Chapter 6

Differential Forms

6.1 Notice that the division operation on scalars is never used in the axioms defining a vector space. The same axioms where the axioms take values in a ring is called a *module*. Its elements are called *vector fields*.

6.2 The dual of a module, tensor products etc. are all defined analogously. There are some important differencers however: there may not exist a basis for example.

6.3 The space of vector fields on a manifold $V(M)$ is a module over the ring of smooth functions $C(M)$. Any vector field can be multiplied on the left by a function: $(fX)(g) = f(Xg)$ in a way that satisfies all the axioms of a module.

6.4 The dual of the module of vector fields is the space of *one-forms* or *co-vector fields*. Thus a one-form $\omega : V(M) \to C(M)$ is a map that associates to every vector field v a function $\omega(v)$ such that $\omega(fv) = f\omega(v)$.

6.5 Sometimes $\omega(v)$ is also denoted by $i_v\omega$ and may be called the *contraction* of the pair (ω, v) .

6.6 In an an analogous way we can define a covariant tensor field T of order r as a map, linear over $C(M)$, of an ordered set of r vector fields, $T(v_1, .. \cdots v_r)$. It is very important that the property of linearity $T(f_1v_1, f_2v_2, \cdots f_rv_r) = f_1f_2 \cdots f_rT(v_1, \cdots v_r)$ hold for arbitrary smooth functions $f_1, \cdots f_r)$ and not just for constants.

6.7 Contravariant tensor fields are $C(M)$ -linear maps of an ordered set of r one-forms; mixed tensor fields of order (r,s) depend linearly on r one-forms and s vector fields. Symmetric and antisymmetric tensor fields are defined as before for tensors.

6.8 Covariant anti-symmetric tensor fields are also called *differential forms*.

6.9 An example of a one-form is the *exterior derivative* of a function, defined by $df(X) = Xf$. Thus the exterior derivative is a generalization of the gradient of a function familiar from vector calculus.

6.10 Recall that there is an associative multiplication on the space $\mathcal{T}(M) = \oplus_{r=0}^{\infty}\mathcal{T}^r(M)$ of all covariant tensors, the direct product:

$$\alpha \otimes \beta(u_1, \cdots u_{r+s}) = \alpha(u_1, \cdots u_r)\beta(u_{r+1} \cdots u_{r+s}).$$

Also there is a projection map $\lambda : \mathcal{T}^r(M) \to \Lambda^r(M)$ which picks out the anti-symmetric part of a covariant tensor:

$$\lambda(\alpha)(u_1, \cdots u_r) = \frac{1}{r!}\sum_{\pi \in S_r} \text{sgn }(\pi)\alpha(u_{\pi(1)} \cdots u_{\pi(r)}).$$

6.11 Conbining these gives us a multiplication on the space of differential forms called the *exterior product* or the *wedge product*:

$$\alpha \wedge \beta = \lambda(\alpha \otimes \beta).$$

The wedge product is graded commutative; i.e.,

$$\alpha \in \Lambda^r(M), \beta \in \Lambda^r(M) \Rightarrow \alpha \wedge \beta \in \Lambda^{r+s}(M) \text{ and } \quad \alpha \wedge \beta = (-1)^r \beta \wedge \alpha$$

Moreover it is associative.

6.12 Every differential form $\alpha \in \Lambda^r(M)$ can be written in the form $\alpha = f_{i_1 \cdots i_r} dg^{i_1} \wedge dg^{i_r}$ for some functions $f_{i_1, \cdots i_r}, g_i \in C(M)$. The number of functions g^i needed to do this can exceed the dimension of the manifold; also, the differential form may not have a unique set of 'components' $f_{i_1, \cdots i_r}$ even for a fixed set of g_i . Within a coördinate neighborhood $U \subset M$, there is always a special class of functions, the coördinates x^i . A differential form can be expressed *uniquely* in terms of these coördinates:

$$\alpha = \alpha_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots dx^{i_r}, \quad \alpha_{i_1 \cdots i_r} \in C(U).$$

6.13 The basis dx^i for one-forms is dual to the basis ∂_j for vector fields in U : $dx^i(\partial_j) = \delta^i_j$.

6.14 Differential forms are special among tensor fields in that there is a natural notion of differentiation , the *exterior derivative*, $d : \Lambda^r(M) \to \Lambda^{r+1}(M)$. It satisfies the axioms

$$d[\alpha + \beta] = d\alpha + d\beta, \quad d[\alpha \wedge \beta] = [d\alpha] \wedge \beta + (-1)^r \alpha \wedge d\beta, \quad d(d\alpha) = 0, \forall \alpha.$$

Moreover, on zero forms it is the same as the derivative defined earlier.

6.15 The explicit formula in terms of components follows easily

$$d\alpha = d\alpha_{i_1 \cdots i_r} \wedge dx^{i_1} \wedge \cdots dx^{i_r}$$

or even

$$d\alpha = \partial_{i_1} \alpha_{i_2 \cdots i_{r+1}} dx^{i_1} \wedge \cdots dx^{i_{r+1}}.$$

6.16 In electrodynamics, the magnetic field is the curl of the vector potential. In our language the magnetic field is really a two-form B in R^3 and its potential a one-form A , with $B = dA$. If we add the derivative of a scalar field to A , the magnetic field remains unchanged: $d[A + d\phi] = dA + d(d\phi) = dA$. This symmetry is called gauge invariance and is of enormous importance.

6.17 In relativistic langauge the electrostatic potential A_0 and the magnetic potential A above combine to a one-form in four-dimensional Minkowski space, $A = A_0 dx^0 + A_i dx^i$; the electric and magnetic fields combine to give a two form in Minkowski space $F = F_{0i} dx^0 \wedge dx^i + F_{ij} dx^i \wedge dx^j$. Here, $E_1 = F_{01}$ etc. and $B_1 = F_{23}$ etc. Then $F = dA$ and we still have a gauge invariance under $A \mapsto A + d\phi$.

6.18 The subset of Maxwell's equations that do not involve sources become now $dF = 0$; we can understand this as a necessary condition for the existence of a potential A such that $F = dA$. We will see later how to understand the rest of Maxwell equations in terms of differential forms.