Chapter 21

Deformation Quantization

21.1 The set of instantaneous states of a quantum system form a complex Hilbert space $\mathcal{H}$; the algebra of linear operators is an associative algebra; the self-adjoint operators are the *observables*. Note that the set of observables is not an algebra, since the product of two observables is not always an observable, because it may not be self-adjoint.

21.2 In the limit as $\hbar \to 0$, the observables must form a Poisson algebra. To see how this happens let us start with a system whose phase space is $R^{2n} = \{(x,p)\}$ with the standard Poisson bracket

$$\{A, B\} = \frac{\partial A}{\partial x^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x^i}$$

21.3 The usual rules of quantum mechanics (Schrodinger quantization) say that the quantum Hilbert space $\mathcal{H} = L^2(R^n)$. An operator $\hat{A}$ in this space can always be represented by an integral kernel

$$\hat{A}\psi(x) = \int \tilde{A}(x,y)\psi(y)dy.$$

Self-adjoint operators have hermitean kernels: $\tilde{A}(x,y) = \tilde{A}^*(y,x)$. The product of operators cooresponds to a multliplication of the integral kernels

$$\widetilde{AB}(x,y) = \int \tilde{A}(x,z)\tilde{B}(z,y)dz.$$

21.4 We expect that corresponding to each operator $\hat{A}$ there is a function $A(x,p)$ in the classical phase space. Given an operator with integral kernel $\tilde{A}(x,y)$ we define its *Weyl symbol*

$$A(x,p) = \int \tilde{A}(x+\frac{z}{2}, x-\frac{z}{2})e^{-\frac{i}{\hbar}p.z}dz.$$

This is a partial Fourier transform of the kernel, where the relative co-ordinate is replaced by its Fourier conjugate. The inverse map associates to every function on the phase space a linear operator with integral kernel) *correspondence principle*)

$$\tilde{A}(x,y) = \int A(\frac{x+y}{2}, p)e^{\frac{i}{\hbar}p\cdot(x-y)}[dp].$$

Here $2\pi\hbar$ is Planck's constant and $[dp]$ stands for $\frac{d^n p}{(2\pi\hbar)^n}$.

21.5 This map from functions on the phase space and operators on the Hilbert space has all the properties we expect of the classical to quantum correspondence. Real functions go to hermitean integral kernels and vice versa. The trace of an operator is the integral of its symbols:

$$\mathrm{tr}\hat{A} = \int \tilde{A}(x,x)dx = \int A(x,p)dx[dp].$$

21.6 A function independent of momenta correspond to a multiplication operator:

$$\hat{A}\psi(x) = A(x)\psi(x)$$

as we expect in the Schrodinger picture of quantum mechanics.

21.7 The operator corresponding to p_i is the derivative operator : $\hat{p}_k = \frac{\hbar}{i}\frac{\partial}{\partial x^k}$. The simplest way to see this to consider the function $A(x,p) = e^{ip_i a^i}$ whose operator kernel is

$$\tilde{A}(x,y) = \delta(x + \hbar a - y)$$

so that the operator $\hat{A} = e^{\hbar a^k \partial_k}$ is the translation by $\hbar a$. For infinitesimally small a we get the derivative operator.

21.8 A polynomial in p with coeffiecients which are functions of x corresponds to the operator in which p is replaced by the derivative operator, with the *symmetric* ordering rule:

$$f(x)p^{l_1}\cdots p^{l_b} \to \frac{1}{b+1}\sum_{a=0}^{b}\hat{p}^{l_1}\cdots\hat{p}^{l_a}f(x)\hat{p}^{l_{a+1}}\cdots\hat{p}^{l_b}.$$

For example,

$$x^k p_l \to -i\hbar\frac{1}{2}[x^k\partial_l + \partial_l x^k].$$

Thus we recover the usual form of the correspondence principle used in elementary quantum mechanics.

21.9 It is important that the Weyl symbol is not just the classical approximation to the quantum operator. It contains *all* the information in the operator, indeed the operator is determined by its symbol by the above Fourier transformation. Thus it should be possible to translate the multiplication of operators to the classical language: $A \circ B$ is the symbol of the operator $\hat{A}\hat{B}$. it must correspond to some multiplication law for function in phase space.

21.10 The above formulae yield the following result:

$$A\circ B(x,p) = \left\{e^{\frac{-i\hbar}{2}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial p_i'}-\frac{\partial}{\partial p_i}\frac{\partial}{\partial x^{i\prime}}\right)}A(x,p)B(x',p')\right\}_{x=x';p=p'}.$$

To understand this better we expand in powers of $\hbar$:

$$\tilde{A}\circ\tilde{B}(x,p) = \sum_{n=0}^{\infty}\left(\frac{-i\hbar}{2}\right)^n\frac{1}{n!}\{\tilde{A},\tilde{B}\}_{(n)}.$$

where

$$\{\tilde{A},\tilde{B}\}_{(n)} = \sum_{r=0}^{n}(-1)^r\tilde{A}^{j_1\cdots j_r}_{i_1\cdots i_{n-r}}\tilde{B}^{i_1\cdots i_{n-r}}_{j_1\cdots j_r}$$

where $\tilde{A}^i = \frac{\partial\tilde{A}}{\partial p_i}$ and $\tilde{A}_i = \frac{\partial\tilde{A}}{\partial x^i}$, $A^{j_1\cdots j_s}_{i_1\cdots i_r} = \frac{\partial^{r+s}A}{\partial p_{j_1}\cdots\partial p_{j_s}\partial x^{i_1}\cdots\partial x^{i_r}}$ etc. It is possible to prove by induction on the order that this multiplication is associative.

21.11 The first order term is just the Poisson bracket:

$$A \circ B(x,p) = A(x,p)B(x,p) - \frac{i\hbar}{2}\{A, B\} + O(\hbar^2)$$

Thus, *the Poisson bracket is the first order deviation way from the pointwise product of observable.* The Poisson bracket is a remnant in the classical theory of the non-commutativity of quantum mechanics.

21.12 Thus quantization is nothing but a change (*deformation*) in the multiplication law for functions on phase space, to a new law that is associative but not commutative. The product involves the value of the function at nearby points as well: it is nonlocal in phase space. This is a consequence of the uncertainty principle, the points in phase space no longer correspond to states in the quantum theory.

21.13 The deformation must be such that to first order it reproduces the Poisson bracket. But this does not fix it uniquely: there may be more than one quantum theory corresponding to a given classical theory. We have given the rules that give the usual Schrodinger quantization. Such ambiguities are present in any quantization scheme.

21.14 Deformation quantization gives a convenient starting point for deriving the semi-classical approximation. A typical question of interest in the quantum mechanics is the determination of the spectrum of an operator $\hat{h}$: i.e., the set of singularities of the resolvent operator $\hat{R}(E) = (\hat{h} - E)^{-1}$. The symbol of the resolvent operator satisfies

$$R(E) \circ (h - E) = 1.$$

21.15 This can be solved as a power series in $\hbar$:

$$R(E) = \sum_{k=0}^{\infty} R_{(k)}(E)\hbar^k,$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{-i\hbar}{2}\right)^n \frac{1}{n!}\hbar^k\{R_{(k)}(E), h - E\}_{(n)} = 1.$$

Equating the powers of $\hbar$ on both sides of this equation, we get a set of recursion relations

$$R_{(0)}(E) = (h - E)^{-1},$$

$$R_{(m)}(E) = -\sum_{n=1}^{m} (\frac{-i}{2})^n \frac{1}{n!} \{R_{(n-m)}(E), h\}_{(n)} (h - E)^{-1}.$$

21.16 Of particular interest is the case where h is of the form:

$$h(x, p) = t(p) + v(x).$$

In this case the mixed derivatives in the generalized Poisson brackets vanish and we get

$$\{R_{(k)}, h\}_{(n)} = R_{(k)i_1\cdots i_n} t^{i_1\cdots i_n} + (-1)^n \tilde{R}_{(k)}^{i_1\cdots i_n} v_{i_1\cdots i_n}.$$

If $t(p) = p_i p_i$ as in nonrelativistic quantum mechanics, there is only one term for $n > 2$,

$$\{R_{(k)}, h\}_{(n)} = (-1)^n R_{(k)}^{i_1\cdots i_n} v_{i_1\cdots i_n}$$

while

$$\{R_{(k)}, h\}_{(1)} = 2p_i R_{(k)i} - R_{(k)}^{i} v_i$$

and

$$\{R_{(k)}, h\}_{(2)} = 2R_{(k)ii} + R_{(k)}^{ij} v_{ij}.$$

In fact, $R_{(1)} = 0$ and

$$R_{(2)}(E) = \frac{1}{2(h - E)^2} [\frac{v_i v_i}{h - E} + 2(p_i v_i)^2 - v_i i].$$

These formalae are useful for example in deriving the Thomas-Fermi theory in atomic physics.

21.17 In general the observables of a classical mechanical system form a Poisson algebra. Kontsevich has found a way of deforming such a general Poisson algebra into an associative algebra, a generalization of the above infinite series. The physical applications are still being worked out.