Chapter 9

RELAXATION METHODS

In the past three chapters, we developed a methodology for designing, analyzing, and choosing time-marching methods. These methods can be used to compute the time-accurate solution to linear and nonlinear systems of ODE's in the general form

$$\frac{d\vec{u}}{dt} = \vec{F}(\vec{u}, t) \tag{9.1}$$

which arise after spatial discretization of a PDE. Alternatively, they can be used to solve for the steady solution of Eq. 9.1, which satisfies the following coupled system of nonlinear algebraic equations:

$$\vec{F}(\vec{u}) = 0 \tag{9.2}$$

In the latter case, the unsteady equations are integrated until the solution converges to a steady solution. The same approach permits a time-marching method to be used to solve a linear system of algebraic equations in the form

$$A\vec{x} = \vec{b} \tag{9.3}$$

To solve this system using a time-marching method, a time derivative is introduced as follows

$$\frac{d\vec{x}}{dt} = A\vec{x} - \vec{b} \tag{9.4}$$

and the system is integrated in time until the transient has decayed to a sufficiently low level. Following a time-dependent path to steady state is possible only if all of the eigenvalues of the matrix A (or $-A$) have real parts lying in the left half-plane. Although the solution $\vec{x} = A^{-1}\vec{b}$ exists as long as A is nonsingular, the ODE given by Eq. 9.4 has a stable steady solution only if A meets the above condition.

The common feature of all time-marching methods is that they are at least first-order accurate. In this chapter, we consider iterative methods which are not time accurate at all. Such methods are known as *relaxation methods*. While they are applicable to coupled systems of nonlinear algebraic equations in the form of Eq. 9.2, our analysis will focus on their application to large sparse linear systems of equations in the form

$$A_b\vec{u} - \vec{f}_b = 0 \tag{9.5}$$

where A_b is nonsingular, and the use of the subscript b will become clear shortly. Such systems of equations arise, for example, at each time step of an implicit time-marching method or at each iteration of Newton's method. Using an iterative method, we seek to obtain rapidly a solution which is arbitrarily close to the exact solution of Eq. 9.5, which is given by

$$\vec{u}_\infty = A_b^{-1}\vec{f}_b \tag{9.6}$$

9.1 Formulation of the Model Problem

9.1.1 Preconditioning the Basic Matrix

It is standard practice in applying relaxation procedures to *precondition* the basic equation. This preconditioning has the effect of multiplying Eq. 9.5 from the left by some nonsingular matrix. In the simplest possible case the conditioning matrix is a diagonal matrix composed of a constant $D(b)$. If we designate the conditioning matrix by C, the problem becomes one of solving for $\vec{u}$ in

$$CA_b\vec{u} - C\vec{f}_b = 0 \tag{9.7}$$

Notice that the solution of Eq. 9.7 is

$$\vec{u} = [CA_b]^{-1}C\vec{f}_b = A_b^{-1}C^{-1}C\vec{f}_b = A_b^{-1}\vec{f}_b \tag{9.8}$$

which is identical to the solution of Eq. 9.5, provided C^{-1} exists.

In the following we will see that our approach to the iterative solution of Eq. 9.7 depends crucially on the eigenvalue and eigenvector structure of the matrix CA_b, and, equally important, does not depend at all on the eigensystem of the basic matrix A_b. For example, there are well-known techniques for accelerating relaxation schemes if the eigenvalues of CA_b are all real and of the same sign. To use these schemes, the conditioning matrix C must be chosen such that this requirement is satisfied. A choice of C which ensures this condition is the negative transpose of A_b.

For example, consider a spatial discretization of the linear convection equation using centered differences with a Dirichlet condition on the left side and no constraint on the right side. Using a first-order backward difference on the right side (as in Section 3.6), this leads to the approximation

$$\delta_x \vec{u} = \frac{1}{2\Delta x}\left\{\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ & & & -2 & 2 \end{bmatrix}\vec{u} + \begin{bmatrix} -u_a \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right\} \tag{9.9}$$

The matrix in Eq. 9.9 has eigenvalues whose imaginary parts are much larger than their real parts. It can first be conditioned so that the modulus of each element is 1. This is accomplished using a diagonal preconditioning matrix

$$D = 2\Delta x \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{9.10}$$

which scales each row. We then further condition with multiplication by the negative transpose. The result is

$$A_2 = -A_1^T A_1 =$$

$$\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ & & & -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ & & & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 & & \\ 0 & -2 & 0 & 1 & \\ 1 & 0 & -2 & 0 & 1 \\ & 1 & 0 & -2 & 1 \\ & & 1 & 1 & -2 \end{bmatrix} \tag{9.11}$$

If we define a permutation matrix P [1] and carry out the process $P^T[-A_1^T A_1]P$ (which just reorders the elements of A_1 and doesn't change the eigenvalues) we find

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 & & \\ 0 & -2 & 0 & 1 & \\ 1 & 0 & -2 & 0 & 1 \\ & 1 & 0 & -2 & 1 \\ & & 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix} \tag{9.12}$$

which has all negative real eigenvalues, as given in Appendix B. Thus even when the basic matrix A_b has nearly imaginary eigenvalues, the conditioned matrix $-A_b^T A_b$ is nevertheless symmetric negative definite (i.e., symmetric with negative real eigenvalues), and the classical relaxation methods can be applied. We do not necessarily recommend the use of $-A_b^T$ as a preconditioner; we simply wish to show that a broad range of matrices can be preconditioned into a form suitable for our analysis.

9.1.2 The Model Equations

Preconditioning processes such as those described in the last section allow us to prepare our algebraic equations in advance so that certain eigenstructures are guaranteed. In the remainder of this chapter, we will thoroughly investigate some simple equations which model these structures. We will consider the preconditioned system of equations having the form

$$A\vec{\phi} - \vec{f} = 0 \tag{9.13}$$

where A is symmetric negative definite.[2] The symbol for the dependent variable has been changed to ϕ as a reminder that the physics being modeled is no longer time

[1] A permutation matrix (defined as a matrix with exactly one 1 in each row and column and has the property that $P^T = P^{-1}$) just rearranges the rows and columns of a matrix.

[2] We use a symmetric negative definite matrix to simplify certain aspects of our analysis. Relaxation methods are applicable to more general matrices. The classical methods will usually converge if A_b is *diagonally dominant*, as defined in Appendix A.

accurate when we later deal with ODE formulations. Note that the solution of Eq. 9.13, $\vec{\phi} = A^{-1}\vec{f}$, is guaranteed to exist because A is nonsingular. In the notation of Eqs. 9.5 and 9.7,

$$A = CA_b \qquad \text{and} \qquad \vec{f} = C\vec{f}_b \tag{9.14}$$

The above was written to treat the general case. It is instructive in formulating the concepts to consider the special case given by the diffusion equation in one dimension with unit diffusion coefficient ν:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - g(x) \tag{9.15}$$

This has the steady-state solution

$$\frac{\partial^2 u}{\partial x^2} = g(x) \tag{9.16}$$

which is the one-dimensional form of the Poisson equation. Introducing the three-point central differencing scheme for the second derivative with Dirichlet boundary conditions, we find

$$\frac{d\vec{u}}{dt} = \frac{1}{\Delta x^2} B(1,-2,1)\vec{u} + (\vec{bc}) - \vec{g} \tag{9.17}$$

where $(\vec{bc})$ contains the boundary conditions and $\vec{g}$ contains the values of the source term at the grid nodes. In this case

$$\begin{aligned} A_b &= \frac{1}{\Delta x^2} B(1,-2,1) \\ \vec{f}_b &= \vec{g} - (\vec{bc}) \end{aligned} \tag{9.18}$$

Choosing $C = \Delta x^2 I$, we obtain

$$B(1,-2,1)\vec{\phi} = \vec{f} \tag{9.19}$$

where $\vec{f} = \Delta x^2 \vec{f}_b$. If we consider a Dirichlet boundary condition on the left side and either a Dirichlet or a Neumann condition on the right side, then A has the form

$$A = B\left(1, \vec{b}, 1\right)$$

$$\vec{b} = [-2, -2, \cdots, -2, s]^T$$

$$s = -2 \qquad \text{or} \qquad -1 \tag{9.20}$$

Note that $s = -1$ is easily obtained from the matrix resulting from the Neumann boundary condition given in Eq. 3.24 using a diagonal conditioning matrix. A tremendous amount of insight to the basic features of relaxation is gained by an appropriate study of the one-dimensional case, and much of the remaining material is devoted to this case. We attempt to do this in such a way, however, that it is directly applicable to two- and three-dimensional problems.

9.2 Classical Relaxation

9.2.1 The Delta Form of an Iterative Scheme

We will consider relaxation methods which can be expressed in the following delta form:

$$H\left[\vec{\phi}_{n+1} - \vec{\phi}_n\right] = A\vec{\phi}_n - \vec{f} \tag{9.21}$$

where H is some nonsingular matrix which depends upon the iterative method. The matrix H is independent of n for *stationary* methods and is a function of n for *nonstationary* ones. The iteration count is designated by the subscript n or the superscript (n). The converged solution is designated $\vec{\phi}_\infty$ so that

$$\vec{\phi}_\infty = A^{-1}\vec{f} \tag{9.22}$$

9.2.2 The Converged Solution, the Residual, and the Error

Solving Eq. 9.21 for $\vec{\phi}_{n+1}$ gives

$$\vec{\phi}_{n+1} = [I + H^{-1}A]\vec{\phi}_n - H^{-1}\vec{f} = G\vec{\phi}_n - H^{-1}\vec{f} \tag{9.23}$$

where

$$G \equiv I + H^{-1}A \tag{9.24}$$

Hence it is clear that H should lead to a system of equations which is easy to solve, or at least easier to solve than the original system. The *error* at the nth iteration is defined as

$$\vec{e}_n \equiv \vec{\phi}_n - \vec{\phi}_\infty = \vec{\phi}_n - A^{-1}\vec{f} \tag{9.25}$$

where $\vec{\phi}_\infty$ was defined in Eq. 9.22. The *residual* at the nth iteration is defined as

$$\vec{r}_n \equiv A\vec{\phi}_n - \vec{f} \tag{9.26}$$

Multiply Eq. 9.25 by A from the left, and use the definition in Eq. 9.26. There results the relation between the error and the residual

$$A\vec{e}_n - \vec{r}_n = 0 \tag{9.27}$$

Finally, it is not difficult to show that

$$\vec{e}_{n+1} = G\vec{e}_n \tag{9.28}$$

Consequently, G is referred to as the basic iteration matrix, and its eigenvalues, which we designate as σ_m, determine the convergence rate of a method.

In all of the above, we have considered only what are usually referred to as stationary processes in which H is constant throughout the iterations. Nonstationary processes in which H (and possibly C) is varied at each iteration are discussed in Section 9.5.

9.2.3 The Classical Methods

Point Operator Schemes in One Dimension

Let us consider three classical relaxation procedures for our model equation

$$B(1,-2,1)\vec{\phi} = \vec{f} \tag{9.29}$$

as given in Section 9.1.2. The Point-Jacobi method is expressed in point operator form for the one-dimensional case as

$$\phi_j^{(n+1)} = \frac{1}{2}\left[\phi_{j-1}^{(n)} + \phi_{j+1}^{(n)} - f_j\right] \tag{9.30}$$

This operator comes about by choosing the value of $\phi_j^{(n+1)}$ such that together with the *old* values of ϕ_{j-1} and ϕ_{j+1}, the jth row of Eq. 9.29 is satisfied. The Gauss-Seidel method is

$$\phi_j^{(n+1)} = \frac{1}{2}\left[\phi_{j-1}^{(n+1)} + \phi_{j+1}^{(n)} - f_j\right] \tag{9.31}$$

This operator is a simple extension of the point-Jacobi method which uses the most recent update of ϕ_{j-1}. Hence the jth row of Eq. 9.29 is satisfied using the *new* values of ϕ_j and ϕ_{j-1} and the *old* value of ϕ_{j+1}. The method of successive overrelaxation (SOR) is based on the idea that if the correction produced by the Gauss-Seidel method tends to move the solution toward $\vec{\phi}_\infty$, then perhaps it would be better to move further in this direction. It is usually expressed in two steps as

$$\begin{aligned}\tilde{\phi}_j &= \frac{1}{2}\Big[\phi_{j-1}^{(n+1)} + \phi_{j+1}^{(n)} - f_j\Big]\\ \phi_j^{(n+1)} &= \phi_j^{(n)} + \omega\Big[\tilde{\phi}_j - \phi_j^{(n)}\Big]\end{aligned} \tag{9.32}$$

where ω generally lies between 1 and 2, but it can also be written in the single line

$$\phi_j^{(n+1)} = \frac{\omega}{2}\phi_{j-1}^{(n+1)} + (1-\omega)\phi_j^{(n)} + \frac{\omega}{2}\phi_{j+1}^{(n)} - \frac{\omega}{2}f_j \tag{9.33}$$

The General Form

The general form of the classical methods is obtained by splitting the matrix A in Eq. 9.13 into its diagonal, D, the portion of the matrix below the diagonal, L, and the portion above the diagonal, U, such that

$$A = L + D + U \tag{9.34}$$

Then the point-Jacobi method is obtained with $H = -D$, which certainly meets the criterion that it is easy to solve. The Gauss-Seidel method is obtained with $H = -(L+D)$, which is also easy to solve, being lower triangular.

9.3 The ODE Approach to Classical Relaxation

9.3.1 The Ordinary Differential Equation Formulation

The particular type of delta form given by Eq. 9.21 leads to an interpretation of relaxation methods in terms of solution techniques for coupled first-order ODE's, about which we have already learned a great deal. One can easily see that Eq. 9.21 results from the application of the explicit Euler time-marching method (with $h = 1$) to the following system of ODE's:

$$H\frac{d\vec{\phi}}{dt} = A\vec{\phi} - \vec{f} \tag{9.35}$$

This is equivalent to

$$\frac{d\vec{\phi}}{dt} = H^{-1}C\Big[A_b\vec{\phi} - \vec{f}_b\Big] = H^{-1}[A\vec{\phi} - \vec{f}] \tag{9.36}$$

In the special case where $H^{-1}A$ depends on neither $\vec{u}$ nor t, $H^{-1}\vec{f}$ is also independent of t, *and* the eigenvectors of $H^{-1}A$ are linearly independent, the solution can be written as

$$\vec{\phi} = \underbrace{c_1 e^{\lambda_1 t}\vec{x}_1 + \cdots + c_M e^{\lambda_M t}\vec{x}_M}_{\text{error}} + \vec{\phi}_\infty \tag{9.37}$$

where what is referred to in time-accurate analysis as the transient solution, is now referred to in relaxation analysis as the error. It is clear that, if all of the eigenvalues of $H^{-1}A$ have negative real parts (which implies that $H^{-1}A$ is nonsingular), then the system of ODE's has a *steady-state* solution which is approached as $t \to \infty$, given by

$$\vec{\phi}_\infty = A^{-1}\vec{f} \tag{9.38}$$

which is the solution of Eq. 9.13. We see that the goal of a relaxation method is to remove the transient solution from the general solution in the most efficient way possible. The λ eigenvalues are fixed by the basic matrix in Eq. 9.36, the preconditioning matrix in 9.7, and the secondary conditioning matrix in 9.35. The σ eigenvalues are fixed for a given λh by the choice of time-marching method. Throughout the remaining discussion we will refer to the independent variable t as "time", even though no true time accuracy is involved.

In a stationary method, H and C in Eq. 9.36 are independent of t, that is, they are not changed throughout the iteration process. The generalization of this in our approach is to make h, the "time" step, a constant for the entire iteration.

Suppose the explicit Euler method is used for the time integration. For this method $\sigma_m = 1 + \lambda_m h$. Hence the numerical solution after n steps of a stationary relaxation method can be expressed as (see Eq. 6.28)

$$\vec{\phi}_n = \underbrace{c_1\vec{x}_1(1+\lambda_1 h)^n + \cdots + c_m\vec{x}_m(1+\lambda_m h)^n + \cdots + c_M\vec{x}_M(1+\lambda_M h)^n}_{\text{error}} + \vec{\phi}_\infty \tag{9.39}$$

The initial amplitudes of the eigenvectors are given by the magnitudes of the c_m. These are fixed by the initial guess. In general it is assumed that any or all of the eigenvectors could have been given an equally "bad" excitation by the initial guess, so that we must devise a way to remove them all from the general solution on an equal basis. Assuming that $H^{-1}A$ has been chosen (that is, an iteration process has been decided upon), the only free choice remaining to accelerate the removal of the error terms is the choice of h. As we shall see, the three classical methods have all been conditioned by the choice of H to have an optimum h equal to 1 for a stationary iteration process.

9.3.2 ODE Form of the Classical Methods

The three iterative procedures defined by Eqs. 9.30, 9.31 and 9.32 obey no apparent pattern except that they are easy to implement in a computer code since all of the data required to update the value of one point are explicitly available at the time of the update. Now let us study these methods as subsets of ODE as formulated in Section 9.3.1. Insert the model equation 9.29 into the ODE form 9.35. Then

$$H\frac{d\vec{\phi}}{dt} = B(1,-2,1)\vec{\phi} - \vec{f} \tag{9.40}$$

As a start, let us use for the numerical integration the explicit Euler method

$$\phi_{n+1} = \phi_n + h\phi'_n \tag{9.41}$$

with a step size, h, equal to 1. We arrive at

$$H(\vec{\phi}_{n+1} - \vec{\phi}_n) = B(1,-2,1)\vec{\phi}_n - \vec{f} \tag{9.42}$$

It is clear that the best choice of H from the point of view of matrix algebra is $-B(1,-2,1)$ since then multiplication from the left by $-B^{-1}(1,-2,1)$ gives the correct answer in one step. However, this is not in the spirit of our study, since multiplication by the inverse amounts to solving the problem by a direct method without iteration. The constraint on H that is in keeping with the formulation of the three methods described in Section 9.2.3 is that all the elements above the diagonal (or below the diagonal if the sweeps are from right to left) are zero. If we impose this constraint and further restrict ourselves to banded tridiagonals with a single constant in each band, we are led to

$$B(-\beta,\frac{2}{\omega},0)(\vec{\phi}_{n+1} - \vec{\phi}_n) = B(1,-2,1)\vec{\phi}_n - \vec{f} \tag{9.43}$$

where β and ω are arbitrary. With this choice of notation the three methods presented in Section 9.2.3 can be identified using the entries in Table 9.1.

TABLE 9.1: VALUES OF β and ω IN EQ. 9.43 THAT LEAD TO CLASSICAL RELAXATION METHODS

β	ω	Method	Equation
0	1	Point-Jacobi	6.2.3
1	1	Gauss-Seidel	6.2.4
1	$2/\left[1+\sin\left(\frac{\pi}{M+1}\right)\right]$	Optimum SOR	6.2.5

The fact that the values in the tables lead to the methods indicated can be verified by simple algebraic manipulation. However, our purpose is to examine the whole procedure as a special subset of the theory of ordinary differential equations. In this light, the three methods are all contained in the following set of ODE's

$$\frac{d\vec{\phi}}{dt} = B^{-1}(-\beta, \frac{2}{\omega}, 0)\left[B(1,-2,1)\vec{\phi} - \vec{f}\right] \tag{9.44}$$

and appear from it in the special case when the explicit Euler method is used for its numerical integration. The point operator that results from the use of the explicit Euler scheme is

$$\phi_j^{(n+1)} = \left(\frac{\omega\beta}{2}\phi_{j-1}^{(n+1)} + \frac{\omega}{2}(h-\beta)\phi_{j-1}^{(n)}\right) - \left((\omega h - 1)\phi_j^{(n)}\right) + \left(\frac{\omega h}{2}\phi_{j+1}^{(n)}\right) - \frac{\omega h}{2} f_j \tag{9.45}$$

This represents a generalization of the classical relaxation techniques.

9.4 Eigensystems of the Classical Methods

The ODE approach to relaxation can be summarized as follows. The basic equation to be solved came from some time-accurate derivation

$$A_b\vec{u} - \vec{f}_b = 0 \tag{9.46}$$

This equation is preconditioned in some manner which has the effect of multiplication by a conditioning matrix C giving

$$A\vec{\phi} - \vec{f} = 0 \tag{9.47}$$

An iterative scheme is developed to find the converged, or steady-state, solution of the set of ODE's

$$H\frac{d\vec{\phi}}{dt} = A\vec{\phi} - \vec{f} \tag{9.48}$$

This solution has the analytical form

$$\vec{\phi}_n = \vec{e}_n + \vec{\phi}_\infty \tag{9.49}$$

where $\vec{e}_n$ is the transient, or error, and $\vec{\phi}_\infty \equiv A^{-1}\vec{f}$ is the steady-state solution. The three classical methods, Point-Jacobi, Gauss-Seidel, and SOR, are identified for the one-dimensional case by Eq. 9.44 and Table 9.1.

Given our assumption that the component of the error associated with each eigenvector is equally likely to be excited, the asymptotic convergence rate is determined by the eigenvalue σ_m of G ($\equiv I + H^{-1}A$) having maximum absolute value. Thus

$$\text{Convergence rate} \quad \sim \quad |\sigma_m|_{max} \quad , \quad m = 1, 2, \cdots, M \tag{9.50}$$

In this section, we use the ODE analysis to find the convergence rates of the three classical methods represented by Eqs. 9.30, 9.31, and 9.32. It is also instructive to inspect the eigenvectors and eigenvalues in the $H^{-1}A$ matrix for the three methods. This amounts to solving the generalized eigenvalue problem

$$A\vec{x}_m = \lambda_m H\vec{x}_m \tag{9.51}$$

for the special case

$$B(1, -2, 1)\vec{x}_m = \lambda_m B(-\beta, \frac{2}{\omega}, 0)\vec{x}_m \tag{9.52}$$

The generalized eigensystem for simple tridigonals is given in Appendix B.2. The three special cases considered below are obtained with $a = 1$, $b = -2$, $c = 1$, $d = -\beta$, $e = 2/\omega$, and $f = 0$. To illustrate the behavior, we take $M = 5$ for the matrix order. This special case makes the general result quite clear.

9.4.1 The Point-Jacobi System

If $\beta = 0$ and $\omega = 1$ in Eq. 9.44, the ODE matrix $H^{-1}A$ reduces to simply $B(\frac{1}{2}, -1, \frac{1}{2})$. The eigensystem can be determined from Appendix B.1 since both d and f are zero. The eigenvalues are given by the equation

$$\lambda_m = -1 + \cos\left(\frac{m\pi}{M+1}\right), \quad m = 1, 2, \ldots, M \tag{9.53}$$

The λ-σ relation for the explicit Euler method is $\sigma_m = 1 + \lambda_m h$. This relation can be plotted for any h. The plot for $h = 1$, the optimum stationary case, is shown in Fig. 9.1. For $h < 1$, the maximum $|\sigma_m|$ is obtained with $m = 1$, Fig. 9.2 and for $h > 1$, the maximum $|\sigma_m|$ is obtained with $m = M$, Fig. 9.3. Note that for $h > 1.0$ (depending on M) there is the possibility of instability, i.e. $|\sigma_m| > 1.0$. To obtain the optimal scheme we wish to minimize the maximum $|\sigma_m|$ which occurs when $h = 1$, $|\sigma_1| = |\sigma_M|$ and the best possible convergence rate is achieved:

$$|\sigma_m|_{max} = \cos\left(\frac{\pi}{M+1}\right) \tag{9.54}$$

For $M = 40$, we obtain $|\sigma_m|_{max} = 0.9971$. Thus after 500 iterations the error content associated with each eigenvector is reduced to no more than 0.23 times its initial level.

Again from Appendix B.1, the eigenvectors of $H^{-1}A$ are given by

$$\vec{x}_j = (x_j)_m = \sin\left[j\left(\frac{m\pi}{M+1}\right)\right], \quad j = 1, 2, \ldots, M \tag{9.55}$$

This is a very "well-behaved" eigensystem with linearly independent eigenvectors and distinct eigenvalues. The first 5 eigenvectors are simple sine waves. For $M = 5$, the eigenvectors can be written as

$$\vec{x}_1 = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \\ 1 \\ \sqrt{3}/2 \\ 1/2 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} \sqrt{3}/2 \\ \sqrt{3}/2 \\ 0 \\ -\sqrt{3}/2 \\ -\sqrt{3}/2 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$

$$\vec{x}_4 = \begin{bmatrix} \sqrt{3}/2 \\ -\sqrt{3}/2 \\ 0 \\ \sqrt{3}/2 \\ -\sqrt{3}/2 \end{bmatrix}, \vec{x}_5 = \begin{bmatrix} 1/2 \\ -\sqrt{3}/2 \\ 1 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \tag{9.56}$$

The corresponding eigenvalues are, from Eq. 9.53

$$\begin{aligned} \lambda_1 &= -1 + \frac{\sqrt{3}}{2} &&= -0.134\cdots \\ \lambda_2 &= -1 + \tfrac{1}{2} &&= -0.5 \\ \lambda_3 &= -1 &&= -1.0 \\ \lambda_4 &= -1 - \tfrac{1}{2} &&= -1.5 \\ \lambda_5 &= -1 - \frac{\sqrt{3}}{2} &&= -1.866\cdots \end{aligned} \tag{9.57}$$

From Eq. 9.39, the numerical solution written in full is

$$\begin{aligned} \vec{\phi}_n - \vec{\phi}_\infty &= c_1[1 - (1 - \frac{\sqrt{3}}{2})h]^n \vec{x}_1 \\ &+ c_2[1 - (1 - \frac{1}{2})h]^n \vec{x}_2 \end{aligned}$$

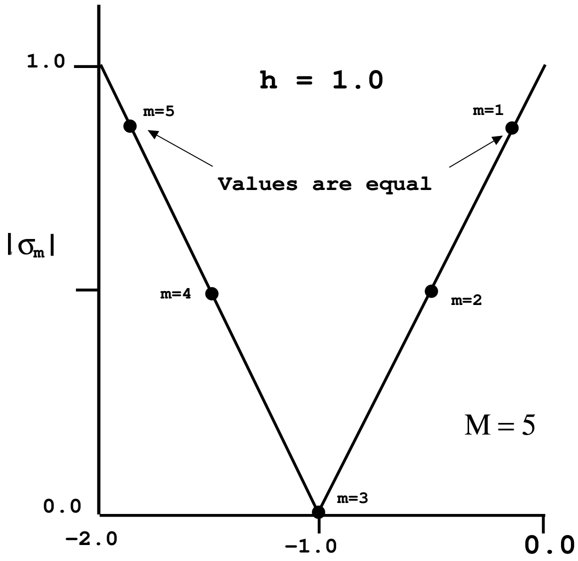

Figure 9.1: The σ, λ relation for Point-Jacobi, $h = 1, M = 5$.

$$
\begin{aligned}
&+ \quad c_3[1 - (1 \qquad\quad)h]^n \vec{x}_3 \\
&+ \quad c_4[1 - (1 + \frac{1}{2})h]^n \vec{x}_4 \\
&+ \quad c_5[1 - (1 + \frac{\sqrt{3}}{2})h]^n \vec{x}_5
\end{aligned} \tag{9.58}
$$

9.4.2 The Gauss-Seidel System

If β and ω are equal to 1 in Eq. 9.44, the matrix eigensystem evolves from the relation

$$
B(1, -2, 1)\vec{x}_m = \lambda_m B(-1, 2, 0)\vec{x}_m \tag{9.59}
$$

which can be studied using the results in Appendix B.2. One can show that the $H^{-1}A$ matrix for the Gauss-Seidel method, A_{GS}, is

$$
A_{GS} \equiv B^{-1}(-1, 2, 0)B(1, -2, 1) =
$$

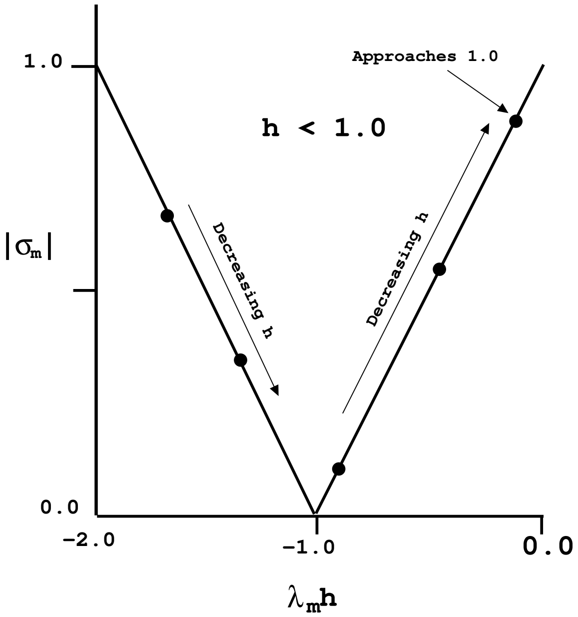

Figure 9.2: The σ, λ relation for Point-Jacobi, $h = 0.9, M = 5$.

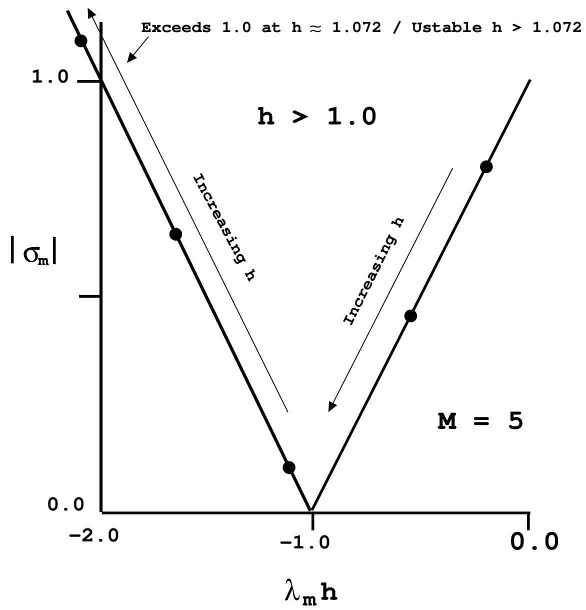

Figure 9.3: The σ, λ relation for Point-Jacobi, $h = 1.1, M = 5$.

$$
\begin{array}{c}
\left[\begin{array}{ccccccc}
-1 & 1/2 & & & & & \\
0 & -3/4 & 1/2 & & & & \\
0 & 1/8 & -3/4 & 1/2 & & & \\
0 & 1/16 & 1/8 & -3/4 & 1/2 & & \\
0 & 1/32 & 1/16 & 1/8 & -3/4 & 1/2 & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 1/2^M & \cdots & & & &
\end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \\ M \end{array}
\end{array}
\tag{9.60}
$$

The eigenvector structure of the Gauss-Seidel ODE matrix is quite interesting. If M is odd there are $(M+1)/2$ distinct eigenvalues with corresponding linearly independent eigenvectors, and there are $(M-1)/2$ defective eigenvalues with corresponding principal vectors. The equation for the nondefective eigenvalues in the ODE matrix is (for odd M)

$$
\lambda_m = -1 + \cos^2\left(\frac{m\pi}{M+1}\right) \quad , \quad m = 1, 2, \ldots, \frac{M+1}{2} \tag{9.61}
$$

and the corresponding eigenvectors are given by

$$
\vec{x}_m = \left[\cos\left(\frac{m\pi}{M+1}\right)\right]^{j-1} \sin\left[j\left(\frac{m\pi}{M+1}\right)\right] \quad , \quad m = 1, 2, \ldots, \frac{M+1}{2} \tag{9.62}
$$

The λ-σ relation for $h = 1$, the optimum stationary case, is shown in Fig. 9.4. The σ_m with the largest amplitude is obtained with $m = 1$. Hence the convergence rate is

$$
|\sigma_m|_{max} = \left[\cos\left(\frac{\pi}{M+1}\right)\right]^2 \tag{9.63}
$$

Since this is the square of that obtained for the Point-Jacobi method, the error associated with the "worst" eigenvector is removed in half as many iterations. For $M = 40$, $|\sigma_m|_{max} = 0.9942$. 250 iterations are required to reduce the error component of the worst eigenvector by a factor of roughly 0.23.

The eigenvectors are quite unlike the Point-Jacobi set. They are no longer symmetrical, producing waves that are higher in amplitude on one side (the updated side) than they are on the other. Furthermore, they do not represent a common family for different values of M.

The Jordan canonical form for $M = 5$ is

$$
X^{-1} A_{GS} X = J_{GS} = \left[\begin{array}{ccc}
\left[\vec{\phi}_1\right] & & \\
 & \left[\vec{\phi}_2\right] & \\
 & & \left[\begin{array}{ccc} \vec{\phi}_3 & 1 & \\ & \vec{\phi}_3 & 1 \\ & & \vec{\phi}_3 \end{array}\right]
\end{array}\right] \tag{9.64}
$$

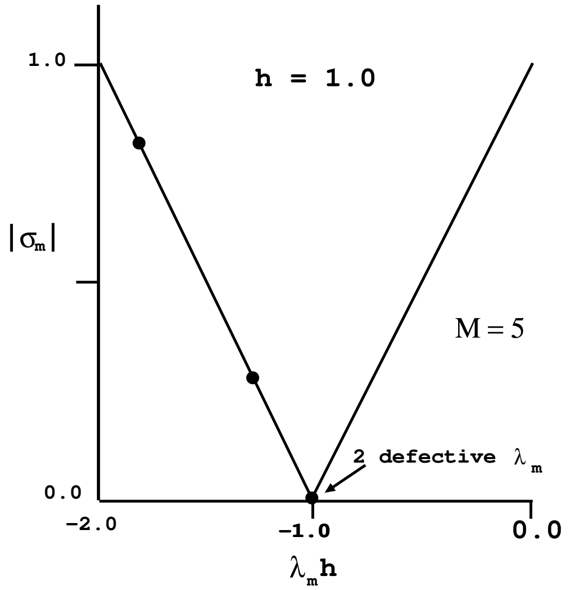

Figure 9.4: The σ, λ relation for Gauss-Seidel, $h = 1.0, M = 5$.

The eigenvectors and principal vectors are all real. For $M = 5$ they can be written

$$\vec{x}_1 = \begin{bmatrix} 1/2 \\ 3/4 \\ 3/4 \\ 9/16 \\ 9/32 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} \sqrt{3}/2 \\ \sqrt{3}/4 \\ 0 \\ -\sqrt{3}/16 \\ -\sqrt{3}/32 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_4 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_5 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ -4 \\ 1 \end{bmatrix} \tag{9.65}$$

The corresponding eigenvalues are

$$\begin{aligned} \lambda_1 &= -1/4 \\ \lambda_2 &= -3/4 \\ \lambda_3 &= -1 \\ \left.\begin{matrix}(4)\\(5)\end{matrix}\right\} & \quad \begin{matrix}\text{Defective, linked to} \\ \lambda_3 \text{ Jordan block}\end{matrix} \end{aligned} \tag{9.66}$$

The numerical solution written in full is thus

$$\begin{aligned} \vec{\phi}_n - \vec{\phi}_\infty &= c_1(1 - \frac{h}{4})^n \vec{x}_1 \\ &+ c_2(1 - \frac{3h}{4})^n \vec{x}_2 \end{aligned}$$

$$
\begin{aligned}
&+ \left[c_3(1-h)^n + c_4 h \frac{n}{1!}(1-h)^{n-1} + c_5 h^2 \frac{n(n-1)}{2!}(1-h)^{n-2}\right]\vec{x}_3 \\
&+ \left[c_4(1-h)^n + c_5 h \frac{n}{1!}(1-h)^{n-1}\right]\vec{x}_4 \\
&+ c_5(1-h)^n \vec{x}_5
\end{aligned}
\tag{9.67}
$$

9.4.3 The SOR System

If $\beta = 1$ and $2/\omega = x$ in Eq. 9.44, the ODE matrix is $B^{-1}(-1, x, 0)B(1, -2, 1)$. One can show that this can be written in the form given below for $M = 5$. The generalization to any M is fairly clear. The $H^{-1}A$ matrix for the SOR method, $A_{SOR} \equiv B^{-1}(-1, x, 0)B(1, -2, 1)$, is

$$
\frac{1}{x^5}\begin{bmatrix}
-2x^4 & x^4 & 0 & 0 & 0 \\
-2x^3 + x^4 & x^3 - 2x^4 & x^4 & 0 & 0 \\
-2x^2 + x^3 & x^2 - 2x^3 + x^4 & x^3 - 2x^4 & x^4 & 0 \\
-2x + x^2 & x - 2x^2 + x^3 & x^2 - 2x^3 + x^4 & x^3 - 2x^4 & x^4 \\
-2 + x & 1 - 2x + x^2 & x - 2x^2 + x^3 & x^2 - 2x^3 + x^4 & x^3 - 2x^4
\end{bmatrix}
\tag{9.68}
$$

Eigenvalues of the system are given by

$$
\lambda_m = -1 + \left(\frac{\omega p_m + z_m}{2}\right)^2 \quad , \quad m = 1, 2, \ldots M \tag{9.69}
$$

where

$$
\begin{aligned}
z_m &= [4(1-\omega) + \omega^2 p_m^2]^{1/2} \\
p_m &= \cos[m\pi/(M+1)]
\end{aligned}
$$

If $\omega = 1$, the system is Gauss-Seidel. If $4(1-\omega) + \omega^2 p_m < 0, z_m$ and λ_m are complex. If ω is chosen such that $4(1-\omega) + \omega^2 p_1^2 = 0, \omega$ is optimum for the stationary case, and the following conditions hold:

1. Two eigenvalues are real, equal and defective.
2. If M is even, the remaining eigenvalues are complex and occur in conjugate pairs.
3. If M is odd, one of the remaining eigenvalues is real and the others are complex occurring in conjugate pairs.

One can easily show that the optimum ω for the stationary case is

$$\omega_{opt} = 2/\left[1 + \sin\left(\frac{\pi}{M+1}\right)\right] \tag{9.70}$$

and for $\omega = \omega_{opt}$

$$\begin{aligned} \lambda_m &= \zeta_m^2 - 1 \\ \vec{x}_m &= \zeta_m^{j-1} \sin\left[j\left(\frac{m\pi}{M+1}\right)\right] \end{aligned} \tag{9.71}$$

where

$$\zeta_m = \frac{\omega_{opt}}{2}\left[p_m + i\sqrt{p_1^2 - p_m^2}\right]$$

Using the explicit Euler method to integrate the ODE's, $\sigma_m = 1 - h + h\zeta_m^2$, and if $h = 1$, the optimum value for the stationary case, the λ-σ relation reduces to that shown in Fig. 9.5. This illustrates the fact that for optimum stationary SOR all the $|\sigma_m|$ are identical and equal to $\omega_{opt} - 1$. Hence the convergence rate is

$$\begin{aligned} |\sigma_m|_{max} &= \omega_{opt} - 1 \qquad (9.72) \\ \omega_{opt} &= 2/\left[1 + \sin\left(\frac{\pi}{M+1}\right)\right] \end{aligned}$$

For $M = 40$, $|\sigma_m|_{max} = 0.8578$. Hence the worst error component is reduced to less than 0.23 times its initial value in only 10 iterations, much faster than both Gauss-Seidel and Point-Jacobi. In practical applications, the optimum value of ω may have to be determined by trial and error, and the benefit may not be as great.

For odd M, there are two real eigenvectors and one real principal vector. The remaining linearly independent eigenvectors are all complex. For $M = 5$ they can be written

$$\vec{x}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \\ 1/6 \\ 1/18 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -6 \\ 9 \\ 16 \\ 13 \\ 6 \end{bmatrix}, \vec{x}_{3,4} = \begin{bmatrix} \sqrt{3}(1 \qquad)/2 \\ \sqrt{3}(1 \pm\ i\sqrt{2})/6 \\ 0 \\ \sqrt{3}(5 \pm\ i\sqrt{2})/54 \\ \sqrt{3}(7 \pm 4i\sqrt{2})/162 \end{bmatrix}, \vec{x}_5 = \begin{bmatrix} 1 \\ 0 \\ 1/3 \\ 0 \\ 1/9 \end{bmatrix} \tag{9.73}$$

The corresponding eigenvalues are

$$\begin{aligned} \lambda_1 &= -2/3 \\ (2) & \quad \text{Defective linked to } \lambda_1 \\ \lambda_3 &= -(10 - 2\sqrt{2}i)/9 \\ \lambda_4 &= -(10 + 2\sqrt{2}i)/9 \\ \lambda_5 &= -4/3 \end{aligned} \tag{9.74}$$

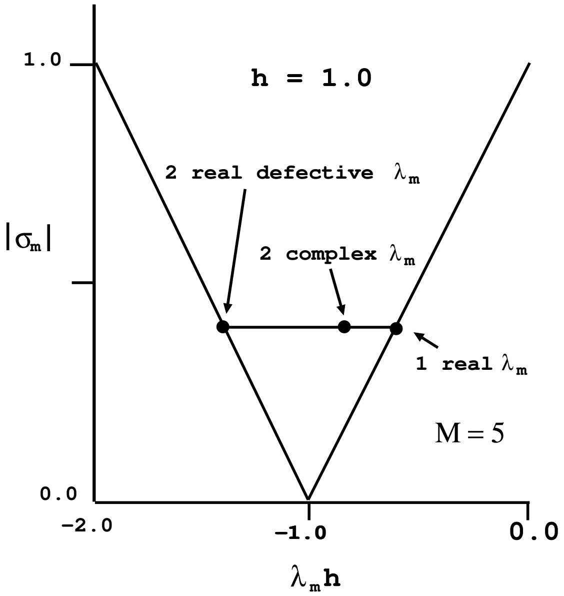

Figure 9.5: The σ, λ relation for optimum stationary SOR, $M = 5$, $h = 1$.

The numerical solution written in full is

$$
\begin{aligned}
\vec{\phi}_n - \vec{\phi}_\infty &= [c_1(1-2h/3)^n + c_2 nh(1-2h/3)^{n-1}]\vec{x}_1 \\
&+ c_2(1-2h/3)^n \vec{x}_2 \\
&+ c_3[1-(10-2\sqrt{2}i)h/9]^n \vec{x}_3 \\
&+ c_4[1-(10+2\sqrt{2}i)h/9]^n \vec{x}_4 \\
&+ c_5(1-4h/3)^n \vec{x}_5 \qquad (9.75)
\end{aligned}
$$

9.5 Nonstationary Processes

In classical terminology a method is said to be nonstationary if the conditioning matrices, H and C, are varied at each time step. This does not change the steady-state solution $A_b^{-1}\vec{f}_b$, but it can greatly affect the convergence rate. In our ODE approach this could also be considered and would lead to a study of equations with nonconstant coefficients. It is much simpler, however, to study the case of fixed H and C but variable step size, h. This process changes the Point-Jacobi method to Richardson's method in standard terminology. For the Gauss-Seidel and SOR methods it leads to processes that can be superior to the stationary methods.

The nonstationary form of Eq. 9.39 is

$$
\begin{aligned}
\vec{\phi}_N \;=\;& c_1\vec{x}_1 \prod_{n=1}^{N}(1+\lambda_1 h_n) + \cdots + c_m\vec{x}_m \prod_{n=1}^{N}(1+\lambda_m h_n) \\
+\;& \cdots + c_M\vec{x}_M \prod_{n=1}^{N}(1+\lambda_M h_n) + \vec{\phi}_\infty
\end{aligned}
\tag{9.76}
$$

where the symbol Π stands for product. Since h_n can now be changed at each step, the error term can theoretically be completely eliminated in M steps by taking $h_m = -1/\lambda_m$, for $m = 1, 2, \cdots, M$. However, the eigenvalues λ_m are generally unknown and costly to compute. It is therefore unnecessary and impractical to set $h_m = -1/\lambda_m$ for $m = 1, 2, \ldots, M$. We will see that a few well chosen h's can reduce whole clusters of eigenvectors associated with nearby λ's in the λ_m spectrum. This leads to the concept of selectively annihilating clusters of eigenvectors from the error terms as part of a total iteration process. This is the basis for the multigrid methods discussed in Chapter 10.

Let us consider the very important case when all of the λ_m are *real* and *negative* (remember that they arise from a conditioned matrix so this constraint is not unrealistic for quite practical cases). Consider one of the error terms taken from

$$
\vec{e}_N \equiv \vec{\phi}_N - \vec{\phi}_\infty = \sum_{m=1}^{M} c_m\vec{x}_m \prod_{n=1}^{N}(1+\lambda_m h_n) \tag{9.77}
$$

and write it in the form

$$
c_m\vec{x}_m P_e(\lambda_m) \equiv c_m\vec{x}_m \prod_{n=1}^{N}(1+\lambda_m h_n) \tag{9.78}
$$

where P_e signifies an "Euler" polynomial. Now focus attention on the polynomial

$$
(P_e)_N(\lambda) = (1+h_1\lambda)(1+h_2\lambda)\cdots(1+h_N\lambda) \tag{9.79}
$$

treating it as a continuous function of the independent variable λ. In the annihilation process mentioned after Eq. 9.76, we considered making the error exactly zero by taking advantage of some knowledge about the discrete values of λ_m for a particular case. Now we pose a less demanding problem. Let us choose the h_n so that the maximum value of $(P_e)_N(\lambda)$ is as small as possible for all λ lying between λ_a and λ_b such that $\lambda_b \leq \lambda \leq \lambda_a \leq 0$. Mathematically stated, we seek

$$
\max_{\lambda_b \leq \lambda \leq \lambda_a} |(P_e)_N(\lambda)| = \text{minimum} \quad , \quad \text{with}(P_e)_N(0) = 1 \tag{9.80}
$$

This problem has a well known solution due to Markov. It is

$$(P_e)_N(\lambda) = \frac{T_N\left(\dfrac{2\lambda - \lambda_a - \lambda_b}{\lambda_a - \lambda_b}\right)}{T_N\left(\dfrac{-\lambda_a - \lambda_b}{\lambda_a - \lambda_b}\right)} \tag{9.81}$$

where

$$T_N(y) = \cos(N \arccos y) \tag{9.82}$$

are the Chebyshev polynomials along the interval $-1 \le y \le 1$ and

$$T_N(y) = \frac{1}{2}\left[y + \sqrt{y^2 - 1}\right]^N + \frac{1}{2}\left[y - \sqrt{y^2 - 1}\right]^N \tag{9.83}$$

are the Chebyshev polynomials for $|y| > 1$. In relaxation terminology this is generally referred to as Richardson's method, and it leads to the nonstationary step size choice given by

$$\frac{1}{h_n} = \frac{1}{2}\left\{-\lambda_b - \lambda_a + (\lambda_b - \lambda_a)\cos\left[\frac{(2n-1)\pi}{2N}\right]\right\}, \quad n = 1, 2, \ldots N \tag{9.84}$$

Remember that all λ are negative real numbers representing the magnitudes of λ_m in an eigenvalue spectrum.

The error in the relaxation process represented by Eq. 9.76 is expressed in terms of a set of eigenvectors, $\vec{x}_m$, amplified by the coefficients $c_m \prod(1 + \lambda_m h_n)$. With each eigenvector there is a corresponding eigenvalue. Eq. 9.84 gives us the best choice of a series of h_n that will minimize the amplitude of the error carried in the eigenvectors associated with the eigenvalues between λ_b and λ_a.

As an example for the use of Eq. 9.84, let us consider the following problem:

Minimize the maximum error associated with the λ eigenvalues in the interval $-2 \le \lambda \le -1$ using only 3 iterations.

(9.85)

The three values of h which satisfy this problem are

$$h_n = 2\Big/\left(3 - \cos\left[\frac{(2n-1)\pi}{6}\right]\right) \tag{9.86}$$

and the amplitude of the eigenvector is reduced to

$$(P_e)_3(\lambda) = T_3(2\lambda + 3)/T_3(3) \tag{9.87}$$

where

$$T_3(3) = \left\{[3 + \sqrt{8}]^3 + [3 - \sqrt{8}]^3\right\}/2 \approx 99 \tag{9.88}$$

A plot of Eq. 9.87 is given in Fig. 9.6 and we see that the amplitudes of all the eigenvectors associated with the eigenvalues in the range $-2 \leq \lambda \leq -1$ have been reduced to less than about 1% of their initial values. The values of h used in Fig. 9.6 are

$$\begin{aligned} h_1 &= 4/(6 - \sqrt{3}) \\ h_2 &= 4/(6 - 0) \\ h_3 &= 4/(6 + \sqrt{3}) \end{aligned}$$

Return now to Eq. 9.76. This was derived from Eq. 9.37 on the condition that the explicit Euler method, Eq. 9.41, was used to integrate the basic ODE's. If instead the implicit trapezoidal rule

$$\phi_{n+1} = \phi_n + \frac{1}{2}h(\phi'_{n+1} + \phi'_n) \tag{9.89}$$

is used, the nonstationary formula

$$\vec{\phi}_N = \sum_{m=1}^{M} c_m \vec{x}_m \prod_{n=1}^{N} \left(\frac{1 + \frac{1}{2}h_n\lambda_m}{1 - \frac{1}{2}h_n\lambda_m} \right) + \vec{\phi}_\infty \tag{9.90}$$

would result. This calls for a study of the rational "trapezoidal" polynomial, P_t:

$$(P_t)_N(\lambda) = \prod_{n=1}^{N} \left(\frac{1 + \frac{1}{2}h_n\lambda}{1 - \frac{1}{2}h_n\lambda} \right) \tag{9.91}$$

under the same constraints as before, namely that

$$\begin{aligned} \max_{\lambda_b \leq \lambda \leq \lambda_a} |(P_t)_N(\lambda)| &= \text{minimum}\ , \\ \text{with } (P_t)_N(0) &= 1 \end{aligned} \tag{9.92}$$

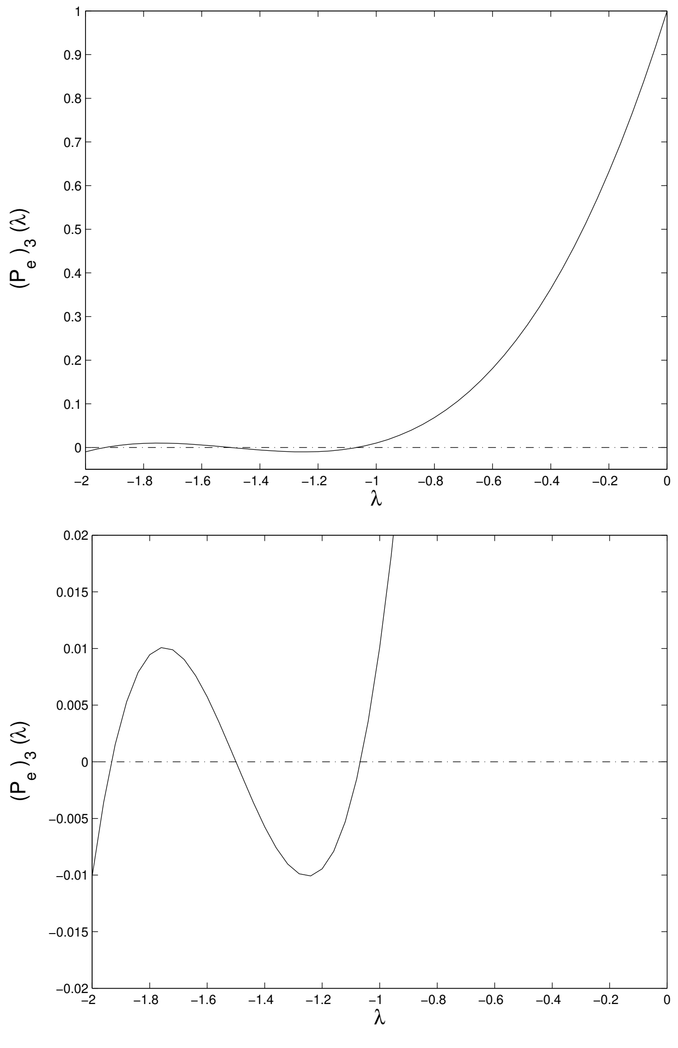

Figure 9.6: Richardson's method for 3 steps, minimization over $-2 \le \lambda \le -1$.

The optimum values of h can also be found for this problem, but we settle here for the approximation suggested by Wachspress

$$\frac{2}{h_n} = -\lambda_b \left(\frac{\lambda_a}{\lambda_b} \right)^{(n-1)/(N-1)} , \quad n = 1, 2, \cdots, N \tag{9.93}$$

This process is also applied to problem 9.85. The results for $(P_t)_3(\lambda)$ are shown in Fig. 9.7. The error amplitude is about 1/5 of that found for $(P_e)_3(\lambda)$ in the same interval of λ. The values of h used in Fig. 9.7 are

$$\begin{aligned} h_1 &= 1 \\ h_2 &= \sqrt{2} \\ h_3 &= 2 \end{aligned}$$

9.6 Problems

1. Given a relaxation method in the form

$$H\Delta\vec{\phi}_n = A\vec{\phi}_n - \vec{f}$$

show that

$$\vec{\phi}_n = G^n\vec{\phi}_0 + (I - G^n)A^{-1}\vec{f}$$

where $G = I + H^{-1}A$.

2. For a linear system of the form $(A_1 + A_2)x = b$, consider the iterative method

$$\begin{aligned} (I + \mu A_1)\tilde{x} &= (I - \mu A_2)x_n + \mu b \\ (I + \mu A_2)x_{n+1} &= (I - \mu A_1)\tilde{x} + \mu b \end{aligned}$$

where μ is a parameter. Show that this iterative method can be written in the form

$$H(x_{k+1} - x_k) = (A_1 + A_2)x_k - b$$

Determine the iteration matrix G if $\mu = -1/2$.

3. Using Appendix B.2, find the eigenvalues of $H^{-1}A$ for the SOR method with $A = B(4 : 1, -2, 1)$ and $\omega = \omega_{opt}$. (You do not have to find H^{-1}. Recall that the eigenvalues of $H^{-1}A$ satisfy $A\vec{x}_m = \lambda_m H\vec{x}_m$.) Find the numerical values, not just the expressions. Then find the corresponding $|\sigma_m|$ values.

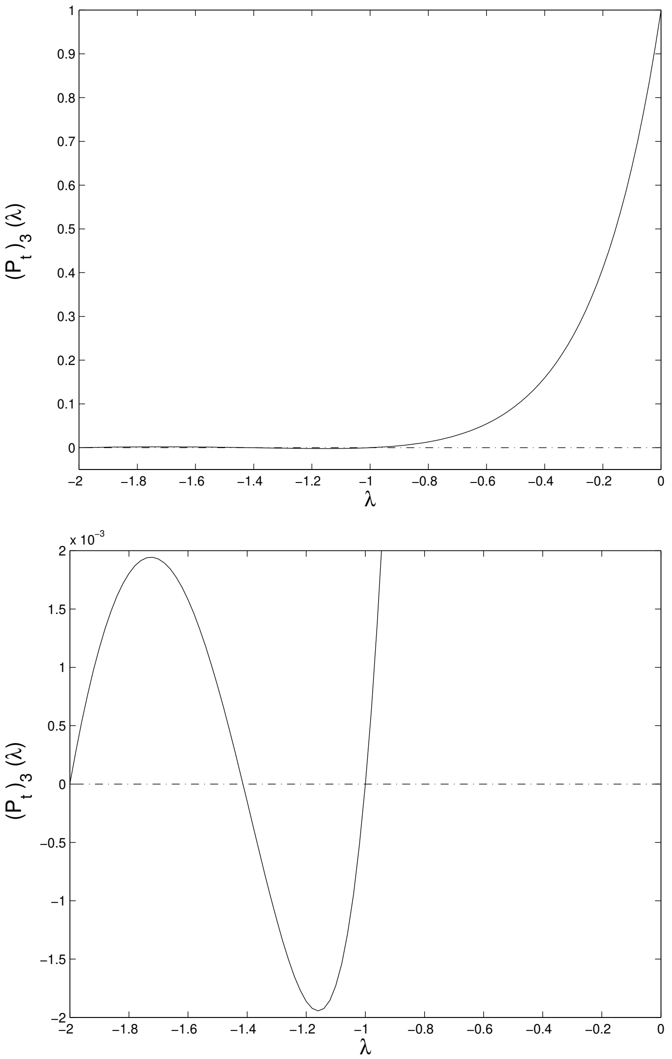

Figure 9.7: Wachspress method for 3 steps, minimization over $-2 \leq \lambda \leq -1$.

4. Solve the following equation on the domain $0 \le x \le 1$ with boundary conditions $u(0) = 0$, $u(1) = 1$:

$$\frac{\partial^2 u}{\partial x^2} - 6x = 0$$

For the initial condition, use $u(x) = 0$. Use second-order centered differences on a grid with 40 cells ($M = 39$). Iterate to steady state using

   (a) the point-Jacobi method,

   (b) the Gauss-Seidel method,

   (c) the SOR method with the optimum value of ω, and

   (d) the 3-step Richardson method derived in Section 9.5.

   Plot the solution after the residual is reduced by 2, 3, and 4 orders of magnitude. Plot the logarithm of the L_2-norm of the residual vs. the number of iterations. Determine the asymptotic convergence rate. Compare with the theoretical asymptotic convergence rate.