Chapter 20

Poisson Brackets

20.1 The set of observables of a classical system forms a *real commutative algebra*. That is, the sum of two observables is an observable, as is its product with a real number. The product of two observables gives another observable, which is independent of the order of multiplication. Also, there is an identity element decribing the trivial observable which is always equal to one. There is one more important operation on a pair of classical observables, that of a Poisson bracket.

20.2 A *Poisson bracket* on a commutative algebra A is a bilinear map $\{,\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, satisfying the conditions

$$\begin{array}{ll} (i) & \{f, g\} = -\{g, f\}, \\ (ii) & \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \\ (iii) & \{f, gh\} = \{f, g\}h + g\{f, h\}. \end{array}$$

A commutative algebra along with a Poisson bracket is a *Poisson algebra*.

20.3 The first two conditions say that the Poisson bracket turns the set of observables into a Lie algebra. The condition (iii) says that this bracket acts as a derivation on the commutative multiplication; i.e., satisfies the Leibnitz rule of derivation.

20.4 The complexification of a Poisson algebra can often be thought of as the limit of a one-parameter family of (complex) associative algebras, when the commutator is taken to be infinitesimally small. (Whether *every* Poisson algebra arises as such a limit is a subtle question which has been studied by kontsevich.) The Poisson bracket is the limiting form of the commutator. The conditions on the Poisson bracket are just the remnants of the associativity of the underlying algebra. This makes sense physically if we remember that the observables of a quantum system are the (self-adjoint) elements of a complex associative algebra. The limit in which the commutator is infinitesimally small is the classical limit. We will see some examples of this idea later.

20.5 Two elements of a Poisson algebra are said to *commute* if their Poisson bracket is zero. The set of elements that commute with *all* the elements of a Poisson algebra is the *center*.

20.6 The set of instantaneous states of a classical system is called the *phase space*. A classical observable assigns a real number to each state, so it is a real valued funcion on the phase space.The sum and product of obervables correspond to the pointwise operations on these functions.

20.7 In many cases the phase space is naturally thought of as a smooth manifold. We will study this case in more detail here. An observable is then a smooth function on the phase space. (Sometimes it is more natural to think of the phase space as an algebraic variety or as as a differentiable manifold of class C^k ; the observables will then to be chosen to be the corresponding category of functions.)

20.8 Any derivation on the space of smooth functions of a manifold is a first order partial differential operator: this is the meaning of the Leibnitz rule for derivations. Explicitly in terms of co-ordinates x^i on the manifold, it takes the form

$$Vf = V^i \frac{\partial f}{\partial x^i}$$

for a set of functions V^i . In fact

$$V(x^i) = V^i.$$

These functions transform as the components of a (contra-variant) vector field under co-ordinate transformations. Thus *a derivation on the algebra of functions on a smooth manifold is the same as a vector field*. Thus a derivation on a commutative algebra is a natural generalization of the notion of a vector field.

20.9 Thus for functions on a phase space, the condition (iii) says that to every observable f there corresponds a derivation V_f :

$$V_f g = \{f, g\}.$$

This is called the *canonical* vector field of the function f .

20.10 In many mathematical texts, V_f is called the *Hamiltonian* vector field of f . However, the word Hamiltonian has a specific meaning in physics, as the generator of time translations. Therefore we will call V_f the canonical vector field of f .

20.11 In tensor notation,

$$V_f g = V_f^i \partial_i g.$$

By reversing the roles of f and g we can see that V_f^i itself is linear in the derivatives of f . Thus there must in fact be a tensor field w^{ij} such that

$$\{f, g\} = w^{ij} \partial_i f \partial_j g.$$

In fact the tensor w^{ij} gives the Poisson brackets of the co-ordinate functions with each other:

$$\{x^i, x^j\} = w^{ij}.$$

The first condition on the Poisson bracket requires that this tensor be anti-symmetric

$$w^{ij} = -w^{ji}.$$

The second condition (the Jacobi identity) then becomes

$$w^{il} \partial_l w^{jk} + w^{jl} \partial_l w^{ki} + w^{kl} \partial_l w^{ji} = 0.$$

20.12 Conversely, if we are given, on any manifold, an anti-symmetric tensor field satisfying this identity we can define a Poisson bracket on the space functions of that manifold.

20.13 An anti-symmetric tensor field w whose components satisfy the Jacobi identity (**??**) is called a *Poisson tensor* or *Poisson structure*. A manifold along with a Poisson structure is called a *Poisson manifold*.

20.14 The only Poisson bracket on the real line R is the trivial one: $\{f, g\} = 0$ for all f, g. The following is a Poisson bracket on R^2 :

$$\{f, g\} = \partial_1 f \partial_2 g - \partial_1 g \partial_2 f.$$

In this case the Poisson tensor w^{ij} has constant components,

$$w^{12} = -w^{21} = 1, \quad w^{11} = w^{22} = 0.$$

Indeed, any anti-symmetric tensor w^{ij} with constant components (in some co-ordinate system) will give a Poisson bracket on a manifold.

20.15 There are many other Poisson tensors in dimension two; see [**?**] for a classification. On three dimensional Euclidean space, we have the Poisson bracket

$$\{f, g\} = \epsilon_{ijk} x^i \partial_j f \partial_k g$$

where ϵ_{ijk} is the Levi-Civita tensor and x^i the Cartesian co-ordinate system.

20.16 There is an important difference between the two examples above: the Poisson tensor w^{ij} is non-degenerate (has non-zero determinant) in the case of R^2 but it clearly vanishes at the origin in the case of R^3 . Even away from the origin, the tensor $w^{ij} = \epsilon_{ijk} x^k$ has a zero eigenvector:

$$w^{ij} x^j = 0.$$

As a result, a function on R^3 that depends only on the radial distance will have zero Poisson bracket with any function: its derivative points in the radial direction. This leads us to a concept of non-degeneracy.

20.17 A Poisson structure is *non-degenerate* if

$$\{f, g\} = 0 \text{ for all } g \Rightarrow f = \text{ constant.}$$

In other words a Poisson structure is non-degenerate if the corresponding Poisson algebra has a trivial center; i.e., consists of just the multiples of the identity.

20.18 A Poisson structure is non-degenerate if and only if the matrix w^{ij} has non-zero determinant everywhere. Thus, non-degenerate Poisson brackets exist only on even-dimensional manifolds: recall that the determinant of any anti-symmetric matrix of odd dimension is zero.

20.19 On R^{2n}, suppose we denote the first n co-ordinates by q^a (for $a = 1, 2, \cdots n$) and the next n by p_a. The following brackets define a non-degenerate Poisson structure on R^{2n}:

$$\{q^a, q^b\} = \{p_a, p_b\} = 0, \quad \{p_a, q^b\} = \delta_a^b.$$

This is called the *standard* Poisson structure on R^{2n}. The Poisson tensor has components $w^{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in this co-ordinate system. A co-ordinate system in which the above Poisson brackets hold is called a *canonical* co-ordinate system.

20.20 If a Poisson-tensor is non-degenerate, there is always a co-ordinate system that makes it a constant within each co-ordinate neighborhood. The Jacobi identity is just the integrability condition that makes this possible. We can even choose the co-ordinate system so that $w^{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus locally every *non-degerate* Poisson structure is equivalent (upto co-ordinate transformations) to the standard Poisson structure on R^{2n}. This result is known as 'Darboux theorem'. Thus there is no local geometric information in a symplectic structure: there is no symplectic analogue of the curvature of Riemannian geometry.

20.21 Let $\underline{G}$ be a Lie algebra with basis e^i and commutation relations

$$[e^i, e^j] = c_k^{ij} e^k.$$

The structure constants are then anti-symmetric

$$c_k^{ij} = -c_k^{ji}$$

and satisfy the Jacobi identity

$$c_l^{ij} c_m^{lk} + c_l^{jk} c_m^{li} + c_l^{ki} c_m^{lj} = 0.$$

Consider the dual vector space $\underline{G}^*$ of $\underline{G}$ and let ξ^i be the co-ordinate defined on it by the basis e^i of $\underline{G}$. Now we can define a Poisson bracket on $\underline{G}^*$ by

$$\{\xi^i, \xi^j\} = c_k^{ij}\xi^k.$$

The Poisson structure $w^{ij} = c_k^{ij}\xi^k$ satisfies the Jacobi identity due to the identity on the structure constants of a Lie algebra.

20.22 If we regard R^3 as the (dual of the) Lie algebra of the rotation group, its structure constants are just the components of the Levi-Civita tensor. Thus our example of a Poisson structure on R^3 is just a special case of the last example.

20.23 A vector field can be thought of as an infinitesimal transformation of a manifold to itself. To visualize this, think of the points of the manifold as though they are particles in a fluid, with the vector field having the meaning of the velocity of the fluid. Suppose the components of a vector field are $V^i(x)$ in some co–ordinate system. An infinitesimal 'time' t later the point x^i will be mapped to the nearby point $x^i + V^i(x)dt$. By repeating this process we can obtain a one-parameter family of transformations ϕ_t of the manifold to itself. They are solutions to the differential equations

$$\frac{d\phi_t^i(x)}{dt} = V^i(\phi_t(x)),$$

with the initial condition that at 'time' $t = 0$ we get the identity map: $\phi_0^i(x) = x^i$. Moreover, these transformations satisfy the composition law

$$\phi_{t_1}(\phi_{t_2}) = \phi_{t_1+t_2}$$

which means that they form a one-parameter group. More precisely, they define an action of the additive group of real numbers on the manifold.

20.24 Now, each function on a Poisson manifold defines a vector field V_f with components $V_f^i = w^{ij}\partial_j f$. Thus, associated to each observable on the phase space of a classical mechanical systems is a one-parameter group of transformations of the manifold to itself. These are the *canonical transformations* generated by f . Of course any vector field generates such a one parameter group of transformations; the canonical transformations are special in that they leave the Poisson structure invariant. In a sense they are the "inner automorphisms" of the Poisson structure.

20.25 Now we can see another interpretation of the Poisson bracket:

$$\{f, g\} = V_f^i \partial_i g$$

is just the infinitesimal change of g under the canonical transformation generated by f.

20.26 Time evolution of a classical mechanical system is a one-parameter group of canonical transformations; the observable that generates this transformation is the *Hamiltonian*. Thus the Hamiltonian is the single most important observable of a mechanical system. It has the physical meaning of energy.

20.27 A free particle of mass m moving in R^3 has a six-dimensional phase space: its instantaneous state is described by the position co-ordinates q^a, for $a = 1, 2, 3$ and the momentum variables p_a. The Poisson structure is the standard one:

$$\{q^a, q^b\} = 0 = \{p_a, p_b\}, \quad \{p_a, q^b\} = \delta_a^b.$$

The Hamiltonian is just the kinetic energy,

$$H = \frac{p_a p_a}{2m}.$$

Thus we have

$$\{H, q^a\} = \frac{1}{m} p^a, \quad \{H, p_a\} = 0$$

The canonical vector field of H is

$$V_H = \frac{1}{m} p^a \frac{\partial}{\partial q^a}.$$

Suppose the canonical transformations generated by the Hamiltonian map the point (q, p) to $(q(t), p(t))$ at time t. Then,

$$q^a(t) = q^a + \frac{1}{m} p^a t, \quad p_a(t) = p_a.$$

This describes the motion of a free particle in a straight line. Thus the momentum is a constant under time evolution and $\frac{p_a}{m}$ has the meaning of velocity.

20.28 An observable whose Poisson bracket with the Hamiltonian vanishes is a *constant of the motion*: it does not change its value under time evolution.

20.29 In the above example, momentum is also a constant of the motion. The Hamiltonian itself is always a constant of the motion; its conservation is just the law of conservation of energy. (We will only consider 'closed' systems for which the Hamltonian and the constraints have no explicit dependence on time.) The Jacobi identity can be used to show that if f and g are constants of the motion, so is $\{f, g\}$. Thus the set of all constants of the motion form a Lie algebra under the Poisson bracket. Since the product of two constants of motion is also one, the set of constants of motion form a sub-Poisson algebra.

20.30 If a Poisson tensor w is non-degenerate (i.e., the Poisson algebra has trivial center), we can find a two-form ω which is the inverse of w:

$$\omega = \omega_{ij} dx^i \wedge dx^j, \quad \omega_{ij} w^{jk} = \delta_i^k.$$

The Jacobi identity on w becomes a much simpler, *linear*, equation in terms of ω:

$$\partial_i \omega_{jk} + \partial_j \omega_{ki} + \partial_k \omega_{ij} = 0.$$

In other words, ω is a closed two-form. Conversely any closed non-degenerate two-form defines a Poisson tensor.

20.31 A *sympletic form* or *symplectic structure* is a closed nondegenerate two-form. A manifold along with a sympletic structure is a symplectic manifold.

20.32 A symplectic manifold is even-dimensional.

20.33 The standard Poisson structure on $R^{2n} = \{(q^i, p_i), i, j = 1, \cdots n$ arises from the sympletic form

$$\omega = dq^i \wedge dp_i.$$

20.34 The volume form is a sympletic structure on the sphere S^2.

20.35 The rigid body is a system which has led to much beautiful mathematics and physics [?]. If the distance between any two points of a body is fixed as it moves, it is a rigid body. The group of transformations on space that preserve the distance between any pair of points is its isometry group. In the case of R^3 this is the semi–direct product of rotations and translations $O(3) \times R^3$. Any configuration of a rigid body can be mapped into any other by an action of the isometry group of space. [1] Since time evolution is continuous, we can restrict to the connected component of the isometry group. Also, it is convenient to consider the center of mass to be at rest, so that we can restrict ourselves to configurations that can be mapped into each other by rotations. We will study the rigid body in the absence of external torques.

20.36 Thus we can describe the time evolution of a rigid body by a curve $g : R \to SO(3)$ in the rotation group: the configuration at time time t is obtained by a rotation $g(t)$ from some arbitrarily chosen reference configuration. We will use the convention that time evolution is a left action by $g(t)$ on this initial point in the configuration space. The angular velocity of the body relative to space (i.e., w.r.t. an observer in the same reference frame as the center of mass) is then the generator of left translations, hence a *right* invariant vector field of the group $SO(3)$. Explicitly, the angular velocity relative to space is

$$\omega_s = \frac{dg}{dt} g^{-1}$$

which can be thought of as an element of the Lie algebra $\underline{SO(3)}$. The angular velocity in a reference frame moving with the body differs from this by a rotation (adjoint action) by $g(t)$:

$$\omega_c = g^{-1} \frac{dg}{dt}.$$

20.37 The kinetic energy of the system is given by a positive quadratic form A on the Lie algebra,

$$H = \frac{1}{2} A(\omega_c, \omega_c).$$

[1] For a rigid body of generic shape, this action will not have fixed points: any rotation or translations will change the configuration

A is the moment of inertia. (See [?] for a derivation from first principles.) The quantity $L = A(g^{-1}\frac{dg}{dt})$ is the angular momentum relative to the body. The angular momentum relative to space differs from this by a rotation by $g(t)$,

$$R = gLg^{-1}.$$

In the absence of external torques, this is a conserved quantity. The conservation of R leads to the equations of motion:

$$\frac{dL}{dt} = [L, g^{-1}\dot{g}].$$

These are the equations of motion of a rigid body.

20.38 We can solve for the angular velocity to express the equations of motion in terms of the angular momentum. It is convenient to do this in a basis that diagonalizes the moment of inertia:

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$$

We will get,

$$\frac{dL_1}{dt} = -b_1 L_2 L_3, \quad \frac{dL_2}{dt} = b_2 L_3 L_1, \quad \frac{dL_3}{dt} = -b_3 L_1 L_2.$$

These are the famous [?] Euler equations of a rigid body. The constants here are defined by

$$b_1 = a_2^{-1} - a_3^{-1}, \quad b_2 = a_1^{-1} - a_3^{-1}, \quad b_3 = a_1^{-1} - a_2^{-1}.$$

20.39 If the constants a_1, a_2, a_3 are all equal, the Euler equations are trivial to solve: each component of L_a is a constant of the motion. If even a pair of the constants are equal, (say $a_1 = a_2$) then L_3 is conserved and the solution in terms of trigonometric functions is elementary. So we will consider the non-isotropic case from now on. We can assume without any loss of generality that $a_1 < a_2 < a_3$; then b_1, b_2, b_3 are positive.

20.40 The equations for L_a do not involve g ; so it is reasonable to solve them first, forgetting about the variables g for the moment. The phase space is now just R^3 , with the components of angular momentum L_i forming the Cartesian co-ordinate system . The hamiltonian can be expressed as a quadratic function:

$$H = \frac{L_1^2}{2a_1} + \frac{L_2^2}{2a_2} + \frac{L_3^2}{2a_3}.$$

If we postulate on R^3 the Poisson brackets of the earlier example,

$$\{L_i, L_j\} = \epsilon_{ijk} L_k$$

and calculate the Poisson brackets of the angular momentum with the Hamiltonian, we get the Euler equations of motion.

20.41 We have two obvious constants of the motion: the hamiltonian H and the central function

$$L^2 = L_1^2 + L_2^2 + L_3^2.$$

In the case of a non–isotropic rigid body these constant of motion are independent of each other. The surfaces of fixed energy are ellipsoids; the surfaces of fixed L^2 are spheres. Thus the time evolution of the system must take place on the intersection of these two surfaces. Thus it shouldn't be surprising that the Euler equations can be solved explicitly in terms of elliptic functions.

20.42 The Jacobi elliptic functions [?] of modulus k satisfy the quadratic constraints

$$\text{sn}^2(u,k) + \text{cn}^2(u,k) = 1, \quad k^2\,\text{sn}^2(u,k) + \text{dn}^2(u,k) = 1$$

and the differential equations

$$\begin{aligned}
\frac{d\,\text{cn}\,(u,k)}{du} &= -\,\text{sn}\,(u,k)\,\text{dn}\,(u,k), \\
\frac{d\,\text{sn}\,(u,k)}{du} &= \text{cn}\,(u,k)\,\text{dn}\,(u,k), \\
\frac{d\,\text{dn}\,(u,k)}{du} &= -k^2\,\text{sn}\,(u,k)\,\text{cn}\,(u,k).
\end{aligned}$$

It is not a coincidence that these differential equations look analogous to the Euler equations of the rigid body; Jacobi [?] discovered these functions by solving the motion of the rigid body.

20.43 The ansatz

$$L_1 = A_1 \text{ cn } (\omega t, k), \quad L_2 = A_2 \text{ sn } (\omega t, k), \quad L_3 = A_3 \text{ dn } (\omega t, k)$$

will then solve the Euler equations. More precisely the differential equations are reduced to algebraic relations for the constants:

$$\omega A_1 = b_1 A_2 A_3, \quad \omega A_2 = b_2 A_1 A_3, \quad k^2 \omega A_3 = b_3 A_1 A_2.$$

We get, upon solving these,

$$A_1^2 = \frac{1}{b_2}(2H - \frac{L^2}{a_3}), \quad A_2^2 = \frac{1}{b_1}(2H - \frac{L^2}{a_3}), \quad A_3^2 = \frac{1}{b_2}(\frac{L^2}{a_1} - 2H),$$

and

$$\omega^2 = b_2(\frac{L^2}{a_1} - 2H), \quad k^2 = \frac{a_1 b_3 (2a_3 H - L^2)}{a_3 \sqrt{(b_1 b_2)} (L^2 - 2a_1 H)}.$$