Chapter 5

Vector Fields

5.1 A *curve* is a smooth function $\gamma : [a, b] \to M$.

5.2 Given any co-ordinate system in a neighborhood $U \subset M$, $x : U \to R^n$, we can think of a curve as a function $x \circ \gamma : [a, b] \to R^n$.

5.3 Consider two curves that pass through the same point $\gamma(t_1) = \tilde{\gamma}(t_2) = p$. We say that they are *equivalent to first order* at p if in any coördinate system, $\lim_{t\to t_1} \frac{x\circ\gamma(t)-x(p)}{t-t_1} = \lim_{t\to t_2} \frac{x\circ\tilde{\gamma}(t)-x(p)}{t-t_2}$ .This relation is independent of the coördinate system.

5.4 A class of curves equivalent to first order at p is a *tangent vector* at p . There is a natural addition and multiplication by scalars on the set of tangent vectors which makes it a real vector space of dimension n , the tangent space T_pM to M at p .

5.5 The point is that a curve is determined to first order in t by its tangent vector. So we turn this idea around and define a tangent vector as the equivalence class of curves that agree to first order.

5.6 Given a coördinate system x , there is a basis for T_pM . The basis vectors are the tangents to the coördinate axes: the curves we get by varying just one coördinate at a time.The tangent vector to the curve γ has components $\lim_{t\to t_1} \frac{x\circ\gamma(t)-x(p)}{t-t_1}$.

5.7 A *derivation* of a ring is a map V such that

$$V(f + g) = V f + V g, \quad V(fg) = (V f)g + fV g.$$

5.8 A derivation can be thought of as an infinitesimal isomorphism of a ring. More precisely, let ϕ_t^* be a one-parameter family of isomorphisms of a ring such that ϕ_0 is the identity. Then the equation $\phi_t^*(fg) = (\phi_t^* f)(\phi_t^* g)$ implies $\frac{d}{dt}\phi_t^*(fg)|_{t=0} = (\frac{d}{dt}\phi_t^* f)|_{t=0} g + f\frac{d}{dt}\phi_t^* g|_{t=0}$.

5.9 For the ring $C(R)$ of smooth functions on the real line, the derivations are the first order differential operators $Vf(x) = v(x)\frac{df}{dx}$. For $C(R^n)$, also, any derivation is of the form $Vf(x) = v^i(x)\frac{\partial f}{\partial x^i}$. Thus derivations of $C(R^n)$ correspond to vector fields on R^n , the components of which are $v^i(x)$.

5.10 A *vector field* on a manifold M is a derivation of its coördinate ring $C(M)$.

5.11 In a coördinate system, a vector field is a first order differential operator $X = X^i(x)\frac{\partial}{\partial x^i}$. The functions X^i are the *components* of X with respect to the coördinate basis $\frac{\partial}{\partial x^i}$.

5.12 A vector field can be thought of as an infinitesimal diffeomorphism of the manifold to itself. Suppose $\phi : [a, b] \times M \to M$, so that $(t, p) \mapsto \phi_t(p)$ is a diffeomorphism for each value of t , and that $\phi_0(p) = p$. This is called a one-parameter family of diffeomorphisms. The infinitesimal effect of this diffeomorphism on functions is the derivation $Xf(p) = \frac{d}{dt} f(\phi_t(p))|_{t=0}$.

5.13 If we hold p fixed and vary t , $\phi_t(p)$ is a curve passing through p ; at p , the vector field X can be identified with the tangent vector to this curve.

5.14 Conversely, given a vector field X , there is a one-parameter family of diffeomorphisms $\phi : [a, b] \times M \to M$ to which it is tangent, for some a and b . In terms of coördinates this means that the system

$$\frac{d\phi_t^i(x)}{dt} = X^i(\phi_t(x)) \quad \phi_{t=0}^i(x) = x^i$$

has a solution that is smooth, and that the functions ϕ_t have smooth inverses. This is a rather difficult theorem on the existence of solutions to ordinary differential equations.

5.15 The solutions to the above equation satisfy $\phi_t(\phi_s(x)) = \phi_{t+s}(x)$. Thus we can regard ϕ_t as depending exponentially on the parameter t . Indeed we can think of ϕ_t as $\exp tX$.

5.16 The curves $\phi_t(p)$ obtained by keeping p fixed and varying t are called the *integral curves* or *characteristic curves* of the vector field X. They can be immensely complicated even for innocent looking vector fields. A whole field of mathematical physics called *chaos* has developed around understanding the long time behavior of integral curves.

5.17 A simple example (not chaotic) is a rotation in a plane:

$$\phi_t(x,y) = (x\cos t - y\sin t, x\sin t + y\cos t).$$

This is the integral curve of the vector field $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

5.18 Consider the torus $S^1 \times S^1$ and the curve $\gamma(t) = (t, \omega t)$, where r is some real number. If ω is a rational number $\omega = \frac{p}{q}$, this is a curve that winds around the torus and returns to the starting point in a time $2\pi q$: it will make q revolutions around the first circle and p revolutions around the second. If ω is irrational, this curve will never return to the starting point. But it will come arbitrarily close to *any* point on the torus. The integral curve fills the whole torus!. This shows how very simple vector fields can have complicated integral curves.

5.19 The book *Differential Equations: Geometric Theory* by S. Lefshetz is a classic text on the subject of integral curves. We will come back to this topic in the context of classical mechanics.

5.20 The product of two derivations is in general not a derivation; however the commutator of two derivations $[U,V]f = U(Vf) - V(Uf)$ is always a derivation. This allows us to define the commutator of vector fields.

5.21 In coördinates $[U,V]f = \left(u^k\frac{\partial v^i}{\partial x^k} - v^k\frac{\partial u^i}{\partial x^k}\right)\frac{\partial f}{\partial x^i}$. The point is that the product of two first order differential operators is in general of second order; but in the commutator the second order terms cancel out so we again get a first order operator.

5.22 The set of vector fields form a Lie algebra under this operation; i.e., $[U,V] = -[V,U]$ and $[U,[V,W]] + [V,[W,U]] + [W,[U,V]] = 0$.

5.23 An infinitesimal rotation around any axis is a vector field on S^2. If we denote the infinitesimal rotation around the first axis by L_1, that around the second by L_2 and so on, we can check that $[L_1, L_2] = L_3, [L_2, L_3] = L_1, [L_3, L_1] = L_2$.

5.24 A smooth function on S^1 can be expanded in a Fourier series $f(\theta) = \sum_n f_n e^{in\theta}$. The coefficients vanish faster than any power of n : $\lim_{n\to\infty} |n|^k |f_n| = 0$. Any vector field can also be expanded: $V = \sum_n v_n e^{in\theta} \frac{d}{d\theta}$. The vector fields $L_n = e^{in\theta} \frac{d}{d\theta}$ form a basis. They satisfy the commutation relation $[L_m, L_n] = i(n-m)L_{m+n}$.

5.25 The sum of vector fields is a vector field; so is its product with a real number.

5.26 We can multiply a vector field *on the left* by a smooth function to get another vector field. The components get multiplied by the function. Thus the set of vector fields is a module over the ring of functions.

5.27 Given a coördinate system x^i on an open neighborhood U , there is a basis of vector fields in U : $\partial_i = \frac{\partial}{\partial x^i}$. These vector fields commute with each other: $[\partial_i, \partial_j] = 0$. Conversely any set of linearly independent (over $C(M)$) vector fields that commute provide a co-ordinate system in some neighborhood. We will see this when we talk about integral curves.

5.28 A smooth map $\phi : [a, b] \times M \to M$ is a one-parameter family of diffeomeorphisms if ϕ_t is a diffeomorphism for each t and $\phi_t \circ \phi_u = \phi_{t+u}$.