Chapter 1

Vector Spaces

1.1 A *vector space* is a set on which the operations of addition and multiplication by a number are defined.

1.2 The numbers can be *real* or *complex*; then we get *real* or *complex vector spaces* respectively. We will (unless said otherwise) work with real vector spaces.

1.3 The elements of a vector space are called *vectors*; the numbers we multiply vectors with are often called *scalars*.

1.4 Addition of vectors must be commutative and associative; there must be a vector 0 which when added to any vector produces itself.

1.5 Multiplication of a vector by a scalar must be distributive with respect to vector addition as well as scalar addition.

1.6 The smallest vector space is the set consisting of just the zero vector; this is called the trivial vector space and is denoted by 0.

1.7 The set of real numbers is itself a real vector space, called R.

1.8 The set of ordered n-tuples of real numbers is a vector space, with addition being defined component-wise and the scalar multiplying each component.

1.9 The above vector space is said to have dimension n; we will see an abstract definition of dimension later.

1.10 There are also vector spaces of infinite dimension; the set of all real valued functions on any set is a vector space. The dimension of this vector space is the cardinality of the set.

1.11 A map $f : V \to W$ between vector spaces is linear if $f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$.

1.12 If a linear map is one-one and onto, it is an *isomorphism*; the corresponding vector spaces are said to be *isomorphic*, $V \sim W$. 'Isomorphic' means 'having the same structure'.

1.13 The set V' of linear maps of a vector space V to R is called its dual. V' is also, of course, a vector space. The elements of V' are often called 1 -forms.

1.14 A *linear operator* is a linear map from V to itself. It makes sense to multiply linear operators: $LM(u) = L(M(u))$. Operator multiplication is associative but not always commutative.

1.15 An *algebra* is a vector space along with a bilinear multiplication $V \times V \to V$;i.e., $(\alpha u + \beta v)w = \alpha uw + \beta vw, u(\alpha v + \beta w) = \alpha uv + \beta uw$.

1.16 An algebra is *commutative* if $uv = vu$ for all pairs; it is *associative* if $u(vw) = (uv)w$ for all triples.

1.17 The set of linear operators on a vector space is an associative algebra; it is commutative only when the the vector space is either 0 or R .

1.18 There are two ways of combining two vector spaces to get a new one: the direct sum and the direct product.

1.19 The *direct sum* $V \oplus W$ of two vector spaces V and W is the set of ordered pairs (v, w) with the obvious addition and scalar multiplication : $(v, w) + (v', w') = (v + v', w + w')$ and $\alpha(v, w) = (\alpha v, \alpha w)$.

1.20 Transposition, $(v, w) \mapsto (w, v)$, is a natural isomorphism between $V \oplus W$ and $W \oplus V$; also $U \oplus (V \oplus W)$ and $(U \oplus V) \oplus W$ can both be thought of as the space of triples (u, v, w) . Hence we will just write $U \oplus V \oplus W$ etc.

1.21 It is clear that $R^{m+n} = R^m \oplus R^n$.

1.22 The *direct* or *tensor* product $V \otimes W$ of two vector spaces is the set of linear maps from V' to W .

1.23 Again $V \otimes W \sim W \otimes V$; $U \otimes (V \otimes W) \sim (U \otimes V) \otimes W$.

1.24 Also, $V \times R = V$; $R^m \otimes R^n = R^{mn}$.

1.25 We can iterate the direct product n times to get $V^{\otimes n} = V \otimes V \otimes \cdots V$. Its elements are called *contravariant tensors of order* n .

1.26 Since $V^{\otimes m} \otimes V^{\otimes n} = V^{\otimes (m+n)}$ it is natural to define $V^{\otimes 0} = R; V^{\otimes 1} = V$. Thus scalars are tensors of order zero while vectors are contravariant tensors of order one.

1.27 We can then take the direct sum of all of these to get the *total tensor space* $\mathcal{T}(V) = \otimes_{n=0}^{\infty} V^{\otimes n} = R \oplus V \oplus V \otimes V \cdots$.

1.28 $V'^{\otimes n}$ can be viewed also as the space of multilinear functions of n vectors. Its elements are also called *covariant tensors*.

1.29 The *direct product* of two 1-forms $u' \otimes v'$ can defined by its action on a pair of vectors: $u' \otimes v'(u, v) = u'(u)v'(v)$. A general element of $V' \otimes V'$ is a linear combination of such 'factorizable' elements.

1.30 More generally we can define the direct product of two covariant tensors of order m and n :

$$t \otimes \tilde{t}(u_1, \cdots u_m, v_1, \cdots v_n) = t(u_1, \cdots, u_m)\tilde{t}(v_1, \cdots v_n).$$

This turns $\mathcal{T}(V')$ into an associative but in general not commutative algebra. It is commutative only if $V \sim 0, R$.

1.31 An element $t \in V'^{\otimes n}$ is symmetric if it is invariant under permutations; e.g., $t(u, v) = t(v, u)$. The subspace of symmetric tensors is denoted by $\mathcal{S}^n(V')$ and $\mathcal{S}(V') = \bigoplus_{n=0}^{\infty} \mathcal{S}^n(V')$.

1.32 Averaging over all possible orderings gives a projection $\sigma : V'^{\otimes n} \to \mathcal{S}^n(V')$; e.g., $\sigma(t)(u, v) = \frac{1}{2}[t(u, v) + t(v, u)]$.

1.33 We can define a 'symmetrized multiplication' $s\tilde{s}$ of two symmetric tensors: $s\tilde{s} = \sigma(s \otimes \tilde{s})$. This turns $\mathcal{S}(V')$ into a commutative associative algebra.

1.34 A tensor $t \in V'^{\otimes n}$ is *antisymmetric* if it changes sign under an odd permutation of its arguments; e.g., $t(u, v) = -t(v, u)$. A covariant anti-symmetric tensor is also called a *form*; the space of anti-symmetric tensors is denoted by $\Lambda^n(V')$.

1.35 Averaging all permutations weighted by the sign of the permutation gives a projection $\lambda : V'^{\otimes n} \to \Lambda^n(V')$ to the space of anti-symmetric tensors.

1.36 The *wedge product* or *exterior product* of forms is defined by $a \wedge b = \lambda(a \otimes b)$. It is skew-commutative: $a \wedge b = (-1)^{mn} b \wedge a$ if $a \in \Lambda^m(V')$ and $b \in \Lambda^n(V')$.

1.37 The wedge product turns $\Lambda(V') = \bigoplus_n \Lambda^n(V')$ into an associative algebra called the *exterior algebra*.

1.38 An *inner product* $< .,. >$ on V is a symmetric bilinear map $V \times V \to R$ which is non-degenerate; i.e., $< u, v >=< v, u >$ and $< u, v >= 0 \forall u \Rightarrow v = 0$. It is positive if $< u, u >\geq 0$.

1.39 Clearly an inner product is a kind of covariant tensor; it is called the metric tensor. We may denote $< u, v >= g(u, v)$.

1.40 A positive inner product gives a notion of length for every vector; the square of the *length* of a vector is just its inner product with itself.

1.41 An inner product can be thought of as an invertible linear map from V to V' ; i.e., it is an isomorphism between V and V' .

1.42 The inverse of the above map gives an inner product on V' given one on V .

1.43 The simplest example of an inner product is $< u, v >= \sum_i u^i v^i$ in R^n . The length of a vector is just its Euclidean distance from the center.

1.44 A *symplectic form* is an anti-symmetric non-degenerate bilinear map $V \times V \to R$. A vector space along with a symplectic form is a symplectic vector space.

1.45 Just as the metric tensor measures the length of a vector, the symplectic form measures the area of the parallelogram formed by a pair of vectors. The sign of the area contains information about the relative orientation of the vectors: if we reverse the direction of one of them the area changes sign.

1.46 A *complex structure* is a linear map $J : V \to V$ which satisfies $J^2 = -1$, where 1 denotes the identity map.

1.47 Let $\zeta = \alpha + i\beta$ be a complex number. Define $\zeta u = \alpha u + \beta J v$. This turns a real vector space with a complex structure into a complex vector space. Every complex vector space can be obtained this way.

1.48 The definition of inner product on complex vector spaces is a bit different. We require the complex number $< u, v >$ to be *hermitean* rather than symmetric: $< u, v >=< v, u >^*$. Moreover it is linear in the second argument but *anti-linear* in the first : $< \alpha u + \beta v, w >= \alpha^* < u, w > +\beta^* < v, w >$.

1.49 A complex vector space with an inner product can be thought of as a real vector space with a complex structure and an inner product, with the inner product satisfying the additional condition that $\omega(u, v) = g(u, Jv)$ is antisymmetric; i.e., ω is a symplectic form.

1.50 Conversely given a complex structure J and a symplectic form ω we have a hermitean inner product if $g(u, v) = \omega(Ju, v)$ is symmetric.

1.51 The elements of the space $V^{\otimes m} \otimes V'^{\otimes n}$ are called *tensors of order* (m, n) . For example, a complex structure is a tensor of order $(1, 1)$.

1.52 The multiplication rule of an algebra can also be viewed as a tensor . For, a bilinear map $V \times V \to V$ can be viewed as a trilinear map $m : V' \times V \times V \to R$. The *structure tensor* m is an element of $V \otimes V'^{\otimes 2}$; i.e., a tensor of order $(1, 2)$.