Chapter 16

Cohomology

16.1 Recall that the exterior derivative $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$ satisfies

$$d(d\omega) = 0$$

for any ω . A form ϕ is *closed* if $d\phi = 0$; it is *exact* if there is a form ω such that $\phi = d\omega$. Clearly all exact forms are closed.

16.2 Let $Z^p(M)$ be the space of closed differential forms, and $B^p(M)$ the subspace of exact forms. The quotient space $H^p(M) = Z^p(M)/B^p(M)$ is called the *cohomology* of the manifold. For compact manifolds $\dim H^p(M)$ is finite dimensional even though $Z^p(M)$ and $B^p(M)$ can both be infnite dimensional. The number $b^p(M) = \dim H^p(M)$ is called the p -th Betti number of the manifold. It depends only on the homotopy type of M .

16.3 For example, let $M = S^1$. The space of functions on a circle can be thought of as the space of smooth functions on the real line which are periodic of period 2π . More generally, a differential form on S^1 is a form on R that is invariant under translations by 2π . Thus the co-ordinate $0 \leq \theta < 2\pi$ itself is not a function on S^1 but $\omega = d\theta$ is a differential form on the circle : $d(\theta) = d(\theta + 2\pi)$. Now, $d\omega = 0$; indeed all 1-forms on S^1 are closed; however it is not exact. The most general 1-form is a periodic function times ω : $f(\theta)\omega$ which is closed for any f . It is exact if and only if the indefinite integral of f is a periodic function. That is, iff

$$\int_0^{2\pi} f(\theta) d\theta = 0.$$

Any 1-form can be written as the sum of an exact 1-form and a multiple of ω :

$$f(\theta)\omega = [f - \bar{f}]\omega + \bar{f}\omega$$

where $\bar{f} = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$ is the mean value of f . Thus $\dim H^1(S^1) = 1$. Similarly, $\dim H^0(S^1) = 1$

16.4 More generally,

$$\dim H^p(S^n) = 1; \mathrm{for} p = 0, n$$

and zero otherwise.

16.5 Recall that d is a graded derivation with respect to exterior product: $d(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^{\deg \omega} \wedge d\phi$. So the product of two closed forms is closed. Also, $\omega \wedge d\phi = d[\omega \wedge \phi]$ if $d\omega = 0$; i.e, the product of a closed form with an exact form is exact. Stated differently, $\oplus_p B^p(M)$ is an ideal in the associative algebra $\oplus_p Z^p(M)$. Thus the quotient $\oplus_p H^p(M)$ is an associative (graded commutative) algebra, under the wedge product.

16.6 Let us give some examples. The *Grassmannian* $Gr_k(n)$ is the set of all k -dimensional subspaces of the complex vector space C^n . It is a compact manifold; also, $\dim Gr_k(n) = 2k(n-k)$. (The special case $k = 1$ is the space of rays in n and is also called the *Complex Projective space* CP^{n-1} .) We can embed the Grassmannian into a Euclidean space, the space of hermitean $n \times n$ matrices of square one and having k negative eigenvalues :

$$Gr_k(n) = \{\Phi^\dagger = \Phi; \Phi^2 = 1;\ \mathrm{tr}\Phi = n - 2k\}.$$

To each such matrix we can associate a subspace of dimension k ; the eignespace with eigenvalue -1 . Conversely given a subspace of dimension k , there is exactly one hermitean matrix with eigenvalue -1 on this subspace and $+1$ on the orthogonal complement. Grassmannians are interesting examples of differential manifolds; they are simple enough to be studied in detail yet complicated enough to serve as 'building blocks' of cohomology theory.

16.7 The differential forms $\omega_{2r} = \mathrm{tr}\Phi(d\Phi)^{2r}$ are closed: $d\omega_{2r} = \mathrm{tr}[d\Phi]^{2r+1} = 0$ since Φ anticommutes with $d\Phi$. These ω_{2r} generate the cohomology algebra of the Grassmannian. There are some relations among them, since the wedge of too high an order should vanish. In the inductive limit $n \to \infty$ these relations dissappear and the cohomology algebra of the Grassmannian $Gr_k(\infty)$ is thus a free commutative algebra generated by ω_{2r} for $r = 1, \cdots \infty$. This is related to the theory of characteristic classes (see the book *Characteristic Classes* by J. Milnor.)