Chapter 14

Electrodynamics

14.1 The equations that determine the electric and magnetic fields in vacuum with electric charge density ρ and current density $\mathbf{j}$ are **Maxwell's equations**:

$$\mathbf{div}\ B = 0, \quad \mathbf{curl}\ E = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

and

$$\mathbf{div}\ E = 4\pi\rho, \quad \mathbf{curl}\ B = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}.$$

14.2 It is a consequence that the electric charge is conserved:

$$\frac{1}{c}\frac{\partial \rho}{\partial t} - \mathbf{div}\ \mathbf{j} = 0.$$

14.3 A charge distribution that is static (i.e., with **j=0**) to one observer will look moving to another. Under Lorentz transformations, ρ and $\mathbf{j}$ transform into each other: they together transform as a four vector $j = (\rho, \mathbf{j})$. The conservation of charge then is the partial differential equation

$$\partial_\mu j^\mu = 0.$$

14.4 Electric and magnetic fields are transformed into each other as well under Lorentz transformations. Together they must transform as a tensor with six independent components. The simplest possibility is that they form a two-form:

$$F = F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad F_{0i} = E_i, \quad F_{12} = B_3, F_{23} = B_1, F_{31} = B_2.$$

This is indeed the correct trasnformation law.

14.5 The pair Maxwell equations that do not involve sources can now be combined into a single equation that is Lorentz invariant:

$$dF = 0.$$

This means within any local neighborhood there is a one-form A such that $F = dA$. In other words, with $A = A_0 dx^0 + A_i dx^i$

$$\mathbf{E} = \frac{\partial \mathbf{A}}{c\partial t} - \mathbf{grad}\ A_0, \quad \mathbf{B} = \mathbf{curl}\ \mathbf{A}.$$

Thus A_0 is the *electrostatic potential* and $\mathbf{A}$ the **vectorpotential** .

14.6 Note that if we add the deirivative of a scalar to A the electromagnetic field is invariant:

$$A \mapsto A + d\lambda, F \mapsto F.$$

Thus only functions of A that are invariant under this *gauge transformation* are of physical significance.

14.7 The remanining Maxwell's equations can be written as

$$\partial^\mu F_{\mu\nu} = j_\nu.$$

The conservation of current is an immediate consequence of the anti-symmetry of $F_{\mu\nu}$.

14.8 We can derive the aove equation from a variational principle, where A is thought as the fundamental variable. The only gauge and Lorentz invariant quantities that can be formed from the first derivatives of A are

14.9 $\int F_{\mu\nu}F_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma}d^4x$ and $\int F\wedge F$. The latter can be excluded because it is a total derivative, $F\wedge F = d(A\wedge F)$: it would lead to trivial equations of motion. The combination $\int j_\mu A_\mu d^4x$ is gauge invariant when the current is conserved. The Maxwell equations are the Euler-Lagrange equations of the action

$$S = -\frac{1}{4}\int F_{\mu\nu}F_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma}d^4x + \int j^\mu A_\mu d^4x.$$

14.10 If we have a particle moving along a worldline $\xi(\tau)$, the quantity $\int_\xi A = \int A_\mu(\xi(\tau))\frac{d\xi^\mu}{d\tau}d\tau$ is gauge invariant. The action principle

$$m\int\sqrt{\left[\eta_{\mu\nu}\dot{\xi}^\mu\dot{\xi}^\nu\right)\right]}d\tau + e\int A_\mu(\xi(\tau))\frac{d\xi^\mu}{d\tau}d\tau$$

is thus gauge, Lorentz and reparametrization invariant. It leads to the equation of motion

$$m\frac{d^2x^\mu}{ds^2} = eF^\mu{}_\nu\frac{d\xi^\nu}{ds}$$

where s is the proper-time parameter. The quantity on the r.h.s. is the force on a particle due to an electromagnetic field.

14.11 In the above we have ignored the fact that an accelerating charge will radiate. This in turn will change its acceleration: there is a recoil due to the momentum carried away by the radiation. Dirac derived a remarkable equation for the motion of a charged particle including these effects. It is a third order non-linear equation. See the book by A. O. Barut for details.