Appendix C

THE HOMOGENEOUS PROPERTY OF THE EULER EQUATIONS

The Euler equations have a special property that is sometimes useful in constructing numerical methods. In order to examine this property, let us first inspect Euler's theorem on homogeneous functions. Consider first the scalar case. If $F(u, v)$ satisfies the identity

$$F(\alpha u, \alpha v) = \alpha^n F(u, v) \tag{C.1}$$

for a fixed n, F is called homogeneous of degree n. Differentiating both sides with respect to α and setting $\alpha = 1$ (since the identity holds for all α), we find

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} = nF(u, v) \tag{C.2}$$

Consider next the theorem as it applies to systems of equations. If the vector $F(Q)$ satisfies the identity

$$F(\alpha Q) = \alpha^n F(Q) \tag{C.3}$$

for a fixed n, F is said to be homogeneous of degree n and we find

$$\left[\frac{\partial F}{\partial q}\right] Q = nF(Q) \tag{C.4}$$

Now it is easy to show, by direct use of eq. C.3, that both E and F in eqs. 2.11 and 2.12 are homogeneous of degree 1, and their Jacobians, A and B, are homogeneous of degree 0 (actually the latter is a direct consequence of the former).[1] This being the case, we notice that the expansion of the flux vector in the vicinity of t_n which, according to eq. 6.105 can be written in general as,

$$\begin{aligned} E &= E_n + A_n(Q - Q_n) + O(h^2) \\ F &= F_n + B_n(Q - Q_n) + O(h^2) \end{aligned} \tag{C.5}$$

can be written

$$\begin{aligned} E = A_n Q + O(h^2) \\ F = B_n Q + O(h^2) \end{aligned} \tag{C.6}$$

since the terms $E_n - A_n Q_n$ and $F_n - B_n Q_n$ are identically zero for homogeneous vectors of degree 1, see eq. C.4. Notice also that, under this condition, the constant term drops out of eq. 6.106.

As a final remark, we notice from the chain rule that for *any* vectors F and Q

$$\frac{\partial F(Q)}{\partial x} = \left[\frac{\partial F}{\partial Q}\right]\frac{\partial Q}{\partial x} = A\frac{\partial Q}{\partial x} \tag{C.7}$$

We notice also that for a homogeneous F of degree 1, $F = AQ$ and

$$\frac{\partial F}{\partial x} = A\frac{\partial Q}{\partial x} + \left[\frac{\partial A}{\partial x}\right]Q \tag{C.8}$$

Therefore, if F is homogeneous of degree 1,

$$\left[\frac{\partial A}{\partial x}\right]Q = 0 \tag{C.9}$$

in spite of the fact that individually $[\partial A/\partial x]$ and Q are not equal to zero.

[1]Note that this depends on the form of the equation of state. The Euler equations are homogeneous if the equation of state can be written in the form $p = \rho f(\epsilon)$, where ϵ is the internal energy per unit mass.

Bibliography

[1] D. Anderson, J. Tannehill, and R. Pletcher. *Computational Fluid Mechanics and Heat Transfer*. McGraw-Hill Book Company, New York, 1984.

[2] C. Hirsch. *Numerical Computation of Internal and External Flows*, volume 1,2. John Wiley & Sons, New York, 1988.