Solution for Chapter 15 (B)

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A.

Exercise 15.1 Fluid motions in gravity waves [by Xinkai Wu/02]

(a) The potential function for the velocity field is given by eq. (15.3)

$$\psi(x,z,t) = \psi_0 cosh[k(z+h_0)]exp[i(kx-\omega t)]$$

from which we find the velocity $\mathbf{v} = \boldsymbol{\nabla}\psi$(upon taking the real part):

$$v_x = -k\psi_0 cosh[k(z+h_0)]sin(kx-\omega t);\ v_y = 0;\ v_z = k\psi_0 sinh[k(z+h_0)]cos(kx-\omega t)$$

Thus we see that at a given depth z, the fluid element undergoes elliptical motion.

(b) From the velocity field found in the previous part, we see that the longitudinal and vertical diameters are given by

$$D_l = 2k\psi_0 cosh[k(z+h_0)];\quad D_v = 2k\psi_0 sinh[k(z+h_0)]$$

with their ratio being

$$\frac{D_v}{D_l} = tanh[k(z+h_0)]$$

(c) For a deep-water wave, $kh_0 >> 1$, and we see that

$$D_l \approx k\psi_0 exp[k(z+h_0)];\quad D_v \approx k\psi_0 exp[k(z+h_0)]$$

namely, the ellipses are all circles with diameters dying out exponentially with depth.

(d) For a shallow-water, $kh_0 << 1$, we see that

$$\begin{aligned} D_l &\approx 2k\psi_0\left[1+\frac{1}{2}k^2(z+h_0)^2\right] \approx 2k\psi_0 \\ D_v &\approx 2k\psi_0 k(z+h_0) \approx 0 \end{aligned}$$

namely the motion is (nearly) horizontal and independent of height z.

(e) The N-S equation in our case is

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{\boldsymbol{\nabla}P}{\rho} + \mathbf{g}_e$$

Write the pressure as the sum of an unperturbed part P_0 and a perturbation δP

$$P(x,z,t) = P_0(z) + \delta P(x,z,t)$$

and plug it into the z component of the N-S equation, we get

$$\begin{aligned}\frac{\partial P_0}{\partial z} &= -\rho g_e \\ \frac{\partial \delta P}{\partial z} &= -\rho \frac{\partial v_z}{\partial t}\end{aligned}$$

Integrating the second equation from $z = 0$ to z gives

$$\delta P(z) = \delta P(z=0) - \int_0^z \rho \frac{\partial v_z}{\partial t}$$

where the first term corresponds to the weight of the overlying fluid, and the second term corresponds to the contribution from the vertical acceleration of the fluid.

Using the expression for v_z worked out in part (a) as well as the boundary condition

$$\delta P(z=0) = \rho g_e \xi = -\rho \frac{\partial \psi}{\partial t}|_{z=0} = -\rho\omega\psi_0 sin(kx - \omega t) cosh[kh_0]$$

where we have used eq. (15.5) for ξ, the surface's vertical displacement from equilibrium, we find that

$$\begin{aligned}\delta P(x,z,t) &= -\rho\omega\psi_0 sin(kx-\omega t) cosh[kh_0] \\ &\quad -\rho\omega\psi_0 sin(kx-\omega t)\left\{cosh[k(z+h_0)] - cosh[kh_0]\right\} \\ &= -\rho\omega\psi_0 sin(kx-\omega t) cosh[k(z+h_0)]\end{aligned}$$

For shallow water, the vertical fluid acceleration term (the second term in the first line of the above expression) is $\propto kh_0$ and can be ignored, and the pressure is basically determined by the first term, i.e. the weight of the overlying fluid. For general depth the second term is no longer negligible.

Exercise 15.2 Surface tension [by Xinkai Wu/02]

(a) Since $\mathbf{g} - \mathbf{e}_z \otimes \mathbf{e}_z$ is the 2-dimensional metric of the surface, we know that the stress tensor must be proportional to it, namely

$$\mathbf{T} = C\delta(z)(\mathbf{g} - \mathbf{e}_z \otimes \mathbf{e}_z)$$

and we only have to determine the multiplicative constant C. Take a small line element $d\mathbf{l}$ on the surface, the force across the line element is $d\mathbf{F} = \mathbf{T} \cdot d\mathbf{l} = C d\mathbf{l}$. On the other hand we know that $d\mathbf{F} = -\gamma d\mathbf{l}$. This tells us that $C = -\gamma$ and thus $\mathbf{T} = -\gamma\delta(z)(\mathbf{g} - \mathbf{e}_z \otimes \mathbf{e}_z)$.

(b) Let the surface be specified by $z = f(x, y)$. Then the total stress tensor (the bulk one plus the surface one) is given by

$$(T_{total})_{ij} = P g_{ij} - \gamma\delta(z - f(x,y))(g_{ij} - n_i n_j)$$

where $\mathbf{n}$ is the unit vector normal to the surface whose components are given by

$$n_x = \frac{-\partial f/\partial x}{\sqrt{1+(\partial f/\partial x)^2+(\partial f/\partial y)^2}}, \; n_y = \frac{-\partial f/\partial y}{\sqrt{1+(\partial f/\partial x)^2+(\partial f/\partial y)^2}}$$
$$n_z = \frac{1}{\sqrt{1+(\partial f/\partial x)^2+(\partial f/\partial y)^2}}$$

Vanishing of net force density requires $\boldsymbol{\nabla}\cdot\mathbf{T}=0$, and let's consider the z component of this:

$$0 = \partial^i (T_{total})_{iz} = \partial_z P - \gamma \partial^i [\delta(z - f(x,y))(g_{iz} - n_i n_z)]$$

Integrating the above expression from $z = f(x,y)-\epsilon$ to $z = f(x,y)+\epsilon$, we find the pressure difference across the surface at location (x,y)

$$\Delta P(x,y) \equiv P(z=f(x,y)+\epsilon) - P(z=f(x,y)-\epsilon) = -\gamma \left\{ \frac{\partial}{\partial x}(n_x n_z) + \frac{\partial}{\partial y}(n_y n_z) \right\}$$

In the vicinity of $(x,y)=(0,0)$, we can set $f(x,y) = x^2/2R_1 + y^2/2R_2$. Plugging this form of $f(x,y)$ into our expression for $\mathbf{n}$ and ΔP, and finally setting (x,y) to $(0,0)$, we get

$$\Delta P = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

B. Exercise 15.5 Shallow-water waves with variable depth [by Xinkai Wu/00]

We can treat this as a 2-dimensional problem, i.e., only consider the horizontal components of the velocity, which are almost independent of z. In what follows, $\boldsymbol{\nabla}$ is the 2-dimensional derivative operator.

(a) The mass per unit area is $\rho(h_0+\xi)$, and the mass flux per unit length is $\rho(h_0+\xi)\mathbf{v} \approx \rho h_0 \mathbf{v}$, to the first order in perturbation. Then by mass conservation,

$$\begin{aligned} &\partial[\rho(h_0+\xi)]/\partial t + \boldsymbol{\nabla}\cdot(\rho h_0 \mathbf{v}) = 0 \\ &\Rightarrow \partial\xi/\partial t + \boldsymbol{\nabla}\cdot(h_0\mathbf{v}) = 0 \end{aligned} \quad (1)$$

(assuming constant ρ)

The Navier-Stokes equation in this case is:

$$\partial\mathbf{v}/\partial t = -\boldsymbol{\nabla}P/\rho + \mathbf{g}_e$$

whose vertical component tells us $P = \rho g_e(\xi - z)$, and whose horizontal components then tell us

$$\partial\mathbf{v}/\partial t = -g_e \boldsymbol{\nabla}\xi \quad (2)$$

Applying ∂_t to both sides of eqn. (1) and then plugging in eqn (2), we get

$$\partial^2\xi/\partial^2 t - g_e \mathbf{\nabla}\cdot(h_0\mathbf{\nabla}\xi) = 0$$

(b) Plugging a plane wave solution to the wave equation, we get the dispersion relation

$$\omega = k\sqrt{g_e h_0}\sqrt{1 - i\frac{\mathbf{\nabla} h_0}{h_0}\cdot\frac{\mathbf{k}}{\mathbf{k}^2}}$$

the imaginary part in the square root is of order $\lambda/L << 1$, with λ being the wave length and L being the scale over which h_0 varies, and we've used the fact the h_0 is varying very slowly. Thus the dispersion relation is

$$\omega = \Omega(\mathbf{k},\mathbf{x}) \approx k\sqrt{g_e h_0(\mathbf{x})}$$

Our knowledge learned in chapter 6 Geometric Optics then immediately tells us that water waves propagate in the ocean with a "refractive index" $n \propto h_0^{-1/2}$. Hence as waves approach a beach, their propagation directions get closer and closer to the normal direction, by Snell's law, since smaller h_0 corresponds to larger n.

(c) What you have to do is just creating some kind of mountain on the ocean floor (which gives smaller h_0 and thus larger n) with the shape of a lens, with Japan and LA at the object point and image point, respectively.

Exercise 15.4 Ship waves [by Xinkai Wu/00]

(a) Consider a plane wave with frequency ω_0 and wave vector $\mathbf{k_0}$ as measured in the water's frame, and ω and $\mathbf{k}$ as measured in the boat's frame. For an observer with position $\mathbf{x_0}$ in the water's frame and $\mathbf{x}$ in the boat's frame, the phase he measures is $\mathbf{k_0}\cdot\mathbf{x_0} - \omega_0 t$ in terms of water's frame variables and $\mathbf{k}\cdot\mathbf{x} - \omega t$ in terms of boat's frame variables.

Since the phase is invariant under the change of reference frame, we can equate the above two expressions and then differentiate both sides with respect to t, noting that for an observer moving together with the boat, $d\mathbf{x_0}/dt = \mathbf{u}, d\mathbf{x}/dt = 0$, we get:

$\omega = \omega_0 - \mathbf{k_0}\cdot\mathbf{u}$

By looking at Fig 15.3 and use $\mathbf{k}$ to denote the wave vector as measured in the water's frame(as the text does), we get

$\omega = \omega_0 + uk\cos\phi$

(b) θ is the angle between $\mathbf{V}_{g0} - \mathbf{u}$ and $\mathbf{u}$, elementary trigonometry then gives,

$\tan\theta = V_{g0}\sin\phi/(u + V_{g0}\cos\phi)$

For stationary wave pattern $\omega = 0$, using the $\omega(k)$ we got in part (a), we see that

$\omega_0(k) = -uk\cos\phi$

(c) For capillary waves, $\omega_0 \approx \sqrt{(\gamma/\rho)k^3}$, $V_{g0} = \partial\omega_0/\partial k = (3/2)\sqrt{(\gamma/\rho)k}$.

Plugging these and $u = -\omega_0/(k\cos\phi)$ into the expression for $\tan\theta$, we get

$\tan\theta = (3\tan\phi)/(1-2\tan^2\phi)$

Capillary wave pattern for a given θ exists only when we can find some $\phi \in (\pi/2, \pi)$ (i.e. only forward waves can contribute to the pattern) satisfying the above equation. And it's easy to show that indeed for any θ we can find such a ϕ given by:

$tan\phi = (-3-\sqrt{9+8\tan^2\theta})/(4\tan\theta)$ when $\theta < \pi/2$

and $\tan\phi = (-3+\sqrt{9+8\tan^2\theta})/(4\tan\theta)$ when $\theta > \pi/2$.

For gravity waves, $\omega_0 \approx \sqrt{g_e k}$, and $V_{g0} = (1/2)\sqrt{g_e/k}$, and we get

$\tan\theta = (-\tan\phi)/(1+2\tan^2\phi)$

Only when $\theta < \arcsin(1/3)$ can we find some $\phi \in (\pi/2, \pi)$ satisfying this equation, which is:

$\tan\phi = (-1 \pm \sqrt{1-8\tan^2\theta})/(4\tan\theta)$(both solutions are valid)

This means that the gravity-wave pattern is confined to a trailing wedge with an opening angle $\theta_{gw} = 2\arcsin(1/3)$.

C. Exercise 15.8 Two-soliton solution [by Xinkai Wu/02]

(a) Using Mathematica one verifies that expression (15.39) indeed satisfies the KdV eqn. (15.35). (My experience is that if I set $\alpha_1 = r\alpha_2$ with r of course being an arbitrary number, then Mathematica works well; without this little trick, Mathematica doesn't really simplify the KdV equation. Maybe your experience is different from mine.)

(b) We don't lose any generality by assuming that $\alpha_1 < \alpha_2$.

Let's first consider the case of early times, namely τ is a large negative number. In this case, we see $f_1 >> f_2$.

When we consider the spatial region near the soliton labeled number one, namely $\eta \sim \eta_1 + \alpha_1^2\tau$, then $f_1 \sim 1$ and $f_2 << 1$. Thus we have

$$F \approx 1 + f_1$$

and

$$\begin{aligned}
\zeta &\approx \frac{\partial^2}{\partial\eta^2}\left[12ln(1+f_1)\right] \\
&= 12\alpha_1^2 \frac{f_1}{(1+f_1)^2} \\
&= 12\alpha_1^2 \left(\frac{1}{\sqrt{f_1}} + \sqrt{f_1}\right)^{-2} \\
\text{(noting} \quad & \frac{1}{\sqrt{f_1}} + \sqrt{f_1} = 2cosh\left[\frac{\alpha_1}{2}(\eta-\eta_1-\alpha_1^2\tau)\right]\text{)} \\
&= 3\alpha_1^2 sech^2\left[\frac{\alpha_1}{2}(\eta-\eta_1-\alpha_1^2\tau)\right]
\end{aligned}$$

which is the form of the single-soliton solution given in eq. (15.36) with the amplitude being $\zeta_{01} = 3\alpha_1^2$, and the location of the peak being $\eta_1(\tau) = \eta_1 + \alpha_1^2\tau$.

When we consider the spatial region near the soliton labeled number two, namely $\eta \sim \eta_2 + \alpha_2^2\tau$, then $f_2 \sim 1$ and $f_1 >> 1$. Keeping the leading order terms, throwing away terms of order one, we see

$$F \approx f_1 \left[1 + \left(\frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \right)^2 f_2 \right]$$

and as a result

$$\begin{aligned} \zeta &\approx 12\alpha_2^2 \left[\frac{1}{\left(\frac{\alpha_2-\alpha_1}{\alpha_2+\alpha_1}\right)\sqrt{f_2}} + \left(\frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \right) \sqrt{f_2} \right]^{-2} \\ &= 3\alpha_2^2 sech^2 \left[\frac{\alpha_2}{2} \left(\eta - \eta_2 - \alpha_2^2 \tau - \frac{2}{\alpha_2} \ln \left(\frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \right) \right) \right] \end{aligned}$$

which is again of the form of a single-soliton solution with the amplitude given by $\zeta_{02} = 3\alpha_2^2$, and the peak location $\eta_2(\tau) = \eta_2 + \alpha_2^2 \tau + \frac{2}{\alpha_2} \ln \left(\frac{\alpha_2-\alpha_1}{\alpha_2+\alpha_1} \right)$.

Now let's consider late times, namely τ being a large positve number. In this case, $f_1 << f_2$.

Analysis similar to the early-time case immediately gives that in the spatial region ner the soliton labeled number one,

$$\zeta \approx 3\alpha_1^2 sech^2 \left[\frac{\alpha_1}{2} \left(\eta - \eta_1 - \alpha_1^2 \tau - \frac{2}{\alpha_1} \ln \left(\frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1} \right) \right) \right]$$

and that in the spatial region near the soliton labeled number two,

$$\zeta \approx 3\alpha_2^2 sech^2 \left[\frac{\alpha_2}{2} (\eta - \eta_2 - \alpha_2^2 \tau) \right]$$

It's interesting to see that the peak location of the solition lagging behind (solition number two at early times and soliton number one at late times) is shifted from the naive location $\eta_i + \alpha_i^2 \tau$ by an amount of $\delta_i = \frac{2}{\alpha_i} \ln \left(\frac{\alpha_2-\alpha_1}{\alpha_2+\alpha_1} \right)$, which is a negative number. This can be interpreted as caused by a repulsion applied to the lagging soliton by the leading soliton. As one can easily see, taking the $\alpha_1 << \alpha_2$ limit of δ_i justifies the above interpretation.

(c) See Fig. 1 to Fig.12. The ordinate is *zeta* and the abscissa is η.

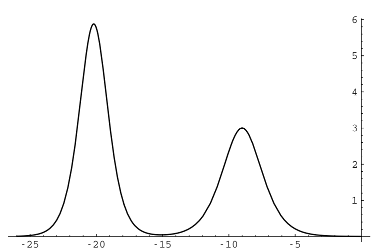

Figure 1: $\tau = -9$

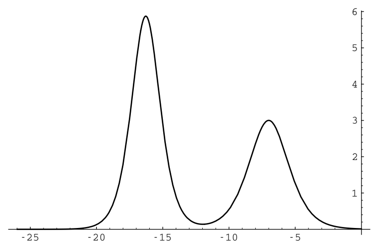

Figure 2: $\tau = -7$

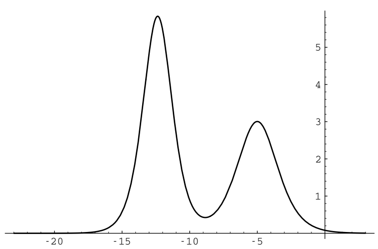

Figure 3: $\tau = -5$

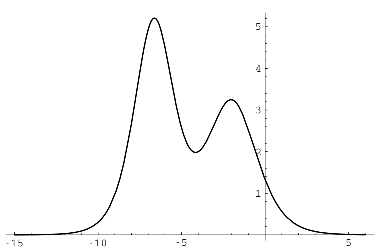

Figure 4: $\tau = -2$

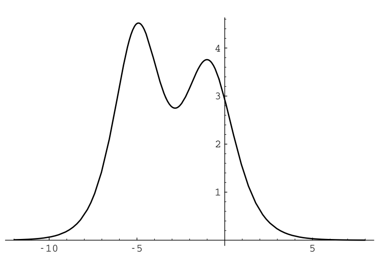

Figure 5: $\tau = -1$

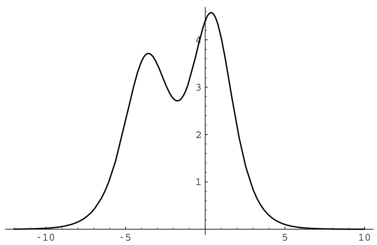

Figure 6: $\tau = 0$

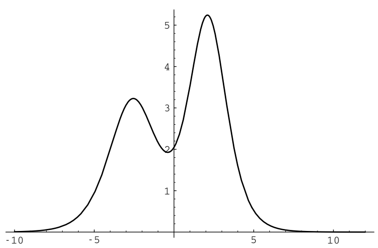

Figure 7: $\tau = 1$

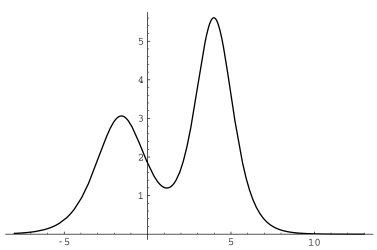

Figure 8: $\tau = 2$

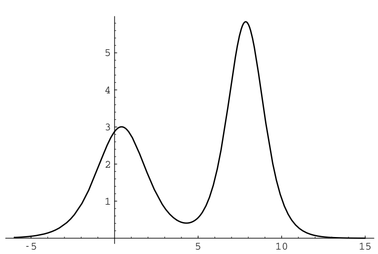

Figure 9: $\tau = 4$

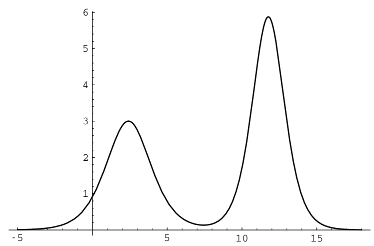

Figure 10: $\tau = 6$

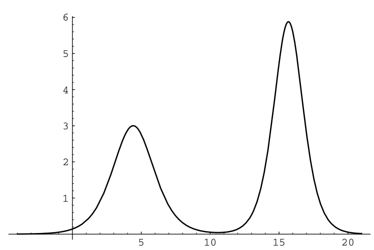

Figure 11: $\tau = 8$

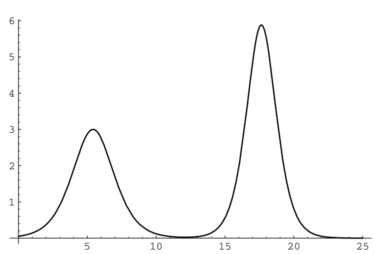

Figure 12: $\tau = 9$