Chapter 13

Special Relativity

13.1 It was discovered at the end of the last century that the velocity of light c in the vacuum is the same for all observers. This was inconsistent with the prevailing view of the time that light is the oscillation of a medium (called ether) that pervaded all matter, just as sound is the oscillatio of air.

13.2 It was also inconsistent with the laws of newtonian mechanics, but was consistent with the laws of electromagnetism (Maxwell's equations). Einstein realized that the laws of mechanics can be changed to fit the new theory of electromagnetism: the departure from newtonian mechanics would be experimentally measurable at velocitie comparable to c.

13.3 For example, the formula for the kinetic energy and momentum of a particle of mass m are

$$E = \frac{mc^2|\mathbf{v}|}{\sqrt{\left[1-\frac{\mathbf{v}2}{c^2}\right]}}, \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{\left[1-\frac{\mathbf{v}2}{c^2}\right]}}.$$

In other words the newtonian relation $E = \frac{\mathbf{p}^2}{2m}$ is replaced by

$$E^2 - \mathbf{p}^2c^2 = m^2c^4.$$

13.4 Moreover, the formulas of transformation between two observers moving at a constant velocity with respect to each other are changed from the *Galilean transformations*

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \quad t' = t$$

to the *Lorentz transformations*

$$\mathbf{x}' = \frac{\mathbf{x} - \mathbf{v}t}{\sqrt{\left[1 - \frac{\mathbf{v}^2}{c^2}\right]}}, \quad t' = \frac{t - \frac{1}{c^2}\mathbf{v}\cdot\mathbf{x}}{\sqrt{\left[1 - \frac{\mathbf{v}^2}{c^2}\right]}}.$$

13.5 In particular, this implies that times measured by different observers are not the same. Two observers can even disagree on whether two events occuring at different points in space are simultaneous.

13.6 Minkowski realized that this *theory of relativity* had a simple meaning in geometric terms. He postulated that space and time be combined into a four dimensional manifold with the metric

$$ds^2 = \mathbf{dx}^2 - c^2 dt^2.$$

This *Minkowski metric* is not positive: vectors of positive length^2 are said to be *space-like*; those of negative length^2 *time–like*; there are non-zero vectors of zero length which are said *null* or *light–like*.

13.7 A basic fact of physics is causality: *it is possible to send a signal from one event only to its future.* This principle continues to hold in relativity with the understanding that *an event* x *is in the future of* y *if* $\eta(x-y, x-y) \leq 0$ *and* $(x^0 - y^0) > 0$. The condition to be in the future is independent of changes of reference frames under Lorentz transformations only if the first condition is also satisfied: the points cannot be space-like separated.

13.8 If they are space-like separate (i.e., if $\eta(x-y, x-y) > 0$) there are reference frames in which $x^0 - y^0$ positive, negative or zero. Thus the principle of causality implies that *it is not possible to send a signal between two events that are space-like separated*. What this means is that *no signal can travel at a velocity greater than that of light*.

13.9 The motion of a particle is given by a curve in space-time called its *world-line*. The principle of causality implies that the tangent vector to the world-line of a particle is either time-like or null.

13.10 Define the length of a time-like worldline (i.e., one whose tangent vector is time–like everywhere) $x : [a, b] \to R^4$ to be

$$S[x] = \int_a^b \left[-\eta(\frac{dx}{d\tau}, \frac{dx}{d\tau}) \right]^{\frac{1}{2}} d\tau.$$

13.11 The length of a time-like curve can be thought of as the time as measured by an observer moving with along the curve, multiplied by c. It is also called the *proper time* of the curve. It is invariant under the changes of the parametrization; i.e., two curves x and $\tilde{x}$ have the same length if there is a monotonic function $\phi : [a, b] \to [a, b]$ preserving endpoints such that

$$\tilde{x}(\tau) = x(\phi(\tau)).$$

Still, the most natural parameter to use in describing the curve is the arc-length, given by the indefinite integral

$$s(\tau) = \int_a^\tau \left[-\eta(\frac{dx}{d\tau}, \frac{dx}{d\tau}) \right]^{\frac{1}{2}} d\tau.$$

The tangent vector with respect to the arc-length is of unit length:

$$\eta(\frac{dx}{ds}, \frac{dx}{ds}) = -1.$$

13.12 *Free particles move along curves for which the arc–length is an extremum.* In other words *free particles move along time–like geodesics of Minkowski metric.* It is easy to see that these are straight-lines with time-like tangents. Thus this principle is the relativistic analogue of Newton's first law.

13.13 The unit tangent vector to the world-line $\frac{v^\mu = dx^\mu}{ds}$ is often called the *four-velocity* of the particle. It is important to remember that its spatial components are *not* the velocity $\mathbf{u} = \frac{d\mathbf{x}}{dt}$ as defined in Newtonian mechanics. Instead,

$$\mathbf{v} = \frac{\mathbf{u}}{\sqrt{[1 - \frac{\mathbf{u}^2}{c^2}]}}.$$

13.14 The relatistic formulae relating momentum and energy to velocity now becomes

$$p_\mu = mc\eta_{\mu\nu}\frac{dx^\nu}{ds};$$

The relation between energy and momentum becomes

$$\eta^{\mu\nu}p_\mu p_\nu = -(mc)^2.$$

13.15 The variational equations that describe a geodesic in arbitrary parameters makes sense even when the tangent is not time–like. Since the length of the tangent is constant, if it is null at one point it will be null everywhere. Null geodesics are defined as the solutions of this equation with null tangents. They are just straight-lines with null tangents. *Light travels along null geodesics*. This is the relativistic analogues of Fermat's principle. The arclength of a null-geodesic is zero, so cannot be used as a parameter on the curve.