Chapter 8

Equations of Motion With Viscosity

Throughout most of the interior of the ocean and atmosphere friction is relatively small, and we can safely assume that the flow is frictionless. At the boundaries, friction, in the form of viscosity, becomes important. This thin, viscous layer is called a *boundary layer*. Within the layer, the velocity of the flow slows from values typical of the interior to zero at a solid boundary. If the boundary is not solid, then the boundary layer is a thin layer of rapidly changing velocity whereby velocity on one side of the boundary changes to match the velocity on the other side of the boundary. For example, there is a boundary layer at the bottom of the atmosphere, the planetary boundary layer we described in Chapter 3. In the planetary boundary layer, velocity goes from many meters per second in the free atmosphere to tens of centimeters per second at the sea surface. Below the sea surface, another boundary layer, the Ekman layer described in Chapter 9, matches the flow at the sea surface to the deeper flow inside the ocean.

In this chapter we consider the role of friction in fluid flows, and the stability of the flows to small changes in velocity or density.

8.1 The Influence of Viscosity

In the last chapter we wrote the x-component of the momentum equation for a fluid in the form:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + 2\Omega v \sin \vartheta + F_x$$
(8.1)

where F_x was a frictional force per unit mass. Now we can consider the form of this term if it is due to viscosity.

Molecules in a fluid close to a solid boundary can strike the boundary and transfer momentum to the boundary (Figure 8.1). Molecules further from the boundary collide with molecules that have struck the boundary, further transferring the change in momentum into the interior of the fluid. This transfer of



Figure 8.1 Molecules colliding with the wall and with each other transfer momentum from the fluid to the wall, slowing the fluid velocity.

momentum is *molecular viscosity*. Molecules, however, travel only micrometers between collisions, and the process is very inefficient for transferring momentum even a few centimeters. Molecular viscosity is important only within a few millimeters of a boundary.

Molecular viscosity is the ratio of the stress T_x tangential to the boundary of a fluid and the shear of the fluid at the boundary. So the stress has the form:

$$T_x = \rho \nu \, \frac{\partial u}{\partial z} \tag{8.2}$$

where ν is the kinematic molecular viscosity. Typical value of ν for water at 20°C is 10⁻⁶ m²/s.

Generalizing (8.2) to three dimensions leads to a stress tensor giving the nine components of stress at a point in the fluid, including pressure, which is a normal stress, and shear stresses. A derivation of the stress tensor is beyond the scope of this book, but you can find the details in Lamb (1945: §328) or Kundu (1990: p. 93). For an incompressible fluid, the frictional force per unit mass in (8.1) takes the from:

$$F_x = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$
(8.3)

8.2 Turbulence

If molecular viscosity is important only over distances of a few millimeters, and if it is not important for most oceanic flows, unless of course you are a zooplankter trying to swim in the ocean, how then is the influence of a boundary transferred into the interior of the flow? The answer is: through turbulence.

Turbulence arises from the non-linear terms in the momentum equation $(u \partial u/\partial x, etc.)$. The importance of these terms is given by a non-dimensional number, the Reynolds Number, which is the ratio of the non-linear terms to the viscous terms:

Reynolds Number = Re =
$$\frac{\text{Non-linear Terms}}{\text{Viscous Terms}} = \frac{\left(u\frac{\partial u}{\partial x}\right)}{\left(\nu\frac{\partial^2 u}{\partial x^2}\right)} \approx \frac{U\frac{U}{L}}{\nu\frac{U}{L^2}}$$



Figure 8.2 **Upper:** Reynolds apparatus for investigating the transition to turbulence in pipe flow. **Lower:** Photographs of near-laminar flow (left) and turbulent flow (right) in a clear pipe much like the one used by Reynolds. (From Binder 1953).

$$Re = \frac{UL}{\nu} \tag{8.4}$$

where, U is a typical velocity of the flow and L is a typical length describing the flow. You are free to pick whatever U, L might be typical of the flow. For example L can be either a typical cross-stream distance, or an along-stream distance. Typical values in the open ocean are U = 0.1 m/s and L = 1 megameter, so $Re = 10^{11}$. Because non-linear terms are important if Re > 10 - 1000, they are certainly important in the ocean. The ocean is turbulent.

The Reynolds number is named after Osborne Reynolds (1842–1912) who conducted experiments in the late 19th century to understand turbulence. In one famous experiment (Reynolds 1883), he injected dye into water flowing at various speeds through a tube (Figure 8.2). If the speed was small, the flow was smooth. This is called *laminar flow*. At higher speeds, the flow became irregular and turbulent. The transition occurred at $\text{Re} = VD/\nu \approx 2000$, where V is the average speed in the pipe, and D is the diameter of the pipe.

As Reynolds number increases above some critical value, the flow becomes more and more turbulent. Note that flow pattern is a function of Reynold's number. All flows with the same geometry and the same Reynolds number have the same flow pattern. Thus flow around all circular cylinders, whether 1 mm or 1 m in diameter, look the same as the flow at the top of Figure 8.3 if the Reynolds number is 20. Furthermore, the boundary layer is confined to a very thin layer close to the cylinder, in a layer too thin to show in the figure.

Turbulent Stresses: The Reynolds Stress Those who studied fluid mechanics in the early 20th century hypothesized that parcels of fluid in a turbulent flow played the same role in transferring momentum within the flow that



Figure 8.3 Flow past a circular cylinder as a function of Reynolds number between one and a million (From Richardson 1961). The appropriate flows are: A—a toothpick moving at 1 mm/s; B—finger moving at 2 cm/s; F—hand out a car window at 60 mph. All flow at the same Reynolds number has the same streamlines. Flow past a 10 cm diameter cylinder at 1 cm/s looks the same as 10 cm/s flow past a cylinder 1 cm in diameter because in both cases Re = 1000.

molecules played in laminar flow. The work led to the idea of turbulent stresses.

To see how these stresses might arise, consider the momentum equation for a flow with mean and a turbulent components of flow:

$$u = U + u'; \quad v = V + v'; \quad w = W + w'; \quad p = P + p'$$
 (8.5)

where the mean value U is calculated from a time or space average:

$$\langle u \rangle = \frac{1}{T} \int_0^T u(t) dt \quad \text{or} \quad \langle u \rangle = \frac{1}{X} \int_0^X u(x) dx$$
 (8.6)

The non-linear terms in the momentum equation can be written:

$$\left\langle (U+u')\frac{\partial(U+u')}{\partial x}\right\rangle = \left\langle U\frac{\partial U}{\partial x}\right\rangle + \left\langle U\frac{\partial u'}{\partial x}\right\rangle + \left\langle u'\frac{\partial U}{\partial x}\right\rangle + \left\langle u'\frac{\partial u'}{\partial x}\right\rangle$$
$$\left\langle (U+u')\frac{\partial(U+u')}{\partial x}\right\rangle = \left\langle U\frac{\partial U}{\partial x}\right\rangle + \left\langle u'\frac{\partial u'}{\partial x}\right\rangle$$
(8.7)

The second equation follows from the first because both $\langle U \partial u' / \partial x \rangle = 0$ and $\langle u' \partial U / \partial x \rangle = 0$, which follow from the definition of U: $\langle U \partial u' / \partial x \rangle = U \partial \langle u' \rangle / \partial x = 0$.

Using (8.7), the continuity equation splits into two equations:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial x} + \frac{\partial W}{\partial x} = 0$$
(8.8a)

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial x} + \frac{\partial w'}{\partial x} = 0$$
(8.8b)

And the x-component of the momentum equation becomes:

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + 2\Omega V \sin \varphi - \frac{\partial}{\partial x} \langle u'u' \rangle - \frac{\partial}{\partial y} \langle u'v' \rangle - \frac{\partial}{\partial z} \langle u'w' \rangle$$
(8.9)

where $2\Omega v' \sin \varphi$ has been dropped because it is small. Thus the turbulent frictional term is

$$F_x = -\frac{\partial}{\partial x} \langle u'u' \rangle - \frac{\partial}{\partial y} \langle u'v' \rangle - \frac{\partial}{\partial z} \langle u'w' \rangle$$
(8.10)

The terms $\rho \langle u'u' \rangle$, $\rho \langle u'v' \rangle$, and $\rho \langle u'w' \rangle$ transfer momentum in the x, y, and z directions. For example, the term $\rho \langle u'w' \rangle$ gives the downward transport of eastward momentum across a horizontal plane. Because they transfer momentum, and because they were first derived by Osborne Reynolds, they are called *Reynolds Stresses*.

8.3 Calculation of Reynolds Stress:

The friction terms such as $\partial \langle u'w' \rangle / \partial z$ are virtual stresses (cf. Goldstein, 1965: 69 & 80). We now assume that they play the same role as the viscous terms in the equation of motion. The problem becomes one of obtaining values or functional form for the Reynolds stress. Several approaches are used.

By Analogy with Molecular Viscosity Let's return to the simple example shown in figure 8.1, which shows a boundary layer above a flat plate in the x, y plane. Now let's assume that the flow above the plate is turbulent. This is a very common type of boundary layer flow, and it a type of flow that we will describe various times in later chapters. It can be wind flow above the sea surface or flow at the bottom boundary layer in the ocean or flow in the mixed layer at the sea surface.

For flow above a boundary, we assume that flow is constant in the x, y direction, that the statistical properties of the flow vary only in the z direction, and that the mean flow is steady. Therefore $\partial/\partial t = \partial/\partial x = \partial/\partial y = 0$. The stress term is:

$$T_x = \rho \langle u'w' \rangle, \quad T_y = \rho \langle v'w' \rangle$$

$$(8.11)$$

and

$$F_x = -\frac{1}{\rho} \frac{\partial T_x}{\partial z} = -\frac{\partial}{\partial z} \langle u'w' \rangle, \quad F_y = -\frac{1}{\rho} \frac{\partial T_y}{\partial z} = -\frac{\partial}{\partial z} \langle v'w' \rangle$$
(8.12)

We now assume, in analogy with (8.2)

$$T_x = \rho A_z \frac{\partial U}{\partial z} \tag{8.13a}$$

$$T_y = \rho A_z \frac{\partial V}{\partial z} \tag{8.13b}$$

where A_z is the *eddy viscosity* which replaces the molecular viscosity ν in (8.2). Then

$$F_x = \frac{1}{\rho} \frac{\partial T_x}{\partial z} = \frac{\partial}{\partial z} \left(A_z \frac{\partial U}{\partial z} \right) \approx A_z \frac{\partial^2 U}{\partial z^2}$$
(8.14a)

$$F_y = \frac{1}{\rho} \frac{\partial T_y}{\partial z} = \frac{\partial}{\partial z} \left(A_z \frac{\partial V}{\partial z} \right) \approx A_z \frac{\partial^2 V}{\partial z^2}$$
(8.14b)

assuming A_z is either constant or that it varies more slowly in the z direction than $\partial U/\partial z$. Thus, we will assume later that $A_z \approx z$.

With these assumptions, the x and y momentum equations are:

$$\rho f V - \frac{\partial T_x}{\partial z} = 0 \tag{8.15a}$$

$$-\rho f U - \frac{\partial T_y}{\partial z} = 0 \tag{8.15b}$$

where $f = 2\omega \sin \varphi$ is the Coriolis parameter.

The assumption that A_z varies with distance from the boundary works well for describing the flow over flat plates where U is a function of distance z from the plate, and W, the mean velocity perpendicular to the plate is zero (See the box *Turbulent Boundary Layer Over a Flat Plate*). This is the classical approach first described in 1925 by Prandtl, who introduced the concept of a boundary layer, and by others. Here A_z is determined by an empirical fit to data collected in wind tunnels or measured in the surface boundary layer at sea. See Hinze (1975, §5–2 and §7–5) and Goldstein (1965: §80) for more on the theory of turbulence flow near a flat plate.

Assumption (8.13) and the classical theory works well only where friction is much larger than the Coriolis force. This is true for air flow within tens of meters of the sea surface and for water flow within a few meters of the surface. The application of the technique to other flows in the ocean is less clear. For example, the flow in the mixed layer at depths below about ten meters is less well described by the classical turbulent theory. Tennekes and Lumley (1970: 57) write:

Mixing-length and eddy viscosity models should be used only to generate analytical expressions for the Reynolds stress and mean-velocity

The Turbulent Boundary Layer Over a Flat Plate

The theory for the mean velocity distribution in a turbulent boundary layer over a flat plate was worked out independently by G.I. Taylor (1886–1975), L. Prandtl (1875–1953), and T. von Karman (1818–1963) from 1915 to 1935. Their empirical theory, sometimes called the *mixing-length theory* predicts well the mean velocity profile close to the boundary. Of interest to us, it predicts the mean flow of air above the sea. Here's a simplified version of the theory applied to a smooth surface.

We begin by assuming that the mean flow in the boundary layer is steady and that it varies only in the z direction. Within a few millimeters of the boundary, friction is important and (8.2) has the solution

$$U = \frac{T_x}{\rho\nu} z \tag{8.16}$$

and the mean velocity varies linearly with distance above the boundary. Usually (8.16) is written in dimensionless form:

$$\frac{U}{u^*} = \frac{u^* z}{\nu} \tag{8.17}$$

where $u^{*2} \equiv T_x / \rho$ is the friction velocity.

Further from the boundary, the flow is turbulent, and molecular friction is not important. In this regime, we can use (8.13), and

$$A_z \frac{\partial U}{\partial z} = u^{*2} \tag{8.18}$$

Prandtl and Taylor assumed that large eddies are more effective in mixing momentum than small eddies, and therefore A_z ought to vary with distance from the wall. Karman assumed that it had the particular functional form $A_z = \kappa z u^*$, where κ is a dimensionless constant. With this assumption, the equation for the mean velocity profile becomes

$$\kappa z u^* \frac{\partial U}{\partial z} = u^{*2} \tag{8.19}$$

Because U is a function only of z, we can write $dU = u^*/(\kappa z) dz$, which has the solution

$$U = \frac{u^*}{\kappa} \ln\left(\frac{z}{z_0}\right) \tag{8.20}$$

where z_0 is distance from the boundary at which velocity goes to zero.

For airflow over the sea, $\kappa = 0.4$ and z_o is given by Charnock's (1955) relation $z_0 = 0.0156 u^{*2}/g$. The mean velocity in the atmospheric boundary layer just above the sea surface described in §4.3 fits well the logarithmic profile of (8.20), as does the mean velocity in the upper few meters of the sea just below the sea surface. Furthermore, using (4.1) in the definition of the friction velocity, then using (8.20) gives Charnock's form of the drag coefficient as a function of wind speed in Figure 4.6.

profile if those are desired for curve fitting purposes in turbulent flows characterized by a single length scale and a single velocity scale. The use of mixing-length theory in turbulent flows whose scaling laws are not known beforehand should be avoided.

Problems with the eddy-viscosity approach:

- 1. Except in boundary layers a few meters thick, geophysical flows may be influenced by several characteristic scales. For example, in the atmospheric boundary layer above the sea, at least three scales may be important: i) the height above the sea z, ii) the Monin-Obukhov scale L discussed in §4.3, and iii) the typical velocity U divided by the Coriolis parameter U/f.
- 2. The velocities u', w' are a property of the fluid, while A_z is a property of the *flow*;
- 3. Eddy viscosity terms are not symmetric:

$$\langle u'v' \rangle = \langle v'u' \rangle;$$
 but
 $A_x \frac{\partial V}{\partial x} \neq A_y \frac{\partial U}{\partial y}$

From a Statistical Theory of Turbulence The Reynolds stresses can be calculated from various theories which relate $\langle u'u' \rangle$ to higher order correlations of the form $\langle u'u'u' \rangle$. The problem then becomes: How to calculate the higher order terms? This is the *closure problem* in turbulence. There is no general solution, but the approach leads to useful understanding of some forms of turbulence such as isotropic turbulence downstream of a grid in a wind tunnel (Batchelor 1967). *Isotropic turbulence* is turbulence with statistical properties that are independent of direction.

The approach can be modified somewhat for flow in the ocean. In the idealized case of high Reynolds flow, we can calculate the statistical properties of a flow in thermodynamic equilibrium. Because the actual flow in the ocean is far from equilibrium, we assume it will evolve towards equilibrium. Holloway (1986) provides a good review of this approach, showing how it can be used to derive the influence of turbulence on mixing and heat transports. One interesting result of the work is that zonal mixing ought to be larger than meridional mixing.

Summary The turbulent eddy viscosities A_x , A_y , and A_z cannot be calculated accurately for most oceanic flows.

- 1. They can be estimated from measurements of turbulent flows. Measurements in the ocean, however, are difficult; and measurements in the lab, although accurate, cannot reach Reynolds numbers of 10^{11} typical of the ocean.
- 2. The statistical theory of turbulence gives useful insight into the role of turbulence in the ocean, and this is an area of active research.

Some Values for Viscosity

$$\nu_{water} = 10^{-6} \text{ m}^2/\text{s}$$
$$\nu_{tar at 15^{\circ}C} = 10^{6} \text{ m}^2/\text{s}$$
$$\nu_{glacier ice} = 10^{10} \text{ m}^2/\text{s}$$
$$A_y = 10^4 \text{ m}^2/\text{s}$$

8.4 Stability

We saw in the last section that fluid flow with a sufficiently large Reynolds number is turbulent. This is one form of instability. Many other types of instability occur in the in the ocean. Here, let's consider three of the more important ones: i) *static stability* associated with change of density with depth, ii) *dynamic stability* associated with velocity shear, and iii) *double-diffusion* associated with salinity and temperature gradients in the ocean.

Static Stability and the Stability Frequency Consider first static stability. If more dense water lies above less dense water, the fluid is unstable. The more dense water will sink beneath the less dense. Conversely, if less dense water lies above more dense water, the interface between the two is stable. But how stable? We might guess that the larger the density contrast across the interface, the more stable the interface. This is an example of static stability. Static stability is important in any *stratified* flow where density increases with depth; and we need some criterion for determining the importance of the stability.



Figure 8.4 Sketch for calculating static stability and stratification frequency.

Consider a parcel of water that is displaced vertically in a stratified fluid (Figure 8.4). The buoyancy force F acting on the displaced parcel is the difference between its weight $Vg\rho'$ and the weight of the surrounding water $Vg\rho_2$, where V is the volume of the parcel:

$$F = V g \left(\rho_2 - \rho'\right)$$

The acceleration of the displaced parcel is:

$$a = \frac{F}{m} = \frac{g(\rho_2 - \rho')}{\rho'}$$
(8.21)

but

$$\rho_2 = \rho + \left(\frac{\partial\rho}{\partial z}\right)_{water} \delta z \tag{8.22}$$

$$\rho' = \rho + \left(\frac{\partial\rho}{\partial z}\right)_{parcel} \delta z \tag{8.23}$$

Using (8.22) and (8.23) in (8.21), ignoring terms proportional to δz^2 , we obtain:

$$a = -\frac{g}{\rho} \left[\left(\frac{\partial \rho}{\partial z} \right)_{water} - \left(\frac{\partial \rho}{\partial z} \right)_{parcel} \right] \delta z$$

We define stability $\equiv E \equiv -a/g$ for $\delta z = 1$:

$$E = -\frac{1}{\rho} \left[\left(\frac{\partial \rho}{\partial z} \right)_{water} - \left(\frac{\partial \rho}{\partial z} \right)_{parcel} \right]$$
(8.24)

In the upper kilometer of the ocean stability is large, and the first term in (8.24) is much larger than the second. The first term is proportional to the rate of change of density of the water column; the second term is proportional to the compressibility of sea water, which is very small. Neglecting the second term, we can write the *stability equation*:

$$E = -\frac{1}{\rho} \frac{\partial \rho}{\partial z}$$
(8.25)

Recalling that $\rho(S, T, 0) - 1000 = \sigma_t$, we can also write:

$$E = -\frac{1}{\rho} \frac{\partial \sigma_t}{\partial z}$$
(8.26)

The approximation used to derive (8.25) and (8.26) is valid for $E > 50 \times 10^{-8}$ /m. Deep in the ocean, the change in density with depth is so small that we must consider the small change in density of the parcel due to changes in pressure as it is moved vertically. At these depths, a more accurate form of (8.26) is:

$$E = -\frac{1}{\rho} \left[\frac{\partial \rho}{\partial S} \frac{\partial S}{\partial z} + \frac{\partial \rho}{\partial \theta} \left(\frac{\partial \theta}{\partial z} + \Gamma \right) \right]$$
(8.27)
$$\Gamma = \left(\frac{\partial \theta}{\partial z} \right)_{parcel}$$

where Γ is the adiabatic temperature gradient. See Sverdrup, Johnson, and Fleming (1942: 416) or Gill (1982: 50) for a more complete derivation.

Stability is defined such that

E > 0 Stable E = 0 Neutral Stability E < 0 Unstable



Figure 8.5. Observed stratification frequency in the Pacific. Left: Stability of the deep thermocline east of the Kuroshio. **Right:** Stability of a shallow thermocline typical of the tropics. Note the change of scales.

In the upper kilometer of the ocean, z < 1,000 m, $E = (100-1000) \times 10^{-8}$ /m, and in deep trenches where z > 7,000 m, $E = 1 \times 10^{-8}$ /m.

The influence of stability is usually expressed by a *stability frequency* N:

$$N^{2} = -\frac{g}{\rho} \frac{\partial \rho}{\partial z} \,(\text{radians/s})^{2} \tag{8.28}$$

The stability frequency is often called the *Brunt-Vaisala frequency* or the *strat-ification frequency*. The frequency quantifies the importance of stability, and it is a fundamental variable in the dynamics of stratified flow. In simplest terms, the frequency can be interpreted as the vertical frequency excited by a vertical displacement of a fluid parcel. Thus, it is the maximum frequency of internal waves in the ocean. Typical values of N are a few cycles per hour (Figure 8.5).

Dynamic Stability and Richardson's Number If velocity changes with depth in a stable, stratified flow, then the flow may become unstable if the change in velocity with depth, the *current shear*, is large enough. The simplest example is wind blowing over the ocean. In this case, stability is very large across the sea surface. We might say it is infinite because there is a step discontinuity in ρ , and (8.28) is infinite. Yet, wind blowing on the ocean creates waves, and if the wind is strong enough, the surface becomes unstable and the waves break.

This is an example of *dynamic instability* in which a stable fluid is made unstable by velocity shear. Another example of dynamic instability, the Kelvin-Helmholtz instability, occurs when the density contrast in a sheared flow is much less than at the sea surface, such as in the thermocline or at the top of a stable, atmospheric boundary layer (Figure 8.6).

The relative importance of static stability and dynamic instability is expressed by the *Richardson Number*:

$$R_i \equiv \frac{g E}{(\partial U/\partial z)^2} \tag{8.29}$$



Figure 8.6 Billow clouds showing a Kelvin-Helmholtz instability at the top of a stable atmospheric boundary layer near Denver, Colorado (From Drazin and Reid 1981). Note that the billows become large enough that more dense air overlies less dense air, and the billows collapse into turbulence.

where the numerator is the strength of the static stability, and the denominator is the strength of the velocity shear.

$$R_i > 0.25$$
 Stable
 $R_i < 0.25$ Velocity Shear Enhances Turbulence

Note that a small Richarson number is not the only criterion for instability. The Reynolds number must be large and the Richardson number must be less than 0.25 for turbulence. These criteria are met in some oceanic flows. The turbulence mixes fluid in the vertical, leading to a vertical eddy viscosity and eddy diffusivity. Because the ocean tends to be strongly stratified and currents tend to be weak, turbulent mixing is intermittent and rare. Measurements of density as a function of depth rarely show more dense fluid over less dense fluid as seen in the breaking waves in Figure 8.6 (Moum and Caldwell 1985).

Double Diffusion and Salt Fingers In some regions of the ocean, less dense water overlies more dense water, yet the water column is unstable even if there are no currents. The instability occurs because the molecular diffusion of heat is about 100 times faster than the molecular diffusion of salt. The instability was first discovered by Melvin Stern in 1960 who quickly realized its importance in oceanography.

Consider two thin layers a few meters thick separated by a sharp interface (Figure 8.7). If the upper layer is warm and salty, and if the lower is colder and less salty than the upper layer, the interface becomes unstable even if the upper layer is less dense than the lower.

Initial Density	Density after a few minutes		
		Warm, Salty p ₁	
Warm, Salty ρ ₁		Cold Salty 0 > 0 o	
<i>Cold, Less Salty</i> ρ ₂			
		Cold. Less Salty on	

Figure 8.7 Left: Initial distribution of density in the vertical. **Right:** After some time, the diffusion of heat leads to a thin unstable layer between the two initially stable layers. The thin unstable layer sinks into the lower layer as salty fingers. The vertical scale in the figures is a few centimeters.

Here's what happens. Heat diffuses across the interface faster than salt, leading to a thin, cold, salty layer between the two initial layers. The cold salty layer is more dense than the cold, less-salty layer below, and the water in the layer sinks. Because the layer is thin, the fluid sinks in fingers 1–5 cm in diameter and 10s of centimeters long, not much different in size and shape from our fingers. This is *salt fingering*. Because two constituents diffuse across the interface, the process is called *double diffusion*.

There are four variations on this theme. Two variables taken two at a time leads to four possible combinations:

- 1. Warm salty over colder less salty: This process is called *salt fingering*. It occurs in central waters of sub-tropical gyres, western tropical North Atlantic, and the North-east Atlantic beneath the outflow from the Mediterranean Sea. Salt fingering eventually leads to density increasing with depth in a series of steps. Layers of constant-density are separated by thin layers with large changes in density, and the profile of density as a function of depth looks like stair steps. Schmitt et al (1987) observed 5–30 m thick steps in the western, tropical North Atlantic that were coherent over 200–400 km and that lasted for at least eight months.
- 2. Colder less salty over warm salty: This process is called *diffusive convection*. It is much less common than salt fingering, and it us mostly found at high latitudes. Diffusive convection also leads to a stair step of density as a function of depth. Here's what happens in this case. Double diffusion leads to a thin, warm, less-salty layer at the base of the upper, colder, less-salty layer. The thin layer of water rises and mixes with water in the upper layer. A similar processes occurs in the lower layer where a colder, salty layer forms at the interface. As a result of the convection in the upper and lower layers, the interface is sharpened; and any small gradients of density in either layer are reduced. Neal et al (1969) observed 2–10 m thick layers in the sea beneath the Arctic ice.
- 3. Cold salty over warmer less salty: Always statically unstable.
- 4. Warmer less salty over cold salty: Always stable and double diffusion diffuses the interface between the two layers.

Double diffusion mixes ocean water, and it cannot be ignored. Merryfield et al (1999), using a numerical model of the ocean circulation that included double diffusion, found that double-diffusive mixing changed the regional distributions of temperature and salinity although it had little influence on large-scale circulation of the ocean.

8.5 Mixing in the Ocean

Instability in the ocean leads to mixing. Because the ocean has stable stratification and any vertical displacement must work against the buoyancy force, vertical mixing requires more energy than horizontal mixing. The larger the stability frequency the greater the work required for vertical mixing. As a result, horizontal mixing along surfaces of constant density is much larger than vertical mixing across surfaces of constant density. The latter, however, usually called *diapycnal mixing*, is very important because it changes the vertical structure of the ocean, and it controls to a large extent the rate at which deep water eventually reaches the surface in mid and low latitudes.

In the ocean, mixing by turbulent eddies is far more important than mixing by molecular diffusion (Munk 1966). The equation for vertical mixing by eddies of a tracer Θ such as salt or temperature is:

$$\frac{\partial\Theta}{\partial t} + W \frac{\partial\Theta}{\partial z} = \frac{\partial}{\partial z} \left(K_z \frac{\partial\Theta}{\partial z} \right) + S \tag{8.30}$$

where K_z is the vertical eddy diffusivity, W is a mean vertical velocity, and S is a source term.

Average Vertical Mixing Average mixing rates in the ocean have been calculated for many years from the distribution of mean properties in the ocean. Munk (1966) considered the important case of the thermocline. He noted that measurements of temperature as a function of depth in the thermocline made decades apart showed the same structure (Figure 8.8). For a steady-state thermocline with no sources or sinks of heat, (8.30) reduces to:

$$W\frac{\partial T}{\partial z} = K_z \frac{\partial^2 T}{\partial z^2} \tag{8.31}$$

where T is temperature as a function of depth in the thermocline. The steadystate thermocline requires that the downward mixing of heat by turbulence be balanced by an upward transport of heat by a mean vertical current W.

The equation has the solution:

$$T \approx T_0 \exp(z/H) \tag{8.32}$$

where $H = K_z/W$ is the scale depth of the thermocline, and T_0 is the temperature near the top of the thermocline. Observations of the shape of the deep thermocline are indeed very close to a exponential function. An exponential function fit through the observations of T(z) gives H, from which K_z can be calculated if W is known.

Munk calculated W from the observed vertical distribution of ¹⁴C, a radioactive isotope of carbon, to obtain a vertical time scale. In this case, S =



Figure 8.8 Potential temperature measured as a function of depth (pressure) near 24.7°N, 161.4° W in the central North Pacific by the *Yaquina* in 1966 (•), and by the *Thompson* in 1985 (\Box). Data from *Atlas of Ocean Sections* produced by Swift, Rhines, and Schlitzer.

 $-1.24\times 10^{-4}~{\rm years^{-1}}.$ The length and time scales gave $W=1.2~{\rm cm/day}$ and

 $\langle K_z \rangle = 1.3 \times 10^{-4} \text{ m}^2/\text{s}$ Average Vertical Eddy Diffusivity (8.33)

where the brackets denote average eddy diffusivity in the thermocline.

Munk also used W to calculate the average vertical flux of water through the thermocline in the Pacific, and the flux agreed well with the rate of formation of bottom water assuming that bottom water upwells almost everywhere at a constant rate in the Pacific.

Measured Vertical Mixing Direct observations of vertical mixing required the development of techniques for measuring: i) the fine structure of turbulence, including probes able to measure temperature and salinity with a spatial resolution of a few centimeters (Gregg 1991), and ii) the distribution of tracers such as sulpher hexafluoride (SF₆) with concentrations of 10^{-15} mole.

Direct measurements of open-ocean turbulence and the diffusion of SF_6 yield an eddy diffusivity:

$$K_z \approx 1 \times 10^{-5} \text{ m}^2/\text{s}$$
 Open-Ocean Vertical Eddy Diffusivity (8.34)

For example, Ledwell, Watson, and Law (1991) injected 139 kg of SF₆ in the Atlantic near 26°N, 29°W 1200 km west of the Canary Islands at a depth of 310 m. They then measured the concentration for five months as it mixed over hundreds of kilometers to obtain a diapycnal eddy diffusivity of $K_z = 1.1 \pm 0.2 \times 10^{-5} \text{ m}^2/\text{s}.$

These and other open-ocean experiments indicate that turbulent mixing is driven by breaking internal waves and shear instability at boundaries. Furthermore, mixing by turbulence seems to be more important than double diffusion (Gregg 1987).

The large discrepancy between the mean eddy diffusivity for vertical mixing and the observed values in the open ocean led to further experiments to resolve the difference. Two recent experiments are especially interesting.

- 1. Polzin et al. (1997) measured the vertical structure of temperature in the Brazil Basin in the South Atlantic. They found $K_z > 10^{-3} \text{ m}^2/\text{s}$ close to the bottom when the water flowed over the western flank of the mid-Atlantic ridge at the eastern edge of the basin.
- 2. Kunze and Toole (1997) calculated enhanced eddy diffusivity as large as $K = 10^{-3} \text{ m}^2/\text{s}$ above Fieberling Guyot in the Northwest Pacific and smaller diffusivities along the flank of the seamount. Summing the influence over all Pacific seamounts, they found, however, that the mixing near seamounts does not account for Munk's basin-wide average.

The experiments indicate that over seamounts and ridges

$$K_z \approx 10^{-3} \text{ m}^2/\text{s}$$
 Rough Bottom Vertical Eddy Diffusivity (8.35)

The results of these and other experiments show that mixing occurs mostly at oceanic boundaries: along continental slopes, above seamounts and mid-ocean ridges, at fronts, and in the mixed layer at the sea surface.

Measured Horizontal Mizing Eddies mix fluid in the horizontal, and large eddies mix more fluid than small eddies. Eddies range in size from a few meters due to turbulence in the thermocline up to several hunderd kilometers for geostrophic eddies discussed in Chapter 10.

In general, mixing depends on Reynolds number R (Tennekes 1990: p. 11)

$$\frac{K}{\gamma} \approx \frac{K}{\nu} \sim \frac{UL}{\nu} = R \tag{8.36}$$

where γ is the molecular diffusivity of heat. Furthermore, horizontal eddy diffusivities are ten thousand to ten million times larger than the average vertical eddy diffusivity.

Equation (8.35) implies $K_x \sim UL$. This functional form agrees well with Joseph and Sender's (1958) analysis, as reported in (Bowden 1962) of spreading of radioactive tracers, optical turbidity, and Mediterranean Sea water in the North Atlantic. They report

$$K_x = PL$$
 (8.37)
10 km < L < 1500 km
 $P = 0.01 \pm 0.005$ m/s

where L is the distance from the source, and U is a constant.

The horizontal eddy diffusivity (8.36) also agrees well with more recent reports of horizontal diffusivity. Work by Holloway (1986) who used satellite altimeter observations of geostrophic currents, Freeland et al. (1975) who tracked SOFAR underwater floats, McWilliams (1976) and Ledwell et al (1998) who used observations of currents and tracers to find

$$K_x \approx 8 \times 10^2 \text{ m}^2/\text{s}$$
 Geostrophic Horizontal Eddy Diffusivity (8.38)

Using (8.37) and the measured K_x implies eddies with typical scales of 80 km, a value near the size of geostrophic eddies responsible for the mixing.

Ledwell, Watson, and Law (1991) also measured a horizontal eddy diffusivity. They found

$$K_x \approx 1 - 3 \text{ m}^2/\text{s}$$
 Open-Ocean Horizontal Eddy Diffusivity (8.39)

over scales of meters due to turbulence in the thermocline probably driven by breaking internal waves. This value, when used in (8.37) implies typical lengths of 100 m for the small eddies responsible for mixing in this experiment.

Comments

1. Water in the interior of the ocean seems to move along sloping surfaces of constant density with little local mixing until it reaches a boundary where it is mixed vertically. The mixed water then moves back into the open ocean again along surfaces of constant density (Gregg 1985).

One particular case is particularly noteworthy. When water mixed downward through the base of the mixed layer flows out into the thermocline along surfaces of constant density, the mixing leads to the *ventilated thermocline* model of oceanic density distributions.

- 2. The observations of mixing in the ocean imply that numerical models of the oceanic circulation should use mixing schemes that have different eddy diffusivities parallel and perpendicular to surfaces of constant density, not parallel and perpendicular to level surfaces of constant z as we used above. Horizontal mixing along surfaces of constant z leads to mixing across layers of constant density because layers of constant density are inclined to the horizontal by about 10^{-3} radians (see §10.7 and figure 10.13). Studies by Danabasoglu, McWilliams, and Gent (1994) show that numerical models using isopycnal and diapycnal mixing leads to much more realistic simulations of the oceanic circulation.
- 3. The observed mixing in the open ocean away from bundaries is too small to account for the mixing calculated by Munk. Recent work reported at the World Ocean Circulation Experiment Conference on Circulation and Climate 1998 and by Munk and Wunsch (1998) indicate thet the dilemma may be resolved several ways:
 - (a) First, separate studies by Gargett, Salmon, and Marotzke show that we must separate the concept of deep convection from that of the meridional overturning circulation (see chapter 13). Deep convection

may mix properties not mass. The mass of upwelled water required by Munk may be overestimated, and the vertical mixing needed to balance the upwelling may be smaller than he calculated.

(b) Second, mixing probably takes place along boundaries or in the source regions for thermocline waters (Gnadadesikan, 1999). For example, water at 1200 m in the central North Atlantic could move horizontally to the Gulf Stream, where it mixes with water from 1000 m. The mixed water may then move horizontally back into the central North Atlantic at a depth of 1100 m. Thus water at 1200 m and at 1100 m may reach their position along entirely different paths.

8.6 Important Concepts

- 1. Friction in the ocean is important only over distances of a few millimeters. For most flows, friction can be ignored.
- 2. The ocean is turbulent for all flows whose typical dimension exceeds a few centimeters, yet the theory for turbulent flow in the ocean is poorly understood.
- 3. The influence of turbulence is a function of the Reynolds number of the flow. Flows with the same geometry and Reynolds number have the same streamlines.
- 4. Oceanographers assume that turbulence influences flows over distances greater than a few centimeters in the same way that molecular viscosity influences flow over much smaller distances.
- 5. The influence of turbulence leads to Reynolds stress terms in the momentum equation.
- 6. The influence of static stability in the ocean is expressed as a frequency, the stability frequency. The larger the frequency, the more stable the water column.
- 7. The influence of shear stability is expressed through the Richardson number. The greater the velocity shear, and the weaker the static stability, the more likely the flow will become turbulent.
- 8. Molecular diffusion of heat is much faster than the diffusion of salt. This leads to a double-diffusion instability which modifies the density distribution in the water column in many regions of the ocean.
- 9. Instability in the ocean leads to mixing. Mixing across surfaces of constant density is much smaller than mixing along such surfaces.
- 10. Calculations of the average eddy diffusivity in the interior of the ocean is much smaller than measured diffusivity.
- 11. Measurements of eddy diffusivity indicate water is mixed vertically near oceanic boundaries such as above seamounts and mid-ocean ridges. This may explain the small measured values of open-ocean diffusivity.