Solution for Chapter 15A

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(A revised version may appear after Kip comments.)

1 Problem A.(BT-15.7)

[by Kip Thorne and Guodong Wang/03]
(a). The balance of the friction of the wind with the ocean and the Coriolis force builds an Ekman layer in the Sargasso Sea area. See BT-15.4.4 for the discussion of the surface currents' direction.

The layer thickness is $\delta_E = \sqrt{\frac{\nu}{\Omega}}$.

$\Omega = \Omega_e \sin(latitute) = \frac{2\pi}{24\times 3600s} \times 0.5 \simeq 3.6 \times 10^{-5}s^{-1}$. The friction of the wind with the water is achieved by turbulence. The largest eddies have size $l \sim 1$ m, and the turnover speeds v_l ~0.1 m/s, which implies a turbulent viscisity $\nu_t \sim \frac{1}{3}v_l l \sim 0.03$ m/s (see BT-14.3.2). $\delta_E = \sqrt{\frac{\nu}{\Omega}} \simeq 30$ m.

When water flows uphill into the gyre, its kinetic energy gets converted into potential energy. Correspondingly, the potential energy per unit mass, gh, at the top of the gyre is equal to the mean kinetic energy $\frac{1}{2}v_e^2$ of the wind-driven surface current in the Ekman layer. The mean flow speed v_e in the Ekman layer is determined by stress balance between the (turbulent) wind and the (turbulent) water in the Ekman layer. We could try to estimate the mean flow speed by thinking through the details of that stress balance, or we can take it from observational data. it is $v_e \sim$ a few m/s. Inserting this into the law of energy conservation, we get $h \sim v_e^2/2g \sim 1$ m.

(Notes: some references say the surface current speed typically are 3% of wind speed (Force 8 wind = 20 m/s, gives current 0.7 m/s). Then it is too small to be responsible for the 1.5 m height. Kip and us will check this issue later.)

(b). When the water moves towards the center of the Sargasso sea,it piles up and builds the gyre with an outward horizontal pressure gradient. this gradient gives rise to the deep ocean geostrophic flow.

$$2\mathbf{\Omega} \times \mathbf{v} = -\frac{\nabla P'}{\rho}, \tag{1}$$

The excess pressure at the center of the gyre is $\Delta P' = \rho g h$, where $h \sim 150$ cm is the height of the gyre. The north-south distance over which the gyre extends is the width of the region where the winds switch from easterly to westerly, which turns out to be about 300 km, so the gyre rises on each side over a distance $\Delta x \sim 150$ km. In this region of rise (which is where the geostrophic flow is being driven), the pressure gradient has a magnitude $\sim \Delta P'/\delta x \sim \rho g h/\Delta x$." The y-component in Eq.(1) is

$$2\rho\Omega v_y = \frac{dP'}{dx} \simeq \frac{\Delta P'}{\delta x} \simeq \frac{\rho g h}{\Delta x}, \qquad \Rightarrow v_y \simeq \frac{g}{2\Omega}\frac{h}{\Delta x} \simeq 1.4m/s \simeq 5km/s \tag{2}$$

v_y is the speed of the deep ocean current.

(c). Without continents we would see the currents travels parallelly to lines of latitude. With the continents as the barriers, there are circular flows in those ocean basins where there is a substantial change in the prevailing wind velocities as one goes across the ocean basin. They flow clockwise in the North Atlantic and North Pacific and flow counterclockwise in the southern hemisphere.

2 Problem B.(BT-15.8)

[by Kip Thorne and Guodong Wang/03]

(a) In the water's rotating reference frame, $\mathbf{v} = \mathbf{0}$, according to the Navier-Stokes equation in the rotating frame(eq. BT-15.43),

$$\nabla P' = 0 \Rightarrow P' = P + \rho[gz - \frac{1}{2}\Omega^2\varpi^2] = constant \qquad (3)$$

taking $z = 0, \varpi = 0$ as the center of the water's top surface, the constant at (0,0) becomes the air pressure above the tea cup, namely P_0. so,

$$P(\varpi, z) = P_0 - \rho g z + \frac{1}{2}\rho\Omega^2\varpi^2, \qquad (4)$$

note z-axis' direction is upward.

At the top surface of the water, $z(\varpi)$, the pressure $P(\varpi, z(\varpi))$ is P_0. Therefore the surface of the water is parabolic,

$$z(\varpi) = \frac{\Omega^2}{2g}\varpi^2. \qquad (5)$$

(b). The thickness of the Ekman layer at the bottom of the cup is

$$\delta_E = \sqrt{\frac{\nu}{\Omega}} \sim \sqrt{\frac{10^{-6}m^2/s}{1/s}} \sim 1mm. \qquad (6)$$

From Eq.(4), we can see the pressure inside this thin layer is

$$P(\varpi) = P_0 - \rho g z_0 + \frac{1}{2}\rho\Omega^2\varpi^2, \qquad (7)$$

Where $z = z_0$ is the bottom surface of the water. So the Ekman layer at the bottom experiences a presssure difference $\frac{1}{2}\rho\Omega^2\varpi^2 \sim \rho L^2\Omega^2$ towards the center, where L is the radius of the tea cup. Applying the Navier-staokes equation(BT-15.43) to the Ekman layer in an inertial reference frame,

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P/\rho + \nu\nabla^2\mathbf{v} \Rightarrow -(\mathbf{v}\cdot\nabla)\mathbf{v} + \nu\nabla^2\mathbf{v} = \Omega^2\varpi\mathbf{e}_\varpi \qquad (8)$$

According to Eq.(8), inside the layer, $\mathbf{v}$ is radial and

$$\frac{\nu}{L^2}\mathbf{v} - \frac{\mathbf{v}^2}{L} \simeq -\frac{\mathbf{v}^2}{L} \simeq \Omega^2 L\mathbf{e}_\varpi \Rightarrow v \sim \Omega L \sim 1s^{-1} \times 0.1m \sim 0.1m/s \qquad (9)$$

The mass flux that is carried by $v_\varpi \sim \Omega L$ is

$$\rho\delta_E 2\pi L v \sim 60g/s \tag{10}$$

(c). When the boundary -layer water nears the center of the cup, it gets pushed upward into the bulk flow. The bulk flow must move geostrophically, so its water gradually moves radially outward, in a (approximately) z-independent manner, from the center of the cup toward the outer wall. At the outer wall (which we idealize as vertical) there is a boundary layer. Does the water get pulled down through this vertical boundary layer into the Ekman layer at the bottom? To answer this, examine the Navier-Stokes equation for this outer boundary layer.

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P/\rho + \nu\nabla^2\mathbf{v} - \nabla\Phi_e \tag{11}$$

Inserting Eq.(4)with $\varpi = L$ and $\Phi_e = \rho g z$ in Eq.(11), $\Phi_e = \rho g z$ has cancelled with the $-\rho g z$ term in $P(\varpi = L, z)$. So there are no unbalanced vertical forces — the boundary layer has only an angular component of velocity, not a vertical component. The angular (theta) velocity slows down to zero at the wall in the standard manner of a Blausius profile.

Thus, it must be that the water is sucked into the Ekman layer at the cup's bottom primarily from the geostrophic bulk flow, and not from the boundary layer at the cup's outer wall.

The Rossby number in the bulk flow is small. Incompressible flow implies $\nabla\cdot\mathbf{v} \simeq 0 \Rightarrow \Omega L/L \simeq v_{bulk}/\delta_E \Rightarrow v_{bulk} = \Omega\delta_E$. v_{bulk} is the flow speed of the large scale circulation pattern. So Ro=$\frac{v_{bulk}}{\Omega L} \simeq \frac{\delta_E}{L} \sim 0.01$.

(d). The total mass of the water in the cup is $M \sim \rho\pi L^2 \cdot L$, so it can mix much of the water in a time scale of

$$t_E \sim \frac{M}{\rho\delta_E 2\pi L v_\varpi} \sim \frac{L\delta_E}{\nu} \sim \frac{L}{\sqrt{\nu\Omega}}. \tag{12}$$

Let $L = 0.1m$, $\Omega = 1s^{-1}$, $\nu = 10^{-6}m^2/s$, $t_E \sim 100s$. The angular momentum of the water is conserved in the bulk. From the discussion in part(c), we know only in the bounary layer at the bottom, the angular momentum loses by the friction. So the time scale of the mass mixing is also the time scale for the bulk flow to slow down.