Chapter 19

Lie Groups

19.1 A *group* is a set G along with a map (*multiplication*) $G \times G \to G, (g, h) \mapsto gh$ such that: (i) it is associative, (i.e., $(gh)k = (g(hk)$) ;(ii) there is an element $1 \in G$ (the *identity*) with $g1 = 1g = g$;(iii) every element g must have an inverse g^{-1} , with $gg^{-1} = 1 = g^{-1}g$. The axioms imply that the inverse of each element is unique.

19.2 A *homomorphism* $f : G \to H$ is a map from a group to another that preserves the multiplication; $f(gh) = f(g)f(h)$. If a homomorphism is a one-one onto map it is an *isomorphism*. Isomorphic groups have exactly the same sructure.

19.3 A *Lie group* is a group which is also a manifold such that the multiplication map and the inverse map are differentiable.

19.4 The simplest example is the real line with the group operation being *addition*; or equivalently the set of positive numbers with multiplication as the group operation. The set of integers is a group but is not a manifold. The sphere S^2 is a manifold on which there exists no differentiable group operation.

19.5 There are three classical series of Lie groups, the compact simple matrix groups. These are the Lie groups of greatest imprtance in physics. The *orthogonal group* $O(n)$ is the set of all linear maps in a real n -dimensional vector space that preserve a positive inner product. (Upto isomorphism this gives a unique definition). The *Unitary group* $U(n)$ is the analogous object for complex vector spaces. It can be thought of the subgroup of $O(2n)$

that preserves a complex structure: $gJg^T = J$. And finally the *compact sympectic group* $USp(n)$ is the set of elements of $O(4n)$ that preserve a quaternionic structure; i.e., $gIg^T = I, gJg^T = J, gKg^T = K$, with $I^2 = J^2 = K^2 = -1, IJ = -JI = K$. Note that the Unitary group is a real manifold (of dimension n) although it is defined in terms of compex vector spaces.

19.6 Some subgroups of the above groups are also worth noting. $SO(n)$ is the group of orthogonal matrices of determinant one. $O(n)$ is the union of two connected components, one with determinant one and the other with determinant -1 .