# Chapter 4

# Manifolds

**4.1** We will now study manifolds, the basic objects of geometry.

**4.2** An *atlas* on a connected topological space $M$ is a collection of open subsets $U_\alpha$ with $M = \bigcup_\alpha U_\alpha$ and continuous injective maps $\phi_\alpha : U_\alpha \to R^n$ such that the *transition functions* $\phi_\alpha \circ \phi_\beta^{-1}$ are smooth.

**4.3** The subsets $U_\alpha$ are called *coördinate charts*, and the functions $\phi_\alpha$ the *coördinates*.

**4.4** Two atlases $M = \bigcup_\alpha U_\alpha, \phi_\alpha : U_\alpha \to R^n$ and $M = \bigcup_a V_a, \psi_a : V_a \to R^n$ are said to be *equivalent* if the functions $\phi_\alpha \circ \psi_a^{-1}$ and $\psi_a \circ \phi_\alpha^{-1}$ are smooth. What this means is that the change of coördinates from one atlas to the other are smooth; or equivalently that appending the subsets $V_a$ and coördinate maps $\psi_a$ to the collection $U_\alpha, \phi_\alpha$ gives another atlas.

**4.5** A *manifold* is an equivalence class of atlases on a toplogical space.

**4.6** This is often called a *smooth manifold*. If the transition functions are only required to be continuous we get a *topological manifold*; if they are only $C^k$ we get a $C^k$ - *manifold*. If they are real analytic we get an *analytic* manifold and so on.

**4.7** If we replace in the above definition, $R^n$ by $C^n$ and require the transition functions to be complex analytic, we get a *complex manifold*.

**4.8** The dimension of the Euclidean space where the coördinates take values is called the *dimension* of the manifold.

**4.9** The simplest examples of manifolds are $R^n$ ; the standard coördinate system on it is the identity map. Other familiar coördinate systems such as polar coördinates may not be defined on all of $R^n$ ; in that case we will need to think of $R^n$ as the union of several open subsets $U_\alpha$ with a coördinate in each subset. The transition functions with the standard coördinate is smooth so these define an equivalent atlas and as a manifold it is the same.

**4.10** It is sometimes possible to have inequivalent manifolds based on the same toplogical space. On $R^n$ for $n \neq 4$ any atlas is equivalent to the standard one, so that there is only one manifold structure. It was discovered by Donaldson in 1984 that on $R^4$ there are inequivalent atlases; i.e., that $R^4$ can be turned into a manifold in many different ways.

**4.11** The sphere $S^2$ is a manifold. Choose a point $n$ on the sphere (call it the North Pole); let $s$ be its antipode ("South Pole"). Let $U_1 = S^2 - \{n\}$ and $U_2 = S^2 - \{s\}$ . For each point $p \in U_1$ we define $\phi_1(p)$ to be the point on the tangent plane at $s$ that lies on the straight-line $np$ . Similarly $\phi_2(p)$ is the point where $sp$ intersects the tangent plane at $n$ . The transition functions are smooth so that we have an atlas on $S^2$ . In fact the transition functions are even analytic so the sphere is a real analytic manifold. If we regard $R^2$ as $C$ in the standard way, the transition functions are complex analytic : $\phi_2(p) = -\frac{1}{\phi_1(p)}$ , so that the sphere is a complex manifold as well.

**4.12** The hyperboloid $\{(x, y, z)|z^2 - x^2 - y^2 = 1; z >\}$ is the same as $R^2$ as a manifold: each pair of numbers $(x, y)$ uniquely specifies a point.

**4.13** A function $f : M \to R$ is smooth if each of its coördinate representatives $f \circ \phi_\alpha^{-1} : R^n \to R$ is smooth.

**4.14** Let $M$ and $N$ be manifolds, with atlases $(U_\alpha, \phi_\alpha)$ and $(V_a, \psi_a)$ respectively. A map $f : M \to N$ is smooth if $\psi_a \circ f \circ \phi_\alpha^{-1} : R^m \to R^n$ is smooth for all $a, \alpha$ .

**4.15** A smooth map $f : M \to N$ is a *diffeomorphism* if it has a smooth inverse. Two manifolds are *diffeomorphic* if there is such a map between them.

**4.16** Given a diffeomorphism $f : M \to N$ , we can identify $M$ and $N$ as manifolds. In this sense a diffeomorphism is the analogue of an isomorphism of vector spaces.

**4.17** A one dimensional sphere $S^1$ is diffeomorphic to the real line modulo translations by $2\pi$ : $\theta \mapsto (\cos\theta, \sin\theta)$ is a diffeomorphism.

**4.18** Given two manifolds $M$ and $N$ we can define their product $M \times N$ in the obvious way. $M \times N$ is the set of ordered pairs $\{(p,q)\}, p \in M, q \in N$ ; the open sets are cartesian products of open sets and the co-ordinate functions are $\chi_{\alpha a} = (\phi_\alpha, \chi_a)$ , where $\phi_\alpha$ and $\chi_a$ are the coördinates on $M$ and $N$ respectively.

**4.19** This allows us to construct more examples of manifolds: the torus $T^2 = S^1 \times S^1$ which is also the quotient of $R^2$ by the action of the group of integers; the cylinder $R \times S^1$ ; the higher dimensional Tori $T^n = S^1 \times \cdots S^1$ ; manifolds such as $S^m \times S^n, R^m \times S^n$ .

**4.20** The set of smooth real functions is denoted by $C^\infty(M)$ or just $C(M)$ . It is a commutative ring under pointwise addition and multiplication, called the *coördinate ring* of the manifold. It is not a field since functions may vanish at some points or subsets so that division is not defined.

**4.21** Each closed subset defines an ideal of $C(M)$ : the set of functions that vanish in that subset; the smaller the subset the larger the ideal. It is possible to think of points of $M$ as maximal ideals of $C(M)$ .

**4.22** There is a sense in which the manifold can be reconstructed from its coördinate ring (with an appropriate topology on $C(M)$ .) This has led to ideas about a Non-commutative geometry, based on non-commutative rings.

**4.23** Given a smooth map $\phi : M \to N$ between manifolds, we have a morphism of rings $\phi^* : C(N) \to C(M)$ : $\phi^* f(p) = f(\phi(p))$ . It preserves addition and multiplication;e.g., $\phi^*(fg) = \phi^* f \phi^* g$ . If $\phi$ is a diffeomorphism this is an isomorphism.