ADDENDUM

General Remarks. First we want to refer to the book of A. Wintner [3] which deals with the analytic aspects of celestial mechanics and contains a large number of references to old and new literature. Secondly Siegel's book [2] contains many topics related to Birkhoff's book and is a very valuable source of information. Finally the second part of Nemytskii's and Stepanov's book [1] deals with the abstract aspects of dynamical systems and has much contact with Chapter 7 of Birkhoff's book. In [1] the reader will find a large number of references about the more recent developments in this area.

Chapter III: In this chapter the formal aspects of trigonometrical expansions of solutions is discussed. The "Hamiltonian multipliers" are of basic importance and it is shown (see p. 78) that at an equilibrium of a Hamiltonian system these multipliers occur in pairs of $\lambda_i, -\lambda_i$ (which was also proven by Liapounov). A similar statement holds near a periodic solution for the so-called Floquet exponents (see Chapter III, Section 9). It is remarkable that another restriction on the Floquet exponents was overlooked and only discovered by M. G. Krein in 1950 (see [22], [23]), namely that for the pairs $\lambda_i, -\lambda_i$ on the circle $|\lambda| = 1$ there is an ordering which is invariant under canonical transformations. In other words, if

$$\lambda_i = \exp(-1)^{1/2}\omega_i$$

one can associate a sign with the frequencies ω_i. This fact is of importance for the stability theory of periodic solutions (see Gelfand and Lidskii [12] for the linear theory and Moser [29] for the nonlinear theory). This phenomenon is related to the different behavior between "difference and sum resonances".

Chapter V: Contains a discussion of Birkhoff's minimax method and an "extension of Morse" (see Section 8). This area expanded to a vast theory, the now well-known Morse theory for which we refer to Morse's book [25]. This theory, which started with Poincaré's study of closed orbits for the three body problem has now taken many new directions and proved so successful in topology (see, for example, Milnor's book [24]). We mention some further developments of the geodesics problem. After Morse's and Lusternik and Schnirelman's study of this problem there appeared recently a long paper by Alber [4] estimating the minimal number of closed geodesics on an n-dimensional sphere which contains further references (see also Klingenberg [19]). However, it should be mentioned also that the general Morse theory has not yet been successfully applied to the problems of dynamics. Even for the restricted three body problem such an application would be of great interest.

Chapter VI: This chapter contains a discussion of the celebrated "Poincaré's geometric theorem", the proof of which was Birkhoff's first work in this subject (1915). This beautiful theorem withstood all attempts of generalizations and still it is not clear whether it has an analogue in higher dimensions, for say, canonical transformations.

We mention a new application of this theorem to the restricted three body problem. In [11] Conley established the existence of infinitely many periodic solutions around the small mass point (lunar orbits). This is a nontrivial extension of Birkhoff's study of 1915 mentioned in the footnote on p. 177.

Chapter VII: The subject of this chapter has become a basis of a very abstract formulation in the book by Gottschalk and Hedlund [14]. Another source of references related to Chapter 7 is the second part of Nemytskii's and Stepanov's book on Differential Equations [1]. We make special mention of a paper by S. Schwartzman [33] in which the concept of rotation numbers is generalized to flows on a compact manifold.

Chapter VIII: In the problems discussed in Chapter VIII many advances have been made. A number of questions have been settled and others have expanded into theories of their own. The example of Section 11 illustrates a transitive flow. The study of the geodesic flow on a manifold of negative curvature has been studied thoroughly in ergodic theory and we refer to Hopf's book [16], his fundamental papers [17] and to Hedlund's paper [15]. Recently Anosov [5] generalized these ideas considerably and studied a class of differential equations (so-called U-systems) for which he proves transitivity. For recent surveys in this direction see Sinai [37], [38].

The question of stability raised in Section 7 has been answered and it is known that every fixed point of general stable type (in the terminology of this book) is stable in the sense of Liapounov. This assertion is contained in the work of Arnol'd [**6**], [**7**], [**8**] and [**9**], Kolmogorov [**20**], [**21**] and Moser [**31**]. The problem is intimately connected with the difficulty of the small divisors. The first definitive results concerning such small divisor problems were found by C. L. Siegel [**34**], [**35**] but his approach did not cover the case in question here. In 1954 Kolmogorov suggested an approach which ultimately led to the stability proof of periodic solutions of general stable type for systems of two degrees of freedom. However, the results of Arnol'd reach much further covering Hamiltonian systems of several degrees of freedom, although in this case stability in the sense of Liapounov cannot be inferred. In fact, Arnol'd [**9a**] proves instability for a system of 3 degrees of freedom and one can say that the concept of stability for Hamiltonian systems has been clarified to a large extent. In [**7**], [**9**] Arnol'd gives a most remarkable application of these results to the n-body problem.

Chapter IX: Concerning Sundman's results we mention the clear and complete exposition in Siegel's book [**2**]. Also Wintner's book on celestial mechanics [**3**] contains a wealth of information on the n-body problem.

FOOTNOTES

1. Page 78; Line 9 after "quantities."

This statement is certainly incorrect as it stands. It can occur that $\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda}$ are four distinct numbers as in the example $H = \mu(p_1 q_1 + p_2 q_2) + \nu(p_1 q_2 - p_2 q_1)$. Incidentally, an equilibrium of this type occurs for the equilateral solution (of Lagrange) of the restricted three body problem, at least for appropriate mass ratios (see Wintner [**3**, §476]).

2. Page 86; Third line from below, after "pure imaginary."

The remark of Footnote 1, p. 78, applies here too.

3. Page 91; Line 8, after "quantities."

See Footnote 1.

4. Page 99; Line 13, after "solution."

This solution will be periodic if the period $T(c)$ of the family of reference solutions is independent of c. Otherwise the solution in question involves a term linear in t, but still contributes a second multiplier zero.

5. Page 116; Line 13, after "definition."

The investigation of complete stability has been carried further by J. Glimm [**13**]. He considered an equilibrium (or a periodic solution) also in the case where the λ_i are rationally dependent. He replaced the power series expansion by expansions in terms of rational functions.

6. Page 165; At the end of Section 4, after "period."

This question was pursued further by G. D. Birkhoff himself in "Une généralisation a n-dimensions due dernier théorème de geometrie de Poincaré," Compt. Rend des Sciences de l'Acad. d. S. 192, p. 196, 1931.

7. Page 211; Line 8 from below, after "or $2\lambda - \sqrt{-1}$."

This case distinction refers to Floquet theory: if the eigenvalues of linearized mapping over one period are real and denoted by $e^{\pm 2\pi\lambda}$, then for $e^{2\pi\lambda} > 0$ one can choose λ real and for $e^{2\pi\lambda} < 0$ one can choose $2\lambda - (-1)^{1/2}$ real.

8. Page 211; At the end of formula (3), line 4 from below, after "($\mu \neq \pm 1$)."

In this case one can actually take $\Phi = \Psi = 0$ as was proven in Moser [**28**]. The transformation into the normal form is indeed convergent. This point was left open in Birkhoff's paper of 1920 (which is cited on p. 211).

9. Page 213; Line 14, after "motion)."

The contents of this parenthesis apparently refers to degenerate cases, illustrated by a transformation $u_1 = u_0 + v_0$, $v_1 = v_0$ where $v_0 = 0$ represents a family of fixed points.

10. Page 214; Line 7 after "from the origin."

This remark has to be qualified. If σ/π is rational one can easily produce unstable examples and for integral $3\sigma/2\pi$ instability is the generic case, see for example [**26**].

11. Page 215; Line 15 after "multiples are simple with $l \neq 0$."

This means that the number σ appearing in equation (2) of p. 211 is assumed to be incommensurable with π. The number l was defined on p. 211 as $l = (-1)^{1/2} s/2\pi$. For the definition of "simple" and "multiple" see p. 142 bottom.

12. Page 217; The sentence starting on line 5 is misleading and should be replaced by:

In this case the corresponding normal form is of type (2) (see p. 211). In the unstable case the normal form is of the form (3) where μ is positive or negative but not ± 1.

13. Page 218; Line 4, replace *period* after "negative" by a *comma*. Replace lines 5 and 6 by:

as was mentioned on p. 215 bottom. Consequently a real negative root μ is not possible, and the case (3) can occur only with $\mu > 0$, i.e. I is a fixed point of stable type.

14. Page 222; Line 3 from bottom, after "is bounded."

The statements of this section are not sufficiently proven and it seems impossible to supply the necessary arguments. It is quite conceivable that such an invariant "curve" is very pathological making the geometrical considerations inadequate.

In fact, N. Levinson [**18**] constructed a second order differential equation where such a pathological invariant set occurs. In Levinson's example the rotation number (which on an invariant curve should be constant) takes on various values on the invariant set which is of measure zero. But it has to be mentioned that Levinson's differential equation is not conservative and can only be considered as an illustration, not as a counterexample.

15. Page 227; Line 12 from below after "such actual stability?"
This question has been answered in the affirmative as was mentioned in my General Remarks to Chapter VIII, (see [**9**], [**31**]).

16. Page 227; Last line after "variable periods."
According to the previous footnote a periodic motion of general stable type is stable, which makes the present assumption as well as Section 8 vacuous!

17. Page 237; After first paragraph, i.e. after "stable type."
Recent work by Smale [**39**] extends these results, to which Birkhoff alludes, considerably. Smale finds infinitely many periodic motions, and even a perfect minimal Cantor set near a "homoclinic" motion, even for several dimensional systems. Unfortunately, his results are not applicable to Hamiltonian systems of more than two degrees of freedom, due to some assumption which fails for Hamiltonian systems.

18. Page 238; Line 3 after title of Section:
A very interesting example of this type had been discussed already in 1924 by E. Artin [**10**] (following a suggestion of Herglotz). It also deals with the geodesic flow on a manifold of two dimensions (the modular region in the upper half plane) and a symbolism for these geodesics is put into correspondence with the continued fraction expansions.

19. Page 245; Line 5 after "of motions."
Extensions of such results are contained in the recent work by Anosov [**5**]. He considers systems of differential equations (so-called U-systems) in several dimensions whose solutions have a similar behavior as the geodesic flow on a manifold with negative curvature.

20. Page 257; Line 12 after "for $m = 1$."

The following argument leaves a number of points unclear. Careful proofs and sharper results have been given by Siegel [**36**], Rüssmann [**32**] and Moser [**27**]. The paper [**30**] contains an explicit class of nonintegrable polynomial transformations.

21. Page 259; Line 13 after "analytic families."

These families lie on the level surface of the Hamiltonian and of the integral I which was assumed to exist. The family could be parametrized by the canonically conjugate variable of I. For this purpose one would have to introduce new variables, say u_1, u_2, v_1, v_2 by a canonical transformation such that $u_1 = I$, say, which can be done. Then v_1 would be a family parameter.

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