Chapter 9

Riemmanian Geometry

9.1 A *Riemannian metric* g on a manifold M is a symmetric non-degenerate covariant tensor field. In other words, given a pair of vector fields u and v, we get a function (their *inner product*) $g(u,v)$ which is bilinear and $g(u,v) = g(v,u)$. Moreover, $g(.,u) = 0 \Rightarrow u = 0$.

9.2 A metric g is *positive* if $g(u,u) \geq 0$ for all u. For positive metrics, the $g(u,u)$ is the square of the length of a vector. The *angle* between two vectors is $\arccos\left(\frac{g(u,v)}{\sqrt{g(u,u)}\sqrt{g(v,v)}}\right)$. If the metric is not positive, the length and angle can become complex numbers.

9.3 The simplest example is the Euclidean metric on R^n : $g(u,v) = u^i v^i$. It is clear that this metric is positive and that the length and angle as defined above have their usual meanings.

9.4 Another example is the Minkowski metric on R^4 : $g(u,v) = u^0 v^0 - u^i v^i$ where $i = 1,2,3$. This describes space-time in the special theory of relativity.

9.5 Recall that a tangent vector to a sphere can be thought of as a vector in R^3 as well. Thus the Euclidean inner product on R^3 induces a metric on S^2, called the *standard metric*.

9.6 In local co-ordinates a metric is described by a set of $\frac{n(n+1)}{2}$ functions, its components: $g = g_{ij} dx^i \otimes dx^j$. Sometimes the product $\otimes$ is omitted and we just write $g = g_{ij} dx^i dx^j$. This can be viewed as the length squared of an infinitesimal arc connecting x^i to $x^i + dx^i$; thus often it is denoted by $ds^2 = g_{ij} dx^i dx^j$.

9.7 Thus in Euclidean space $ds^2 = dx^i dx^i$. The standard metric on the sphere is, in spherical polar coördinates, $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. Another interesting case is the Poincare metric on the upper half plane: $ds^2 = \frac{dx^2+dy^2}{y^2}$.

9.8 An *immersion* is a smooth map $\phi : N \to M$ whose Jacobian matrix $d\phi$ is of constant rank. An *embedding* is an immersion which is injective; i.e., its image doesnt intersect itself anywhere. Given a metric g on M , and an embdedding $\phi : N \to M$ we can get a metric (the *induced metric* $\phi^* g$)on N . Since N can be identified as a subset of M , (the image of ϕ) its tangent vectors can be thought of as tangent vectors of M ; the induced metric is just the inner product taken with g . We already saw an example of this: the metric on S^2 induced by its embedding to R^3 .

9.9 A diffeomorphism $\phi : N \to M$ is an *isometry* between Riemannian manifolds (N, h) and (M, g) if $h = \phi^* g$. Isometric manifolds are essentially the same from the point of view of Riemannian geometry.

9.10 A deep theorem of Nash and Whitney is that any Riemannian manifold with positive metric is isometric to a submanifold of some Euclidean space. This generalizes the way we obtained the metric on S^2 . In general, the dimension of the ambient Euclidean space is much higher than the dimension of the original Riemannian manifold.

9.11 Let us restrict for the rest of this section to the case of positive metrics; all the ideas can be generalized to the non-positive case as well.

9.12 The *length* of a curve $\gamma : [a, b] \to M$ is the integral of the length of its tangent vectors: $l(\gamma) = \int_a^b \sqrt{g(\dot\gamma, \dot\gamma)}dt$. It is invariant under changes of the parameter describing the curve: if $\phi : [a, b] \to [c, d]$ is a diffeomorphism of intervals, $\tilde\gamma(t) = \gamma(\phi(t))$ has the same length as γ .

9.13 The arclength $s(t) = \int_a^t \sqrt{g(\dot\gamma, \dot\gamma)}dt$ is the most natural choice of parameter along a curve. In this parametrization, the tangent vector is always of unit length so that $\sqrt{\left[g(\frac{d}{ds}\gamma, \frac{d}{ds}\gamma)\right]}\, ds = g(\frac{d}{ds}\gamma, \frac{d}{ds}\gamma)ds$.

9.14 A *geodesic* is curve for which the length does not change under infinitesimal variations that fix the initial and final points. We can get a differential equation for a geodesic by the Euler-Lagrange method. It is most convenient to use the arclength as the paremeter so that we minimize the functional $\int_0^l g(\frac{d}{ds}\gamma, \frac{d}{ds}\gamma)ds$. (If we use a generic parametrization we will get the same answer, after some more calculations.)

9.15 It is best to use a co-ordinate system; i.e., consider the special case where the curve lies within one co-ordinate neighborhood and is given by some functions $x^i(t)$ for $i = 1, \cdots n$:

$$\begin{aligned}
\delta \int_0^l g_{ij}(x(s)) \frac{dx^i}{ds}\frac{dx^j}{ds} ds &= \int_0^l \partial_k g_{ij} \delta x^k \frac{dx^i}{ds}\frac{dx^j}{ds} ds + 2\int_0^l g_{ij} \frac{d\delta x^i}{ds}\frac{dx^j}{ds} ds \\
&= \int_0^l \partial_k g_{ij} \delta x^k \frac{dx^i}{ds}\frac{dx^j}{ds} ds - 2\int_0^l g_{ij} \delta x^i \frac{d^2 x^j}{ds^2} ds \\
&- 2\int_0^l \partial_k g_{ij} \frac{dx^k}{ds} \delta x^i \frac{dx^j}{ds} ds
\end{aligned}$$

9.16 The boundary term in the integration by parts can be ignored because the endponts are fixed. Relabelling indices,

$$\delta \int_0^l g_{ij}(x(s)) \frac{dx^i}{ds}\frac{dx^j}{ds} ds =$$
$$(-2)\int_0^l \delta x^k \left[g_{kj} \frac{d^2x^j}{ds^2} + \frac{1}{2}\left\{\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}\right\} \frac{dx^i}{ds}\frac{dx^j}{ds} \right] ds$$

9.17 Thus the differential equation for a geodesic is

$$\frac{d^2x^j}{ds^2} + \Gamma^j_{ik} \frac{dx^i}{ds}\frac{dx^k}{ds} = 0$$

where Γ^j_{ik} is a quantity called the *Christoffel symbol*:

$$\Gamma^j_{ik} = \frac{1}{2} g^{lj}\left\{\partial_i g_{kl} + \partial_i g_{kl} - \partial_l g_{ik}\right\}.$$

9.18 The geodesic equation is invariant under arbitrary changes of co-ordinates. But, the Christoffel symbols do *not* transform as tensors. For example, even if they are zero in one co-ordinate system (e.g., cartesian co-ordinates in Euclidean space) it may not be zero in a general co-ordinate system.

9.19 It follows that the geodesics of Euclidean space are straight lines; those of the sphere are the great circles. The geodesics of the Poincare metric of the upper half plane are circles that intersect with the real line at right angles.

9.20 Let us fix a point $O \in M$. There is a neighborhood U of O where each point is connected to the origin by a unique geodesic. Define a function $S : U \to R$ to be the square of the distance of the point from O. Surfaces of constant S are analogous to spheres.

9.21 The vector $g^{ij}\partial_i S\partial_j$ is orthogonal to the surface of constant S. In fact it is just the tangent vector of unit length to the geodesic connecting the point to the origin. Thus

$$g^{ij}(x)\partial_i S\partial_j S = 1.$$

This first order partial differential equation is called the *eikonal* equation. It is just another way of describing geodesics of a manifold: the geodesics are its characteristic curves. It is analogous to the Hamilton-Jacobi equation in Mechanics. Often it is easier to solve this equation than the ordinary differential equation above.

9.22 There is a natural co-ordinate system in the neighborhood of each point in a Riemannian manifold, called the *Riemann normal co-ordinates*. Choose a point $O \in M$ as the origin. There is a neighborhood U of O such that each point $P \in U$ is connected to O by a unique geodesic. Let $s(P)$ be the length of geodesic and $u(P)$ the unit tangent vector *at the origin* of the geodesic connecting O to P. Then $s(P)u(P)$ which is a vector in the tangent space to M at the origin. We can now choose an orthonormal basis in $T_O M$ and get a set of components $x^i(P)$ of this vector, which will be the normal co-ordinate of the point P. A single such co-ordinate system may not cover M; but we can choose a number of such points spread out over the manifold with overlaping neighborhoods as usual to cover M.

9.23 The co-ordinate axes of the normal system are geodesics originating at O pointing along orthogonal directions.The system is valid as long these geodesics dont intersect. In Euclidean space, the normal co-ordinates are thus just the Cartesian co-ordinates.