CHAPTER II

VARIATIONAL PRINCIPLES AND APPLICATIONS

1. **An algebraic variational principle.** On the formal side of dynamics it has proved to be a fact of fundamental importance that the differential equations can in general be obtained by demanding that the 'variation' of some definite integral vanishes.

To make clear the essential nature of the variational method, we may consider an analogous question concerning ordinary maxima and minima.

Let there be given n equations in n unknown quantities.

$$f_i(x_1, \cdots, x_n) = 0 \qquad (i = 1, \cdots, n),$$

in which the left hand-members are expressible as the partial derivatives of a single unknown real analytic function F,

$$f_i \equiv \partial F / \partial x_i \qquad (i = 1, \cdots, n)$$

The n equations are then of the special type which arises in the determination of the maxima and minima of F, and they may be combined in one symbolic equation $dF = 0$. Their significance is that for the values $x_1^0, \cdots, x_n^0$ under consideration, the function F is 'stationary'.

Now suppose that the variables x_i are changed to y_i in the n equations, where the relation between x_i and y_i is one-to-one and analytic. Since the phenomenon of a stationary value of F is clearly independent of the particular variables in terms of which F is expressed, the solutions of the original equations can be expressed in the characteristic differential form $dF = 0$, in the new as well as the old variables. This furnishes a means of obtaining an equivalent system of equations in the new variables, which is in general simpler than that of direct substitution in the original equations.

In cases when it is not possible to write the given equations in the special form, it is frequently possible to find combinations of these equations which may be so written.

Moreover any non-specialized set of n equations in $x_1, \cdots, x_n$ of the form first written is equivalent to $2n$ equations obtained from $dF = 0$ where

$$F = \sum_{j=1}^{n} f_j x_{n+j},$$

at least provided that the determinant $|\partial f_i/\partial x_j| \neq 0$. For we find that $x_{n+1}, \cdots, x_{2n}$ are 0, while $x_1, \cdots, x_n$ must satisfy the required equations.

From these circumstances it is easy to conjecture that the significance of the analogous variational principles of dynamics is largely formal.

2. Hamilton's principle. Let us formulate the concept of a 'stationary integral'. Suppose that the equations

$$x_i = x_i(t, \lambda) \qquad (i = 1, \cdots, m)$$

represent a family of functions depending on the parameter λ in such wise that for $\lambda = 0$ we have a given set of functions,

$$x_i(t, 0) = x_i^0(t) \qquad (i = 1, \cdots, m).$$

We shall assume that the functions $x_i(t, \lambda)$ are continuous with continuous first and second partial derivatives in t and λ, and also that these functions of t and λ vanish identically sufficiently near to the two ends of the interval (t_0, t_1) under consideration,

$$x_i(t, \lambda) = 0 \qquad (t_0 \leqq t \leqq t_0 + \varepsilon,\ t_1 - \varepsilon \leqq t \leqq t_1).$$

Under these conditions the integral

$$I = \int_{t_0}^{t_1} F(x_1, \cdots, x_m, x_1', \cdots, x_m')\, dt,$$

where F and its partial derivatives of the first two orders are taken continuous, is said to be 'stationary' for $x_i = x_i^0(t)$, if for every such family of functions we have

$$\delta I = \left.\frac{\partial I}{\partial \lambda}\right|_{\lambda=0} \delta\lambda = 0.$$

This amounts to the equation for $\lambda = 0$,

$$\int_{t_0}^{t_1} \sum_{j=1}^{m} \left(\frac{\partial F}{\partial x_j}\frac{\partial x_j}{\partial \lambda} + \frac{\partial F}{\partial x_j'}\frac{\partial x_j'}{\partial \lambda}\right) dt = 0.$$

Integrating by parts and noting that δx_i vanishes at the end points, we obtain the equivalent equations

$$\int_{t_0}^{t_1} \sum_{j=1}^{m} \left[\frac{\partial F}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial F}{\partial x_j'}\right)\right] \delta x_j \, dt = 0.$$

In particular we may take

$$x_i(t, \lambda) = x_i^0(t) + \lambda \delta x_i \qquad (i = 1, \cdots, m)$$

where the functions δx_i are arbitrary continuous functions of t with a continuous first and second derivative except that they are to vanish near to t_0 and t_1.

In this way the condition that the integral be stationary is found to be equivalent to the system of m differential equations of Euler in $x_1^0, \cdots, x_m^0$,

$$\frac{d}{dt}\left(\frac{\partial F}{\partial x_i'}\right) - \frac{\partial F}{\partial x_i} = 0 \qquad (i = 1, \cdots, m).$$

In fact the above integral can not vanish for all possible admissible functions $x_i(t, \lambda)$ unless this condition is satisfied.*

But the m equations just written are identical in form with the Lagrangian equations except that L is replaced by F. Hence we obtain the following important result:

* See, for example, O. Bolza, *Vorlesungen über Variationsrechnung*, chap. 1, for fuller statements and arguments.

The Lagrangian equations may be given the variational form known as Hamilton's principle,

$$\delta \int_{t_0}^{t_1} L\, dt = 0 \tag{1}$$

According to the principle which led us to introduce the concept of variation, we may affect any desired change of variables in the given Lagrangian equations by introducing the new variables in the function L. To this fact is due much of the convenience of the Lagrangian form.

3. **The principle of least action.** There is a second well-known variational form for the Lagrangian equations termed the 'principle of least action', and we proceed to clarify the relation of this principle to the one just formulated. We assume that $L = L_2 + L_1 + L_0$ is quadratic in the velocities, and recall that the Lagrangian equations admit the energy integral

$$W \equiv \sum_{j=1}^{m} \left(q_j' \frac{\partial L}{\partial q_j'} \right) - L = L_2 - L_0 = c.$$

It is on this fact that our considerations will be based.

Let us confine attention to the case where the energy constant c has a specified value, say $c = 0$. Hence we have $L_2 = L_0$ along the motion $q_i = q_i^0(t)$, $(i = 1, \cdots, m)$, considered.

Now define I^* as follows·

$$I^* = I - \int_{t_0}^{t_1} (\sqrt{L_2} - \sqrt{L_0})^2\, dt = \int_{t_0}^{t_1} (2\sqrt{L_0 L_2} + L_1)\, dt.$$

This yields

$$\delta I^* = \delta I - 2 \int_{t_0}^{t_1} (\sqrt{L_2} - \sqrt{L_0})\, (\delta \sqrt{L_2} - \delta \sqrt{L_0})\, dt.$$

Accordingly, if the $q_i^0(t)$ satisfy the assumed energy condition we shall have

$$\delta I^* = \delta I$$

for *all* variations of the q_i. Hence if the q_i^0 in addition satisfy the Lagrangian equations, so that $\delta I = 0$, we shall have $\delta I^* = 0$ also.

The integrand of I^* is positively homogeneous of dimensions unity in the derivatives q_i'. Consequently the numerical value of this integral I^* is independent of the parameter t used along the path of integration, and the value of the integral depends only on the path in $q_1, \cdots, q_m$ space;† for variations of the admitted type the end points of the path are fixed. Thus the integral of energy can be regarded as merely determining the parameter t, since if we write

$$\bar{t} = \int^t \sqrt{L_2}/\sqrt{L_0}\, dt,$$

the integral relation is satisfied in the new parameter $\bar{t}$.

Consequently, if we have $\delta I^* = 0$ for $q_i = q_i^0(t)$ and if the new parameter t is chosen in this manner, we have $\delta I = 0$ also for $q_i = q_i^0(t)$.

An alternative variational form for the equations of motion of such a Lagrangian system is $\delta I^* = 0$, *or more explicitly,*

$$(2) \qquad \delta \int_{t_0}^{t_1} (2\sqrt{L_0 L_2} + L_1)\, dt = 0$$

provided that L_0 *is so chosen that the energy constant vanishes, and the parameter* t *is determined as specified.*

The equation $\delta I^* = 0$ constitutes the 'principle of least action' for this problem, and is usually given for the case where the linear term L_1 in the velocities is not present.

By means of this principle not only the variables q_i but also the variable t may be transformed with facility. Indeed, it is obvious that the condition $\delta I^* = 0$ is invariant in form under a transformation of the dependent variables q_i to new variables $\bar{q}_i$. For along the transformed curve the same variational condition will be satisfied, except that L is replaced by its expression in terms of the new variables, while t has the same meaning as before. Consequently in order to transform these variables, it is sufficient to effect the transformation of L directly. The corresponding trans-

† See O. Bolza, *Vorlesungen über Variationsrechnung*, chap. 5.

formed equations are then obtained by the use of the new expression for L.

The allowable type of transformation of the independent variable t is the following:

$$dt = \mu(q_1, \cdots, q_m)\, d\bar{t}.$$

In other words, the differential element of time is divided by a factor μ depending upon the coördinates. We may determine the nature of the modification which the Langrangian equations undergo as a result of this transformation as follows: We note that the integral I^* may be written equally well

$$I^* = \int_{\bar{t}_0}^{\bar{t}_1} (2\sqrt{\mu L_0 \cdot \mu L_2} + \mu L_1)\, d\bar{t}.$$

This modified integral is of the same form as before if we set

$$\bar{L} = \mu L.$$

Furthermore δI^* vanishes along the curve whether t or $\bar{t}$ be regarded as parameter. By this transformation of t, then, the equations of Lagrange and the given integral condition go over into other equations of the same type with the principal function L multiplied by μ.

The differential form $L\,dt$ is invariant under transformations of either type. We conclude therefore the following fact:

By a transformation

$$q_i = f_i(\bar{q}_1, \cdots, \bar{q}_m) \quad (i = 1, \cdots, m), \qquad dt = \mu(\bar{q}_1, \cdots, \bar{q}_m)\, d\bar{t},$$

the Lagrangian equations with energy constant 0 *go over into a like set of equations with energy constant* 0 *in which* $\bar{L}$ *is obtained from the formula*

$$L\,dt = \bar{L}\,d\bar{t}.$$

In the reversible case we have $L_1 = 0$, and thus

$$I^* = \int_{t_0}^{t_1} 2\sqrt{L_0 L_2}\, dt = 2\int_{t_0}^{t_1} ds,$$

where $ds^2 = L_0 L_2 (dt)^2$ is the squared element of arc on a surface with coördinates $q_1, \cdots, q_m$.

Thus in the reversible case with fixed energy constant the curves of motion may be interpreted as geodesics on the m-dimensional surface with squared element of arc

$$ds^2 = L_0 L_2 (dt)^2.$$

This result indicates the degree of generality which attaches to the geodesic problem on an m-dimensional surface.

4. Normal form (two degrees of freedom). The transformations deduced above admit of particularly elegant application of the case of two degrees of freedom.* In this case the differential element

$$L_2\, dt^2 = \frac{1}{2}(a_{11}\, dq_1^2 + 2\, a_{12}\, dq_1\, dq_2 + a_{22}\, dq_2^2)$$

may be regarded as the squared element of arc length of a certain two-dimensional surface. By choosing $\overline{q}_1$ and $\overline{q}_2$ to be the coördinates of an isothermal net on the surface, the squared element of arc is given the form

$$\frac{1}{2}\lambda(d\overline{q}_1^2 + d\overline{q}_2^2).$$

Consequently if we choose the function μ as $1/\lambda$, and make the transformation of t above, λ reduces to 1.

For a given Lagrangian system with two degrees of freedom and given energy constant 0, *there exist variables of the above type for which the principal function L has the form*

$$L = \frac{1}{2}(q_1'^2 + q_2'^2) + \alpha q_1' + \beta q_2' + \gamma.$$

The equations and condition then take the normal form

$$q_1'' + \lambda q_2' = \partial\gamma/\partial q_1, \qquad q_2'' - \lambda q_1' = \partial\gamma/\partial q_2$$
$$(\lambda = \partial\alpha/\partial q_2 - \partial\beta/\partial q_1),$$
$$\frac{1}{2}(q_1'^2 + q_2'^2) = \gamma.$$

* See my paper *Dynamical Systems with Two Degrees of Freedom*, Trans. Amer. Math. Soc., vol. 18 (1917), sections 2–5.

Now if we regard q_1, q_2 as the rectangular coördinates of a particle of unit mass in the plane, it is seen that the above equations express the fact that the particle moves subject to a field of force derived from a potential energy $-\gamma$ and a force of magnitude λv perpendicular to the direction of motion, where v denotes velocity.

Any such Lagrangian system with two degrees of freedom can be regarded as that of a mass particle in the q_1, q_2*-plane, subject to a conservative field of force derived from a potential energy* $-\gamma$, *and a non-energic force* λv (*v, velocity*) *acting in a direction perpendicular to the direction of motion.*

5. Ignorable coördinates. The search for integrals is a task of fundamental importance in connection with differential systems. The question as to whether integrals of a particular type exist or not can usually be answered by formal methods. Their determination has been considered in many cases. In order to refer somewhat to this phase of dynamics, we consider briefly integrals of Lagrangian systems which are either linear or quadratic in the velocities. The variables $q_1, \cdots, q_m$ are confined to the small neighborhood of a point $q_1^0, \cdots, q_m^0$ while $q_1', \cdots, q_m'$ are arbitrary for the integrals treated.

We shall assume that L is quadratic in the velocities with the homogeneous quadratic component L_2 a positive definite form.

There is one very simple case in which a particular integral of the Lagrangian equations linear in the velocities can be found immediately, namely the case in which one of the coördinates, as q_1, does not appear explicitly in the principal function L. In this case, the corresponding differential equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_1'}\right) = 0,$$

so that

$$\partial L/\partial q_1' = c$$

is an integral linear in the velocities. The coördinate q_1 is then said to be an 'ignorable coördinate'.

It may be proved by the method of variation that the $m-1$ equations remaining, which give a system of $m-1$ equations of the second order in $q_2, \cdots, q_m$ after the above integral has been used to eliminate q_1', can be expressed in Lagrangian form. Let us denote by $\bar{L}$ the function of $q_2, \cdots, q_m, q_2', \cdots, q_m'$ obtained from L by this elimination. If $q_1^0, \cdots, q_m^0$ satisfy the given Lagrangian equations, we find for an arbitrary variation of $q_2, \cdots, q_m$

$$\delta \int_{t_0}^{t_1} \bar{L}\, dt = \sum_{j=1}^{m} \frac{\partial L}{\partial q_j'} \delta q_j \Big|_{t_0}^{t_1}$$

after an integration by parts; here q_1' is determined by the integral relation, although q_1 is not determined up to an additive constant. If the $\delta q_2, \cdots, \delta q_m$ vanish near the end points, this reduces to

$$\delta \int_{t_0}^{t_1} \bar{L}\, dt = c\, \delta q_1 \Big|_{t_0}^{t_1} \quad \text{or} \quad \delta \int_{t_0}^{t_1} (\bar{L} - c q_1')\, dt = 0.$$

If q_1 is an ignorable coördinate, the Lagrangian equations can be replaced by a set of Lagrangian equations in $q_2, \cdots, q_m$ only, with modified principal function

$$L - \frac{\partial L}{\partial q_1'} q_1',$$

in which the known integral is used to eliminate q_1'.

We sketch the above reduction of the number of degrees of freedom by use of such an integral because it is typical of the kind of reduction aimed at in many dynamical problems, namely a reduction maintaining the general form of the equations.

6. The method of multipliers. Let us ask next the following question: Under what conditions is it possible to find m 'multipliers' M_i, depending upon the coördinates and the velocities, such that when the Lagrangian equations are multiplied by $M_1, \cdots, M_m$ respectively and added, the left-hand member of the resulting equation is the exact derivative of a function V linear in the velocities? If a set of such multipliers exist, we have

$$\sum_{j=1}^{m} M_j \left[\frac{d}{dt}\left(\frac{\partial L}{\partial q_j'}\right) - \frac{\partial L}{\partial q_j} \right] \equiv \frac{dV}{dt}.$$

Evidently this will lead to a generalization of the notion of ignorable coördinates, in which special case we have $M_i = 1$ for some i while $M_j = 0$ for $i \neq j$.

On comparing coefficients of q_i'' we derive first

$$\sum_{j=1}^{m} M_j \frac{\partial^2 L}{\partial q_i' \partial q_j'} = \frac{\partial V}{\partial q_i'} \quad (i = 1, \cdots, m).$$

Here, because of the assumption on L, the coefficients of M_j are functions of the coördinates only. The right-hand member is also a function of the coördinates only, since V is linear in the q_i'. Hence the functions M_i must involve only the coördinates, and partial integration with respect to q_i' yields

$$V = \sum_{j=1}^{m} M_j \frac{\partial L}{\partial q_j'} + S(q_1, \cdots, q_m).$$

For a given V only one such a set of functions M_i, S exist, since the coefficients $\partial L / \partial q_j'$ of M_j are linearly independent expressions in the velocities, q_i'. Furthermore, this type of relation will persist if the variables are changed, since an integral linear in the velocities remains linear under any change of variable. Making then, a change from q_i to $\bar{q}_i$, we find

$$V = \sum_{j,k=1}^{m} M_j \frac{\partial L}{\partial \bar{q}_k'} \frac{\partial \bar{q}_k}{\partial q_j} + S.$$

Thus the new coefficients are given by

$$\bar{M}_i = \sum_{j=1}^{m} M_j \frac{\partial \bar{q}_i}{\partial q_j}.$$

From the known theory of linear partial differential equations of the first order, we can determine m functionally independent functions $\bar{q}_i$ such that we have the relations:

$$\bar{M}_1 = 1, \qquad \bar{M}_2 = \cdots = \bar{M}_m = 0.$$

On making this change, we obtain

$$V = \partial L / \partial \bar{q}_1' + S.$$

Differentiating with respect to t, and using the first Lagrangian equation, we find the identity

$$\frac{\partial L}{\partial q_1} + \sum_{j=1}^{m} \frac{\partial S}{\partial q_j} q_j' \equiv 0.$$

Hence $\partial L / \partial q_1$ is linear in the velocities. Consequently the quadratic terms in L must have the form

$$L_2 = \sum_{j,k=1}^{m} a_{jk}(q_2, \cdots, q_m) q_j' q_k'.$$

Now let us write

$$L_1 = \sum_{j=1}^{m} b_j(q_1, \cdots, q_m) q_j', \qquad L_0 = e(q_1, \cdots, q_m).$$

Then the above identity simplifies to

$$\sum_{j=1}^{m} \frac{\partial b_j}{\partial q_1} q_j' + \frac{\partial e}{\partial q_1} + \sum_{j=1}^{m} \frac{\partial S}{\partial q_j} q_j' \equiv 0.$$

We infer at once that e is independent of q_1, and that if we write $S^* = \int S\, dq_1$, then L_1 is given by

$$L_1 = -\sum_{j=1}^{m} \frac{\partial S^*}{\partial q_j} q_j' + \sum_{j=2}^{m} b_j^*(q_2, \cdots, q_m) q_j',$$

i. e., by an exact differential augmented by a linear expression in $q_2', \cdots, q_m'$ with coefficients depending only upon $q_2, \cdots, q_m$.

Since the L function may be modified by an exact derivative without affecting the variation and the Lagrangian equations, we may omit the first term in L_1. Hence L may be written so as not to involve the coördinate q_1 directly.

The most general case, in which multipliers $M_i(q_1, \cdots, q_m)$ of the various Lagrangian equations exist, by the aid of which the left-hand members of these equations may be combined to form an exact derivative of a function V linear in the velocities $q_1', \cdots, q_m'$, reduces by change of variable to the case of an ignorable coördinate q_1 in which all of the multipliers but one are zero and that one is unity.

The existence of such linear integrals can be determined by purely geometric methods. We observe that in the derivation of the result above, only transformations involving $q_1, \cdots, q_m$ were made so that t was unchanged. Hence the quadratic differential form $ds^2 = L_2\, dt^2$ is an invariant, which in the final variables has coefficients only involving $q_2, \cdots, q_m$. But of course this analytic property merely means that the surface with differential element belonging to this form admits of one-parameter continuous group of transformations into itself,

$$\bar{q}_1 = q_1 + c,\ \bar{q}_2 = q_2,\ \cdots,\ \bar{q}_m = q_m.$$

A necessary condition for the existence of such a generalized ignorable coördinate is that the surface $ds^2 = L_2\, dt^2$ admits of a one-parameter continuous group of transformations into itself.

We shall not attempt to develop such necessary conditions further.

7. The general integral linear in the velocities. So far as our reasoning above is concerned, we cannot as yet infer that all integrals linear in the velocities can be obtained by the method of generalized ignorable coördinates. However, this may be demonstrated to be the case as follows.

Since L_2 is by assumption a positive definite form, we may write the integral in the form used in the preceding section,

$$V = \sum_{j=1}^{m} M_j \frac{\partial L}{\partial q_j'} + S$$

where M_i and S are functions of the coördinates only. Employing exactly the method of that section, it appears that by a suitable change of variables we can take $M_1 = 1$, $M_2 = \cdots = M_m = 0$, and then by differentiation as to t it appears just as there, that L is essentially independent of q_1, so that q_1 is ignorable.

The method of multipliers specified yields all integrals of the Lagrangian equations which are linear in the velocities.

8. Conditional integrals linear in the velocities. In the preceding section we have considered integrals linear in the velocities which hold for *all* values of the energy constant. A more difficult problem is that of obtaining the conditional integral, holding for a *specified* particular value of the energy constant c, say for $c = 0$. We proceed to treat this problem for the case of two degrees of freedom. Here, by the use of the normalizing variables obtained earlier, we may write the equations of motion and the energy integral in the form:

$$x'' + \lambda y' = \gamma_x, \quad y'' - \lambda x' = \gamma_y; \qquad x'^2 + y'^2 = \gamma,$$

where γ_x, for instance, denotes $\partial \gamma / \partial x$.

Moreover, since any change of variables leaves the linear nature of the integral unaltered, the integral may be written

$$V = l x' + m y' + n = k,$$

where it is understood that this relation is required to hold only when the energy constant vanishes.

If the linear integral be differentiated as to the time, the equation which results must be an identity in virtue of the differential equations of motion written above and the energy relation. The differential equations may be employed to eliminate x'', y''. When this has been done, an equation quadratic in x', y' is obtained, which must be an identity in virtue of the integral relation alone. The quadratic terms are

$$l_x x'^2 + (l_y + m_x)\, x' y' + n_y y'^2.$$

In order that this sum shall combine with those of lower degrees in x', y' by use of the integral relation, it must be of the form $\varrho(x'^2 + y'^2)$. This implies

$$l_x = m_y, \qquad l_y = -m_x$$

i. e., that

$$l = u_y, \qquad m = u_x$$

where u is a harmonic function.

The integral can now be written:

$$u_y x' + u_x y' + n = k.$$

According to the principles outlined above in section 4, a further arbitrary conformal transformation of the x, y-plane, joined with the appropriate change in t, will leave the differential equation and integral relation in the normal form. In order to simplify further the linear integral, we shall choose the transformation to $\bar{x}$, $\bar{y}$ defined by

$$\bar{x} + i\bar{y} = \int \frac{dx + i\,dy}{u_y + i\,u_x} \qquad (i = \sqrt{-1}).$$

This is evidently conformal in type. The inverse transformation,

$$x + iy = f(\bar{x} + i\bar{y}),$$

is also conformal, and we have

$$|f'(\bar{x} + i\bar{y})|^2 = \left| \frac{dx + i\,dy}{d\bar{x} + i\,d\bar{y}} \right|^2 = u_y^2 + u_x^2.$$

Now let the transformed value of t be defined by

$$dt = (u_y^2 + u_x^2)\, d\bar{t}$$

From this last equation we find at once

$$\bar{x}' + i\bar{y}' = (u_y - iu_x)(x' + iy')$$

where $\bar{x}' = d\bar{x}/d\bar{t}$, $\bar{y}' = d\bar{y}/d\bar{t}$. Thus we have in particular

$$\bar{x}' = u_y x' + u_x y'.$$

Consequently when such a further transformation has been made, the above integral is simplified to

$$x' + n = k.$$

Now let this integral be differentiated as to t and let x'' be eliminated by means of the first Lagrangian equation. There results

$$n_x x' + (n_y - \lambda) y' + \gamma_x = 0,$$

which must vanish identically in virtue of the integral relation. Therefore, we conclude that the left-hand member vanishes identically in x', y'. But this will happen only if λ and γ are functions of y only. In this case the equation can be made to vanish identically by a proper choice of n, namely $\int \lambda\, dy$.

If such a dynamical system with two degrees of freedom with energy constant 0 *admits of a conditional integral linear in the velocities, then by means of a suitable transformation of the coördinates and the time, the equations can be taken in normal form with*

$$L = \frac{1}{2}(x'^2 + y'^2) + n(y)x' + \gamma(y),$$

so that the system contains the ignorable coördinate x. In this integrable case the curves of motion are given by

$$x = \int \frac{(c_1 - n)\, dy}{\sqrt{2\gamma - (c_1 - n)^2}} + c_2,$$

$$t = \int \frac{dy}{\sqrt{2\gamma - (c_1 - n)^2}} + c_3.$$

9. Integrals quadratic in the velocities. The energy integral is a known integral which is quadratic in the velocities. Furthermore it is well known that dynamical systems of the so-called Liouville type with L of the form

$$L = \frac{1}{2} U \sum_{j=1}^{m} v_j(q_j)\, q_j'^2 - (W/U),$$

$$U = \sum_{j=1}^{m} u_j(q_j), \qquad W = \sum_{j=1}^{m} w_j(q_j)$$

admit of m integrals quadratic in the velocities, in particular

$$\frac{1}{2} U^2 v_i\, q_i'^2 - c u_i + w_i = c_i \qquad (i = 1, \cdots, m),$$

and can be completely integrated.

We propose here only to discuss a special converse problem: to determine the conditions under which a Lagrangian system with two degrees of freedom and of reversible type, with energy constant 0, admits of a conditional integral

$$\frac{1}{2}(a x'^2 + 2 b x' y' + c y'^2) + d x' + e y' + f = k$$

where $a, \cdots, f$ are functions of x and y, and where a, b, c are not all identically zero.

If such an integral exists, any transformation of x, y, t of the type discussed in section 3 leaves the form of the integral unaltered. Hence we may transform the equations to the normal form for which

$$L = \frac{1}{2}(x'^2 + y'^2) + \gamma.$$

Differentiating the above assumed integral relation, and making use of the Lagrangian equations to eliminate x'', y'', we obtain a polynomial of the third degree in x', y' at most, which must vanish identically in virtue of the above integral relation. Now the third degree terms are

$$\frac{1}{2} a_x x'^3 + \left(b_x + \frac{1}{2} a_y\right) x'^2 y' + \left(b_y + \frac{1}{2} c_x\right) x' y'^2 + \frac{1}{2} c_y y'^3,$$

and these must combine with those of lower degree by virtue of the integral relation. This can only happen if this polynomial is divisible by $x'^2 + y'^2$, i. e., if

$$a_x - c_x = 2 b_y, \qquad a_y - c_y = -2 b_x.$$

These are the Cauchy-Riemann differential equations for the conjugate harmonic functions $a - c$, $2b$, and we may write

$$a - c = 2 u_y, \qquad b = u_x,$$

where u is a harmonic function.

Our conclusion is that the hypothetical integral has quadratic terms

$$\frac{1}{2} u_y x'^2 + u_x x' y' - \frac{1}{2} u_y y'^2 + \varrho (x'^2 + y'^2).$$

Taking account of the energy relation we may replace the last term by $2 \varrho \gamma$. The remaining quadratic terms may be written

$$\frac{1}{2} \Re \, [(u_y - i u_x). \, (x' + i y')^2]$$

where $\Re$ stands for 'the real part of'.

Now write

$$f'^2 = 1 / (u_y + i u_x)$$

so that f is an analytic function of $x + iy$. Make the change of variables

$$\bar{x} + i \bar{y} = f(x + iy), \qquad d\bar{t} = |f'|^2 \, dt,$$

which leaves the normal form of the equations unaltered. We find that the above quadratic terms, which may be written

$$\frac{1}{2} \Re \left[\frac{f'^2 (dx + i\,dy)^2}{|f'|^4 \, dt^2} \right]$$

become

$$\frac{1}{2} (\bar{x}'^2 - \bar{y}'^2)$$

in the new variables. Hence, dropping the bars, the integral relation takes the simplified form

$$\frac{1}{2}(x'^2-y'^2)+dx'+ey'+f = k.$$

Again if this be differentiated with respect to t as before, there is obtained

$$d_x x'^2+(d_y+e_x)x'y'+e_y y'^2$$
$$+(f_x+\gamma_x)x'+(f_y-\gamma_y)y'+d\gamma_x+e\gamma_y = 0.$$

The linear terms must vanish so that we find

$$\gamma = \varphi(x)+\psi(y), \qquad f = -\varphi(x)+\psi(y).$$

But for this value of γ the differential equations are of immediately integrable type:

If a reversible Lagrangian system with two degrees of freedom and with the energy constant 0 *admits of a conditional integral quadratic in the velocities and distinct from the energy integral, then, by a transformation of variables, the equations and integral take the form*

$$x'' = \varphi'(x), \quad y'' = \psi'(y), \quad \frac{1}{2}(x'^2+y'^2) = \varphi(x)+\psi(y).$$

A special quadratic integral is then

$$\frac{1}{2}(x'^2-y'^2) = \varphi(x)-\psi(y)+k$$

and the equations are integrable with

$$t = \frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\varphi+k}} = \frac{1}{\sqrt{2}}\int\frac{dy}{\sqrt{\psi-k}}.$$

The Liouville type of equations is essentially an equivalent case.

10. The Hamiltonian equations. Next we proceed to formulate another important type of variational principle, which leads to the so-called Hamiltonian or canonical form of the equations of dynamics.

Let us write

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{m} p_j q_j' - \sum_{j=1}^{m} p_j r_j + L(q_1, \cdots, q_m, r_1, \cdots, r_m) \right] dt = 0,$$

in which the r_i are the functions of $p_1, \cdots, p_m, q_1, \cdots, q_m$ properly defined by the m equations

$$p_i = \partial L / \partial r_i \qquad (i = 1, \cdots, m),$$

and where $p_1, \cdots, p_m, q_1, \cdots, q_m$ are to be varied independently. The first m equations, obtained from the variation of $p_1, \cdots, p_m$, are of course

$$\frac{d}{dt}\left(\frac{\partial F}{\partial p_i'}\right) - \frac{\partial F}{\partial p_i} \equiv -q_i' + r_i + \sum_{j=1}^{m} \left(p_j \frac{\partial r_j}{\partial p_i} - \frac{\partial L}{\partial r_j} \frac{\partial r_j}{\partial p_i}\right)$$
$$\equiv -q_i' + r_i = 0,$$

where F stands for the integrand. The second set of m equations can be likewise obtained and may be written

$$p_i' + \partial H / \partial q_i = 0,$$

if we introduce the abbreviation H for

$$\sum_{j=1}^{m} p_j r_j - L.$$

It is important to observe that the $2m$ differential equations so obtained are each only of the first order, with the general solution containing only $2m$ arbitrary constants.

The first set of equations show that the functions p_i^0, q_i^0 which make the integral stationary are such that $r_i^0 = q_i^{0\prime}$. Now let r_i be fixed as q_i', so that the integral reduces to the Lagrangian integral

$$\int_{t_0}^{t_1} L(q_1, \cdots, q_m, q_1', \cdots. q_m')\, dt.$$

The variation of $q_1, \cdots, q_m$ is still arbitrary, but the variation of $p_1, \cdots, p_m$ is determined. Furthermore if the variations of $q_1, \cdots, q_m$ vanish near to t_0 and t_1, so will the variations of $p_1, \cdots, p_m$. Hence we have

$$\delta \int_{t_0}^{t_1} L\,dt = 0,$$

along $q_i = q_i^0(t)$, and we conclude that q_i^0 satisfy the associated Lagrangian equations, with $r_i^0 = q_i^{0\prime}$, thus determining the corresponding p_i^0.

Thus each solution of the proposed variational problem leads to a solution of the associated Lagrangian equations. The converse is also true, since the choice of p_i, q_i at any time t is arbitrary and leads to an arbitrary set of values of q_i, q_i'.

If the principal function for a Lagrangian system is $L(q_1, \cdots, q_m, q_1', \cdots, q_m')$ *and we form the function of* $p_1, \cdots, p_m, q_1, \cdots, q_m$ *defined by*

$$H = -L + \sum_{j=1}^{m} p_j q_j', \tag{3}$$

where the variables q_i' *are to be eliminated by means of the equations*

$$p_i = \partial L / \partial q_i' \qquad (i = 1, \cdots, m), \tag{4}$$

the original equations $\delta \int L\,dt = 0$ *may be replaced by the equivalent system in* p_i, q_i

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{m} p_j q_j' - H \right] dt = 0, \tag{5}$$

or, more explicitly,

$$dp_i/dt = -\partial H/\partial q_i, \quad dq_i/dt = \partial H/\partial p_i \quad (i = 1, \cdots, m). \tag{6}$$

The equations (6) are the 'Hamiltonian' equations, and the variables p_i are called the 'generalized momenta'. A pair of variables p_i, q_i are called 'conjugate'. Furthermore it is to be noted that the Hamiltonian 'principal function' H is the total energy expressed in terms of the generalized coördinates and momenta. The energy integral $H =$ const. follows at once from the canonical equations.

It may be observed here that the above variational principle leads to the same canonical equations even if L and H involve the time t.

Conversely, any Hamiltonian system (5), (6), *H being arbitrary, can be reduced to a Lagrangian system.*

To prove this statement we need only define L by the equation

$$L(q_1, \cdots, q_m, r_1, \cdots, r_m) = -H + \sum_{j=1}^{m} p_j r_j$$

where $p_1, \cdots, p_m$ are functions of q_i and r_i given by the implicit relations

$$r_i = \partial H / \partial p_i \qquad (i = 1, \cdots, m).$$

It is obvious that the Lagrangian system with this principal function L is associated with the prescribed function H in the way desired.

If H contains t, so will L of course, and the same method is applicable.

11. Transformation of the Hamiltonian equations. The variational principle (5) is remarkable in that it only involves the second half of the derivatives $p_1', \cdots, p_m', q_1', \cdots, q_m'$ under the integral sign, and those linearly with coefficients precisely the conjugate variables. A general point transformation from $p_1, \cdots, q_m$ to $\bar{p}_1, \cdots, \bar{q}_m$ will yield a form linear in $p_1', \cdots, q_m'$ but not of this special type. We shall desire in the next section to consider the corresponding Pfaffian type of equation so obtained, which has certain advantages over the Hamiltonian type.

A general 'contact transformation' preserving the canonical form is the following

$$(7) \qquad p_i = \partial K / \partial q_i, \ \bar{p}_i = -\partial K / \partial \bar{q}_i \qquad (i = 1, \cdots, m),$$

where K is an arbitrary function of $q_1, \cdots, q_m, \bar{q}_1, \cdots, \bar{q}_m, t$ except it must be such as to define a proper transformation from $p_1, \cdots, q_m$ to $\bar{p}_1, \cdots, \bar{q}_m$ by means of the above equations. We shall not undertake to explain the apparent artificiality in these equations, but proceed to prove that such transformations do indeed leave invariant the canonical form. By use of the first m of these equations we modify the variational problem to the form

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{m} \frac{\partial K}{\partial q_j} q_j' - H \right] dt = 0$$

where the independent variables are now taken as $\overline{p}_1, \cdots, \overline{q}_m$. But for these same variables, we have

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{m} \left(\frac{\partial K}{\partial q_j} q_j' + \frac{\partial K}{\partial \overline{q}_j} \overline{q}_j' + \frac{\partial K}{\partial t} \right) \right] dt = 0,$$

since the expression under the integral sign is an exact derivative. By subtraction and use of the second set of m equations of transformation we deduce

$$\delta \int_{t_0}^{t_1} \left[\sum_{j=1}^{m} \overline{p}_j q_j' - \overline{H} \right] dt = 0 \quad (\overline{H} = H + \partial K / \partial t).$$

The transformation (7) *preserves the Hamiltonian form with* $\overline{H} = H + \partial K / \partial t$, *in case the arbitrary function* K *yields a proper transformation.*

Similarly we may write

$$(8) \qquad p_i = \partial K / \partial q_i, \qquad \overline{q}_i = \partial K / \partial \overline{p}_i \quad (i = 1, \cdots, m),$$

and find a corresponding result.

The transformation (8) *also preserves the Hamiltonian form with* $\overline{H} = H + \partial K / \partial t$.

It deserves to be remarked that transformations of type (8) form a group. In fact such a transformation is characterized by the fact that

$$\sum_{j=1}^{m} (p_j \, dq_j + \overline{q}_j \, d\overline{p}_j)$$

is an exact differential, dK. For a second such transformation from $\overline{p}_1, \cdots, \overline{q}_m$ to $\overline{\overline{p}}_1, \cdots, \overline{\overline{q}}_m$, there is a second characteristic $d\overline{K}$. By addition we infer

$$\sum_{j=1}^{m}(p_j\,dq_j+\overline{\overline{p}}_j\,d\overline{\overline{q}}_j) = d(K+\overline{K}-\sum_{j=1}^{m}\overline{p}_j\,d\overline{q}_j),$$

so that the compound transformation is of the same type. Similarly the inverse of a transformation (7), or the resultant of an odd number of transformations is of the same type, while the resultant of an even number of transformations (7) is of type (8).*

12. The Pfaffian equations. It is clear that Hamiltonian equations can be regarded as a special type arising from the more general Pfaffian variational principle,

$$(9)\qquad \delta\int_{t_0}^{t_1}\left[\sum_{j=1}^{n}P_j p_j'+Q\right]dt = 0,$$

in which the integral is linear in all of the first derivatives with arbitrary functions $P_1, \cdots, P_n$, Q of $p_1, \cdots, p_n$ as coefficients, and n is even.

If we develop these equations explicitly they become

$$(10)\qquad \sum_{j=1}^{n}\left(\frac{\partial P_i}{\partial p_j}-\frac{\partial P_j}{\partial p_i}\right)\frac{dp_j}{dt}-\frac{\partial Q}{\partial p_i} = 0 \quad (i=1,\cdots,n).$$

Furthermore these equations are evidently those of a degenerate Lagrangian problem with $L_2 = 0$, $L_1 = \sum P_j p_j'$, $L_0 = Q$, so that there is the particular integral $Q = \text{const.}$ This reduces to the energy integral in the Hamiltonian case.

These equations admit of an arbitrary point transformation of all of the variables without losing their form. It is only necessary to determine the modified linear differential form under the integral sign by direct substitution. Thus the Pfaffian equations admit of perfect flexibility of transformation, and in this respect are easier to deal with than either the Lagrangian or Hamiltonian equations.

13. On the significance of variational principles. Since the variational principles have taken an important

* For the applications of the theory of contact transformations and for consideration of the associated Hamiltonian partial differential equation, the reader is referred to Whittaker, *Analytical Dynamics*, chaps. 10, 11, 12.

part in dynamical theory, it is of especial interest to determine their real significance for dynamics. In other words, what especial properties are possessed by the Lagrangian, Hamiltonian or Pfaffian equations arising from the respective variational principles treated above? All of these can be regarded as systems of $n = 2m$ equations of the first order if we introduce the new variables $r_i = q_i'$ in the Lagrangian equations.

Let us first remark that so long as these equations are considered in the vicinity of a point in the corresponding space of n dimensions not an equilibrium point, there are no especial characteristics to be found.

Indeed if we take a dynamical system as defined by any set of n equations

$$dx_i/dt = X_i(x_1, \cdots, x_n) \quad (i = 1, \cdots, n),$$

it will in general remain of the same type under an arbitrary point transformation

$$x_i = \varphi_i(y_1, \cdots, y_n) \quad (i = 1, \cdots, n)$$

under certain conditions. Two systems of this kind will naturally be termed 'equivalent' if it is possible to pass from one to the other by an admissible point transformation of this kind. If we confine attention to the neighborhood of a point $x_1^0, \cdots, x_n^0$ at which not all of the X_i vanish, so that this is not an equilibrium point, the equivalence with other such systems is unrestricted, and the new equations may be taken to be

$$dy_1/dt = 1, \quad dy_i/dt = 0 \quad (i = 2, \cdots, n),$$

for instance. This is readily seen as follows. Conceive of the given differential system as defining a steady fluid motion in $x_1, \cdots, x_n$ space so that the curves of motion are defined by the solution $x_i = x_i(t)$, $(i = 1, \cdots, n)$. These curves which have a definite direction with direction cosines proportional to $X_1, \cdots. X_n$ may be deformed into the straight lines

$$y_1 = t, \qquad y_2 = c_2, \cdots, y_n = c_n$$

of a $y_1, \cdots, y_n$ space by one-to-one analytic deformation. Consequently the transformed equations have as general solution

$$y_1 = t + c_1, \qquad y_2 = c_2, \cdots, y_n = c_n,$$

whence it follows immediately that these equations have the desired normal form.

Hence in such a domain there is no distinction between equations derived from a variational principle and the most general equation.

In the following chapter we shall see that variational principles play an important role in connection with the formal stability of dynamical systems near equilibrium or periodic motion. Indeed this appears to be their principal significance for dynamics.

One further interesting remark concerning variational principles may be made here. Suppose that we start with n arbitrary equations of the form

$$dx_i/dt = X_i(x_1, \cdots, x_n, t) \qquad (i = 1, \cdots, n). \tag{11}$$

The equations of variation are

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} \frac{\partial X_i}{\partial x_j} y_j \qquad (i = 1, \cdots, n).$$

There can be formally integrated at once if the general solution

$$x_i = f_i(t, c_1, \cdots, c_n) \qquad (i = 1, \cdots, n)$$

is at hand, namely

$$y_i = k_1 \frac{\partial f_i}{\partial c_1} + \cdots + k_n \frac{\partial f_i}{\partial c_n} \qquad (i = 1, \cdots, n)$$

where $k_1, \cdots, k_n$ are arbitrary constants.

Similarly the adjoint system to the equations of variation

$$\frac{dz_i}{dt} = -\sum_{j=1}^{n} \frac{\partial X_j}{\partial x_i} z_j \qquad (i = 1, \cdots, n) \tag{12}$$

can be integrated explicitly by taking

$$\frac{\partial f_1}{\partial c_i} z_1 + \cdots + \frac{\partial f_n}{\partial c_i} z_n = k_i \quad (i = 1, \cdots, n).$$

Hence the given system (11) of equations of the first order can be called 'equivalent' to that of the extended system (11), (12) of twice the order in the $2n$ variables $x_1, \cdots, x_n, z_1, \cdots, z_n$, since the explicit solution of either system involves that of the other. But the extended system (11), (12) is Hamiltonian with conjugate variables x_i, z_i, and with

$$H = -\sum_{j=1}^{m} X_j z_j,$$

as may be directly verified.

These remarks serve to indicate the care necessary in assigning to the variational principles their true significance.