# Chapter 10

# Parallel Transport

**10.1** The components of the metric tensor $g = g_{ij}dx^i dx^j$ are constant in cartesian co-ordinates in Euclidean space. In general, there will be no such co-ordinate system . The necessary and sufficient condition for the existence of a local co-ordinate system, in which the metric tensor has constant components, is the vanishing of a quantity called the curvature or Riemann tensor.

**10.2** Consider a geodesic $\gamma : [a, b] \to M$ in a two-dimensional Riemannian manifold (of positive metric). We define an operation called *parallel transport* $P(\gamma) : T_{\gamma(a)}M \to T_{\gamma(b)}M$ which takes each vector from the beginning of the curve to one at the endpoint: $P(\gamma)v$ has the same length as $v$ and makes the same angle with the final tangent vector $\dot{\gamma}(b)$ as the $v$ makes with the initial tangent vector $\dot{\gamma}(a)$ .

**10.3** If each segment of a piecewise smooth continuous curve $\gamma : [a, b] \to M$ is a geodesic, we can define the parallel transport operator on it as the product of parallel transport along each segment. Note that the parallel transport around a closed curve may not be the identity .

**10.4** Any smooth curve can be approximated by geodesic segments; by choosing the segments small enough we can make the approximation as good as necessary. Thus we can define parallel transport on an arbitrary smooth curve as the limit of the transport on these approximations.

**10.5** Thus a geodesic is a curve for which the tangent vector at any point is the parallel transport of the initial tangent vector.

**10.6** Consider two vectors $u, v$ at some point $p \in M$ . These define a parallelogram whose sides are geodesics of infinitesimal lengths $\epsilon_1$ and $\epsilon_2$ . The parallel transport around this parallelogram is $P \sim 1 + R(u, v)\epsilon_1\epsilon_2$ . The operator $R(u, v)$ is bilinear in $u$ and $v$ ; interchanging $u, v$ reverses the direction in which the parallelogram is traversed, so $R(u, v) = -R(v, u)$ . Thus $R$ is a two-form valued in the vector space of linear operators on $T_pM$ . In other words it is a tensor (called the *curvature* or *Riemman tensor*) with one contravariant index and three covariant indices.

**10.7** Riemann found a generalization to higher dimensions of these ideas.

**10.8** The infinitesimal change of the components of vector under parallel transport from $x^i$ to $x^i + dx^i$ is $\delta v^i = -dx^j\Gamma^i_{jk}v^k$ .

**10.9** The difference between the value of the vector field at $x^i + dx^i$ and the parallel tranpsort of $v^j(x)\partial_j$ to $x^i + dx^i$ gives us the notion of the *covariant derivative*:

$$\nabla_j v^i = \frac{\partial v^i}{\partial x^j} + \Gamma^i_{jk}v^k$$

These are the components of a tensor field $\nabla v = \nabla_j v^i dx^j \otimes \partial_i$ of type $(1, 1)$ .

**10.10** *The partial derivatives* $\frac{\partial v^i}{\partial x^j}$ *do not form the components of a tensor.* Neither do the Christoffel symbols $\Gamma^i_{jk}$ . But the covariant derivative is a tensor because the inhomogenous terms in the transformation law for each term cancel out.

**10.11** The parallel transport operator $P(t) : T_{\gamma(t)}M \to T_{\gamma(t)}M$ can be thought of as a matrix once we have a co-ordinate basis $\partial_i$ on the tangent spaces. It is the solution of the ordinary differential equation

$$\frac{dP^i_j(t)}{dt} + \Gamma^i_{jk}\frac{d\gamma^k(t)}{dt} = 0$$

with the initial condition $P^i_j(0) = \delta^i_j$ .

**10.12** A *geodesic vector field* is one whose covariant derivative along itself is zero:

$$v^j \nabla_j v^i = 0.$$

The integral curves of a geodesic vector field are geodesics.

**10.13** The partial derivatives satisfy the symmetry relation $\partial_j \partial_k v^i = \partial_k \partial_j v^i$ . The covariant derivatives do not; the difference $\nabla_j \nabla_k v^i - \nabla_k \nabla_j v^i$ however does not involve a derivative of $v^i$ :

$$\nabla_j \nabla_k v^i - \nabla_k \nabla_j v^i = \left[\partial_j \Gamma^i_{kl} - \partial_k \Gamma^i_{jl} + \Gamma^i_{jm}\Gamma^m_{kl} - \Gamma^i_{jm}\Gamma^m_{kl}\right] v^l.$$

Since the covariant derivative is a tensor, the term within square brackets must be a tensor as well. It is the *Riemann tensor* or *curvature tensor*:

$$R^i_{jkl} = \partial_j \Gamma^i_{kl} - \partial_k \Gamma^i_{jl} + \Gamma^i_{jm}\Gamma^m_{kl} - \Gamma^i_{jm}\Gamma^m_{kl}$$

**10.14** There is another way to understand the curvature tensor. Given three vector fields $u, v, w$ , imagine parallel transporting $u$ along the integral curve of $v$ by a distance $\epsilon_1$ ; then along $w$ by an amount $\epsilon_2$ . Now do the same parallel transports but in reverse order. In general we wont arrive at the same points: they will differ by $[v, w]\epsilon_1\epsilon_2$ . We do one more parallel transport along this curve to arrive at the same point. The difference between the two parallel transports is a tensor:

$$\nabla_v \nabla_w u - \nabla_v \nabla_w u - \nabla_{[v,w]} u = R(v, w)u$$

The components of this tensor $R = R^i_{jkl} dx^j \wedge dx^k \otimes dx^l \otimes \partial_i$ are as given above.

**10.15** A vector field $u$ is *covariantly constant* if

$$\nabla_v u = 0, \forall v.$$

Then

$$R(v, w)u = 0 \forall v, w.$$

Or, $R^i_{jkl} u^l = 0$ . It means that the parallel transport of $u$ along a curve depends only on the endpoint.

**10.16** If and only if the curvature tensor is zero, we can find a system of $n$ linearly independent (over $R$) vector fields in a small enough neighborhood of any point. The metric tensor is then constant in the Riemann normal co-ordinates. Thus the curvature tensor measures locally the departure from Euclidean geometry.

**10.17** The simplest example of a Riemannian manifold of non-zero curvature is a sphere.