CHAPTER IV

STABILITY OF PERIODIC MOTIONS

1. On the reduction to generalized equilibrium. For motion near equilibrium of a Hamiltonian or, more generally, of a Pfaffian system, the stable case is naturally defined as that in which the multipliers $\lambda_1, \cdots, \lambda_m$ are pure imaginaries, at least provided that there are no linear commensurability relations between these multipliers.

In this chapter, however, we shall limit attention to the analogous but somewhat more complicated question of stability for motion near a periodic motion of such a system.* The method employed involves a reduction to the case of generalized equilibrium. In the more general Pfaffian case this can be accomplished by a change of variables

$$x_i = \bar{x}_i + \varphi_i(t) \qquad (i = 1, \cdots, 2m),$$

in which the periodic functions $\varphi_i(t)$ of period τ are the coördinates of the given periodic motion. By this means the functions $X_1, \cdots, X_{2m}, Z$ are modified (see (12), page 89), since they are no longer independent of t but periodic of period τ; and the given motion now corresponds to generalized equilibrium at the origin in the new $x_1, \cdots, x_{2m}$ space. Hence we are led to consider the question of motion near such a point of generalized equilibrium.

There is, however, a difficulty associated with this reduction to generalized equilibrium which was first signalized by Poincaré for Hamiltonian systems, and which it is desirable to explain briefly.

Following the analogy with the case of ordinary equilibrium, the stable case is defined as that in which the multipliers $\lambda_1, \cdots, \lambda_m$ are pure imaginaries, at least provided that there

* Cf. my article *Stability and the Equations of Dynamics*, Amer. Journ. Math., vol. 49 (1927) for a treatment of the equilibrium problem.

are no linear commensurability relations between these multipliers and $2\pi\sqrt{-1}/\tau$. If such relations exist the questions to be considered become more complicated in character.

Unfortunately, for a point of generalized equilibrium obtained by the above method of reduction, the multipliers will not satisfy this condition; more specifically, there will always be a multiplier 0, which is double of course. This may be readily seen. The Pfaffian system admits of the integral $Z =$ const. in the original variables, and therefore admits the integral

$$Z(x_1+\varphi_1, \cdots, x_{2m}+\varphi_{2m}) = \text{const.}$$

in the modified variables. By differentiation with respect to the $2m$ arbitrary constants in the general solution $x_1, \cdots, x_{2m}$, it appears that the linear relation

$$\frac{\partial Z}{\partial x_1} y_1 + \cdots + \frac{\partial Z}{\partial x_{2m}} y_{2m} = \text{const.}$$

subsists for $2m$ linearly independent solutions $y_1, \cdots, y_{2m}$ of the equations of variation, and so for the most general solution; it is understood that $\partial Z/\partial x_i$, $(i = 1, \cdots, 2m)$, have $\varphi_1, \cdots, \varphi_{2m}$ as arguments. Now if the $2m$ multipliers $\pm\lambda_1, \cdots, \pm\lambda_m$ are distinct, a complete set of $2m$ solutions

$$y_i = p_{ik} e^{\lambda_k t} \qquad (i = 1, \cdots, 2m)$$

for $k = 1, \cdots, 2m$ exists $(\lambda_{m+i} = -\lambda_i)$, in which p_{ij} are of period τ in t. Since $\partial Z/\partial x_i$ are also periodic, substitution of these solutions in the linear integral relations in the y_i's leads immediately to the conclusion that the constants on the right-hand side must vanish, at least for $\lambda_k \neq 0$. But if these constants vanished for such a complete set of solutions, the constants would vanish for every solution $y_1, \cdots, y_{2m}$. This cannot be the case since $y_1, \cdots, y_{2m}$ can be taken arbitrarily for any particular value of t.*

* It is not possible for $\partial Z/\partial x_i$ to vanish simultaneously for $i = 1, \cdots, 2m$ along the original motion, since the Pfaffian equations then yield $dx_i/dt = 0$, $(i = 1, \cdots, 2m)$ which is impossible, the case of ordinary equilibrium being excluded.

There is then a pair of solutions of the equations of variation which belong to the multiplier 0. Now

$$x_i = \varphi_i(t+k) - \varphi_i(t) \quad (i = 1, \cdots, 2m)$$

for any k defines a solution of the given equations after the reduction, so that by differentiation with respect to k, one solution of the equations of variation

$$y_1 = \varphi_1', \cdots, y_{2m} = \varphi_{2m}'$$

is obtained. This has periodic components and so belongs to the multiplier 0. On the other hand the periodic motion with which we start is not isolated, but varies analytically with the constant c in the known integral (i. e., with the energy constant in the Hamiltonian case). This yields the second periodic solution 4

$$y_1 = \frac{\partial \varphi_1'}{\partial c}, \cdots, y_{2m} = \frac{\partial \varphi_{2m}'}{\partial c},$$

belonging to the multiplier 0. In general there will be no others.

The difficulty may be turned in the following manner. The variable Z may be taken as one of the dependent variables $x_1, \cdots, x_{2m}$, say as x_{2m}, in the original $x_1, \cdots, x_{2m}$ space. Furthermore the variable $\theta = x_{2m-1}$ may be selected as the single angular coördinate, which increases by 2π when a circuit of the curve of periodic motion is made. The remaining coördinates $x_1, \cdots, x_{2m-2}$ may be made to vanish along this curve. Now let us restrict attention to those motions near the given periodic motion for which

$$Z = c$$

has the same value as along this motion. With this understanding, the Pfaffian system becomes of order $2m - 1$ in $x_1, \cdots, x_{2m-2}, \theta$, and may be written in the variational form

$$\delta \int_{t_0}^{t_1} \left(\sum_{j=1}^{2m-2} X_j x_j' + X_{2m-1} \theta' \right) dt = 0,$$

to which set of equations must be added the last equation of the first set. But the integrand is positively homogeneous of dimensions unity in $x_1', \cdots, x_{2m-2}', \theta'$, so that θ may be taken as parameter instead of t. Then the variational principle takes the form

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{2m-2} X_j x_j' + X_{2m-1} \right] d\theta = 0.$$

Hence we obtain a Pfaffian system of even order $2m-2$ only instead of $2m$, in which the coefficients are periodic in a variable θ of period 2π, and the known periodic motion corresponds to the origin in $x_1, \cdots, x_{2m-2}$ space.

By this second method of reduction to a generalized equilibrium problem, the formal difficulties referred to above are avoided.

For these reasons, in dealing with the applications we can restrict attention to the case of generalized equilibrium of stable type as above defined.

2. Stability of Pfaffian systems. Our starting point is furnished by the equations of motion, normalized to terms of an arbitrary degree s by means of an appropriate transformation defined by convergent series, according to the method of the preceding chapter. The equations are thus given the form

$$\begin{aligned} \frac{dp_i}{dt} &= -\frac{\partial H}{\partial \pi_i} p_i + L_{i,s+1}, \\ \frac{dq_i}{dt} &= \frac{\partial H}{\partial \pi_i} q_i + M_{i,s+1} \end{aligned} \qquad (i = 1, \cdots m) \tag{1}$$

where we may write

$$H = \sum_{j=1}^{m} \lambda_j p_j q_j + H_4 \cdots + H_{\bar{s}} \qquad (\bar{s} = s \text{ or } s+1),$$

in which H_k involves only the m products $\pi_i = p_i q_i$, of total degree $k/2$ in $\pi_i, \cdots, \pi_m$, while $L_{i,s+1}$, $M_{i,s+1}$ are convergent power series in $p_1, \cdots, q_m$ which commence with terms of degree not lower than $s+1$, the coefficients being of course analytic and periodic in t of period τ.

Suppose that we write

$$u^2 = \sum_{j=1}^{m} p_j q_j.$$

Evidently u can be appropriately regarded as measuring the distance of a point from equilibrium at any instant t; for, in terms of the original real variables $x_1, \cdots, x_{2m}$, the function u^2 is given by a real power series in $x_1, \cdots, x_{2m}$ which begins with a positive definite quadratic form in these variables,

$$u^2 = \sum_{j,k=1}^{2m} a_{jk}(t)\, x_j x_k + \cdots,$$

for all t, whence

$$k \sum_{j=1}^{2m} x_j^2 \leqq u^2 \leqq K \sum_{j=1}^{2m} x_j^2, \qquad K > k > 0,$$

in a certain neighborhood of the origin.

It is obvious then that we can choose N so large that

$$|L_{i,s+1}|,\ |M_{i,s+1}| \leqq N u^{s+1} \quad (i = 1, \cdots, m)$$

within a sufficiently small distance of the origin.

Multiplying the first of the partially normalized equations (1) by q_i, the second by p_i, and adding, we conclude

$$\left| \frac{d\pi_i}{dt} \right| \leqq 2 N u^{s+2} \qquad (i = 1, \cdots, m).$$

From the definition of u, the inequality

$$\left| u \frac{du}{dt} \right| \leqq m N u^{s+2}$$

then follows, so that

$$-mN \leqq \frac{1}{u^{s+1}} \frac{du}{dt} \leqq mN.$$

Integrating from t_0 to t, we deduce from this last inequality

$$\left| \frac{1}{u_0^s} - \frac{1}{u^s} \right| \leqq msN \, | t - t_0 | .$$

Now let us ask in how short an interval of time u can exceed $2u_0$. At the corresponding t we obtain

$$\left| \frac{1}{u_0^s} - \frac{1}{2^s u_0^s} \right| < msN \, | t - t_0 |,$$

whence obviously since $s \geqq 1$, this cannot happen for

$$(2) \qquad | t - t_0 | \leqq \frac{1}{2msN u_0^s} .$$

Hence the minimum time interval which must elapse before the initial distance u_0 can double in magnitude is of the s-th order in the reciprocal distance.

In this same interval of time we obtain

$$\left| \frac{d\pi_i}{dt} \right| \leqq 2^{s+3} N u_0^{s+2},$$

whence by integration

$$(3) \qquad | \pi_i - \pi_i^0 | \leqq 2^{s+3} N u_0^{s+2} \, | t - t_0 | \quad (i = 1, \cdots, m).$$

Also since H and its partial derivatives are polynomials, we have

$$\left| \frac{\partial H}{\partial \pi_i} - \frac{\partial H^0}{\partial \pi_i} \right| \leqq P \sum_{j=1}^{m} | \pi_i - \pi_i^0 | \leqq 2^{s+3} m N P u_0^{s+2} \, | t - t_0 | \qquad (i = 1, \cdots, m)$$

for π_i, π_i^0 small. On the other hand from the normalized differential equations we find in this interval

$$\left| \frac{dp_i}{dt} + \frac{\partial H}{\partial \pi_i} p_i \right|, \quad \left| \frac{dq_i}{dt} - \frac{\partial H}{\partial \pi_i} q_i \right| \leqq 2^{s+1} N u_0^{s+1} \qquad (i = 1, \cdots, m).$$

Combining these inequalities with the preceding set, there results

$$\left|\frac{dp_i}{dt}+\frac{\partial H^0}{\partial \pi_i}p_i\right|,\ \left|\frac{dq_i}{dt}-\frac{\partial H^0}{\partial \pi_i}q_i\right|$$
$$\leqq 2^{s+1}Nu_0^{s+1}+2^{s+4}mNPu_0^{s+3}\,|t-t_0|$$

for $i = 1, \cdots, m$. These are essentially the same as the following inequalities

$$\left|\frac{d}{dt}(p_i e^{\gamma_i t})\right|,\ \left|\frac{d}{dt}(q_i e^{-\gamma_i t})\right|$$
$$\leqq 2^{s+1}Nu_0^{s+1}+2^{s+4}mNPu_0^{s+3}\,|t-t_0|$$

where $\gamma_i = \partial H^0/\partial \pi_i$ are pure imaginary constants. The fact here made use of, namely that H and its partial derivatives as to π_i are pure imaginaries, can easily be verified: if $p_i, q_i, (i = 1, \cdots, m)$, be interchanged and H be changed to its conjugate in (1), these equations are not altered; but this means that the conjugate of H coincides with its negative, i. e. that H is a pure imaginary function. By integration the above inequalities give

$$(4)\qquad |p_i - \overset{0}{p_i}\, e^{-\gamma_i(t-t_0)}|,\ |q_i - \overset{0}{q_i}\, e^{\gamma_i(t-t_0)}|$$
$$\leqq 2^{s+1}Nu_0^{s+1}\,|t-t_0| + 2^{s+3}mNPu_0^{s+3}\,|t-t_0|^2$$

for $i = 1, \cdots, m$.

Now if we return to the convergent power series expressing $x_1, \cdots, x_{2m}$ in terms of $p_1, \cdots, q_m$ and if we replace $p_1, \cdots, q_m$ by

$$\overset{0}{p_1}\, e^{-\gamma_1(t-t_0)}, \cdots, \overset{0}{q_m}\, e^{\gamma_m(t-t_0)}$$

respectively, the series obtained agree with the formal series solutions up to terms of the $(s+1)$-th order in the $2m$ arbitrary constants $\overset{0}{p_1}, \cdots, \overset{0}{q_m}$. But the error committed in so doing is of the order of the differences appearing in (4). Hence if we express $x_1, \cdots, x_{2m}$ by means of the formal series solutions derived from the normal form, broken off after the terms of degree s in the $2m$ arbitrary constants $p_1^0, \cdots, q_m^0$, the error committed will not exceed in numerical value an expression

$$Au_0^{s+1} + Bu_0^{s+1} \, | t - t_0 | + Cu_0^{s+3} \, | t - t_0 |^2$$

during the interval (2), where A, B, C are suitably chosen positive constants.

On account of the fact that s is an arbitrary positive integer in the above inequalities, these can be given a still more simple form. Suppose that $| t - t_0 |$ is even more severely restricted than in (2), namely to be of the order at most $(s/3)+1$ in the reciprocal distance. Then the constitutents of the sum above are clearly of order exceeding $s/3$ in the distance u_0 itself. Consequently if all the terms of the formal series solutions of degree exceeding $s/3$ are discarded, the order of the error will exceed $s/3$. But $s/3$ is arbitrary, whence the conclusion:

If this formal series solution of the generalized Pfaffian equilibrium problem of stable type is written to terms of an arbitrary degree s in the initial values $p_1^0, \cdots, q_m^0$ of the arbitrary constants, the $2m$ trigonometric sums so obtained will have coefficients of at most the first order in u_0, and will represent the coördinates $x_1, \cdots, x_{2m}$ with an error of order u_0^{s+1} at most during a time interval of at least the reciprocal order. Here u_0 represents the distance to the origin for $t = t_0$ in $x_1, \cdots, x_{2m}$ space.

When written out explicitly these trigonometric sums for $x_1, \cdots, x_{2m}$ have the real form

$$A_0 + \sum_j (A_j \cos l_j t + B_j \sin l_j t)$$

where

$$l_i \sqrt{-1} = i_1 \frac{\partial H^0}{\partial \pi_1} + \cdots + i_m \frac{\partial H^0}{\partial \pi_m}, \qquad d = \sum_{j=1}^{m} | i_j | \leqq s,$$

in which $i_1, \cdots, i_m$ are integers, and where A_i, B_i are polynomials in $p_1^0, \cdots, q_m^0$ whose terms are of degree at least d and not more than s.

3. Instability of Pfaffian systems. In the case when some of the multipliers λ_i are real, the situation is entirely altered. If we assume that there are positive and negative multipliers $\pm\lambda_1, \cdots, \pm\lambda_k$, there will be a real k-dimensional, analytic manifold of curves of motion approaching the curve of periodic motion. Points on these curves near to the periodic motion leave its vicinity in a relatively short interval of time. More exactly, the distance will exceed

$$u_0 e^{\lambda(t-t_0)},$$

if u_0 denotes the initial distance from the motion at $t = t_0$ and λ is a positive constant less than the least positive multiplier. Similarly if t decreases the distance u_0 may increase in a like manner along a second real analytic manifold of curves.

Evidently such a situation is entirely different from that found in the stable case and is properly termed unstable.

We shall not enter upon a derivation of results of this sort, the first of which were obtained by Poincaré.*

4. Complete stability. The work of section 2 makes it clear that Pfaffian and Hamiltonian systems possess a species of complete formal or trigonometric stability, in case $\lambda_1, \cdots, \lambda_m, 2\pi\sqrt{-1}/\tau$ are pure imaginary quantities without linear relations of commensurability. Let us elaborate this concept of 'complete stability'.

Consider a differential system of even order $2m$,

$$dx_i/dt = X_i(x_1, \cdots, x_{2m}, t) \quad (i = 1, \cdots, 2m), \tag{5}$$

for which the origin is a point of generalized equilibrium. Suppose that for $t = t_0$ the point x_i^0 is at a distance ϵ from the origin. Let T be any fixed time interval, f any positive integer, and $P_s(x_1, \cdots, x_{2m}, t)$ any polynomial with terms of lowest degree s in the coördinates and with coefficients analytic and of period τ in t. If then it is possible always

* For some of the fundamental results see Picard, *Traité d'Analyse*, vol. 3, chap. 1.

to approximate to P_s for $|t-t_0| \leqq T$ with an error less numerically than

$$M\varepsilon^{f+s+1}$$

by a trigonometric sum of order N

$$\sum_{j=0}^{N}(A_j \cos l_j t + B_j \sin l_j t) \qquad (|l_i - l_j| > l > 0),$$

where M, N, l depend only on f and P_s, and where $l_0 = 0$, the equations (5) will be said to be 'completely stable'.

As a very simple example consider the pair of equations

$$dx_1/dt = kx_2, \quad dx_2/dt = -kx_1,$$

of which the general solution is

$$x_1 = A\cos kt + B\sin kt, \quad x_2 = -A\sin kt + B\cos kt,$$

so that the coördinates x_1, x_2 are represented by trigonometric sums of the first order. Any polynomial P_s of degree $s_1 \geqq s$ can also be exactly represented by a sum of order N not exceeding 2^{s_1+1}. Hence the conditions of the definition are satisfied.

The results of section 2 show that in the case of Hamiltonian or Pfaffian systems, there will be complete stability if there is ordinary stability as defined earlier.

This is obvious since the differences $l_i - l_j$ which enter in the trigonometric sums of section 2 are nearly given by a certain limited number of integral linear combinations of the $m+1$ quantities $\lambda_1/\sqrt{-1}, \cdots, \lambda_m/\sqrt{-1}, 2\pi/\tau$, and no such combination vanishes.

In case of complete stability, the solutions of the normalized equations of variation (chapter III, section 5) are limits of trigonometric sums of the specified type, and are trigonometric by the lemma on trigonometric sums of sections 5,6. Hence the multipliers are pure imaginaries.

It is important to establish that this definition of complete stability is independent of the particular coördinates $x_1, \cdots, x_{2m}$ selected. In fact, suppose that the given system is completely stable. Let us make the admissible change of variables

$$\bar{x}_i = \varphi_i(x_1, \cdots, x_{2m}, t) \qquad (i = 1, \cdots, 2m)$$

in which φ_i are analytic in $x_1, \cdots, x_{2m}, t$, vanish at the origin, and are such that the determinant $|\partial \varphi_i / \partial x_j|$ is not 0 there, while the coefficients in φ_i are analytic periodic functions of t of period τ. Then the two variables

$$\bar{\varepsilon} = [\bar{x}_1^2 + \cdots + \bar{x}_{2m}^2]^{1/2}\Big|_{t=t_0}$$

and

$$\varepsilon = [x_1^2 + \cdots + x_{2m}^2]^{1/2}\Big|_{t=t_0}$$

evidently serve equally well to measure the distance from the origin at $t = t_0$, since we have

$$0 < d < \bar{\varepsilon}/\varepsilon < D$$

in the neighborhood of the origin.

Now consider any polynomial $P_s(\bar{x}_1, \cdots, \bar{x}_{2m}, t)$ which can obviously be written

$$P^*(x_1, \cdots, x_{2m}, t) + Q(x_1, \cdots, x_{2m}, t)$$

where P^* is a polynomial in $x_1, \cdots, x_{2m}$ with terms of lowest degree s while Q is given by a power series commencing with terms of degree at least $f + s + 1$. It is clear that the polynomial P^* can be represented by a trigonometric sum of the specified type with an error of order $f + s + 1$ in ε, by the condition for complete stability, while it is clear furthermore that Q is of order $f + s + 1$. Hence it is plain that $P_s(\bar{x}_1, \cdots, \bar{x}_{2m})$ can be represented by the same trigonometric sum in the desired manner. This establishes the complete stability in the new variables.

The mere fact that the multipliers of the system of $2m$ equations of the first order fall into pure imaginary pairs by no means ensures complete stability in the above sense.

A sufficiently simple example is furnished by the pair of equations

$$dx/dt = ky + x(x^2 + y^2), \qquad dy/dt = -kx + y(x^2 + y^2),$$

in which k is positive, and the fundamental period is taken as 2π. The multipliers are then pure imaginaries, namely $\pm k\sqrt{-1}$. But if the first of these equations be multiplied by $2x$, the second by $2y$, and the two equations so modified be added, there results

$$du/dt = 2u^2 \qquad (u = x^2 + y^2),$$

whence, by a further integration

$$u = \frac{u_0}{1 - 2u_0(t - t_0)}.$$

But, if there were complete stability, it would be possible to find a fixed integer N so large that, for some constant K, the inequality

$$|u - S_N| \leqq Ku_0^3$$

held, in which S_N represents a trigonometric expression of order N of the specified type; this follows from the fact that u is a homogeneous polynomial of the second degree in x and y, while u_0 is the squared distance ϵ^2. This inequality may be written

$$\left| \frac{u - u_0}{u_0^2} - \frac{S_N - u_0}{u_0^2} \right| \leqq Ku_0.$$

Now let u_0 approach 0. It is obvious that

$$\lim_{u_0=0} \frac{u - u_0}{u_0^2} = 2(t - t_0).$$

We infer then that

$$\lim_{u_0=0} \frac{S_N - u_0}{u_0^2} = t - t_0.$$

But the expression on the left is a trigonometric sum of order at most N of the specified form, and approaches its limit uniformly. Consequently by the lemma on trigonometric sums considered in sections 5, 6, the limit of this sum is necessarily a sum of the same type. However it is impossible that $2(t - t_0)$ should be so represented. Hence in this case there is not complete stability.

The condition that the multipliers be pure imaginaries has been seen to be necessary for complete stability even if not sufficient. Henceforth we shall assume that, if the m pairs of pure imaginary multipliers be denoted by $\pm\lambda_1, \cdots, \pm\lambda_m$, there are no linear commensurability relations between $\lambda_1, \cdots, \lambda_m$ and $2\pi\sqrt{-1}/\tau$. Of course by so doing certain exceptional cases are excluded which require further study.

For complete stability an infinite number of conditions besides that of pure imaginary multipliers will be found to be requisite.

5. Normal form for completely stable systems. We have already seen that Pfaffian and Hamiltonian systems of equations possess the property of complete stability, in case the characteristic numbers are pure imaginaries. It becomes a very interesting question to determine the most general case in which there is complete stability and to find the characteristics of motion near generalized equilibrium in this case. This we shall do by establishing a suitable normal form for equations of completely stable type.

Since the multipliers $\lambda_1, \cdots, \lambda_m$ are of the stated type, we may transform the variables $x_1, \cdots, x_{2m}$ to $p_1, \cdots, q_m$ by a linear transformation so that the transformed system is

$$dp_i/dt = -\lambda_i p_i + P_i, \quad dq_i/dt = \lambda_i q_i + Q_i \quad (i = 1, \cdots, m)$$

with p_i, q_i conjugate, and P_i, Q_i beginning with terms of at least the second degree.

Now change the variables once more by writing

$$p_i = \bar{p}_i + \bar{\varphi}_{i2}, \quad q_i = \bar{q}_i + \bar{\psi}_{i2} \quad (i = 1, \cdots, m).$$

It is readily found that the equations preserve their form, with the new P_i, Q_i having homogeneous quadratic terms

$$P_{i2} + \sum_{j=1}^{m} \left(p_j \frac{\partial \varphi_{i2}}{\partial p_j} - q_j \frac{\partial \varphi_{i2}}{\partial q_j} \right) \lambda_j - \lambda_i \varphi_{i2} + \frac{\partial \varphi_{i2}}{\partial t},$$

$$Q_{i2} + \sum_{j=1}^{m} \left(p_j \frac{\partial \psi_{i2}}{\partial p_j} - q_j \frac{\partial \psi_{i2}}{\partial q_j} \right) \lambda_j + \lambda_i \psi_{i2} + \frac{\partial \psi_{i2}}{\partial t},$$

respectively. On inspecting these terms and making use of the incommensurability of the multipliers, it appears at once that these new expressions can be made to vanish in one and only one way. In fact let

$$P(t)\, p_1^{\alpha_1} \cdots p_m^{\alpha_m} q_1^{\beta_1} \cdots q_m^{\beta_m} \qquad (\alpha_1 + \cdots + \beta_m = 2$$

be such a term in P_i while the corresponding term in φ_{i2} has a coefficient $\varphi(t)$. By comparison there is obtained the differential equation for φ

$$P(t) + \left[\sum_{j=1}^{m} (\alpha_j - \beta_j) \lambda_j - \lambda_i \right] \varphi + \frac{d\varphi}{dt} = 0,$$

which can be satisfied by a periodic function φ of period τ unless the coefficient of φ is an integral multiple of $2\pi\sqrt{-1}/\tau$. This is not possible because of the hypothesis of incommensurability. Moreover the periodic solution is unique (cf. chapter III, section 9).

Thus all of the second degree terms in P_i, Q_i may be removed.

By a precisely similar method all of the third degree terms in P_i, Q_i may be removed by a further transformation

$$p_i = \bar{p}_i + \bar{\varphi}_{i3}, \qquad q_i = \bar{q}_i + \bar{\psi}_{i3} \quad (i = 1, \cdots, m)$$

except when the analogous coefficients

$$\sum_{j=1}^{m} (\alpha_j - \beta_j) \lambda_j - \lambda_i, \qquad \sum_{j=1}^{m} (\alpha_j - \beta_j) \lambda_j + \lambda_i$$

$$(\alpha_1 + \cdots + \beta_m = 3)$$

vanish. Such exceptional terms will have the form

$$P(t)\, p_i p_j q_j, \qquad Q(t)\, q_i p_j q_j \quad (j = 1, \cdots, m).$$

But even in these terms the functions φ, ψ may be so selected as to make the new coefficients, namely

$$P(t) + \frac{d\varphi}{dt}, \qquad Q(t) + \frac{d\psi}{dt},$$

reduce to constants (cf. chapter III, section 9). Hence we may normalize P_i, Q_i so that

$$\begin{aligned} P_i &= p_i\,(c_{i1}\, p_1 q_1 + \cdots + c_{im}\, p_m q_m) + \cdots, \\ Q_i &= q_i\,(d_{i1}\, p_1 q_1 + \cdots + d_{im}\, p_m q_m) + \cdots, \end{aligned}$$

where the complete terms P_{i3}, Q_{i3} of the third degree appear explicitly written in the right-hand members.

Our next step will be to show that in the event of complete stability we must have the further relations

$$q_i P_{i3} + p_i Q_{i3} = 0 \qquad (i = 1, \cdots, m),$$

i. e., $c_{ij} + d_{ij} = 0$, $(i, j = 1, \cdots, m)$. In order to establish this fact we employ the following lemma which is almost self-evident:

Lemma on trigonometric sums. If a sequence of trigonometric sums of the type

$$\sum_{j=0}^{N} (A_j \cos l_j t + B_j \sin l_j t) \qquad (|l_i - l_j| > l > 0),$$

with N, l fixed while A_i, B_i, l_i vary except that $l_0 = 0$, approaches a limit $\varphi(t)$ uniformly in some interval, then $\varphi(t)$ is itself a trigonometric sum of order at most N in this interval.

The proof of this simple lemma is deferred to the following section.

Consider the quadratic polynomials $p_i q_i$ in the coördinates. We find from the given equations

$$d(p_i q_i)/dt = q_i P_{i3} + p_i Q_{i3} + \cdots, \quad (i = 1, \cdots, m),$$

where the terms not explicitly indicated are at least of the fifth degree, and where the terms written explicitly are

$$p_i q_i [(c_{i1} + d_{i1}) p_1 q_1 + \cdots + (c_{im} + d_{im}) p_m q_m] \quad (i = 1, \cdots, m).$$

We desire to prove that these terms vanish identically.

Now the differential equations above lead at once to the inequalities

$$|d\pi_i/dt| \leqq K(\pi_1 + \cdots + \pi_m)^2 \qquad (\pi_i = p_i q_i)$$

for $i = 1, 2, \cdots, m$, where $\pi_1, \cdots, \pi_m$ are of course positive or 0. From these last inequalities it follows that we have

$$|du/dt| \leqq mKu^2 \qquad \left(u = \sum_{j=1}^{m} \pi_j\right),$$

and thence

$$|u - u_0| \leqq 4mKu_0^2 |t - t_0| \leqq 2mKTu_0^2$$

for any given interval of time $|t - t_0| \leqq T$, provided that u_0 is sufficiently small. This follows by the methods of section 2. Hence $u - u_0$ is of the second order in u_0 throughout this interval, while the inequalities for $d\pi_i/dt$ show that $\pi_i - \pi_i^0$ is also. Thus

$$q_i P_{i3} + p_i Q_{i3},$$

which is a quadratic polynomial in $\pi_1, \cdots, \pi_m$, differs from its value at $t = t_0$ by terms of the third order in u_0, and the differential equations above give

$$\left| \frac{d\pi_i}{dt} - (q_i^0 P_{i3}^0 + p_i^0 Q_{i3}^0) \right| \leqq L u_0^{5/2} \qquad (i = 1, \cdots, m).$$

By integration there results

$$|\pi_i - \pi_i^0 - (q_i^0 P_{i3}^0 + p_i^0 Q_{i3}^0)(t - t_0)| \leqq LTu_0^{5/2} \quad (i = 1, \cdots, m)$$

in the interval under consideration.

Suppose now that we write

$$p_i^0 = \alpha_i \varepsilon, \quad q_i^0 = \beta_i \varepsilon \quad (i = 1, \cdots, m),$$

$\alpha_1, \cdots, \beta_m$ being m arbitrary pairs of conjugate imaginaries, and suppose that the positive quantity ε approaches 0. The inequality last written in which u_0 is to be regarded as a constant multiple of ε^2, shows that we have

$$\lim_{\varepsilon=0} (\pi_i - \pi_i^0)/\varepsilon^4 = \alpha_i \beta_i [(c_{i1} + d_{i1}) \alpha_1 \beta_1 + \cdots (c_{im} + d_{im}) \alpha_m \beta_m](t - t_0),$$

where the limit is approached uniformly in the interval under consideration. On the other hand π_i can be approximated to by a trigonometric sum of the specified type to terms of order ε^5 in this interval, and consequently $(\pi_i - \pi_i^0)/\varepsilon^4$ can be approximated to terms of order ε. Thus the left-hand member is the uniform limit of a trigonometric sum having the properties specified in the lemma, and must therefore itself be trigonometric. This can only be true if the sums $c_{ij} + d_{ij}$ vanish for all values of i and j, as we desired to prove.

Thus we have to terms of the third order for $i = 1, \cdots, m$,

$$dp_i/dt = -p_i \left[\lambda_i - \sum_{j=1}^{m} c_{ij} p_j q_j\right] + \cdots,$$

$$dq_i/dt = \quad p_i \left[\lambda_i - \sum_{j=1}^{m} c_{ij} p_j q_j\right] + \cdots.$$

Evidently we have begun a process which enables us to remove terms of higher and higher degree in P_i, Q_i except for terms with factors p_i, q_i respectively, and coefficients which are polynomials in the m products $p_i q_i$, one being precisely the negative of the other.

Any completely stable system of equations (5) *may be reduced formally to the normal form*

(6) $$d\xi_i/dt = -M_i \xi_i, \quad d\eta_i/dt = M_i \eta_i \quad (i = 1, \cdots, m),$$

where $M_1, \cdots, M_m$ are pure imaginary power series in the m variables ξ_i, η_i, i. e.,

$$M_i = \lambda_i - \sum_{j=1}^{m} c_{ij}\,\xi_j\,\eta_j + \cdots \qquad (i = 1, \cdots, m),$$

and ξ_i, η_i are conjugate pairs of variables.

Conversely, if any set of equations have this normal form, the argument of section 2 is available to show that there is complete stability.

6. Proof of the lemma of section 5. Let us consider a sequence of trigonometric sums $\psi(t)$ of the type prescribed in the lemma to be proved. For such a sum we have

$$[D\,(D^2 + l_1^2) \cdots (D^2 + l_N^2)]\,\psi = 0$$

where D indicates ordinary differentiation with respect to t in the symbolic differential operator on the left. Direct integration $2N+1$ times gives

$$\psi + \left(\sum_{j=1}^{N} l_j^2\right) \int^t \int_0^t \psi(t)\,dt^2 + \cdots$$
$$+ \left(\prod_{j=1}^{N} l_j^2\right) \int_0^t \cdots \int_0^t \psi(t)\,dt^{2N} = P(t)$$

where $P(t)$ is a polynomial of degree at most $2N$.

Now all of the l_i exceed l in absolute value, for by hypothesis

$$|l_i - l_0| = |l_i| \geqq l \qquad (i = 1, \cdots, N).$$

It is clear then that, by suitable choice of a sub-sequence $\psi(t)$, the reciprocals $m_i = 1/l_i$, which are less numerically than $1/l$, will approach limits m_i^* with $|m_i^*| \leqq 1/l$. Any two of the quantities m_i^* will be distinct unless both are 0 of course. Now divide both members of the above integral equation by the product $l_1^2 \cdots l_N^2$, and pass to the limit. Since ψ approaches φ uniformly, we obtain at once

$$\left(\prod_{j=1}^{N} m_j^{*2}\right)\varphi + \cdots + \int_0^t \cdots \int_0^t \varphi(t)\,dt^{2N} = Q(t),$$

where $Q(t)$, being the uniformly approached limit of a sequence of polynomials $P(t)$ of degree not exceeding $2N$, is itself such a polynomial. This leads at once to the conclusion that φ satisfies the linear differential equation with constant coefficients

$$[D(m_1^{*2} D^2 + 1) \cdots (m_N^{*2} D^2 + 1)] \varphi = 0,$$

with general solution a trigonometric sum

$$C_0 + \sum_{j=1}^{N'} [C_j \cos (t/m_j^*) + D_j \sin (t/m_j^*)]$$

where the sum is only extended over those values of j for which m_j^* is not 0. Hence φ is of the stated type.

7. Reversibility and complete stability. It would be possible to show further how intimately the variational principle and the requirement of complete stability are interrelated.† Instead I prefer to follow another direction of thought in order to show that the requirement of complete stability is also very intimately connected with that of reversibility in time of the given differential system, provided that the ordinary definition of reversibility is suitably generalized.

We shall say that a system (5) with generalized equilibrium point at the origin is 'reversible' if when t is changed to $-t$ the system then obtained is equivalent to (5) under the formal group.

By this change of sign of t, the multipliers λ_i are changed to their negatives $-\lambda_i$. Hence it is obvious at the outset that in the reversible case of even order, these multipliers are grouped in pairs, each member of a pair being the negative of the other. We are primarily interested in the case when these multipliers are furthermore pure imaginaries and without linear commensurability relations. For this reason we shall assume that these conditions for first order stability are satisfied.

† See my paper, *Stability and the Equations of Dynamics*, loc. cit.

It is clear that this definition of reversibility is independent of the dependent variables employed. Hence if we have a completely stable system to begin with, we may take it in the normal form (6). The change of t to $-t$ gives a modified system,

$$d\bar{\xi}_i/dt = \bar{M}_i\,\bar{\xi}_i, \qquad d\bar{\eta}_i/dt = -\bar{M}_i\,\bar{\eta}_i \qquad (i = 1, \cdots, m)$$

where we introduce the dashes to avoid confusion. But it is possible to pass from one set of equations to the other by the aid of the transformation of the formal group

$$\xi_i = \bar{\eta}_i, \qquad \eta_i = \bar{\xi}_i \quad (i = 1, \cdots, m).$$

Therefore if there is stability of the first order, a necessary condition for complete stability is that (5) *is reversible in the*
5 *sense of the above definition.*

It remains to prove that this simple necessary condition of reversibility, together with the requirement that the multipliers are of the prescribed type, is also sufficient.

The same process of normalization used in section 5 leads us to the normal form of more general type

$$(7) \quad d\xi_i/dt = U_i\xi_i, \qquad d\eta_i/dt = V_i\eta_i \quad (i = 1, \cdots, m),$$

where U_i, V_i are functions of the m products $\xi_1\eta_1, \cdots, \xi_m\eta_m$ with initial terms λ_i, $-\lambda_i$ respectively. This may be obtained without the hypothesis of complete stability.

Now if we change t to $-t$ these normalized equations become

$$(8) \quad d\xi_i/dt = -U_i\xi_i, \quad d\eta_i/dt = -V_i\eta_i \quad (i = 1, \cdots, m).$$

These are to be equivalent, by hypothesis, to the original equations (7). It is to be observed that the equations (8) are the same in form as (7) save that the roles of ξ_i and η_i are interchanged, while $-U_i$, $-V_i$ take the place of V_i, U_i respectively.

But it is readily proved that the most general transformation preserving this normal form (7) is of the type

$$(9) \qquad \xi_i = \bar{\xi}_i \bar{f}_i, \quad \eta_i = \bar{\eta}_i \bar{g}_i \quad (i = 1, \cdots, m),$$

where $\bar{f}_i$ and $\bar{g}_i$ are arbitrary power series in the m products $\bar{\xi}_1 \bar{\eta}_1, \cdots, \bar{\xi}_m \bar{\eta}_m$ not lacking constant terms, and with coefficients independent of t.

The fact that these transformations do preserve the normal form is obvious upon direct substitution. In the first place, we note that the inverse relations are of the same type

$$\bar{\xi}_i = \xi_i h_i, \quad \bar{\eta}_i = \eta_i k_i \quad (i = 1, \cdots, m)$$

with

$$\bar{f}_i h_i = \bar{g}_i k_i = 1 \quad (i = 1, \cdots, m).$$

Hence we find

$$d\bar{\xi}_i/dt = \bar{U}_i \bar{\xi}_i$$

where

$$\bar{U}_i = \bar{f}_i \left[h_i U_i + \sum_{j=1}^{m} \frac{\partial h_i}{\partial u_j} (U_j + V_j) \xi_j \eta_j \right] \quad (u_i = \xi_i \eta_i),$$

together with like expressions for $d\bar{\eta}_i/dt$ and $\bar{V}_i$, for $i = 1, \cdots, m$.

In order to establish the fact that this group of transformations is the most general preserving the normal form, we shall proceed step by step.

Consider the terms of the first degree in any such series for $\bar{\xi}_i, \bar{\eta}_i$. These may be written

$$a\xi_i + b\eta_i, \qquad c\xi_i + d\eta_i$$

respectively, so that we must have, for instance,

$$\begin{aligned} \frac{d}{dt}(a\xi_i + b\eta_i) &= a\lambda_i \xi_i - b\lambda_i \eta_i + \xi_i \frac{da}{dt} + \eta_i \frac{db}{dt} + \cdots \\ &\equiv \lambda_i (a\xi_i + b\eta_i) + \cdots, \end{aligned}$$

if the normal form is to be preserved to terms of the first degree only. Hence we infer that b vanishes and a is a constant. Similarly c vanishes and d is a constant, conjugate to a.

Of course this means that the transformations from ξ_i, η_i to $\bar{\xi}_i, \bar{\eta}_i$ are of the specified form as far as the first degree terms are concerned.

Hence the most general transformation which preserves the normal form may be obtained by the composition of the special linear transformation of the group

$$\bar{\xi}_i = a\xi_i, \qquad \bar{\eta}_i = d\eta_i \qquad (i = 1. \cdots, m),$$

and a transformation of the form

$$\bar{\xi}_i = \xi_i + F_i, \qquad \bar{\eta}_i = \eta_i + G_i$$

in which F_i, G_i begins with terms of at least the second degree.

Denote the quadratic terms in F_i and G_i by F_{i2} and G_{i2} respectively. Thus we have to consider

$$\bar{\xi}_i = \xi_i + F_{i2} + \cdots, \qquad \bar{\eta}_i = \eta_i + G_{i2} + \cdots \qquad (i = 1, \cdots, m)$$

with inverse transformation

$$\xi_i = \bar{\xi}_i - \bar{F}_{i2} + \cdots, \qquad \eta_i = \bar{\eta}_i - \bar{G}_{i2} + \cdots \qquad (i = 1, \cdots, m),$$

in which $\bar{F}_{i2}, \bar{G}_{i2}$ are merely F_{i2}, G_{i2} respectively with ξ_i, η_i replaced by $\bar{\xi}_i, \bar{\eta}_i$ respectively. We are to determine what is the most general form of F_{i2}, G_{i2} which can preserve the normal form. Now we have

$$\begin{aligned} \frac{d\bar{\xi}_i}{dt} &= U_i \xi_i + \sum_{j=1}^{m} \lambda_j \left(\xi_j \frac{\partial F_{i2}}{\partial \xi_j} - \eta_j \frac{\partial F_{i2}}{\partial \eta_j} \right) + \frac{\partial F_{i2}}{\partial t} + \cdots \\ &\equiv \bar{U}_i \bar{\xi}_i \end{aligned}$$

for $i = 1, \cdots, m$, whence by comparison of second degree terms

$$-\lambda_i F_{i2} + \sum_{j=1}^{m} \lambda_j \left(\xi_j \frac{\partial F_{i2}}{\partial \xi_j} - \eta_j \frac{\partial F_{i2}}{\partial \eta_j} \right) + \frac{\partial F_{i2}}{\partial t} = 0 \quad (i = 1, \cdots, m).$$

The constituent terms which may occur in F_{i2} can be discussed by the methods of section 5, and this leads to the conclusion that F_{i2} must vanish. Likewise G_{i2} is found to vanish. Thus the transformation has the stated form to terms of the second degree inclusive, and it is necessary to consider next a transformation

$$\bar{\xi}_i = \xi_i + F_{i3} + \cdots, \qquad \bar{\eta}_i = \eta_i + G_{i3} + \cdots \qquad (i = 1, \cdots, m)$$

with inverse transformation

$$\xi_i = \bar{\xi}_i - \bar{F}_{i3} + \cdots, \qquad \eta_i = \bar{\eta}_i - \bar{G}_{i3} + \cdots \qquad (i = 1, \cdots, m).$$

Here we are led to m equations

$$-\lambda_i F_{i3} + \sum_{j=1}^{m} \lambda_j \left(\xi_j \frac{\partial F_{i3}}{\partial \xi_j} - \eta_j \frac{\partial F_{i3}}{\partial \eta_j} \right) + \frac{\partial F_{i3}}{\partial t} = \xi_i \Delta U_{i2}. \qquad (i = 1, \cdots, m),$$

where ΔU_{i2} denotes the difference between the second degree components of $\bar{U}_i$ and U_i, when $\xi_1 \eta_1, \cdots, \xi_m \eta_m$ replace $\bar{\xi}_1 \bar{\eta}_1, \cdots, \bar{\xi}_m \bar{\eta}_m$ in $\bar{U}_i$. Thus ΔU_{i2} is a linear function of these m products with constant coefficients. But the method of section 5 shows then that F_{i3} contains a factor ξ_i in every term, these terms being of the form

$$\xi_i \sum_{j=1}^{m} c_{ij} \xi_j \eta_j \qquad (i = 1, \cdots, m).$$

Of course G_{i3} has a corresponding similar form in which η_i appears as a factor.

It follows that the transformation has the stated form to terms of the third degree inclusive. But then the most general transformation can be expressed as one of the specified group followed by a further transformation

$$\bar{\xi}_i = \xi_i + F_{i4} + \cdots, \qquad \bar{\eta}_i = \eta_i + G_{i4} + \cdots \qquad (i = 1, \cdots, m),$$

and we can continue the above method of treatment to the terms of fourth degree and of all higher degrees. Thus we arrive at the conclusion desired that the most general type of formal transformation preserving the normal form is given by (9).

It remains to consider in what cases it is possible to pass from (7) to (8) by a transformation of the type (9), where now we shall introduce dashes so as to distinguish the two sets of variables $\xi_1, \cdots, \eta_m$ and $\bar{\xi}_1, \cdots, \bar{\eta}_m$. If we write $u_i = \xi_i \eta_i$, $\bar{u}_i = \bar{\xi}_i \bar{\eta}_i$, $W_i = U_i + V_i$, we obtain the two associated sets of equations in $u_1, \cdots, u_m$ and $\bar{u}_1, \cdots, \bar{u}_m$

$$(10) \qquad du_i/dt = W_i(u_1, \cdots, u_m)u_i \quad (i = 1, \cdots, m),$$

$$(11) \qquad d\bar{u}_i/dt = -W_i(\bar{u}_1, \cdots, \bar{u}_m)\bar{u}_i \quad (i = 1, \cdots, m),$$

while we have the relations

$$(12) \quad \bar{u}_i = u_i h_i(u_1, \cdots, u_m) k_i(u_1, \cdots, u_m) = u^i l_i(u_1, \cdots, u_m) \quad (i = 1, \cdots, m).$$

Furthermore the constant term in l_i is ϱ_i, a real positive constant, for reasons given above. It is very easy to show that it is impossible that (10) and (11) are related by (12) unless $W_i \equiv 0$.

To begin with, we recall that U_i and V_i have constant terms which are the negatives of one another. In consequence W_i starts off with terms of positive degree r in $u_1, \cdots, u_m$, the aggregate of which we designate by W_{ir}. If we perform the indicated change of variables, we obtain the identities

$$\frac{d\bar{u}_i}{dt} = -\varrho_i W_{ir}(\varrho_1 u_1, \cdots, \varrho_m u_m) u_i + \cdots$$
$$= \varrho_i W_{ir}(u_1, \cdots, u_m) u_i + \cdots,$$

in which only the terms of lowest degree $r+1$ are explicitly indicated. Hence we obtain by comparison

$$W_{ir}(u_1, \cdots, u_m) + W_{ir}(\varrho_1 u_1, \cdots, \varrho_m u_m) = 0.$$

But consider some term of W_{ir}, say

$$c_i u_1^{\alpha_1} \cdots u_m^{\alpha_m} \quad (\alpha_1 + \cdots + \alpha_m = r).$$

This identity yields

$$c_i(1 + \varrho_1^{\alpha_1} \cdots \varrho_m^{\alpha_m}) = 0$$

which is impossible for $c_i \neq 0$. Hence every term of W_{ir} must vanish, which is contrary to assumption. In consequence, we must have $W_i \equiv 0, (i = 1, \cdots, m)$. In other words the requirement of reversibility necessitates that the normal form (8) has the special property characteristic of the case of complete stability.

If there is stability of the first order, reversibility is a sufficient condition for complete stability in the generalized equilibrium problem.

The case of ordinary equilibrium is of course still simpler than that of generalized equilibrium, and the results are entirely analogous.

8. Other types of stability. We have already defined stability of the first order, and complete or trigonometric stability. It was proved in section 2 that, for the equations of dynamics (taken as of Hamiltonian or Pfaffian type), first order stability necessitated complete stability. Other types of stability also possess interest.

In the first place as of the greatest theoretic importance may be mentioned 'permanent stability', for which small displacements from equilibrium or periodic motion remain small for all time. This is the kind of stability of ordinary equilibrium when the potential energy is a minimum. The equations of dynamics are of the type for which this stability *may* obtain, although in general the problem of determining whether or not it does obtain is one of extraordinary difficulty, and constitutes the so-called 'problem of stability'. Thus far the problem has only been solved when a known convergent integral guarantees such actual stability of permanent type.

Another type of stability is that in which these displacements remain small for a very long interval of increasing and decreasing time. A sufficient condition for such 'semi-permanent stability' is the existence of a formal series integral starting off with a homogeneous polynomial of least degree constituting a definite form in the dependent variables. It seems likely that a slight extension of this sufficient condition will turn out to be necessary. Complete stability necessitates semi-permanent stability of course.

Finally a type of 'unilateral stability' in which the displacements remain small for $t > 0$, and in general tend to vanish as t increases indefinitely, has been considered by Liapounoff and others.* It is easy to demonstrate that if the m multipliers possess negative real parts, this kind of stability will obtain. Furthermore it is necessary for this kind of stability that none of these real parts are positive. In the case of the equations of dynamics, however, the real parts of the multipliers can not all be negative, since with every multiplier λ_i is associated its negative. Thus the only possibility of unilateral stability in dynamics is seen to arise when the multipliers are pure imaginaries. In this case the proof of unilateral stability would lead to the proof of permanent stability.

Thus for the problems of dynamics the important types of stability are complete or trigonometric stability, and the permanent stability mentioned above. We shall recur later (chapter VIII) to the important problem of stability concerned with the interrelation of these two types.

* See, for instance, Picard, *Traité d'Analyse*, vol. 3, chap. 8.