Chapter 15

General Relativity

15.1 *General Relativity* is the most accurate theory of gravity, due to Albert Einstein.

15.2 We saw that in special relativity, space-time is described as a flat Riemannian manifold of signature $-+++$. The co-ordinate systems in this manifold correspond physically to reference frames. The special co-ordinate systems in which the metric has the constant form

$$ds^2 = -c^2 dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

correspond to *inertial reference frames*; i.e., those that move at a constant velocity.

15.3 However, in geometry we do not allow any special significance to a particular class of co-ordinate systems; we therefore seek a theory of relativity which is valid in any reference frame, even accelerating ones. That is, we postulate the *principle of general relativity*:

The laws of Physics are the same in all reference frames.

15.4 Such a theory must necesarily include a description of gravity. For, we also have the *Principle of Equivalence*, justified by direct observation:

All test-particles have the same acceleration due to gravity, regardless of their composition.

Here, a test-particle is one size and mass are small compared to the size and mass of the body creating the gravitational field: for example an artificial satellite would be a test-particle for the Earth's gravitational field; the earth itself is a test–particle for the sun; the sun is one for the cosmological gravitational field due to all the galaxies in the universe and so on.

15.5 Thus by local meausrements we cannot distinguish between a gravitational field and a reference which is accelerating at a constant rate: a uniform gravitational field is just flat space-time viewed in a non-inertial co-ordinate system.

15.6 We have not succeeded yet in producing a quantum theory that satisfies the above principles (perhaps they need to be modified.) Einstein produced a theory of gravity that satisfied both of these principles.

15.7 The first postulate of this theory is

Axiom 15.1 Space-time is a four-dimensional Riemannian manifold, whose metric g is of signature $-+++$.

If the metric is the Minkowski metric on R^4, there is no gravitational field. Thus the curvature of the metric is a measure of the strength of the gravitational field.

15.8 The next axiom satisfies the equivalence principle since it treats all test-particles the same way, regardless of constitution:

Axiom 15.2 The metric tensor g of space-time describes the gravitational field; in the absence of forces other than gravity, test-particles move along time-like geodesics of g.

Thus if the path of a particle is described by a function $x^\mu(s)$ of arc-length s, it satisfies the differential equation

$$\frac{d^2x^\mu(s)}{ds^2} + \Gamma^\mu_{\nu\rho}(x(s))\frac{dx^\nu(s)}{ds}\frac{dx^\rho(s)}{ds} = 0.$$

15.9 Thus in Minkowski space particles will move along time-like straight lines.

15.10 In the limit when the velocities of particles are small compared to the velocity of light, this should reduce to the newtonian equations of motion

$$\frac{d^2\mathbf{x}(t)}{dt^2} + \mathbf{grad}\ \phi(x(t)) = 0$$

where ϕ is the gravitational potential. In this limit,

$$s \sim ct = x^0, \quad \frac{dx^0}{ds} \sim 1, \quad \frac{d\mathbf{x}}{ds} \sim \frac{d\mathbf{x}}{cdt}$$

so that we recover the newtonian limit if we identify

$$g_{00} \sim 1 + c^{-2}\phi.$$

15.11 The source of the gravitational field in the newtonian limit is mass-density ρ :

$$-\nabla^2\phi = 4\pi G\rho$$

where G is Newton's constant of gravitation.

15.12 Already in special relativity, mass and energy are interchangeable: they are not separately conserved.

15.13 Moreover, energy has to combined with momentum to obtain a Lorentz co-variant notion, that of four-momentum. The density of energy-momentum is the *stress-energy* energy tensor $T_{\mu\nu}$: the mass-energy density is $\rho = T_{00}$ and momentum density is T_{0i} for $i = 1, 2, 3$. The components T_{ij} describe the stress: the pressure exterted by matter in the direction i on a surface whose normal points along the co-ordinate direction j . The conservation of energy-momentum requires this tensor to satisfy

$$\partial^\mu T_{\mu\nu} = 0$$

in special relativity (i.e., in flat space-time). Moreover, conservation of angular momentum requires it to be symmetric:

$$T_{\mu\nu} = T_{\nu\mu}.$$

15.14 In general relativity, matter should described a symmetric tensor satisfying the covariant conservation law

$$\nabla^\mu T_{\mu\nu} = 0.$$

15.15 The gravitationl field must be described by an equation of the form $G_{\mu\nu} = 4\pi G T_{\mu\nu}$, where $G_{\mu\nu}$ must be a tensor built from the metric tensor which is (i) symmetric,, (ii) satisfies the identity $\nabla^\mu G_{\mu\nu} = 0$, (iii) must involve at most second derivatives of the components of the metric tensor. The latter requirement is suggested by the fact that the newtonian equation for the gravitational potential is a second order partial differential equation.

15.16 The curvature tensor $R^{\mu}_{\nu\rho\ \sigma}$ involved second derivatives of the metric tensor. From it we can construct the symmetric tensor $R_{\mu\nu} = R^{\rho}_{\mu\rho\ \nu}$ called the *Ricci tensor*; any other contraction will produce the same tensor upto a sign or zero. Further contraction yields the *Ricci scalar*, $R = g^{\mu\nu} R_{\mu\nu}$. Thus it is reasonable to suppose that $G_{mu\nu}$ is a linear combination $G_{\mu\nu} = aR_{\mu\nu} + bg_{\mu\nu} R$. The Bianchi identity on the curvature tensor gives $2\nabla^\mu R_{\mu\nu} = \partial_\mu R$. Thus a possible choice satisfying the identity $\nabla^\mu G_{\mu\nu} = 0$ is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

This is called the *Einstein tensor*.

Axiom 15.3 The field equations of general relativity that determine the metric tensor are $G_{\mu\nu} = 8\pi G T_{\mu\nu}$.

15.17 It is possible to modify the above equations while retaining the property that they are second order partial differential equations:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

This modification of the theory depends on the *cosmological constant* Λ ; (barring some recent unconfirmed observations) this parameter is found to be zero and we will from now on let it vanish.

15.18 Einsteins equations also follow from a variational principle, due to Hilbert:

$$S = \int \surd[-\det g] R d^4 x + 8\pi G S_m$$

where S_m is the action of the matter field. The point is that the variational derivative of the first term is the Einstein tensor and that of the second term the stress-energy tensor.