# Chapter 12

# Geometrical Optics

**12.1** When the wavelength of light is small, it is possible to approximate its propagation by rays. This is the limit of *geometrical optics*.

**12.2** The speed $v$ of propagation of light in a medium is less than that in vacuum, $c$ . The ratio $\frac{c}{v} = n$ is called the *refractive index*. It can depend on position. For glass it varies in the range $n \sim 1.3 - 1.8$ .

**12.3** Given a curve $\mathbf{r} : [a, b] \to R^3$ in space, its *optical length* is defined to be $\int_a^b n(\mathbf{r}(t))|\dot{\mathbf{r}}(t)|dt$ . This is the length of the curve with respect to the *optical metric* $ds_o^2 = n(x)^2[dx^2 + dy^2 + dz^2]$ . This is a Riemann metric on $R^3$ which can differ from the Euclidean metric by the *conformal factor* $n(x)$ . The angles between vectors with respect to the optical metric are the same as in Euclidean geometry, but the lengths of vectors (and hence curves) are different.

**12.4** The basic law of geometrical optics is *Fermat's principle*: *light propagates along curves of extremal optical length*. In other words *light propagates along the geodesics of the optical metric*.

**12.5** If we parametrize the curve $\mathbf{r}(s)$ by the *Euclidean arc length* $s$ , the tangent vector of unit $\dot{\mathbf{r}}(s)$ *Euclidean length*. Then the *optical length* of the curve is given by $\int n(\mathbf{r}(s))\dot{\mathbf{r}}(s)^2 ds$ ; it is an extremum when the ordinary differential equation

$$\frac{d}{ds}\left(n\frac{d\mathbf{r}}{ds}\right) = \mathbf{grad}\ n$$

is satisfied.

**12.6** Given a function $S : R^3 \to R$, we have a finally of surfaces on which $S$ is a constant. Given the Euclidean metric on $R^3$, we have a family of curves ( *characteristic curves*), which are orthogonal to these surfaces.

**12.7** If we choose $S$ to be the optical distance from a fixed point (say the origin) the characteristic curves are the light rays. Since $\frac{d\mathbf{r}}{ds} \cdot \mathbf{grad}\ S = n$ is the difference between the optical distances between two points, we have

$$(\mathbf{grad}\ S)^2 = n^2.$$

This *eikonal equation* contains the same information as the geodesic equation. The surfaces of constant $S$ are called *wavefronts*.

**12.8** The fundamental physical laws of optics are the Maxwell equations, since light is just a kind of electromagnetic field. It should be possible to derive the eikonal equation and the Fermat principle from the Maxwell equations,

$$\mathbf{curl\ H} = \frac{1}{c}\dot{\mathbf{D}}, \quad \mathbf{curl\ E} = -\frac{1}{c}\dot{\mathbf{B}}$$

$$\mathbf{div\ E} = 0, \quad \mathbf{div\ B} = 0.$$

In addition we have the *constitutive relations*

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}(x).$$

Here we have assumed that the medium is time static and isotropic and that there are no electric charges or currents . The function $\epsilon$ is *dielectric permittivity* and $\mu$ is the *magnetic permeability*.

**12.9** If $\epsilon$ and $\mu$ are constants, there are plane wave solutions: $\mathrm{E}(x,t) = \mathrm{Re}\ \mathbf{e}\ e^{i\mathrm{k}\cdot\mathrm{x}-\omega t}, \mathrm{H}(x,t) = \mathrm{Re}\ \mathbf{h}\ e^{i\mathrm{k}\cdot\mathrm{x}-\omega t}$ . The phase of the wave is constant along the planes $\mathbf{k}\cdot\mathbf{x} = \text{constant}$ .

**12.10** We will assume that $\epsilon$ and $\mu$ are slowly varying functions: i.e., over one wavelength they change by a small amount. In this case, there should be a modification of the plane wave solution, where the wavefronts are some curved surfaces, and the amplitudes $\mathbf{e}(x)$ and $\mathbf{h}(x)$ are slowly varying functions of $x$ :

$$\mathrm{E}(x,t) = \mathrm{Re}\ \mathbf{e}(x)\ e^{ik_0 S(x)-\omega t}, \quad \mathrm{H}(x,t) = \mathrm{Re}\ \mathbf{h}\ e^{ik_0 S(x)-\omega t}$$

Here, $k_0 = \frac{\omega}{c}$ . The wavefronts are the surfaces of constant $S$ .

**12.11** We now put this into Maxwell's equations and ignore derivatives of $\epsilon, \mu, \mathbf{e}, \mathbf{h}$ to get

$$\textbf{grad}\ S \times \mathbf{h} + \epsilon \mathbf{e} = 0, \quad \textbf{grad}\ \times \mathbf{e} - \mu \mathbf{h} = 0.$$

The remaining equations follow from these.

**12.12** This system of algebraic equations have a solution only if

$$(\textbf{grad}\ S)^2 = \epsilon\mu$$

as we can see by eliminating $\mathbf{h}$ and using $\mathbf{e} \cdot \textbf{grad}\ S = 0$ . If we identify $n = \sqrt{[}\epsilon\mu]$ with the refractive index, we get the eikonal equation.

**12.13** How do the amplitudes $\mathbf{e}$ and $\mathbf{h}$ change as a wave propagates along the geodesics?. Clearly they are orthogonal to the direction of propagation $\textbf{grad}\ S$ and to each other. From Maxwell's equations we can derive in a similar way

$$\frac{\partial}{\partial\tau}\mathbf{e} + [\mathbf{e} \cdot \textbf{grad}\ \ln n]\textbf{grad}\ S = 0, \quad \frac{\partial}{\partial\tau}\mathbf{h} + [\mathbf{h} \cdot \textbf{grad}\ \ln n]\textbf{grad}\ S = 0$$

where $\frac{\partial}{\partial\tau} = \textbf{grad}\ S \cdot \textbf{grad}$ is the derivative along the light ray. These are precisely the equations of parallel transport with respect to the optical metric.

**12.14** Thus if two points are connected by two different optical geodesics, there is not only a phase difference between the two paths due to their different optical lengths, but also a relative rotation of polarization due to the parallel transport. This rotation of the polarization (often called the *Pancharatnam*, *geometric* or *Berry phase*) can be observed easily using laser beams propagating along two different optical fibers.

**12.15** Maxwell proposed an optical instrument called the *fish-eye*. It has the refractive index

$$n(x) = \frac{n_0}{1 + \frac{\mathbf{x}^2}{a^2}}.$$

It is an example of an *absolute* optical instrument; i.e., from every point outside the circle of radius $a$ , there are an infinite number of rays that connect it to an image point inside the circle. (This image point is the inversion with respect to the circle of the original point.) The rays are circles orthogonal to the circle of radius $a$. These facts follow easily once we realize that the optical metric corresponding to the fish-eye is the *standard metric on a three-dimensional sphere* of radius $a$ :

$$ds_o^2 = \frac{1}{1 + \frac{\mathbf{x}^2}{a^2}}[dx^2 + dy^2 + dz^2].$$