Chapter 2

CONSERVATION LAWS AND THE MODEL EQUATIONS

We start out by casting our equations in the most general form, the integral conservation-law form, which is useful in understanding the concepts involved in finite-volume schemes. The equations are then recast into divergence form, which is natural for finite-difference schemes. The Euler and Navier-Stokes equations are briefly discussed in this Chapter. The main focus, though, will be on representative model equations, in particular, the convection and diffusion equations. These equations contain many of the salient mathematical and physical features of the full Navier-Stokes equations. The concepts of convection and diffusion are prevalent in our development of numerical methods for computational fluid dynamics, and the recurring use of these model equations allows us to develop a consistent framework of analysis for consistency, accuracy, stability, and convergence. The model equations we study have two properties in common. They are linear partial differential equations (PDE's) with coefficients that are constant in both space and time, and they represent phenomena of importance to the analysis of certain aspects of fluid dynamic problems.

2.1 Conservation Laws

Conservation laws, such as the Euler and Navier-Stokes equations and our model equations, can be written in the following integral form:

$$\int_{V(t_2)} Q dV - \int_{V(t_1)} Q dV + \int_{t_1}^{t_2} \oint_{S(t)} \mathbf{n}.\mathbf{F} dS dt = \int_{t_1}^{t_2} \int_{V(t)} P dV dt \tag{2.1}$$

In this equation, Q is a vector containing the set of variables which are conserved, e.g., mass, momentum, and energy, per unit volume. The equation is a statement of the conservation of these quantities in a finite region of space with volume $V(t)$ and surface area $S(t)$ over a finite interval of time $t_2 - t_1$. In two dimensions, the region of space, or cell, is an area $A(t)$ bounded by a closed contour $C(t)$. The vector $\mathbf{n}$ is a unit vector normal to the surface pointing outward, $\mathbf{F}$ is a set of vectors, or tensor, containing the flux of Q per unit area per unit time, and P is the rate of production of Q per unit volume per unit time. If all variables are continuous in time, then Eq. 2.1 can be rewritten as

$$\frac{d}{dt}\int_{V(t)} Q dV + \oint_{S(t)} \mathbf{n}.\mathbf{F} dS = \int_{V(t)} P dV \tag{2.2}$$

Those methods which make various numerical approximations of the integrals in Eqs. 2.1 and 2.2 and find a solution for Q on that basis are referred to as *finite-volume methods*. Many of the advanced codes written for CFD applications are based on the finite-volume concept.

On the other hand, a partial derivative form of a conservation law can also be derived. The divergence form of Eq. 2.2 is obtained by applying Gauss's theorem to the flux integral, leading to

$$\frac{\partial Q}{\partial t} + \nabla.\mathbf{F} = P \tag{2.3}$$

where $\nabla.$ is the well-known divergence operator given, in Cartesian coordinates, by

$$\nabla. \equiv \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right). \tag{2.4}$$

and $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are unit vectors in the x, y, and z coordinate directions, respectively. Those methods which make various approximations of the derivatives in Eq. 2.3 and find a solution for Q on that basis are referred to as *finite-difference methods*.

2.2 The Navier-Stokes and Euler Equations

The Navier-Stokes equations form a coupled system of nonlinear PDE's describing the conservation of mass, momentum and energy for a fluid. For a Newtonian fluid in one dimension, they can be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0 \tag{2.5}$$

with

$$Q = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(e+p) \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{4}{3}\mu\frac{\partial u}{\partial x} \\ \frac{4}{3}\mu u\frac{\partial u}{\partial x} + \kappa\frac{\partial T}{\partial x} \end{bmatrix} \tag{2.6}$$

where ρ is the fluid density, u is the velocity, e is the total energy per unit volume, p is the pressure, T is the temperature, μ is the coefficient of viscosity, and κ is the thermal conductivity. The total energy e includes internal energy per unit volume $\rho\epsilon$ (where ϵ is the internal energy per unit mass) and kinetic energy per unit volume $\rho u^2/2$. These equations must be supplemented by relations between μ and κ and the fluid state as well as an equation of state, such as the ideal gas law. Details can be found in Anderson, Tannehill, and Pletcher [1] and Hirsch [2]. Note that the convective fluxes lead to first derivatives in space, while the viscous and heat conduction terms involve second derivatives. This form of the equations is called *conservation-law* or *conservative* form. Non-conservative forms can be obtained by expanding derivatives of products using the product rule or by introducing different dependent variables, such as u and p. Although non-conservative forms of the equations are *analytically* the same as the above form, they can lead to quite different *numerical* solutions in terms of shock strength and shock speed, for example. Thus the conservative form is appropriate for solving flows with features such as shock waves.

Many flows of engineering interest are steady (time-invariant), or at least may be treated as such. For such flows, we are often interested in the steady-state solution of the Navier-Stokes equations, with no interest in the transient portion of the solution. The steady solution to the one-dimensional Navier-Stokes equations must satisfy

$$\frac{\partial E}{\partial x} = 0 \tag{2.7}$$

If we neglect viscosity and heat conduction, the Euler equations are obtained. In two-dimensional Cartesian coordinates, these can be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = 0 \tag{2.8}$$

with

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(e+p) \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(e+p) \end{bmatrix} \tag{2.9}$$

where u and v are the Cartesian velocity components. Later on we will make use of the following form of the Euler equations as well:

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} + B\frac{\partial Q}{\partial y} = 0 \tag{2.10}$$

The matrices $A = \frac{\partial E}{\partial Q}$ and $B = \frac{\partial F}{\partial Q}$ are known as the flux Jacobians. The flux vectors given above are written in terms of the *primitive* variables, ρ, u, v, and p. In order to derive the flux Jacobian matrices, we must first write the flux vectors E and F in terms of the *conservative* variables, q_1, q_2, q_3, and q_4, as follows:

$$E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} q_2 \\ (\gamma-1)q_4 + \frac{3-\gamma}{2}\frac{q_2^2}{q_1} - \frac{\gamma-1}{2}\frac{q_3^2}{q_1} \\ \frac{q_3 q_2}{q_1} \\ \gamma\frac{q_4 q_2}{q_1} - \frac{\gamma-1}{2}\left(\frac{q_2^3}{q_1^2} + \frac{q_3^2 q_2}{q_1^2}\right) \end{bmatrix} \tag{2.11}$$

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} q_3 \\ \frac{q_3 q_2}{q_1} \\ (\gamma-1)q_4 + \frac{3-\gamma}{2}\frac{q_3^2}{q_1} - \frac{\gamma-1}{2}\frac{q_2^2}{q_1} \\ \gamma\frac{q_4 q_3}{q_1} - \frac{\gamma-1}{2}\left(\frac{q_2^2 q_3}{q_1^2} + \frac{q_3^3}{q_1^2}\right) \end{bmatrix} \tag{2.12}$$

We have assumed that the pressure satisfies $p = (\gamma-1)[e - \rho(u^2+v^2)/2]$ from the ideal gas law, where γ is the ratio of specific heats, c_p/c_v. From this it follows that the flux Jacobian of E can be written in terms of the conservative variables as

$$A = \frac{\partial E_i}{\partial q_j} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_{21} & (3-\gamma)\left(\frac{q_2}{q_1}\right) & (1-\gamma)\left(\frac{q_3}{q_1}\right) & \gamma-1 \\ -\left(\frac{q_2}{q_1}\right)\left(\frac{q_3}{q_1}\right) & \left(\frac{q_3}{q_1}\right) & \left(\frac{q_2}{q_1}\right) & 0 \\ a_{41} & a_{42} & a_{43} & \gamma\left(\frac{q_2}{q_1}\right) \end{bmatrix} \tag{2.13}$$

where

$$a_{21} = \frac{\gamma-1}{2}\left(\frac{q_3}{q_1}\right)^2 - \frac{3-\gamma}{2}\left(\frac{q_2}{q_1}\right)^2$$

$$
\begin{aligned}
a_{41} &= (\gamma-1)\left[\left(\frac{q_2}{q_1}\right)^3+\left(\frac{q_3}{q_1}\right)^2\left(\frac{q_2}{q_1}\right)\right]-\gamma\left(\frac{q_4}{q_1}\right)\left(\frac{q_2}{q_1}\right) \\
a_{42} &= \gamma\left(\frac{q_4}{q_1}\right)-\frac{\gamma-1}{2}\left[3\left(\frac{q_2}{q_1}\right)^2+\left(\frac{q_3}{q_1}\right)^2\right] \\
a_{43} &= -(\gamma-1)\left(\frac{q_2}{q_1}\right)\left(\frac{q_3}{q_1}\right)
\end{aligned}
\tag{2.14}
$$

and in terms of the primitive variables as

$$
A=\begin{bmatrix}
0 & 1 & 0 & 0 \\
a_{21} & (3-\gamma)u & (1-\gamma)v & (\gamma-1) \\
-uv & v & u & 0 \\
a_{41} & a_{42} & a_{43} & \gamma u
\end{bmatrix}
\tag{2.15}
$$

where

$$
\begin{aligned}
a_{21} &= \frac{\gamma-1}{2}v^2-\frac{3-\gamma}{2}u^2 \\
a_{41} &= (\gamma-1)u(u^2+v^2)-\gamma\frac{ue}{\rho} \\
a_{42} &= \gamma\frac{e}{\rho}-\frac{\gamma-1}{2}(3u^2+v^2) \\
a_{43} &= (1-\gamma)uv
\end{aligned}
\tag{2.16}
$$

Derivation of the two forms of $B = \partial F/\partial Q$ is similar. The eigenvalues of the flux Jacobian matrices are purely real. This is the defining feature of *hyperbolic* systems of PDE's, which are further discussed in Section 2.5. The homogeneous property of the Euler equations is discussed in Appendix C.

The Navier-Stokes equations include both convective and diffusive fluxes. This motivates the choice of our two scalar model equations associated with the physics of convection and diffusion. Furthermore, aspects of convective phenomena associated with coupled systems of equations such as the Euler equations are important in developing numerical methods and boundary conditions. Thus we also study linear hyperbolic systems of PDE's.

2.3 The Linear Convection Equation

2.3.1 Differential Form

The simplest linear model for convection and wave propagation is the linear convection equation given by the following PDE:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{2.17}$$

Here $u(x,t)$ is a scalar quantity propagating with speed a, a real constant which may be positive or negative. The manner in which the boundary conditions are specified separates the following two phenomena for which this equation is a model:

(1) In one type, the scalar quantity u is given on one boundary, corresponding to a wave entering the domain through this "inflow" boundary. No boundary condition is specified at the opposite side, the "outflow" boundary. This is consistent in terms of the well-posedness of a 1^{st}-order PDE. Hence the wave leaves the domain through the outflow boundary without distortion or reflection. This type of phenomenon is referred to, simply, as the convection problem. It represents most of the "usual" situations encountered in convecting systems. Note that the left-hand boundary is the inflow boundary when a is positive, while the right-hand boundary is the inflow boundary when a is negative.

(2) In the other type, the flow being simulated is periodic. At any given time, what enters on one side of the domain must be the same as that which is leaving on the other. This is referred to as the *biconvection* problem. It is the simplest to study and serves to illustrate many of the basic properties of numerical methods applied to problems involving convection, without special consideration of boundaries. Hence, we pay a great deal of attention to it in the initial chapters.

Now let us consider a situation in which the initial condition is given by $u(x,0) = u_0(x)$, and the domain is infinite. It is easy to show by substitution that the exact solution to the linear convection equation is then

$$u(x,t) = u_0(x - at) \tag{2.18}$$

The initial waveform propagates unaltered with speed $|a|$ to the right if a is positive and to the left if a is negative. With periodic boundary conditions, the waveform travels through one boundary and reappears at the other boundary, eventually returning to its initial position. In this case, the process continues forever without any change in the shape of the solution. Preserving the shape of the initial condition $u_0(x)$ can be a difficult challenge for a numerical method.

2.3.2 Solution in Wave Space

We now examine the biconvection problem in more detail. Let the domain be given by $0 \le x \le 2\pi$. We restrict our attention to initial conditions in the form

$$u(x, 0) = f(0)e^{i\kappa x} \tag{2.19}$$

where $f(0)$ is a complex constant, and κ is the *wavenumber*. In order to satisfy the periodic boundary conditions, κ must be an integer. It is a measure of the number of wavelengths within the domain. With such an initial condition, the solution can be written as

$$u(x, t) = f(t)e^{i\kappa x} \tag{2.20}$$

where the time dependence is contained in the complex function $f(t)$. Substituting this solution into the linear convection equation, Eq. 2.17, we find that $f(t)$ satisfies the following ordinary differential equation (ODE)

$$\frac{df}{dt} = -ia\kappa f \tag{2.21}$$

which has the solution

$$f(t) = f(0)e^{-ia\kappa t} \tag{2.22}$$

Substituting $f(t)$ into Eq. 2.20 gives the following solution

$$u(x, t) = f(0)e^{i\kappa(x-at)} = f(0)e^{i(\kappa x-\omega t)} \tag{2.23}$$

where the frequency, ω, the wavenumber, κ, and the phase speed, a, are related by

$$\omega = \kappa a \tag{2.24}$$

The relation between the frequency and the wavenumber is known as the dispersion relation. The linear relation given by Eq. 2.24 is characteristic of wave propagation in a nondispersive medium. This means that the phase speed is the same for all wavenumbers. As we shall see later, most numerical methods introduce some dispersion; that is, in a simulation, waves with different wavenumbers travel at different speeds.

An arbitrary initial waveform can be produced by summing initial conditions of the form of Eq. 2.19. For M modes, one obtains

$$u(x, 0) = \sum_{m=1}^{M} f_m(0)e^{i\kappa_m x} \tag{2.25}$$

where the wavenumbers are often ordered such that $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_M$. Since the wave equation is linear, the solution is obtained by summing solutions of the form of Eq. 2.23, giving

$$u(x,t) = \sum_{m=1}^{M} f_m(0) e^{i\kappa_m(x-at)} \tag{2.26}$$

Dispersion and dissipation resulting from a numerical approximation will cause the shape of the solution to change from that of the original waveform.

2.4 The Diffusion Equation

2.4.1 Differential Form

Diffusive fluxes are associated with molecular motion in a continuum fluid. A simple linear model equation for a diffusive process is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \tag{2.27}$$

where ν is a positive real constant. For example, with u representing the temperature, this parabolic PDE governs the diffusion of heat in one dimension. Boundary conditions can be periodic, Dirichlet (specified u), Neumann (specified $\partial u/\partial x$), or mixed Dirichlet/Neumann.

In contrast to the linear convection equation, the diffusion equation has a nontrivial *steady-state* solution, which is one that satisfies the governing PDE with the partial derivative in time equal to zero. In the case of Eq. 2.27, the steady-state solution must satisfy

$$\frac{\partial^2 u}{\partial x^2} = 0 \tag{2.28}$$

Therefore, u must vary linearly with x at steady state such that the boundary conditions are satisfied. Other steady-state solutions are obtained if a source term $g(x)$ is added to Eq. 2.27, as follows:

$$\frac{\partial u}{\partial t} = \nu \left[\frac{\partial^2 u}{\partial x^2} - g(x) \right] \tag{2.29}$$

giving a steady state-solution which satisfies

$$\frac{\partial^2 u}{\partial x^2} - g(x) = 0 \tag{2.30}$$

In two dimensions, the diffusion equation becomes

$$\frac{\partial u}{\partial t} = \nu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - g(x,y) \right] \tag{2.31}$$

where $g(x,y)$ is again a source term. The corresponding steady equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - g(x,y) = 0 \tag{2.32}$$

While Eq. 2.31 is parabolic, Eq. 2.32 is elliptic. The latter is known as the Poisson equation for nonzero g, and as Laplace's equation for zero g.

2.4.2 Solution in Wave Space

We now consider a series solution to Eq. 2.27. Let the domain be given by $0 \leq x \leq \pi$ with boundary conditions $u(0) = u_a$, $u(\pi) = u_b$. It is clear that the steady-state solution is given by a linear function which satisfies the boundary conditions, i.e., $h(x) = u_a + (u_b - u_a)x/\pi$. Let the initial condition be

$$u(x,0) = \sum_{m=1}^{M} f_m(0) \sin \kappa_m x + h(x) \tag{2.33}$$

where κ must be an integer in order to satisfy the boundary conditions. A solution of the form

$$u(x,t) = \sum_{m=1}^{M} f_m(t) \sin \kappa_m x + h(x) \tag{2.34}$$

satisfies the initial and boundary conditions. Substituting this form into Eq. 2.27 gives the following ODE for f_m:

$$\frac{df_m}{dt} = -\kappa_m^2 \nu f_m \tag{2.35}$$

and we find

$$f_m(t) = f_m(0) e^{-\kappa_m^2 \nu t} \tag{2.36}$$

Substituting $f_m(t)$ into equation 2.34, we obtain

$$u(x,t) = \sum_{m=1}^{M} f_m(0) e^{-\kappa_m^2 \nu t} \sin \kappa_m x + h(x) \tag{2.37}$$

The steady-state solution ($t \to \infty$) is simply $h(x)$. Eq. 2.37 shows that high wavenumber components (large κ_m) of the solution decay more rapidly than low wavenumber components, consistent with the physics of diffusion.

2.5 Linear Hyperbolic Systems

The Euler equations, Eq. 2.8, form a hyperbolic system of partial differential equations. Other systems of equations governing convection and wave propagation phenomena, such as the Maxwell equations describing the propagation of electromagnetic waves, are also of hyperbolic type. Many aspects of numerical methods for such systems can be understood by studying a one-dimensional constant-coefficient linear system of the form

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \tag{2.38}$$

where $u = u(x,t)$ is a vector of length m and A is a real $m \times m$ matrix. For conservation laws, this equation can also be written in the form

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{2.39}$$

where f is the flux vector and $A = \frac{\partial f}{\partial u}$ is the flux Jacobian matrix. The entries in the flux Jacobian are

$$a_{ij} = \frac{\partial f_i}{\partial u_j} \tag{2.40}$$

The flux Jacobian for the Euler equations is derived in Section 2.2.

Such a system is hyperbolic if A is diagonalizable with real eigenvalues.[1] Thus

$$\Lambda = X^{-1}AX \tag{2.41}$$

where Λ is a diagonal matrix containing the eigenvalues of A, and X is the matrix of right eigenvectors. Premultiplying Eq. 2.38 by X^{-1}, postmultiplying A by the product XX^{-1}, and noting that X and X^{-1} are constants, we obtain

$$\frac{\partial X^{-1}u}{\partial t} + \frac{\partial \overbrace{X^{-1}AX}^{\Lambda} X^{-1}u}{\partial x} = 0 \tag{2.42}$$

With $w = X^{-1}u$, this can be rewritten as

$$\frac{\partial w}{\partial t} + \Lambda\frac{\partial w}{\partial x} = 0 \tag{2.43}$$

When written in this manner, the equations have been decoupled into m scalar equations of the form

$$\frac{\partial w_i}{\partial t} + \lambda_i\frac{\partial w_i}{\partial x} = 0 \tag{2.44}$$

[1]See Appendix A for a brief review of some basic relations and definitions from linear algebra.

The elements of w are known as *characteristic variables*. Each characteristic variable satisfies the linear convection equation with the speed given by the corresponding eigenvalue of A.

Based on the above, we see that a hyperbolic system in the form of Eq. 2.38 has a solution given by the superposition of waves which can travel in either the positive or negative directions and at varying speeds. While the scalar linear convection equation is clearly an excellent model equation for hyperbolic systems, we must ensure that our numerical methods are appropriate for wave speeds of arbitrary sign and possibly widely varying magnitudes.

The one-dimensional Euler equations can also be diagonalized, leading to three equations in the form of the linear convection equation, although they remain nonlinear, of course. The eigenvalues of the flux Jacobian matrix, or wave speeds, are $u, u+c$, and $u-c$, where u is the local fluid velocity, and $c = \sqrt{\gamma p/\rho}$ is the local speed of sound. The speed u is associated with convection of the fluid, while $u+c$ and $u-c$ are associated with sound waves. Therefore, in a supersonic flow, where $|u| > c$, all of the wave speeds have the same sign. In a subsonic flow, where $|u| < c$, wave speeds of both positive and negative sign are present, corresponding to the fact that sound waves can travel upstream in a subsonic flow.

The signs of the eigenvalues of the matrix A are also important in determining suitable boundary conditions. The characteristic variables each satisfy the linear convection equation with the wave speed given by the corresponding eigenvalue. Therefore, the boundary conditions can be specified accordingly. That is, characteristic variables associated with positive eigenvalues can be specified at the left boundary, which corresponds to inflow for these variables. Characteristic variables associated with negative eigenvalues can be specified at the right boundary, which is the inflow boundary for these variables. While other boundary condition treatments are possible, they must be consistent with this approach.

2.6 Problems

1. Show that the 1-D Euler equations can be written in terms of the *primitive* variables $R = [\rho, u, p]^T$ as follows:

$$\frac{\partial R}{\partial t} + M\frac{\partial R}{\partial x} = 0$$

where

$$M = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \rho^{-1} \\ 0 & \gamma p & u \end{bmatrix}$$

Assume an ideal gas, $p = (\gamma - 1)(e - \rho u^2/2)$.

2. Find the eigenvalues and eigenvectors of the matrix M derived in question 1.

3. Derive the flux Jacobian matrix $A = \partial E/\partial Q$ for the 1-D Euler equations resulting from the conservative variable formulation (Eq. 2.5). Find its eigenvalues and compare with those obtained in question 2.

4. Show that the two matrices M and A derived in questions 1 and 3, respectively, are related by a similarity transform. (Hint: make use of the matrix $S = \partial Q/\partial R$.)

5. Write the 2-D diffusion equation, Eq. 2.31, in the form of Eq. 2.2.

6. Given the initial condition $u(x,0) = \sin x$ defined on $0 \le x \le 2\pi$, write it in the form of Eq. 2.25, that is, find the necessary values of $f_m(0)$. (Hint: use $M = 2$ with $\kappa_1 = 1$ and $\kappa_2 = -1$.) Next consider the same initial condition defined only at $x = 2\pi j/4$, $j = 0, 1, 2, 3$. Find the values of $f_m(0)$ required to reproduce the initial condition at these discrete points using $M = 4$ with $\kappa_m = m - 1$.

7. Plot the first three basis functions used in constructing the exact solution to the diffusion equation in Section 2.4.2. Next consider a solution with boundary conditions $u_a = u_b = 0$, and initial conditions from Eq. 2.33 with $f_m(0) = 1$ for $1 \le m \le 3$, $f_m(0) = 0$ for $m > 3$. Plot the initial condition on the domain $0 \le x \le \pi$. Plot the solution at $t = 1$ with $\nu = 1$.

8. Write the classical wave equation $\partial^2 u/\partial t^2 = c^2 \partial^2 u/\partial x^2$ as a first-order system, i.e., in the form

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0$$

where $U = [\partial u/\partial x, \partial u/\partial t]^T$. Find the eigenvalues and eigenvectors of A.

9. The Cauchy-Riemann equations are formed from the coupling of the steady compressible continuity (conservation of mass) equation

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

and the vorticity definition

$$\omega = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

where $\omega = 0$ for irrotational flow. For isentropic and homenthalpic flow, the system is closed by the relation

$$\rho = \left(1 - \frac{\gamma - 1}{2}\left(u^2 + v^2 - 1\right)\right)^{\frac{1}{\gamma - 1}}$$

Note that the variables have been nondimensionalized. Combining the two PDE's, we have

$$\frac{\partial f(q)}{\partial x} + \frac{\partial g(q)}{\partial y} = 0$$

where

$$q = \begin{pmatrix} u \\ v \end{pmatrix}, \quad f = \begin{pmatrix} -\rho u \\ v \end{pmatrix}, \quad g = \begin{pmatrix} -\rho v \\ -u \end{pmatrix}$$

One approach to solving these equations is to add a time-dependent term and find the steady solution of the following equation:

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

(a) Find the flux Jacobians of f and g with respect to q.

(b) Determine the eigenvalues of the flux Jacobians.

(c) Determine the conditions (in terms of ρ and u) under which the system is hyperbolic, i.e., has real eigenvalues.

(d) Are the above fluxes homogeneous? (See Appendix C.)