Chapter 2

Index Notation for Tensors

2.1 Just as it is useful to represent numbers in the decimal (or other base) to perform calculations, it is useful to represent vectors by their components with respect to a basis.

2.2 A set of vectors $e_1, \cdots e_k \in V$ is *linearly independent* if $\sum_i \alpha_i e_i = 0 \Rightarrow \alpha_i = 0$.

2.3 A *basis* is a maximal set of linearly independent vectors.

2.4 That is, any vector $v \in V$ can be expressed as $v = \sum_i v^i e_i$ for some unique n -tuple of real numbers $(v^1, \cdots v^n)$. These numbers are called the *components of* v *with respect to* e_i .

2.5 We will see soon why we place the indices on the components as superscripts and not subscripts.

2.6 The maximum number of linearly independent vectors in V is called the *dimension* of V . We will consider the case of finite dimensional vector spaces unless stated otherwise.

2.7 Two bases $e_i, \tilde{e}_i$ are related by a linear transformation $\tilde{e}_i = A_i^j e_j$. A_i^j are the components of an invertible matrix.

2.8 The components of a vector with respect to these bases are also related by a linear transformation: $v = \sum_i v^i e_i = \sum_i \tilde{v}^i \tilde{e}_i, v^i = \sum_j A_j^i \tilde{v}^j$.

2.9 To simplify notation we will often drop the summation symbol. Any index that appears more than once in a factor will be assumed to be summed over. This convention was introduced by Einstein.

2.10 It will be convenient to introduce the Kronecker delta symbol: δ^i_j =0 if $i \neq j$ and $\delta^i_j = 0$ if $i = j$.

2.11 Given a basis e_i in V , we can construct its *dual basis* in V' : $e'^i(e_j) = \delta^i_j$.

2.12 A form can be expanded in terms of this dual basis: $\phi = \phi_i e'^i$. These components transform in the opposite way to the components of a vector: $\tilde{\phi}_i = A^j_i \phi_j$.

2.13 Although the components depend on the basis, the scalar $\phi(u) = \phi_i u^i$ is invariant under changes of basis.

2.14 More generally, the collection $e_{i_1} \otimes \cdots e_{i_m} \otimes e'^{j_1} \otimes \cdots e'^{j_n}$ is a basis in the space of tensors of order (m, n) : $t = t^{i_1 \cdots i_m}_{j_1 \cdots j_n} e_{i_1} \otimes \cdots e_{i_m} \otimes e'^{j_1} \otimes \cdots e'^{j_n}$.

2.15 Each upper index transforms contravariantly under changes of basis; each lower index transforms covariantly. This is the reason for distinguishing between the two types of indices. A contraction of an upper index with a lower index is invariant under changes of basis.

2.16 The metric tensor of an inner product space has components that satisfy $g_{ij} = g_{ji}$. Moreover, the determinant of the matrix g_{ij} is non-zero.

2.17 Given an inner product in V , we can construct one on V' ; its components g^{ij} form the matrix inverse to g_{ij} : $g^{ij} g_{jk} = \delta^i_k$.

2.18 We call the g^{ij} the contravariant components of the metric tensor and g_{ij} the covariant components.

2.19 Because the metric tensor can be thought of as an isomorphism between a vector space and its dual, it can be used to convert a contravariant index into a covariant index and vice-versa: $v^i \to g_{ij} v^j$ or $w_i \to g^{ij} w_j$. This is called lowering and raising of indices.

2.20 A symplectic structure ω has components ω_{ij} which form an invertible anti-symmetric matrix. Since the determinant of an odd dimensional anti-symmetric matrix is zero, symplectic forms exist only in even dimensional vector spaces.

2.21 A complex structure J has components J^i_j satisfying the condition $J^i_j J^j_k = -\delta^i_k$.

2.22 The multiplication rule of an algebra can be thought of as a tensor of order $(1,2)$. It has components m^i_{jk} ; the components of the product of u and v is $m^i_{jk}u^jv^k$. The m^i_{jk} are called the *structure constants* of the algebra.

2.23 If the algebra is commutative, the structure constants are symmetric in the two lower indices: $m^i_{jk} = m^i_{kj}$. If it is associative, it satisfies the quadratic relation $m^p_{ij}m^q_{pk} = m^q_{ip}m^p_{jk}$.

2.24 A symmetric tensor $t \in \mathcal{S}^n(V')$ can be thought of as defining a polynomial of order n in the components of a vector: $t(v, v, \cdots v) = t_{i_1 i_2 \cdots i_n} v^i_1 \cdots v^{i_n}$. In fact this polynomial completely determines the symemtric tensor. The mutplication law for symmetric tensors we defined earlier is just the ordinary multiplication of polynomials: this explains why it is commutative.

2.25 The wedge product of forms can be thought of in a similar way as the polynomials in some 'anti-commuting' variable. This is a suggestive notation, particularly favored by physicists.

2.26 We define some abstract variables (*Grassmann variables*) ψ^i satisfying $\psi^i\psi^j = -\psi^j\psi^i$ for all i, j . In particular $(\psi^i)^2 = 0$. A polynomial or order n in these variables will be $t_{i_1\cdots i_n}\psi^{i_1}\cdots\psi^{i_n}$. The multiplication of these polynomials is equivalent to the wedge prodcut of forms.

2.27 More precisely, the exterior algebra is the free algebra generated by the ψ^i quotiented by the ideal generated by the relations $\psi^i\psi^j = -\psi^j\psi^i$.

2.28 The free algebra on n variables is just the tensor algebra on an n -dimensional vector space.

2.29 No physicist can work with symmetric and anti-symmetric tensors without being reminded of bosonic and fermionic quantum systems. Indeed if V is the space of states of a particle, the space of states of an n particle system is $\mathcal{S}^n(V)$ for bosons and $\Lambda^n(V)$ for fermions. The only point to remember is that the vector spaces of interest in quantum mechanics are over the field of complex numbers.

2.30 The components of a vector can be thought of as the Cartesian co-ordinates on V. Given a function $f: V \to R$ and a basis e_i in V we can construct a function $\tilde{f}: R^n \to R, \tilde{f}(x_1, x_2, \cdots x_n) = f(x^i e_i)$. We can now define a f to be *differentiable* if $\tilde{f}$ has continuous partial derivatives as a function on R^n .This class of functions is called $C^1(V)$.

2.31 Analogously we define $C^k(V)$, $C^\infty(V)$, $C^\omega(V)$. They are respectively, the class of functions whose k th derivatives are continuous, then those with continuous partial derivatives of any order and finally functions which have a convergent Taylor series expansion. C^∞ functions are called *smooth* and the C^ω are *analytic*.

2.32 Analytic functions are completely defined by their derivatives at the origin; these derivatives are symmetric tensors. Thus, each element of $\mathcal{T}(V)$ for which the infinite series $\sum_{n=0}^{\infty} t_{i_1 \cdots i_n} x^{i_1} \cdots x^{i_n}$ converges characterizes an analytic function. There is no similar characterization of smooth or differentiable functions. For polynomials this series terminates.