Chapter 18

Supersymmetry

18.1 The set of instaneous states of a q uantum mechanical system form a complex Hilbert space $\mathcal{H}$; its observables are self-adjoint operators. The spectrum of this operator is the set of outcomes when a meaurement is made of the observable. Of special importance is the hamiltonian $H : \mathcal{H} \to \mathcal{H}$, a self-adjoint operator which represents the energy of the system. Moreover, a system which is in state $\psi \in \mathcal{H}$ at time t_0 will be at time t in the state $e^{-iH(t-t_0)}\psi$.

18.2 A *conserved quantity* is an observable that commutes with the hamiltonian: $[H, Q] = 0$. It does not change under time evolution.

18.3 A system is *supersymmetric* if there are self-adjoint operators $Q_i, i = 1, \cdots N$ such that

$$Q_iQ_j + Q_jQ_i = 2\delta_{ij}H.$$

Thus, H is the square of any of the operators Q_i ; hence they are all conserved.

18.4 The case $N = 1$ is of special interest. We can write the supersymmetry algebra as

$$Q^2 = Q^{\dagger 2} = 0, \quad QQ^\dagger + Q^\dagger Q = H$$

where $Q = \frac{1}{2}(Q_1 + iQ_2)$.

18.5 The algebra generated by Q_i has an automorphism, $Q_i \to -Q_i$. There is an operator F that generates this automorphism $FQ_iF^\dagger = -F, F^2 = 1$ or equivalently

$$FQ_i + Q_iF = 0, \quad F = F^\dagger, \quad F^2 = 1.$$

The subspace of $\mathcal{H}$ with eigenvalue 1 for F is called the *bosonic* subspace and that with eigenvalue -1 the *fermionic* subspace: $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$. It is clear that $Q, Q^\dagger : \mathcal{H}_B \to \mathcal{H}_F$ and vice versa.

18.6 Within an eigenspace of $\mathcal{H}$ with non-zero eigenvalue E , the number of bosonic and fermionic states are equal; i.e, $\mathrm{tr}_E F = 0$. To see this we note that $\mathrm{tr}_E F = \frac{1}{E}\,\mathrm{tr}_E Q_1^2 F = -\frac{1}{E}\,\mathrm{tr}_E Q_1 F Q_1 = -\,\mathrm{tr}_E F$.

18.7 The states of zero energy must satisfy

$$Q_i|\psi >= 0$$

for all i . The number $W = \mathrm{tr}_0 F$ of bosonic zero energy states minus fermionic zero energy states, is called the *Witten index*. We can see that

$$W = \mathrm{tr} e^{-tH} F$$

for all $t > 0$ since only zero energy states contribute to this sum.

18.8 As an example, let $\mathcal{H}$ be the space of (complex-valued) differential forms on a real manifold M with positive Riemannian metric. The inner product is

$$< \omega, \phi >= \sum_{p=0}^{\dim M} \int \omega^*_{i_1\cdots i_p}\phi_{j_1\cdots j_p} g^{i_1j_1}\cdots g^{i_pj_p}\sqrt{\det g}d^n x.$$

Then the bosonic subspace consists of forms of even order and fermionic states those of odd order. We choose $Q = d$, the exterior derivative.

18.9 Then the hamiltonian is the laplacian; the zero energy states are harmonic forms; i.e, forms that satisfy

$$d\omega = 0, \quad d^\dagger\omega = 0.$$

Recall that each cohomology class has a unique harmonic representative. For, if ω is closed and belongs to the orthogonal complement of the space of hamrmonic forms, then it is exact: $\omega = d(\Delta^{-1}[d^\dagger\omega])$. Thus the Witten index is the Euler characteristic of the manifold:

$$\mathrm{tr} e^{-t\Delta}(-1)^p = \sum_{p=0}^{\dim M}(-1)^p \dim H^p(M).$$