Chapter 11

NUMERICAL DISSIPATION

Up to this point, we have emphasized the second-order centered-difference approximations to the spatial derivatives in our model equations. We have seen that a centered approximation to a first derivative is nondissipative, i.e., the eigenvalues of the associated circulant matrix (with periodic boundary conditions) are pure imaginary. In processes governed by nonlinear equations, such as the Euler and Navier-Stokes equations, there can be a continual production of high-frequency components of the solution, leading, for example, to the production of shock waves. In a real physical problem, the production of high frequencies is eventually limited by viscosity. However, when we solve the Euler equations numerically, we have neglected viscous effects. Thus the numerical approximation must contain some inherent dissipation to limit the production of high-frequency modes. Although numerical approximations to the Navier-Stokes equations contain dissipation through the viscous terms, this can be insufficient, especially at high Reynolds numbers, due to the limited grid resolution which is practical. Therefore, unless the relevant length scales are resolved, some form of added numerical dissipation is required in the numerical solution of the Navier-Stokes equations as well. Since the addition of numerical dissipation is tantamount to intentionally introducing nonphysical behavior, it must be carefully controlled such that the error introduced is not excessive. In this Chapter, we discuss some different ways of adding numerical dissipation to the spatial derivatives in the linear convection equation and hyperbolic systems of PDE's.

11.1 One-Sided First-Derivative Space Differencing

We investigate the properties of one-sided spatial difference operators in the context of the biconvection model equation given by

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} \tag{11.1}$$

with periodic boundary conditions. Consider the following point operator for the spatial derivative term

$$\begin{aligned} -a(\delta_x u)_j &= \frac{-a}{2\Delta x}[-(1+\beta)u_{j-1} + 2\beta u_j + (1-\beta)u_{j+1}] \\ &= \frac{-a}{2\Delta x}[(-u_{j-1} + u_{j+1}) + \beta(-u_{j-1} + 2u_j - u_{j+1})] \end{aligned} \tag{11.2}$$

The second form shown divides the operator into an antisymmetric component $(-u_{j-1}+u_{j+1})/2\Delta x$ and a symmetric component $\beta(-u_{j-1} + 2u_j - u_{j+1})/2\Delta x$. The antisymmetric component is the second-order centered difference operator. With $\beta \neq 0$, the operator is only first-order accurate. A *backward* difference operator is given by $\beta = 1$ and a *forward* difference operator is given by $\beta = -1$.

For periodic boundary conditions the corresponding matrix operator is

$$-a\delta_x = \frac{-a}{2\Delta x}B_p(-1-\beta, 2\beta, 1-\beta)$$

The eigenvalues of this matrix are

$$\lambda_m = \frac{-a}{\Delta x}\left\{\beta\left[1-\cos\left(\frac{2\pi m}{M}\right)\right] + i\sin\left(\frac{2\pi m}{M}\right)\right\} \quad \text{for} \quad \text{m} = 0,1,\ldots,\text{M}-1$$

If a is positive, the forward difference operator $(\beta = -1)$ produces $Re(\lambda_m) > 0$, the centered difference operator $(\beta = 0)$ produces $Re(\lambda_m) = 0$, and the backward difference operator produces $Re(\lambda_m) < 0$. Hence the forward difference operator is inherently unstable while the centered and backward operators are inherently stable. If a is negative, the roles are reversed. When $Re(\lambda_m) = 0$, the solution will either grow or decay with time. In either case, our choice of differencing scheme produces nonphysical behavior. We proceed next to show why this occurs.

11.2 The Modified Partial Differential Equation

First carry out a Taylor series expansion of the terms in Eq. 11.2. We are lead to the expression

$$(\delta_x u)_j = \frac{1}{2\Delta x}\left[2\Delta x\left(\frac{\partial u}{\partial x}\right)_j - \beta\Delta x^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{\Delta x^3}{3}\left(\frac{\partial^3 u}{\partial x^3}\right)_j - \frac{\beta\Delta x^4}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_j + \ldots\right]$$

We see that the antisymmetric portion of the operator introduces odd derivative terms in the truncation error while the symmetric portion introduces even derivatives. Substituting this into Eq. 11.1 gives

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \frac{a\beta\Delta x}{2}\frac{\partial^2 u}{\partial x^2} - \frac{a\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3} + \frac{a\beta\Delta x^3}{24}\frac{\partial^4 u}{\partial x^4} + \ldots \tag{11.3}$$

This is the partial differential equation we are really solving when we apply the approximation given by Eq. 11.2 to Eq. 11.1. Notice that Eq. 11.3 is *consistent* with Eq. 11.1, since the two equations are identical when $\Delta x \to 0$. However, when we use a computer to find a numerical solution of the problem, Δx can be small but it is *not* zero. This means that each term in the expansion given by Eq. 11.3 is excited to some degree. We refer to Eq. 11.3 as the *modified partial differential equation.* We proceed next to investigate the implications of this concept.

Consider the simple linear partial differential equation

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^3 u}{\partial x^3} + \tau\frac{\partial^4 u}{\partial x^4} \tag{11.4}$$

Choose periodic boundary conditions and impose an initial condition $u = e^{i\kappa x}$. Under these conditions there is a wave-like solution to Eq. 11.4 of the form

$$u(x,t) = e^{i\kappa x}e^{(r+is)t}$$

provided r and s satisfy the condition

$$r + is = -ia\kappa - \nu\kappa^2 - i\gamma\kappa^3 + \tau\kappa^4$$

or

$$r = -\kappa^2(\nu - \tau\kappa^2), \qquad s = -\kappa(a + \gamma\kappa^2)$$

The solution is composed of both amplitude and phase terms. Thus

$$u = \underbrace{e^{-\kappa^2(\nu-\tau\kappa^2)}}_{\text{amplitude}}\ \underbrace{e^{i\kappa[x-(a+\gamma\kappa^2)t]}}_{\text{phase}} \tag{11.5}$$

It is important to notice that the amplitude of the solution depends only upon ν and τ, the coefficients of the even derivatives in Eq. 11.4, and the phase depends only on a and γ, the coefficients of the odd derivatives.

If the wave speed a is positive, the choice of a backward difference scheme ($\beta = 1$) produces a modified PDE with $\nu - \tau\kappa^2 > 0$ and hence the amplitude of the solution decays. This is tantamount to *deliberately adding* dissipation to the PDE. Under the same condition, the choice of a forward difference scheme ($\beta = -1$) is equivalent to *deliberately adding* a destabilizing term to the PDE.

By examining the term governing the phase of the solution in Eq. 11.5, we see that the speed of propagation is $a + \gamma\kappa^2$. Referring to the modified PDE, Eq. 11.3 we have $\gamma = -a\Delta x^2/6$. Therefore, the phase speed of the numerical solution is *less* than the actual phase speed. Furthermore, the numerical phase speed is dependent upon the wavenumber κ. This we refer to as *dispersion*.

Our purpose here is to investigate the properties of one-sided spatial differencing operators relative to centered difference operators. We have seen that the three-point centered difference approximation of the spatial derivative *produces a modified PDE that has no dissipation* (or amplification). One can easily show, by using the antisymmetry of the matrix difference operators, that the same is true for *any* centered difference approximation of a first derivative. As a corollary, *any departure from antisymmetry in the matrix difference operator must introduce dissipation* (or amplification) into the modified PDE.

Note that the use of one-sided differencing schemes is not the only way to introduce dissipation. Any symmetric component in the spatial operator introduces dissipation (or amplification). Therefore, one could choose $\beta = 1/2$ in Eq. 11.2. The resulting spatial operator is not one-sided but it is dissipative. Biased schemes use more information on one side of the node than the other. For example, a third-order backward-biased scheme is given by

$$\begin{aligned}
(\delta_x u)_j &= \frac{1}{6\Delta x}(u_{j-2} - 6u_{j-1} + 3u_j + 2u_{j+1}) \\
&= \frac{1}{12\Delta x}[(u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}) \\
&+ (u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2})]
\end{aligned} \tag{11.6}$$

The antisymmetric component of this operator is the fourth-order centered difference operator. The symmetric component approximates $\Delta x^3 u_{xxxx}/12$. Therefore, this operator produces fourth-order accuracy in phase with a third-order dissipative term.

11.3 The Lax-Wendroff Method

In order to introduce numerical dissipation using one-sided differencing, backward differencing must be used if the wave speed is positive, and forward differencing must be used if the wave speed is negative. Next we consider a method which introduces dissipation independent of the sign of the wave speed, known as the Lax-Wendroff method. This explicit method differs conceptually from the methods considered previously in which spatial differencing and time-marching are treated separately.

Consider the following Taylor-series expansion in time:

$$u(x, t+h) = u + h\frac{\partial u}{\partial t} + \frac{1}{2}h^2\frac{\partial^2 u}{\partial t^2} + O(h^3) \tag{11.7}$$

First replace the time derivatives with space derivatives according to the PDE (in this case, the linear convection equation $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$). Thus

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}, \qquad \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \tag{11.8}$$

Now replace the space derivatives with three-point centered difference operators, giving

$$u_j^{(n+1)} = u_j^{(n)} - \frac{1}{2}\frac{ah}{\Delta x}(u_{j+1}^{(n)} - u_{j-1}^{(n)}) + \frac{1}{2}\left(\frac{ah}{\Delta x}\right)^2 (u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}) \tag{11.9}$$

This is the Lax-Wendroff method applied to the linear convection equation. It is a fully-discrete finite-difference scheme. There is no intermediate semi-discrete stage.

For periodic boundary conditions, the corresponding fully-discrete matrix operator is

$$\vec{u}_{n+1} = B_p\left(\frac{1}{2}\left[\frac{ah}{\Delta x} + \left(\frac{ah}{\Delta x}\right)^2\right], 1 - \left(\frac{ah}{\Delta x}\right)^2, \frac{1}{2}\left[-\frac{ah}{\Delta x} + \left(\frac{ah}{\Delta x}\right)^2\right]\right)\vec{u}_n$$

The eigenvalues of this matrix are

$$\sigma_m = 1 - \left(\frac{ah}{\Delta x}\right)^2\left[1 - \cos\left(\frac{2\pi m}{M}\right)\right] - i\frac{ah}{\Delta x}\sin\left(\frac{2\pi m}{M}\right) \quad \text{for} \quad m = 0, 1, \ldots, M-1$$

For $|\frac{ah}{\Delta x}| \leq 1$ all of the eigenvalues have modulus less than or equal to unity and hence the method is stable *independent of the sign of* a. The quantity $|\frac{ah}{\Delta x}|$ is known as the Courant (or CFL) number. It is equal to the ratio of the distance travelled by a wave in one time step to the mesh spacing.

The nature of the dissipative properties of the Lax-Wendroff scheme can be seen by examining the modified partial differential equation, which is given by

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{a}{6}(\Delta x^2 - a^2h^2)\frac{\partial^3 u}{\partial x^3} - \frac{a^2 h}{8}(\Delta x^2 - a^2h^2)\frac{\partial^4 u}{\partial x^4} + \ldots$$

This is derived by substituting Taylor series expansions for all terms in Eq. 11.9 and converting the time derivatives to space derivatives using Eq. 11.8. The two leading error terms appear on the right side of the equation. Recall that the odd derivatives on the right side lead to unwanted dispersion and the even derivatives lead to dissipation (or amplification, depending on the sign). Therefore, the leading error term in the Lax-Wendroff method is dispersive and proportional to

$$-\frac{a}{6}(\Delta x^2 - a^2h^2)\frac{\partial^3 u}{\partial x^3} = -\frac{a\Delta x^2}{6}(1 - C_n^2)\frac{\partial^3 u}{\partial x^3}$$

The dissipative term is proportional to

$$-\frac{a^2 h}{8}(\Delta x^2 - a^2h^2)\frac{\partial^4 u}{\partial x^4} = -\frac{a^2 h\Delta x^2}{8}(1 - C_n^2)\frac{\partial^4 u}{\partial x^4}$$

This term has the appropriate sign and hence the scheme is truly dissipative as long as $C_n \leq 1$.

A closely related method is that of MacCormack. Recall MacCormack's time-marching method, presented in Chapter 6:

$$\begin{aligned} \tilde{u}_{n+1} &= u_n + hu'_n \\ u_{n+1} &= \frac{1}{2}[u_n + \tilde{u}_{n+1} + h\tilde{u}'_{n+1}] \end{aligned} \tag{11.10}$$

If we use first-order backward differencing in the first stage and first-order forward differencing in the second stage,[1] a dissipative second-order method is obtained. For the linear convection equation, this approach leads to

$$\begin{aligned} \tilde{u}_j^{(n+1)} &= u_j^{(n)} - \frac{ah}{\Delta x}(u_j^{(n)} - u_{j-1}^{(n)}) \\ u_j^{(n+1)} &= \frac{1}{2}[u_j^{(n)} + \tilde{u}_j^{(n+1)} - \frac{ah}{\Delta x}(\tilde{u}_{j+1}^{(n+1)} - \tilde{u}_j^{(n+1)})] \end{aligned} \tag{11.11}$$

which can be shown to be identical to the Lax-Wendroff method. Hence MacCormack's method has the same dissipative and dispersive properties as the Lax-Wendroff method. The two methods differ when applied to nonlinear hyperbolic systems, however.

[1] Or vice-versa; for nonlinear problems, these should be applied alternately.

11.4 Upwind Schemes

In Section 11.1, we saw that numerical dissipation can be introduced in the spatial difference operator using one-sided difference schemes or, more generally, by adding a symmetric component to the spatial operator. With this approach, the direction of the one-sided operator (i.e., whether it is a forward or a backward difference) or the sign of the symmetric component depends on the sign of the wave speed. When a *hyperbolic system* of equations is being solved, the wave speeds can be both positive and negative. For example, the eigenvalues of the flux Jacobian for the one-dimensional Euler equations are $u, u+a, u-a$. When the flow is subsonic, these are of mixed sign. In order to apply one-sided differencing schemes to such systems, some form of splitting is required. This is avoided in the Lax-Wendroff scheme. However, as a result of their superior flexibility, schemes in which the numerical dissipation is introduced in the spatial operator are generally preferred over the Lax-Wendroff approach.

Consider again the linear convection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{11.12}$$

where we do not make any assumptions as to the sign of a. We can rewrite Eq. 11.12 as

$$\frac{\partial u}{\partial t} + (a^+ + a^-)\frac{\partial u}{\partial x} = 0 \quad ; \quad a^{\pm} = \frac{a \pm |a|}{2}$$

If $a \geq 0$, then $a^+ = a \geq 0$ and $a^- = 0$. Alternatively, if $a \leq 0$, then $a^+ = 0$ and $a^- = a \leq 0$. Now for the a^+ (≥ 0) term we can safely backward difference and for the a^- (≤ 0) term forward difference. This is the basic concept behind upwind methods, that is, some decomposition or splitting of the fluxes into terms which have positive and negative characteristic speeds so that appropriate differencing schemes can be chosen. In the next two sections, we present two splitting techniques commonly used with upwind methods. These are by no means unique.

The above approach to obtaining a stable discretization independent of the sign of a can be written in a different, but entirely equivalent, manner. From Eq. 11.2, we see that a stable discretization is obtained with $\beta = 1$ if $a \geq 0$ and with $\beta = -1$ if $a \leq 0$. This is achieved by the following point operator:

$$-a(\delta_x u)_j = \frac{-1}{2\Delta x}[a(-u_{j-1} + u_{j+1}) + |a|(-u_{j-1} + 2u_j - u_{j+1})] \tag{11.13}$$

This approach is extended to systems of equations in Section 11.5.

In this section, we present the basic ideas of flux-vector and flux-difference splitting. For more subtle aspects of implementation and application of such techniques to nonlinear hyperbolic systems such as the Euler equations, the reader is referred to the literature on this subject.

11.4.1 Flux-Vector Splitting

Recall from Section 2.5 that a linear, constant-coefficient, hyperbolic system of partial differential equations given by

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \tag{11.14}$$

can be decoupled into characteristic equations of the form

$$\frac{\partial w_i}{\partial t} + \lambda_i \frac{\partial w_i}{\partial x} = 0 \tag{11.15}$$

where the wave speeds, λ_i, are the eigenvalues of the Jacobian matrix, A, and the w_i's are the characteristic variables. In order to apply a one-sided (or biased) spatial differencing scheme, we need to apply a backward difference if the wave speed, λ_i, is positive, and a forward difference if the wave speed is negative. To accomplish this, let us split the matrix of eigenvalues, Λ, into two components such that

$$\Lambda = \Lambda^+ + \Lambda^- \tag{11.16}$$

where

$$\Lambda^+ = \frac{\Lambda + |\Lambda|}{2}, \qquad \Lambda^- = \frac{\Lambda - |\Lambda|}{2} \tag{11.17}$$

With these definitions, Λ^+ contains the positive eigenvalues and Λ^- contains the negative eigenvalues. We can now rewrite the system in terms of characteristic variables as

$$\frac{\partial w}{\partial t} + \Lambda\frac{\partial w}{\partial x} = \frac{\partial w}{\partial t} + \Lambda^+\frac{\partial w}{\partial x} + \Lambda^-\frac{\partial w}{\partial x} = 0 \tag{11.18}$$

The spatial terms have been split into two components according to the sign of the wave speeds. We can use backward differencing for the $\Lambda^+\frac{\partial w}{\partial x}$ term and forward differencing for the $\Lambda^-\frac{\partial w}{\partial x}$ term. Premultiplying by X and inserting the product $X^{-1}X$ in the spatial terms gives

$$\frac{\partial Xw}{\partial t} + \frac{\partial X\Lambda^+X^{-1}Xw}{\partial x} + \frac{\partial X\Lambda^-X^{-1}Xw}{\partial x} = 0 \tag{11.19}$$

With the definitions[2]

$$A^+ = X\Lambda^+X^{-1}, \qquad A^- = X\Lambda^-X^{-1} \tag{11.20}$$

and recalling that $u = Xw$, we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial A^+ u}{\partial x} + \frac{\partial A^- u}{\partial x} = 0 \tag{11.21}$$

Finally the split flux vectors are defined as

$$f^+ = A^+ u, \qquad f^- = A^- u \tag{11.22}$$

and we can write

$$\frac{\partial u}{\partial t} + \frac{\partial f^+}{\partial x} + \frac{\partial f^-}{\partial x} = 0 \tag{11.23}$$

In the linear case, the definition of the split fluxes follows directly from the definition of the flux, $f = Au$. For the Euler equations, f is also equal to Au as a result of their homogeneous property, as discussed in Appendix C. Note that

$$f = f^+ + f^- \tag{11.24}$$

Thus by applying backward differences to the f^+ term and forward differences to the f^- term, we are in effect solving the characteristic equations in the desired manner. This approach is known as flux-vector splitting.

When an implicit time-marching method is used, the Jacobians of the split flux vectors are required. In the nonlinear case,

$$\frac{\partial f^+}{\partial u} \neq A^+, \qquad \frac{\partial f^-}{\partial u} \neq A^- \tag{11.25}$$

Therefore, one must find and use the new Jacobians given by

$$A^{++} = \frac{\partial f^+}{\partial u}, \qquad A^{--} = \frac{\partial f^-}{\partial u} \tag{11.26}$$

For the Euler equations, A^{++} has eigenvalues which are all positive, and A^{--} has all negative eigenvalues.

[2]With these definitions A^+ has all positive eigenvalues, and A^- has all negative eigenvalues.

11.4.2 Flux-Difference Splitting

Another approach, more suited to finite-volume methods, is known as flux-difference splitting. In a finite-volume method, the fluxes must be evaluated at cell boundaries. We again begin with the diagonalized form of the linear, constant-coefficient, hyperbolic system of equations

$$\frac{\partial w}{\partial t} + \Lambda \frac{\partial w}{\partial x} = 0 \tag{11.27}$$

The flux vector associated with this form is $g = \Lambda w$. Now, as in Chapter 5, we consider the numerical flux at the interface between nodes j and $j+1$, $\hat{g}_{j+1/2}$, as a function of the states to the left and right of the interface, w_L and w_R, respectively. The centered approximation to $g_{j+1/2}$, which is nondissipative, is given by

$$\hat{g}_{j+1/2} = \frac{1}{2}\left(g(w_L) + g(w_R)\right) \tag{11.28}$$

In order to obtain a one-sided *upwind* approximation, we require

$$(\hat{g}_i)_{j+1/2} = \begin{cases} \lambda_i (w_i)_L \text{ if } \lambda_i > 0 \\ \lambda_i (w_i)_R \text{ if } \lambda_i < 0 \end{cases} \tag{11.29}$$

where the subscript i indicates individual components of w and g. This is achieved with

$$(\hat{g}_i)_{j+1/2} = \frac{1}{2}\lambda_i\left[(w_i)_L + (w_i)_R\right] + \frac{1}{2}|\lambda_i|\left[(w_i)_L - (w_i)_R\right] \tag{11.30}$$

or

$$\hat{g}_{j+1/2} = \frac{1}{2}\Lambda\left(w_L + w_R\right) + \frac{1}{2}|\Lambda|\left(w_L - w_R\right) \tag{11.31}$$

Now, as in Eq. 11.19, we premultiply by X to return to the original variables and insert the product $X^{-1}X$ after Λ and $|\Lambda|$ to obtain

$$X\hat{g}_{j+1/2} = \frac{1}{2}X\Lambda X^{-1}X\left(w_L + w_R\right) + \frac{1}{2}X|\Lambda|X^{-1}X\left(w_L - w_R\right) \tag{11.32}$$

and thus

$$\hat{f}_{j+1/2} = \frac{1}{2}\left(f_L + f_R\right) + \frac{1}{2}|A|\left(u_L - u_R\right) \tag{11.33}$$

where

$$|A| = X|\Lambda|X^{-1} \tag{11.34}$$

and we have also used the relations $f = Xg$, $u = Xw$, and $A = X\Lambda X^{-1}$.

In the linear, constant-coefficient case, this leads to an upwind operator which is identical to that obtained using flux-vector splitting. However, in the nonlinear case, there is some ambiguity regarding the definition of $|A|$ at the cell interface $j + 1/2$. In order to resolve this, consider a situation in which the eigenvalues of A are all of the same sign. In this case, we would like our definition of $\hat{f}_{j+1/2}$ to satisfy

$$\hat{f}_{j+1/2} = \begin{cases} f_L \text{ if all } \lambda_i\text{'s} > 0 \\ f_R \text{ if all } \lambda_i\text{'s} < 0 \end{cases} \tag{11.35}$$

giving pure upwinding. If the eigenvalues of A are all positive, $|A| = A$; if they are all negative, $|A| = -A$. Hence satisfaction of Eq. 11.35 is obtained by the definition

$$\hat{f}_{j+1/2} = \frac{1}{2}(f_L + f_R) + \frac{1}{2}|A_{j+1/2}|(u_L - u_R) \tag{11.36}$$

if $A_{j+1/2}$ satisfies

$$f_L - f_R = A_{j+1/2}(u_L - u_R) \tag{11.37}$$

For the Euler equations for a perfect gas, Eq. 11.37 is satisfied by the flux Jacobian evaluated at the Roe-average state given by

$$u_{j+1/2} = \frac{\sqrt{\rho_L}u_L + \sqrt{\rho_R}u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{11.38}$$

$$H_{j+1/2} = \frac{\sqrt{\rho_L}H_L + \sqrt{\rho_R}H_R}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{11.39}$$

where u and $H = (e + p)/\rho$ are the velocity and the total enthalpy per unit mass, respectively.[3]

11.5 Artificial Dissipation

We have seen that numerical dissipation can be introduced by using one-sided differencing schemes together with some form of flux splitting. We have also seen that such dissipation can be introduced by adding a symmetric component to an antisymmetric (dissipation-free) operator. Thus we can generalize the concept of upwinding to include any scheme in which the symmetric portion of the operator is treated in such a manner as to be truly dissipative.

[3]Note that the flux Jacobian can be written in terms of u and H only; see problem 6 at the end of this chapter.

For example, let

$$(\delta_x^a u)_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \qquad (\delta_x^s u)_j = \frac{-u_{j+1} + 2u_j - u_{j-1}}{2\Delta x} \tag{11.40}$$

Applying $\delta_x = \delta_x^a + \delta_x^s$ to the spatial derivative in Eq. 11.15 is stable if $\lambda_i \geq 0$ and unstable if $\lambda_i < 0$. Similarly, applying $\delta_x = \delta_x^a - \delta_x^s$ is stable if $\lambda_i \leq 0$ and unstable if $\lambda_i > 0$. The appropriate implementation is thus

$$\lambda_i \delta_x = \lambda_i \delta_x^a + |\lambda_i| \delta_x^s \tag{11.41}$$

Extension to a hyperbolic system by applying the above approach to the characteristic variables, as in the previous two sections, gives

$$\delta_x(Au) = \delta_x^a(Au) + \delta_x^s(|A|u) \tag{11.42}$$

or

$$\delta_x f = \delta_x^a f + \delta_x^s(|A|u) \tag{11.43}$$

where $|A|$ is defined in Eq. 11.34. The second spatial term is known as *artificial dissipation*. It is also sometimes referred to as artificial diffusion or artificial viscosity. With appropriate choices of δ_x^a and δ_x^s, this approach can be related to the upwind approach. This is particularly evident from a comparison of Eqs. 11.36 and 11.43.

It is common to use the following operator for δ_x^s

$$(\delta_x^s u)_j = \frac{\epsilon}{\Delta x}(u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2}) \tag{11.44}$$

where ϵ is a problem-dependent coefficient. This symmetric operator approximates $\epsilon \Delta x^3 u_{xxxx}$ and thus introduces a third-order dissipative term. With an appropriate value of ϵ, this often provides sufficient damping of high frequency modes without greatly affecting the low frequency modes. For details of how this can be implemented for nonlinear hyperbolic systems, the reader should consult the literature. A more complicated treatment of the numerical dissipation is also required near shock waves and other discontinuities, but is beyond the scope of this book.

11.6 Problems

1. A second-order backward difference approximation to a 1st derivative is given as a point operator by

$$(\delta_x u)_j = \frac{1}{2\Delta x}(u_{j-2} - 4u_{j-1} + 3u_j)$$

(a) Express this operator in banded matrix form (for periodic boundary conditions), then derive the symmetric and skew-symmetric matrices that have the matrix operator as their sum. (See Appendix A.3 to see how to construct the symmetric and skew-symmetric components of a matrix.)

(b) Using a Taylor table, find the derivative which is approximated by the corresponding symmetric and skew-symmetric operators and the leading error term for each.

2. Find the modified wavenumber for the first-order backward difference operator. Plot the real and imaginary parts of $\kappa^* \Delta x$ vs. $\kappa \Delta x$ for $0 \leq \kappa \Delta x \leq \pi$. Using Fourier analysis as in Section 6.6.2, find $|\sigma|$ for the combination of this spatial operator with 4th-order Runge-Kutta time marching at a Courant number of unity and plot vs. $\kappa \Delta x$ for $0 \leq \kappa \Delta x \leq \pi$.

3. Find the modified wavenumber for the operator given in Eq. 11.6. Plot the real and imaginary parts of $\kappa^* \Delta x$ vs. $\kappa \Delta x$ for $0 \leq \kappa \Delta x \leq \pi$. Using Fourier analysis as in Section 6.6.2, find $|\sigma|$ for the combination of this spatial operator with 4th-order Runge-Kutta time marching at a Courant number of unity and plot vs. $\kappa \Delta x$ for $0 \leq \kappa \Delta x \leq \pi$.

4. Consider the spatial operator obtained by combining second-order centered differences with the symmetric operator given in Eq. 11.44. Find the modified wavenumber for this operator with $\epsilon = 0, 1/12, 1/24$, and $1/48$. Plot the real and imaginary parts of $\kappa^* \Delta x$ vs. $\kappa \Delta x$ for $0 \leq \kappa \Delta x \leq \pi$. Using Fourier analysis as in Section 6.6.2, find $|\sigma|$ for the combination of this spatial operator with 4th-order Runge-Kutta time marching at a Courant number of unity and plot vs. $\kappa \Delta x$ for $0 \leq \kappa \Delta x \leq \pi$.

5. Consider the hyperbolic system derived in problem 8 of Chapter 2. Find the matrix $|A|$. Form the plus-minus split flux vectors as in Section 11.4.1.

6. Show that the flux Jacobian for the 1-D Euler equations can be written in terms of u and H. Show that the use of the Roe average state given in Eqs. 11.38 and 11.39 leads to satisfaction of Eq. 11.37.