Chapter 11

Connection and Curvature

11.1 We can define the notion of covariant derivative in an axiomatic way, analogous to the way we introduced the exterior derivative.

11.2 A *covariant derivative* or *connection* on a differential manifold is a map $\nabla : \mathcal{T}^r_s(M) \to \mathcal{T}^r_{s+1}(M)$ such that

$$\nabla(S \otimes T) = \nabla S \otimes T + S \otimes \nabla T \tag{11.1}$$
$$\nabla f = df, \forall f \in \mathcal{T}^0_0(M) = C(M) \tag{11.2}$$
$$d\left[\phi(u)\right] = \nabla\phi(u) + \phi\left[\nabla u\right] \tag{11.3}$$

11.3 We can translate these into tensor index notation

$$\nabla_m(S^{i_1\cdots i_p}_{j_1\cdots j_q} T^{k_1\cdots k_r}_{l_1\cdots l_s}) = \left[\nabla_m S^{i_1\cdots i_p}_{j_1\cdots j_q}\right] T^{k_1\cdots k_r}_{l_1\cdots l_s} + S^{i_1\cdots i_p}_{j_1\cdots j_q}\left[\nabla_m T^{k_1\cdots k_r}_{l_1\cdots l_s}\right] \tag{11.4}$$
$$\nabla_i f = \partial_i f, \forall f \in \mathcal{T}^0_0(M) = C(M) \tag{11.5}$$
$$\partial_i\left[\phi_j u^j\right] = \left[\nabla_i \phi_j\right] u^j + \phi_j\left[\nabla_i u^j\right] \tag{11.6}$$

11.4 The covariant derivative of a vectorfield satisfies as a consequence

$$\nabla_i\left[f u^j\right] = \partial_i f u^j + f\nabla_i u^j, \forall f \in C(M).$$

It follows that the covariant derivative must be involve at most first derivatives of the components of u :

$$\nabla_i u^j = \partial_i u^j + \Gamma^j_{ik} u^k$$

for some some set of quantities Γ^j_{ik} called the *connection coefficients*.

11.5 The covariant derivatives on one-forms has a similar form. The third axiom now determines the connection coefficients of the one-forms in terms of the very same Γ :

$$\nabla_i \phi_j = \partial_i \phi_j - \Gamma_{ij}^k \phi_k.$$

11.6 The first axiom now gives the general formula for covariant derivative (a tensor is a linear combination of tensor products of vector fields and one-forms)

$$\begin{aligned} \nabla_k T^{i_1 \cdots i_p}_{j_1 \cdots j_q} = & \\ & \nabla_k T^{i_1 \cdots i_p}_{j_1 \cdots j_q} + \\ \Gamma^{i_1}_{kl} T^{l i_2 \cdots i_p}_{j_1 \cdots j_q} \quad + & \quad \Gamma^{i_2}_{kl} T^{i_1 l i_3 \cdots i_p}_{j_1 \cdots j_q} + \cdots \Gamma^{i_p}_{kl} T^{i_1 i_2 \cdots l}_{j_1 \cdots j_q} \\ -\Gamma^{l}_{k j_1} T^{i_1 i_2 \cdots i_p}_{l j_2 \cdots j_q} \quad - & \quad \Gamma^{l}_{k j_2} T^{i_1 i_2 \cdots i_p}_{j_1 l j_3 \cdots j_q} - \cdots \Gamma^{l}_{k j_q} T^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots l}. \end{aligned} \tag{11.7}$$

11.7 The covariant derivative along a vector u of a tensor is defined by contracting it with the connection:

$$\nabla_v T^{i_1 \cdots i_p}_{j_1 \cdots j_q} = v^k \nabla_k T^{i_1 \cdots i_p}_{j_1 \cdots j_q}$$

11.8 The parallel transport operator $P(t) : T_{\gamma(a)} M \to T_{\gamma(t)} M$ along a curve $\gamma : [a, b] \to M$ is given by the solution of the ordinary differential equation

$$\frac{dP^i_j(t)}{dt} + \Gamma^i_{jk} \frac{d\gamma^k(t)}{dt} = 0$$

with the initial condition

$$P^i_j(a) = \delta^i_j.$$

The parallel transport depends on the curve chosen in general, not just on the endpoints.

11.9 The anti-symmetric part of the second covariant derivative on functions defines a tensor called *Torsion*

$$\nabla_i \nabla_j f - \nabla_i \nabla_j f = T_{ij}^k \partial_k f, \quad T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

Although Γ_{ij}^k do not transform as the components of a tensor, its antisymmetric part does.

11.10 We will only consider connections for which the torsion tensor vanishes. This will be an extra axiom:

$$\nabla_u \nabla_v f - \nabla_v \nabla_u f - \nabla_{[u,v]} f = 0, \forall f \in C(M).$$

11.11 Given a Riemann metric $g = g_{ij} dx^i \otimes dx^j$ it is natural to restrict ourselves to connections that preserve the inner prodcuts of vector fields. This is the axiom

$$\nabla g = 0$$

11.12 There is a unique connection on a Riemann manifold (M, g) satisfying the above axioms. To see this we use the symmetry of the connection coefficients $\Gamma_{jk}^i = \Gamma_{kj}^i$ and the equation

$$\nabla_i g_{jk} = \partial_i g_{jk} - \Gamma_{ij}^l g_{lk} - \Gamma_{ik}^l g_{jl} = 0$$

to solve for the Γ_{jk}^i :

$$g_{il} \Gamma_{jk}^l = \frac{1}{2} \left[\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk} \right].$$

11.13 The difference of covariant derivatives taken in two different order gives the *curvature tensor* or *Riemann tensor*:

$$\nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w = R(u, v) w.$$

It is a non-trivial calculation to check that this is indeed a tensor:

$$R(fu, v) w = f R(u, v) w, \quad R(u, v)[fw] = f R(u, v) w.$$

This can also be thought of the difference between parallel transports along two different infinitesimally small curves, with sides $u\epsilon_1, v\epsilon_2$ and $v\epsilon_2, u\epsilon_1, [u, v]\epsilon_1\epsilon_2$ respectively.

11.14 In terms of a co-ordinates basis, we have $R = R^k_{ijl} dx^i \otimes dx^j \otimes \partial_k \otimes dx^l$
.

$$R^k_{ijl} = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{im}\Gamma^m_{jl}$$

The curvature tensor satisfies the identities

$$R^k_{ijl} = -R^k_{jil}, \quad R^k_{ijl} + R^k_{jli} + R^k_{lij} = 0.$$

as well as the *Bianchi identity*

$$\nabla_m R^i_{jkl} + \nabla_k R^i_{mjl} + \nabla_j R^i_{kml} = 0.$$

11.15 The parallel transport operator preserves the length of a vector. The curvature tensor, being the infinitesimal parallel transport operator around a closed curve, must preserve the length of a vector:

$$g(R(v,w)u, r) + g(u, R(v,w)r) = 0.$$

If we define $g_{km} R^m_{ijl} = R_{ijkl}$, we have thus $R_{ijkl} = -R_{ijlk}$.

11.16 Finally, we can show that $R_{ijkl} = R_{klij}$.

11.17 From the Riemann tensor we can construct some other standard objects of Riemannian geometry. The *Ricci tensor* is

$$R^j_{ijl} = R_{il}$$

and its trace is the *Ricci scalar*

$$R = g^{jk} R_{jk}.$$

11.18 A Riemannian manifold is *flat* if there is a co-ordinate system such that the metric tensor has constant components. The basic theorem of Riemannian geometry is that *a Riemannian manifold is flat if and only if the curvature tensor vanishes everywhere.* In fact the metric tensor has constant components in Riemann normal co-ordinates.