Chapter 17

Morse Theory

17.1 In physics and geometry the problem of finding the extrema of a function arise often. For example the time evolution of a classical system is given by the extremum of the action; its static solutions are the extrema of potential energy; a geodesic is an extremum of the distance; the path of a light ray is the extremum of the optical distance.

17.2 Morse theory examines the question of whether a smooth function on a given manifold must have an extremum, even if the function does not have any special symmetries.

17.3 A point $p \in M$ is a *critical point* for a function $f : M \to R$ if $df(p) = 0$; thus a critical point is the same as an extremum. A critical point p is *isolated* if the second derivative $\frac{\partial^2 f}{\partial x^i \partial x^i}$ is a non-degenerate matrix at p . This property is independent of the choice of co-ordinates, at a critical point. For, the matrix of second derivatives $\partial^2 f$ transforms homogenously under co-ordinate transformations: $\partial^2 f \to S\partial^2 f S^T$, where S the matrix of first derivatives of the transformation at p . The matrix $\partial^2 f$ is called the *hessian*.

17.4 A smooth function $f : M \to R$ is a *Morse function* if all its critical points are isolated.

17.5 In the vector space of all smooth functions, the Morse functions form an open subset; the complement of this set has non-zero co-dimension; thus the space of Morse functions is like the set of vectors in R^3 with non-zero component along the x -axis. Thus a Morse function is 'generic'. Other examples of generic subsets are knots (one-dimensional submanifolds in the space of all curves in a 3-manifolds); plane curves that intersect transversally; the orbits on a chaotic system that are not periodic.

17.6 The hessian matrix of a function at a critical point can be brought to the form $\text{diag}\,(+ + + \cdots + - - - \cdots)$ by a choice of co-ordinates. Thus the only co-ordinate invariant information in the Hessian is the number of negative eigenvalues, which is called the *index* of a critical point. If M is compact, any smooth function f on it has a finite number $C_p(f)$ of critical points of index p.

17.7 The basic theorems of the subject are the *Weak Morse Inequalities*: For any Morse function f,

$$C_p(f) \geq \dim H^p(M)$$

and the *Morse index theorem*:

$$\sum_{p=0}^{\dim M} (-1)^p C_p(f) = \sum_{p=0}^{\dim M} (-1)^p \dim H^p(M).$$

17.8 In the other words, the number of critical points of index p is at least equal to the p -th Betti number. Also, the alternating sum of the number of critical points is equal to the Euler characteristic.

17.9 Examples: Any periodic function has a minimum and a maximum within a period, since S^1 has Betti numbers $B^p = 1$ for $p = 0, 1$. Any function on the torus must have at least two saddle points since $b_1(T^2) = 2$. What is the minimum number of critical points of index p of a smooth function on the Grassmannian $Gr_k(C^\infty)$?.