Chapter 4

The Sphere and the Hyperboloid

4.1 We now begin the study of manifolds. We will look at a few examples before presenting a definition of a manifold.

4.2 Next to the plane, the most familiar surface is a sphere. The surface of the earth approximates a sphere. The sky can also be thought of as a sphere, the celestial sphere.

4.3 The geometry of spheres was first developed by ancient astronomers. The **Aryabhatiya** (written in 499 A.D.) contains results on non-Euclidean geometry.

4.4 The two dimensional *sphere* S^2 is the set of points at unit distance from the origin in R^3 .

4.5 From each point in S^2 we can draw a line connecting it to the origin. The distance between two points can be defined to be the angle (in radians) at which these radial lines intersect at the origin.

4.6 If two points are at zero distance they are identical. The maximum distance between points is π ; this happens when the points are *anti-podal* to each other; i.e., when the straight-line connecting them passes through the center of the sphere.

4.7 A *great circle* on S^2 is the intersection of the sphere with a plane that passes through the center. Any two distinct points lie along a great circle; for these two points and the center define a plane in R^3 (if the points are not antipodal). If the points are antipodal they lie on an infinite number of great circles.

4.8 If two points are not antipodal, (the short arc of) the great circle is the shortest path on the sphere between the two points. Such curves are called *geodesics*.

4.9 Thus longitudes (also called meridians) are geodesics but the only latitude (or parallel) that is a geodesic is the equator.

4.10 A *spherical triangle* consists of three distinct points connected pairwise by geodesics. The angle at each vertex is defined to be the angle between the tangents at that point. The sum of these angles need not be π ; indeed it is always greater than π . This shows that the axioms of Euclid cannot hold for spherical geometry.

4.11 For example we can draw a spherical triangle connecting the north pole to two points on the equator where each of the angles is a right angle.

4.12 In the *Aryabhatiya* the following problem is solved: given two sides of a spherical triangle and the angle at which they intersect, determine the remaining side and angles. Can you solve this?.

4.13 Geodesics are the closest analogues to straight lines on the surface of a sphere. But there are important differences between spherical and Euclidean (planar) geometry: any two geodesics will meet.

4.14 An axiom of Euclidean geometry is that *given a straight line L and a point P not on it, there is exactly one straight line passing through P but not intersecting L .* This axiom does not hold in spherical geometry. This is due to the curvature of the surface of the sphere.

4.15 Geodesics on a sphere tend to 'focus': even though two geodesics starting at a point will move away from each other at first, they will eventually converge and meet at the anti-podal point. This is the feature of positive curvature.

4.16 A plane that intersects the sphere at exactly one point is called the *tangent plane* at that point. Being a vector space it is also called the *tangent space* . The tangent plane is the best planar approximation of the sphere at that point.

4.17 The tangent space at $p \in S^2$ is also the set of all vectors in R^3 orthogonal to the radius vector at p .

4.18 Elements of the tangent space at p are called *tangent vectors*.

4.19 The *length* of a tangent vector is just its length thought as an element of R^3 . Thus on each tangent space of S^2 there is a metric tensor, called the *induced metric* .

4.20 A curve in S^2 is a smooth function $f : R \to R^3$ that lies on the sphere: $< f(t), f(t) >= 1$.

4.21 The derivative of $f(t)$ is a vector in R^3 orthogonal to $f(t)$; hence it can be thought of as an element of the tangent space at $f(t)$. This is called the *tangent vector* to the curve.

4.22 The *length* of a curve is the integral of the lengths of its tangent vectors: $l(f) = \int_R \surd < \dot{f}(t), \dot{f}(t) > dt$. (The dot denotes differentiation with respect to t .)

4.23 The shortest curve connecting two points is a geodesic; the length of the geodesic is the distance between points as defined previously. These facts can be proved using variational calculus.

4.24 A function $f : S^2 \to R$ is *smooth* if it is the restriction to the sphere of some smooth function on R^3 . The set of smooth functions is a *commutative ring*: it is closed under addition and multiplication. Division is not well-defined since functions may vanish.

4.25 The set of all tangent vectors at the various points on the sphere is called its *tangent bundle* $TS^2 = \{(r, u) \in R^3 \times R^3 | r^2 = 1; r \cdot u = 0\}$.

4.26 A *vector field* on the sphere is a smooth function that assigns to each point in S^2 a tangent vector at that point. More precisely, it is a smooth function $u : S^2 \to R^3$ such that $r \cdot u(r) = 0$.

4.27 Given any non-zero constant $a \in R^3$, its cross product with the radius vector gives a vector field on the sphere. Even if $a \neq 0$, the vector field $a \times r$ vanishes at two points on the sphere: $\frac{a}{|a|}$ and its antipodal point, $-\frac{a}{|a|}$.

4.28 In fact this is a general fact: any smooth vector field on the sphere vanishes somewhere. We will not prove this statement in this course.

4.29 A vector field can be thought of as an infinitesimal displacement of each point. It is useful to picture it as the velocity of some fluid attached to the surface of the sphere.

4.30 A vector field can also be thought of as a first order differential operator acting on the ring of smooth functions. We will examine this point of view later.

4.31 It is useful to establish a co-ordinate system: a pair of real numbers that will specify a point on the sphere.

4.32 An example is the spherical polar co-ordinate system. We fix a point ('North Pole') and a geodesic ('standard meridian') that connects it to its antipode ('South Pole'). The *latitude* of point θ is its distance from the origin; i.e, the length of the geodesic connecting the point to the North Pole. The *longitude* ϕ is the angle this geodesic makes with the standard meridian at the North Pole.

4.33 Excepting the points on the standard meridian, each point is uniquely specified by the ordered pair (θ, ϕ) . There is no single co-ordinate system that can cover all of the sphere.

4.34 A curve (that doesn't intersect the meridian) is given by two functions $\theta(t)$ and $\phi(t)$. Its length is given by $\int \sqrt{[\dot{\theta}^2(t) + \sin^2\theta(t)\dot{\phi}^2(t)]dt}$.

4.35 The *hyperboloid* is the set of points in R^3 satisfying $z^2 - x^2 - y^2 = 1; z > 0$.

4.36 It is possible to define the notions of tangent space, vector field, curves, length of curves and geodesics for this case by analogy to the sphere.

4.37 The resulting geometry (studied first by Lobachewsky, Bolyai and Poincare) has negative curvature; two geodesics emanating from a point diverge from each other and never meet.

4.38 We will return to these examples after developing the general theory of manifolds.