling that $\beta y \ll f_0$

$$f_0 \frac{\partial w_G}{\partial z} = \beta \, v \tag{12.19}$$

where we have used the subscript G to emphasize that (12.19) applies to the ocean's interior, geostrophic flow. Thus the variation of Coriolis force with latitude allows vertical velocity gradients in the geostrophic interior of the ocean, and the vertical velocity leads to north-south currents. This explains why Sverdrup and Stommel both needed to do their calculations on a β -plane. Wind stress curl cannot produce vertical currents unless f varies with latitude.

Ekman Pumping in the Ocean In Chapter 9, we saw that the curl of the wind stress T produced a divergence of the Ekman transports leading to a vertical velocity $w_E(0)$ at the top of the Ekman layer.

$$w_E(0) = -\operatorname{curl}\left(\frac{T}{\rho f}\right) \tag{12.20}$$

where ρ is density and f is the Coriolis parameter. Because the vertical velocity at the sea surface must be zero, the Ekman vertical velocity must be balanced by a vertical geostrophic velocity $w_G(0)$.

$$w_E(0) = -w_G(0) = -\operatorname{curl}\left(\frac{T}{\rho f}\right)$$
(12.21)



Figure 12.6 Winds at the sea surface drive Ekman transports to the right of the wind in this northern hemisphere example (bold arrows in shaded Ekman layer). The converging Ekman transports driven by the trades and westerlies drives a downward geostrophic flow just below the Ekman layer (bold vertical arrors), leading to downward bowing constant density surfaces ρ_i . Geostrophic currents associated with the warm water are shown by bold arrows. (After Tolmazin, 1985)

To see how this works in practice, let's look at how Ekman pumping drives the geostrophic flow in the interior of the ocean. Consider the mid-latitude winds in an ocean basin in the northern hemisphere (Figure 12.6). Wind stress at the sea surface drives an Ekman mass transport to the right of the wind. The westerlies drive a southward transport, the trades drive a northward transport. The converging Ekman transports must be balanced by downward velocity at the top of the geostrophic layer just under the Ekman layer.

Because the water just below the Ekman layer is warmer than the deeper water, the vertical velocity produces a dome of warm water shown by the downward bending surfaces of constant density. The density distribution produces north-south pressure gradients that must be balanced by east-west geostrophic currents. In short, the divergence of the Ekman transports redistributes mass within the frictionless interior of the ocean leading to the wind-driven geostrophic currents.

The wind-driven geostrophic currents are baroclinic. Their influence does not extend much below 1 km. As a result, there is a level of no motion indicated by a level density surface in the figure, and the sea surface must bow upward above the warm water.

We finish this example by noting that conservation of vorticity within the geostrophic interior of the ocean in this example leads to a Sverdrup transport to the south. Why this is so is explained by Niiler (1987: 16).

Let us postulate there exists a deep level where horizontal and vertical motion of the water is much reduced from what it is just below the mixed layer [Figure 12.7]... Also let us assume that vorticity is conserved there (or mixing is small) and the flow is so slow that accelerations over the



Figure 12.7 Ekman pumping that produces a downward velocity at the base of the Ekman layer forces the fluid in the interior of the ocean to move southward. (From Niiler, 1987)

earth's surface are much smaller than Coriolis accelerations. In such a situation a column of water of depth H will conserve its spin per unit volume, f/H (relative to the sun, parallel to the earth's axis of rotation). A vortex column which is compressed from the top by wind-forced sinking (H decreases) and whose bottom is in relatively quiescent water would tend to shorten and slow its spin. Thus because of the curved ocean surface it has to move southward (or extend its column) to regain its spin. Therefore, there should be a massive flow of water at some depth below the surface to the south in areas where the surface layers produce a sinking motion and to the north where rising motion is produced. This phenomenon was first modeled correctly by Sverdrup (1947) (after he wrote "Oceans") and gives a dynamically plausible explanation of how wind produces deeper circulation in the ocean.

Ekman Pumping: An Example Let's now apply Ekman pumping ideas to flow in the North Pacific to see how winds can produce currents flowing upwind. We read in §11.1 how Sverdrup's theory led to a description of currents in the northeast Pacific, including the north equatorial countercurrent flows upwind. The description, however, was theoretical, and it provided little insight into the mechanisms at work.

To gain insight, look at Figure 12.8. It shows the mean zonal winds in the Pacific, together with the north-south Ekman transports driven by the zonal winds. Notice that convergence of transport leads to downwelling, which produces a thick layer of warm water in the upper kilometer of the water column, and high sea level. Figure 12.6, which is a schematic cross section of the region between 10°N and 60°N shows the pool of warm water in the upper kilometer of the Pacific centered on 30°N. Conversely, divergent transports leads to low sea level. The mean north-south pressure gradients are balanced by the Coriolis force of east-west geostrophic currents in the upper ocean.



Figure 12.8 An example of how winds produce geostrophic currents running upwind. Ekman

transports due to winds in the north Pacific (Left) lead to Ekman pumping (Center), which sets up north-south pressure gradients in the upper ocean. The pressure gradients are balanced by the Coriolis force due to east-west geostrophic currents (**Right**). Horizontal lines indicate regions where the curl of the zonal wind stress changes sign.

12.4 Important Concepts

- 1. Vorticity strongly constrains ocean dynamics.
- 2. The oceans have large vorticity because they are on Earth, which rotates once per day. This planetary vorticity is much larger than other sources of vorticity.
- 3. Taylor and Proudman showed that vertical velocity is impossible in a uniformly rotating flow. Hence Ekman pumping requires that vorticity vary with latitude. This explains why Sverdrup and Stommel found that realistic oceanic circulation, which is driven by Ekman pumping, requires that f vary with latitude.
- 4. The curl of the wind stress adds relative vorticity central gyres of each ocean basin. For steady state circulation in the gyre, the ocean must lose vorticity in western boundaru currents.
- 5. Positive wind stress curl leads to divergent flow in the Ekman layer. The ocean's interior geostrophic circulation adjusts through a northward mass transport.
- 6. Conservation of absolute vorticity in an ocean with constant density leads to the conservation of potential vorticity. Thus changes in depth in an ocean of constant density requires changes of latitude of the current.