BRIOT AND BOUQUET'S

ELEMENTS

OF

ANALYTICAL GEOMETRY

OF TWO DIMENSIONS

The Fourteenth Edition

TRANSLATED AND EDITED

BY

JAMES HARRINGTON BOYD

INSTRUCTOR IN MATHEMATICS IN THE UNIVERSITY OF CHICAGO

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PREFACE.

This translation has been made with the hope that the high scientific character of Briot et Bouquet's *Leçons de Géométrie Analytique* may contribute something toward the improvement of the standard of instruction in the elements of analytical geometry.

The translator leaves for the second edition the addition of notes which will bring some of the topics treated in the text down to the present scientific development of the subject. A note has been added with the object of furnishing the more elementary courses with simple exercises.

I wish to thank Professor E. Hastings Moore, Professor Oscar Bolza, Professor Henry S. White, Dr. Harris Hancock, Dr. T. J. J. See, for valuable suggestions and assistance in making this translation.

JAMES HARRINGTON BOYD.

University of Chicago, July 1, 1896.

AUTHORIZED ENGLISH TRANSLATION.

LIBRAIRIE CHARLES DELAGRAVE

15, Rue Soufflot, à Paris

Adresse Télégraphique :
DELAGRAVE PARIS

Paris, le 4 Xbre – 96

Monsieur

Moyennant le versement de ce droit d'auteur vous pouvez considérer comme seule autorisée la traduction que vous ferez de l'ouvrage en langue anglaise.

Veuillez agréer, monsieur, l'assurance de mes sentiments les plus distingués,

C Delagrave

(*Translation.*)

(DEAR SIR:

As agreed to by both parties your translation will be regarded as the only authorized translation of this work (Briot et Bouquet Geometrie Analytique) in the English language.

Accept, dear sir, my highest compliment,

C. DELAGRAVE.)

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ANALYTICAL GEOMETRY

ANALYTICAL GEOMETRY has for its object the study of figures through the methods of algebraic calculation or analysis.

The representation of figures by algebraic symbols is due to Descartes, who established a general method for the resolution of geometrical questions.

In this treatise, plane figures, or those of two dimensions, are considered.

PLANE GEOMETRY

BOOK I

CHAPTER I

CONCERNING CO-ORDINATES.

The position of a point in a plane is determined by means of two magnitudes, which are called the co-ordinates of that point.

There can be an infinity of *systems of co-ordinates.* An exposition of those systems only is given which are most simple and most used.

RECTILINEAR CO-ORDINATES.

1. Let there be two non-parallel straight lines or fixed axes $X'X$ and $Y'Y$ traced in a plane (Fig. 1); the position of any point M of the plane will be determined by the intersection of the two lines $G'G$, $H'H$ parallel to the axes. The position of the parallel $H'H$ is defined by the segment OP, which it intercepts on $X'X$. It is necessary to indicate the direction in which the length OP is measured. For this purpose it will be convenient to give the sign + to the distance OP, if it is measured on OX, for example; the sign −, if it is measured on OX'. In like manner, the position of the parallel $G'G$ is defined by the length OQ affected with the + sign or the − sign, according as it is measured on OY or OY'.

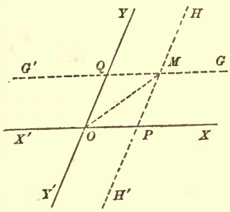

Fig. 1.

The two lengths OP and OQ (each affected with the proper sign), which determine thus the position of the two parallels, and consequently the point M, are the **rectilinear co-ordinates** of M. They are usually represented by the letters x and y. Further, the co-ordinate designated by x is given the name **abscissa**; the other, y, that of **ordinate.** The two fixed right lines $X'X$ and $Y'Y$ are called the **axes of co-ordinates**; the first is the axis of the x's, and the second the axis of the y's. The point O from which we measure the co-ordinates on each axis, in the one direction or in the other, is called the **origin of co-ordinates.**

If all possible values, positive or negative, be assigned to x and to y, — in other terms, if x and y be made to vary from $-\infty$ to $+\infty$, — all points of the plane are obtained; otherwise, each pair of values gives a point, and one only.

The two co-ordinates of the point M are the projections of the line OM, taken in the direction OM, on the axes OX and OY, the projection on each axis being taken parallel to the other. The projection on the axis of x is the length OP, identical with the co-ordinate x, affected with the + sign or − sign, according as it is measured in the direction OX or in the opposite direction OX'; the projection on the axis of y is the length OQ identical with the co-ordinate y, affected with the + sign or − sign, according as it is measured in the direction OY or in the opposite direction OY'.

RECTILINEAR RECTANGULAR CO-ORDINATES.

2. Usually the fixed axes are drawn perpendicularly to each other; in this case, the two co-ordinates of the point M (Fig. 2) are the distances of this point from the two axes: they are also the orthogonal projections of the line OM on the two axes.

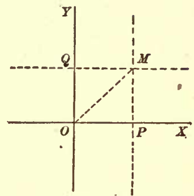

Fig. 2.

POLAR CO-ORDINATES.

3. Let O be a fixed point called the **pole,** OX a fixed axis (Fig. 3). We can determine the position of a point M by the length ρ, the **radius vector** OM, and by the angle ω, which the radius vector makes with the axis.

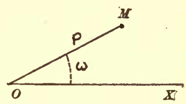

Fig. 3.

The position of the point M is determined by the intersection of a circle of radius ρ, having the pole for center, and the half-line OL drawn from the pole and making the angle ω with the axis OX (Fig. 4); but it is necessary to define the direction in which we reckon the angle ω, namely counter clockwise from the axis OX. All the points of the plane are obtained if ρ vary from 0 to $+\infty$, and ω from 0 to 2π. In fact if, ω remaining constant, one makes ρ vary from 0 to $+\infty$, one has all the points of the half-line OL; if, therefore, ω vary from 0 to 2π, the half-line OL moves from the position OX and describes the entire plane.

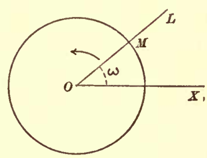

Fig. 4.

BI-POLAR CO-ORDINATES.

4. The position of a point may also be defined by the distances u and v from two fixed points F and F' (Fig. 5). The position of the point M is then determined by the intersection of the two circles described about the points F and F' as centers with the radii u and v. However, this system does not offer the same theoretical perfection as the two preceding; for every couple of values of u and v is not admissible; it is necessary that the distance between the poles be less than their sum and greater than their difference. When this condition is fulfilled, the two circumferences intersect in two points, and a troublesome ambiguity arises.

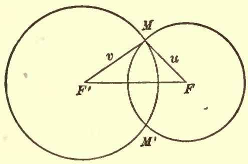

Fig. 5.

The position of the point M may still be determined by aid of the angles MFF', $MF'F$; we designate these angles, reckoned in a definite direction, by α and β; each of them varying from 0 to 2π: to every couple of values of α and β corresponds one point of the plane, and one only.

5. The number of systems of co-ordinates is infinite. In general, the position of a point in a plane is determined by the intersection of two lines traced in this plane. Let A', A'', A''', $\cdots$ (Fig. 6) be a first system of lines of the same kind, corresponding to the several values u', u'', u''', $\cdots$ of the variable u; B', B'', B''', $\cdots$ a second system of lines of the same kind, corresponding to the several values v', v'', v''', $\cdots$ of the variable v; any arbitrary point of the plane is defined by the two lines which meet in this point, and the particular values which it is necessary to give to the variables u and v, in order to determine these two lines, are called the co-ordinates of the point. The totality of these two series of lines constitutes a system of co-ordinates.

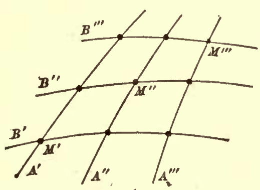

Fig. 6.

In the first system which we have studied, each series of lines is composed of right parallel lines; hence the co-ordinates were given the name rectilinear co-ordinates.

In the polar system, the first series of lines is composed of half-lines emanating from the pole O, and positioned by the variable angle ω, which they make with the axis OX (Fig. 4); the second system of concentric circles described about the point O as center with the variable radius ρ.

In the first bi-polar system each series is composed of concentric circles (Fig. 5). In the second, each series is composed of half-lines emanating from one of the points F or F'.

Representation of Lines in a Plane by Equations.

6. Let AB be any line whatever, straight or curved, in the plane (Fig. 7); draw in the plane two straight lines OX and OY, and designate by x and y the two co-ordinates OP and PM of any point M of the line; when the point M is moved along the line, the two co-ordinates vary simultaneously; if an arbitrary value be assigned to OP, the magnitude of the corresponding ordinate MP is perfectly determined, and the variation of the abscissa controls that of the ordinate. That is, the ordinate MP is a function of the abscissa OP; the *character* of this function depends on that of the line. If the line is defined geometrically, an equation between x and y, serving to define the function y, can be deduced from the geometric definition of the line. The equation which is found in this manner is called the equation of the line.

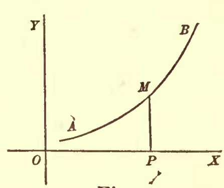

Fig. 7.

7. Conversely, let there be given an equation

$$F(x, y) = 0,$$

between the variables x and y; each pair of real values of x and y, satisfying this equation, determine a point of the plane. Let x_0 and y_0 be a pair of real values of x and y satisfying the equation; if x begin with the value x_0, and vary in a continuous manner, one of the values of y, beginning with y_0, will also vary in a continuous manner, and will in general be real as long as x is restricted to varying between certain limits: the point, of which the co-ordinates are x and y, will describe in the plane a continuous line. Thus, the totality of the real solutions of an equation in two variables is, in general, represented by a line in a plane.

8. What has been said concerning rectilinear co-ordinates is applicable to every other system of co-ordinates. In the polar system, where the point M is on the given line, the radius vector ρ varies with the angle ω; it is a function of ω, and the line is represented by some equation between ρ and ω.

9. The representation of figures by equations is the object of Analytical Geometry, and in it the results of algebraic calculation are applied to their study. In Analytical Geometry the student is occupied with three fundamental questions: given a figure defined geometrically, determine its equation; conversely, given an equation, determine the figure which corresponds to this equation; finally, study the relations which exist between the geometric properties of the figures and the analytic properties of the equations.

The examples which are given in the following chapter will show how lines may be represented by equations.

CHAPTER II

EXAMPLES.

In general, the geometric definition of a curve determining each of the points corresponds to a certain system of co-ordinates; if the particular system implied by the definition be chosen, the equation of the curve is the immediate algebraic translation of its geometric definition.

Circle.

10. *The circle is the locus of all points equally distant from a fixed point called the center.* It is described by means of a compass; one foot being placed at the center, the other will trace the circumference.

If the center O be taken as pole, and any right line OX as polar axis (Fig. 8), and r represent the length of the radius, the equation of the circumference in polar co-ordinates is

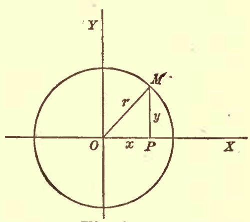

Fig. 8.

(1) $$\rho = r,$$

since the length of the radius vector is constant and equal to r whatever value the angle ω may have.

Let us now seek the equation of the circle in rectilinear co-ordinates. If two rectangular axes OX and OY passing through the center be taken, the right triangle OMP gives immediately the relation

(2) $$x^2 + y^2 = r^2,$$

which exists between the two co-ordinates x and y of any point M of the circumference. This is the equation of the circumference in this system of co-ordinates.

Ellipse.

11. *The ellipse is a curve such that the sum of the distances of each of its points from two fixed points is constant.* The two fixed points are the *foci* of the ellipse.

Let $2a$ represent the sum of the distances of any point of the ellipse to the foci, and $2c$ the distance FF' between them. The points of the ellipse can be constructed by describing a circle with arbitrary radius u about one of the foci as center, and a second circle about the other focus as center with radius v equal to $2a - u$. The points of intersection M and M' of the two circles belong to the ellipse. In order that the two circles intersect, it is necessary that the longest radius be no longer than $a + c$, and the shortest no shorter than $a - c$.

The points M and M' being symmetrical with respect to the line FF', this line is an axis of curve. The right line BB', perpendicular to FF' at its mid-point O, is a second axis.

The points in which the axes cut the curve are called *summits.* The summits A and A' are obtained by taking the distances FA, $F'A'$ equal to $a - c$. The summits B and B', situated on the second axis, are determined by describing a circle with radius a about one of the foci as center of a circle. The distance OA is equal to a, and the distance OB, which is designated by b, is equal to $\sqrt{a^2 - c^2}$. Instead of defining the ellipse by the lengths $2a$ and $2c$, as in the preceding, it can be defined by means of the lengths $2a$ and $2b$; hence, $c = \sqrt{a^2 - b^2}$. The point O, the mid-point of FF', is the center of the curve.

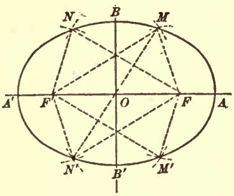

Fig. 9.

12. We now derive the equation of the ellipse. The system of co-ordinates used is the first bi-polar system (§ 4); if the position of each of the points of the plane is determined by its distances from two fixed points F and F', the ellipse will have for its equation,

$$u + v = 2a. \tag{1}$$

In the second bi-polar system the equation has also a very simple form; if α and β represent the two co-ordinate angles MFF', $MF'F$, and $2p$ the perimeter of the triangle MFF', then

$$\tan\frac{\alpha}{2}=\sqrt{\frac{(p-2c)(p-u)}{p(p-v)}},\quad \tan\frac{\beta}{2}=\sqrt{\frac{(p-2c)(p-v)}{p(p-u)}};$$

whence,

(2) $$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{p-2c}{p}=\frac{a-c}{a+c}.$$

13. Finally, the equation in rectilinear co-ordinates is considered. Take the two axes of the curve as axes of co-ordinates (Fig. 10); the lengths PF and PF' being equal to $c-x$ and $c+x$, the right-angled triangles FMP, $F'MP$, give

$$u=\sqrt{y^2+(c-x)^2},$$

$$v=\sqrt{y^2+(c+x)^2}.$$

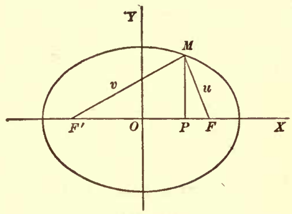

Fig. 10.

By substituting the values of u and v in equation (1), we obtain the equation

(3) $$\sqrt{y^2+(c-x)^2}+\sqrt{y^2+(c+x)^2}=2a.$$

Transposing the first radical to the second member and squaring, gives

$$y^2+(c+x)^2=4a^2+y^2+(c-x)^2-4a\sqrt{y^2+(c-x)^2};$$

or, simplifying,

$$a\sqrt{y^2+(c-x)^2}=a^2-cx.$$

Squaring and transposing lead to the equation,

(4) $$a^2y^2+(a^2-c^2)x^2=a^2(a^2-c^2).$$

However, equation (4) is not equivalent to equation (3); it is equivalent to the four equations

$$u+v=2a,\quad u-v=2a,\quad -u+v=2a,\quad -u-v=2a,$$

which are obtained from equation (3) by changing the sign of the radicals. The equation $-u-v=2a$ has no real solution.

B

The equation $u - v = 2a$, and $-u + v = 2a$, do not have real solutions if one suppose $2a > 2c$; because the quantities u and v represent the distances of the points F and F' from a point in the plane whose co-ordinates are x and y, and the difference of these distances cannot be equal to the length $2a$, greater than the distance $2c$ or FF'. Thus, when real solutions only are considered, the equation (4) can be regarded as equivalent to equation (3). The constant sum $2a$ being greater than the distance between the foci $2c$, one can put $a^2 - c^2 = b^2$, and the equation of the ellipse reduces to the form $a^2y^2 + b^2x^2 = a^2b^2$, or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{5}$$

HYPERBOLA.

14. *The hyperbola is a curve such that the difference of the distances of each of its points from two fixed points is constant.* The two fixed points F and F' are the foci of the hyperbola.

The hyperbola, like the ellipse, has two axes of symmetry, the right line, FF' (Fig. 11), and the perpendicular, BB', to this line at its mid-point O. It is composed of two distinct branches. The points of a branch of a hyperbola can be constructed by describing a circle with an arbitrary radius u about F as a center, and a second circle with a radius v equal to $2a + u$ about F' as a center. In order that these circles intersect, it will be necessary that u be greater than $c - a$. In a similar manner a second branch may be found. The point O, the middle of FF', is the center of the curve.

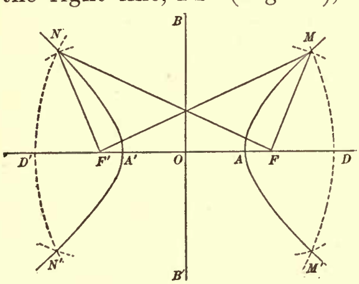

Fig. 11.

The first axis intersects the curve in two points only, namely A and A', which are its vertices and are determined by taking $OA = OA' = a$; this axis is for this reason called the transverse axis.

In the first bi-polar system, if u and v represent the distances of any point of the curve from the foci F and F', the two branches of the curve have respectively for their equations

(1) $$v - u = \pm 2a.$$

In the second bi-polar system the equations of the two branches are

(2) $$\frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}} = \frac{c+a}{c-a}, \quad \frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}} = \frac{c-a}{c+a}.$$

15. If the two axes of the curve are taken as axes of co-ordinates, the equation of the hyperbola in rectangular co-ordinates will be

$$\sqrt{y^2 + (c+x)^2} - \sqrt{y^2 + (c-x)^2} = \pm 2a.$$

By repeating the transformation of (§ 13), one obtains the integral equation $a^2y^2 + (a^2 - c^2)\,x^2 = a^2\,(a^2 - c^2)$, which we have already obtained for the ellipse.

This equation, as has been remarked, is equivalent to four distinct equations $v - u = \pm 2a$, $u + v = \pm 2a$; but in the given case $2a$ being smaller than $2c$ the last two equations have no real solution. Placing $c^2 - a^2 = b^2$ the equation becomes

(3) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

It is well to observe that, in the rectilinear system, the two branches of the hyperbola are embraced in the same equation (3), while in the first bi-polar system, one of the branches is represented by the equation $v - u = 2a$, the other by $u - v = 2a$. It is also necessary to have two distinct equations in the second bi-polar system.

PARABOLA.

16. *The parabola is a curve every point of which is equally distant from a fixed point called the focus and a fixed line called the directrix.*

The perpendicular drawn through the focus to the directrix is an axis of symmetry of the curve. The point A, middle of DF, is the vertex of the parabola. The curve lies wholly to the right of a line drawn through A, parallel to the directrix. Any point of the curve can be found by drawing a line MM' parallel to the directrix, at the distance DP greater than AD and describing a circle with a radius equal to this distance DP about the focus as center (Fig. 12).

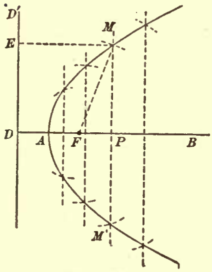

Fig. 12.

17. The definition of the parabola suggests a system of co-ordinates, which has not yet been considered. Any point, M, of the plane can be determined by the distances MF and ME from the fixed point F and fixed line DD' (Fig. 13). The position of the point M will be determined by the intersection of a circle described about F as a center and a right line parallel to DD'. If we call u and v the two co-ordinates of the point M, the parabola will have for its equation, in this system,

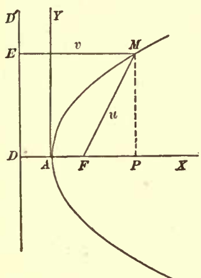

Fig. 13.

$$u = v.$$

18. Let A, the vertex of the parabola, be taken for the origin of rectangular co-ordinates, the axis of the parabola for the x-axis and the perpendicular AY for the y-axis. Represent the distance FD of the focus from the directrix by p: then is

$$(1) \quad v = AP + AD = x + \frac{p}{2}, \quad u = \sqrt{y^2 + \left(x - \frac{p}{2}\right)^2},$$

and the equation of the parabola becomes

$$\sqrt{y^2 + \left(x - \frac{p}{2}\right)^2} = x + \frac{p}{2};$$

or (2)
$$y^2 = 2px.$$

19. Before proceeding further the definition of a tangent to any curve whatever will be given. In elementary geometry, a line is said to be tangent to a circle when it has but one point in common with the circumference, but this definition cannot be generalized and it will be convenient to define a tangent in another manner. Let M be a given point on a curve (Fig. 14); through this point and a neighboring point M' draw an indefinite right line; in the figure which we study, the direction MM' has, in general, a limiting position MT, as the point M' approaches the point M as a limit. The right line MT is called a *tangent* to the curve at the point M. The perpendicular to the tangent at this point M is called the *normal* to the curve.

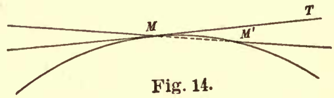

Fig. 14.

From this definition, it follows that the tangent to the circle at the point M is perpendicular to the radius OM at its extremity (Fig. 15); because, in the isosceles triangle MOM', the angle OMM' is equal to a right angle less half the angle MOM'. When the point M' approaches continuously toward the point M, the angle at the center approaches zero, and the angle OMM' becomes right-angled. The normal to the circle in M is the radius MO.

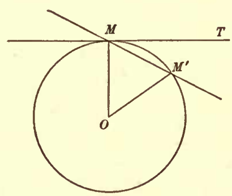

Fig. 15.

Cissoid of Diocles.

20. If one be given a circle, a *diameter* AB, a tangent BC at the extremity of this diameter (Fig. 16), and if a secant AE be made to revolve about the point A, on which a length AM be then equal to the distance DE comprised between the circle and the fixed tangent, the locus of M is a curve which is called the *cissoid.*

If the movable secant start from the position AB and revolves about the point A, from AX toward the perpendicular AY, the length DE, and consequently AM, increase indefinitely, and the point M will describe an infinite branch MM' of the curve. By revolving the movable secant from the other side of AB, a second branch of the curve equal to the first is obtained. The line AB is an axis of the curve, since the two branches are symmetrical to this right line.

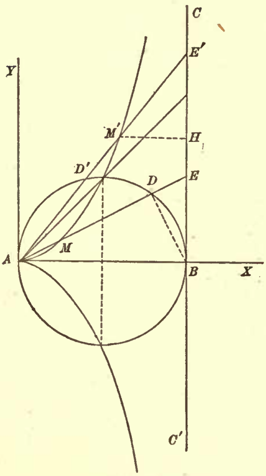

Fig. 16.

The tangents to the two branches at the point A coincide with the axis. Because, if the secant revolves about the point A in such a manner that the chord AM or DE becomes zero, it tends toward the limiting position AB; therefore AB is the tangent at A. The point A is called a cusp or turning point. It is also apparent that the two branches of the curve continually approach the line CC'. In fact, consider the secant in the position AE'; if from the line AE' the equal lengths AM' and $D'E'$ be subtracted alternatively, then will $M'E' = AD'$. The chord AD' diminishes continually and approaches zero; it is equal in length to $M'E'$, therefore for a greater reason does the perpendicular $M'H$ approach zero. The right line CC', which the curve continually approaches, is called the *asymptote.*

The cissoid was conceived by the Greek geometer, Diocles, to solve the problem, to construct two mean proportionals between two given lines.

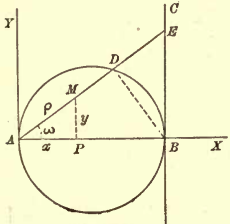

Fig. 17.

21. Let us seek the equation of the cissoid in polar co-ordinates; take the point A as pole and the right line AB for the polar axis. Call a the diameter of the given circle, ρ and ω the co-ordinates of any point M of the curve (Fig. 17). In the right-angled triangles ABE, ABD, one has

$$AE = \frac{a}{\cos\omega}, \quad AD = a\cos\omega;$$

whence, $\rho = DE = AE - AD = \dfrac{a}{\cos\omega} - a\cos\omega = \dfrac{a\sin^2\omega}{\cos\omega}.$

Hence the cissoid has for its polar equation,

(1) $$\rho = \frac{a \sin^2 \omega}{\cos \omega}.$$

Let us now derive the equation in rectangular co-ordinates; take the point A for the origin, the right line AB for the x-axis, and a perpendicular for the y-axis. From the triangle MAP, one gets

$$x = \rho \cos \omega,\ y = \rho \sin \omega,\ \rho^2 = x^2 + y^2;$$

if, in equation (1) $\cos \omega$ be replaced by $\frac{x}{\rho}$, $\sin \omega$ by $\frac{y}{\rho}$, it becomes $\rho^2 x = ay^2$, then ρ^2 by $x^2 + y^2$, the equation of the cissoid in rectangular co-ordinates will be

(2) $$y^2(a - x) - x^3 = 0.$$

22. Having already derived the equation of the cissoid in rectangular co-ordinates from its geometric definition, it is proposed to construct the curve from its equation. Solving the equation (2) with respect to y, one has

$$y = \pm x \sqrt{\frac{x}{a - x}}.$$

The ordinate is real for all values of the abscissa comprised between $x = 0$ and $x = a$ and for those values only; therefore the curve lies wholly between the y-axis and the parallel CC' erected at the distance a (Fig. 16). As x increases from 0 to a, the numerical value of y increases from 0 to ∞, which determines a branch of the curve beginning at the origin A and ascending indefinitely. At the same time the distance $M'H = a - x$ of a point on the curve to the line BC approaches zero, which shows that the line BC is an asymptote of the curve. Since to each value of x there corresponds two equal values of y opposite in sign, the curve is composed of two branches symmetrical with respect to the axis AX.

Strophoid.

23. A right angle YOX (Fig. 18) and a fixed point A on one of the legs, being given in a plane, draw from the fixed point *A* any line *AD*, which cuts the side *OY* in *D*, and beginning at *D*, lay off to the one side and to the other on this line, the lengths *DM* and *DN* equal to *OD*; the locus of the points *M* and *N* is the *strophoid.*

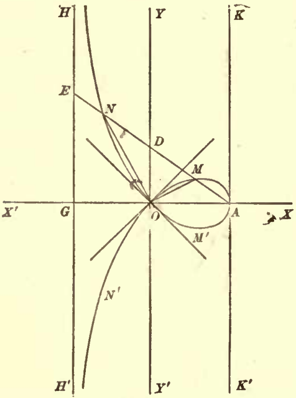

Fig. 18.

When the movable line occupies the position *AO*, the two points *M* and *N* fall together in *O*. If the line moves so that the point *D* ascends continually on *OY*, *OD* increases and the point *N* describes the infinite branch *ON* of the curve. The point *M* approaches continually the point *A*, because the points *M* and *N* are obtained by describing a circle with radius *DO* about *D* as a center; as the point *D* recedes continually from *O*, the angle *OAM* approaches a right angle and the point *M* coincides with *A*. The curve has evidently another branch symmetrical to the first with respect to the axis *OX*.

The point *O*, in which the two branches of the curve cross, is a double point. The tangents to the two branches of the curve at this point coincide with the bisectors of the angles *YOX* and *YOX'*. Because the angle *ODE*, exterior to the isosceles triangle *DOM*, is equal to the sum of the two opposite interior angles and, consequently, to two times the angle *DOM*; similarly the angle *ODA* is equal to two times the angle *DON*. As the line *AD* approaches *OA*, the obtuse angle *ODE* decreases and tends toward a right angle; therefore the half angle *YOM* decreases and tends toward $\frac{\pi}{4}$. The acute angle *ODA* increases and tends toward a right angle; hence the half angle *YON* increases and approaches $\frac{\pi}{4}$ as its limit. Whence it follows that *OM* and *ON* are perpendicular to each other in their limiting positions as tangents at *O*. It is to be noticed further that the arc *OMA* is below, while the arc *AON* is above its tangent.

The tangent at A is perpendicular to the axis OX, because as the point D is continually elevated, the chord AM becomes finally perpendicular to OX.

On AO produced take $OG = OA$, and at the point G erect the perpendicular HH'. This line is an asymptote to each of the infinite branches of the curve, for the distance NE, equal to AM, approaches zero.

24. To derive the equation of the curve in polar co-ordinates, take the point O as pole and the line OA as polar axis; the polar co-ordinates of the point M are $\rho = OM$, $\omega = MOA$; in the isosceles triangle DOM, each of the angles DOM, DMO is equal to $\frac{\pi}{2} - \omega$, and the angle ODM to 2ω; the angle OAM, complement of the preceding, equals $\frac{\pi}{2} - 2\omega$.

If a represent the length OA, it follows from the triangle OMA that

$$\frac{\rho}{a} = \frac{\sin\left(\frac{\pi}{2} - 2\omega\right)}{\sin\left(\frac{\pi}{2} - \omega\right)},$$

whence

$$(1) \qquad \rho = \frac{a\cos 2\omega}{\cos\omega}.$$

The co-ordinates of the point N satisfy the same equation.

To derive the equation of the curve in rectangular co-ordinates, take for axes the two lines OX and OY. If, in the preceding equation, put under the form $\rho\cos\omega = a(\cos^2\omega - \sin^2\omega)$, $\cos\omega$ and $\sin\omega$ be replaced by their values $\frac{x}{\rho}$, $\frac{y}{\rho}$, one gets $x\rho^2 = a(x^2 - y^2)$; putting in the place of ρ^2 its value $x^2 + y^2$, the following equation of the third degree is obtained:

$$(2) \qquad x(x^2 + y^2) - a(x^2 - y^2) = 0.$$

25. The curve can now be constructed by means of its equation in rectangular co-ordinates. Equation (2), solved with respect to y, becomes

$$y = \pm x\sqrt{\frac{a - x}{a + x}}.$$

In order that the ordinate y be real, it is necessary that the quantity under the radical be positive. If x be given positive values, the denominator being positive, the numerator will also be positive so long as x is less than a. If x be given negative values, the numerator being positive, the denominator will be positive so long as the absolute value of x is less than a. Thus the abscissa x can vary from $-a$ to $+a$. If, therefore, we begin at the origin and lay off on the x-axis, to the right and to the left, the distances OA and OG equal to a, and at the points G and A erect HH', KK' parallel to the y-axis, the curve will be wholly comprised between these two parallels. The form of the curve will vary in accordance with the variation of the function

$$y = x\sqrt{\frac{a-x}{a+x}}.$$

As x varies from 0 to a, the ordinate y takes finite values. It is zero for $x = 0$ and also for $x = a$. This furnishes the branch OMA of the curve, beginning at the point O and ending in the point A. As x varies from 0 to $-a$, the ordinate y is negative and varies from 0 to $-\infty$. This furnishes the branch ON', which begins at the origin and descends continually, approaching indefinitely the line HH', which is an asymptote. This branch ON' is a continuation of the branch AMO.

By changing the sign of the radical, the branch $AM'ON$, symmetrical to the first with respect to the x-axis, is obtained.

Limaçon of Pascal.

26. Through a point A on a circle, draw any secant AD, on which beginning at D, where it cuts the circle again, lay off a constant length DM or DN; the locus of the points M and N (Fig. 19) is a curve called the *Limaçon of Pascal.*

The entire curve will be traced by supposing the radius vector to coincide with the diameter AB of the circle and then to revolve through an angle π in either direction. The whole curve will also be traced by giving the radius vector a complete revolution, and laying off a constant length in the direction of radius vector, beginning with the point in which this radius vector or its prolongation intersects the circle. The curve takes three different forms according as the constant length a is greater, equal to, or less than the diameter b of the circle.

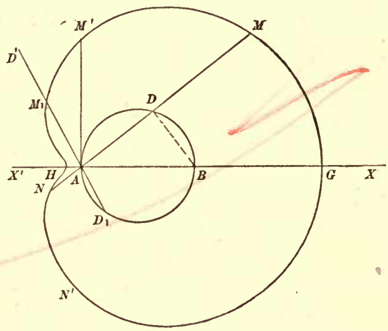

Fig. 19.

1° The first case considered is when the length a is greater than b. If the radius vector coincides with AB, it will be necessary to start from the point B. Construct on AB a length BG equal to a, which determines the point G of the locus (Fig. 19). If the radius vector revolves from the point A and takes the direction AD, the point M is determined. When the radius vector has revolved through a right angle, the point D coincides with A, and the point M with M'. Continuing the revolution of the radius vector to the position AD' and prolonging it, it intersects the circle in D_1; it is necessary to start from this point D_1, and lay off in the same direction AD', a length D_1M_1 equal to a. When the radius vector having revolved through the two right angles occupies the position AX', the point D_1 coincides with B and the point M_1 with H; thus the arc, $M'M_1H$, a continuation of GMM', is constructed, and is, moreover, exterior to the circle. The radius vector revolving beyond AX' through two more right angles returns to its initial position AX; the moving point describes the arc $HN'G$, symmetrical to the arc GMH, with respect to the line XX'. In this manner, by a continuous movement, the point describes the entire curve.

2° Suppose that the length a be equal to b. When the radius vector, starting from the initial position AX, moves through two right angles, the point M describes the arc $GMM'A$ (Fig. 20), which ends in the point A. The tangent at A is the right line AX', limiting position of the secant AM_1. The point A is called a cusp.

3° As the last case to be considered, let a be less than b. When the radius vector, starting from the initial position, has

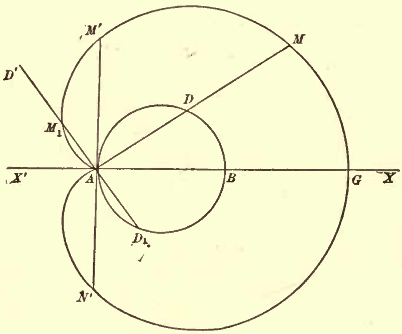

Fig. 20.

revolved through one right angle, the point M describes the arc GMM' (Fig. 21). If now the radius vector takes the direction AD', the point D moves to D_1, the point M falls in M_1. But when the radius vector assumes a direction AD'', such that the chord D_2A is equal to a, the point M_1 falls then in A,

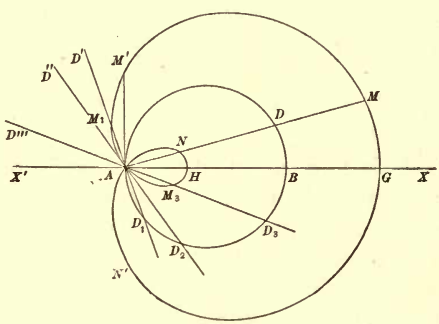

Fig. 21.

and the curve will be tangent to the right line AD''. As the radius vector continues its motion, the chord D_3A becomes greater than a, and, if the length D_3M_3 is taken equal to a, one has a point M_3 situated within the circle. Finally, as the radius vector takes the direction AX', the point M_3 falls in H. Thus the interior arc AM_3H is the prolongation of the exterior arc $GM'A$. The other half of the rotation gives the arc $HNAN'G$ symmetrical to the first with respect to the line $X'X$, and completes the curve.

27. Let us obtain now the equation of the curve in polar co-ordinates. Take the point A as pole, and the line AX as polar axis. Call ω the angle which the radius vector makes with the direction AX. When the radius vector has the position of the line ADM, the right triangle ADB gives $AD = b \cos \omega$, and, consequently,

$$\rho = DM + AD = a + b \cos \omega.$$

When the prolongation of the radius vector intersects the circumference, as is the case in the position AD', the angle ω is the angle XAD'; the right triangle BAD_1 gives

$$D_1A = - b \cos \omega.$$

and hence,

$$\rho = D_1M_1 - D_1A = a + b \cos \omega.$$

But, if the radius vector, in Fig. 21, has the direction AD''', the length is measured in the opposite direction to AD'''. Therefore, the radius vector of the point M will be AM_3 affected with the $-$ sign; whence, one has

$$\rho = - AM_3 = D_3M_3 - AD_3 = a + b \cos \omega.$$

Thus, the entire curve is, in any case, represented by the equation

$$(1) \qquad \rho = a + b \cos \omega.$$

If the point A is taken as origin, the diameter AB as axis of x, and a perpendicular to it at A as the axis of y, then the equation of the curve in rectilinear co-ordinates will be

$$(2) \qquad (x^2 + y^2 - bx)^2 = a^2 (x^2 + y^2).$$

Equation (2) is derived from (1) by putting $\frac{x}{\rho}$ for $\cos \omega$ and $x^2 + y^2$ for ρ^2 (§ 20), and squaring in order to remove the radical.

28. The same curve may be obtained by another process. Being given a circle GH and a fixed point A, think of a movable tangent CM revolving on the circumference of the circle, and drop from the point A a perpendicular AM on this tangent (Fig. 22); find the locus of the point M. There will be three cases to consider, according as the point lies within, on, or without the circumference of the circle. Suppose, for example, that the point A lies without the circumference. When the tangent touches the circumference at G, the perpendicular from A coincides with the diameter AG and the point G is a point of the locus. As the tangent revolves about the quadrant GCC', the point M describes the arc GMM' of the curve. When the tangent descends to the position $C''A$, the point M describes the arc $M'A$. The tangent continuing its motion along $C''H$, the foot of the perpendicular falls below the diameter and describes the arc ANH of the curve. The tangent has revolved about the semi-circumference $GC'H$; when the tangent revolves about the lower semi-circumference, the point M will trace a portion of the curve symmetrical to the first half.

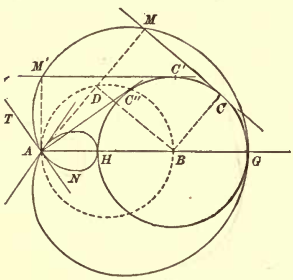

FIG. 22.

In order to get the equation of this curve in polar co-ordinates, represent the radius of the given circle by a, the distance AB by b, and draw through the center B, of the circle, a line BD parallel to the tangent CM. Whence it follows

$$\rho = AD + DM = b \cos \omega + a.$$

This equation is identical with equation (1) of § 26; therefore the curves which they represent are identical.

Moreover it is easy to verify geometrically this identity. The angle D being a right angle, the locus of the point D is the circumference described on AB as a diameter. The point M will therefore be obtained by prolonging the chord AD till DM is equal to BC.

THE ROSE OF FOUR BRANCHES.

29. Being given two lines OX and OY at right angles to each other, on which the extremities of a right line PQ of constant length are free to move, find the locus of the foot of the perpendicular OM drawn from O to PQ (Fig. 23). When the line PQ coincides with OY, the point M coincides with O and the chord OM takes the direction OX; therefore the tangent to the arc OM at O coincides with OX. The point I, the mid-point of PQ, describes a circumference of which the center is O, and the radius equal to a; if the constant length be represented by $2a$, the perpendicular OM is less than the oblique line OI; therefore the distance OM is a maximum when the right line PQ is perpendicular to the bisector OA. As the movable line PQ continues its motion, it will assume a position $P'Q'$ symmetrical to PQ with respect to the bisector OA, and one finds the arc $OM'A$ the *symmetrique* of the arc OMA with respect to OA. The same curve is reproduced in each of the other right angles. Hence the curve has four axes, the two fixed right lines OX, OY, and the two bisectors $A'A$, $B'B$. The point O is the *center* of the curve.

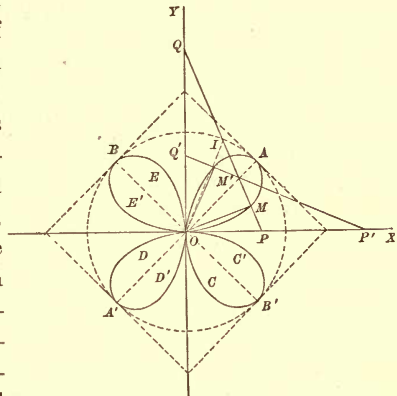

Fig. 23.

30. If the point O be taken as pole and OX as the polar axis, it follows from the right-angled triangles OMP, OPQ, that

$$\rho = OP \cos\omega, \quad OP = 2a\sin\omega; \quad \text{therefore}$$

$$(1) \qquad \rho = a \sin 2\omega.$$

In rectangular co-ordinates the curve is represented by an equation of the sixth degree

$$(2) \qquad (x^2+y^2)^3 - 4a^2x^2y^2 = 0,$$

which follows at once from equation (1) by substituting respectively for $\cos \omega$, $\sin \omega$ and ρ, $\frac{x}{\rho}$, $\frac{y}{\rho}$, and $\rho = \sqrt{x^2 + y^2}$, squaring and transposing.

TANGENTS.

31. The preceding examples show how to construct a curve from its geometric definition and to derive finally its equation. It is possible also in some cases to deduce from the geometric definition of a curve a simple construction of a tangent to it. Two remarkable examples will be given, — the curves described by the various points of a plane figure, which moves in a plane, and the locus of the feet of perpendiculars drawn from a fixed point to the tangents to a given curve. The construction belonging to the first class of curves depends on the following proposition:

LEMMA. — *Every plane figure can be brought from one position to another in its plane by a rotation about a fixed point.*

It is first to be noted that the position of a plane figure in a plane is determined when one knows the position of two of its points. Let A and B be the two points of the figure in its first position (Fig. 24), A' and B' the same points in a second position; the line AB, of constant length, is transferred to $A'B'$. Erect perpendiculars to AA', BB' at their midpoints; the perpendiculars intersect in a point I. The two triangles AIB, $A'IB'$ are equal, since their sides are equal each to each, AB equal to $A'B'$, IA and IA' are equal, being oblique lines drawn from a point in a perpendicular cutting off equal distances from its foot, and similarly IB and IB' are equal; therefore the two angles AIB and $A'IB'$ are equal; by subtracting the common angle $A'IB$, it follows that the angles AIA', BIB' are equal. Suppose now that the figure is revolved about the fixed point I, through the angle AIA', the radius IA will fall on IA' and the point A on A'; in the same manner

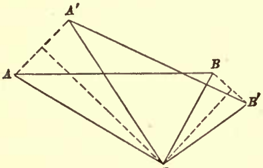

Fig. 24.

the line IB, revolving through the angle BIB' equal to AIA', will fall on IB' and the point B on B'. Therefore this rotation about the point I brings the figure from its first position to the second.

32. THEOREM. — *If one considers the curves described by the different points A, B, C, ⋯, of an invariable plane figure which moves in a plane, the normals to these curves, at points which correspond to the same position of the figure, intersect in the same point.*

Suppose A, B, C, ⋯, to be different points of the figure in any position whatever, A', B', C', ⋯, these same points in a new position. After what has been said, we can bring the figure from the first position to the second by revolving it about a certain point I_1; by this motion the lines I_1A, I_1B, I_1C ⋯, describe angles respectively equal, and finally coincide with I_1A', I_1B', I_1C', ⋯ (Fig. 25). The lines MI_1, NI_1, PI_1, ⋯, perpendicular to the chords AA', BB', CC', ⋯, at their mid-points, all intersect in the point I_1.

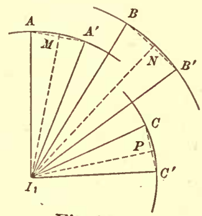

Fig. 25.

Suppose now that the second position approaches continually the first, and that the point I_1 tends toward a limiting position I; the chords AA', BB', CC', ⋯, prolonged, become tangents to the curves in A, B, C, ⋯ ; the perpendiculars MI_1, NI_1, PI_1, ⋯, to the chords coincide with the perpendiculars to the tangents at A, B, C, ⋯ ; that is, with the normals to the curves. Hence the normals to the curves described by A, B, ⋯, at these points all intersect in the same point I.

COROLLARY. — If one could draw the normals to the curves described by the two points A and B of the movable figure, these two normals determine by their intersection the point I; by joining the point I to any third point C, one will have a normal to the curve described by C; a perpendicular to the normal at C will be a tangent. This is the case if the two points describe straight lines or circumferences of circles. In the next section some applications of this method will be given.

33. In case two points of the movable figure describe right lines, it will be shown that the curve described by any other point is an ellipse; the preceding method will enable one to construct a tangent to the ellipse.

Suppose, then, that the two extremities of a straight line CD of constant length lie on two lines OX, OY at right angles to each other, find the locus traced by a point M of this line (Fig. 26).

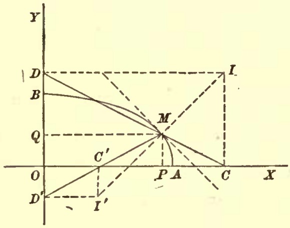

Fig. 26.

When the line CD coincides with OX, the point M will fall on A at a distance OA equal to DM; as the extremity D slides along OY starting from the point O, the point C approaches the point O, and the point M describes the arc AMB. When the line CD coincides with OY, the point M will fall on B at a distance MB equal to CM. The same arc is reproduced in each of the four right angles, and the curve thus described is an ellipse.

For, take the two fixed lines OX, OY as axes of co-ordinates, and call a and b the two constant lengths DM and CM, x and y the co-ordinates of the point M, then the similar triangles MPC, DQM give

$$\frac{MP}{DQ}=\frac{CM}{DM}, \text{ or } \frac{y}{\sqrt{a^2-x^2}}=\frac{b}{a}, \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

which is equation (5) of § 13. Thus, the curve is an ellipse whose axes $2a$ and $2b$ coincide with the two given rectangular axes.

34. It is not necessary that the point M be restricted to lying on the movable line between the points C and D; it can be situated on the prolongation. Consider the line $C'D'$ of which the two points C' and D' slide on the two perpendicular lines OX and OY, and find the locus described by the point M. If a and b be put for the distances $D'M$ and $C'M$, the similar triangles MPC', $D'QM$ will give, as in the preceding,

$$\frac{MP}{D'Q}=\frac{MC'}{MD'}, \text{ or } \frac{y}{\sqrt{a^2-x^2}}=\frac{b}{a}.$$

The construction of a small instrument called an *elliptical compass* depends upon this property.

The two feet are placed on the points C and D, taken at wish on the line CD, and a pencil point at the point M; the two feet slide in grooves placed on the perpendicular lines OX and OY; the pencil point M describes by a continuous movement an ellipse.

It is evident that a straight line is its own proper tangent. The points C and D of the movable plane describe the lines OX and OY; the perpendiculars CI and DI to these lines determine the point I through which pass the normals to the curves described by the various points of the movable figure for every position of this figure. The line IM is therefore normal to the ellipse at M described by this point; the line drawn through M perpendicular to the normal to the ellipse at that point will be a tangent.

35. Imagine two points E and F of the movable plane to slide on any two fixed straight lines OA and OB (Fig. 27). The perpendiculars to these lines at the points E and F determine the point of intersection I of the normals. The circle described on OI as a diameter passes through the points E and F; the line EF and the angle EOF being constant, the diameter of the circle is constant. Suppose that the circle is situated in the movable plane, and controlled

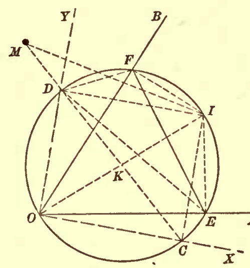

Fig. 27.

in its movement by the motion of the line EF; this circle will always pass through the point O; every point D of the circumference will describe a straight line OY, since the inscribed angle FOD, which corresponds to the constant arc FD, is itself constant.

Consider any point M of the movable plane; draw a line through this point and the center K of the circle; the two points, C and D, the extremities of the diameter MK, describe two perpendicular lines OX and OY; whence it follows that the point M describes an ellipse whose axes are two times the distances CM and DM, and have the same directions as OX and OY. The line IM is normal to the ellipse at M.

Conchoid.

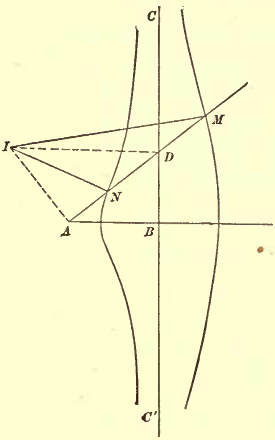

Fig. 28.

36. Being given a point A and a straight line CC', draw through the point A any secant AD, and, beginning at the point D where it meets the line CC', lay off on either side a given length DM and DN; the locus of the points M and N is the *conchoid* (Fig. 28). It can easily be seen that this curve has two infinite branches, one on each side of the line CC', and asymptotic to this line. The left branch will have different forms according as the given length DM is less, equal to, or greater than the perpendicular drawn from A to the line CC'.

This curve belongs to the preceding category: one can regard, in fact, the line AD as revolving in the plane, in the following manner, one of its points D describes the line CC', while the line itself passes through the point A, about which it revolves; a point M of this line describes a branch of the conchoid. Consider the point of the movable line which is in A, when the line occupies the position AD; this point describes a branch of the conchoid passing through the point A and tangent to the line AD in this point; the normal to this particular branch of the curve is the line AI, perpendicular to AD. The normal to the curve described by the point D is the line DI, perpendicular to the line CC'; by drawing a straight line from the point of intersection I of the two normals to the point M, one obtains the normal IM to the curve described by the point M; the perpendicular to IM at M is a tangent to the curve.

The limaçon (§ 26) is a curve analogous to the conchoid; it is sufficient to replace the line CC' on which the point D slides, by the circumference of a circle (Fig. 29). Consider then the point of the movable line which is in A, when the line occupies the position AD. This point describes a curve passing through the point A and tangent to the straight line AD at this point; the normal to this curve is the perpendicular AI. The normal to the circumference described by the point D is the diameter DI; the point of intersection I of the normals is therefore the extremity of the diameter which passes through the point D; the straight line IM is a normal to the curve described by the point M.

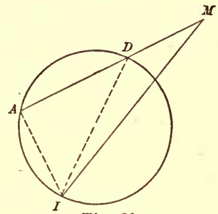

Fig. 29.

37. The same construction is also applicable to the cissoid and strophoid; but it is necessary beforehand to give these curves another geometrical definition. Consider a right angle ABC (Fig. 30), of which a side BA passes through a fixed point A, and a point C on the other side slides on the line EE'; it is further supposed that the length BC is equal to the distance AO of the point A from the line EE'; the point M, mid-point of BC, describes a cissoid, and the vertex B of the right angle a strophoid.

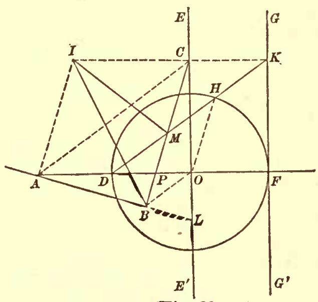

Fig. 30.

In fact, the two right triangles ABC, AOC being equal, the angles CAL, ACL are equal, and the triangle ALC isosceles; since AB is equal to CO, one has also $LB = LO$; therefore the locus of the point B is a strophoid (§ 23).

The triangle ACP is also isosceles; join the point M to the mid-point D of AO and prolong this line till it is intersected in K by CK, drawn parallel to AO. The triangle CMK being isosceles, it follows that $CK = CM = AD$. Finally, describe a circumference about the point O as a center with a radius OD; let F be the extremity of the diameter DO and H the point of intersection of the circumference with DK. The isosceles triangle MCK is equal to DOH, $DH = MK$, whence $DM = KH$; moreover, the line FG is tangent to the circumference at F. Therefore the locus of the point M is a cissoid having the point M for vertex and the line GG' for an asymptote (§ 20).

Consider now the point of the movable figure which is in A, when the right angle occupies the position ABC; this point describes a curve passing through the point A and tangent to the line AB at this point; the line AI, perpendicular to AB, will be a normal to this curve. Further, the line CI, perpendicular to EE', is a normal to the curve described by the point C; the point of intersection I of the two normals is the common point of intersection of all the normals. Therefore the lines IB and IM are normals, the one to the strophoid, the other to the cissoid.

Pedals.

38. The pedal of a given curve AB is the locus of the foot P of the perpendicular dropped from a fixed point O upon any line MP tangent to this curve (Fig. 31). A neighboring tangent $M'P'$ will give a second point P' of the pedal. Let D be the point of intersection of these two tangents; the circle described on OD as a diameter passes through the points P and P', and the line PP' is a secant of the circle. Suppose now that the point M' approaches indefinitely the point M, the point D will ultimately coincide with M,

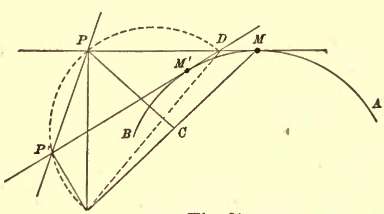

Fig. 31.

and the diameter OD with OM; the secant PP' will at the same time become tangent to the circle and to the pedal; the normal to the pedal will therefore coincide with the normal to the circle constructed on OM as a diameter, and this normal may be found by joining the point P to the point C, the mid-point of OM.

This construction may also be applied to the limaçon, which is the pedal of a circle (§ 28). But it is seen later (§ 307) that the construction of the tangents to pedals is reduced to the general method sketched in § 32.

EXERCISES.

1. A variable triangle ABC, whose vertex A is fixed and the angle A constant, is inscribed in a given circle. Show that the locus of the center of the circle inscribed in and escribed about the triangle is represented by two limaçons.

2. Show that the locus of the vertices of angles of given magnitude, whose sides are tangents to two given circles, is represented by two limaçons.

3. A variable circle touches a given circle in a given point, and a tangent is drawn common to the two circles. Show that the locus of the point of contact of this tangent with the variable circle is a cissoid.

4. A variable plane moves in a fixed plane in such a manner that two straight lines of the variable plane remain respectively tangent to two circles of the fixed plane. Show that a point on the fixed plane traces an ellipse on the movable plane.

5. Construct the curves which, in the first system of bi-polar co-ordinates, are defined by the equation $u + nv = a$. Show that of the three equations $u + nv = a$, $u - nv = a$, $-u + nv = a$, in which the two constants a and n have the same values, two alone define geometrical loci. These loci are closed curves, the one within the other; one calls them the conjugate ovals of Descartes. They are represented by the same integral algebraic equations in rectangular co-ordinates. On the line which passes through the two poles there exists a third point, such that by taking this point and one of the first as poles, the equation preserves its form.

6. If, being given two circles, any secant be drawn through a fixed point taken on the line of centers, and each center be joined to one of the points of intersection of the secant with the circle, show that the point of intersection of these two lines describes the ovals of Descartes.

7. The projection of the curve of intersection of two cones of revolution, whose axes are parallel to a plane perpendicular to the axes, is a system of the ovals of Descartes.

8. Construct the curve which, in the first bi-polar system, is represented by the equation $u \cdot v = a^2$, $2a$ being the distance between the two poles. This curve is called the *lemniscate.*

9. Find the locus of the vertex of a triangle whose base a remains fixed, and in which the other two sides b, c, and the corresponding median m satisfies the relation $b - c = m\sqrt{2}$ (*lemniscate*).

10. A straight line and a circumference each revolve with a uniform motion about a fixed point common to the two lines, the ratio m of the two angular velocities, is affected with the $+$ or $-$ sign, according as the rotations are in the same or opposite sense; required to find the locus described by the second point of intersection of the two lines.

Discuss the following particular cases:

$m = \frac{3}{4}$, or $m = \frac{3}{2}$, the limaçon of Pascal;

$m = -1$, or $m = \frac{1}{3}$, rose of four branches;

$m = -\frac{1}{2}$, or $m = \frac{1}{4}$;

$m = 2$, or $m = \frac{2}{3}$.

11. Solve the same problems, taking for the revolving curves two equal circumferences which revolve about a fixed point common to them.

Discuss the cases:

$m = 2$, limaçon of Pascal;

$m = 3$, rose of four branches;

$m = -2$;

$m = -3$.

CHAPTER III*

CONCERNING HOMOGENEITY.

39. Definition. — The function $f(a, b, c, \cdots)$ is said to be homogeneous with respect to the letters $a, b, c, \cdots$, when, on replacing a by ka, b by kb, $\cdots$, one has

$$f(ka, kb, kc, \cdots) = k^m f(a, b, c, \cdots);$$

the exponent m being the degree of the homogeneous function.

The following are examples of such functions:

$$a^2 + 2ab, \quad \frac{a\sqrt{b} + b\sqrt{c}\sin\frac{c}{a}}{a+b}, \quad \frac{a+\sqrt{ab}}{a+c}, \quad \frac{a}{a^3+b^3},$$

the degree of the first is 2, of the second $\frac{1}{2}$, of the third 0, of the fourth -2.

One can easily see:

1° That the sum or difference of two homogeneous functions of the same degree is a function of the same degree as the given function;

2° That the product of several homogeneous functions of any degree whatever is a function whose degree is equal to the sum of the degrees of the given functions;

3° That the quotient of two homogeneous functions is a homogeneous function whose degree is equal to the excess of the degree of the dividend over that of the divisor;

4° That the power of a homogeneous function is a homogeneous function whose degree is equal to the degree of the given function times the exponent of the power;

5° That the root of any homogeneous function is a homogeneous function whose degree is equal to the degree of the given function divided by the index of the root;

6° That a transcendental function of a homogeneous function of degree 0, is a homogeneous function, and of the degree 0. For example, the functions

$$\sin\left(\frac{ab}{a^2+b^2}\right),\quad \log\left(\frac{b+\sqrt{a^2+b^2}}{a+b}\right)$$

are homogeneous and of the degree 0; because if a and b are replaced by ka and kb, the letter k disappears under the transcendental sign. But if the quantity placed under the transcendental sign, though homogeneous, were not of the degree 0, the letter k could not be removed from under the transcendental sign and the function would not be homogeneous.

Thus, the function $\sin(a+\sqrt{bc})$ is not homogeneous, for here $\sin(ak+\sqrt{bck^2}) = \sin(a+\sqrt{bc})k$.

When a monomial is rational and integral with respect to the letters $a, b, c, \cdots$, the degree of the monomial with respect to a letter will be the exponent of this letter in the monomial: the degree of the monomial with respect to several letters is the sum of the exponents of these letters. A monomial is always a homogeneous function, of a degree equal to the degree of the monomial; therefore the sum of several monomials of the same degree is a homogeneous polynomial of this same degree. For example, the polynomial

$$a^3 - 4\,a^2b + 5\,ab^2 - 2\,b^3$$

is a homogeneous function of the third degree, with respect to the letters a and b.

40. In seeking the relations which exist between the lengths of the various lines $A, B, C, \cdots$, of a figure, one thinks of these lines as being expressed in terms of a unit of length, which usually is not specified and remains to be chosen at will. Represent by $a, b, c, \cdots$, the numbers which thus express the measures of the lines of the figure and suppose that one has found between the numbers the relation

$$(1) \qquad f(a, b, c, \cdots) = 0.$$

The steps made in arriving at this result being independent of the unit of length, it is evident that this relation exists whatever be the unit of length. Call $\alpha, \beta, \gamma, \cdots$, the particular values of $a, b, c, \cdots$, for the first unit; $\alpha', \beta', \gamma', \cdots$, the values of these same quantities for another unit; the two sets of numbers satisfy the relations

(2) $$f(\alpha, \beta, \gamma, \cdots) = 0,$$

(3) $$f(\alpha', \beta', \gamma', \cdots) = 0.$$

But as the unit is changed the numbers vary proportionately, of the sort that, if k designate the ratio of the first unit to the second, one has

$$\frac{\alpha'}{\alpha} = \frac{\beta'}{\beta} = \frac{\gamma'}{\gamma} = \cdots = k;$$

whence $$\alpha' = k\alpha, \quad \beta' = k\beta, \quad \gamma' = k\gamma \cdots.$$

If these values are substituted in the relation (3) it becomes

(4) $$f(k\alpha, k\beta, k\gamma, \cdots) = 0.$$

Consider that the first unit is fixed and the second varies; $\alpha, \beta, \gamma, \cdots$, will be constant numbers and equation (4) will be satisfied whatever this number k may be.

Thus, *if equation* (1) *is satisfied when the letters* $a, b, c, \cdots$, *are replaced by* $\alpha, \beta, \gamma, \cdots$, *it will be satisfied when the letters are replaced by* $k\alpha, k\beta, k\gamma, \cdots$, *whatever the number* k *may be.*

41. The preceding condition is evidently fulfilled when the first member of equation (1) is a homogeneous function of the letters $a, b, c, \cdots$; because then one has

$$f(k\alpha, k\beta, k\gamma, \cdots) = k^m f(\alpha, \beta, \gamma, \cdots);$$

if the expression $f(\alpha, \beta, \gamma, \cdots)$ is zero, the same will be true of $f(k\alpha, k\beta, k\gamma, \cdots)$ whatever k may be.

Conversely, in order that the previous condition be fulfilled, it is necessary that the equation be homogeneous. The only case considered here is that in which the equation is algebraic.

Suppose that $f(a, b, c, \cdots)$ be an integral polynomial; if all the terms are not of the same degree, there will be groups of them which will be of the same degree; call $\phi(a, b, c, \cdots)$ the collection of all the terms of the degree m, the highest, $\psi(a, b, c, \cdots)$ the collection of all terms of the next degree n, etc., the equation (4) becomes

$$k^m\phi(\alpha, \beta, \gamma, \cdots) + k^n\psi(\alpha, \beta, \gamma, \cdots) + \cdots = 0.$$

In order that this equation be verified, k being arbitrary, it is necessary that there exist separately

$$\phi(\alpha, \beta, \gamma, \cdots) = 0, \quad \psi(\alpha, \beta, \gamma, \cdots) = 0, \cdots.$$

If the unit to which the numbers α, β, γ, $\cdots$, are referred is arbitrary, then there must exist between the lines of the figures the homogeneous relations

$$\phi(\alpha, \beta, \gamma, \cdots) = 0, \quad \psi(a, b, c, \cdots) = 0, \cdots.$$

Therefore, *if equation* (1) *is not homogeneous it is equivalent to several equations, separately homogeneous.*

42. It can happen that a homogeneous equation may be satisfied, where a particular unit has been chosen, without the parts which compose it being zero separately; however, if the unit be changed, the equation will no longer be satisfied.

This is illustrated by the example: Determine the dimensions of a cylinder whose total surface shall be equivalent to that of a sphere of radius A and its volume to that of a sphere of radius B.

Let X be the radius and Y the height of the cylinder; call a, b, x, y, the measure of the lines A, B, X, Y, referred to any unit whatever; the unknown quantities will satisfy the two equations:

(5) $$2x^2 + 2xy - 4a^2 = 0,$$

(6) $$x^2y - \tfrac{4}{3}b^3 = 0.$$

Each of these equations is homogeneous; the one is of the second degree, the other is of the third. If they are satisfied when the lines are measured in a certain unit, the same will be true when they are referred to another unit.

The unknowns x and y satisfy also the non-homogeneous equation

(7) $$(2x^2 + 2xy - 4a^2) + (x^2y - \tfrac{4}{3}b^3) = 0,$$

which is obtained by adding equations (5) and (6) member to member.

Consider now equation (7), disregarding its origin. Four lines, A, B, X, Y, can be found such that, if they be measured in a particular unit, the numbers obtained verify this equation without annulling separately the two parts. Suppose, for example, that the lines referred to a first unit have for measures the four members $a = 1$, $b = 3$, $x = 2$, $y = 4$, of which three have been taken arbitrarily and the fourth determined by equation (7); if the lines are measured in a unit half as large, then one gets the numbers twice as large, $a = 2$, $b = 6$, $x = 4$, $y = 8$, which do not satisfy the equation. The cylinder constructed with the lines X and Y thus determined enjoys the property, that the sum of the numbers, which, with the unit chosen, express the measures of its surface and of its volume, is equal to the sum of the numbers which express the measure of the surface of a sphere and of the volume of another sphere; but the same relation does not exist when the linear unit changes. Equation (7) can only be satisfied by the measures of the same lines when the unit of length is changed arbitrarily, provided these lines satisfy equations (5) and (6) taken singly. In the solution of problems of geometry, one never uses combinations of equations analogous to the preceding. The equations which give immediately the theorems of elementary geometry, are homogeneous; and when equations are added member to member it is to obtain a new equation more simple than the proposed; for this it is necessary that the equations added be of the same degree. The principle of homogeneity serves in each instance to verify the algebraic transformation deduced.

43. In case one of the lines of the figure is taken as the unit of length, the equations cease to be homogeneous; but it is easy to re-establish homogeneity. Let

(8) $$F(b', c', \cdots) = 0$$

be the equation which is obtained when a line A is taken for the unit; the letters b', c', $\cdots$, represent the measures of the lines B, C, $\cdots$, with respect to A. Choose an arbitrary unit, and call a, b, c, $\cdots$, the measures of the lines A, B, C, $\cdots$; then will

$$\frac{1}{a}=\frac{b'}{b}=\frac{c'}{c}=\cdots,$$

whence $$b'=\frac{b}{a},\quad c'=\frac{c}{a},\ \cdots,$$

and equation (8) is transformed into the following:

$$F\left(\frac{b}{a},\frac{c}{a},\cdots\right)=0, \tag{9}$$

which is homogeneous.

Thus, for example, if the sides of the right angle of a right triangle be referred to the hypotenuse of this triangle taken for the unit of measure, the measures of the sides satisfy the non-homogeneous equation

$$b'^2+c'^2=1,$$

from which is deduced the homogeneous equation

$$\frac{b^2}{a^2}+\frac{c^2}{a^2}=1,\ \text{ or }\ b^2+c^2=a^2,$$

by replacing b' by $\frac{b}{a}$ and c' by $\frac{c}{a}$.

The curves, ellipse, hyperbola, parabola, cissoid, etc., studied in the preceding chapter, are represented by homogeneous equations. Any homogeneous equation

$$f(x,y,a,b,c,\cdots)=0,$$

between the variable co-ordinates x and y of a point of the plane and the lengths a, b, c, $\cdots$ of the various given lines, determine a curve, of which the position and dimensions are independent of the unit with which the lines are measured. Consider, on the contrary, a numerical equation in x and y,

$$f(x,y)=0;$$

that is, an equation which does not involve other letters than x and y, and suppose their equation to be non-homogeneous. In order to represent by points of the plane the real solution of this equation, it is necessary to begin by choosing arbitrarily a scale, or the line to be employed as the unit. When the scale varies, the curve is no longer the same. It will be seen later that the various curves obtained in this manner have remarkable analogies; they are called *homothetic* curves.

44. REMARK I. — It frequently happens that one considers the numbers which represent the measures of lines, surfaces, and volumes. The units of surface and of volume, as well as the unit of length, remain indeterminate; but one habitually assumes that there exists among them this relation, that the unit of surface is the square constructed on the unit of length, and the unit of volume is the cube constructed on the same line. In this case, in order to verify the homogeneity of a relation in which certain letters S and V represent the measure of a surface and a volume, these letters are replaced by p^2 and q^3, where p and q represent a side of the square and an edge of the cube equivalent to the surface and the volume considered. By this change the equation will contain only the lines. Moreover, their substitution may be dispensed with, namely, in evaluating the degree of each term the exponent of a letter which designates a surface is doubled, and a letter representing a volume is tripled.

REMARK II. — In general, when angles enter into a calculation, these angles are referred to a unit definitely determined, and their measures are fixed numbers. In evaluating an angle, an arc of a circle is described about its vertex as a center with an arbitrary radius, and the ratio of this arc to the radius is taken as the measure of the arc; the unit arc is the arc which is equal to the radius. The trigonometric functions of angles are therefore numbers. In the application of the principle of homogeneity, one introduces the abstraction of letters which represent the angles or their trigonometric functions.

Construction of Formulas.

45. In solving, if it be possible, the equations of a definite problem, one determines the formulas which represent the arithmetical operations that it is necessary to perform on these numbers which measure the known magnitudes in order to find the numerical values of the unknown. But can one not deduce from each formula, or what is the same from each equation, an appropriate graphical construction to give, not merely the numerical value of the unknown, but the unknown itself? In a word, is it possible to replace the numerical operations by graphical? In elementary geometry, the constructions are considered which can be accomplished by means of a limited number of straight lines and circles, and which, consequently, can be made with the use of a rule and a compass. Since the circle is the most simple of curves and the most easily constructed, the ancient geometers set a great price on this sort of construction; on the other hand, being ignorant of algebraic analysis, they did not have the means to decide if the questions which they had in view were susceptible of this kind of a solution, and it was not until they had made many fruitless efforts that they decided finally to investigate other curves. Their investigations have made certain problems celebrated which can be shown to-day not to be solvable by the straight line and circle. Examples of such are the duplication of the cube, the trisection of an angle, etc.

The unknown quantity is assumed to be a straight line; when the unknown is a surface or a volume, it is represented by ax or a^2x, a being a line taken arbitrarily; the construction of the line x gives a rectangle or a parallelopiped equivalent to the surface or volume sought. The determination of an angle given by one of its trigonometrical lines is reduced also to that of a straight line. It can be assumed then that every letter, such as x, designates a straight line.

46. Rational Formula. — The formula which gives the unknown x ought to be homogeneous and of the first degree;

it can, however, be integral, rational, or irrational. When it is integral, it takes the form

$$x = a + b + c + \cdots,$$

and the length x is found by measuring one after the other, in one direction or in another, the lengths a, b, c, $\cdots$.

The fractional formula is the most simple:

$$x = \frac{ab}{c}.$$

The unknown is a fourth proportional, which can be constructed by two parallels or by a circle.

In the same manner may be constructed the formula

$$x = \frac{abcd}{a'b'c'} \text{ or } x = \frac{a}{a'} \times \frac{b}{b'} \times \frac{cd}{c'}$$

by means of the series of fourth proportionals,

$$\gamma = \frac{cd}{c'}, \quad \beta = \frac{b\gamma}{b'}, \quad x = \frac{a\beta}{a'}.$$

By the aid of the preceding construction, a monomial $\frac{abc \cdots ghi \cdots l}{a'b'c' \cdots g'}$, of the degree m, may be reduced to the form $ai \cdots l$, or, further, to the form $\lambda^{m-1}t$, λ being any length and t a line determined by the formula

$$t = \frac{ai \cdots l}{\lambda^{m-1}}.$$

Consider now the formula

$$x = \frac{A - B + C}{A' + B' - C'},$$

in which A, B, C designate monomials of the degree $m + 1$, A', B', C' monomials of the degree m: each monomial may be reduced to the simple forms

$$\lambda^m a,\ \lambda^m b,\ \lambda^m c,\ \cdots, \quad \lambda^{m-1}a',\ \lambda^{m-1}b',\ \lambda^{m-1}c';$$

whence it follows that

$$x = \frac{\lambda(a - b + c)}{a' + b' - c} = \lambda\frac{\alpha}{\beta}.$$

That is, the unknown x can be determined by a fourth proportional between the lines β, α, λ.

If the fraction were of the degree m, the preceding operations would reduce it to the form

$$\lambda^{m-1}\frac{\lambda\alpha}{\beta} = \lambda^{m-1}t.$$

47. Irrational Formula of the Second Degree. — Let the formula in this case be

$$x = \sqrt{ab}, \text{ or } \frac{a}{x} = \frac{x}{b}.$$

The unknown x is a mean proportional between the lines a and b; it is constructed by a right triangle, or by a tangent to a circle. When the quantity under the radical is a rational function of the degree m, the formula is transformed as follows:

$$\sqrt{\lambda^{m-1}t} = \sqrt{\lambda^{m-2}\lambda t} = \lambda^{\frac{m-2}{2}}u.$$

Consider next an irrational formula of the second degree, in which the quantities are supposed to be connected by the $+$ or $-$ sign, are homogeneous, and of the same degree. For the sake of clearness, suppose that the value of x is reduced to the form

$$x = \frac{N}{D},$$

N and D representing functions in which the sign of division does not enter, neither do fractional nor negative exponents; it can also be assumed that neither the product of two radicals nor the product of a radical by an integral quantity enters the expression. In order to find the value of the numerator N, it is necessary to perform certain operations in a definite order; the first radical sign affects an integral expression, it will reduce to the form $\lambda^{\frac{m}{2}}u$; if this quantity be added to the others, they will be reduced to the same form, and consequently their sum also. A new radical sign may now be introduced affecting either an integral quantity, or a quantity with the exponent $\frac{m}{2}$,

m being odd. In every case, the radical will reduce to the form $\lambda^{\frac{m}{4}}v$; this term is added to the others of the same form, and so on. Thus, it is seen that the numerator N will take the form $\lambda^{\frac{m}{2^p}}t$. The denominator can be discussed in the same manner. The unknown x being of the first degree, it can be found as a fourth proportional.

One can demonstrate that the hypotheses which have been assumed in constructing the formula are necessary in order that it be homogeneous.

Thus, *every homogeneous expression of the first degree constructed in any arbitrary manner by means of the symbols of the simple operations, addition, subtraction, multiplication, division, involution to an integral power, the extraction of a square root; in a word, every expression, rational and irrational, containing square roots only, can be constructed by means of a finite number of straight lines and circles.*

It can also be shown that only expressions of this sort are susceptible of construction by the method just indicated; but this demonstration cannot appropriately be given here. For example, the edge x of a cube which is the double of another whose edge is a, is given by the formula

$$x = \sqrt[3]{2a^3},$$

and cannot be constructed by a rule and a compass. In like manner, it is, in general, true of roots of equations of the third and fourth degree, since cubical radicals enter in the expression of these roots.

48. CONSTRUCTION OF THE ROOTS OF THE EQUATION OF THE SECOND DEGREE. — The equation of the second degree in one unknown quantity is reducible to the form $x^2 + px + q = 0$; in order that it be homogeneous, it is necessary that the quantity p be of the first degree, and q of the second; whether these quantities be rational or irrational of the second degree, it will be possible to construct a straight line a equivalent to the first and a square b^2 equivalent to the second, and the equation of the second degree will assume one of the four following forms:

$$x^2 + ax + b^2 = 0,$$
$$x^2 + ax - b^2 = 0,$$
$$x^2 - ax + b^2 = 0,$$
$$x^2 - ax - b^2 = 0.$$

The roots of the first and second equation are equal to those of the third and fourth, taken with contrary signs; it suffices, therefore, to consider those of the latter; if they be put under the form

$$x(a - x) = b^2, \ x(x - a) = b^2,$$

it is evident that it suffices to construct a rectangle equivalent to a square b^2, and of which the sum or difference of the edges is equal to a given line a, problems which can be solved by elementary geometry. The solution of equations and the construction of formulas necessitate the discovery of theorems of geometry.

The bi-quadratic equation may be reduced in a similar manner to one of the types

$$x^4 + abx^2 - c^2d^2 = 0,$$
$$x^4 - abx^2 + c^2d^2 = 0,$$
$$x^4 - abx^2 - c^2d^2 = 0;$$

because it is useless to consider the equation $x^4 + abx^2 + c^2d^2 = 0$, which has imaginary roots. If one put $x^2 = cz$, these equations become

$$z^2 + \frac{ab}{c}z - d^2 = 0, \ z^2 - \frac{ab}{c}z + d^2 = 0, \ z^2 - \frac{ab}{c}z - d^2 = 0.$$

One solves these equations for z, then finds x by means of a mean proportional between c and z.

CHAPTER IV

TRANSFORMATION OF CO-ORDINATES.

The equation of a curve in terms of certain co-ordinates being given, it is important to be able to deduce the equation of the same curve in terms of other co-ordinates.

In order to discuss the problem in a general manner, it is necessary to deduce the formulas which express the co-ordinates of any point of the plane in a certain system in terms of the co-ordinates of the same point in another system. These formulas are, moreover, useful in the investigation of a large number of other questions.

First will be discussed the transformation of rectilinear co-ordinates of one kind into other rectilinear co-ordinates.

TRANSPOSITION OF THE ORIGIN.

49. Suppose that the two axes OX and OY be replaced by other axes $O'X'$ and $O'Y'$, which are respectively parallel to the first (Fig. 32) and have the same direction. The position of the new axes will be determined by the co-ordinates a and b of the new origin with respect to the primitive axes. Let x and y be the co-ordinates of any point M of the plane with respect to the primitive axes; x' and y' the co-ordinates of the same point with respect to the new axes. Imagine the point O to be moved along the straight line OM or the broken line $OO'M$ to M, and project, parallel to OY, these two lines upon the axes OX. The projection of the line OM with the proper sign is the abscissa x of the point M; the projection of the line OO' is the abscissa a of the point O; the projection of the line $O'M$

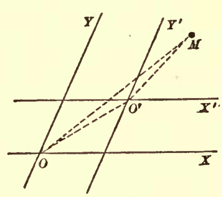

Fig. 32.

on OX, or on the parallel axis $O'X'$, is the new abscissa x'. The projections of the two lines OM, $OO'M$, being equal, one has $x = a + x'$. By projection parallel to OX on the axis OY, one has in a similar manner $y = b + y'$. Thus are obtained the two relations,

$$(1) \qquad x = a + x', \; y = b + y',$$

between the old and the new co-ordinates of the point M. These relations are satisfied, whatever be the position of the point M in the plane. One may deduce from (1),

$$(2) \qquad x' = x - a, \; y' = y - b.$$

Change in the Direction of the Axes.

50. Preserving the same origin, suppose now that the direction of the axis is changed. Consider a particular case which has frequent application, — the case when the two axes are rectangular. Suppose that the direction of the axes is changed by revolving the right angle XOY (Fig. 33) through an angle α about the origin till it attains the position $X'OY'$, and consider the angle α as positive if the rotation takes place from OX toward OY, and negative if the rotation be accomplished in an opposite direction.

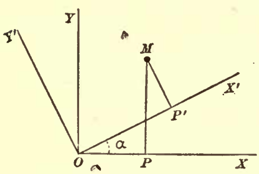

Fig. 33.

Through any point M of the plane draw MP and MP' parallel respectively to OY and OY'; let x and y be the co-ordinates of the point M with respect to the first axes, and x' and y' the co-ordinates of the same point with respect to the new axes. The projections of the two paths OPM, $OP'M$ on any axis are equal. Project then these two paths on the axis OX; the projection of the length OP is the line itself, affected with the $+$ or $-$ sign, according as it is measured in the direction OX, or in the opposite direction; that is, in every case, the abscissa x; PM being perpendicular to OX, its projection is zero; the projection of the first path reduces, therefore, to x. Project now the path $OP'M$, projecting first the portion OP';

if the length OP' is measured on OX', it is necessary to multiply by $\cos\alpha$, which gives for the projection $OP'\cos\alpha$; should this length be measured in an opposite direction, it is necessary to multiply by $\cos(\pi+\alpha)$, which gives $OP'\cdot\cos(\pi+\alpha)$ or $-OP'\cdot\cos\alpha$; but in the first case one has $x'=OP'$, and in the second $x'=-OP'$: thus the projection of the line OP' is always expressed by $x'\cos\alpha$. Consider the second line $P'M$. If it be constructed in the direction OY', it makes the angle $\alpha+\frac{\pi}{2}$ with OX, and its projection is $P'M\cdot\cos\left(\alpha+\frac{\pi}{2}\right)$; if it be measured in an opposite direction, it makes the angle $\alpha+\frac{\pi}{2}+\pi$ with OX, and its projection is $-P'M\cdot\cos\left(\alpha+\frac{\pi}{2}\right)$; but one has, in the first case $y'=P'M$, in the second, $y'=-P'M$; hence, the projection of $P'M$ is always expressed by $y'\cos\left(\alpha+\frac{\pi}{2}\right)$. Consequently, the projection of the path $OP'M$ is always $x'\cos\alpha+y'\cos\left(\alpha+\frac{\pi}{2}\right)$, or $x'\cos\alpha-y'\sin\alpha$. By equating the projection of the two paths OPM, $OP'M$, one gets the relation $x=x'\cos\alpha-y'\sin\alpha$.

Project now the two paths on OY. The projection OP is zero; that of PM, affected with the proper sign, is y; thus the projection of the first path reduces to y. The two directions OX' and OY' make respectively the angles $-\frac{\pi}{2}+\alpha$ and $+\alpha$ with OY, which furnishes for the projection of the second path $x'\cos\left(-\frac{\pi}{2}+\alpha\right)+y'\cos\alpha$, or $x'\sin\alpha+y'\cos\alpha$, and one has the relation $y=x'\sin\alpha+y'\cos\alpha$. Therefore the formulas sought are

$$(3)\qquad x=x'\cos\alpha-y'\sin\alpha,\quad y=x'\sin\alpha+y'\cos\alpha,$$

which express the old co-ordinates as functions of the new.

51. Next the general question will be investigated. Let OX and OY be any two axes inclosing an angle θ, OX' and OY', two new axes whose directions are defined by the angles α and β, which they make with OX (Fig. 34); one considers the angles α and β as positive, when a movable straight line, starting from the position OX, generates them in revolving from OX toward OY, and as negative in case the line revolves in an opposite direction. From any point M of the plane draw the lines MP and MP' respectively parallel to the axes OY and OY'. To get x, project the two paths OPM, $OP'M$ on OH, perpendicular to OY, so that a line, starting from the position OY, revolving

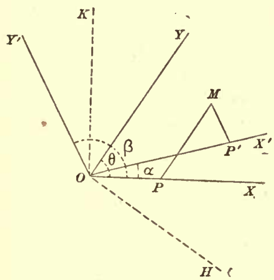

Fig. 34.

in the direction OX through an angle equal to $\frac{\pi}{2}$, will arrive finally in the position OH. Since the line OX makes with OH the angle $\frac{\pi}{2} - \theta$, and the direction OY is perpendicular to OH, the projection of the first path reduces to $x \sin \theta$. The line OX' makes with OH an angle equal to the angle HOX increased by the angle XOX', which together make $\left(\frac{\pi}{2} - \theta\right) + \alpha$. In the same manner the line OY' makes with OH an angle $\left(\frac{\pi}{2} - \theta\right) + \beta$; one has, therefore, for the projection of the second path

$$x' \cos\left(\frac{\pi}{2} - \theta + \alpha\right) + y' \cos\left(\frac{\pi}{2} - \theta + \beta\right),$$

or

$$x' \sin(\theta - \alpha) + y' \sin(\theta - \beta),$$

which furnishes the relation

$$x \sin \theta = x' \sin(\theta - \alpha) + y' \sin(\theta - \beta).$$

To calculate y, project the two paths OPM, $OP'M$ on a line OK perpendicular to OX, so that a straight line starting from OX and revolving through the angle $\frac{\pi}{2}$ toward OY will coincide with OK. Since the line OX is perpendicular to OK and the line OY makes with this line the angle $-\frac{\pi}{2} + \theta$, the projection of the first path reduces to $y \sin \theta$. The angles which the lines

OX' and OY' form with OK are equal to the angles which they make with OX diminished respectively by $\frac{\pi}{2}$, which gives $-\frac{\pi}{2}+\alpha$, and $-\frac{\pi}{2}+\beta$; the projection of the second path is therefore $x'\cos\left(-\frac{\pi}{2}+\alpha\right)+y'\cos\left(-\frac{\pi}{2}+\beta\right)$,

or
$$x'\sin\alpha+y'\sin\beta,$$

and one has the relation

$$y\sin\theta=x'\sin\alpha+y'\sin\beta.$$

Thus are derived the formulas

$$(4)\qquad \begin{cases} x=\dfrac{x'\sin(\theta-\alpha)+y'\sin(\theta-\beta)}{\sin\theta}, \\[2ex] y=\dfrac{x'\sin\alpha+y'\sin\beta}{\sin\theta}, \end{cases}$$

for the transformation of oblique co-ordinates into other oblique co-ordinates.

It is a simple process to deduce the formulas serving to return from the new to the old co-ordinates. The angle between the new axes is $\beta-\alpha$; the axes OX and OY form with OX' the angles $-\alpha$ and $-\alpha+\theta$; it suffices therefore to replace in the preceding formulas the angle θ by $\beta-\alpha$, α by $-\alpha$, β by $\theta-\alpha$, which gives

$$(5)\qquad \begin{cases} x'=\dfrac{x\sin\beta+y\sin(\beta-\theta)}{\sin(\beta-\alpha)}, \\[2ex] y'=\dfrac{-x\sin\alpha+y\sin(\theta-\alpha)}{\sin(\beta-\alpha)}. \end{cases}$$

Let the angle $\beta-\alpha$ between the new axes be represented by θ'; then the determinant of the coefficients x' and y' in formulas (4) is

$$\frac{\sin\beta\sin(\theta-\alpha)-\sin\alpha\sin(\theta-\beta)}{\sin^2\theta}=\frac{\sin\theta'}{\sin\theta},$$

and the determinant of the coefficients x and y in formulas (5) is the reciprocal of the preceding, $\dfrac{\sin\theta}{\sin\theta'}$.

52. The general formulas furnish certain special formulas which are of frequent use.

1° *The case when the primitive axes are rectangular.* Here θ will be equal to $\frac{\pi}{2}$, and formulas (4) will become

$$(6)\qquad \begin{cases} x = x' \cos\alpha + y' \cos\beta, \\ y = x' \sin\alpha + y' \sin\beta. \end{cases}$$

2° *The case when the new axes are rectangular.* Let $\beta = \alpha + \frac{\pi}{2}$, then formulas (4) reduce to

$$(7)\qquad \begin{cases} x = \dfrac{x' \sin(\theta - \alpha) - y' \cos(\theta - \alpha)}{\sin\theta}, \\ y = \dfrac{x' \sin\alpha + y' \cos\alpha}{\sin\theta}. \end{cases}$$

One could also put $\beta = \alpha - \frac{\pi}{2}$, which would amount to changing the direction of the axis OY', and, consequently, the sign of y' in formulas (7).

3° *The case when the two systems of axes are rectangular.* If, in formulas (6), one put $\beta = \alpha + \frac{\pi}{2}$, one deduces formulas (3), already found,

$$(3)\qquad \begin{aligned} x &= x' \cos\alpha - y' \sin\alpha, \\ y &= x' \sin\alpha + y' \cos\alpha. \end{aligned}$$

These formulas can also be derived by putting in formulas (7) $\theta = \frac{\pi}{2}$.

General Transformation.

53. Suppose that the origin and the direction of the axes are changed at the same time. The new system of axes will be determined by the co-ordinates a and b of the new origin O', with respect to the old axes, and by the angles α and β which the new axes $O'X'$ and $O'Y'$ make with OX (Fig. 35). Through the point O' draw the two axes OX_1 and OY_1 respectively parallel to OX and OY. Then will in one case

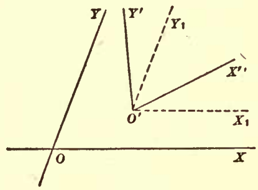

Fig. 35.

$$x = a + x_1, \quad y = b + y_1;$$

and in the other case, by virtue of formulas (4),

$$x_1 = \frac{x' \sin(\theta - \alpha) + y' \sin(\theta - \beta)}{\sin\theta}, \quad y_1 = \frac{x' \sin\alpha + y' \sin\beta}{\sin\theta};$$

substituting the values of x_1 and y_1, the general formulas of transformation become

$$(8) \quad \begin{cases} x = a + \dfrac{x' \sin(\theta - \alpha) + y' \sin(\theta - \beta)}{\sin\theta}, \\ y = b + \dfrac{x' \sin\alpha + y' \sin\beta}{\sin\theta}. \end{cases}$$

The old co-ordinates x and y are expressed as linear integral functions of the first degree in the new co-ordinates x' and y'.

The Transformation of Rectilinear Co-ordinates into Polar Co-ordinates.

54. Let OX and OY be the rectangular axes; take the origin as pole, and the x-axis as the polar axis (Fig. 36); by projecting the line OM on the axes OX and OY, one obtains the relations

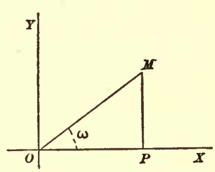

Fig. 36.

$$(9) \quad x = \rho \cos\omega, \quad y = \rho \sin\omega.$$

Conversely, one can pass from polar co-ordinates to rectangular co-ordinates by means of the formulas

$$\rho = \sqrt{x^2 + y^2}, \quad \tan \omega = \frac{y}{x}.$$

Several transformations of this kind have been made, namely, when the equations of the cissoid, strophoid, limaçon of Pascal, and rose (§§ 21, 24, 27, 30) were derived in rectangular co-ordinates.

Distance between Two Points.

55. Assume the axes to be rectangular and seek the distance of the origin from the point M, whose co-ordinates are x and y. From the right triangle OPM (Fig. 37) one has

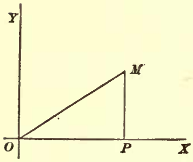

Fig. 37.

$$\overline{OM}^2 = \overline{OP}^2 + \overline{PM}^2 = x^2 + y^2,$$

whatever be the position of the point M in the plane; whence it follows, by putting l for the distance OM,

$$l = \sqrt{x^2 + y^2}. \qquad (10)$$

Seek, next, the distance between two points M and M', situated anywhere in the plane; call x and y the co-ordinates of the point M, x' and y' those of the point M' with respect to the rectangular axes OX, OY. Through the point M (Fig. 38) draw the axes MX', MY' parallel to the given axes. The co-ordinates of the point M' with respect to the new axes are equal to $x' - x$, $y' - y$, by virtue of formulas (2) of § 49. The distance of the new origin M from M' will therefore be, owing to formula (10),

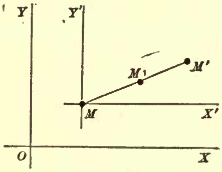

Fig. 38.

$$l = \sqrt{(x' - x)^2 + (y' - y)^2}. \qquad (11)$$

56. In case the axes are oblique and the angle included by them is represented by θ, the expression will be somewhat more complicated.

Seek now the distance of the origin O from any point M of the plane. In the triangle OPM (Fig. 39), whatever be the position of the point M, one has

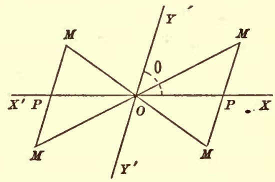

Fig. 39.

$$\overline{OM}^2 = \overline{OP}^2 + \overline{PM}^2 - 2 \cdot OP \cdot PM \cos OPM.$$

In case the point M is situated within the angle YOX, the co-ordinates x and y of this point are equal to $+OP$ and $+PM$, and the angle OPM is the supplement of θ; one has therefore

$$(12) \qquad l = \sqrt{x^2 + y^2 + 2xy \cos \theta}.$$

If the point M is situated within the angle $Y'OX'$, the co-ordinates x and y being equal to $-OP$ and $-PM$, and the angle OPM, the supplement of θ, the same formula (12) is deduced. When the point M is situated within one of the angles YOX', $Y'OX$, the angle OPM is equal to θ, but one of the co-ordinates is positive and the other negative, which reproduces formula (12). This formula is, therefore, universal.

In order to obtain the distance between two points M and M', one imagines, as above, axes drawn through the point M parallel to the first, and obtains the formula

$$(13) \qquad l = \sqrt{(x' - x)^2 + (y' - y)^2 + 2\,(x' - x)\,(y' - y) \cos \theta}.$$

57. It is frequently useful to know the co-ordinates of a point which divides the distance between two given points in a given ratio. In case several segments are situated on the same line, one calls *the direction* of the segment the direction in which a movable point travels that starts from the first point M and goes toward the second M'. The algebraic value of the ratio of two segments is then the absolute value of their ratio, preceded by the + or − sign, according as the two

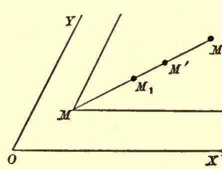

Fig. 40.

segments are measured in the same or opposite direction. Thus, in Fig. 40, the ratio $\frac{MM_1}{M'M_2}$ is positive, the ratios $\frac{MM_1}{M_2M'}$, $\frac{M_1M}{M_1M'}$, are negative.

Being given two points M and M', on an indefinite straight line, having the co-ordinates x and y, x' and y', find on this line a point M_1 with the co-ordinates x_1 and y_1, such that the ratio $\frac{M_1M}{M_1M'}$ has in magnitude and sign the value $\frac{m'}{m}$. If the axes are transferred parallel to themselves to the point M_1, the new co-ordinates of the points M and M' will be $x - x_1$ and $y - y_1$, $x' - x_1$ and $y' - y_1$. In case the given ratio $\frac{m'}{m}$ is negative, the point sought, M_1, ought to lie between M and M'; this is the case in the figure. The differences $x - x_1$ and $x' - x_1$ or $y - y_1$ and $y' - y_1$ have opposite signs; their ratio is negative, and the absolute value of their ratio is equal to the absolute value of $\frac{M_1M}{M_1M'}$ or $\frac{m'}{m}$. One has, therefore, in magnitude and in sign

$$(14) \qquad \frac{x - x_1}{x' - x_1} = \frac{y - y_1}{y' - y_1} = \frac{m'}{m}.$$

When $\frac{m'}{m}$ is positive, the point sought, M_1, lies without the segment MM'; the differences $x - x_1$ and $x' - x_1$ or $y - y_1$ and $y' - y_1$ have the same sign; their ratio is plus and equal to the ratio $\frac{M_1M}{M_1M'}$ or to $\frac{m'}{m}$. Therefore equations (14) are also applicable to this case. Whence one has the following formulas which solve the problem for every value of the ratio $\frac{m'}{m}$,

$$x_1 = \frac{mx - m'x'}{m - m'},\ y_1 = \frac{my - m'y'}{m - m'}.$$

Remark. — The co-ordinates of the point M_2, x_2 and y_2, may be deduced from the preceding by changing the sign of m'.

$$x_2 = \frac{mx + m'x'}{m + m'},\ y_2 = \frac{my + m'y'}{m + m'}.$$

From the position of this point it follows that

$$\frac{M_1M}{M_1M'} = -\frac{M_2M}{M_2M'} = \frac{m'}{m};$$

the points M_1 and M_2, corresponding to values, equal and opposite in sign, of the ratio $\frac{m'}{m}$, are called *harmonic conjugates* with respect to the segment MM'. In case the given ratio $\frac{m'}{m}$ is equal to -1, the point M_1 will bisect the segment MM' and has the co-ordinates

$$x_1 = \frac{x+x'}{2}, \quad y_1 = \frac{y+y'}{2};$$

the point M_2 is removed to infinity.

If $\frac{m'}{m}$ is put equal to $-\lambda$, it follows at once that the two conjugate points have respectively the co-ordinates

$$x_1 = \frac{x+\lambda x'}{1+\lambda}, \quad y_1 = \frac{y+\lambda y'}{1+\lambda};$$

$$x_2 = \frac{x-\lambda x'}{1-\lambda}, \quad y_2 = \frac{y-\lambda y'}{1-\lambda}.$$

Classification of Plane Curves.

58. Rectilinear co-ordinates are especially adapted to the study of the general properties of plane curves. In this system plane curves are classified in the following manner: They are distinguished as *algebraic* and *transcendental*, according as the equations which represent them are algebraic or transcendental. An equation is said to be algebraic when the co-ordinates x and y enter affected only by the symbols of algebraic operations. If, however, one of the co-ordinates enters affected by a transcendental symbol, as a *sin*, *logarithm*, *tan*, etc., the equation is said to be transcendental. Algebraic equations can always be put under an integral form by removing the radicals and the denominators.

One classifies algebraic curves according to the degree of their equations. Curves of the first degree (straight lines) are those which are represented by equations of the first degree in x and y; the equation of the second degree furnishes curves of the second degree, etc.

It is very plain that the degree of any curve remains unaltered whatever may be the position of the axes of co-ordinates in the plane. In fact, let $f(x, y) = 0$ be the equation of a curve referred to certain axes OX and OY, m the degree of this equation supposed to be integral. To refer this curve to other axes $O'X'$ and $O'Y'$, it is necessary to substitute for x and y in the proposed equation the values given by the formulas of transformation (8); these formulas being of the first degree in the co-ordinates x' and y', it is impossible that the equation in x' and y' be of a degree greater than m. The equation will not be of a degree less, because in that case the inverse transformation would increase the degree, which is impossible. Thus, the new equation is of the same degree as the primitive.

The degree of a curve is the same as the number of points of its intersection with a straight line. In fact, let m be the degree of a curve whose equation is $f(x, y) = 0$ when the straight line has been chosen as the x-axis; if in this equation one makes $y = 0$, the equation thus obtained in x will give the abscissas of the points common to the curve and the x-axis. Since the first member of the equation is not identically zero, and is at most of the degree m, the equation cannot have more than m roots, and consequently the line has at most m points in common with the line. If the equation were satisfied by more than m values of x, the first member would be identically zero, and consequently the line would be a part of the locus; in this case, the polynomial $f(x, y)$ vanishing identically when y is put equal to zero would contain y as a factor, and the equation $f(x, y) = 0$ could be decomposed into two equations, one $y = 0$ of the first degree, the other of the degree $m-1$.

Accordingly, curves of the first degree cannot be cut by a straight line in more than one point; therefore the curves are straight lines. Curves of the second degree cannot be cut by a straight line in more than two points; those of the third degree, in more than three points. The circle, ellipse, hyperbola, and the parabola are curves of the second degree (§§ 10, 13, 15, 18). These curves can be cut by a straight line in two points. The cissoid and strophoid (§§ 21 and 24) are of the third degree. They can be cut in three points by a straight line. The limaçon of Pascal (§ 27) is of the fourth degree; the rose of four branches (§ 30) is of the sixth degree.

First, one studies curves of the first, then those of the second, and finally those of any degree whatever.

When an algebraic integral equation of the degree m is said to represent a curve of the degree m, it is assumed that the first member cannot be decomposed into integral factors; otherwise the equation could represent two or a greater number of curves of lower degrees. Thus, for example, an equation of the second degree, whose first member is the product of two integral factors of the first degree, represents two lines of the first degree; that is, two straight lines. Similarly, an equation of the third degree may represent three straight lines, or one curve of the second degree and a straight line. It is for this reason that certain properties of curves of the mth order are applicable to a system of m straight lines; that is, to a polygon of m sides. Thus is learned that the properties of curves of the second degree are applicable to a system of two straight lines, since this system can be considered as a locus of the second degree.

Book II

STRAIGHT LINE AND CIRCLE

CHAPTER I

STRAIGHT LINE.

Construction of the Equation of the First Degree.

59. The general equation of the first degree between two variables x and y has the form

(1) $$Ax + By + C = 0.$$

It has already been noticed that the line represented by this equation cannot be cut by a straight line in more than one finite point, and is necessarily straight. However, it is best to show directly that this equation represents a straight line. It is impossible that the coefficients A and B be zero at the same time, for then C must also be zero, and the equation is reduced to an identity. But it is possible that one of the coefficients be zero. If, for example, the coefficient A be zero, the equation takes the form $By + C = 0$, whence $y = -\frac{C}{B} = b$. This equation represents the locus of a point M whose ordinate is constant and equal to b, whatever the abscissa may be; the locus is a straight line parallel to the

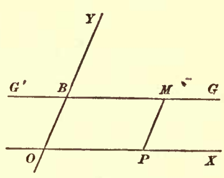

Fig. 41.

axis OX (Fig. 41). This line is constructed by laying off on OY, beginning at the origin, a length equal in absolute value to b, in one direction or the opposite, according to the sign of b, then drawing GG' through the point B parallel to the axis OX. As a special case, the equation $y = 0$ represents the axis OX.

When the coefficient B is equal to zero, the equation reduces to $Ax + C = 0$, or $x = -\frac{C}{A} = a$. This equation represents the locus of the point M, whose abscissa is constant and equal to a, whatever the ordinate may be. It is a straight line HH' parallel to the axis OY (Fig. 42). This line can be constructed by laying off on the axis OX, beginning at the origin, a length OA equal to the absolute value of a, in one direction or the opposite, according to the sign of a, then drawing HH' through the point A parallel to OY. As a special case, the equation $x = 0$ represents the axis OY.

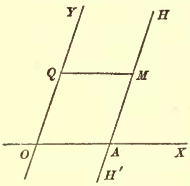

Fig. 42.

In case the coefficient B is not zero, all the terms of the equation can be divided by B and it may be written

$$y = -\frac{A}{B}x - \frac{C}{B}$$

or (2) $$y = ax + b,$$

by putting, for brevity, $a = -\frac{A}{B}$, $b = -\frac{C}{B}$.

Consider next the particular case when $b = 0$.

The equation then reduces to the form

$$y = ax, \text{ or } \frac{y}{x} = a.$$

If a be a positive number, every point of the locus, having co-ordinates with the same sign, lies in the angle YOX or its vertically opposite (Fig. 43). Take an arbitrary abscissa OP, and draw through the point P a line parallel to the axis of y; if a point M can be found on this parallel, such that $\frac{MP}{OP} = a$, it will be a point of the locus. Let M, M', M'', $\cdots$, be points of the locus constructed by the preceding rule; it follows from the equal ratios

$$\frac{MP}{OP}=\frac{M'P'}{OP'}=\frac{-M''P''}{-OP''}=\cdots=a,$$

that the triangles OPM, $OP'M'$, $OP''M''$, $\cdots$, are similar, and hence the angles MOP, $M'OP'$, $M''OP''$, $\cdots$, are equal; therefore the points M, M', M'', $\cdots$, all lie on the straight line $A'A$ passing through the origin. If x varies continuously from $-\infty$ to $+\infty$, the point M will move continuously and describe an indefinite straight line AA'.

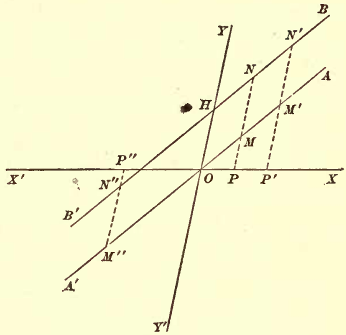

Fig. 43.

When a is negative, all points of the locus, having co-ordinates of opposite signs, lie in the angles YOX' and $Y'OX$ (Fig. 44). Let M, M', M'', $\cdots$, be different points of the locus; then, as above, it follows from the relations

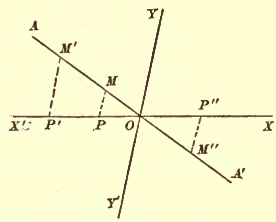

Fig. 44.

$$\frac{MP}{-OP}=\frac{M'P'}{-OP'}=\frac{-M''P''}{OP''}=\cdots=a,$$

that all these points are on the same straight line $A'A$ passing through the origin. Thus, in every case, the equation $y=ax$ represents a straight line $A'A$ passing through the origin.

Let us return now to the equation $y=ax+b$. By comparing the two equations $y=ax+b$, $y=ax$, one sees that the ordinates corresponding to the same abscissa differ by a constant b; if therefore the ordinates of all the points of the straight line $A'A$ are increased or diminished according to the sign of b by the lengths MN, $M'N'$, $M''N''$, $\cdots$, equal to the absolute value of b (Fig. 43), the points N, N', N'', $\cdots$, thus obtained, form evidently the right line $B'B$ parallel to $A'A$.

It follows from what precedes that *any equation of the first degree between two variables x and y represents a straight line.*

60. It can be shown, reciprocally, that any straight line is represented by an equatin of the first degree. If the straight line be parallel to the axis OX, then all of its points have the same ordinate, and the equation has the form $y = b$ (Fig. 41). If it be parallel to the axis OY, all of the points have the same abscissa and the equation will have the form $x = a$ (Fig. 42). In case the straight line passes through the origin, it occupies one or the other of the two positions indicated in the figures 43 and 44, and the similar triangles give

$$\frac{MP}{OP} = \frac{M'P'}{OP'} = \frac{-M''P''}{-OP''} = \cdots,$$

or

$$\frac{MP}{-OP} = \frac{M'P'}{-OP'} = \frac{-M''P''}{OP''} = \cdots.$$

If a be this constant ratio, the equation of the right line is $\frac{y}{x} = a$, or $y = ax$. Suppose, finally, that the straight line is not parallel to either of the axes nor passes through the origin (Fig. 43); according to what precedes, a line drawn through the origin parallel to this straight line will have the equation $y = ax$; now the excess of the ordinate of a point on the proposed line over the ordinate of the corresponding point on the parallel is a constant quantity b; therefore the proposed straight line has for its equation $y = ax + b$.

Meaning of the Coefficients.

61. The equation of every straight line which is not parallel to the axis of y can be put in the form

(2) $$y = ax + b.$$

The constant b is the ordinate of the point H (Fig. 43) where the straight line cuts the axis of y; it is called the *ordinate of the origin.*

The constant a determines the direction of the line; it is the same for all parallel straight lines and is called the *angular coefficient* or *coefficient of direction.*

Draw through the origin a line $A'OA$ parallel to the proposed straight line and situated with respect to the axis XX' on the same side as the line OY. Let θ be the angle XOY, α the angle AOX, an angle which can vary from 0 to π; it follows from Fig. 43 that

$$a=\frac{y}{x}=\frac{MP}{OP}=\frac{\sin MOP}{\sin OMP}=\frac{\sin\alpha}{\sin(\theta-\alpha)},$$

and from Fig. 44 that

$$a=\frac{y}{x}=\frac{MP}{-OP}=\frac{\sin MOP}{-\sin OMP}=\frac{\sin(\pi-\alpha)}{-\sin(\alpha-\theta)}=\frac{\sin\alpha}{\sin(\theta-\alpha)};$$

one has, therefore, in every case,

$$(3)\qquad \frac{\sin\alpha}{\sin(\theta-\alpha)}=a.$$

If the axes are rectangular, this relation reduces to

$$(4)\qquad \tan\alpha=a,$$

and determines the angle α which OA makes with the axis OX.

When the axes are oblique, one deduces from the relation (3) the formula

$$\sin\alpha=a\sin\theta\cos\alpha-a\cos\theta\sin\alpha,$$

or (5)
$$\tan\alpha=\frac{a\sin\theta}{1+a\cos\theta}.$$

In order that this formula may be solved by logarithms, the following transformation is made. It follows from (3),

$$\frac{a-1}{a+1}=\frac{\sin\alpha-\sin(\theta-\alpha)}{\sin\alpha+\sin(\theta-\alpha)}=\frac{\tan\left(\alpha-\frac{\theta}{2}\right)}{\tan\frac{\theta}{2}},$$

or (6)
$$\tan\left(\alpha-\frac{\theta}{2}\right)=\frac{a-1}{a+1}\tan\frac{\theta}{2}.$$

62. In constructing the straight line represented by the equation of the first degree, with numerical coefficients, one usually seeks the points in which the straight line cuts the axes and draws a straight line through them.

Suppose that the equation $2x - 3y = 5$ be given; for $y = 0$, one has $x = \frac{5}{2}$; for $x = 0$, $y = -\frac{5}{3}$; starting from the origin, one lays off on the x-axis the length $\frac{5}{2}$ in the direction OX, on the y-axis the length $\frac{5}{3}$ in the direction OY'; through these two points the line is to be drawn. If the equation be free from an additive constant, the straight line passes through the origin. One determines then a second point, by giving to x a particular value; let, for example, $2y + 3x = 0$; the equation being satisfied for $x = 0$, $y = 0$, the line passing through the origin; if one makes $x = 2$, then one has $y = -3$; construct the point whose co-ordinates are $x = 2$, $y = -3$, and draw a line through it to the origin.

63. The general equation of the straight line,

$$Ax + By + C = 0,$$

contains but two arbitrary coefficients or parameters; because one can divide the equation by one of the coefficients, then the other two will be replaced by their ratios to the divisor. When the equation is put under the form $y = ax + b$, the two parameters are a and b. In order to fix the position of the straight line in the plane, it will be necessary to give a value to each of the two parameters or to be given two relations between them.

64. PROBLEM I. — *To find the general equation of straight lines which pass through a given point.*

Let x' and y' be the co-ordinates of the given point M. The equation of any line is

$$y = ax + b.$$

If this line pass through the given point M, the co-ordinates of this point must satisfy the equation to the line; if therefore the variable co-ordinates x and y are replaced by the co-ordinates x' and y' of the point M, one will have the equation of condition,

$$y' = ax' + b.$$

This relation between the two parameters a and b determines one of them as a function of the other; for example, the parameter b as a function of a. By replacing b in the equation of the straight line by its value $y' - ax'$ deduced from the equation of condition, one obtains the equation

$$(7) \qquad y - y' = a(x - x').$$

Equation (7), in which the angular coefficient a is arbitrary, represents all the straight lines which pass through the point M. When the parameter a is varied, the line revolves about the point M.

It has been assumed that every straight line is represented by an equation of the form $y = ax + b$, whatever its position in the plane may be. But there is one exception, viz., when the straight line is parallel to the y-axis; because, in this case, the angular coefficient a is infinite, and the ordinate at the origin is b. Accordingly, if in equation (7) a is replaced by the ratio $\frac{m}{n}$, the equation may be put under the form

$$n(y - y') = m(x - x');$$

and letting $n = 0$, one gets the equation $x = x'$, which represents a straight line, drawn through the point M, parallel to the y-axis.

65. Problem II. — *Through a given point draw a straight line parallel to a given straight line.*

Let $y = ax + b$ be the equation of the given straight line AB, x' and y', the co-ordinates of the given point M (Fig. 45). Since the line is to pass through the given point M, its equation, as we have seen above, will have the form

$$y - y' = a'(x - x').$$

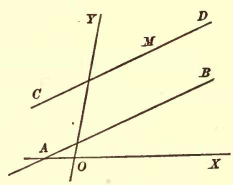

Fig. 45.

This line will be parallel to the line AB when the angular coefficient a' is equal to the angular coefficient of the line AB. One will have, therefore, $a' = a$, and the parallel required will have for its equation

$$y - y' = a(x - x').$$

66. Problem III. — *Draw a straight line through two given points.*

Let M and M' (Fig. 46) be the two given points, x' and y' the co-ordinates of M, x'' and y'' those of M'. The line MM', passing through the point M, is represented by an equation of the form

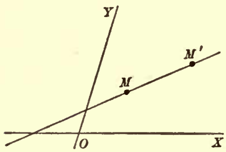

Fig. 46.

$$(7) \qquad y - y' = a(x - x').$$

It is a simple matter to determine the coefficient a so that this line may pass through the point M'. For this, it is necessary that the co-ordinates of the point M' satisfy equation (7), which gives the relation

$$y'' - y' = a(x'' - x'),$$

whence one deduces

$$a = \frac{y'' - y'}{x'' - x'}.$$

Thus, the angular coefficient of the line MM' is equal to the ratio of the difference of the ordinates to the difference of the abscissas of the two given points. If in equation (7) a be replaced by its value, one obtains the equation of the line MM',

$$(8) \qquad y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

an equation which can be written in the form

$$\frac{x - x'}{x'' - x'} = \frac{y - y'}{y'' - y'}.$$

When the point M is at the origin, one has $x' = 0$, $y' = 0$, and equation (8) reduces to

$$y = \frac{y''}{x''}x.$$

67. It is sometimes useful to define a line by the points where it cuts the axes (Fig. 47). Call a the abscissa of the first point, b the ordinate of the second, and let

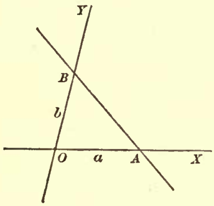

Fig. 47.

$$Ax + By + C = 0$$

be the equation of the line sought. If one makes successively $y = 0$ and $x = 0$, one obtains the points where the line cuts the axes; one has $a=-\frac{C}{A}$, $b=-\frac{C}{B}$; whence $A=-\frac{C}{a}$, $B=-\frac{C}{b}$. By replacing A and B by their values, the equation takes the simple form,

(9) $$\frac{x}{a}+\frac{y}{b}=1.$$

68. Problem IV. — *Find the point of intersection of two given lines.*

Let $$Ax+By+C=0,$$

$$A'x+B'y+C'=0,$$

be the equations of the two given lines AB and CD (Fig. 48), M the point of intersection of these two lines. The point M being common to each of the two lines, its co-ordinates will satisfy at the same time the two equations; if, therefore, one solves these two simultaneous equations for the two unknown quantities, x and y, we obtain the co-ordinates of the point M,

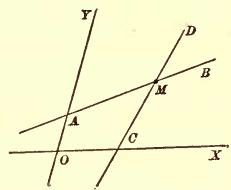

Fig. 48.

$$x=\frac{BC'-CB'}{AB'-BA'},\qquad y=\frac{CA'-AC'}{AB'-BA'}.$$

When the denominator $AB'-BA'$ is different from zero, the formulas furnish finite and determinate values for x and y, and the two lines intersect in a finite point M. But when the denominator is zero and the numerators different from zero, the values of x and y are infinite; in this case, the two lines are parallel, and, in fact, they have equal angular coefficients $-\frac{A}{B}=-\frac{A'}{B'}$. If one has $\frac{A'}{A}=\frac{B'}{B}=\frac{C'}{C}$, the two numerators and denominators will be zero at the same time, and the values of x and y will take the form $\frac{0}{0}$; the intersection will be indeterminate, and, in fact, the two proposed lines coincide; because if one puts $\frac{A'}{A}=\frac{B'}{B}=\frac{C'}{C}=K$, then is $A'=AK$, $B'=BK$,

$C' = CK$; substitute these values in the second equation, and divide by K, the resulting equation will be identical with the first.

69. Problem V. — *To find the general equation of a straight line which passes through the point of intersection of two given straight lines.* Let

$$(10) \qquad Ax + By + C = 0,$$

$$(11) \qquad A'x + B'y + C' = 0,$$

be the equations of two given straight lines. One could first find the point of intersection of the proposed lines by solving equations (10) and (11); then find the equation to any line through this point (§ 64). But one can arrive at the same result in a more rapid manner.

If one multiplies equation (11) by an arbitrary quantity, then adds it member by member to equation (10), one gets an equation of the first degree,

$$(12) \qquad (Ax + By + C) + \lambda(A'x + B'y + C') = 0,$$

which represents a third line passing through the point of intersection of the first two; for, in fact, the co-ordinates of this point satisfy the two equations (10) and (11), annulling the two quantities put in parentheses, and consequently, satisfy equation (12). This equation (12), in which the coefficient λ is arbitrary, represents any straight line which passes through the point of intersection of the two given lines; because one can determine this coefficient λ so that the line may pass through any point M of the plane having as co-ordinates x' and y'; for this it suffices that the equation of condition,

$$(Ax' + By' + C) + \lambda(A'x' + B'y' + C') = 0,$$

be satisfied, which gives

$$(13) \qquad \lambda = -\frac{Ax' + By' + C}{A'x' + B'y' + C'}.$$

In case one makes $\lambda = 0$, the equation (12) becomes

$$Ax + By + C = 0;$$

this is the first straight line. If one replace λ by $\frac{m}{n}$, after having multiplied by n, place $n=0$, one gets the second line, $A'x+B'y+C'=0$.

If in equation (12) one replace λ by the value (13), one gets the equation

$$(14) \qquad \frac{Ax+By+C}{Ax'+By'+C}=\frac{A'x+B'y+C'}{A'x'+B'y'+C'},$$

which represents the line passing through the point M and the point of intersection of the two given lines. The numerators are the first members of the given lines, the denominators are these same polynomials when x and y are replaced by the co-ordinates of the given point. One recognizes at once, from inspecting this equation, that the line it represents passes through the given point and through the point of intersection of the two given lines.

When the two lines (10) and (11) are parallel, equation (12) represents all the lines parallel to them.

An equation of the first degree in x and y, which contains an arbitrary constant λ, represents an infinity of lines; when this parameter appears in the first degree in the equation, one can put the equation in the form (12); one sees then that all the lines pass through the same point, — the point of intersection of the lines (10) and (11).

Remark. — Suppose that four concurrent lines d, d', d_1, and d_2 are given; then the lines d_1 and d_2 are called *harmonic conjugates* of the lines d and d', when the two points where a secant cuts the lines d_1 and d_2 are harmonic conjugates of the two points where it cuts d and d' (§ 57). It is easy to see then that the two lines d_1 and d_2, whose equations are:

$$(d_1) \quad Ax+By+C+\lambda(A'x+B'y+C')=0,$$

$$(d_2) \quad Ax+By+C-\lambda(A'x+B'y+C')=0,$$

are *harmonic conjugates* of the given lines (10) and (11). In fact, cut the three lines (10), (11), and (12) by a secant having the equation $y=mx+n$ and meeting these lines in the points M, M', and M_1.

The abscissas of the three points are

$$x = -\frac{Bn + C}{A + Bm},\quad x' = -\frac{B'n + C'}{A' + B'm},\quad x_1 = -\frac{Bn + C + \lambda(B'n + C')}{A + Bm + \lambda(A' + B'm)},$$

and, according to the formulas of § 57, one has in magnitude and in sign,

$$\frac{M_1M}{M_1M'} = \frac{x - x_1}{x' - x_1} = -\lambda\frac{A' + B'm}{A + Bm}.$$

Similarly, calling M_2 the point where the same secant cuts the line d_2,

$$\frac{M_2M}{M_2M'} = \lambda\frac{A' + B'm}{A + Bm},$$

as one can easily see by changing λ into $-\lambda$. Therefore, finally,

$$\frac{M_1M}{M_1M'} = -\frac{M_2M}{M_2M'};$$

which shows that the two points M_1, M_2 are harmonic conjugates of the points M, M' (§ 57).

70. Problem VI. — *The condition that three lines pass through a point.*

Let

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$
$$A''x + B''y + C'' = 0,$$

be the equations of three given lines. One finds the point of intersection of the first two lines, and substitutes the co-ordinates of this point in the third equation. This furnishes the equation of condition

$$A''(BC' - CB') + B''(CA' - AC') + C''(AB' - A'B) = 0,$$

or $C''(AB' - A'B) + C'(A''B - AB'') + C(A'B'' - A''B') = 0.$

The lines will not only intersect, but also be parallel if

$$AB' - A'B,\quad A'B'' - A''B',\quad A''B - AB''$$

are all three zero.

Otherwise, the general equation of the lines which pass through the point of intersection of the first two is

$$(Ax + By + C) + \lambda (A'x + B'y + C') = 0,$$

or $$(A + \lambda A')x + (B + \lambda B')y + (C + \lambda C') = 0.$$

If the lines have a common point, by assigning a suitable value to λ, this equation will represent the third line; therefore we ought to have

$$A + \lambda A' = KA'', \quad B + \lambda B' = KB'', \quad C + \lambda C' = KC'',$$

where K is arbitrary,

or $$\frac{A + \lambda A'}{A''} = \frac{B + \lambda B'}{B''} = \frac{C + \lambda C'}{C''}.$$

By eliminating λ, one gets the equation already obtained.

71. Example. — Consider the three medians of a triangle OAB (Fig. 49); take the vertex O as origin, the two sides OA and OB as co-ordinate axes, and designate by a and b the two lengths OA and OB. The median AE, cutting the axes at the distances a and $\frac{b}{2}$ from the origin, has for its equation,

$$\frac{x}{a} + \frac{2y}{b} = 1;$$

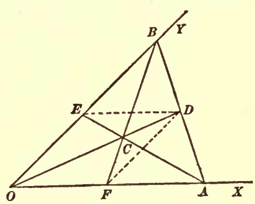

Fig. 49.

similarly, the median BF has for its equation,

$$\frac{2x}{a} + \frac{y}{b} = 1.$$

The mid-point of AB has the co-ordinates $OF = \frac{a}{2}$, $OE = \frac{b}{2}$; the line OD, which joins the origin and this point, has the equation,

$$y = \frac{bx}{a}.$$

By solving the first two equations, one gets the co-ordinates

$$x = \frac{a}{3}, \quad y = \frac{b}{3},$$

of the point C the intersection of AE and BF. These co-ordinates satisfy the third equation; hence the third median OD passes through the point C.

By applying the second method, we see at once that the three medians pass through the same point; for, by subtracting the second equation, member by member, from the first, we get the third equation.

72. Problem VII. — *Find the condition that three points lie on a straight line.*

Let x' and y', x'' and y'', x''' and y''', be the co-ordinates of three given points M', M'', M'''. If the points lie on a line, the preceding pairs of co-ordinates satisfy the equation $Ax + By + C = 0$, and the determinant

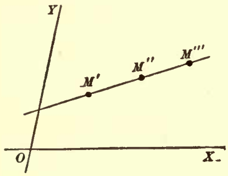

Fig. 50.

$$D = \begin{vmatrix} x', & y', & 1 \\ x'', & y'', & 1 \\ x''', & y''', & 1 \end{vmatrix}$$

is zero.

The lines $M'M''$, $M'M'''$, coincide, and their angular coefficients are equal, then will

$$\frac{y'' - y'}{x'' - x'} = \frac{y''' - y'}{x''' - x'}.$$

73. Example. — If the four sides of a quadrilateral $OACB$ are prolonged (Fig. 51), a complete quadrilateral $OACBA'B'$ is formed; the sides intersect two by two in six points or vertices; by joining the vertices one obtains the diagonals AB, $A'B'$, OC; it will be proven that the mid-points F, E, D, of the three diagonals OC, $A'B'$, AB, lie on a straight line.

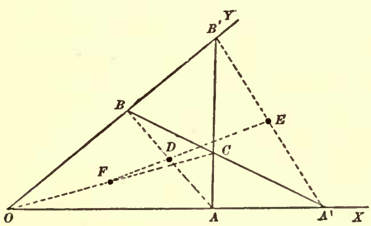

Fig. 51.

Choose the sides OA and OB as co-ordinate axes; represent by a and a' the abscissas of the points A and A'; by b and b' the ordinates of the points B and B'. The point D, middle of AB, has the co-ordinates $x' = \frac{a}{2}$, $y' = \frac{b}{2}$. The point E, middle of $A'B'$, has the co-ordinates $x'' = \frac{a'}{2}$, $y'' = \frac{b'}{2}$.

In order to get the co-ordinates of the point F, the middle of AC, seek those of the point C, which is the intersection of the lines AB', $A'B$, whose equations are

$$\frac{x}{a} + \frac{y}{b'} = 1, \quad \frac{x}{a'} + \frac{y}{b} = 1.$$

By solving these equations, the co-ordinates of the point C are found to be

$$x = \frac{aa'(b - b')}{ab - a'b'}, \quad y = \frac{bb'(a - a')}{ab - a'b'}.$$

The point F being the middle of the line OC, its co-ordinates x''', y''' are the halves of those of the point C; one has, therefore,

$$x''' = \frac{aa'(b-b')}{2(ab-a'b')}, \quad y''' = \frac{bb'(a-a')}{2(ab-a'b')}.$$

Having the co-ordinates of the points D, E, F one can easily show that they lie on a straight line. The lines DE and DF have the following angular coefficients:

$$\frac{y''-y'}{x''-x'} = \frac{b'-b}{a'-a}, \quad \frac{y'''-y'}{x'''-x'} = \frac{\frac{bb'(a-a')}{ab-a'b'} - b}{\frac{aa'(b-b')}{ab-a'b'} - a} = \frac{b'-b}{a'-a};$$

these two angular coefficients being equal to each other, it follows that the points D, E, F lie on a straight line.

74. PROBLEM VIII. — *Find the angle between two lines.*

Let $y = ax + b$, $y = a'x + b'$, be the equations of two given lines. Draw through the origin, and on the same side of the axis as OY, two lines OA and OA' parallel to the given lines (Fig. 52); call α and α' the angles which they form with OX, V the angle which they inclose, and, to be definite, let $\alpha' > \alpha$. Evidently one has $V = \alpha' - \alpha$, whence

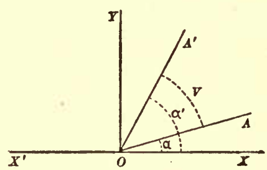

Fig. 52.

$$(15) \qquad \tan V = \frac{\tan\alpha' - \tan\alpha}{1 + \tan\alpha\tan\alpha'}.$$

When the axes are rectangular, one knows that

$$\tan\alpha = a, \quad \tan\alpha' = a',$$

if those values be substituted in the preceding formula,

$$(16) \qquad \tan V = \frac{a'-a}{1+aa'}.$$

In case the axes are oblique, one has (§ 61):

$$\tan \alpha = \frac{a \sin \theta}{1 + a \cos \theta}, \quad \tan \alpha' = \frac{a' \sin \theta}{1 + a' \cos \theta};$$

and, hence,

(17) $$\tan V = \frac{(a' - a) \sin \theta}{1 + aa' + (a + a') \cos \theta}.$$

One can deduce from these formulas the relation which must exist between the angular coefficients of two lines which are perpendicular to each other. In fact, in case the angle V is right, its tangent becomes infinite; one has, if the axes are rectangular,

(18) $$1 + aa' = 0,$$

and, if they are oblique,

(19) $$1 + aa' + (a + a') \cos \theta = 0.$$

75. Problem IX. — *From a given point draw a perpendicular to a given line, and find the length of this perpendicular.*

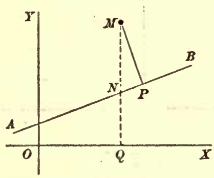

Fig. 53.

Let (2) $$y = ax + b$$

be the equation of the given line AB, x' and y' the co-ordinates of the given point M (Fig. 53). Suppose the axes to be rectangular. Any line passing through the point M has an equation of the form (§ 64)

$$y - y' = a'(x - x').$$

In order that this line be perpendicular to the line AB, it is necessary that the relation $1 + aa' = 0$, be satisfied (§ 74); whence it follows that $a' = -\frac{1}{a}$. On replacing a' by its value, one gets the equation of the perpendicular MP

(20) $$y - y' = -\frac{1}{a}(x - x').$$

The co-ordinates x and y of the foot P of the perpendicular, or the point of intersection of the two lines AB and MP, are found by solving the simultaneous equations (2) and (20); but it is necessary to calculate the differences $x - x'$ and $y - y'$ in terms of quantities which do not contain x and y (§ 55). Equation (2) can be written in the form

$$y - y' = a(x - x') - (y' - ax' - b);$$

if, in this equation, $y - y'$ is replaced by its value derived from equation (20), one finds

$$x - x' = \frac{a(y' - ax' - b)}{1 + a^2},$$

and hence, by virtue of equation (20),

$$y - y' = -\frac{y' - ax' - b}{1 + a^2}.$$

By applying the formula for the distance between two points (§ 55), one gets the length l of the perpendicular MP,

$$l = \sqrt{(x - x')^2 + (y - y')^2} = \sqrt{\frac{(y' - ax' - b)^2 (1 + a^2)}{(1 + a^2)^2}},$$

whence (21) $$l = \pm \frac{y' - ax' - b}{\sqrt{1 + a^2}}.$$

The sign is so chosen that l will have a positive value. It is easy to see that the numerator is positive or negative, according as the point M is situated on the opposite or origin side of the line AB. For, let N be the point where the line AB is intersected by a line drawn from the point M parallel to the axis of y; the point N being on the line AB, the ordinate y_1 of this point will equal $ax' + b$, so that the formula (21) becomes

$$l = \pm \frac{y' - y_1}{\sqrt{1 + a^2}}.$$

The difference, $y' - y_1$, is positive in the first case and negative in the second.

It is to be noticed that the length of the perpendicular under this last form may be obtained immediately, by noticing that the right-angled triangle MNP gives

$$MP = MN \sin MNP = \pm (y' - y_1) \cos \alpha$$

$$= \pm \frac{y' - y_1}{\sec \alpha} = \pm \frac{y' - y_1}{\sqrt{1 + a^2}}.$$

76. Suppose that the axes are oblique; the lines AB and MP will be perpendicular if their angular coefficients a and a' satisfy the relation $1 + aa' + (a + a') \cos \theta = 0$ (by 17); whence $a' = -\frac{1 + a \cos \theta}{a + \cos \theta}$. Therefore the equation of the perpendicular MP is

(22) $$y - y' = -\frac{1 + a \cos \theta}{a + \cos \theta}(x - x').$$

By solving the two simultaneous equations (2) and (22), one gets the co-ordinates x and y of the point P. If, as above, one seeks the differences $x - x'$, $y - y'$, one finds

$$x - x' = \frac{(y' - ax' - b)(a + \cos \theta)}{1 + a^2 + 2a \cos \theta},$$

$$y - y' = -\frac{(y' - ax' - b)(1 + a \cos \theta)}{1 + a^2 + 2a \cos \theta};$$

substituting these differences in the formula for the distance between two points (§ 56),

$$l = \pm \sqrt{(x - x')^2 + (y - y')^2 + 2(x - x')(y - y') \cos \theta},$$

one gets

$$l = \pm \frac{\sqrt{(a + \cos \theta)^2 + (1 + a \cos \theta)^2 - 2(a + \cos \theta)(1 + a \cos \theta) \cos \theta}}{1 + 2a \cos \theta + a^2} \times (y' - ax' - b)$$

By developing, one remarks that the quantity under the radical contains the factor $1 - \cos^2 \theta$ or $\sin^2 \theta$, and is equal to

$$(1 + 2a \cos \theta + a^2) \sin^2 \theta;$$

hence (23) $$l = \pm \frac{(y' - ax' - b) \sin \theta}{\sqrt{1 + 2a \cos \theta + a^2}}.$$

The numerator will be positive or negative according as the point M is situated on the side of AB opposite to, or on the same side as the origin. The sign is so chosen that l is positive.

77. In what precedes, we have supposed that the equation of the given line has the form $y = ax + b$. If the equation has the general form

(1) $$Ax + By + C = 0,$$

the angular coefficient a of the given line being equal to $-\frac{A}{B}$, one will have, in case of rectangular co-ordinates, $a' = -\frac{1}{a} = \frac{B}{A}$, and the perpendicular let fall from M will be represented by the equation

$$y - y' = \frac{B}{A}(x - x'),$$

or (24)
$$\frac{x - x'}{A} = \frac{y - y'}{B}.$$

Formula (21), in which one substitutes for a and b their values $-\frac{A}{B}$, $-\frac{C}{B}$, becomes

(25)
$$l = \pm \frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}.$$

This formula is an expression for the distance of a point from a straight line in rectangular co-ordinates: the numerator is the first member of the equation of the line, in which x and y are replaced by the co-ordinates of the point; the denominator is the square root of the sum of the squares of the coefficients of x and y.

When the axes are oblique, one has

$$a' = \frac{B - A\cos\theta}{A - B\cos\theta};$$

the equation of the perpendicular will be

(26)
$$\frac{x - x'}{A - B\cos\theta} = \frac{y - y'}{B - A\cos\theta},$$

and formula (23) becomes

(27)
$$l = \pm \frac{(Ax' + By' + C)\sin\theta}{\sqrt{A^2 + B^2 - 2AB\cos\theta}}.$$

It is easy to determine the sign of the numerator, according to the position of the point M with respect to the line AB. Let N (Fig. 53) be the point where the line AB is intersected by MQ drawn parallel to the axis of y; imagine a movable point, having x and y for co-ordinates, to travel along this parallel, and consider the values of the polynomial $Ax+By+C$ for the various positions of the movable point. If the movable point be at N, the value of the polynomial is zero. If the coefficient B is positive when one travels in the direction of positive y's, the term By increases, and the function takes greater and greater positive values; when one travels in the opposite direction, it takes negative values; the contrary is true when B is negative.

78. Problem X. — *Through the point of intersection of two given lines, draw a line perpendicular to a given line.*

Let

$$Ax + By + C = 0,$$
$$A'x + B'y + C' = 0,$$
$$A''x + B''y + C'' = 0,$$

be the equation of three lines in rectangular co-ordinates. Every line passing through the intersection of the first two is represented by an equation of the form

$$(Ax + By + C) + \lambda(A'x + B'y + C') = 0;$$

in order that it be perpendicular to the third, one must have

$$1 + \frac{A''(A + \lambda A')}{B''(B + \lambda B')} = 0;$$

whence one finds

$$\lambda = -\frac{AA'' + BB''}{A'A'' + B'B''}.$$

On replacing λ by its value, one obtains the equation sought,

$$(28)\quad (A'A'' + B'B'')(Ax + By + C) = (A''A + B''B)(A'x + B'y + C').$$

79. The three given lines form a triangle whose vertices are the intersections of these lines two by two. Equation (28) represents the perpendicular let fall from one of the vertices to the side opposite. By permuting the accents, one gets the equation of the perpendiculars let fall from each of the other two vertices to its opposite side, *i.e.*,

$$(A''A + B''B)(A'x + B'y + C') = (AA' + BB')(A''x + B''y + C''),$$
$$(AA' + BB')(A''x + B''y + C'') = (A'A'' + B'B'')(Ax + By + C).$$

By adding the first two of these equations member to member, one obtains the third. Hence one infers (§ 70) that the three altitudes of a triangle pass through a common point.

80. PROBLEM XI. — *To find the locus of all points equally distant from two given points.*

Suppose the axes to be rectangular, and let x' and y', x'' and y'' be the co-ordinates of the two given points. If x and y are the co-ordinates of any point whatever of the locus, the equation of the locus will be

$$(x - x')^2 + (y - y')^2 = (x - x'')^2 + (y - y'')^2,$$

or, more simply,

$$(29) \qquad (x'' - x')\left(x - \frac{x' + x''}{2}\right) + (y'' - y')\left(y - \frac{y' + y''}{2}\right) = 0.$$

This locus is a straight line perpendicular to the line joining the two given points at its mid-point.

81. PROBLEM XII. — *To find the locus of all points which are equally distant from two given lines.*

Let us suppose the axes to be rectangular. Let

$$Ax + By + C = 0,$$

$$A'x + B'y + C' = 0,$$

be the equations of the two given lines. If one represent the co-ordinates of any point of the locus by x and y, the equation of the locus will be

$$(30) \qquad \frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \pm \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}}.$$

Owing to the double sign, this equation represents two lines, which are the bisectors of the angles which are formed by the given lines.

Equation of the Straight Line in Polar Co-ordinates.

82. Let O be the pole and OX the polar axis. The position of a line AB can be determined by the length a of the perpendicular let fall from the origin on this line, and by the angle α which this perpendicular makes with the polar axis, this angle having the limits 0 and 2π. Let ρ and ω be the co-ordinates of any point of this line; by projecting the radius vector OM on the perpendicular OD, one has

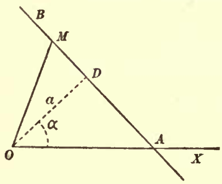

Fig. 54.

$$(31)\qquad \rho\cos(\omega-\alpha)=a;\quad \text{or}\quad \rho=\frac{a}{\cos(\omega-\alpha)}.$$

Since a and α are constants, this equation can be given the form, by developing $\cos(\omega-\alpha)$,

$$(32)\qquad \rho=\frac{C}{A\cos\omega+B\sin\omega}.$$

Conversely, every equation of this form represents a straight line; for, by referring it to rectangular co-ordinates, *i.e.*, by taking the polar axis as the x-axis, and a perpendicular to it at the pole as the y-axis, then using the transformation formulas $x=\rho\cos\omega$, $y=\rho\sin\omega$, the new equation is $Ax+By=C$.

Remark. — If the line pass through the pole, then $a=0$ and ρ, in equation (31), is not zero; therefore $\cos(\omega-\alpha)=0$, or

$$\omega=\alpha+\frac{\pi}{2},\ \alpha+\frac{3\pi}{2},\cdots,$$

i.e., $$\omega=\text{constant}.$$

Another Form of the Equation to a Straight Line.

83. Equation (31) developed becomes:

$$\rho \cos \omega \cos \alpha + \rho \sin \omega \sin \alpha = a,$$

or, in rectangular co-ordinates,

(33) $$x \cos \alpha + y \sin \alpha - a = 0.$$

The equation of the line being put under this form, its first member has a very simple geometric meaning. Let any point M of the plane, whose polar co-ordinates are ρ and ω, and rectangular co-ordinates x and y, be considered; from this point drop a perpendicular MP on the line AB (Fig. 55). The projection of the radius vector OM on the line OD is $\rho \cos (\omega - \alpha)$; but this projection is equal to OD, increased or diminished by the perpendicular PM, according as the point M and the origin O are situated on opposite sides or on the same side of the line; if therefore this perpendicular be represented by p, affected with the + sign in the first case, and by the − sign in the second, one will have, in general,

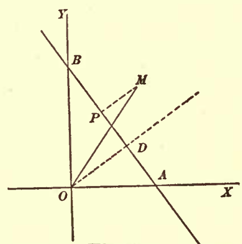

Fig. 55.

$$a + p = \rho \cos (\omega - \alpha) = x \cos \alpha + y \sin \alpha,$$

whence $$p = \pm (x \cos \alpha + y \sin \alpha - a).$$

Thus, the first member of the equation (33) represents the distance from any point of the plane, whose co-ordinates are x and y, to the line represented by this equation, this distance affected with the proper sign.

It is easy to deduce the co-ordinates x_1 and y_1 of the foot P of the perpendicular; the differences $x - x_1$, $y - y_1$, being the projections of the line PM on the two axes, we have

$$x - x_1 = p \cos \alpha = (x \cos \alpha + y \sin \alpha - a) \cos \alpha,$$

$$y - y_1 = p \sin \alpha = (x \cos \alpha + y \sin \alpha - a) \sin \alpha.$$

The form (33), under which the equation of the line can always be put, is useful in a large number of investigations.

83. 2. *The equation of a line passing through two points.* Let

$$(\rho_1, \omega_1),\ (\rho_2,\ \omega_2)$$

be the co-ordinates of the two points; the equation of the line joining these two points is

$$\begin{vmatrix} \frac{1}{\rho} & \cos\omega & \sin\omega \\ \frac{1}{\rho_1} & \cos\omega_1 & \sin\omega_1 \\ \frac{1}{\rho_2} & \cos\omega_2 & \sin\omega_2 \end{vmatrix} = 0;$$

in fact, this equation represents a straight line since it can be put under the form of (32), and it is evidently verified by $\rho = \rho_1$, $\omega = \omega_1$, and $\rho = \rho_2$, $\omega = \omega_2$.

CHAPTER II

THE CIRCLE.

84. We seek first the equation of the circumference of a circle in rectangular co-ordinates. Represent by a and b the co-ordinates of the center C (Fig. 56), and by r the radius; the circumference, being the locus of the points whose distance from the center is equal to the radius, has for its equation

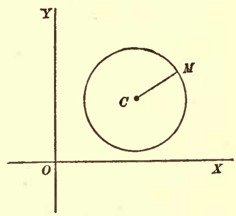

Fig. 56.

(1) $$(x-a)^2+(y-b)^2=r^2;$$

this equation developed may be written

(2) $$A(x^2+y^2)+2\,Dx+2\,Ey+F=0.$$

Hence, *the equation of the circle, in rectangular co-ordinates, is an equation of the second degree, which does not involve the product xy of the variables, and in which the terms in x^2 and y^2 have the same coefficient.*

85. Conversely, every equation of this form, in rectangular co-ordinates, represents a circumference of a circle, if it represents a locus. In fact, equation (2) can be written in the form

$$\left(x+\frac{D}{A}\right)^2+\left(y+\frac{E}{A}\right)^2=\frac{D^2+E^2}{A^2}-\frac{F}{A}.$$

The center C will, therefore, by (1), have the co-ordinates $-\frac{D}{A}$ and $-\frac{E}{A}$; the first member represents the square of the distance of any point M of the plane, having the co-ordinates x and y, from the point C; if the second member is positive, the equation will be satisfied by the co-ordinates of all the points of the plane whose distance from C is equal to $\sqrt{\frac{D^2+E^2}{A^2}-\frac{F}{A}}$; it represents therefore a circumference of a circle. When the second member is zero, the distance MC becomes zero, the point M will coincide with the point C, and the equation will still be satisfied by the co-ordinates of this point; the locus will be reduced therefore to a single point.

Finally, in case the second member is negative, the equation cannot be satisfied by any point of the plane; because the square of the distance of the point M from the point C is a positive quantity; the equation cannot therefore, in this case, represent a geometrical locus.

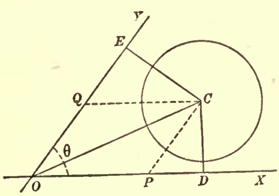

Fig. 57.

86. Suppose now that the co-ordinate axes are oblique, and inclose an angle θ (Fig. 57); by expressing that the distance of any point of the locus from the center is equal to the radius, one will have the equation of the circumference,

(3) $$(x-a)^2+(y-b)^2+2\,(x-a)\,(y-b)\cos\theta=r^2.$$

This equation may be written in the form

(4) $$A(x^2+y^2+2\,xy\cos\theta)+2\,Dx+2\,Ey+F=0.$$

Hence, *the equation of a circle, in oblique co-ordinates, is an equation of the second degree, in which the terms in x^2, in y^2, and in $2\,xy\cos\theta$ have the same coefficients.*

On dividing by A, one reduces, as in (3), the coefficients of x^2, y^2, $2\,xy\cos\theta$ to unity.

87. Conversely, every equation of this form represents a circumference of a circle, if it represent a locus. In fact, one can determine the three constants, a, b, and r^2, by comparing equations (3) and (4). Equation (3), developed, becomes

$$x^2+y^2+2\,xy\cos\theta-2\,(a+b\cos\theta)x-2\,(b+a\cos\theta)y$$
$$+\,a^2+b^2+2\,ab\cos\theta-r^2=0.$$

Equations (3) and (4) will become identical by placing

$$a + b\cos\theta = -\frac{D}{A},\quad b + a\cos\theta = -\frac{E}{A},$$

$$a^2 + b^2 + 2\,ab\cos\theta - r^2 = \frac{F}{A}.$$

The first two relations give finite values for a and b, since the determinant $1 - \cos^2\theta$ or $\sin^2\theta$ is different from zero. The third gives

$$r^2 = a^2 + b^2 + 2\,ab\cos\theta - \frac{F}{A}.$$

Notice the point C, which has the co-ordinates a and b. The first member of equation (3) represents the square of the distance of any point M of the plane, whose co-ordinates are x and y, from the point C. If r^2 is found to be a positive quantity, the equation will be satisfied for every point of the plane whose distance from C is equal to r; it represents therefore the circumference of a circle. If r^2 has the value zero, then the distance MC equals zero, and the equation will be satisfied by the co-ordinates of the point C; it will represent a single point. Finally, if r^2 have a negative value, the equation will not be satisfied by a single point of the plane.

Instead of determining the center C of the circle by its co-ordinates a and b, it is more convenient to determine it by the orthogonal projections of the line OC on the two axes. Call these two projections OD and OE, a' and b' (Fig. 57), affected by proper signs, and express the fact that the projection of the line OC on the one or on the other axis is equal to that of the broken line OPC or OQC; one has

$$a' = a + b\cos\theta,\quad b' = b + a\cos\theta;$$

whence, $a' = -\frac{D}{A}$, $b' = -\frac{E}{A}$. After having laid off the lengths a' and b' on the axes, beginning at the origin, one erects perpendiculars to the axes at the points D and E; the intersection of the two perpendiculars will determine the center C.

88. The equation of the circumference of a circle, as has been found, is

$$(5)\quad (x-a)^2+(y-b)^2+2(x-a)(y-b)\cos\theta - r^2 = 0.$$

The first member has a geometrical signification which it is well to notice. Consider a point M of the plane having the co-ordinates x and y; the expression

$$(x-a)^2+(y-b)^2+2(x-a)(y-b)\cos\theta$$

represents the square of the line MC, which joins the point M with the center (Fig. 58); the first member of the equation is, therefore, equal to $\overline{MC}^2 - r^2$, that is, to the product of the two factors $MC + r$ and $MC - r$, which are the two segments MA and MB of the diameter drawn through the point M, the segments being affected by the same or contrary signs, according as they are measured in the same or in opposite directions. Thus the first member of equation (5) represents the product of two segments of any secant drawn from the point M, that is, the power of this point with respect to the circle. When the point M is without the circumference, this product is equal to the square of the tangent.

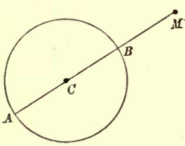

Fig. 58.

89. Problem I. — *To find the equation of the tangent to any curve.*

We have already given the definition of a tangent at a point M of a curve (§ 19). Through the point M and a neighboring point M' on the curve draw a secant MM', and allow the point M' to continually approach the point M. The secant MM' will revolve about the point M, and if it tends toward a limiting position MT, this line MT is called the *tangent* to the curve at the point M (Fig. 59).

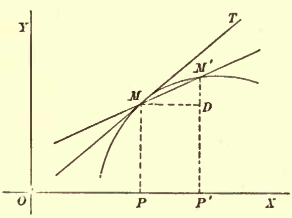

Fig. 59.

Let x and y be the co-ordinates of the point of contact M; $x+h$ and $y+k$ those of the neighboring point M'; the angular coefficient of the secant MM' is the ratio $\frac{k}{h}$ of the difference of the ordinates of the two points M and M' to the difference of their abscissas. As the point M' approaches indefinitely toward the point M, the two increments h and k tend simultaneously toward zero; we study here curves defined by equations such that the ratio $\frac{k}{h}$ tends toward a limit, which is the derivative of the ordinate regarded as a function of its abscissa.

If the equation of the curve is solved with respect to y, and put under the form $y=f(x)$, the tangent will have for its angular coefficient $y'=f'(x)$. In case the equation of the curve $f(x, y)=0$ cannot be solved, the derivative y' of the implicit function y can be derived from the equation $f'_x + y' \cdot f'_y = 0$, in which f'_x and f'_y represent the partial derivatives of the function $f(x, y)$, with respect to x and y. Whence it follows

$$(6) \qquad y' = -\frac{f'_x}{f'_y}.$$

Thus, if X and Y be the co-ordinates of any point of the tangent, the equation of this line will be

$$(7) \quad Y - y = -\frac{f'_x}{f'_y}(X-x), \text{ or } (X-x)f'_x + (Y-y)f'_y = 0.$$

90. Problem II. — *To find the equation of the tangent to the circle.*

Let the preceding formula be applied to the circle, supposing that the axes are rectangular and the origin is at the center of the circle. The equation of the circle is

$$(8) \qquad x^2 + y^2 - r^2 = 0.$$

The equation, solved for y, becomes $y = \pm\sqrt{r^2 - x^2}$; on taking the derivative of this function, one has

$$y' = \frac{-x}{\pm\sqrt{r^2-x^2}} = -\frac{x}{y}.$$

By leaving the equation unsolved and applying formula (6), the same value $y' = -\frac{x}{y}$ is obtained. Thus the equation of the tangent is

$$Y - y = -\frac{x}{y}(X - x), \text{ or } xX + yY = x^2 + y^2.$$

Since the point M is on the circle, its co-ordinates satisfy the equation of the circle, and one has $x^2 + y^2 = r^2$. The equation of the tangent is simplified and becomes

$$(9) \qquad xX + yY = r^2.$$

Since the angular coefficient of the radius drawn to the point of contact is $\frac{y}{x}$, it follows that the tangent is perpendicular to this radius.

91. Problem III. — *To draw a tangent to a circle from an exterior point.*

Suppose that the circle is always referred to rectangular axes drawn through the center, and represented by the equation

$$(8) \qquad x^2 + y^2 = r^2;$$

call x_1 and y_1 the co-ordinates of the given point P (Fig. 60). Let MP be a tangent drawn from this point; the question is now to determine the point M, whose unknown co-ordinates are assumed to be x and y. The point M being on the circle, its co-ordinates satisfy equation (8). The tangent at the point M has the equation $xX + yY = r^2$. This tangent passes through the point P, and the equation must be satisfied by the co-ordinates of this point, which furnishes the relation

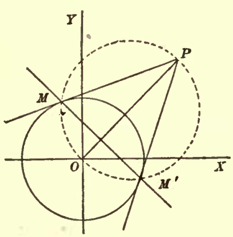

Fig. 60.

$$(10) \qquad xx_1 + yy_1 = r^2.$$

By solving the two simultaneous equations (8) and (10), the values of the unknown x and y will be obtained.

The solving of the two equations (8) and (10) results in finding the point of intersection of the two lines. The first equation represents the given circle; the second a straight line. To find the values of x and y, which satisfy at the same time these two equations, is to find the points of intersection of the line and circle. This line cuts the circle in two points M and M' and is the line of contact. It is to be noticed that equation (10) of the line of contact has the same form as equation (9) of the tangent; only, that the co-ordinates of the point of contact are replaced by those of point P.

92. One knows that, in case one has two equations

$$A = 0,\ B = 0,$$

simultaneous with respect to two unknown quantities x and y, if one of the equations be replaced by $mA + nB = 0$, which is obtained by multiplying the equations by the arbitrary quantities m and n, and then adding them member to member, one forms a new system of equations

$$A = 0,\ Am + Bn = 0,$$

equivalent to the given system. This signifies geometrically that the points of intersection of the two curves represented by the two given equations are the same as the points of intersection of one of them with the third curve.

It has been stated that the points of contact M and M' are given by the intersection of the given circle and the line of contacts. By subtracting the two equations (8) and (10) member from member, one obtains the new equation

$$x^2 + y^2 - x_1x - y_1y = 0,$$

or

$$\left(x - \frac{x_1}{2}\right)^2 + \left(y - \frac{y_1}{2}\right)^2 = \frac{x_1^2 + y_1^2}{4},$$

which may replace equation (10). This new equation represents a circle whose center is the mid-point of the line OP and has the co-ordinates $\frac{x_1}{2}$ and $\frac{y_1}{2}$. Since the equation has no constant term, the circle passes through the origin and is, therefore, described on OP as a diameter. The points in which the circle cuts the given circle are the points of contact. In this manner the construction of elementary geometry is verified.

93. Problem IV. — *To draw a tangent parallel to a given line.* To the circle

(8) $$x^2 + y^2 = r^2,$$

it is required to draw a tangent parallel to the line OA, which is supposed to be drawn through the origin, and to be represented by the equation $y = mx$ (Fig. 61). If x and y be the co-ordinates of the point of contact M, one knows that the angular coefficient of the tangent is equal to $-\frac{x}{y}$. In order that the tangent MT be parallel to the given line, the angular coefficient must be equal to m, *i.e.* $-\frac{x}{y} = m$, or

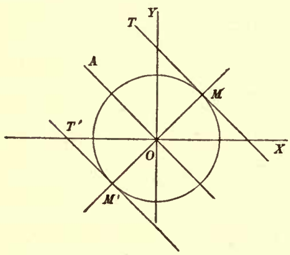

Fig. 61.

(11) $$y = -\frac{1}{m}x.$$

Further, the co-ordinates of the point M satisfy the equation of the circle. These co-ordinates are therefore determined by the two simultaneous equations (9) and (11), and, consequently, the points of contact M and M' are given by the points of intersection of the circle and the line represented by equation (11). It may easily be shown that the line MM' is perpendicular to the line OA.

94. This problem may be discussed in another manner, and this will give us the opportunity of presenting the equation of the tangent to the circle in a new form. Let us therefore seek the points of intersection of the circle $x^2 + y^2 = r^2$, by the line $y = mx + k$. On eliminating y, one gets the equation of the second degree, $x^2 + (mx + k)^2 = r^2$, or

$$(m^2 + 1)x^2 + 2mkx + k^2 - r^2 = 0.$$

When this equation has real roots, the line cuts the circle in two real points whose abscissas are the roots of the equation. In case the roots of the equation are equal, the points of intersection will coincide, and the line becomes a tangent to the circle. Finally, when the roots are imaginary, the line does not cut the circle.

Thus, the condition that the line be tangent to the circle is

$$m^2k^2 = (m^2 + 1)(k^2 - r^2), \text{ or } k^2 = r^2(m^2 + 1).$$

Substituting for k its value, the equation of the line becomes

(12) $$y = mx \pm r\sqrt{m^2 + 1}.$$

This equation, which involves a single arbitrary parameter m, represents all the tangents to the circle.

95. Problem V.— *To find the locus of the points whose distances from two fixed points are in a given ratio.*

Let A and B be the two given points (Fig. 62). Take the line AB as the x-axis, and for the y-axis a perpendicular to AB at its middle point. If one calls $2a$ the distance AB, $\frac{m}{n}$ the given ratio, and if x and y designate the co-ordinates of any point in the locus, the equation of this locus will be

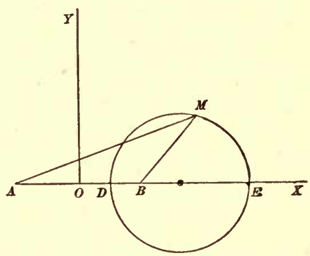

Fig. 62.

$$\frac{y^2 + (x + a)^2}{y^2 + (x - a)^2} = \frac{m^2}{n^2},$$

or (13) $$x^2 + y^2 - 2ax\frac{m^2 + n^2}{m^2 - n^2} + a^2 = 0.$$

This is a circle whose center lies on the axis of X. The two extremities of the diameter DE are the points which divide the line AB in the ratio m to n.

96. Problem VI. — *To find the points of intersection of two circles.*

Let (14) $$x^2 + y^2 + 2Dx + 2Ey + F = 0,$$

(15) $$x^2 + y^2 + 2D'x + 2E'y + F' = 0,$$

be the equations of two circles in rectangular co-ordinates, the coefficients of $x^2 + y^2$ being equal to unity. The points of intersection will be given by these two simultaneous equations. One can replace the second circle by the line

$$(16) \qquad 2(D-D')x+2(E-E')y+(F-F')=0,$$

which is obtained by subtracting the equations member from member, and the question is reduced to finding the points of intersection of the first circle with this line. If the line cuts the circle, the circles have two points of intersection, and equation (16) represents the common secant. If the line becomes tangent to the circle, the points of intersection coincide, and the two circles will be tangent; equation (16) will in this case represent the common tangent. Finally, when the line does not cut the circle, the two circles do not have a common point.

Moreover, the equation (16) has in every case a remarkable geometrical signification. The first members of equations (14) and (15) represent respectively (§ 88) the powers of any point M of the plane, having the co-ordinates x and y, with respect to the two given circles; whence equation (16) may be obtained by equating these two expressions, the terms of the second degree canceling; equation (16) represents, therefore, the locus of points of equal power with respect to two circles; this locus is a straight line, which is called the *radical axis* of the two circles. The portion of this line external to the circles is the locus of the points from which any pair of tangents drawn to the two circles are equal each to each. It is clear that the radical axes of three circles meet in a point; this point is called the *radical center* of the three circles. When it is exterior to the three circles, the tangents emanating from this point have the same lengths. The circle described about this point as center with a radius equal to the common length of the tangents is orthogonal to the three circles considered.

Remark. — If the coefficients of x^2+y^2 be not equal to unity, and if the equation of the two circles are of the form

$$(17) \qquad \begin{aligned} f(x, y) &\equiv A\,(x^2+y^2)+2\,Dx\,+2\,Ey\,+F=0, \\ \phi(x, y) &\equiv A'(x^2+y^2)+2\,D'x+2\,E'y+F'=0, \end{aligned}$$

the equation of the radical axis can be derived by eliminating the terms of the second degree between the two equations, that is to say, by multiplying the first by $-A'$ the second by A, and adding; thus is found the equation

(18) $$A\phi - A'f \equiv 2(AD' - DA')x + 2(AE' - EA')y + AF' - FA' = 0.$$

More clearly does this equation represent the radical axis, because the power of the point (x, y) with respect to the first circle is $\frac{f(x, y)}{A}$, with respect to the second, $\frac{\phi(x, y)}{A'}$; by equating these two powers and clearing of fractions, one gets equation (18).

97. PROBLEM VII. — *To find the general equation of the circles which pass through the points of intersection of two given circles.*

The totality of these circles is called a *pencil of circles.* Their equation may be found by a method identical with that employed in the analogous problem of straight lines (§ 69). Let the two circles be represented by equations (17), the equation

(19) $$f(x, y) + \lambda\phi(x, y) = 0,$$

that is to say,

$$(A + \lambda A')(x^2 + y^2) + 2(D + \lambda D')x + 2(E + \lambda E')y + F + \lambda F' = 0,$$

where λ plays the rôle of an arbitrary constant, represents a circle passing through the points of intersection of the two given circles; for the co-ordinates of each of the points of intersection reducing f and ϕ to zero, evidently make $f + \lambda\phi = 0$. Equation (19) is the most general equation of the circle sought, that is to say, for every value of λ, it represents some circle S passing through the points common to the given circles. In fact, choose a point (x_1, y_1) on the circle S, and determine λ by the equation of the first degree

$$f(x_1, y_1) + \lambda\phi(x_1, y_1) = 0,$$

which expresses the condition that the circle (19) passes through the point (x_1, y_1). The coefficient λ being thus determined, the circle (19) and the circle S coincide, because they have in common three points at finite distances, namely, the two points of intersection of the given circles and the given point (x_1, y_1).

All of the circles of the pencil (19), taken two by two, have the same radical axis, which is no other than the radical axis of the given circles (18). This radical axis is to be reckoned among the circles of the pencil, for it is derived by giving the particular value $-\frac{A}{A'}$ to λ, which causes the terms of the second degree to disappear.

LIMITING POINTS. — Take, for simplicity, the line through the centers of the given circles as the x-axis and their radical axis as the y-axis; the equations of the two circles take the form

$$(20)\qquad x^2 + y^2 - 2\,ax + c = 0,$$

$$x^2 + y^2 - 2\,a'x + c = 0,$$

a and a' being the abscissas of the centers of the two circles and c the power of the origin with respect to each of the two circles, the power being the same for the two circles, because by hypothesis the origin is on the radical axis. The general equation of the circles passing through the points common to the two circles is

$$(1 + \lambda)(x^2 + y^2) - 2(a + \lambda a')x + (1 + \lambda)c = 0,$$

or more simply on dividing by $(1 + \lambda)$ and calling the ratio

$$\frac{a + \lambda a'}{1 + \lambda} = \mu,$$

$$(21)\qquad x^2 + y^2 - 2\,\mu x + c = 0,$$

where μ represents an arbitrary coefficient. This last equation could have been deduced *a priori*, because it is the general equation of the circles, which, associated with either of the given circles, has OY as the radical axis.

Among the circles (21), there are *two*, each of which reduces to a point real or imaginary, or, in other words, has a radius equal to zero; these two circles are the *limiting points.* Equation (21) can be written

$$(x-\mu)^2+y^2=\mu^2-c.$$

Therefore, for $\mu=\pm\sqrt{c}$, this circle reduces to the point $x=\mu$, $y=0$. If c is positive, that is, if the origin is *exterior* to the two circles, or what amounts to the same thing, if the two circles intersect in imaginary points, the values of μ *are real* and the two limiting points are real. In this case, $\sqrt{c}$ represents the common length of the two tangents drawn from the point O to the given circles; the limiting points are therefore the intersections of the line of centers, Ox, with the circle described about the foot O of the radical axis as center, with a radius equal to the length of the tangent drawn from O to any one of the given circles. If, on the other hand, the two given circles (20) intersect in real points, O being within the two circles, c is negative and the two limiting points are imaginary. If the two given circles are *tangent*, O is their point of contact, $c=0$, $\mu=0$, and the two limiting points coincide with O

97. 2. PROBLEM VII. — *Find the condition that two circles intersect orthogonally.*

When two circles intersect at right angles, the radii drawn to the point of intersection M are perpendicular, because they are perpendicular to tangents which are perpendicular by hypothesis. The triangle, which has as vertices the point M and the two centers, is therefore right angled at M, and the square of the distance between the centers is equal to the sum of the squares of the radii. Suppose that the two circles are represented by equation (17); then by using the expressions given in § 85 for the co-ordinates of the center of a circle and the square of its radius, the condition that the two circles cut orthogonally in rectangular co-ordinates will be

$$(22) \qquad AF'+FA'-2(DD'+EE')=0.$$

The same result can be derived without the assistance of geometry. Let (x, y) be a point common to the two circles (17): the angular coefficients of the tangents to the two circles at this point being respectively (§ 89) $-\frac{f'_x}{f'_y}$ and $-\frac{\phi'_x}{\phi'_y}$, the necessary and sufficient condition that the two circles be orthogonal at the point (x, y) is

$$f'_x \phi'_x + f'_y \phi'_y = 0,$$

which becomes by substituting and developing

$$(23) \qquad AA'(x^2 + y^2) + (AD' + DA')x + (AE' + A'E)y + DD' + EE' = 0.$$

If x, y be regarded as current co-ordinates, this last equation represents a circle, and, as it should be satisfied by the points of intersection of the two given circles (17), it ought to represent a circle passing through the points of intersection of these two circles. Moreover, the three circles (17) and (23), taken two by two, should have the same radical axis. The radical axis of circle (23) and the first circle, $f = 0$, of (17) has the equation

$$(AD' - DA')x + (AE' - EA')y + DD' + EE' - FA' = 0;$$

the equation of the radical axis of circle (23) and the second circle $\phi = 0$, of (17), is

$$(AD' - DA')x + (AE' - EA')y - DD' - EE' + AF' = 0.$$

Expressing the condition that these two equations should be identical, one gets equation (22).

The condition expressed in (22) may be verified by supposing $A' = 0$; the second circle becomes a straight line, and the condition of orthogonality ought to express the condition that this straight line pass through the center of the first circle.

Remark. — The condition of orthogonality is linear and homogeneous with respect to the coefficients of each of the circles. Conversely, if between the coefficients A, D, E, F of the equation of a circle

$$A(x^2+y^2)+2Dx+2Ey+F=0,$$

any linear and homogeneous equation

$$LA+MD+NE+PF=0$$

be established, this relation compared with condition (22) shows that the circle considered is orthogonal to the fixed circle

$$P(x^2+y^2)-Mx-Ny+L=0.$$

APPLICATIONS. — To find the equation of a circle which cuts orthogonally three given circles

$$f(x,\ y)\equiv A(x^2+y^2)+2Dx+2Ey+F=0,$$

$$\phi(x,\ y)\equiv A'(x^2+y^2)+2D'x+2E'y+F'=0,$$

$$\psi(x,\ y)\equiv A''(x^2+y^2)+2D''x+2E''y+F''=0$$

Let (24) $$a(x^2+y^2)+2dx+2ey+f=0$$

be the circle sought; one should have, after condition (22) has been applied to circle (24), associated with each of the other three in order and been arranged with respect to a, d, e, and f.

(25) $$aF-2dD-2eE+fA=0,$$

$$aF'-2dD'-2eE'+fA'=0,$$

$$aF''-2dD''-2eE''+fA''=0.$$

If the three given circles taken two by two do not have the same radical axis, these equations give a single system of values for the ratio of any one of the coefficients a, d, e, f to the other three; there is, therefore, one circle *only* cutting the proposed circles at right angles; it is called the *orthotomic* circle. Its equation is obtained by eliminating a, $2d$, $2e$, f between equations (24) and (25), which gives in determinant form

$$\begin{vmatrix} x^2+y^2, & -x, & -y, & 1 \\ F, & D, & E, & A \\ F', & D', & E', & A' \\ F'', & D'', & E'', & A'' \end{vmatrix}=0.$$

It is admissible to suppose that any one of the coefficients A, A', A'' is zero; the corresponding circle is then replaced by a straight line.

NET OF CIRCLES. — Let $f(x, y)$, $\phi(x, y)$, $\psi(x, y)$, be the first members of the equation of the three preceding circles, which, taken two by two, do not have the same radical axis; the equation

$$\lambda f(x, y) + \mu\phi(x, y) + \nu\psi(x, y) = 0, \tag{26}$$

where λ, μ, ν are arbitrary coefficients, represents an infinitude of circles, forming what is called a *net.* It is desired to determine the condition that every circle of the net be orthogonal to some fixed circle, that is, *orthotomic* to a circle. In fact, by adding equations (25), member to member, after having multiplied the first by λ, the second by μ, the third by ν, a relation is obtained which expresses precisely that the circle (26) is orthogonal to the orthotomic circle (24).

Conversely, the totality of the circles which are orthogonal to a fixed circle forms a *net.* For the condition that a circle S be orthogonal to a fixed circle leads to a linear homogeneous relation with respect to the four coefficients of the equation of the circle S. One of these coefficients is therefore a linear homogeneous function of the other three, which are arbitrary and which may be called λ, μ, ν; the equation of the circle S arranged with respect to λ, μ, ν takes then the form of (26), and the circle S forms a net.

EQUATION OF A CIRCLE IN POLAR CO-ORDINATES.

97. 3. Let O be the pole and OX the polar axis (Fig. 63); call a and α the co-ordinates of the center C, r the radius, and ρ and ω the co-ordinates of any point M of the circumference. In the triangle OCM, one has

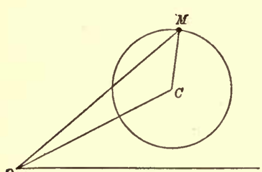

Fig. 63.

$$\rho^2 - 2a\rho\cos(\omega - \alpha) + a^2 - r^2 = 0. \tag{27}$$

When the pole O is situated on the circumference, one has $a = r$, and the equation reduces to

$$\rho = 2r\cos(\omega - \alpha). \tag{28}$$

As an application of this equation, consider two circles which intersect; through one of the points of intersection O, draw any secant; this secant meets the circles in two other points M and M'; find the locus of the mid-point of the line MM'. If the point O be taken as pole, the two circles are represented by the equation

$$\rho = 2r \cos(\omega - \alpha), \quad \rho = 2r' \cos(\omega - \alpha'),$$

and one obtains immediately the equation of the locus

$$\rho = r \cos(\omega - \alpha) + r' \cos(\omega - \alpha');$$

this equation can be put under the form

$$\rho = 2r_1 \cos(\omega - \alpha_1),$$

and the locus is a circle passing through the point of intersection O of the two given circles.

CHAPTER III*

GEOMETRICAL LOCI.

98. Geometrical loci may be defined in various ways. Whenever a property common to all points of a locus is given, it is by interpreting this property by means of algebraic symbols that the equation of the locus is obtained. In this manner, the circumference of a circle was defined as the locus of points whose distances from a given point are equal to the same given quantity; it was by expressing this property, common to every point of the locus, that the equation of the circumference was obtained (§ 84). Thus, also, has been found the locus of the points whose distances from two given points are in a given ratio (§ 95); the expressing of this property gives the equation of the locus. Likewise, by the same process, the equation of the perpendiculars erected at the mid-point of the straight line which joins two given points (§ 80), and those of the bisectors of the angles formed by two given lines (§ 81).

But, usually, a curve PQ (Fig. 64) is defined by the motion of a point in the plane. Each position of the variable point M is determined by the construction of a figure whose various parts depend on an arbitrary parameter a. Consequently the two co-ordinates x and y of the point M are functions of this variable parameter a: let

$$x = f(a),\ y = f_1(a),$$

be the two functions; one sees that the equation of the locus described by the point M is found by eliminating the parameter a between the two equations.

More generally, the geometric construction determines every point of the locus by the intersection of the variable curves which depend on the parameter a; let

(1) $$F(x, y, a) = 0,$$

(2) $$F_1(x, y, a) = 0,$$

be the equations of the two curves. If a particular value be assigned to this parameter, two curves A and B are obtained which intersect in a point M, whose co-ordinates x and y satisfy the two simultaneous equations (1) and (2). If another value a' be assigned to the parameter, the two lines will occupy the positions A' and B', and the point of intersection will be at M'; a third value a'' assigned to the parameter will give the two curves A'' and B'' and their point of intersection M'', and so on. Allow the parameter a to vary in a continuous manner; then the two curves A and B will be displaced in the plane in a continuous manner, and the point of intersection M will describe the line PQ.

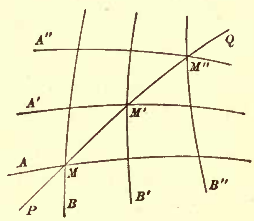

Fig. 64.

The equation of the curve PQ, the locus of the point M, will be found by eliminating the parameter a between the two equations (1) and (2). In fact, the elimination of a between the two equations (1) and (2) amounts to finding a system of two equations

(3) $$F_2(x, y, a) = 0,$$

(4) $$f(x, y) = 0,$$

equivalent to the system of two equations (1) and (2), and such that one of them does not contain the symbol a. Two systems of equations are said to be equivalent, when they are satisfied by the same assigned values of the variables. When a particular value is assigned to a, the co-ordinates x and y of the point M associated with this value of a form a system of three quantities, x, y, a, which satisfy at the same time the two equations (1) and (2); since the system of equations (3) and (4) is equivalent to the preceding, these values satisfy also equations (3) and (4); equation (4), which does not involve a, is therefore satisfied by the co-ordinates of every point of the locus.

Conversely, every point M, whose co-ordinates x and y satisfy equation (4) belongs to the locus. Because, if one determines a value a which satisfies equation (3), in which one gives to x and y the preceding values, one gets a system of values of three quantities, x, y, a, satisfying the system of equations (3) and (4). Equations (1) and (2), constituting a system equivalent to this system, will be satisfied by the same values; one will thus obtain two lines A and B passing through the point M.

It can happen, moreover, that to a system of real values of x and y satisfying equation (4) corresponds a value of a, for which equations (1) and (2) do not represent real curves; one will have this kind of a locus, for example, if the value of a were imaginary. But, in every case, the values of x, y, a will satisfy the two equations (1) and (2).

99. Although the construction of each of the positions of the figure, which gives the various points of the locus, depends upon the value given to the arbitrary parameter, it is frequently more convenient to introduce into the discussion several variable parameters $a, b, c, \cdots$; but these parameters are then so connected with one another that the value of one only is arbitrary, and that the variation of this parameter determines moreover the value of the others. If these parameters are n in number, they will be connected by $n-1$ equations of condition.

Suppose, for example, that only two variable parameters a and b connected by the equation of condition

$$(5) \qquad \phi(a, b) = 0$$

are employed, and let

$$(6) \qquad F(x, y, a, b) = 0,$$

$$(7) \qquad F_1(x, y, a, b) = 0,$$

be the equations of two variable curves A and B, whose intersections furnish every point of the locus. If the parameter a varies in a continuous manner, the parameter b which depends upon a by reason of the relation (5) will vary also in a continuous manner; the two curves A and B, whose equations contain the two parameters, vary also in a continuous manner, and their point of intersection M will describe a curve PQ.

The equation of this curve will be obtained by eliminating the two parameters a and b between the three equations (5), (6), (7). In fact, to eliminate a and b between these three equations is to find a system of three equations

(8) $$F_2(x, y, a, b) = 0,$$

(9) $$F_3(x, y, a, b) = 0,$$

(10) $$f(x, y) = 0,$$

equivalent to the given system, and such that one of them no longer contains a and b. When values are assigned to a and b which satisfy equation (5), the co-ordinates x and y of the point M, associated with these values of a and b, form a system of values of four quantities x, y, a, b, satisfying at the same time the three equations (5), (6), (7). The three equations (8), (9), (10) forming a system equivalent to the preceding system will also be satisfied by the same values; equation (10), being independent of a and b, will therefore be satisfied by the co-ordinates x and y of each point of the locus.

Conversely, every point M whose co-ordinates x and y satisfy equation (10) belongs to the locus; because if one determines the values of a and b which satisfy the two equations (8) and (9), in which x and y have been assigned the preceding values, one has a system of values of four quantities x, y, a, b satisfying the system of three equations (8), (9), (10). The three equations (5), (6), (7), forming a system equivalent to the former, will also be satisfied, and one will have two curves A and B passing through the point M.

100. Suppose, in general, that one employs n variable parameters $a, b, c, \ldots, h$ connected by $n-1$ equations of condition,

$$(11) \quad \begin{cases} \phi_1(a, b, c, \cdots, h) = 0, \\ \phi_2(a, b, c, \cdots, h) = 0, \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \phi_{n-1}(a, b, c, \cdots, h) = 0, \end{cases}$$

and let

$$(12) \qquad F(x, y, a, b, c, \cdots, h) = 0,$$

$$(13) \qquad F_1(x, y, a, b, c, \cdots, h) = 0,$$

be the equations of two variable curves A and B, whose intersections give every point of the locus. When the parameter a is allowed to vary, the remaining parameters vary simultaneously, and the point M describes the locus. The equation of this locus is obtained by eliminating the n parameters between $n+1$ equations (11), (12), (13).

101. It has been asserted that the construction of the figure depends upon a single arbitrary parameter a. If the figure should depend upon two arbitrary parameters a and b, the two co-ordinates x and y of the point M would be functions of these two parameters, *i.e.*,

$$x = f(a, b), \quad y = f_1(a, b).$$

Such values could be assigned to these parameters that the point M might be made to coincide with any point of the plane, having the co-ordinates x_1 and y_1. To accomplish this, it suffices to determine a and b by means of the equations

$$x_1 = f(a, b), \quad y_1 = f_1(a, b).$$

The point M may describe the entire plane and not any definite curve in the plane.

One sees very clearly, then, why it is necessary, when n variable parameters are employed, that these n parameters be connected by $n-1$ distinct equations of condition; because, if these equations of condition could be reduced to a smaller number, two parameters at least would be arbitrary.

It is possible that the two variable curves A and B intersect in several points; the preceding process gives the locus described by the totality of these points.

102. Remark. — It often happens that one of the two variable curves A and B, whose intersection furnishes a point M of the locus, passes through a fixed point P. In this case, the co-ordinates of this point P satisfy the equation found by elimination. In fact, suppose that the equations of the two curves contain n variable parameters connected by $n-1$ relations (§ 100); if the co-ordinates x_1 and y_1 of a fixed point P satisfy the equation of the line A, whatever be the values of the parameters, by replacing x and y in the equation of the curve B by x_1 and y_1, one will get an equation, which, combined with the $n-1$ equations of condition between the parameters, will form a system of n equations which will determine the values of these parameters. This point P will, properly speaking, be foreign to the geometrical locus, if imaginary curves correspond to the values found.

In this case, it frequently happens that the point P enters in the equation through a particular factor which can be removed. After this factor has been suppressed, the equation represents the geometrical locus itself. But often it is impossible to decompose the first member of the equation into two factors, and the point P must be considered as an isolated point connected with the curve.

103. Problem I. *Being given in the plane* (Fig. 65) *an angle XOY, and a fixed point P, draw through the point P the fixed secant PBA and the variable secant PDC; draw also the lines AD, BC; find the locus of their point of intersection M.*

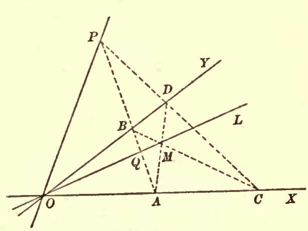

Fig. 65.

Take the lines OX and OY as co-ordinate axes, and represent by x_1 and y_1 the co-ordinates of the point P. The fixed secant PBA will have an equation of the form

$$y - y_1 = a(x - x_1),$$

in which the parameter a has a constant value. Similarly, the variable secant PDC will be represented by the equation

$$y - y_1 = m(x - x_1),$$

in which m is a variable parameter. If one puts successively in these equations $y = 0$, $x = 0$, one gets the co-ordinate of the points in which these lines intersect the axes of co-ordinates.

$$A,\quad y = 0,\quad x = x_1 - \frac{y_1}{a},$$

$$B,\quad x = 0,\quad y = y_1 - ax_1,$$

$$C,\quad y = 0,\quad x = x_1 - \frac{y_1}{m},$$

$$D,\quad x = 0,\quad y = y_1 - mx_1.$$

By applying the formula of § 67, one gets the equations of the lines AD, CB,

(1) $$\frac{x}{x_1 - \frac{y_1}{a}} + \frac{y}{y_1 - mx_1} = 1,$$

(2) $$\frac{x}{x_1 - \frac{y_1}{m}} + \frac{y}{y_1 - ax_1} = 1.$$

The values of x and y, which satisfy the two simultaneous equations (1) and (2), are the co-ordinates of the point of intersection M of the two lines AD and BC; these co-ordinates vary with the arbitrary parameter m. By subtracting the equations member from member one obtains the equation

$$x\left(\frac{m}{y_1 - mx_1} - \frac{a}{y_1 - ax_1}\right) + y\left(\frac{1}{y_1 - mx_1} - \frac{1}{y_1 - ax_1}\right) = 0,$$

or more simply

(3) $$\frac{m - a}{(y_1 - mx_1)(y_1 - ax_1)}(y_1x + x_1y) = 0,$$

which, combined with equation (1), forms a system equivalent to the system of two equations (1) and (2). So long as the parameter m has a value different from a, the first factor being different from zero, the co-ordinates x and y of the point M must reduce the second factor to zero. Therefore the co-ordinates of each of these points of the locus satisfy the equation

$$y_1x + x_1y = 0,$$

or

(4) $$\frac{y}{x} = -\frac{y_1}{x_1}.$$

This locus is a straight line passing through the origin.

When $m = a$, the system of two equations (1) and (2) reduces to equation (1); the two lines AD and BC coincide, and their point of intersection is any point of the fixed secant PA.

Suppose the elimination had been made in another manner; if for example the value of m deduced from equation (1) were substituted in equation (2), an equation of the second degree would be obtained, the first member of which would be decomposable into two linear factors of the first degree, and which, consequently, would represent two straight lines, the locus OL and the straight line PA. This equation would have the form

$$(y_1x + x_1y)[y - y_1 - a(x - x_1)] = 0.$$

It is to be noticed that equation (4) does not contain the constant parameter a; therefore the locus is independent of the particular position assigned to the fixed secant PA. Whence the following theorem may be deduced: When an angle XOY and a fixed point P, in the same plane are given, if any two secants PA, PC be drawn through this point P, the point of intersection M of the two straight lines AD and BC is always situated on the same straight line OL.

Further, it is to be noticed that equation (4) depends upon the ratio $\frac{y_1}{x_1}$, that is, upon the angular coefficient of the straight line OP. Hence, the locus OL will remain the same, if the point P be moved along the line OP passing through the origin.

104. This question can be discussed more quickly in another manner. Suppose that any two axes have been drawn in the plane. Represent, for brevity, the equations of the given straight lines OA and OB by $\alpha = 0$, $\beta = 0$, and the fixed secant PA by $\gamma = 0$. The given point P will no longer be determined by its co-ordinates, but by the intersection of the two given straight lines PA and OP; the latter, passing through the point of intersection O of the lines OA and OB, has an equation of the form $\beta + a\alpha = 0$. The movable secant PC, drawn through the point of intersection P of the two lines $\beta + a\alpha = 0$, $\gamma = 0$, is represented by an equation of the form

(1) $$\beta + a\alpha + m\gamma = 0,$$

in which m represents an arbitrary parameter. The point C, in which this secant cuts the line OA, is given by the two simultaneous equations $\alpha = 0$, $\beta + a\alpha + m\gamma = 0$, or more simply $\alpha = 0$, $\beta + m\gamma = 0$; the last equation represents a line passing through the point C, and also through the point of intersection B of the lines $\beta = 0$, $\gamma = 0$; it is, therefore, the equation of the line BC. Similarly, the point D, where the movable secant intersects the line OB, is given by the two simultaneous equations $\beta = 0$, $\beta + a\alpha + m\gamma = 0$, or more simply $\beta = 0$, $a\alpha + m\gamma = 0$; the line represented by the last equation, passing also through the point of intersection A of the lines $\alpha = 0$, $\gamma = 0$, is none other than the line AD. The two movable lines BC and AD, whose intersection determines the point M of the locus, have therefore the equations,

(2) $$\beta + m\gamma = 0,$$

(3) $$a\alpha + m\gamma = 0.$$

The equation of the locus will be found by eliminating m between the two equations; if the equations be subtracted member from member, one obtains the equation

(4) $$\beta - a\alpha = 0.$$

Whence it follows that the locus is a straight line passing through the point O. This line is independent of γ, that is, of the fixed secant PA, and is the same whatever be the position of the point P on the line OP.

It has been assumed that the parameter m has a finite value; if m be replaced by $\frac{p}{q}$, and after multiplying by q, one makes $q = 0$, the equations (1), (2), (3) reduce to $\gamma = 0$; the movable secant coincides with the fixed secant PA, so also the two lines BC and AD.

105. Problem II. — *The sides of a variable triangle ABC revolve about three fixed points P, P', P'', situated in a straight line, while the two vertices A and B slide on the two fixed lines ID and IE; find the locus described by the third vertex C* (Fig. 66).

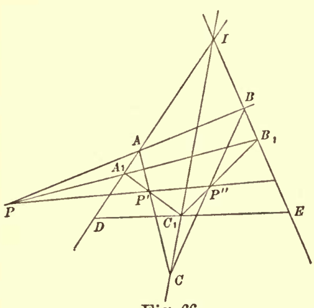

Fig. 66.

Draw in the plane any two axes, and, for brevity, represent, as in the preceding discussion, the equations of the given lines ID, IE, by $\alpha = 0$, $\beta = 0$, and the line $PP'P''$ by $\gamma = 0$; each of the fixed points P, P', P'', can be defined by the intersection of this line and of a line passing through the point I; the point I being the point of intersection of the lines $\alpha = 0$, $\beta = 0$, the lines IP, IP', IP'' have equations of the form

$$\beta + a\alpha = 0,\ \beta + a'\alpha = 0,\ \beta + a''\alpha = 0,$$

in which a, a', a'' designate constant coefficients. In order to construct a particular position of the variable figure, draw through the point P an arbitrary line PA, then construct the lines AP' and BP'', whose intersection will determine a point C of the locus. The point P being the intersection of the two lines $\gamma = 0$, $\beta + a\alpha = 0$, the line PA, drawn through this point, will have an equation of the form

(1) $$\beta + a\alpha + m\gamma = 0,$$

in which m is an arbitrary parameter. The point A, in which the line PA cuts the line ID, is given by the two simultaneous equations

$$\alpha = 0, \ \beta + a\alpha + m\gamma = 0, \text{ or more simply } \alpha = 0, \ \beta + m\gamma = 0.$$

The line AP', passing through this point, has an equation of the form $\beta + m\gamma + m'\alpha$; it is necessary to determine the coefficient m' in such a manner that the line passes also through the point P' determined by the two equations $\gamma = 0$, $\beta + a'\alpha = 0$; if γ be put equal to zero in the equation of this line and $\beta = -a'\alpha$, then will $(m' - a')\,\alpha = 0$; as α is not zero, since the point P' is not on the line $\alpha = 0$, therefore must $m' - a' = 0$, or $m' = a'$. Thus the line AP' will be represented by the equation

$$(2) \qquad \beta + a'\alpha + m\gamma = 0.$$

Similarly, the point B, in which PB cuts the line IE, is given by the two equations $\beta = 0$, $\beta + a\alpha + m\gamma = 0$, or more simply $\beta = 0$, $a\alpha + m\gamma = 0$; the line BP'', passing through this point, has an equation of the form $a\alpha + m\gamma + m''\beta = 0$; determine now the coefficient m'' in such a manner that this line may pass through the point P'', the intersection of the lines $\gamma = 0$, $\beta + a''\alpha = 0$; if in this equation γ be put equal to zero and $\beta = -a''\alpha$, then will $(a - m''a'')\,\alpha = 0$; therefore, choose $m'' = \dfrac{a}{a''}$; hence the line BP'' is represented by the equation

$$(3) \qquad \frac{a}{a''}(\beta + a''\alpha) + m\gamma = 0.$$

Equations (2) and (3) are the equations of the two movable lines AP' and BP'', whose intersection is any point C of the locus; the equation of the locus will be obtained by eliminating m between these two equations; subtracting them member from member, one gets the equation

$$(4) \qquad (a' - a)\,a''\alpha + (a'' - a)\,\beta = 0.$$

Therefore, it follows that the locus is a straight line passing through the point I.

106. Corollary I. — The solution of the following problem may be deduced from what precedes. *Inscribe in a triangle IED a second triangle whose edges pass respectively through the three given points P, P', P'' lying in the same straight line.*

If a variable triangle be constructed whose sides are conditioned to pass through the points P', P'', P''', while the two vertices A and B slide on the straight lines ID and IE, the locus of the third vertex is a straight line IC. The point of intersection C_1 of the lines IC and DE is therefore one of the vertices C_1 of the triangle sought; the lines C_1P', C_1P'' give the other two vertices A_1 and B_1. It is worthy of notice that this solution requires the use of no other instrument than the rule.

Corollary II. — The preceding problem may be easily generalized. Consider a quadrilateral whose four sides pass through the four points P, P', P'', P''' lying in a straight line, in such a manner that the three vertices A, B, C slide on three fixed straight lines R, S, T; find the locus described by the fourth vertex (Fig. 67).

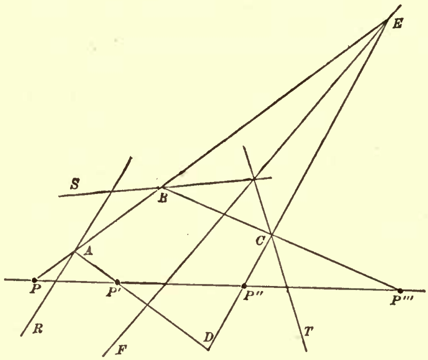

Fig. 67.

The three sides of the triangle BCE passing through the three fixed points P, P'', P''', revolve in such a way that the two vertices B and C slide on the fixed lines S and T; the vertex E describes therefore a straight line EF. Accordingly, the three sides of the triangle AED passing through the three fixed points P, P', P'', revolve in such a manner that the two vertices A and E slide on the two lines R and EF; the vertex D describes therefore a straight line.

From the quadrilateral one may pass to the pentagon. Moreover, when the n sides of a polygon pass through n fixed points lying in a straight line and revolve so that $n - 1$ of its vertices slide on $n - 1$ fixed lines, the nth vertex will describe a straight line.

107. Problem III. — *Being given a triangle ABA', draw through O taken on the side AA' a variable secant OCC'; pass a circumference of a circle through the three points O, A, C, and a second through the three points O, A', C'; find the locus of the point of intersection M of these two circumferences* (Fig. 68).

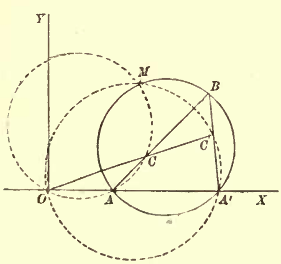

Fig. 68.

Take the line OA' for the x-axis and a perpendicular OY, drawn through O, for the y-axis. If a and a' be chosen as the abscissas of the points A and A', the two fixed lines AB and $A'B$ will have the equations

(1) $$y = c(x - a),$$

(2) $$y = c'(x - a'),$$

and the variable secant the equation

(3) $$y = mx,$$

in which m represents a variable parameter. The co-ordinates of the point C are found by solving the two simultaneous equations (1) and (3), which gives

$$x = \frac{ca}{c - m}, \quad y = \frac{mca}{c - m}.$$

Every circle passing through the points O and A has an equation of the form

$$x^2 + y^2 - ax - by = 0,$$

in which the parameter b is arbitrary. This parameter is determined by the condition that the circle passes through the point C, which gives

$$b = \frac{a(cm + 1)}{c - m};$$

the circle which passes through the three points O, A, C has therefore the equation

(4) $$x^2 + y^2 - ax - \frac{a(cm + 1)}{c - m} y = 0.$$

If, in this equation, a and c be replaced by a' and c', the equation of the circle which passes through the three points O, A', C' will evidently be

(5) $$x^2 + y^2 - a'x - \frac{a'(c'm + 1)}{c' - m} y = 0.$$

In order to find the equation of the locus of the point of intersection M of the two circles, it is necessary to eliminate the variable parameter m between the two equations (4) and (5). By equating the values of m deduced from (4) and (5), one gets the equation

$$\frac{c(x^2 + y^2 - ax) - ay}{(x^2 + y^2 - ax) + cay} = \frac{c'(x^2 + y^2 - a'x) - a'y}{(x^2 + y^2 - a'x) + c'a'y},$$

which may be written

$$(c - c')[(x^2 + y^2 - ax)(x^2 + y^2 - a'x) + aa'y^2] +$$
$$(1 + cc')y[a'(x^2 + y^2 - ax) - a(x^2 + y^2 - a'x)] = 0,$$

or $$(c - c')[(x^2 + y^2)^2 - (a + a')x(x^2 + y^2) + aa'(x^2 + y^2)]$$
$$+ (1 + cc')(a' - a)y(x^2 + y^2) = 0;$$

by putting $(x^2 + y^2)$ without a bracket, and dividing by $c - c'$, one obtains the equation

(6) $$(x^2 + y^2)\left[x^2 + y^2 - (a + a')x - \frac{(1 + cc')(a - a')}{c - c'} y + aa'\right] = 0.$$

This equation is decomposed into two: the one $x^2 + y^2 = 0$ gives the fixed point O in which the two variable circles intersect; the other

$$(7) \qquad x^2 + y^2 - (a + a')\,x - \frac{(1 + cc')(a - a')}{c - c'}\,y + aa' = 0$$

is the equation of the locus of the point M. This locus is a circle.

It can be seen *à priori* that the three points B, A, A' belong to the locus. Because, if the variable secant pass through the point B, the two circles intersect in B; this point constitutes a part of the locus. Suppose now that the secant becomes parallel to the line $A'B$; the point C' is removed to infinity, the second circle coincides with the line OA', which cuts the first circle in A. In a similar manner the point A' is found by supposing the secant to be made parallel to AB. It is also easy to show that the co-ordinates of the points B, A, A' satisfy equation (7). Thus the locus required is a circle circumscribing the triangle ABA.

108. Problem IV. — *Being given a circle and a fixed point P, revolve about the fixed point P a right angle APB; join by a straight line the two points A and B, in which the sides of the right angle produced meet the circle, and draw from the point P a perpendicular PM to the line AB; find the locus of the foot M of the perpendicular* (Fig. 69).

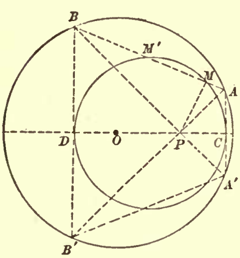

Fig. 69.

Take the diameter OP for the x-axis and the diameter perpendicular to OP for the y-axis; the given circle is represented by the equation

$$(1) \qquad x^2 + y^2 = r^2.$$

Let

$$(2) \qquad y = ax + b$$

be the equation of the secant AB. If y be eliminated between the two equations (1) and (2), one gets an equation of the second degree,

$$(3) \qquad (1 + a^2)\,x^2 + 2\,abx + b^2 - r^2 = 0,$$

whose roots are the abscissas x' and x'' of the points A and B and the values of the ordinates will be $ax' + b$, $ax'' + b$. If c represent the constant length OP, the two lines PA, PB have the angular coefficients

$$\frac{y'}{x' - c}, \quad \frac{y''}{x'' - c}, \quad \text{or} \quad \frac{ax' + b}{x' - c}, \quad \frac{ax'' + b}{x'' - c};$$

the angle APB being right, one has the condition

$$\frac{(ax' + b)(ax'' + b)}{(x' - c)(x'' - c)} = -1,$$

which may be written

$$(1 + a^2)x'x'' + (ab - c)(x' + x'') + b^2 + c^2 = 0\,;$$

if the values of $x' + x''$ and $x'x''$ be replaced by their values deduced from equation (3), one obtains the relation

$$(4) \qquad (1 + a^2)(c^2 - r^2) + 2b(ac + b) = 0,$$

which connects the two parameters a and b.

The perpendicular PM, drawn from the point P to the line AB, has the equation

$$(5) \qquad y = -\frac{1}{a}(x - c).$$

The point M is determined by the equations (2) and (5), in which the variable parameters a and b satisfy equation (4); the equation of the locus of the point M is found by eliminating these two parameters between the three equations (2), (4), (5). From equation (5) it follows that $a = -\frac{x - c}{y}$; whence from equation (2) one deduces $b = \frac{y^2 + (x - c)x}{y}$. On substituting these values in equation (4), one gets the equation

$$(6) \qquad [y^2 + (x - c)^2](x^2 + y^2 - cx + \frac{c^2 - r^2)}{2} = 0,$$

which decomposes into two: the one, $y^2 + (x - c)^2 = 0$, gives the point P; the other,

$$(7) \qquad x^2 + y^2 - cx + \frac{c^2 - r^2}{2} = 0,$$

represents the locus sought.

It is evident that the point P does not belong to the geometrical locus according to its definition; but it is easy to understand how analysis has introduced it into the result. The co-ordinates $x = c$, $y = 0$ of the point P satisfy equation (5), whatever the parameters may be; one could therefore deduce from equations (2) and (4) the corresponding values of the two parameters a and b; thus one finds $a = \pm i$, $b = -ac$. This is an application of the remark made in § 102.

Equation (7) shows that the locus is a circle having its center on the line OP. To construct it, it suffices to determine the extremities of the diameter CD; if AB', BA' be drawn making angles of 45° with the diameter OP, the chords AA', BB', being perpendicular to this diameter, will give the two points C and D.

109. The same circle may be found by seeking the locus of the mid-point M' of the chord AB. In fact, the mid-point is determined by the intersection of the chord AB and the perpendicular drawn from the center to this chord. Since these two lines have the equations

$$y = ax + b, \quad y = -\frac{1}{a}x,$$

the equation of the locus will be obtained by eliminating the two variable parameters a and b between these two equations and equation (4). We thus have the equation

$$(x^2 + y^2)\left(x^2 + y^2 - cx + \frac{c^2 - r^2}{2}\right) = 0,$$

which decomposes into two, the one giving the point O foreign to the geometrical locus, the other the circle.

110. Problem V.—*A circumference and a fixed point P are given, a right angle revolves about its vertex placed in P; find the locus of the point of concurrence M of the tangents drawn to the circumference at the points of intersection A and B with the sides of the right angle* (Fig. 70).

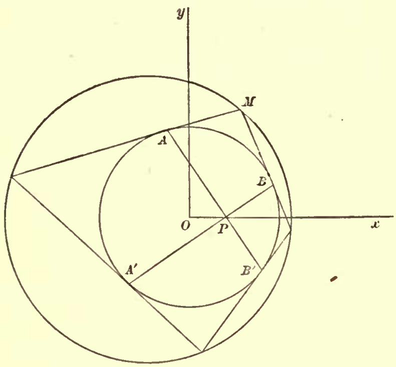

Fig. 70.

Take the diameter OP as the x-axis and the diameter perpendicular to it as the y-axis; let r be the radius of the circumference and c the distance OP; the equation of the given circumference is

(1) $$x^2 + y^2 = r^2.$$

Represent by x_1 and y_1 the co-ordinates of any point M of the plane. The chord of contact of the tangents drawn from this point will have the equation

(2) $$x_1x + y_1y = r^2.$$

The co-ordinates of the points of contact will be found by solving the simultaneous equations (1) and (2). If x, y be considered a solution of this system, the value m of the angular coefficient of the line which joins the corresponding point to the point P has the equation

(3) $$m = \frac{y}{x - c}.$$

The elimination of x and y from the equations (1), (2), (3) gives the equation which determines the angular coefficients of the two lines drawn from the point P to the points of intersection of line (2) with the circumference. In order to accomplish this elimination, solve equations (2) and

(3) for x and y, and substitute their values in (1); thus is found the equation of the second degree

(4) $$[(r^2 - cx_1)^2 + (c^2 - r^2)y_1^2]\,m^2 + 2\,r^2y_1(c - x_1)m + r^2(r^2 - x_1^2) = 0.$$

In order that the point M, chosen arbitrarily in the plane, be a point of the locus, it is necessary and sufficient that the directions, which correspond to the two roots of equation (4), be rectangular. On expressing that the product of the roots is equal to -1, the equation of the locus is found to be

$$(x_1^2 + y_1^2)(r^2 - c^2) + 2\,r^2cx_1 - 2\,r^4 = 0,$$

which, suppressing the indices, may be written

(5) $$\left(x + \frac{r^2c}{r^2 - c^2}\right)^2 + y^2 = \frac{r^4(2\,r^2 - c^2)}{(r^2 - c^2)^2}.$$

The locus is a circle which can be constructed by the method indicated in the preceding problem.

The radii R and r of the two circumferences and the distance D between their centers satisfy the relation

(6) $$(R^2 - D^2)^2 = 2\,r^2(R^2 + D^2).$$

If the sides of the right angle APB be prolonged, and tangents be drawn at A' and B', the points of intersection of the consecutive tangents are the vertices of a variable quadrilateral, which is at one and the same time circumscribed about the given circle and inscribed in circle (5). Hence, *when the radii R and r of the two circles O_1 and O and the distance D between their centers satisfy relation* (6), *a quadrilateral can be constructed, inscribed in O_1 and circumscribed about O, by taking as an edge of the quadrilateral any tangent to the circle O.*

111. Problem VI. — *Find the locus of the points, such that the feet of the perpendiculars drawn from each of them to the sides of a triangle lie in a straight line.*

Let

(1) $$\begin{cases} x\cos\alpha + y\sin\alpha - p_1 = 0, \\ x\cos\beta + y\sin\beta - p_2 = 0, \\ x\cos\gamma + y\sin\gamma - p_3 = 0, \end{cases}$$

be the equation of the three sides of the triangle, referred to any two rectangular axes, and, for the sake of brevity, represent by α_1, β_1, γ_1, the first members of these equations. Calling x and y the co-ordinates of the point M of the locus, x_1 and y_1, x_2 and y_2, x_3 and y_3 those of the feet of the perpendiculars drawn from the point M to the sides of the triangle, one has (§ 83)

$$x - x_1 = \alpha_1\cos\alpha, \quad x - x_2 = \beta_1\cos\beta, \quad x - x_3 = \gamma_1\cos\gamma,$$

$$y - y_1 = \alpha_1\sin\alpha, \quad y - y_2 = \beta_1\sin\beta, \quad y - y_3 = \gamma_1\sin\gamma.$$

The equation of the locus will be found by expressing the condition that the three points lie on a straight line. For this purpose it is necessary to equate the two ratios $\frac{y_2 - y_1}{x_2 - x_1}$ and $\frac{y_3 - y_1}{x_3 - x_1}$, which can be put under the form

$$\frac{(y_2 - y) - (y_1 - y)}{(x_2 - x) - (x_1 - x)} = \frac{(y_3 - y) - (y_1 - y)}{(x_3 - x) - (x_1 - x)}.$$

By substituting from the preceding equation, this equation becomes

$$\frac{\beta_1 \sin\beta - \alpha_1 \sin\alpha}{\beta_1 \cos\beta - \alpha_1 \cos\alpha} = \frac{\gamma_1 \sin\gamma - \alpha_1 \sin\alpha}{\gamma_1 \cos\gamma - \alpha_1 \cos\alpha},$$

or (2) $\alpha_1 \beta_1 \sin(\beta - \alpha) + \beta_1 \gamma_1 \sin(\gamma - \beta) + \gamma_1 \alpha_1 \sin(\alpha - \gamma) = 0.$

The letters α_1, β_1, γ_1, representing polynomials of the first degree in x and y, it follows that equation (2) is of the second degree. The coefficient of xy is

$$\sin(\alpha + \beta)\sin(\beta - \alpha) + \sin(\beta + \gamma)\sin(\gamma - \beta) + \sin(\gamma + \alpha)\sin(\alpha - \gamma);$$

if the product of sines be transformed into the difference of cosines, this coefficient becomes

$$\frac{(\cos 2\alpha - \cos 2\beta) + (\cos 2\beta - \cos 2\gamma) + (\cos 2\gamma - \cos 2\alpha)}{2},$$

it is identically zero. The coefficients of x^2 and y^2 are

$$M = \cos\alpha\cos\beta\sin(\beta - \alpha) + \cos\beta\cos\gamma\sin(\gamma - \beta) + \cos\gamma\cos\alpha\sin(\alpha - \gamma),$$
$$N = \sin\alpha\sin\beta\sin(\beta - \alpha) + \sin\beta\sin\gamma\sin(\gamma - \beta) + \sin\gamma\sin\alpha\sin(\alpha - \gamma).$$

If their sum and difference be calculated, one has

$$M - N = \cos(\alpha + \beta)\sin(\beta - \alpha) + \cos(\beta + \gamma)\sin(\gamma - \beta) + \cos(\gamma + \alpha)\sin(\alpha - \gamma)$$
$$= \frac{\sin 2\beta - \sin 2\alpha + \sin 2\gamma - \sin 2\beta + \sin 2\alpha - \sin 2\gamma}{2} = 0,$$
$$M + N = \cos(\alpha - \beta)\sin(\beta - \alpha) + \cos(\beta - \gamma)\sin(\gamma - \beta) + \cos(\gamma - \alpha)\sin(\alpha - \gamma)$$
$$= \frac{\sin 2(\beta - \alpha) + \sin 2(\gamma - \beta) + \sin 2(\alpha - \gamma)}{2}$$
$$= -2\sin(\beta - \alpha)\sin(\gamma - \beta)\sin(\alpha - \gamma);$$

whence it follows that

$$M = N = -\sin(\beta - \alpha)\sin(\gamma - \beta)\sin(\alpha - \gamma).$$

Therefore the locus is the circumference of a circle. Equation (2) being satisfied when one puts $\beta_1 = \gamma_1 = 0$, it follows that the point A belongs to the locus; similarly with the points B and C; the locus is therefore the circle circumscribed about the triangle ABC.

112. From equation (2), which represents the circle circumscribed about a triangle whose sides are represented by equations (1), may easily be deduced an important property of a special system of two circles. Suppose that the sides (1) be tangents to a circle of radius r having its center at the origin O of co-ordinates. It will be necessary in equations (1) to make $p_1 = p_2 = p_3 = r$. If equation (2) of the circle be developed, it may be written

$$(3) \qquad M(x^2 + y^2) - Px - Qy + F = 0.$$

Let R be the radius of this circle and D be the distance of its center O_1 from the center O of the first circle; one will have

$$R^2 = \frac{P^2 + Q^2}{4M^2} - \frac{F}{M}, \quad D^2 = \frac{P^2 + Q^2}{4M^2},$$

whence
$$D^2 - R^2 = \frac{F}{M}.$$

The radii of the circle O, determined by the angles α, β, γ, form two by two the supplementary angles of the angles A, B, C of the triangle formed by the three tangents. One has, therefore,

$$M = -\sin A \sin B \sin C = -\frac{S}{2R^2},$$

$$F = r^2(\sin A + \sin B + \sin C) = 4r^2 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2} = \frac{r}{R}S,$$

S representing the area of the triangle ABC. From these results follows that $\frac{F}{M} = -2Rr$, and consequently

$$(4) \qquad D^2 = R^2 - 2Rr.$$

Now it is proposed to determine all the triangles which are at the same time inscribed in the circle O_1 and circumscribed about the circle O, whose radii and the distance between the centers satisfy relation (4). It will be no restriction to suppose that the point O_1 is situated on the x-axis, and the angles α, β, γ to fulfill the conditions

$$Q = 0, \quad R^2 = \frac{P^2}{4M^2} - \frac{F}{M}, \quad D^2 = \frac{P^2}{4M^2}.$$

But, owing to relation (4), which the given quantities R, r, and D satisfy, these three relations may be replaced by the two following:

$$(5) \qquad Q = 0, \quad \frac{F}{M} = -2Rr.$$

Let, in fact, R' be the radius of a circle circumscribed about the triangle $A'B'C'$, determined by the angles α', β', γ', which satisfy equations

(5), D' the distance of its center O' from the point O. From the preceding, it will follow

$$-\frac{F}{M}=2R'r, \quad D'^2=R'^2-2R'r.$$

Moreover, by hypothesis

$$-\frac{F}{M}=2Rr, \quad D^2=R^2-2Rr;$$

whence it follows that R' is equal to R, and D' equal to D. One of the three angles α', β', γ', which must fulfill only the two conditions (5), can therefore be taken arbitrarily. Hence, *when the radii R and r of the two circles O_1 and O and the distance D between their centers satisfy relation* (4), *a triangle can be constructed inscribed in O_1 and circumscribed about O by taking any tangent to the circle O as a side of the triangle.*

The theorems analogous to the preceding and to § 110 exist for polygons of any number of sides.

113. Form (2) of the equation of the circle circumscribing a triangle is worthy of notice. The first member has a very simple geometrical meaning. To be precise, suppose that the origin of co-ordinates be situated within the triangle ABC (Fig. 71), and that the angles α, β, γ, varying between 0 and 2π, be arranged according to their increasing order of magnitude. Consider a point M having the co-ordinates x and y and situated also within the triangle; draw from this point perpendiculars to the sides, and join the feet of these perpendiculars forming the triangle DEF. The letters α_1, β_1, γ_1 designate the length of these perpendiculars affected in this position by the $-$ sign; these perpendiculars are constructed in the same direction as those which have been drawn from the origin, and which have served to determine the angles α, β, γ. The term $\alpha_1\beta_1 \sin(\beta-\alpha)$ being equal to $MD \cdot ME \cdot \sin DME$ represents double the area of the triangle DME; the two remaining terms represent in a similar manner double the triangles EMF, FMD; thus the first member of equation (2) represents double the area of the triangle DEF.

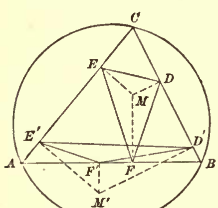

Fig. 71.

Consider next a point M' situated without the triangle ABC. It follows from the figure that $\alpha_1=-M'D'$, $\beta_1=-M'E'$, $\gamma_1=+M'F'$; the first member of the equation represents double the difference between the triangle $D'M E'$ and the sum of the two triangles $E'M'F'$, $F'M'D'$; which is, moreover, double the area of the triangle $D'E'F'$. Whatever the position of the point M in the plane may be, the first member of the equation represents double the area of the triangle DEF affected by the $+$ or $-$

sign. Equation (2) expresses, therefore, that the area of the triangle DEF is zero; that is, that the three points D, E, F lie in a straight line.

If r be the radius of the circle circumscribing the triangle ABC, and d the distance of a point, whose co-ordinates are x and y, from the center of the circle, the first member of equation (2) can be written in the form

$$A(x^2+y^2+\cdots),$$

and is equal to $A(d^2-r^2)$. This expression preserves the same sign, so long as the distance d is less than r, that is, while the point M lies within the circle, and takes the contrary sign as soon as the point M falls without.

It follows from the preceding that the locus of points such that the area of the triangle whose vertices are the feet of the three perpendiculars is a constant quantity k, is represented by two circles whose equations are

$$\alpha_1\beta_1 \sin(\beta-\alpha)+\beta_1\gamma_1 \sin(\gamma-\beta)+\gamma_1\alpha_1 \sin(\alpha-\gamma)=\pm 2k^2.$$

These two circles are concentric to the circle circumscribing the triangle ABC: the one lies without and is always real, whatever be the given area; the other lies within, and is not real unless the given area is less than the absolute value of $\frac{Ar^2}{2}$.

EXERCISES.

1. Express the area of a triangle and of any polygon as a function of the co-ordinates of its vertices.

2. Find the area of a triangle formed by lines whose equations are given.

3. Being given n points A, B, C, $\cdots$ in a plane and n quantities m', m'', m''', $\cdots$ which correspond to these n points; on the line AB take a point N_1, so that the distances from this point to the first two points are in the ratio m'' to m'; then on the line N_1C, which joins N_1 to the third, take a point N_2, so that its distances from the points N_1 and C are in the ratio m''' to $m'+m''$; further, on the line N_2D which joins the point N_2 to the fourth point D, a point N_3, so that its distances from the points N_2 and D are in the ratio m'''' to $m'+m''+m'''$, and so on, till the last given point is reached. Find the co-ordinates of the last point of division, which is called the *center of proportional distances.*

When the multipliers m', m'', m''' $\cdots$ are all equal to the same quantity, the last point of division is called the *center of mean distances.*

As an application, find the quantities m', m'', m''', which give, in case of a triangle, the center of gravity, the center of the inscribed circle, the point of intersection of the three altitudes, the center of the circumscribed circle.

4. Find the locus of the points such that the sum of the products of the squares of the distances of each of them from n given points, by the quantities m', m'', $m''' \cdots$, is equal to a given quantity.

5. Find the locus of the centers of circles which, viewed from two fixed points, subtend constant angles.

6. Find the locus of the centers of circles which intersect each of two given circles in diametrically opposite points.

7. Find the locus of points such that the sum of the distances of each of them from two given straight lines, and in general from several given straight lines, is constant.

8. Construct on two perpendicular lines OX, OY a variable rectangle $OABC$ having a given perimeter $2a$. Show that the perpendicular drawn from the vertex C to the diagonal AB passes through a fixed point.

9. Being given the figure used in demonstrating the theorem concerning the square of the hypotenuse of a right triangle, show that the two straight lines, which join the extremities of the hypotenuse to the vertices of the squares constructed on the opposite sides, meet in a point on the perpendicular drawn from the vertex of the right angle to the hypotenuse.

10. From a fixed point P draw tangents to the circles which pass through two given points; find the locus of the point in which the chord of contacts intersects the diameter which passes through P.

11. Being given a regular hexagon $ABCDEF$, draw the straight lines AC and AE; through the center draw any secant which cuts the two straight lines AC and AE in G and H; draw BG and FH; find the locus of the point of intersection of these two lines.

12. The circumferences described on the three diagonals of a complete quadrilateral as diameters, have two by two the same radical axis.

13. Being given five straight lines, four are chosen to form a complete quadrilateral, and the mid-points of its three diagonals are in a straight line; the five lines thus obtained meet in the same point.

14. Being given three points A, B, C and two straight lines X, Y; on AB as a diagonal, construct a parallelogram whose sides are parallel to X and Y; proceed in the same manner with B, C and C, A; the second diagonals of the three parallelograms pass through the same point.

15. Being given four straight lines A, B, C, D, construct a triangle with any three and determine the common point of intersection of its altitudes; the four points thus determined lie in a straight line.

16. Two variable circles, which are tangent to each other, are tangent to two given circles; find the locus of the point of contact of the two variable circles.

17. Four points are chosen arbitrarily on the circumference of a circle; the bisectors of the three pairs of angles formed by the lines which pass through these four points are parallel two by two.

18. Find the locus of the point such that the chords of contact of the tangents drawn from this point to three given circles meet in the same point.

19. One is given a fixed angle AOA' and a fixed point C on its bisector. An angle of constant magnitude revolves about its vertex placed at C; join by a straight line the points of intersection B and B' of the sides of the movable angle with the sides of the fixed angle and drop a perpendicular from the point C upon BB'; find the locus of the foot of the perpendicular.

20. One is given four straight lines A, B, C, D, which taken three by three form four triangles. The line A belongs to three of these triangles; the center of the circle circumscribed about each of them is joined to the vertex which is not situated on A; the three lines thus constructed intersect in the same point I; the four points analogous to I and the centers of the four circles lie on the same circumference.

21. A series of circles are given, which taken two by two have the same radical axis; if a variable circle cut two of these circles with constant angles, it will cut similarly each of the remaining circles with a constant angle.

22. The locus of the centers of circles orthogonal to two fixed circles is the radical axis.

23. Show that the circle, cutting orthogonally three given circles

$$f=0, \quad \phi=0, \quad \psi=0,$$

which, taken two by two, do not have the same radical axis, is the locus of the points of which the polars, with respect to these three circles, are concurrent.

24. Show that each of the limiting points of a pencil of circles and also the point at infinity on the radical axis, has the same polar with respect to all circles of the pencil.

Book III

CURVES OF THE SECOND DEGREE

CHAPTER I

CONSTRUCTION OF CURVES OF THE SECOND DEGREE.

114. The general equation of the second degree between the variables x and y is of the form

(1) $$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0;$$

it involves five arbitrary parameters, the ratios of five coefficients to the sixth.

In order to give an account of the different forms of the curves which can be represented by this equation, solve it with respect to y.

Two cases are to be distinguished, according as y appears in the equation to the second, or only to the first degree; that is, according as C is different from zero or equal to zero.

Suppose that the coefficient C is not zero, and solve the equation with respect to y; one gets the equation

(2) $$y = -\frac{Bx + E}{C} \pm \frac{1}{C}\sqrt{Mx^2 + 2Nx + P},$$

by putting $M = B^2 - AC$, $N = BE - CD$, $P = E^2 - CF$.

Construct the straight line DD' represented by the equation

$$y = -\frac{Bx + E}{C}.$$

In order now to construct the points of the locus represented by equation (2) for each value of x, it is necessary, starting from the straight line DD', to lay off from either side along the ordinate a length equal to

$$Y = \frac{1}{C}\sqrt{Mx^2 + 2Nx + P}.$$

The line DD' (Fig. 72), which bisects the chords parallel to the axis OY, is a *diameter* of the curve; the quantity Y is the length of the ordinate measured from the diameter. The construction of the locus is thus reduced to the study of the trinomial

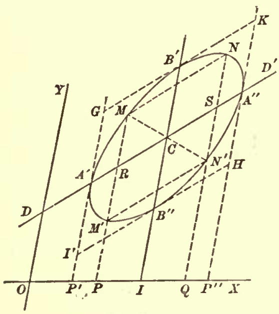

Fig. 72.

$$Mx^2 + 2Nx + P$$

and, as the form of the locus depends principally on the sign of the coefficient M, there will be three principal cases to be discussed.

Genus Ellipse.

115. Consider the case when the coefficient M, that is, $B^2 - AC$, has a negative value. The ordinate is not real unless the trinomial has a positive value. The case investigated here is subdivided into three others, according to the nature of the roots of this trinomial.

1° $N^2 - MP > 0$. The two roots of the trinomial are real and unequal. Represent by x' the smaller, and by x'' the larger root; the trinomial can be written

$$M(x - x')(x - x''), \text{ or } -M(x - x')(x'' - x);$$

the trinomial is positive, and, consequently, the ordinate Y is real, for every value of x taken between the limits x' and x''; the trinomial is, on the contrary, negative, and the ordinate imaginary for every value of x less than x' or greater than x''.

Take on the x-axis two points P' and P'' having the abscissas x' and x'', and draw through these points the lines $P'A'$, $P''A''$

parallel to the y-axis; the curve will lie wholly between these two parallels. As the abscissa x varies from x' to x'', the ordinate Y preserves a finite value, and begins with the value zero and returns to zero; the locus is therefore a closed curve, which passes through the points A' and A'', and to which has been given the name *ellipse.*

A value of x taken between x' and x'' will be the abscissa of a point P situated between P' and P'', and the corresponding value of Y will be equal to

$$\frac{1}{C}\sqrt{(-M)\,PP'\cdot PP''}.$$

The variable product $PP'\cdot PP''$ of the two segments of the line $P'P''$ is equal to the square of the rectangular ordinate of the circle described on $P'P''$ as a diameter; when the point P moves from P' to I, the mid-point of $P'P''$, the rectangular ordinate of the circle, and consequently the quantity Y, which is proportional to it, will continually increase; it diminishes continually, on the contrary, as the point moves from I to P''. The quantity Y has therefore a maximum value, when the point P is at I, that is, when $x=\frac{x'+x''}{2}=-\frac{N}{M}$; this maximum value is equal to $\frac{(x''-x')\sqrt{-M}}{2C}$. Beginning at the point C, the middle of the diameter $A'A''$, lay off along the ordinate, in opposite directions, a length equal to this maximum value; two points B' and B'' of the curve will be found, and, by drawing through these points parallels to the diameter, a parallelogram will be formed, which will circumscribe the ellipse.

It is clear that to the two points P and Q, equally distant from the mid-point I, correspond equal values of Y; these values, laid off in opposite directions from the diameter DD', give the four points M, M', N, N'. The two triangles CRM, CSN' being equal, the three points M, C, N' lie in a straight line, and the point C is the mid-point of MN'; hence all the points of the curve are two by two symmetrical with respect to the point C, the mid-point of the diameter $A'A''$; the point C is, therefore, the *center* of the ellipse. It follows also that the lines MN, $M'N'$ are parallel to the diameter $A'A''$, and each is divided into two equal parts by the line $B'B''$; this line is a second diameter. The diameters $A'A''$, $B'B''$, each of which bisects the chords parallel to the other, are called *conjugate diameters* of the ellipse.

2° $N^2 - MP = 0$. The two roots x' and x'' are equal, and it follows that

$$x' = x'' = -\frac{N}{M}, \quad Y = \frac{x - x'}{C}\sqrt{M};$$

the coefficient M being negative, the quantity Y is imaginary for all values of x excepting $x = x'$, and then $Y = 0$; the equation has no longer a real solution, excepting the single point C situated on the straight line DD'.

3° $N^2 - MP < 0$. The trinomial

$$Mx^2 + 2\,Nx + P = M\left(x + \frac{N}{M}\right)^2 + \frac{MP - N^2}{M}$$

is negative, and consequently Y is imaginary for every value of x; the equation, having no real solution, does not represent a geometrical locus.

Genus Hyperbola.

116. Consider next the case when the coefficient M has a positive value; this case subdivides into three.

1° $N^2 - MP > 0$. The trinomial

$$Mx^2 + 2\,Nx + P,$$

which one writes in the form

$$M(x - x')(x - x''),$$

is positive, and consequently Y is real as x varies from x'' to $+\infty$, and from x' to $-\infty$; moreover, Y varies at the same time from 0 to ∞. Choose, as before, on the x-axis two points P' and P'' with the abscissas x' and x'', and draw through these two points the lines $P'A'$, $P''A''$ parallel to the y-axis; the curve will be situated to the right and left of these parallels; it is composed of two distinct branches extending to infinity (Fig. 73). This curve has been given the name *hyperbola*. If, beginning at the point I, the mid-point of $P'P''$, one lay off in opposite directions on the x-axis two equal lengths IP and IQ, the corresponding values of Y are equal; the point C, the mid-point of $A'A''$, is the center of the curve, and the two lines DD' and IC are two conjugate diameters.

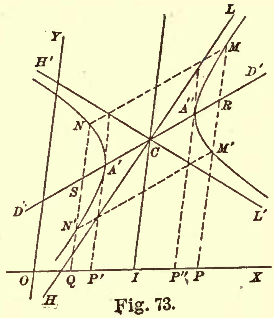

Fig. 73.

117. Consider the following value of y:

$$y = -\frac{Bx+E}{C} \pm \frac{1}{C}\sqrt{M\left(x+\frac{N}{M}\right)^2 + \frac{MP-N^2}{M}}.$$

In case x has a very large numerical value, the first term of the quantity under the radical sign is very large as compared with the absolute value of the second. If the first term only of the quantity be considered, an approximate value of y will be

$$(3) \qquad y_1 = -\frac{Bx+E}{C} \pm \frac{1}{C}\left(x+\frac{N}{M}\right)\sqrt{M}.$$

The preceding equation defines two distinct straight lines which intersect in a point of the diameter DD', whose abscissa is equal to $-\frac{N}{M}$; that is, equal to the half-sum of the abscissas of the points P' and P''; this point is, therefore, the center C of the curve. Consider the branch $A''M$ of the curve; if C be positive, this branch is represented by the equation

$$y = -\frac{Bx+E}{C} + \frac{1}{C}\sqrt{M\left(x+\frac{N}{M}\right)^2 + \frac{MP-N^2}{M}},$$

in which allow x to vary from x'' to $+\infty$; consider at the same time the line CL, which has the equation

$$y_1 = -\frac{Bx+E}{C} + \frac{1}{C}\left(x+\frac{N}{M}\right)\sqrt{M}.$$

For any value of x greater than x'', the ordinate of the curve is less than the corresponding ordinate of the straight line; hence the branch $A''M$ is comprised within the angle LCD'. The difference $y_1 - y$ of the ordinates which correspond to the same abscissa has the value

$$y_1 - y = \frac{1}{C}\left[\left(x+\frac{N}{M}\right)\sqrt{M} - \sqrt{M\left(x+\frac{N}{M}\right)^2 + \frac{MP-N^2}{M}}\right]$$

$$= \frac{N^2 - MP}{CM} \cdot \frac{1}{\left(x+\frac{N}{M}\right)\sqrt{M} + \sqrt{M\left(x+\frac{N}{M}\right)^2 + \frac{MP-N^2}{M}}}.$$

As x increases indefinitely, the denominator increases indefinitely, and, consequently, the difference $y_1 - y$ approaches the limit zero. The straight line CL, which continually approaches the branch $A''M$ of the curve, is called the asymptote of this branch of the curve, which is comprised within the angle LCD'. In a similar manner it can be shown that the branches $A''M'$, $A'N$, $A'N'$ are comprised within the angles $L'CD'$, $H'CD$, HCD, and have as asymptotes the straight lines CL', CH', CH, and each of the indefinite lines HL, $H'L'$ is asymptotic to two branches of the curve.

It is well to notice that the angular coefficients of the asymptotes are given by the equation

(4) $$m = \frac{-B \pm \sqrt{M}}{C},$$

or (5) $$Cm^2 + 2Bm + A = 0,$$

which may be obtained by substituting in the terms of the second degree of equation (1), 1 for x and m for y.

118. 2° $N^2 - MP < 0$. The trinomial

$$Mx^2 + 2Nx + P = M\left(x + \frac{N}{M}\right)^2 + \frac{MP - N^2}{M}$$

being the sum of two positive quantities, the value of Y is real for every value of x, and never becomes zero; Y attains its minimum value $\frac{1}{C}\sqrt{\frac{MP-N^2}{M}}$ when $x=-\frac{N}{M}$. Let I (Fig. 74) be the point of the x-axis whose abscissa is $-\frac{N}{M}$; draw IC parallel to the axis OY, and take the lengths CB' and CB'' equal to the minimum value of Y; the two points B' and B'' belong to the locus. As x varies from $-\frac{N}{M}$ to $+\infty$, or from $-\frac{N}{M}$ to $-\infty$, the value of Y increases indefinitely. If, therefore, one draw through the points B' and B'' parallels to the diameter DD', the curve is composed of two distinct portions, situated respectively above and below the parallels, and extending to infinity in opposite directions. The name *hyperbola* is also given to this curve.

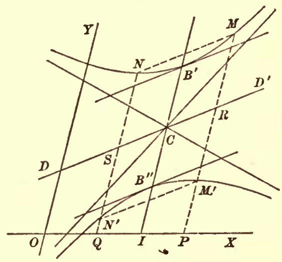

Fig. 74.

If the two values $x=-\frac{N}{M}\pm\alpha$ be assigned to x, and the two distances $IP=IQ=\alpha$ be laid off, starting from I, the corresponding values of Y are equal; whence it follows that the point C is the center of the curve, and that the two straight lines DD', IC are conjugate diameters.

It is also easily seen that the two straight lines

$$y=-\frac{Bx+E}{C}\pm\frac{1}{C}\left(x+\frac{N}{M}\right)\sqrt{M},$$

which intersect at the center, are asymptotes of the two infinite branches.

119. 3° $N^2-MP=0$. One has then

$$Y=\frac{1}{C}\sqrt{M\left(x+\frac{N}{M}\right)^2}=\frac{\sqrt{M}}{C}\left(x+\frac{N}{M}\right),$$

and $$y=-\frac{Bx+E}{C}\pm\frac{\sqrt{M}}{C}\left(x+\frac{N}{M}\right).$$

The locus is represented by two straight lines which intersect on the diameter DD'.

Genus Parabola.

120. Suppose finally that the coefficient M or $B^2 - AC$ be zero. The value of Y reduces to

$$Y = \frac{1}{C}\sqrt{2Nx + P}.$$

This case may be subdivided into several others.

1° $N > 0$. By putting $-\frac{P}{2N} = x'$, the expression for Y may be written

$$Y = \frac{1}{C}\sqrt{2N(x - x')}.$$

When x varies from x' to $+\infty$, the quantity Y is real and varies from 0 to $+\infty$; but it is imaginary for all values of x less than x'. If, therefore, the line $P'A'$ be drawn through the point P', whose abscissa is x', parallel to the y-axis, the curve is situated wholly to the right of this parallel; it passes through the point A' and extends, on either side of the diameter DD', to infinity (Fig. 75). This curve is given the name *parabola*.

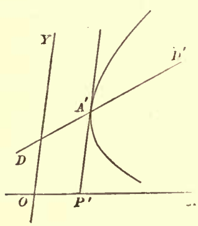

Fig. 75.

2° $N < 0$. The quantity Y is real when x varies from x' to $-\infty$; the curve passes through the point A', lies wholly to the left of the parallel $P'A'$, and extends to infinity; this curve is also called the parabola.

3° $N = 0$. The value of y reduces to

$$y = -\frac{Bx + E}{C} \pm \frac{1}{C}\sqrt{P}.$$

If P is positive, this equation represents two real straight lines, parallel to the diameter DD', and situated at equal distances from this diameter. If $P = 0$, these two parallels coincide with the diameter; finally, if P is negative, the equation does not have a real solution.

121. In what precedes, it has been assumed that the coefficient C differed from zero. In case the coefficient C is zero and the coefficient A different from zero, one can solve the equation with respect to x and construct the locus as in the preceding discussion; the first term of the trinomial under the radical has the coefficient $M = B^2$, a positive quantity or zero, and the locus belongs to the genus hyperbola or the genus parabola. In case a variable appears in the first degree, it is preferable to solve the equation with respect to it; moreover, this method is only applicable when the two coefficients A and C are zero at the same time.

It follows, by arranging equation (1) with respect to y, that

$$2(Bx + E)\,y + Ax^2 + 2\,Dx + F = 0,$$

whence
$$y = -\frac{Ax^2 + 2\,Dx + F}{2(Bx + E)}.$$

Suppose now that B be different from zero, and after arranging with respect to the decreasing powers of x, that one divides till a remainder is found which does not contain x. Two cases are distinguished, according as the remainder is different from or equal to zero. In the first case one will obtain a result of the form

$$y = ax + b + \frac{r}{2(Bx + E)} = ax + b + \frac{c}{x - d}.$$

In order to fix the ideas, assume $c > 0$. Construct the auxiliary locus defined by the equation $y = ax + b$, and put $Y = \dfrac{c}{x - d}$. The equation $y = ax + b$ represents a straight line HL (Fig. 76); for each value of x, it is necessary to increase the ordinate of this line by a quantity QM equal to the value of Y. This quantity becomes infinite for $x = d$; take, therefore, a point I having the abscissa d and draw $H'L'$ parallel to OY. If a value $d + x'$ be given to x, x' being positive, Y will have a positive value, and as x' tends toward zero, Y will increase

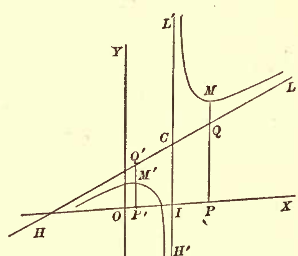

Fig. 76.

indefinitely; if, on the contrary, x' increase indefinitely, Y tends towards zero; thus is obtained a curve comprised within the angle $L'CL$ and composed of two infinite branches, asymptotic to the two lines CL, CL'. To values of x less than d correspond negative values of Y, and a second curve is obtained which is comprised within the angle HCH', and composed of two infinite branches asymptotic to the lines CH and CH'. To two equal values of x' with contrary signs, correspond two values of Y which are also equal and of contrary signs, and, consequently, two points M and M' symmetrical with respect to the point C, which is the center of the curve. If the constant c were negative, one would still obtain a curve consisting of two distinct parts, situated in the angles HCL', $H'CL$. The curve is a hyperbola in both cases.

If the remainder after division be zero, one has

$$Ax^2 + 2\,Dx + F = -\,2\,(Bx + E)\,(ax + b),$$

and the equation takes the form $(y - ax - b)\,(Bx + E) = 0$; it resolves into two others, $y - ax - b = 0$, $Bx + E = 0$, which represent two lines, one of which is parallel to the y-axis.

When A and C are zero at the same time, it is sufficient to put $a = 0$ in the preceding discussion; the line DD' becomes parallel to the x-axis; thus is found, in one case the hyperbola having its asymptotes parallel to the co-ordinate axes, in the other two straight lines respectively parallel to the axes.

In case the coefficients B and C are zero, the value of y has the form $y = ax^2 + bx + c$; it is real whatever real value x may have; by causing x to vary from $-\infty$ to $+\infty$, one gets a curve which extends to infinity in two directions; this is a parabola.

122. Résumé. — In discussing the equation of the second degree, three species of curves have been found; closed curves, curves composed of two distinct parts extending to infinity in two directions, curves composed of a single branch extending to infinity in two directions. To these three species of curves have been given the names ellipse, hyperbola, and parabola.

In the beginning of this work (Book I., Chapter II.) it has been noticed that the curves designated by the same names in elementary geometry are represented by equations of the second degree. Conversely we shall see hereafter that all the curves represented by the equation of the second degree possess the properties which are characteristic of the definitions in elementary geometry, and hence that the two modes of definitions are equivalent.

On reviewing the discussion, it is seen that it is the sign of the quantity $M = B^2 - AC$ which determines the species of the curve represented by the equation of the second degree; the curve is an ellipse, a hyperbola, or a parabola, according as the quantity M is negative, positive, or zero.

Moreover, it is important to recall that the equation does not always represent a curve, or what is the same, a locus; when the quantity M is negative, the equation represents an ellipse or a point, or does not admit of a real solution; in case this quantity is positive, the equation represents an hyperbola, or two straight lines which intersect; finally, when $M = 0$, the equation represents either a parabola, or two parallel straight lines, or a single straight line, or it does not have a real solution.

Various Forms of the Polynomial of the Second Degree in two Variables.

123. The preceding discussion shows that the first member of the equation of the curve can be put under various forms which it is important to characterize. Two principal cases are distinguished, according as C is different from zero or equal to zero.

1° $C \lessgtr 0$. By solving the equation with respect to y, as has been done, transposing and removing the radical from $\sqrt{Mx^2 + 2Nx + P}$ by squaring, the equation can be put under the form

$$(6) \qquad (Cy + Bx + E)^2 - (Mx^2 + 2Nx + P) = 0.$$

If M differ from zero (genus ellipse or hyperbola), and the trinomial $Mx^2 + 2Nx + P$ be resolved into a square, one has the form

$$(7) \qquad (Cy + Bx + E)^2 - M\left(x + \frac{N}{M}\right)^2 + \frac{N^2 - MP}{M} = 0.$$

If one suppose $M = 0$ (genus parabola), it follows from (6) that

$$(8) \qquad (Cy + Bx + E)^2 - (2Nx + P) = 0.$$

Thus, when $M \lessgtr 0$, the first member of the equation is decomposed into three squares (7), of which the last is a constant, these squares being affected by the signs $+$ or $-$, the first square is affected by the $+$ sign, the second is multiplied by $-M$, which can be positive or negative, the third can be positive or negative. The different combinations of signs correspond to the cases which have been met in the preceding general discussion. They are classified in the table given below, where the positive square

$$(Cy + Bx + E)^2$$

is represented by α^2, the square $-M\left(x + \frac{N}{M}\right)^2$ by $+\beta^2$ or $-\beta^2$, according as the coefficient $-M$ is positive or negative; finally, the constant $\frac{N^2 - MP}{M}$ by $+k^2$ or $-k^2$, according as it is positive or negative.

When $M = 0$, the equation takes the form (8) of a square α^2 followed by a linear function $(2Nx + P)$, which is represented by γ when N is different from zero; if N be zero, this linear function is reduced to a constant P, which is designated by $+k^2$ or $-k^2$, according as it is positive or negative.

Thus will arise the following table, in which it is not necessary for the moment to regard the results written in the third column which refer to the case $C = 0$ examined farther on.

The constant c, which appears in the third column, has the value $\frac{1}{2B^3}(-AE^2 - FB^2 + 2BDE)$.

This table shows that, if the inequalities $M < 0$, and $N^2 - MP < 0$ exist at the same time, the equation does not

GENUS	$C \gtrless 0$	$C = 0$	CURVE	FORM OF THE EQUATION
$M < 0$, Ellipse.	$N^2 - MP > 0$	If $C = 0$ the curve is never an ellipse.	Real ellipse.	$\alpha^2 + \beta^2 - k^2 = 0$
	$N^2 - MP < 0$		Imaginary ellipse.	$\alpha^2 + \beta^2 + k^2 = 0$
	$N^2 - MP = 0$		Point.	$\alpha^2 + \beta^2 = 0$
$M > 0$, Hyperbola.	$N^2 - MP \gtrless 0$	$c \gtrless 0$	Hyperbola.	$\alpha^2 - \beta^2 \pm k^2 = 0$
	$N^2 - MP = 0$	$c = 0$	Two straight lines which intersect.	$\alpha^2 - \beta^2 = 0$
$M = 0$, Parabola.	$N \gtrless 0$	$B = 0, E \gtrless 0$	Parabola.	$\alpha^2 - \gamma = 0$
	$N = 0, P > 0$	$B = 0$, $E = 0$: $D^2 - AF > 0$	Real parallel straight lines.	$\alpha^2 - k^2 = 0$
	$N = 0, P < 0$	$B = 0$, $E = 0$: $D^2 - AF < 0$	Id. imaginary.	$\alpha^2 + k^2 = 0$
	$N = 0, P = 0$	$B = 0$, $E = 0$: $D^2 - AF = 0$	Coincident straight lines.	$\alpha^2 = 0$

represent a locus, because the first member of the equation is then the sum of three squares $\alpha^2 + \beta^2 + k^2$, which cannot be zero for any real values of the co-ordinates: in this case the curve is said to represent an imaginary ellipse. Similarly, one sees at once that, if $M < 0$ and $N^2 - MP = 0$, the equation represents a point, because its first member $\alpha^2 + \beta^2$, being the sum of two squares, can only be reduced to zero by the co-ordinates of the point whose co-ordinates reduce to zero at the same time these two squares, that is, $Cy + Bx + E$ and $x + \dfrac{N}{M}$. In case of the genus hyperbola, the equation represents always a locus: if $N^2 - MP = 0$, it represents, as has been seen, two straight lines which intersect; this follows at once from the preceding equation, because the equation, having then the form $\alpha^2 - \beta^2 = 0$, decomposes into a product of two real factors of the first degree $(\alpha + \beta)(\alpha - \beta) = 0$; it is therefore equivalent to the system of two equations

$$\alpha + \beta = 0, \quad \alpha - \beta = 0,$$

which represents two straight lines passing through the point of intersection of $\alpha = 0$ and $\beta = 0$. In analogy with this case, it is sometimes said, that a point-ellipse is an *ensemble* * of

* The French expresses the idea better than a translation.

imaginary straight lines which intersect, because the equation takes then the form $\alpha^2 + \beta^2 = 0$, and is algebraically equivalent to the *ensemble* of two linear equations

$$\alpha + \beta\sqrt{-1} = 0, \quad \alpha - \beta\sqrt{-1} = 0,$$

which represent nothing more than what one is accustomed for convenience to call the equations of conjugate imaginary straight lines. These two equations are satisfied by the co-ordinates of the point which reduce at the same time α and β to zero, that is, of the point to which the ellipse is reduced. The two imaginary straight lines are said to intersect in this point.

2° $C = 0$. The curve can never belong to the genus ellipse.

If B be different from zero, the equation can, as has been seen in § 121, be put under the form

$$y = ax + b + \frac{c}{x - d},$$

where the constant c, which is the remainder of the division of

$$-(Ax^2 + 2Dx + F)$$

by $2B\left(x + \frac{E}{B}\right)$, has the value

$$c = \frac{1}{2B^3}(-AE^2 - FB^2 + 2BDE);$$

by clearing of fractions the equation may therefore be written

$$(y - ax - b)(x - d) - c = 0.$$

The first member is a product of linear factors in x and y. It can also be thrown into the form of a difference of squares $\alpha^2 - \beta^2$ by writing

$$\left(\frac{y - ax - b + x - d}{2}\right)^2 - \left(\frac{y - ax - b - x + d}{2}\right)^2 - c = 0,$$

an equation of the form $\alpha^2 - \beta^2 - c = 0$, c designating a constant which may be positive, negative, or zero.

If c be different from zero, one has a real hyperbola. If $c = 0$, the first member of the equation resolves into the product of two linear factors, and the curve is represented by two straight lines which intersect.

If B be zero at the same time as C, E being different from zero, the equation becomes

$$Ax^2 + 2\,Dx + 2\,Ey + F = 0.$$

It represents a parabola, and can be written in the form $\alpha^2 - \gamma = 0$, α and γ being two linear functions, the first of which reduces to x. If, further, E be zero, the equation is a trinomial in x equal to zero, and can be written

$$\left(x + \frac{D}{A}\right)^2 - \frac{D^2 - AF}{A^2} = 0.$$

It represents two parallel straight lines real, imaginary, or coincident, according as $D^2 - AF$ is positive, negative, or zero. The term $x + \frac{D}{A}$ is a linear function α; the constant $\frac{D^2 - AF}{A^2}$ is of the form $\pm k^2$. The equation takes therefore the form

$$\alpha^2 \pm k^2 = 0.$$

These results are given in the table on page 143; the different hypotheses corresponding to the case $C = 0$ are arranged in the third column.

REMARK. — If the quantity $N^2 - MP$ be constructed by replacing M, N, P by their values as functions of the coefficients A, B, C, D, E, F, it follows that

(9) $N^2 - MP = -C(ACF - AE^2 - CD^2 - FB^2 + 2\,BDE).$

The quantity within the parenthesis, which plays an important rôle in the theory, is called the *discriminant* of the curve: it is designated by Δ:

$$\Delta = ACF - AE^2 - CD^2 - FB^2 + 2\,BDE.$$

It follows from the discussion which has been given, and the results of which have been arranged in the preceding table, that *the necessary and sufficient condition in order that the curve be a system of two real, imaginary, or coincident straight lines is*

$$\Delta = 0.$$

In fact, if neither C nor M be zero, the necessary and sufficient condition in order that the equation may represent two straight lines is $N^2 - MP = 0$, this is, according to (9), $\Delta = 0$; if C be different from zero and M zero, this condition is $N = 0$, that is, still $\Delta = 0$.

If $C = 0$, and B be different from zero, the condition is $C = 0$, that is, by reason of the value of c, $\Delta = 0$.

If C and B are zero, this condition is $E = 0$, that is, still $\Delta = 0$.

124. Seek directly the condition necessary and sufficient, in order that the general equation of the second degree may represent two real or imaginary straight lines, that is, in order that its first member may be resolved into a product of factors linear in x and y:

$$(10) \qquad Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F \equiv (lx + my + p)(l'x + m'y + p').$$

Substitute in this identity for x and y, $\frac{x}{z}$ and $\frac{y}{z}$, then by removing the denominator z^2, one will get a new identity of the form

$$(11) \qquad f(x, y, z) = QR,$$

where $\quad f(x, y, z) = Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2,$

$$Q = lx + my + pz, \quad R = l'x + m'y + p'z.$$

Conversely, in case the identity (11) is given, one may return to the identity (10) by making $z = 1$. Take the successive partial derivatives of the two members of the identity (11) with respect to x, y, z, then one has

$$(12) \qquad \begin{aligned} f'_x &= 2(Ax + By + Dz) = lR + l'Q, \\ f'_y &= 2(Bx + Cy + Ez) = mR + m'Q, \\ f'_z &= 2(Dx + Ey + Fz) = pR + p'Q. \end{aligned}$$

There exists evidently at least one system of values of x, y, z, $x = a$, $y = b$, $z = c$, which reduces to zero at the same time the linear functions R and Q, a, b, c not all three being zero. According to the identities (12), the same values a, b, c reduce simultaneously f'_x, f'_y, f'_z to zero. Hence, when the conic is decomposed into two straight lines, the three linear and homogeneous equations in x and y

$$(13)\qquad \begin{aligned} Ax + By + Dz &= 0, \\ Bx + Cy + Ez &= 0, \\ Dx + Ey + Fz &= 0, \end{aligned}$$

admit at least of one solution $x = a$, $y = b$, $z = c$, in which the three unknown quantities are not zero at the same time. Therefore the determinant of the coefficients

$$\Delta = \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$$

is zero. This determinant is none other than the discriminant written above (REMARK) in a developed form.

The condition $\Delta = 0$ is therefore necessary. It is sufficient. In fact, suppose it fulfilled: there exists then a system of values a, b, c, of x, y, z, of which all three are not zero, satisfying equations (13), that is, reducing to zero f'_x, f'_y, f'_z. Let, for example, c differ from zero: making the change of variables,

$$(14)\qquad \begin{aligned} x &= az' + x', \\ y &= bz' + y', \\ z &= cz', \end{aligned}$$

the function $f(x, y, z)$ will become

$$f(az' + x',\ bz' + y',\ cz'),$$

that is, by developing and recalling that the function is homogeneous, and of the second degree in x, y, z, and that consequently its derivatives are homogeneous and of the first degree,

$$f(x, y, z) = z'^2 f(a, b, c) + x'z'f'_a + y'z'f'_b + Ax'^2 + 2\,Bx'y' + Cy'^2.$$

The derivatives f'_a and f'_b are zero: $f(a, b, c)$ is also zero by virtue of the identity easily verified (theorem of homogeneous functions),

$$2f(a, b, c) = af'_a + bf'_b + cf'_c.$$

The function $f(x, y, z)$ is therefore identical with the expression $Ax'^2 + 2Bx'y' + Cy'^2$, which evidently decomposes into the product of two linear factors $(\lambda x' + \mu y')(\lambda' x' + \mu' y')$, and one has identically

$$f(x, y, z) = (\lambda x' + \mu y')(\lambda' x' + \mu' y').$$

Returning to the variables x, y, z by aid of equations (14), which give

$$z' = \frac{z}{c}, \quad x' = x - \frac{a}{c}z, \quad y' = y - \frac{b}{c}z,$$

one gets for f an expression of the form

$$f(x, y, z) = (lx + my + pz)(l'x + m'y + p'z).$$

Remark. — Designate by a, b, c, d, e, f the minors of the discriminant Δ, with respect to the elements A, B, C, D, E, F; thus

$$\mathsf{a} = CF - E^2, \quad \mathsf{b} = DE - BF, \quad \mathsf{c} = AF - D^2,$$

$$\mathsf{d} = BE - CD, \quad \mathsf{e} = BD - AE, \quad \mathsf{f} = AC - B^2.$$

The genus of the conic depends on the sign of f: when f is zero, the curve belongs to the genus parabola.

From this notation, one has, on developing, the determinant Δ with respect to the elements of a row,

$$\Delta = A\mathsf{a} + B\mathsf{b} + D\mathsf{d} = B\mathsf{b} + C\mathsf{c} + E\mathsf{e} = D\mathsf{d} + E\mathsf{e} + F\mathsf{f}.$$

124. 2. As a special case, determine the necessary and sufficient conditions, in order that the conic be formed of two coincident straight lines. The first member of the equation is then a perfect square of a linear function of the co-ordinates, and one has the identity

$$(15) \qquad f(x, y, z) = (lx + my + pz)^2.$$

On taking the partial derivatives of this identity, it is seen immediately that the three linear equations (13) are replaced by a single equation,

$$(16) \qquad lx + my + pz = 0.$$

Their coefficients are therefore proportional, which shows that all the minors of Δ are zero.

(17) $$\mathsf{a}=0,\ \mathsf{b}=0,\ \mathsf{c}=0,\ \mathsf{d}=0,\ \mathsf{e}=0,\ \mathsf{f}=0.$$

For example, the conditions

$$\frac{A}{B}=\frac{B}{C}=\frac{D}{E}$$

give, when the denominators are removed, $\mathsf{f}=0$, $\mathsf{d}=0$; etc. ⋯

These conditions may also be verified directly, because the identity (15) gives

$$A=l^2,\quad C=m^2,\quad F=p^2,$$

$$B=lm,\quad D=lp,\quad E=mp.$$

It will be found on constructing the minors $\mathsf{a}, \mathsf{b}, \mathsf{c}, \cdots$, that they are all zero.

These conditions are, moreover, sufficient in order that the first member of the equation be a perfect square. In fact, if they be fulfilled, the three coefficients, A, C, F, cannot all be zero, because conditions (17) require that B, D, E should also be zero and all the coefficients would be zero. Assume then that A be different from zero, one will have

$$Af(x,y,z)=A^2x^2+2ABxy+ACy^2+2ADxz+2AEyz+AFz^2.$$

Supposing that the conditions are fulfilled, one has

$$AC=B^2,\ AE=BD,\ AF=D^2;$$

and the preceding relation gives

$$Af(x,\ y,\ z)=(Ax+By+Cz)^2.$$

This subject will be considered more in detail at the end of Book III.

Tangent to Curves of the Second Degree.

125. Let $f(x, y) = 0$ be the equation of a curve; if x and y be the co-ordinates of the point of contact M, X and Y the co-ordinates of any variable point of the tangent, one has seen (§ 89) that the tangent is represented by the equation

$$(X - x)f'_x + (Y - y)f'_y = 0,$$

or

$$Xf'_x + Yf'_y - (xf'_x + yf'_y) = 0.$$

When the curve is of the second degree, one has

$$f(x, y) = Ax^2 + 2\,Bxy + Cy^2 + 2\,Dx + 2\,Ey + F;$$

$$f'_x = 2\,(Ax + By + D),\quad f'_y = 2\,(Bx + Cy + E);$$

$$xf'_x + yf'_y = 2\,(Ax^2 + 2\,Bxy + Cy^2 + Dx + Ey).$$

The point of contact M being situated on the curve, its co-ordinates x and y satisfy the equation

$$(1) \qquad Ax^2 + 2\,Bxy + Cy^2 + 2\,Dx + 2\,Ey + F = 0.$$

It follows that

$$Ax^2 + 2\,Bx + Cy^2 = -\,(2\,Dx + 2\,Ey + F),$$

and, consequently,

$$xf'_x + yf'_y = -\,2\,(Dx + Ey + F).$$

The equation of the tangent, at the point whose co-ordinates are x and y, becomes

$$(2) \quad (Ax + By + D)X + (Bx + Cy + E)\,Y + (Dx + Ey + F) = 0.$$

One notices that the co-ordinates x and y of the point of contact enter only to the first degree. As this equation can be put in the form

$$(3) \quad (AX + BY + D)\,x + (BX + CY + E)\,y + (DX + EY + F) = 0,$$

it is to be noticed that it does not change, in case X and x, Y and y are permuted.

It is proposed now to draw tangents to the curve from a given point P, not situated on the curve, and having the co-ordinates x_1 and y_1. Call x and y the unknown co-ordinates of one of the points of contact M. These co-ordinates should satisfy equation (1). The tangent at the point M is represented by equation (2). Since this tangent passes through the point P, the co-ordinates of this point satisfy equation (2) or equation (3), and one will have

$$(4)\quad (Ax_1 + By_1 + D)\,x + (Bx_1 + Cy_1 + E)\,y + Dx_1 + Ey_1 + F = 0.$$

The co-ordinates x and y are therefore determined by the two simultaneous equations (1) and (4). The one being of the second, the other of the first degree, this system of two equations has two solutions, and two tangents can be drawn from a given point P to a curve of the second degree. The solution of these two equations amounts to finding the points of intersection of the curves defined by each of them; the first is the given curve, the second, a straight line passing through the two points of contact. One notices that equation (4) of the chord of contacts has the same form as equation (2) of the tangent. It is sufficient to replace in the latter the co-ordinates of the point of contact by those of the point P.

126. Find the condition that the straight line $y = mx + k$ be tangent to a curve of the second degree. If y in equation (1) be replaced by $mx + k$, an equation will be found of the second degree in x, which furnishes the abscissas of the points of intersection of the straight line and the curve. The straight line will be a tangent when the two roots are equal. Thus is found the equation of condition

$$\mathrm{a}m^2 - 2\,\mathrm{b}m + \mathrm{c} + 2\,\mathrm{d}mk - 2\,\mathrm{e}k + \mathrm{f}k^2 = 0.$$

When the equation of the line has the form

$$ux + vy + 1 = 0,$$

the equation of condition becomes

$$(5)\qquad \mathrm{a}u^2 + 2\,\mathrm{b}uv + \mathrm{c}v^2 + 2\,\mathrm{d}u + 2\,\mathrm{e}v + \mathrm{f} = 0.$$

The calculation may be made more symmetrical by proceeding as follows:

Consider the line $uX + vY + 1 = 0,$

and suppose that it be tangent to the curve at the point whose co-ordinates are x and y. Then the latter equation ought to be identical with that of the tangent at this point, and one should have, on representing a coefficient of proportionality by λ:

$$
(6) \qquad
\begin{aligned}
u &= \lambda(Ax + By + D),\\
v &= \lambda(Bx + Cy + E),\\
1 &= \lambda(Dx + Ey + F).
\end{aligned}
$$

By multiplying the first of these equations by x, the second by y, and adding to the third, one has

$$ux + vy + 1 = \lambda(Ax^2 + 2\,Bxy + Cy^2 + 2\,Dx + 2\,Ey + F);$$

since the point x, y is on the curve, the second member is zero, and one has

$$ux + vy + 1 = 0,$$

or

$$\lambda(ux + vy + 1) = 0.$$

This equation, combined with equations (6), gives a system of four equations of the first degree in λx, λy, and λ. The elimination of these three quantities gives the condition sought in the form of a determinant:

$$
\begin{vmatrix}
A & B & D & u\\
B & C & E & v\\
D & E & F & 1\\
u & v & 1 & 0
\end{vmatrix} = 0,
$$

whose development leads to equation (5).

CHAPTER II

CENTER, DIAMETERS, AND AXES OF CURVES OF THE SECOND DEGREE.

127. The center of a curve has been defined as a fixed point C, with respect to which all the points of the curve are symmetrical two by two. In the discussion of the general equation of the second degree, it was found that the ellipse and hyperbola have a center. It is proposed now to determine directly the center of a curve of the second degree without solving the equation. The method which will be used depends on the theorem: when the origin of co-ordinates is the center of a curve of the second degree, the equation of the curve does not contain the terms of the first degree.

Let

(1) $$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

be the equation of a curve of the second degree having the origin at the center (Fig. 77); the equation of a straight line MM' drawn through the origin has the form $y = mx$. The elimination of y between this equation and that of the curve gives the equation

(2) $$(A + 2Bm + Cm^2)x^2 + 2(D + Em)x + F = 0,$$

which determines the abscissas of the two points of intersection.

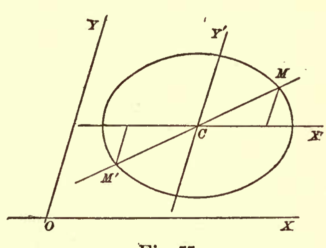

Fig. 77.

The origin being the mid-point of the line MM', the preceding equation ought to have equal roots with contrary signs, and this will be the case if the coefficient of the first power of x be zero; one has, therefore, $D + Em = 0$, and, since this condition should hold for an infinity of values of m, one should have separately $D=0$, $E=0$. Conversely, when these conditions are fulfilled, equation (2) has two equal roots with contrary signs, whatever be the value of m, and, consequently, the origin is the center of the curve.

128. In order to know if a locus of the second degree have a center, keeping the axes parallel to themselves, transfer the origin to an arbitrary point whose co-ordinates are a and b, then examine whether these quantities can be so determined that the new equation does not contain the terms of the first degree.

The formulas for transferring the axes parallel to themselves are $x=a+x'$, $y=b+y'$. On substituting in equation (1), the new equation will be

$$(3)\quad Ax'^2+2Bx'y'+Cy'^2+2(Aa+Bb+D)x'+2(Ba+Cb+E)y' \\ +Aa^2+2Bab+Cb^2+2Da+2Eb+F=0,$$

whose composition should be carefully noted. Represent, for brevity, the first member of equation (1) by $f(x, y)$, which is an integral function of the second degree in x and y; in equation (3), the terms of the second degree are the same as in equation (1); the terms of the first degree have as coefficients the partial derivatives of the function $f(x, y)$, taken with respect to the variables x and y, and in which the variables have been replaced by a and b; finally, the constant term is the value which the polynomial $f(x, y)$ takes for $x=a$ and $y=b$, and equation (2) can therefore be written

$$(4)\quad Ax'^2+2Bx'y'+Cy'^2+f'_a(a, b)x'+f'_b(a, b)y'+f(a, b)=0.$$

On equating to zero the coefficients of x' and y', one obtains the two equations of the first degree,

$$(5)\qquad \begin{cases} Aa+Bb+D=0, \\ Ba+Cb+E=0. \end{cases}$$

It follows, therefore, *that the center of a curve of the second degree is determined by solving the two equations which are found by equating to zero the partial derivatives of the first member of the given equation, taken with respect to x and y.*

129. If a and b be regarded as variable co-ordinates, each of the equations (5) defines a straight line, and there is occasion to distinguish several cases, according as the denominator, common to the values of the unknown quantities, or the determinant $AC - B^2$, which has been represented by $-M$ or $\mathbf{f}$, is different from zero or equal to zero.

1° When the determinant $\mathbf{f}$ is different from zero, the system of equations is satisfied by one system of values of a and b and by one only; the two straight lines intersect; the curve has a center and a definite center, whose co-ordinates are, according to § 124,

$$(6) \qquad \begin{cases} a = \dfrac{\mathbf{d}}{\mathbf{f}}, \\ b = \dfrac{\mathbf{e}}{\mathbf{f}}. \end{cases}$$

2° In case the determinant $\mathbf{f}$ is equal to zero (genus parabola), the lines are parallel, or coincide. In the first case, the curve does not have a center; in the second case, every point of the straight line defined by one of equations (5) is a center. It is easy to see that, in the latter case, the locus, if it exist, is necessarily composed of two parallel straight lines. Let, in fact, CC' be the straight line which is the locus of the centers (Fig. 78), and M a point belonging to the locus; join the point M to the various points of the straight line CC', and prolong each of these straight lines till IN is equal to IM, etc. The points $N, N', N'', \cdots,$ thus obtained, will belong to the locus.

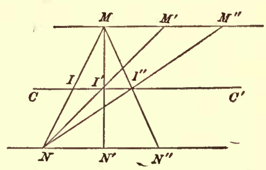

Fig. 78.

Now all of these points are situated on a line parallel to CC'. Proceeding in the same manner with the point N, a second parallel MM' will be determined. Moreover, equation (1) cannot represent other points than those of these straight lines; otherwise a straight line could intersect the locus in more than two points. If the point M were situated on the line CC', the two parallels would coincide with the locus of the centers.

130. If the curve have a center, and the origin be transformed to this point while the axes remain parallel to themselves, the equation simplifies and becomes

(7) $$Ax'^2 + 2\,Bx'y' + Cy'^2 + H = 0,$$

since the terms of the first degree disappear. The constant term H of the new equation has the value

$$H = Aa^2 + 2\,Bab + Cb^2 + 2\,Da + 2\,Eb + F,$$

a and b representing the co-ordinates of the center. But the quantities a and b satisfy equations (5): if the members of each of them be multiplied respectively by a and b, and added together, they become

$$Aa^2 + 2\,Bab + Cb^2 + Da + Eb = 0,$$

whence $$Aa^2 + 2\,Bab + Cb^2 = -(Da + Eb),$$

and, consequently, $H = Da + Eb + F$;

and by replacing a and b by their values (6)

(8) $$H = \frac{\Delta}{\mathrm{f}}.$$

When the discriminant Δ is zero, the equation reduces to

(9) $$Ax'^2 + 2\,Bx'y' + Cy'^2 = 0,$$

whence

(10) $$y' = \frac{-B \pm \sqrt{B^2 - AC}}{C}\,x'.$$

If the quantity $B^2 - AC$ be negative, the equation has the real solution $x' = 0$, $y' = 0$. If it be positive, the equation represents two straight lines passing through the origin. In this case equation (7), in which any arbitrary value may be assigned to H, defines a hyperbola; it has been found (§ 117) that the asymptotes of a hyperbola pass through its center, and that their angular coefficients are given by the formula

$$m = \frac{-B \pm \sqrt{B^2 - AC}}{C};$$

these asymptotes are none other than the straight lines represented by equation (10) or by equation (9). Thus, when an equation of the second degree represents a hyperbola referred to its center, the equation of the asymptotes is found by suppressing the constant term in the given equation.

From this it follows that if the general equation of the second degree,

$$f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

represent a hyperbola, the equation

$$(11) \qquad f(x, y) - \frac{\Delta}{\mathbf{f}} = 0$$

represents the *ensemble* of two asymptotes. In fact, if the origin be transferred to the center of the curve, $f(x, y)$ becomes

$$Ax'^2 + 2Bx'y' + Cy'^2 + \frac{\Delta}{\mathbf{f}};$$

therefore, equation (11) becomes

$$Ax'^2 + 2Bx'y' + Cy'^2 = 0,$$

which is the equation of the asymptotes.

DIAMETERS.

131. If a curve of the second degree be cut by a system of parallel straight lines, the locus of the mid-points I of the chords MM', determined by the two points of intersection, is a *diameter* of the curve. Let m be the angular coefficients of the chords, and

$$(1) \quad f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

be the equation of the curve. If the axes be kept parallel to themselves, and the origin be transferred to the arbitrary point I of the plane, whose co-ordinates are a and b, the equation of the curve becomes (§ 128)

$$(4) \quad Ax'^2 + 2Bx'y' + Cy'^2 + f'_a(a, b)x' + f'_b(a, b)y' + f(a, b) = 0.$$

Draw through this point I a line MM' parallel to the given direction; the equation of this parallel is $y' = mx'$. The elimination of y' between this equation and that of the curve leads to the equation of the second degree,

(11) $$[A+2Bm+Cm^2]x'^2+[f'_a(a,b)+f'_b(a,b)m]x' + f(a,b)=0,$$

which gives the abscissas of the points of intersection. So long as the value assigned to m does not reduce $A+2Bm+Cm^2$, the coefficient of x^2, to zero, each of the secants intersects the curve in two points; if it be assumed that the origin I be placed at the mid-point of the chord MM' (Fig. 79), equation (11) having its roots equal with contrary signs, one has the relation

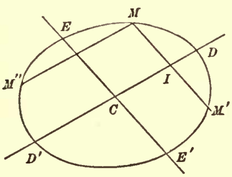

Fig. 79.

(12) $$f'_a(a,b)+f'_b(a,b)m=0;$$

this equation being satisfied by the co-ordinates of the mid-point of any of the chords considered, is the equation of the locus. If a and b be replaced by x and y, it becomes

(13) $$f'_x(x,y)+mf'_y(x,y)=0,$$

or

(14) $$(Ax+By+D)+m(Bx+Cy+E)=0.$$

Since this equation is of the first degree, it follows that the diameter, which corresponds to any system of parallel chords, is a straight line DD'. Call m' the angular coefficient of the diameter; we shall have the relation

(15) $$m'=-\frac{A+Bm}{B+Cm},$$

or

(16) $$Cmm'+B(m+m')+A=0.$$

132. Remark I. — The values of x and y, which satisfy the simultaneous equations

$$Ax+By+D=0,\ Bx+Cy+E=0,$$

satisfy equation (14), whatever be the value of m; therefore if the locus have a definite center, all of the diameters pass through the center, and, if it have an infinity of centers, all of the diameters coincide with the locus of the centers.

The two equations, which determine the center, represent two diameters; the first corresponds to the chords parallel to the x-axis, the second to chords parallel to the y-axis. They are formed by putting $m = 0$ or $m = \infty$.

133. Remark II. — In case the curve be an ellipse, the trinomial $A + 2Bm + Cm^2$, which has imaginary roots, is always different from zero; to every direction of the chords corresponds a diameter defined by equation (14).

This equation (14), if m be regarded as an arbitrary parameter, represents all of the straight lines which pass through the center; whence it follows that every straight line passing through the center is a diameter.

Remark III. — In case of the hyperbola, the trinomial $A + 2Bm + Cm^2$ becomes zero for two real values of m which are the angular coefficients of the asymptotes. If one of these values be given to m, equation (11) is depressed to the first degree; each of the secants intersects the curve in but one point. If, further, the co-ordinates a and b of the point I, through which the secant is drawn, satisfy relation (12), equation (11), having its first two coefficients zero, has no longer a solution; the straight line represented by equation (14) is then the locus of the points I such that the parallels drawn through each of its points with the given direction do not intersect the curve; but, by reason of relation (16), the value of m' being equal to m, all of these parallels coincide with line (14) itself. Since this line passes through the center, it is one of the asymptotes.

If m be regarded as an arbitrary parameter, equation (14) represents all of the straight lines which pass through the center; whence it follows that all of these straight lines, excepting the two asymptotes, are diameters.

134. Remark IV. — In case of the parabola, we have $AC - B^2 = 0$, or $\frac{A}{B} = \frac{B}{C}$; whence it follows that the value of m', given by equation (15), is independent of m and equal to

$-\frac{B}{C}$; thus all of the diameters of the parabola are parallel to each other.

The trinomial $A + 2Bm + Cm^2$ has its two roots equal to $-\frac{B}{C}$, the angular coefficient of the diameters. If parallel secants be drawn in this direction, each of them will intersect the curve in but one point. On the other hand, if the value $-\frac{B}{C}$ be assigned to m, the coefficients of x and y in equation (14) become zero and the equation ceases to represent a straight line.

Equation (14), in which m is regarded as an arbitrary parameter, represents all of the straight lines parallel to the direction $-\frac{B}{C}$; whence it follows that every straight line parallel to this direction is a diameter of the parabola.

If at the same time $AC - B^2 = 0$ and $BE - CD = 0$, or $\frac{A}{B} = \frac{B}{C} = \frac{D}{E}$, the locus of the second degree is represented by two parallel straight lines; if $-m'$ represent the common value of the preceding ratios,

$$Ax + By + D = -m'(Bx + Cy + E),$$

and equation (14) reduces to

$$(m - m')(Bx + Cy + E) = 0.$$

Thus, in this case, all of the diameters coincide.

Conjugate Diameters.

135. Assume that $AC - B^2$ differs from zero. The two coefficients m, and m' are connected by the relation

$$Cmm' + B(m + m') + A = 0. \tag{16}$$

Imagine secants to be so drawn that the chord MM'' be parallel to the diameter DD' (Fig. 79); let m'' be the angular coefficient of the diameter EE' which bisects these chords, then the relation between the direction m' of the chords and the direction m'' of the corresponding diameter EE', will be

$$Cm'm'' + B(m' + m'') + A = 0;$$

this and the preceding equation being of the first degree with respect to m'' and m, it follows that $m'' = m$. The two diameters DD' and EE', whose angular coefficients are m' and m, have the property, that each of them bisects the chords parallel to the other; they are for this reason called *conjugate diameters.*

The ellipse and the hyperbola have an infinity of systems of conjugate diameters. One can take for the first diameter any straight line drawn through the center, provided that, if the curve be a hyperbola, it does not coincide with one of the asymptotes.

136. It has been seen (§ 130) that the equation of the curve, referred to axes parallel to the primitive axes and drawn through the center, is

$$(17) \qquad Ax^2 + 2Bxy + Cy^2 + H = 0.$$

If any two diameters be taken as axes of co-ordinates, and as this transformation may be accomplished by aid of formulas (4) of § 51, the homogeneous polynomial of the second degree $Ax^2 + 2Bxy + Cy^2$ transforming into a homogeneous polynomial of the second degree $A'x'^2 + 2B'x'y' + C'y'^2$, the equation becomes

$$A'x'^2 + 2B'x'y' + C'y'^2 + H = 0.$$

In case the two diameters are conjugate, since to each value of x' two equal values of y' with opposite signs correspond, the coefficient B' is zero, and the equation reduces to the simple form

$$(18) \qquad A'x'^2 + C'y'^2 + H = 0.$$

In case of the parabola, if a point on the curve be taken as the origin, which causes the constant term to vanish, the diameter which passes through this point as the x'-axis, a line through this point parallel to the chords which the diameter bisects as the y'-axis, since to each value of x' correspond two equal values of y' with opposite signs, one ought to have $B'x' + E' = 0$, and, consequently, separately $B' = 0$, $E' = 0$; on the other hand, since the curve is a parabola, the condition $A'C' - B'^2 = 0$ ought to be satisfied, which gives $A' = 0$; thus the equation reduces to the simple form

$$(19) \qquad C'y'^2 + 2D'x' = 0.$$

From which it is seen that the y'-axis coincides with the tangent at the origin.

Axes.

137. In curves of the second degree, the diameters perpendicular to the chords which they bisect are the *axes* of symmetry.

The parabola having all its diameters parallel, if one imagine a series of chords MM' (Fig. 80) perpendicular to the common direction of the diameters, the diameter AA', which bisects the chords, will be an axis of the curve and it will be the only one. The angular coefficient of the diameter is $-\frac{B}{C}$; therefore, if the co-ordinates are rectangular, the axis of the curve is the diameter of the chords having the angular coefficient $\frac{C}{B}$; its equation (§ 131) is

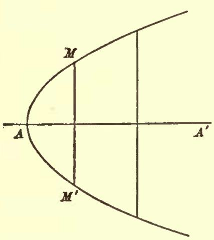

Fig. 80.

$$(20) \qquad B(Ax + By + D) + C(Bx + Cy + E) = 0.$$

In case of oblique co-ordinates, the angular coefficient of the chords perpendicular to the axis being

$$\frac{C - B\cos\theta}{B - C\cos\theta},$$

this straight line is determined by the equation

$$(Ax + By + D)(B - C\cos\theta) + (Bx + Cy + E)(C - B\cos\theta) = 0.$$

The equation of the parabola referred to its axis AA' and to the tangent at the *vertex* A, is of the form (19).

When the curve is an ellipse or a hyperbola, referred to the axis AA' (Fig. 81), and a second BB' corresponding to it, forming with the first a system of conjugate diameters, the question is then reduced to finding a pair of perpendicular conjugate diameters. If the co-ordinates be rectangular, the angular coefficients of the axes are found by combining the relation $mm' = -1$ with

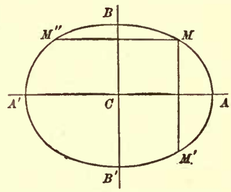

Fig. 81.

equation (16) which gives $m + m' = \frac{C - A}{B}$; thus, m and m' are the roots of an equation of the second degree,

$$(21) \qquad Bu^2 + (A - C)u - B = 0.$$

Should the origin of co-ordinates coincide with the center, the equation

$$(22) \qquad By^2 + (A - C)xy - Bx^2 = 0,$$

which is deduced from equation (21) by substituting $\frac{y}{x}$ for u, represents the *ensemble* of the two axes.

In case of oblique co-ordinates, the angular coefficients of the axes are the roots of the equation

$$(23) \qquad (B - C\cos\theta)u^2 + (A - C)u - (B - A\cos\theta) = 0.$$

The equation of the curve, referred to these two axes, is of the form (18).

Let u be a root of one of the equations (21) or (23); the equation of the corresponding axes will be

$$f'_x + uf'_y = 0.$$

Therefore, the origin of co-ordinates being chosen in any manner, one will have an equation of the second degree representing the *ensemble* of the axes on replacing u by $-\frac{f'_x}{f'_y}$, in (21) or (23), which gives, when the axes are rectangular (eq. 21),

$$(24) \qquad B(f'^2_x - f'^2_y) - (A - C)f'_x f'_y = 0.$$

137. 2. *To determine the position of a point with respect to a conic.*

Let $f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F$ be the first member of the equation of a conic. If the point $M(x, y)$ be displaced in the plane in a continuous manner by making it follow any arbitrary path, the function $f(x, y)$ varies in a continuous manner, and can only change sign when it becomes zero; that is, when the point M crosses the curve. The sign of $f(x, y)$ is therefore the *same* for all points of the plane situated on the *same side* of the curve; moreover, this sign changes when the point M crosses a simple branch of the curve. In fact, let $y = mx + h$ be the equation of a secant $M'M''$ cutting the curve in two real distinct points M', M'', with the abscissas x' and x''; displacing the point M on this secant, one will have

$$f(x, y) = f(x, mx + h);$$

the function $f(x, mx + h)$ is a trinomial of the second degree in x having the roots x' and x''. The trinomial has a certain sign when x is situated without the interval $x'x''$, and the opposite sign when x lies within this interval. Therefore, when the point M is displaced on this indefinite straight line $M'M''$, the function $f(x, mx + h)$, or its equal $f(x, y)$, has a certain sign so long as the point M is exterior to the segment $M'M''$, and the opposite sign when the point is on this segment. The sign of $f(x, y)$ changes, therefore, when the point M crosses the curve on a secant.

Accordingly, it suffices to know the sign of $f(x, y)$ for one point of the plane not situated on the curve in order to know its sign for any portion of the plane. Take, for example, the function

$$f(x, y) = x^2 + xy + y^2 - 2x + y,$$

which, equated to zero, represents a real ellipse. If one take a point $M(x, y)$, situated at a sufficient distance, it will be exterior to the curve; take, for example, on the axis Oy ($x = 0$, y sufficiently large); then $f(x, y)$, which is reduced to a trinomial in y, is evidently positive. Therefore, in this example, $f(x, y)$ is positive without the ellipse, and consequently negative within. One can perceive immediately the position of a point with respect to the curve from the sign of $f(x, y)$.

We come now to the general case, and seek to give simple rules for the different cases.

1° If the equation $f(x, y) = 0$ represent an imaginary ellipse or a point-ellipse, or two imaginary parallel straight lines, or two coincident straight lines, the function $f(x, y)$ has the same sign for every point of the plane; because, in case of the three first hypotheses, it reduces to zero for but one point at most, and in the last (coincident straight lines), the function $f(x, y)$ is a perfect square.

2° If $f(x, y) = 0$ represent a real ellipse or a hyperbola, it is convenient to call the region of the plane which contains the center the interior of the curve; the remainder of the plane, the exterior. The signs of $f(x, y)$ are different for the exterior and the interior. Let a and b be the co-ordinates of the center, the function $f(x, y)$ takes for the center the value (§ 130)

$$f(a, b) = H = \frac{\Delta}{\mathrm{f}} = \frac{\Delta}{AC - B^2};$$

the sign of this quantity furnishes therefore the sign of $f(x, y)$ for the interior of the conic; the sign will be the opposite for the exterior.

3° If $f(x, y) = 0$ represent a parabola or two real parallel straight lines, the interior of the curve is the region which contains the focus of the parabola or the region comprised between the two straight lines. The sign of $f(x, y)$ may be obtained immediately for the exterior of the curve by taking the sign of $f(x, y)$ for a point at infinity in a direction *not parallel* to the direction of the axis or to that of the two lines. This particular direction is obtained by equating to zero all of the terms of the second degree $Ax^2 + 2Bxy + Cy^2$, which is a perfect square in the case considered. Since one of the two co-ordinate axes at least is not parallel to this particular direction, it will suffice to take the sign of $f(x, y)$ for infinity on this co-ordinate axis; that is, the sign of the coefficient A or C which is not zero.

4° If the curve $f(x, y) = 0$ be composed of two straight lines which intersect, the sign of $f(x, y)$ will be the same in the vertical angles; the signs of $f(x, y)$ are opposite in the adjacent angles.

CHAPTER III

REDUCTION OF THE EQUATION OF THE SECOND DEGREE.

138. In order to study with the most facility the properties of a curve of the second degree, it is important to simplify as much as possible its equation by referring it to co-ordinate axes suitably chosen. It has been seen, in the preceding chapter, that the equation of the second degree can always be reduced to one of the two forms

$$(\alpha)\ Ax^2 + Cy^2 + H = 0, \qquad (\beta)\ Cy^2 + 2\,Dx = 0.$$

In case the curve is an ellipse or a hyperbola, its equation is reduced to the form (α), on taking any two conjugate diameters for a system of co-ordinate axes; in general, the co-ordinates will be oblique; they will be rectangular, if the curve be referred to its axes. In case the curve is a parabola, its equation is reduced to the form (β), on taking any diameter as the axis of x, and a tangent at the extremity of this diameter as the axis of y; in case the co-ordinates are rectangular, one takes the axis of the curve as the x-axis.

It is by means of these equations (α) and (β), in rectangular co-ordinates, that one demonstrates in most part the properties of the curve of the second degree. One applies now the method used to accomplish the reduction of the equation. Let

$$(1) \qquad Ax^2 + 2\,Bxy + Cy^2 + 2\,Dx + 2\,Ey + F = 0$$

be the given equation of the second degree referred to rectangular axes; if they were not, one would first render them such by a transformation. On retaining the x-axis and taking for the y-axis the perpendicular erected to the x-axis at the origin, the formulas of transformation are

$$x = x' - y'\frac{\cos\theta}{\sin\theta}, \quad y = \frac{y'}{\sin\theta}.$$

Ellipse and Hyperbola.

139. Consider now the case where the quantity $AC - B^2$ is different from zero; the curve has a definite center, whose co-ordinates a and b are given by the formulas (§ 129)

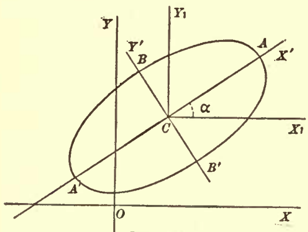

Fig. 82.

$$a = \frac{\mathrm{d}}{\mathrm{f}}, \quad b = \frac{\mathrm{e}}{\mathrm{f}}.$$

Keeping the axes parallel to themselves, transfer the origin to the center C (Fig. 82). One knows that the terms of the second degree do not change, that those of the first degree disappear, and that the constant term H of the new equation is given by the formula $\frac{\Delta}{\mathrm{f}}$. The equation of the curve, by this change of co-ordinates, simplifies and becomes

(2) $$Ax_1^2 + 2\,Bx_1y_1 + Cy_1^2 + H = 0.$$

Rotate now the co-ordinate axes, supposed rectangular, about the center C through the angle α, in order that they may coincide with the axes of the curves. The formulas of transformation are

$$x_1 = x' \cos \alpha - y' \sin \alpha, \quad y_1 = x' \sin \alpha + y' \cos \alpha.$$

Substituting in equation (2), one gets the new equation

(3) $$(A \cos^2 \alpha + C \sin^2 \alpha + 2\,B \sin \alpha \cos \alpha)\,x'^2$$
$$+ (A \sin^2 \alpha + C \cos^2 \alpha - 2\,B \sin \alpha \cos \alpha)\,y'^2$$
$$+ 2\,[(C - A) \sin \alpha \cos \alpha + B(\cos^2 \alpha - \sin^2 \alpha)]\,x'y' + H = 0.$$

The angle α may be so determined that the coefficient of the term $x'y'$ will be null; for this purpose, one will put

(4) $$(C - A) \sin \alpha \cos \alpha + B(\cos^2 \alpha - \sin^2 \alpha) = 0,$$

or

(5) $$B \tan^2 \alpha + (A - C) \tan \alpha - B = 0.$$

This equation of the second degree is the same as equation (21), § 137, by which the directions of the axes of the curve are determined. Equation (4) can be solved more simply by putting it under the form

$$(C-A)\sin 2\alpha + 2B\cos 2\alpha = 0,$$

whence

(6) $$\tan 2\alpha = \frac{2B}{A-C}.$$

If the case of the circle be excluded, where one has at the same time $B=0$ and $A=C$, equation (6) gives for 2α a positive value ω less than π, and the various values of 2α which satisfy this equation are represented by the formula

$$2\alpha = \omega + k\pi,$$

where k designates any integral number positive or negative; whence one deduces $\alpha = \frac{\omega}{2} + k\frac{\pi}{2}.$

The different values of α furnish no more than four different directions for the axis CX'; these four directions are two by two opposite, and determine two perpendicular straight lines. One gives to α the value $\frac{\omega}{2}$, which is always positive and less than $\frac{\pi}{2}$.

140. The term in $x'y'$ disappears from equation (3); it remains to calculate the value of the coefficients of the terms in x'^2 and y'^2. If one put

$$A' = A\cos^2\alpha + C\sin^2\alpha + 2B\sin\alpha\cos\alpha,$$

$$C' = A\sin^2\alpha + C\cos^2\alpha - 2B\sin\alpha\cos\alpha,$$

one will have

(7) $$\begin{cases} A' + C' = A + C, \\ A' - C' = (A-C)(\cos^2\alpha - \sin^2\alpha) + 4B\sin\alpha\cos\alpha, \\ \qquad\quad = (A-C)\cos 2\alpha + 2B\sin 2\alpha. \end{cases}$$

Equation (6) gives

$$\sin 2\alpha = \frac{2B}{\pm\sqrt{4B^2+(A-C)^2}}, \quad \cos 2\alpha = \frac{A-C}{\pm\sqrt{4B^2+(A-C)^2}};$$

it follows $$A' - C' = \pm\sqrt{4B^2+(A-C)^2}.$$

The two coefficients A' and C' can be calculated by means of the formulas

(8) $$\begin{cases} A' + C' = A + C, \\ A' - C' = R, \end{cases}$$

on putting $$R = \sqrt{4B^2 + (A-C)^2}.$$

The value of 2α was taken positive and less than π. Sin 2α having a positive value, it will be necessary to give the radical the sign which B has. In this manner the equation of the curve is reduced to the simple form

(9) $$A'x'^2 + C'y'^2 + H = 0.$$

It represents an ellipse or a hyperbola, according as the two coefficients A' and C' have the same or opposite signs.

The preceding formulas (8), squared and subtracted, furnish the relation

$$A'C' = AC - B^2.$$

The coefficients A' and C' of the equation of the curve referred to its axes are the roots of the equation

(10) $$S^2 - (A+C)S + (AC - B^2) = 0.$$

The dimensions of the curve defined by equation (9) depend on the two parameters $-\frac{H}{A'}$, $-\frac{H}{C'}$. In case of the ellipse, these two quotients, which have the same sign, are positive; if they be represented by a^2 and b^2, a and b will be the segments of CX' and CY' comprised between the center and the curve. The lengths $2a$ and $2b$ are called the axes of the ellipse. The quantities a^2 and b^2 are the roots of the equation of the second degree

(11) $$(AC - B^2)u^2 + (A+C)Hu + H^2 = 0,$$

which is obtained by substituting $-\frac{H}{u}$ for S in equation (10).

In case of the hyperbola, the two quotients $-\frac{H}{A'}$, $-\frac{H}{C'}$ have opposite signs. If they be represented by a^2 and $-b^2$, or $-a^2$ and b^2, according to the two cases which can occur, these two quantities are still the roots of equation (11). The quantities $2a$ and $2b$ are called the *axes* of the hyperbola.

PARABOLA.

141. When $AC - B^2 = 0$, the terms of the second degree in the given equation form a perfect square. One has, in fact, on replacing A by its value $\frac{B^2}{C}$,

$$Ax^2 + 2Bxy + Cy^2 = C\left(y^2 + \frac{2B}{C}xy + \frac{B^2}{C^2}x^2\right) = C\left(y + \frac{B}{C}x\right)^2,$$

and the equation can be written

$$C\left(y + \frac{B}{C}x\right)^2 + 2Dx + 2Ey + F = 0.$$

Rotate the axes of co-ordinates about the origin through an angle α (Fig. 83) by means of the formulas of transformation,

$$x = x_1 \cos\alpha - y_1 \sin\alpha, \quad y = x_1 \sin\alpha + y_1 \cos\alpha;$$

the proposed equation becomes

$$(12) \quad C\left[\left(\cos\alpha - \frac{B}{C}\sin\alpha\right)y_1 + \left(\sin\alpha + \frac{B}{C}\cos\alpha\right)x_1\right]^2$$
$$+ 2(D\cos\alpha + E\sin\alpha)x_1 + 2(E\cos\alpha - D\sin\alpha)y_1 + F = 0.$$

One can determine the value of α so that the coefficient of x_1 or y_1 is zero in that part of the polynomial which is squared. Put, for example,

$$\sin\alpha + \frac{B}{C}\cos\alpha = 0;$$

whence

$$(13) \qquad \tan\alpha = -\frac{B}{C},$$

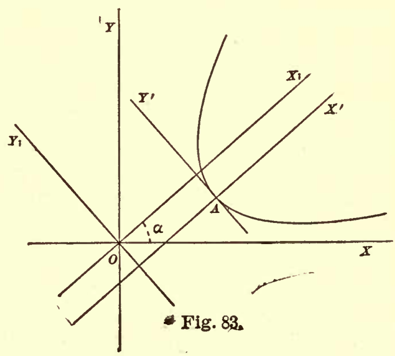

Fig. 83.

$$\sin \alpha = \frac{-B}{\pm\sqrt{B^2+C^2}}, \quad \cos \alpha = \frac{C}{\pm\sqrt{B^2+C^2}},$$

then equation (12) will simplify and become

(14) $$C'y_1^2 + 2D'x_1 + 2E'y_1 + F = 0,$$

where

$$C' = C\left(\cos \alpha - \frac{B}{C}\sin \alpha\right)^2 = C(\cos \alpha + \tan \alpha \sin \alpha)^2 = \frac{C}{\cos^2 \alpha},$$

and, consequently,

$$C' = \frac{B^2 + C^2}{C} = A + C.$$

The coefficients D' and E' are obtained by replacing $\sin \alpha$ and $\cos \alpha$ by their values, which give

$$D' = \frac{CD - BE}{\pm\sqrt{C(A+C)}}, \quad E' = \frac{CE + BD}{\pm\sqrt{C(A+C)}}.$$

One of the values of α given by equation (13) is positive and less than π. If one take this value, $\sin \alpha$ will be positive, and it will be necessary to give the radical a sign opposite to that of B. If the coefficient D' were zero, equation (14) would no longer contain x_1, and would represent two straight lines parallel to the axis OD_1. In case this coefficient is different from zero, one transfers the axes parallel to themselves by putting

$$x_1 = a + x', \quad y_1 = b + y';$$

equation (14) becomes

$$C'y'^2 + 2D'x' + 2(C'b + E')y' + (C'b^2 + 2D'a + 2E'b + F) = 0.$$

The co-ordinates of the new origin A are so determined that the coefficient of y' and the constant term are zero,

$$C'b + E' = 0, \quad C'b^2 + 2D'a + 2E'b + F = 0,$$

which give finite values for a and b, and the equation will reduce to the simple form

(15) $$C'y'^2 + 2D'x' = 0.$$

The dimensions of the curve depend on the numerical value of the quotient $\frac{D'}{C'}$, or of that of

$$\frac{BE - CD}{(A+C)\sqrt{C(A+C)}}.$$

This quantity is called the parameter of the parabola.

142. The coefficients of the reduced equations can be easily calculated by employing certain functions of the coefficients and of the angle between the axes, which do not change in value when any change whatever of the axes is made. To form these functions, take formulas (4) of § 51:

$$(16) \quad \begin{cases} x = \dfrac{x' \sin(\theta - \alpha) + y' \sin(\theta - \beta)}{\sin\theta}, \\ y = \dfrac{x' \sin\alpha + y' \sin\beta}{\sin\theta}, \end{cases}$$

which serve for the transformation of co-ordinates, when the direction of the axes is changed and the origin remains fixed; these formulas express x and y as homogeneous functions of the first degree in x' and y'. If these values be substituted for x and y in the homogeneous polynomial of the second degree

$$Ax^2 + 2Bxy + Cy^2,$$

the result will be a homogeneous polynomial of the second degree in x' and y':

$$A'x'^2 + 2B'x'y' + C'y'^2.$$

In particular, the trinomial

$$x^2 + 2xy\cos\theta + y^2$$

is transformed into

$$x'^2 + 2x'y'\cos\theta' + y'^2,$$

θ' being the angle between the new axes; because each of these trinomials gives the square of the distance of the origin from the same point of the plane.

Consider the polynomial

$$Ax^2 + 2Bxy + Cy^2 - S(x^2 + 2xy\cos\theta + y^2),$$

or (17) $\quad (A - S)x^2 + 2(B - S\cos\theta)xy + (C - S)y^2,$

in which the letter S designates an arbitrary constant; it will evidently furnish the transformed polynomial

$$A'x'^2 + 2B'x'y' + C'y'^2 - S(x'^2 + 2x'y'\cos\theta' + y'^2),$$

or (18) $$(A' - S)x'^2 + 2(B' - S\cos\theta')x'y' + (C' - S)y'^2.$$

One notices now that the values of S, for which one of the polynomials is the square of an integral function of the first degree in the variables which it involves, are the same. Assume, for example, that the first polynomial takes for a certain value of S the form $(ax + by)^2$, a and b being constants; when x and y are replaced by their values (16), the function of the first degree, $ax + by$, changes into a function of the first degree $a'x' + b'y'$, and the second polynomial takes the form $(a'x' + b'y')^2$. When the polynomials are squares, the equations which are found by equating their roots to zero represent the same straight line referred to the two systems of axes YOX, $Y'OX'$.

The values of S, for which the polynomial (17) is a square, are the roots of the equation of the second degree

$$(A - S)(C - S) - (B - S\cos\theta)^2 = 0,$$

or (19) $$S^2\sin^2\theta - (A + C - 2B\cos\theta)S + AC - B^2 = 0.$$

The roots of this equation are represented by S_1 and S_2; similarly, the values of S, for which the polynomial (18) is a square, are the roots of the equation

$$(A' - S)(C' - S) - (B' - S\cos\theta')^2 = 0,$$

or (20) $$S^2\sin^2\theta' - (A' + C' - 2B'\cos\theta')S + A'C' - B'^2 = 0.$$

The two equations (19) and (20) have the same roots, whence it follows

(21) $$\begin{cases} \dfrac{A + C - 2B\cos\theta}{\sin^2\theta} = \dfrac{A' + C' - 2B'\cos\theta'}{\sin^2\theta'}, \\[2ex] \dfrac{AC - B^2}{\sin^2\theta} = \dfrac{A'C' - B'^2}{\sin^2\theta'}. \end{cases}$$

Therefore, the two functions

$$\frac{A + C - 2B\cos\theta}{\sin^2\theta}, \quad \frac{AC - B^2}{\sin^2\theta},$$

of the coefficients of the equation of a conic and the angle between the axes preserve the same values when a transformation of co-ordinates is made.

The quantity $\frac{\Delta}{\sin^2\theta}$ possesses the same property. In fact, suppose $\mathbf{f}$ different from zero; then, by transforming the origin of co-ordinates to the center of the conic, one has an equation whose constant term H has the value (§ 130) $H = \frac{\Delta}{\mathbf{f}}$.

Since this constant term H remains the same whatever be the orientation and the angle between the axes and as $\frac{\mathbf{f}}{\sin^2\theta}$ remains constant when a change of axes is made, it follows that the same is true of $\frac{\Delta}{\sin^2\theta}$. Thus, the conic being referred to the axes xOy which include an angle θ, if it be referred to new axes $x'O'y'$ including an angle θ', and if the new equation be called

$$A'x'^2 + 2B'x'y' + C'y'^2 + 2D'x' + 2E'y' + F' = 0,$$

and the new value of Δ, $\Delta' = A'(C'F' - E'^2) + \cdots$, one has

$$\frac{\Delta'}{\sin^2\theta'} = \frac{\Delta}{\sin^2\theta}. \tag{22}$$

This relation will be satisfied if A', B', C', D', E', F', be replaced by their values as functions of A, B, C, D, E, F, which result from the formulas of transformation of co-ordinates. It still holds whatever A, B, C, D, E, F may be; that is, will be the same in the case where $\mathbf{f}$ is zero, although the argument used to establish this relation can no longer be applied to this case.

The three quantities

$$\frac{A + C - 2B\cos\theta}{\sin^2\theta}, \quad \frac{\mathbf{f}}{\sin^2\theta}, \quad \frac{\Delta}{\sin^2\theta} \tag{23}$$

are homogeneous with respect to the coefficients A, B, C, D, E, F; the first is of the first, the second of the second, the third of the third degree. If, therefore, the first member of the conic

$$Ax^2 + 2\,Bxy + Cy^2 + 2\,Dx + 2\,Ey + F = 0$$

be multiplied by a constant K, that is, if A, B, C, D, E, F be replaced by KA, KB, KC, $\cdots$, the first of the three quantities (23) will be multiplied by K, the second by K^2, the third by K^3; whence it follows that a combination of quantities (23), homogeneous and of the zero degree with respect to A, B, C, $\cdots$, does not change if the factor K be introduced. Such, for example, are the two combinations

$$(24) \qquad \frac{\mathrm{f}\sin^2\theta}{(A + C - 2\,B\cos\theta)^2}, \quad \frac{\Delta\sin^4\theta}{(A + C - 2\,B\cos\theta)^3},$$

found by dividing the second and third of quantities (23) by the square and cube of the first. One has then the two expressions (24), which do not change when the axes are changed and all of the coefficients are multiplied or divided by the same factor.

The condition $\Delta = 0$ expresses that the conic is reduced to two straight lines; the condition $\mathrm{f} = 0$, that it belongs to the genus parabola; the condition $A + C - 2\,B\cos\theta = 0$, that it is an *equilateral* hyperbola, that is, a hyperbola whose asymptotes are perpendicular. In fact, call m' and m'' the angular coefficients of the asymptotes; the condition of perpendicularity is

$$(25) \qquad 1 + (m' + m'')\cos\theta + m'm'' = 0;$$

further, these angular coefficients are roots of the equation

$$Cm^2 + 2\,Bm + A = 0,$$

which gives $\qquad m' + m'' = -\dfrac{2\,B}{C}, \quad m'm'' = \dfrac{A}{C},$

values which, substituted in relation (25), give the required condition.

143. The magnitude of an ellipse or of a hyperbola depends on two numbers which are the lengths of the axes of the curves; the magnitude of a parabola depends on a single number, the *parameter;* finally, the magnitude of a conic reduced to two straight lines depends on a number which is their angle of intersection if they intersect, and the perpendicular distance between them if they be parallel. It is next proposed to calculate these different quantities.

Let $$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

be the equation of a curve of the second degree referred to axes inclosing an angle θ. If one put

$$\epsilon = A + C - 2B\cos\theta, \quad \mathrm{f} = AC - B^2, \quad \Delta = A(CF - E^2) + \cdots,$$

the three quantities

(1) $$\frac{\epsilon}{\sin^2\theta}, \quad \frac{\mathrm{f}}{\sin^2\theta}, \quad \frac{\Delta}{\sin^2\theta}$$

possess this property that any combination of these three quantities, homogeneous and of the degree zero with respect to the coefficients A, B, C, $\cdots$, has a constant value when the co-ordinate axes are changed, and the coefficients of the equation of the curve are multiplied or divided by the same factor, as has already been demonstrated.

Assume, now, that the curve be an ellipse or a hyperbola; on referring the curve to its center and its axes, its equation may be written

$$\alpha x^2 + \beta y^2 - \alpha\beta = 0,$$

where, in the case of a real ellipse, $\alpha = b^2$, $\beta = a^2$; in that of a hyperbola, $\alpha = b^2$, $\beta = -a^2$; in that of an imaginary ellipse, $\alpha = -b^2$, $\beta = -a^2$. It is said in these three cases that α and β are the squares of the lengths of the axes. The two combinations

(2) $$\frac{\epsilon\Delta}{\mathrm{f}^2}, \quad \frac{\Delta^2\sin^2\theta}{\mathrm{f}^3}$$

of quantities (1) being homogeneous and of the degree zero with respect to the coefficients, will have the same constant value if they be constructed for the reduced equation. In case of the reduced equation will

$$\theta = \frac{\pi}{2}, \quad \epsilon = \alpha + \beta, \quad \mathrm{f} = \alpha\beta, \quad \Delta = -\alpha^2\beta^2;$$

therefore $$\frac{\epsilon\Delta}{\mathrm{f}^2} = -(\alpha + \beta), \quad \frac{\Delta^2\sin^2\theta}{\mathrm{f}^3} = \alpha\beta,$$

and α and β are roots of the equation of the second degree,

(3) $$\rho^2 + \rho\frac{\epsilon\Delta}{\mathbf{f}^2} + \frac{\Delta^2 \sin^2\theta}{\mathbf{f}^3} = 0,$$

which is called the *equation of the squares of the axes.* From the nature of the problem, this equation should have real roots; this is easy to verify, because the quantity $\frac{\epsilon^2\Delta^2}{\mathbf{f}^4} - \frac{4\,\Delta^2 \sin^2\theta}{\mathbf{f}^3}$ may then be written

$$\frac{\Delta^2}{\mathbf{f}^4}(\epsilon^2 - 4\,\mathbf{f}\sin^2\theta) = \frac{\Delta^2}{\mathbf{f}^4}\{(A - C)^2 \sin^2\theta + [(A + C)\cos\theta - 2B]^2\},$$

which is necessarily a positive quantity. The roots are equal when

$$A = C, \quad B = A\cos\theta;$$

the curve is then a circle. The roots will be equal and of contrary signs when $\epsilon = 0$; the curve is then an equilateral hyperbola.

Suppose now that the curve of the second degree be a parabola; by referring it to its axis, and to a tangent at its vertex, its equation will take the form

$$y^2 - 2px = 0.$$

What is the value of the parameter p? The second of the three quantities (1) is zero. A homogeneous combination of the degree zero with respect to the coefficients can be formed from the other expressions in (1) by dividing the last by the cube of the first, which gives

$$\frac{\Delta \sin^4\theta}{\epsilon^3}.$$

This last quantity constructed for the reduced equation is $-p^2$; the equation which gives p is therefore

$$\frac{\Delta \sin^4\theta}{\epsilon^3} = -p^2.$$

144. In case the conic is reduced to a system of two straight lines which intersect, its equation can be written in the form

$$y^2 + mx^2 = 0;$$

then $\Delta = 0$, and one has, on forming the expression $\frac{\epsilon^2}{\mathrm{f}\sin^2\theta}$ for the reduced equation,

$$\frac{\epsilon^2}{\mathrm{f}\sin^2\theta} = \frac{(1+m)^2}{m};$$

whence may be deduced two values of m which are reciprocals. If f be negative, the two straight lines are real and the values of m are negative: call ϕ the angle formed by the two straight lines, then will

$$m = -\tan^2\frac{\phi}{2} \quad \text{and} \quad -\frac{\epsilon^2}{\mathrm{f}\sin^2\theta} = 4\cot^2\phi,$$

an equation which determines ϕ.

Finally, suppose that the curve is reduced to two parallel straight lines, and calculate the distance between them. In this case, two of the three quantities (1), f and Δ, are zero; the preceding combinations of the three quantities can no longer be employed. This case will be considered as a limiting case of the case when the curves have a unique center, and this in the following fashion. Let

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

be a conic reduced to two parallel straight lines; since A and C cannot be zero at the same time, on account of the condition $\mathrm{f} = 0$, suppose that C differs from zero, and consider the auxiliary curve

$$(A+\lambda)x^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

which involves the parameter λ. In this curve will

$$\epsilon_1 = A + C + \lambda - 2B\cos\theta, \quad \mathrm{f}_1 = (A+\lambda)C - B^2,$$

$$\Delta_1 = (A+\lambda)(CF - E^2) + \cdots.$$

The expressions Δ_1 and f_1 reduce for $\lambda = 0$ to Δ and f, that is, to zero. When λ is different from zero, the auxiliary curve has a definite center; its equation reduces to the form

$$\alpha_1 x^2 + \beta_1 y^2 - \alpha_1\beta_1 = 0,$$

where α_1 and β_1 are roots of equation (3), which is written

$$\mathrm{f}_1\rho^2 + \frac{\epsilon_1\Delta_1}{\mathrm{f}_1}\rho + \left(\frac{\Delta_1}{\mathrm{f}_1}\right)^2\sin^2\theta = 0.$$

When λ approaches zero, f_1 and Δ_1 approach zero, and their ratio $\frac{\Delta_1}{f_1}$ approaches the limit $\frac{CF-E^2}{C}$. Therefore one of the roots β_1 of the preceding equation increases indefinitely, and the other approaches the limit

$$\alpha = -\frac{CF-E^2}{\epsilon C}\sin^2\theta.$$

Further, the reduced equation may be written under the form

$$\frac{\alpha_1}{\beta_1}x^2 + y^2 - \alpha_1 = 0,$$

which becomes $y^2 - \alpha = 0$, and the value found for α is the *square of half of the distance* between the two parallel straight lines to which the auxiliary conic is reduced for $\lambda = 0$.

Examples.

I. $$2x^2 - 3xy + 3y^2 + x - 7y + 1 = 0.$$

The curve is an ellipse, since the quantity $AC - B^2$ is positive. In order to obtain the co-ordinates of the center, equate to zero the two partial derivatives

$$4x - 3y + 1 = 0, \quad -3x + 6y - 7 = 0,$$

whence $$x = 1, \; y = \tfrac{5}{3}, \; H = -\tfrac{13}{3}.$$

If, keeping the axes parallel to themselves, the origin be transferred to the center C (Fig. 82), the equation becomes

$$2x_1^2 - 3x_1y_1 + 3y_1^2 - \tfrac{13}{3} = 0.$$

Rotate now the axes through the angle α given by the formula

$$\tan 2\alpha = \frac{2B}{A-C} = 3.$$

The equation solved by the tables gives

$$2\alpha = 71^\circ\, 33'\, 54'', \text{ or } \alpha = 35^\circ\, 46'\, 57''.$$

The angle α can also be found by a graphical construction; lay off on the axes of x and y, beginning at the origin C, two lengths respectively equal to 1 and 3; the diagonal of the rectangle constructed on these two lengths makes with the axis of x an angle whose tangent is 3; the axis CX' is therefore the bisector of this angle. Whence A' and C' are obtained by the formulas

$$A'+C'=5,\ A'-C'=-\sqrt{10},$$

since B is negative. One has then

$$A'=\frac{5-\sqrt{10}}{2},\ C'=\frac{5+\sqrt{10}}{2},$$

and the equation of the curve becomes

$$(5-\sqrt{10})x'^2+(5+\sqrt{10})y'^2=\tfrac{26}{3};$$

the intercepts of the curve on the axes are

$$CA=\sqrt{\frac{26}{3(5-\sqrt{10})}},\ CB=\sqrt{\frac{26}{3(5+\sqrt{10})}}.$$

II. $$2x^2-5xy+5y-1=0.$$

The curve is a hyperbola (Fig. 84). The co-ordinates of the center which are given by the equations

$$4x-5y=0,\ -5x+5=0$$

are $x=1,\ y=\frac{4}{5}$,

whence $H=1$.

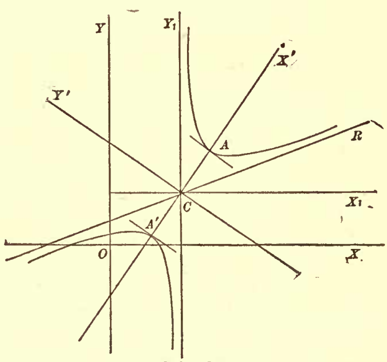

Fig. 84.

By transferring the origin to the center, the equation becomes

$$2x_1^2-5x_1y_1+1=0.$$

The angle α is given by the formula $\tan 2\alpha=-\frac{5}{2}$, and one has $A'+C'=2,\ A'-C'=-\sqrt{29}$;

whence $$A'=\frac{2-\sqrt{29}}{2},\ C'=\frac{2+\sqrt{29}}{2}.$$

The equation of the curve referred to its axes is

$$(2-\sqrt{29})x'^2+(2+\sqrt{29})y'^2+2=0.$$

The primitive equation does not contain a term in y^2. One of the asymptotes is parallel to the axis OY.

III. $$4x^2 - 12xy + 9y^2 - 36x + 100 = 0.$$

The curve is a parabola (Fig. 83). The terms of the second degree form a perfect square, and the equation can be written

$$9(y - \tfrac{2}{3}x)^2 - 36x + 100 = 0.$$

Rotate the axes through an angle α given by the formula $\tan\alpha = -\frac{B}{C} = \frac{6}{9} = \frac{2}{3}$, whence $\alpha = 33° \; 41' \; 25''$, one will have $C' = 13$,

$$\cos\alpha = \frac{3}{\sqrt{13}}, \qquad \sin\alpha = \frac{2}{\sqrt{13}},$$

$$D' = -\frac{54}{\sqrt{13}}, \qquad E' = \frac{36}{\sqrt{13}}.$$

The equation of the curve referred to the axes OX_1 and OY_1 is therefore

$$13y_1^2 - \frac{108}{\sqrt{13}}x_1 + \frac{72}{\sqrt{13}}y_1 + 100 = 0.$$

The co-ordinates of the vertex are found by combining with this equation the following

$$26y_1 + \frac{72}{\sqrt{13}} = 0;$$

whence one finds

$$y_1 = -\frac{36}{13\sqrt{13}}, \quad x_1 = \frac{3901}{27.13\sqrt{13}}.$$

If the origin be transferred to this point, the equation becomes

$$13y'^2 - \frac{108}{\sqrt{13}}x' = 0.$$

CHAPTER IV

THE ELLIPSE.

145. It is proposed to construct the curve represented by the equation

$$A'x^2 + C'y^2 + H = 0,$$

in which the coefficients A' and C' have the same sign.

When the constant H is zero, the equation, being satisfied by $x = 0$, $y = 0$, represents a single point, the origin of coordinates.

If the coefficients A' and C' have the same sign as H, the equation cannot be satisfied by real values of x and y, and does not represent a geometrical locus.

Consider finally the case where these two coefficients have signs contrary to that of H, and put

$$a^2 = -\frac{H}{A'}, \quad b^2 = -\frac{H}{C'};$$

the equation becomes

$$(1) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

On solving it with respect to y, one gets

$$(2) \qquad y = \pm\frac{b}{a}\sqrt{a^2 - x^2}.$$

The ordinate y is real so long as the values of x are comprised between $-a$ and $+a$, and the same is true of x so long as the values of y are comprised between $-b$ and $+b$; if, therefore, starting from the origin, one lay off on the x-axis to the right and left two lengths OA, OA'

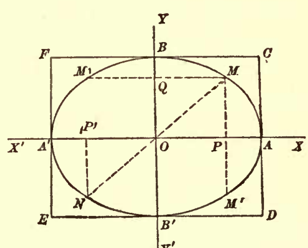

Fig. 85.

equal to a, and on the y-axis two lengths OB, OB' equal to b, the curve is situated wholly within the rectangle $CDEF$ constructed on the two straight lines AA', BB' (Fig. 85).

As x increases from 0 to a, y decreases in absolute value from b to 0, which, on account of the double sign, furnishes the two equal arcs BMA, $B'M'A$. The same is true when x varies from 0 to $-a$, which gives the two equal arcs BM_1A', $B'NA'$, equal to the preceding. These four equal arcs form the ellipse.

146. The straight line $A'A$ is the axis of the ellipse, because to each abscissa OP correspond two ordinates PM, PM', equal and of contrary signs. The straight line BB' is also an axis of the ellipse; because, if the equation be solved with respect to x, one can verify in a similar manner that to each ordinate OQ correspond two abscissas QM, QM_1 equal and of contrary signs. The points A, A', B, B', where the axes intersect the ellipse, are the *vertices* of the ellipse. The lengths $A'A$, $B'B$ of the two axes are respectively equal to $2a$ and $2b$.

The ellipse becomes a circle when the axes are equal.

It is easy to see that the origin O is the center of the ellipse; in fact, let x, y be the co-ordinates of any point M of the ellipse; it is evident that equation (1) is also satisfied by the values $-x$, $-y$; there is, consequently, a second point N of the ellipse which has the co-ordinates $-OP'$, $-P'N$ respectively equal to the co-ordinates OP, PM of the point M, but measured in opposite directions; the triangles OPM, $OP'N$ are equal; therefore $OM = ON$, and the line MON is straight because the angles POM, $P'ON$ are equal. Thus the points M and N of the ellipse are two by two symmetrical with respect to the point O; therefore the point O is the center of the ellipse.

147. In order to study how the distance from the center to different points of the ellipse varies, or the radius vector of the ellipse, find the equation of the ellipse in polar co-ordinates, when the center O is taken as the pole and the axis OA of the curve as the polar axis. If in equation (1) x and y be replaced by $\rho \cos \omega$ and $\rho \sin \omega$, one has

$$\frac{1}{\rho^2} = \frac{\cos^2 \omega}{a^2} + \frac{\sin^2 \omega}{b^2}. \tag{3}$$

Suppose $a > b$ and write the equation in the form

$$\frac{1}{\rho^2} = \frac{1}{a^2} + \left(\frac{1}{b^2} - \frac{1}{a^2}\right) \sin^2 \omega.$$

If ω vary from 0 to $\frac{\pi}{2}$, the quantity $\frac{1}{\rho^2}$ increases, and, consequently, ρ decreases continually from a to b. The maximum value of ρ is a, the minimum is b.

148. Represent by x and y the co-ordinates of any point whatever of the plane and consider the polynomial

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1.$$

The polynomial is equal to zero for a point situated on the ellipse (Fig. 86). Imagine that a movable point P starts from the point M and moves along the prolongation of the radius vector OM: the two co-ordinates x and y increasing in absolute value, the polynomial must increase indefinitely; it takes, therefore, greater and greater positive values. On the contrary, if the movable point travels toward the center, the polynomial diminishes and takes negative values. Thus, the polynomial

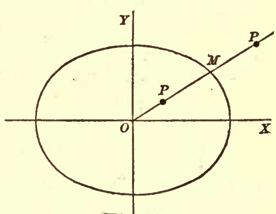

Fig. 86.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

is negative for every point situated within the ellipse, zero for points on the ellipse, and positive for every point situated without the ellipse.

149. *The squares of the ordinates perpendicular to an axis of the ellipse are proportional to the products of the corresponding segments formed on this axis.*

In fact, if x and y designate the co-ordinates of any point M on the ellipse (Fig. 85), one has, on account of equation (2)

$$\frac{y^2}{a^2 - x^2} = \frac{b^2}{a^2}, \text{ or } \frac{y^2}{(a-x)(a+x)} = \frac{b^2}{a^2}.$$

But the two segments AP, $A'P$ of the axis AA' are equal respectively to $a - x$ and $a + x$; one has, therefore,

$$\frac{MP^2}{AP \times A'P} = \frac{b^2}{a^2}.$$

Hence the square of the ordinate is to the product of the segments formed on the axis in a constant ratio.

150. *The ordinates perpendicular to the major axis of an ellipse are to the corresponding ordinates of the circle constructed on this axis as a diameter in the constant ratio of the minor to the major axis.*

Let AA' be the major axis of the ellipse (Fig. 87); on this major axis as a diameter construct a circle; to the ordinate MP of the ellipse corresponds the ordinate M_1P of the circle. Equation (2) may be written

$$\frac{y}{\sqrt{a^2 - x^2}} = \frac{b}{a};$$

but $\sqrt{a^2 - x^2}$ represents the ordinate M_1P of the circle; one has, therefore,

$$\frac{MP}{M_1P} = \frac{b}{a}.$$

The minor axis enjoys the same property; the ordinate MQ, perpendicular to the minor axis, is to the corresponding ordinate M_2Q of the circle constructed on this axis as a diameter in the constant ratio of the major to the minor axis.

The ellipse is the orthogonal projection of a circle. Imagine that the circle AB_1A' be revolved about the axis AA' through an angle ϕ, such that $\cos\phi = \frac{b}{a}$, the ordinate PM_1 of the circle will revolve about the point P, always remaining perpendicular to the axis AA'; in this position MP will be the projection of M_1P. In order to get the length of the projection, it suffices to multiply the length PM_1 by $\cos\phi$, or by $\frac{b}{a}$, which gives the ordinate PM of the ellipse. Thus the projection of the point M_1 of the circle is the point M of the ellipse. Each point of the circle projecting thus into the corresponding point of the ellipse, it follows that the ellipse is the projection of the circle.

One can also consider the circle as the orthogonal projection of an ellipse. Imagine the ellipse to be revolved about the axis BB' through an angle ϕ whose cosine is $\frac{b}{a}$, the ordinate QM of the ellipse will have for its projection the ordinate QM_2 of the circle described on BB' as a diameter, and the small circle will be the projection of the ellipse.

151. *The construction of the ellipse by points.* From what precedes may be deduced a very simple method for constructing the ellipse by points. Construct on each of the axes of the ellipse, as diameters, a circle (Fig. 87); draw from the center an arbitrary secant intersecting the two circles in M_1, M_2; draw through the point M_1 a line parallel to the minor axis; through the point M_2 a line parallel to the major axis. The point of intersection M of these two lines belongs to the ellipse. After having determined in this manner a sufficient number of points, one connects them by a continuous line, and the ellipse is thus constructed.

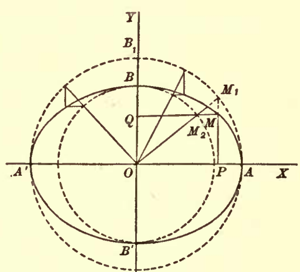

Fig. 87.

152. *Construct the points of intersection of an ellipse and a straight line.* It is useful to be able to construct the points in which a given straight line MM' intersects an ellipse defined by its two axes AA', BB' (Fig. 88) without tracing the ellipse. Thus, as has been seen, the ellipse can be considered as the orthogonal projection of the circle AB_1A', described on the major axis AA' as a diameter, the circle being revolved about AA' through an angle ϕ whose cosine is $\frac{b}{a}$. Find in the plane of the circle the straight line M_1M_1', whose projection in the plane of the ellipse is MM'; let N be any point of the line MM'; prolong the straight line BN till it intersects the axis AA' in H; the line B_1H is projected upon BH; consequently the point N_1, where the line B_1H intersects the ordinate QN, is projected into N. Similarly, any other point of the line M_1M_1' could be found; but it is more simple to begin with the point S, where the line MM' intersects the axis; the line SN_1 has the given line in the plane of the ellipse as its projection. This line SN_1 cuts the circle in two points M_1, M_1'; the ordinates M_1P, $M_1'P'$ will determine in the given line the two points M, M' where this line intersects the ellipse.

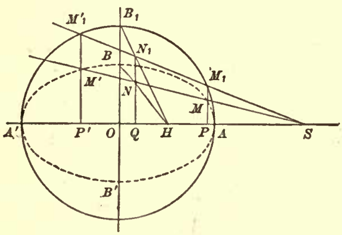

Fig. 88.

TANGENTS.

153. The equation of the tangent to a curve of the second degree has already been found (§ 125); when the equation of the ellipse is put under the simple form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

the equation of the tangent at the point M, whose co-ordinates are x and y, becomes

$$(4) \qquad \frac{xX}{a^2}+\frac{yY}{b^2}-1=0.$$

The angular coefficient of the tangent has the value $-\frac{b^2x}{a^2y}$.

One sees that at the vertices A and A' the tangent is perpendicular to the axis $A'A$, that at B and B' it is parallel, and that, as the point of contact moves along the ellipse from A to B, the tangent makes with the axis $A'A$ an obtuse angle, which increases from $\frac{\pi}{2}$ to π.

The normal, being perpendicular to the tangent, has the equation

$$Y - y = \frac{a^2 y}{b^2 x}(X - x).$$

154. *The construction of the tangent at a point of the ellipse.* If in the equation of the tangent one put $Y = 0$, one obtains the abscissa $X = \frac{a^2}{x}$ of the point T where the tangent intersects the prolongation of the major axis (Fig. 89). Since this value of OT is independent of the minor axis $2b$ and of the ordinate y of the point of contact, it follows, that if several ellipses be constructed on the axis AA', the tangents at the points which have the same abscissas pass through the same point T situated on the prolongation of the axis $A'A$. Among these ellipses is the circle AB_1A'; to construct the tangent to the ellipse at the point M, draw a tangent to the circle at the point M_1 situated on the same ordinate; join the point M with the point T, where the tangent to the circle intersects the prolongation of the axis $A'A$; the straight line MT, thus constructed, is the tangent to the ellipse.

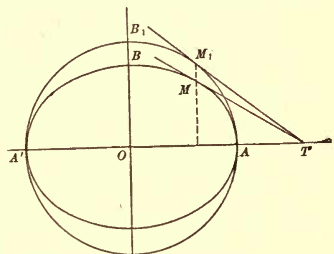

Fig. 89.

This construction is equivalent to regarding the tangent to the ellipse at the point M as the projection of the tangent to the circle at the corresponding point M_1. In fact, when the plane of the circle is made to revolve about the axis AA' through an angle ϕ, the point T, where the tangent M_1T meets the axis, remains fixed; the point M_1 projects into M, the line M_1T has for its projection MT; it is the tangent to the ellipse.

155. *To draw a tangent through an exterior point P.* Let x and y be the co-ordinates of the point P (Fig. 90). The equation of the chord of contact $M'M$ has been found (§ 126). The determination of the points of contact depends therefore on the solution of the two simultaneous equations

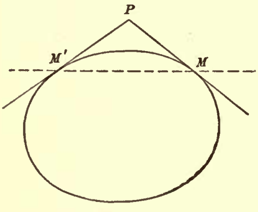

Fig. 90.

$$(1)\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad (5)\quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

By eliminating y, one gets the equation of the second degree

$$\frac{x^2}{a^2}\left(\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}\right)-2\frac{xx_1}{a^2}+1-\frac{y_1^{\,2}}{b^2}=0,$$

of which the roots are the abscissas of the points of contact M and M' of the two tangents drawn from the point P. This equation, in which $\frac{x}{a}$ can be regarded as the unknown, will have real roots if the condition $\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}-1>0$ be satisfied; that is, if the point P be without the ellipse.

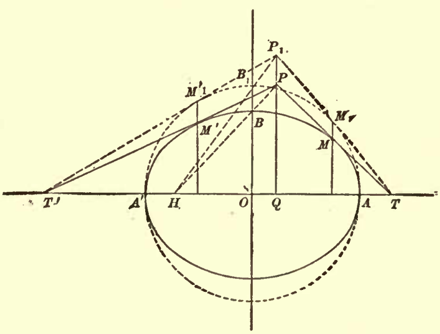

Fig. 91.

It is easy to construct geometrically the tangents drawn from the point P, by regarding the ellipse as the projection of the circle AB_1A' (Fig. 91). Seek in the plane of the circle the point P_1, whose projection in the plane of the ellipse is the point P. Draw in the plane of the ellipse the straight line PB, which is prolonged till it intersects the axis in H; the straight line HB_1, having HB for its projection, will pass through the point P_1 and determine this point. Draw from the point P_1 to the circle the tangents P_1M_1, P_1M_1', which one prolongs till they intersect the axis in T and T'; the straight lines PT, PT', projections of the tangents to the circle, will be tangents to the ellipse, and the points of contact M and M' will be situated on the ordinates of the points M_1, M_1'. In order that these constructions be accomplished, it is not necessary that the ellipse be drawn.

156. *To draw a tangent parallel to a given straight line.* Let $y = mx$ be the equation of the given straight line OL, which may be supposed to be drawn through the center (Fig. 92). Call x and y the unknown co-ordinates of the point of contact M; this point being on the ellipse, one has the equation

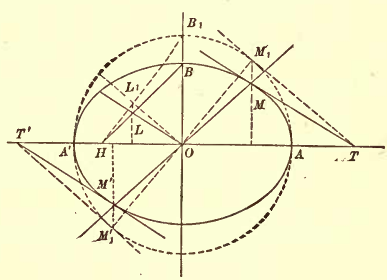

Fig. 92.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

the angular coefficient of the tangent being equal to m, one has a second equation

$$-\frac{b^2x}{a^2y} = m.$$

These two simultaneous equations determine the two unknown quantities x and y; the first represents the given ellipse; the second a straight line passing through the center; the points where this straight line meets the ellipse are the points of contact.

It is easy to construct these tangents geometrically. Determine first in the plane of the circle the diameter OL_1, whose projection in the plane of the ellipse is OL; it is sufficient to join the point B with any point L of the line OL, and prolong the line BL till it intersects the axis in H; then draw B_1H

and locate the point of intersection of this line with the ordinate of the point L; the point L being the projection of the point L_1, the line OL is the projection of OL_1. One draws to the circle the tangents M_1T, M'_1T', parallel to OL_1, and through the points T and T', where three tangents intersect the axis, the lines TM, $T'M'$ parallel to the line OL. One has the tangents required; because the projections OL, TM of the parallel straight lines OL_1, TM_1 are also parallel. The points of contact M and M' are determined by the ordinates of the points M_1 and M'_1.

157. The equation of a tangent to the ellipse may be found in other forms which it will be useful to know.

If one designate by α the angle which the perpendicular let fall from the center to the tangent makes with the axis of x and by p the length of this perpendicular, the tangent will be represented by the equation (§ 83)

$$X\cos\alpha + Y\sin\alpha - p = 0;$$

or, comparing it with equation (4), one has the relations

$$\frac{\frac{x}{a}}{a\cos\alpha} = \frac{\frac{y}{b}}{b\sin\alpha} = \frac{1}{p} = \frac{1}{\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}};$$

whence $$p = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}.$$

Then the tangent will have the equation

(6) $$X\cos\alpha + Y\sin\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}.$$

The equation of the tangent may also be found by seeking the points of intersection of an ellipse and of a straight line, and then expressing the condition that these two points should coincide, as has been done in case of the circle (§ 94). One obtains in this way the equation of the tangent in the form

(7) $$y = mx \pm \sqrt{a^2m^2 + b^2}.$$

158. As an application, it is proposed to find the locus of the vertex of a right angle which circumscribes the ellipse. Suppose tnat one draw through an exterior point P (Fig. 93), whose co-ordinates are x' and y', tangents to the ellipse; on account of the tangent passing through the point P, one will have the equation of condition

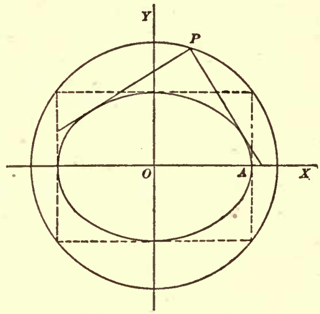

Fig. 93.

$$y' = mx' \pm \sqrt{a^2m^2 + b^2},$$

in which the angular coefficient m is unknown. This equation, written in an integral form,

$$(a^2 - x'^2)\, m^2 + 2\, x'y'm + (b^2 - y'^2) = 0,$$

is of the second degree; its two roots determine the directions of the two tangents drawn from the point P to the ellipse, and, consequently, determine these tangents. The two tangents drawn from the point P will be rectangular if the product of the two values of m be equal to -1, which will be the case if the co-ordinates of the point P satisfy the relation

$$\frac{b^2 - y'^2}{a^2 - x'^2} = -1, \text{ or } x'^2 + y'^2 = a^2 + b^2.$$

Hence the locus of the vertex of a right triangle circumscribed about an ellipse is the circle circumscribed about a rectangle constructed on the axes.

DIAMETERS.

159. The general equation of a diameter of a curve of the second degree has been found in § 131. On representing by

$$f(x, y) = 0$$

the equation of the curve, and by m the angular coefficient of the chords parallel to MM' (Fig. 94), one has seen that the equation of the diameter

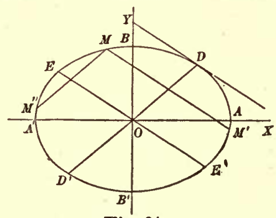

Fig. 94.

DD' may be written in the form $f'_x + mf'_y = 0$. The equation of the ellipse being referred to its axes, the equation of the diameter reduces to

$$\frac{2x}{a^2} + \frac{2my}{b^2} = 0, \text{ or } y = -\frac{b^2}{a^2 m}x.$$

On representing by m' the angular coefficient of the diameter DD', one has, between the direction of the chord and that of the diameter, the relation

$$(8) \qquad mm' = -\frac{b^2}{a^2}.$$

It has also been shown that if the chord MM'' be drawn parallel to the diameter DD', the diameter OE, which bisects this chord, has the angular coefficient m; the two diameters DD', EE' form a system of conjugate diameters, and their angular coefficients m' and m are connected by relation (8).

This relation shows that the two angular coefficients m and m' have opposite signs, and, consequently, that the two semi-conjugate diameters OD and OE, situated on the same side of the major axis, are situated on opposite sides of the minor axis. If the first start from OA and revolve from OA toward OB, the second starts from OB and revolves toward OA'.

160. The tangent at any point D of the ellipse is parallel to the diameter EE', the conjugate of the diameter DD', which passes through the point of contact. In fact, if one call x and y the co-ordinate of the point D, the diameter OD has the angular coefficient $m = \frac{y}{x}$; the coefficient of the tangent at the point D is $m' = -\frac{b^2x}{a^2y}$; these two coefficients satisfy the relation $mm' = -\frac{b^2}{a^2}$.

This property may be described more clearly by imagining that the secant MM', moving parallel to the diameter EE', recedes continually from the center; the two points of intersection M and M' approach more and more the middle of the chord, and end by coinciding with D; then the secant becomes a tangent at D.

161. The properties of conjugate diameters are exhibited at once on considering the ellipse as the projection of the circle. Two rectangular diameters OD_1, OE_1 (Fig. 95), in the plane of the circle, form a system of conjugate diameters, because each of them bisects the chords parallel to the other; the parallel chords are projected into parallel chords in the plane of the ellipse; the mid-point of the chord has for its projection the mid-point of the projection of the chord;

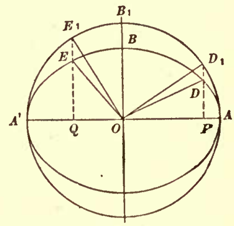

Fig. 95.

each of the diameters OD, OE, the projections of the diameters OD_1, OE_1, bisects therefore the chord parallel to the other; they are therefore conjugate diameters of the ellipse. It is easy to deduce the relation which exists between the angular coefficients m and m' of the two conjugate diameters. If m_1 and m_2 be called the angular coefficients of the two conjugate diameters OD_1, OE_1 of the circle, one has $m = \frac{b}{a} m_1$, $m' = \frac{b}{a} m'_1$; whence $mm' = \frac{b^2}{a^2} m_1 m'_1$; since the conjugate diameters of the circle are perpendicular, one has $m_1 m'_1 = -1$; it follows then that $mm' = -\frac{b^2}{a^2}$.

Being given OD, one can find its conjugate OE, without drawing the ellipse. One constructs the diameter OD_1 whose projection is OD; and draws the diameter OE_1 perpendicular to OD_1, and projects OE_1; the projection OE will be the diameter required.

162. *The ellipse referred to two conjugate diameters.* Owing to what has been said in § 136, when two conjugate diameters OD, OE (Fig. 96) are taken as axes of co-ordinates, the equation of the ellipse can be written

$$A''x'^2 + C''y'^2 + H = 0.$$

Since the coefficients A'' and C'' have the same sign, contrary to that of H, if one put

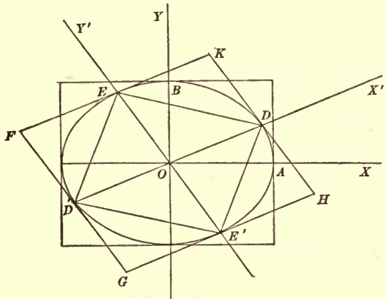

Fig. 96.

$$a'^2 = -\frac{H}{A''},\ b'^2 = -\frac{H}{C''},$$

the equation takes the form

$$(9) \qquad \frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1,$$

which is the same form as that of the curve referred to its axes.

It follows that the calculations employed in demonstrating the properties of the ellipse, when the equation of the curve was referred to its axes, and in which the co-ordinates were not supposed orthogonal, could be repeated with the equation of the curve referred to a system of conjugate diameters. Thus, the ellipse being referred to a system of conjugate diameters OD and OE, the tangent will have the equation

$$\frac{x'X'}{a'^2} + \frac{y'Y'}{b'^2} = 1.$$

However, the equation of the normal does not preserve the form which corresponds to the axes OA and OB.

Theorem of Apollonius.

163. The theorem of Apollonius admits of an easy demonstration by the method of § 142. Imagine the ellipse referred successively to its two axes and to a system of conjugate diameters forming an angle θ. By the formulas of transformation of co-ordinates, the binomial

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

is transformed into $$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2}.$$

Similarly the binomial $\qquad x^2 + y^2$

becomes $\qquad x'^2 + y'^2 + 2x'y' \cos\theta,$

since each of the two expressions represents the square of the distance of the origin from the same point of the plane. Whence it follows that the polynomial

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{1}{\lambda}(x^2 + y^2),$$

or

$$(10) \qquad \left(\frac{1}{a^2} - \frac{1}{\lambda}\right)x^2 + \left(\frac{1}{b^2} - \frac{1}{\lambda}\right)y^2,$$

in which λ plays the rôle of an arbitrary constant, is transformed into

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} - \frac{1}{\lambda}(x'^2 + y'^2 + 2x'y'\cos\theta),$$

or

$$(11) \qquad \left(\frac{1}{a'^2} - \frac{1}{\lambda}\right)x'^2 - 2\frac{\cos\theta}{\lambda}x'y' + \left(\frac{1}{b'^2} - \frac{1}{\lambda}\right)y'^2.$$

The values of λ, which make one of the polynomials (10) or (11) a perfect square, being the same, the two equations

$$\left(\frac{1}{a^2} - \frac{1}{\lambda}\right)\left(\frac{1}{b^2} - \frac{1}{\lambda}\right) = 0,$$

or

$$(12) \qquad (\lambda - a^2)(\lambda - b^2) = 0,$$

and

$$\left(\frac{1}{a'^2} - \frac{1}{\lambda}\right)\left(\frac{1}{b'^2} - \frac{1}{\lambda}\right) - \frac{\cos^2\theta}{\lambda^2} = 0,$$

or

$$(13) \qquad \lambda^2 - (a'^2 + b'^2)\lambda + a'^2b'^2\sin^2\theta = 0,$$

have the same roots. It follows, therefore, that the two roots of equation (13) are equal respectively to a^2 and b^2, whence follow the two relations:

$$(14) \qquad a'^2 + b'^2 = a^2 + b^2;$$

$$(15) \qquad a'^2b'^2\sin^2\theta = a^2b^2, \text{ or } a'b'\sin\theta = ab.$$

The preceding equations furnish the following theorems:

1° *The sum of the squares of any two conjugate diameters of an ellipse is constant and equal to the sum of the squares of the axes.*

2° *The area of the parallelogram constructed on two conjugate diameters is constant and equal to that of the rectangle constructed on the axes.*

Relations (21) of § 142 give immediately the two equations (14) and (15).

164. These theorems may easily be demonstrated by considering the ellipse as the projection of a circle.

Two conjugate diameters OD, OE of the ellipse are the projections of two perpendicular diameters OD_1, OE_1 of the circle (Fig. 95). The angles D_1OP, E_1OQ being complementary, the right triangles D_1OP, E_1OQ are equal, and one has

$$OQ = D_1P; \text{ but } \overline{OD_1}^2 = \overline{OP}^2 + \overline{D_1P}^2;$$

it follows that
$$\overline{OP}^2 + \overline{OQ}^2 = a^2.$$

The lengths OP and OQ being the projections of the two semi-conjugate diameters OD and OE on the major axis of the ellipse, the sum of the squares of these two projections is constant, and one has

$$a'^2 \cos^2 \alpha + b'^2 \cos^2 \beta = a^2,$$

on representing by α and β the angles which the semi-diameters OD and OE make with the axis OA.

Similarly for the other axis, one has the projection of the two semi-conjugate diameters on the minor axis equal to the ordinates DP and EQ. $DP = \frac{b}{a} D_1P$, $EQ = \frac{b}{a} E_1Q$, and, consequently,

$$\overline{DP}^2 + \overline{EQ}^2 = \frac{b^2}{a^2}(\overline{D_1P}^2 + \overline{E_1Q}^2).$$

The lengths E_1Q and OP being equal, one has

$$\overline{D_1P}^2 + \overline{E_1Q}^2 = \overline{D_1P}^2 + \overline{OP}^2 = a^2,$$

and, consequently, $\overline{DP}^2 + \overline{EQ}^2 = b^2,$

or $$a'^2 \sin^2 \alpha + b'^2 \sin^2 \beta = b^2.$$

On adding member to member the two preceding relations, one obtains
$$a'^2 + b'^2 = a^2 + b^2.$$

165. In order to demonstrate the property respecting the area of the parallelogram, one makes use of the following theorem :

The projection of a plane area upon any plane is equal to the projected area multiplied by the cosine of the angle between the planes.

For this purpose consider a triangle ABC (Fig. 97) having an edge AB parallel to the plane of projection; one can assume that the plane of projection passes through this edge AB; from the vertex C, drop upon this plane a perpendicular CC', and, in this plane, draw $C'D$ perpendicular to AB; the straight line CD will also be perpendicular to AB and the angle CDC' is the measure of the dihedral angle of the two planes. From the construction it follows that

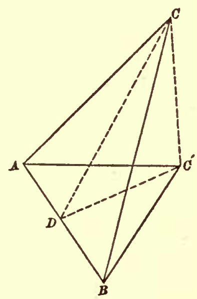

Fig. 97.

$$C'D = CD \cos \phi,$$

whence $$\frac{AB \cdot C'D}{2} = \frac{AB \cdot CD}{2} \cos \phi,$$

and, consequently,
$$AC'B = ACB \times \cos \phi.$$

Thus the area of the triangle $AC'B$ is equal to that of the triangle ACB multiplied by $\cos \phi$.

Suppose now that the triangle ABC (Fig. 98) has no side parallel to the plane of projection; this plane can be passed through a vertex A, in such a way that the other two vertices may lie on the same side; the plane of the triangle produced intersects the plane of projection in a straight line AI, and the line CB intersects this plane in the point I; the triangles AIC, AIB project into AIC', AIB', and one has, after what has just been proven,

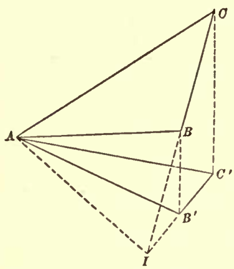

Fig. 98.

$$AIC' = AIC \cos\phi,$$

$$AIB' = AIB \cos\phi,$$

whence, by subtraction,

$$AB'C' = ABC \cos\phi.$$

The theorem, being demonstrated for a triangle, may be extended to a plane polygon, since it can always be decomposed into triangles, and similarly to a plane area bounded by any closed curve; because this plane area may be regarded as the limit of the area of an inscribed polygon, of which the number of sides is increased indefinitely, in such a way that each approaches the limit zero.

When the ellipse is regarded as the projection of a circle, the parallelogram constructed on the two conjugate diameters is the projection of a square circumscribed about the circle; the square having a constant area equal to $4a^2$, that of the parallelogram is also constant and equal to $4a^2 \cos\phi$, that is, to $4ab$.

Area of the Ellipse.

166. The same theorem furnishes immediately the area of the ellipse. The ellipse being the projection of a circle, its area is equal to that of the circle πa^2, multiplied by $\cos\phi$ or by $\frac{b}{a}$, which gives πab.

EQUAL CONJUGATE DIAMETERS.

167. It has been noticed (§ 159) that the two semi-conjugate diameters *OD, OE* lie on opposite sides of the minor axis *OB* (Fig. 99). One knows that the radius vector of the ellipse increases in length as it is rotated farther from the minor axis; in order that two conjugate diameters may become equal, it is therefore necessary that they make equal angles with the minor

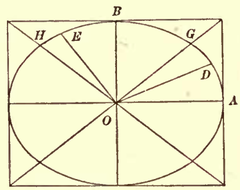

Fig. 99.

axis *OB*, which will take place when the angles α and β are supplementary. One has, therefore, $\tan^2\alpha = \frac{b^2}{a^2}$ and, consequently, $\tan\alpha = \frac{b}{a}$; hence the equal conjugate diameters *OG* and *OH* coincide with the diagonals of the rectangle constructed on the axes.

It follows from the relation $a'^2 + b'^2 = a^2 + b^2$ that

$$a'^2 = \frac{a^2 + b^2}{2},$$

and the equation of the ellipse, referred to its equal conjugate diameters, is

$$x^2 + y^2 = \frac{a^2 + b^2}{2};$$

it has the same form as the equation of a circle, only the co-ordinates are oblique.

This equation shows that the sum of the squares of the distances of each of the points of the ellipse from two equal conjugate diameters is constant. In fact, let θ be the angle between the two equal conjugate diameters; *MP* and *MQ* the co-ordinates of the point *M* (Fig. 100); *ME* and *MF* the perpendiculars dropped from *M* upon these diameters;

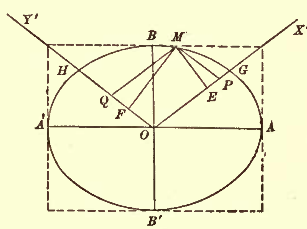

Fig. 100.

one has $ME = y' \sin\theta$, $MF = x' \sin\theta$; whence

$$\overline{ME}^2 + \overline{MF}^2 = (x'^2 + y'^2)\sin^2\theta = \frac{(a^2 + b^2)\sin^2\theta}{2} = \frac{2a^2b^2}{a^2 + b^2}.$$

SUPPLEMENTARY CHORDS.

168. Two chords MC, MC' in an ellipse are called *supplementary chords*, if they be drawn from any point of the ellipse to the extremities of a diameter CC' (Fig. 101).

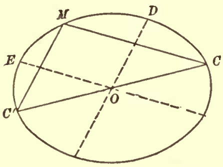

Fig. 101.

Two supplementary chords are parallel to the corresponding conjugate diameters. Draw, in fact, the diameters OD and OE parallel to the supplementary chords MC', MC. In the triangle CMC', the two sides CC' and CM are divided by the line OD, parallel to $C'M$, into parts which are proportional; the center O being the mid-point of CC', it follows that the diameter OD divides the chord CM into two equal parts, and, consequently, every chord parallel to the diameter OE. Similarly, the diameter OE bisects the chord $C'M$, and, consequently, every chord parallel to OD. Therefore the two diameters OD, OE, parallel to the supplementary chords MC', MC, are conjugate.

Conversely, if straight lines be drawn from the extremities of a diameter CC' parallel to two conjugate diameters OD, OE, these straight lines intersect on the ellipse; draw $C'M$; the supplementary chords MC, MC' being parallel to the two conjugate diameters, the second chord $C'M$ will be parallel to OD.

169. The study of the variation of the angle formed by two conjugate diameters is thus reduced to the study of the variation of the angle formed by two supplementary chords, that is, of the angle inscribed in a semi-ellipse. In order to simplify the discussion, one draws the two supplementary chords through the extremities of the major axis (Fig. 102). The angle AMA', represented by θ, is equal to the difference between the angles MAX,

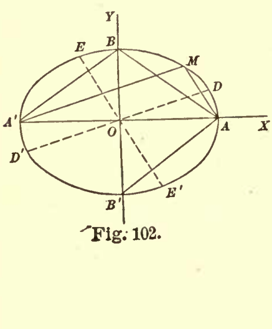

Fig. 102.

MA'X. Since the two straight lines *AM*, *A'M* have the angular coefficients $\frac{y}{x-a}$, $\frac{y}{x+a}$, one has

$$\tan\theta = \frac{\frac{y}{x-a} - \frac{y}{x+a}}{1 + \frac{y^2}{x^2 - a^2}} = \frac{2\,ay}{x^2 + y^2 - a^2},$$

and, by replacing x^2 by its value deduced from the equation of the ellipse,

$$\tan\theta = -\frac{2\,ab^2}{(a^2 - b^2)\,y}.$$

If the point *M* describe the upper portion of the ellipse *ABA'*, the tangent being negative, the angle is obtuse; when the point *M* is at the point *A*, that is, when $y = 0$, the angle is right; the point *M* traveling from *A* toward *B*, *y* increases; the absolute value of $\tan\theta$ diminishes; the obtuse angle ϕ increases also, and acquires its maximum value at *B*; thus one has $y = b$ and $\tan\theta = -\frac{2\,ab}{a^2 - b^2}$. When the point *M* passes the point *B* and traces the elliptical quadrant *BA'*, the angle θ diminishes from its maximum value to a right angle.

Whence it follows that the angle between the semi-conjugate diameters *OD*, *OE*, situated on the same side of the major axis, is obtuse, and varies from a right angle to the maximum value *ABA'*; the conjugate diameters, which embrace the maximum angle, being respectively parallel to the supplementary chords *A'B*, *AB*, and, consequently, forming equal angles with the minor axis *OB*, are equal.

The variation of the obtuse angle *DOE* of two conjugate diameters has been studied; the acute angle *DOE'* varies in an inverse manner. This angle is obtained directly by drawing the corresponding supplementary chords through the extremities of the minor axis *BB'*. When the point *M* describes the quadrant of the ellipse *BA*, the inscribed angle diminishes from a right angle to the minimum value *BAB'*, the supplement of the obtuse maximum value *ABA'*.

170. When the ellipse has been drawn, the center and the axes can be determined graphically. In order to find the center, one draws two parallel chords sufficiently distant from each other, and then joins the mid-points of these chords, which will determine a diameter, whose mid-point will be the center. If on this diameter a semi-circle be constructed, and the point where this semi-circle intersects the semi-ellipse be joined to the extremities of the diameter, one will have two supplementary chords which are perpendicular; the parallel diameters, forming a system of perpendicular conjugate diameters, will be the axes of the ellipse.

In a similar manner, the two systems of conjugate diameters which include a given angle having as limits the minimum and maximum values, can be constructed; it will suffice to construct on a diameter a segment which will circumscribe an angle equal to the given angle.

171. *Being given two conjugate diameters, construct the corresponding ellipse.* Let DD', EE' (Fig. 103) be the given conjugate diameters, whose lengths are represented by $2a'$ and $2b'$. The equation of the ellipse, referred to these two conjugate diameters, is

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1.$$

Draw through the center the line $E_1E'_1$, perpendicular to DD', and take $OE_1=OE$; the ellipse which has the axes DD', $E_1E'_1$, referred to these axes, is represented by the same equation. Whence it follows that the ordinates MP, M_1P, which correspond to the same abscissa OP, are equal to each other. Imagine that different points of the ellipse DE_1D', whose axes are known, are constructed by the process described in § 149; let M_1 be one of these points, M_1P its ordinate; if one draw through the point P, PM parallel to OE and equal to PM_1, one

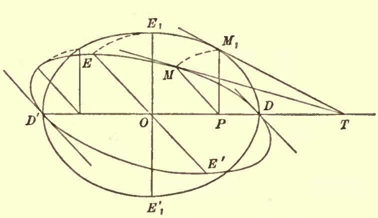

Fig. 103.

will have the point M of the ellipse required. Each point of the first ellipse will give a corresponding point of the second. The first ellipse is deformed into the second by revolving each ordinate PM_1 about its foot P through a constant angle.

The same method of transformation can be applied to the tangent. The tangent at the point M is represented by the equation

$$\frac{xX}{a'^2}+\frac{yY}{b'^2}=1, \qquad (4)$$

written in oblique co-ordinates; this equation represents also the tangent at the point M_1 if written in rectangular co-ordinates. These two tangents intersect the prolongation of the diameter DD' at the same point T, the abscissa of which is found by making $Y=0$.

Instead of constructing the ellipse by points, as has been explained, the axes of the ellipse can first be constructed, and then the ellipse itself by means of its axes. The determination of the axes depends upon the following theorem:

172. *Any two conjugate diameters determine on a fixed tangent PQ two segments DP, DQ, whose product is constant and equal to the square of the semi-diameter OE parallel to the conjugate* (Fig. 104). If one take as axes of co-ordinates the diameter OD, which passes through the point of contact and its conjugate OE, and if one calls a' and b' the lengths of these semi-diameters, the equation of the ellipse is

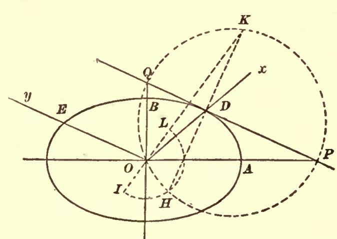

Fig. 104.

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1.$$

Let $$y=mx, \quad y=m'x,$$

be the equations of two conjugate diameters OA, OB; according to the remark made in § 160, the angular coefficients will be connected by the relation $mm' = -\frac{b'^2}{a'^2}$. If in these equations one put $x = a'$, one finds $DP = -ma'$, $DQ = m'a'$; whence

$$DP \cdot DQ = -mm'a'^2 = b'^2.$$

173. This theorem may easily be demonstrated by considering the ellipse as the projection of a circle. Let OA_1, OB_1 (Fig. 105) be two perpendicular diameters of the circle, P_1Q_1 the tangent at any point M_1; draw the radius OM_1 and the radius ON_1 parallel to the tangent; in the right triangle P_1OQ_1, one has

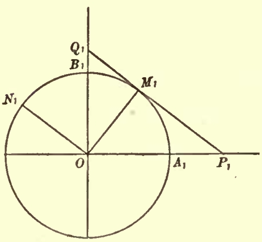

Fig. 105.

$$M_1P_1 \cdot M_1Q_1 = OM_1^2 = ON_1^2.$$

When the figure is projected, the diameters OA_1, OB_1 furnish two conjugate diameters of the ellipse, the tangent P_1Q_1 a tangent to the ellipse, and the line ON_1 a parallel to this tangent; the lines M_1P_1, M_1Q_1, ON_1, being parallels, have the projections MP, MQ, ON, which are proportional to them; there also exist, therefore, between these projections the relation

$$MP \cdot MQ = ON^2.$$

174. Suppose that the two conjugate diameters OA and OB be the axes of the ellipse (Fig. 104). The circle described on PQ as a diameter passes through the point O, and the ordinate DH, perpendicular to PQ, is equal to OE. Whence follows a simple device for constructing the directions of the axes, when one knows the two conjugate diameters OD and OE. One draws through the point D a line parallel to OE; this parallel will be tangent at the point D; on this line one erects a perpendicular DH equal to OE, and describes a circle having its center on PQ and passing through the points O and H; the straight lines OP and OQ which connect the center with the two points P and Q, where the circle intersects the tangent, will give the direction of the axes.

175. It remains to determine the magnitudes of the axes. From the relations

$$a^2 + b^2 = a'^2 + b'^2,\ ab = a'b' \sin\theta,$$

established in § 163, one deduces

$$(a-b)^2 = a'^2 + b'^2 - 2\,a'b'\sin\theta = a'^2 + b'^2 - 2\,a'b'\cos\left(\frac{\pi}{2} - \theta\right),$$

$$(a+b)^2 = a'^2 + b'^2 + 2\,a'b'\sin\theta = a'^2 + b'^2 - 2\,a'b'\cos\left(\frac{\pi}{2} + \theta\right).$$

Since one can suppose that θ designates the angle included by the conjugate diameters, one sees from these formulas that $a - b$ is the third side of a triangle of which the other two sides are a' and b' and the included angle $\frac{\pi}{2} - \theta$. This triangle is the triangle ODH (Fig. 104); because the angle ODH is equal to $\frac{\pi}{2} - \theta$, and the two sides DO and DH are equal to a' and b'; thus the third side OH will be equal to $a - b$. Similarly $a + b$ is the third side of the triangle of which the other two sides are a' and b', and the included angle the supplement of the preceding; this triangle is the triangle ODK, which is obtained by prolonging the perpendicular DH till the prolongation is equal to itself; the third side OK will determine $a + b$. If about the point O as center, with OH as a radius, one describe a circle, the length KI will be equal to the major axis $2\,a$, the length KL to the minor axis $2\,b$.

One remarks that the major axis, which should lie within the angle formed by the conjugate diameters, has the direction OA, the bisector of the angle HOK, the minor axis is the bisector of the supplementary angle.

EXERCISES.

1. Find the locus of the vertices of the parallelograms constructed on the conjugate diameters of an ellipse.

2. Find the locus of the mid-points of chords drawn through the same point in an ellipse.

3. A chord of a circle moves parallel to itself; straight lines parallel to two given straight lines are drawn through the extremities; find the locus of the point of intersection of the parallels.

4. Of all the parallelograms circumscribed about the same ellipse, the parallelograms constructed on two conjugate diameters have a minimum area.

5. Of all the parallelograms inscribed in the same ellipse, those whose diagonals form a system of conjugate diameters have a maximum area.

6. Of all the ellipses inscribed in the same parallelogram, find the greatest.

7. Find the smallest of all the ellipses circumscribed about the same parallelogram.

8. Among all the systems of conjugate diameters of an ellipse, the axes form a minimum sum and the equal conjugate diameters a maximum sum.

9. Inscribe in an ellipse a chord with a given direction such that the sum of its length and of the distance of its mid-point from the center be a maximum; find the locus of the mid-point of this chord when the direction varies.

10. A straight line moves parallel to itself in the plane of two others; one takes on it a point such that the sum of the squares of its distances from the intersections with the fixed lines be constant; what is the locus described by the point?

11. Being given any two ellipses, one can determine two directions parallel at the same time to two conjugate diameters of each of the ellipses; pass a third ellipse of which the equal conjugate diameters are parallel to these two directions through the points common to the two curves.

12. An ellipse revolves about its center; one draws tangents to the ellipse at the points in which it intersects a fixed straight line; find the locus of the point of intersection of these tangents.

13. Being given a circle and a fixed straight line passing through its center; a movable straight line equal to the radius is supported by one of its extremities on the circum-

ference, by the other on the line; find the locus of a point on the movable straight line.

14. Find the area of the ellipse defined by the equation

$$Ax^2 + 2\,Bxy + Cy^2 = 1.$$

15. A triangle being inscribed in an ellipse, if one call R the radius of the circumscribed circle and d, d', d'' the semi-diameters parallel to the sides, one has

$$R = \frac{dd'd''}{ab}.$$

16. Any rectangle being circumscribed about an ellipse, the parallelogram whose vertices are the points of contact has a constant perimeter, and two consecutive sides make, with the tangent, equal angles.

17. Beginning at any point on the ellipse, one lays off on the normal a length equal to $\frac{k^2}{p}$, k being a constant and p the perpendicular dropped from the center upon the tangent; find the locus of the extremity of this line.

18. Being given an ellipse and the circle constructed on its major axis or diameter, one draws normals to the circle and to the ellipse at points situated on the same perpendicular to the major axis; find the locus of the point of intersection of the normals.

O

CHAPTER V

THE HYPERBOLA.

176. Construct the locus defined by the equation

$$A'x^2 + C'y^2 + H = 0,$$

in which A' and C' have contrary signs.

When the constant H is zero, the equation, solved with respect to y, gives

$$y = \pm\sqrt{-\frac{A'}{C'}}x;$$

it represents two straight lines passing through the origin.

One gives the coefficient C' the same sign as H, and the coefficient A' the opposite. If one put

$$a^2 = -\frac{H}{A'}, \quad b^2 = \frac{H}{C'},$$

the equation becomes

(1) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Solving the equation with respect to y, one has

(2) $$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}.$$

The ordinate y is real for values of x greater than a in absolute value. If, therefore, beginning at the origin, one lay off on the axis of x, to the right and the left, two lengths OA, OA' equal to a, and draw through the points A and A' lines parallel to the axis of y, no point of the curve will lie between these parallels.

When x increases from a to $+\infty$, y increases from 0 to $+\infty$ in absolute value, which, on account of the double sign, furnishes two infinite arcs AD, AD', symmetrical with respect to the axis of x. Similarly, when x varies from $-a$ to $-\infty$, one gets two infinite arcs $A'E$, $A'E'$, symmetrical with respect to the axis of y. These four equal arcs form the two branches of the hyperbola.

The hyperbola has a center and two axes. The axis AA' only intersects the curve; for this reason it is called the real or *transverse axis;* the other axis does not meet the curve; one calls it the non-transverse or *imaginary axis;* the length AA' of the transverse axis is $2a$; by analogy, the length of the non-transverse axis is called $2b$, and on this axis one lays off OB and OB' equal in absolute length to b. The points A and A' are the two vertices of the hyperbola.

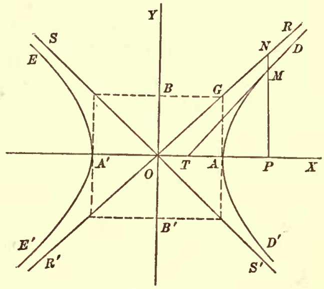

Fig. 106.

177. *The squares of the ordinates perpendicular to the transverse axis are proportional to the products of the corresponding segments on this axis.*

In fact, from equation (1) it follows that

$$\frac{y^2}{x^2 - a^2} = \frac{b^2}{a^2}, \quad \text{or} \quad \frac{y^2}{(x+a)(x-a)} = \frac{b^2}{a^2};$$

therefore
$$\frac{\overline{MP}^2}{A'P \times AP} = \frac{b^2}{a^2}.$$

178. Asymptotes.—It has been found (§ 130) that, when the origin of co-ordinates coincides with the center of the hyperbola, the equation of the asymptotes is found by suppressing the constant term in the equation of the curve. The two asymptotes RR', SS' will have in this case the equations

(3)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \quad \text{or} \quad y = \pm \frac{b}{a} x.$$

It can be easily verified that the difference MN of the ordinates of the straight line OR and the arc AD, has the limit zero; because this difference can be expressed by

$$\left(\frac{b}{a}\right)(x - \sqrt{x^2 - a^2}) = \frac{ab}{x + \sqrt{x^2 - a^2}}.$$

The arc AD lies wholly within the angle ROX and approaches indefinitely the line OR, which is its asymptote. The lines OR', OS, OS' are for a similar reason the asymptotes of the arcs $A'E'$, $A'E$, AD'. According to equation (3), the asymptotes $R'R$, $S'S$ are the diagonals of the rectangle constructed on the axes.

179. Conjugate Hyperbolas.—Two hyperbolas are said to be *conjugate*, when they have the same center and the same axes, the real axis of the one being the imaginary of the other. Thus the proposed hyperbola has as conjugate another hyperbola whose transverse axis is $2b$ and imaginary axis $2a$ (Fig. 107). The equation of this second hyperbola is

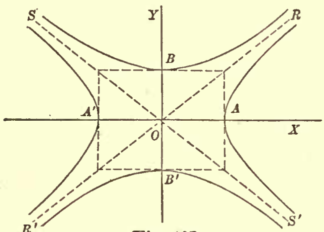

Fig. 107.

(4) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

Two conjugate hyperbolas have the same asymptotes, since the rectangle constructed on the axes is the same for both curves. One of the curves lies wholly within the vertical angles ROS', $R'OS$, the second within the other vertical angles ROS, $R'OS'$.

180. The Equilateral Hyperbolas.—A hyperbola is said to be equilateral when the axes $2a$ and $2b$ have the same length. In this case, the rectangle of the axes becomes a square, and the asymptotes are perpendicular to each other; the conjugate hyperbola is equal to the first; for the two curves will coincide when one revolves the latter through a right angle about its center.

The condition that the general equations of the second degree represents an equilateral hyperbola, has been previously given (§ 144); this condition is $A + C - 2B\cos\theta = 0$.

The hyperbola whose axes are a and b can be constructed by means of the equilateral hyperbola whose axis is a, just as the ellipse having the axes a and b was constructed by means of the circle of radius a; that is, the first hyperbola can be regarded as the orthogonal projection of the second. But this construction has no practical utility in the graphical construction of the hyperbola, inasmuch as the trace of an equilateral hyperbola is not more simple than that of any other hyperbola.

181. Let x and y be the co-ordinates of any point of the plane; consider the expression

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-1.$$

This polynomial is equal to zero for a point M belonging to the curve; if a point P starting from M travels along a line drawn parallel to the transverse axis AA' (Fig. 108), the term $\frac{y^2}{b^2}$ remains constant, while the term $\frac{x^2}{a^2}$ diminishes or increases, according as the point P approaches or recedes from the y-axis. Whence it follows that the polynomial has a negative value for every point situated between the two branches of the hyperbola, and positive for all other points of the plane.

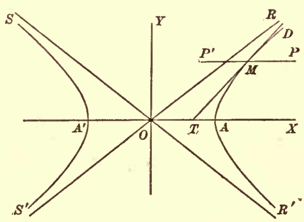

Fig. 108.

THE TANGENT.

182. The equation of the tangent at the point M, whose co-ordinates are x and y, is

$$(5)\qquad \frac{xX}{a^2}-\frac{yY}{b^2}-1=0.$$

In order to construct this line, one can determine the point T (Fig. 108), where it intersects the axis OX. If, in equation (5) one make $Y=0$, it becomes $X=OT=\frac{a^2}{x}$; this length OT can be found by a third proportional.

183. The angular coefficient of the tangent has the value

$$\frac{b^2x}{a^2y}=\frac{b}{a\sqrt{1-\frac{a^2}{x^2}}}.$$

Suppose that the point M describes the arc AD; at A the angular coefficient is infinity, and the tangent perpendicular to the transverse axis; as x increases, the angular coefficient diminishes constantly and approaches the limit $\frac{b}{a}$, the angular coefficient of the asymptote OR; the angle MTX diminishes therefore from $\frac{\pi}{2}$ to ROX; at the same time the value of OT diminishes from a to 0; whence it follows that the asymptote is the limiting position of the tangent, when the point of contact is indefinitely removed.

184. To Draw a Tangent Through an Exterior Point P. — If the co-ordinates of the point P be x_1 and y_1, the points of contact are determined by the equation of the chord of contacts

$$(6)\qquad \frac{x_1x}{a^2}-\frac{y_1y}{b^2}-1=0,$$

combined with equation (1) of the hyperbola.

By eliminating y, one gets the equation of the second degree

$$\frac{x^2}{a^2}\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right)-2\frac{x_1x}{a^2}+\left(1+\frac{y_1^2}{b^2}\right)=0,$$

whose roots are the abscissas of the points of contact M and M' of the two tangents drawn from the point P. The condition that the roots are real is

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1<0;$$

that is, that the point P should be situated between the two branches of the curve. If the point P lie in the angles of the asymptotes which embrace the curve, the coefficient $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}$ being positive, the product of the roots is positive; consequently the two roots have the same sign, and the two points of contact on the same branch of the curve. On the contrary, if the point P be in one of the angles ROS, $R'OS'$, there will be a point of contact on each of the branches.

185. Tangents Parallel to a Given Straight Line. — It is to be noticed that the equation of the hyperbola referred to its axes differs only from that of the ellipse in that b^2 is replaced by $-b^2$; if this change be made in equation (7) of § 157 of the tangent to the ellipse, one gets the equation of the tangent to the hyperbola

$$(7) \qquad y = mx \pm \sqrt{a^2m^2 - b^2}.$$

In order that the problem be possible, it is necessary that the value of m^2 be greater than $\frac{b^2}{a^2}$; that is, that in case the given line passes through the origin, it lie within the angle ROS. It has already been shown (§ 183) that the numerical value of the angular coefficient of a tangent is greater than $\frac{b}{a}$.

186. One can draw to a hyperbola two perpendicular tangents so long as the angle ROR' is less than a right angle, that is, when a is greater than b; when this condition is satisfied, the locus of the vertex of a right angle circumscribed about a hyperbola has the equation

$$x^2 + y^2 = a^2 - b^2;$$

that is, a circle concentric with the curve.

Diameters.

187. When the hyperbola is referred to its axes, the diameter which bisects parallel chords whose angular coefficient is m has the equation

$$\frac{2x}{a^2} - \frac{2my}{b^2} = 0,$$

or

$$y = \frac{b^2}{a^2m}x.$$

If one designate by m' the angular coefficient of the diameter, there will exist, between the direction of the chords and that of the diameter, the relation

$$(8) \qquad mm' = \frac{b^2}{a^2}.$$

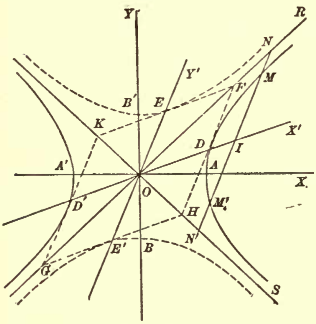

Fig. 109.

This relation shows that if one take m' as the angular coefficient of the chords, one will find m for the angular coefficient of the diameter; that is, in case the line DD' bisects the chords parallel to EE' (Fig. 109), reciprocally the line EE' bisects the chords parallel to DD'. Thus the two diameters DD', EE' are such that each bisects the chords parallel to the other; they are two conjugate diameters.

The hyperbola has an infinity of systems of conjugate diameters, since one can choose at will one of the diameters. Relation (8) shows that m and m' have the same sign; if one suppose them positive, m varies from 0 to $\frac{b}{a}$, m' will vary from ∞ to $\frac{b}{a}$; the diameter DD' revolves from OA toward the asymptote OR, and the diameter EE' from OB toward the same asymptote. One sees thus, that of the two diameters, one always intersects the curve while the other never meets it. The axes form the only perpendicular system of conjugate diameters, and the angle included between the two conjugate diameters varies from $\frac{\pi}{2}$ to 0.

It can be shown, as in the case of the ellipse, that the tangent FH at the point D of the hyperbola is parallel to the diameter EE', the conjugate of the diameter DD' which is drawn to the point of contact (§ 160).

188. Two conjugate hyperbolas and the system of their asymptotes possess the same diameter for the same series of chords; because the equations of the three loci

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\pm 1, \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0,$$

differ only by the constant term which does not enter in the equation of the diameter $f'_x + mf'_y = 0$. The three loci possess also the same systems of conjugate diameters.

If the hyperbola is equilateral, the relation $mm' = \frac{b^2}{a^2}$ becomes $mm' = 1$, which shows that the angles DOX, EOX are complementary, and, consequently, that the asymptotes are the bisectors of the angles of the conjugate diameters.

189. The Hyperbola referred to Two Conjugate Diameters. — When two conjugate diameters OD, OE (Fig. 109) are chosen as co-ordinate axes, the equation of the hyperbola becomes (§ 136)

$$A''x^2 + C''y^2 + H = 0.$$

The coefficients A'' and C'' have contrary signs, for example, C'' has the sign of H, and A'' the contrary sign; if one put

$$a'^2 = -\frac{H}{A''}, \quad b'^2 = \frac{H}{C''},$$

the equation takes the form

$$\frac{x'^2}{a'^2}-\frac{y'^2}{b'^2}=1, \tag{9}$$

which is of the same form as that of the curve referred to its axes.

Since one has, by the transformation of co-ordinates,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{x'^2}{a'^2}-\frac{y'^2}{b'^2}$$

for every point of the plane, it follows that the equation of the conjugate hyperbola, referred to the same diameters OD, OE, is

$$\frac{x'^2}{a'^2}-\frac{y'^2}{b'^2}=-1.$$

The diameter OE, which does not meet the first hyperbola, meets the second in the point E, and the length b' of this semi-conjugate diameter of the first hyperbola is equal to the length OE of the real semi-diameter of the second.

190. Equation (3) of the asymptotes transforms into the equation

$$\frac{x'^2}{a'^2}-\frac{y'^2}{b'^2}=0, \text{ or } y'=\pm\frac{b'}{a'}x.$$

One deduces, therefore, that the diagonals of the parallelogram $FHGK$, constructed on any two conjugate diameters, coincide with the asymptotes of the hyperbola.

The sides FH, GK, of the parallelogram are tangents to the first hyperbola, and the sides FK, GH, to the conjugate, in such a way that the parallelogram is circumscribed to the curves of the two systems.

Theorem of Apollonius.

191. It is sufficient to repeat the reasoning of § 163.

By the formulas of transformation of co-ordinates, the two binomials

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}, \quad x^2+y^2,$$

are changed into

$$\frac{x'^2}{a'^2}-\frac{y'^2}{b'^2}, \quad x'^2+y'^2+2\,x'y'\cos\theta.$$

The polynomial

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{1}{\lambda}(x^2+y^2),$$

or (10)
$$\left(\frac{1}{a^2}-\frac{1}{\lambda}\right)x^2-\left(\frac{1}{b^2}+\frac{1}{\lambda}\right)y^2,$$

is transformed into

$$\frac{x'^2}{a'^2}-\frac{y'^2}{b'^2}-\frac{1}{\lambda}(x'^2+y'^2+2\,x'y'\cos\theta),$$

or (11)
$$\left(\frac{1}{a'^2}-\frac{1}{\lambda}\right)x'^2-2\frac{\cos\theta}{\lambda}x'y'-\left(\frac{1}{b'^2}+\frac{1}{\lambda}\right)y'^2.$$

The two polynomials (10) and (11) being perfect squares for the same values of λ, the two equations

$$(12) \qquad (\lambda - a^2)(\lambda + b^2) = 0,$$

and

$$(13) \qquad \lambda^2 - (a'^2 - b'^2)\lambda - a'^2 b'^2 \sin^2\theta = 0,$$

have the same roots; whence it follows that the two roots of equation (13) are equal respectively to a^2 and $-b^2$; one deduces the relations

$$(14) \qquad a'^2 - b'^2 = a^2 - b^2,$$

$$(15) \qquad a'^2 b'^2 \sin^2\theta = a^2 b^2, \text{ or } a'b' \sin\theta = ab,$$

and, therefore, the two following theorems:

1° *The difference of the squares of any two conjugate diameters is constant and equal to the difference of the squares of the axes.*

2° *The area of the parallelogram constructed on two conjugate diameters is constant and equal to the area of the rectangle constructed on the axes.*

It follows from the relation $a'^2 - b'^2 = a^2 - b^2$ that if a be different from b, one cannot have $a' = b'$; the hyperbola cannot have equal conjugate diameters. If, however, the hyperbola be equilateral, one always has $a' = b'$; every system of conjugate diameters is equal; this agrees with the remark of § 188, for then the two diameters make equal angles with the asymptote.

192. Since the hyperbola and its two asymptotes have the same diameter for the same system of parallel chords, the mid-point I of the chord MM' is also the mid-point of the chord NN' (Fig. 109). Therefore *the portions MN, $M'N'$ of a secant comprised between the hyperbola and its asymptotes are equal.*

If the secant become tangent, one has $DF = DH$. *The portions of a tangent comprised between the point of contact and the asymptotes are equal.*

193. Suppose that the hyperbola is referred to two conjugate diameters DD', EE', of which EE' is parallel to a given secant MM'; the curve will have the equation

$$y'^2 = \frac{b'^2}{a'^2}(x'^2 - a'^2),$$

and the asymptotes $y'^2 = \frac{b'^2}{a'^2}x'^2$. In Fig. 109 the secant MM' intersects the same branch of the curve in two points, while the parallel diameter EE' does not meet the curve; and one has

$$\overline{MI}^2 = \frac{b'^2}{a'^2}(\overline{OI}^2 - a'^2),\quad \overline{NI}^2 = \frac{b'^2}{a'^2}\overline{OI}^2,$$

and, consequently,

$$\overline{NI}^2 - \overline{MI}^2 = b'^2, \text{ or } (NI - MI)(NI + MI) = b'^2;$$

but $$NI - MI = MN,\quad NI + MI = MN';$$

therefore $$MN \cdot MN' = b'^2.$$

In case the secant intersects the two branches of the hyperbola, the parallel diameter meets the curve, and one will arrive at an analogous result. Thus, *the product of the segments of a secant, comprised between a point of the curve and the asymptotes, is equal to the square of the semi-diameter parallel to the secant.*

194. Being given the asymptotes RR', SS', and a point M of the hyperbola, one can obtain as many points of the curve as one wishes (Fig. 110). Draw, in fact, through the point M any straight line NMN'; this line intersects the asymptotes in N and N'; if one take on this line a length $N'M'$ equal to NM, one will have a second point M' of the hyperbola. The direction and lengths of the axes may also be determined. The curve being comprised within the angles ROS, $R'OS'$, the bisector OA of these two angles will be the transverse axis, and the perpendicular OB the imaginary axis. Draw QMQ' perpendicular

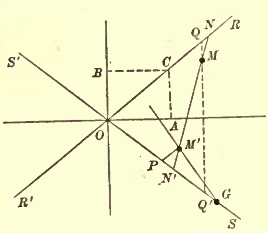

Fig. 110.

to OA; the imaginary semi-axis b will be a mean proportional between MQ and MQ'. On laying off on OB a length OB equal to b, and drawing BC parallel to OA, BC will be the real semi-axis a. In order to construct a tangent at a point M' of the curve, one will draw through this point $M'P$ parallel to an asymptote, taking $OG = 2\ OP$; the straight line $M'G$ will be the tangent required.

195. When one knows the positions and the magnitudes of two conjugate diameters, one can easily find the axes. Let, in fact, DD', EE' (Fig. 111) be the two diameters, of which the first is real. The diagonals of the parallelogram constructed on the two diameters are the asymptotes. Knowing the asymptotes and a point D, one is led to the preceding construction.

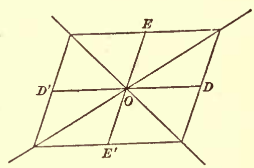

Fig. 111.

196. Supplementary Chords.—Two chords, MC, MC', are called supplementary chords if they, starting from the same point of the curve, be drawn to the extremities of the same diameter CC' (Fig. 112). One can demonstrate, as has been done in § 168 for the ellipse, that *two supplementary chords are parallel to a system of conjugate diameters*, and that, reciprocally, *if straight lines parallel to two conjugate diameters be drawn through the extremities of a diameter, these lines will intersect on the hyperbola and form a system of supplementary chords.*

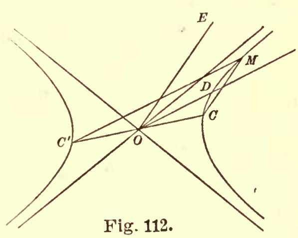

Fig. 112.

THE HYPERBOLA REFERRED TO ITS ASYMPTOTES.

197. If, after having transferred the origin to the center, which makes the terms of the first degree disappear, one take for new axes of co-ordinates the two asymptotes OX, OY (Fig. 113), a line parallel to an asymptote will not meet the curve in more than one point; the equation should be reduced to the first degree in y and also in x; that is, that the coefficients of y^2 and x^2 are zero. The equation will, therefore, have the form

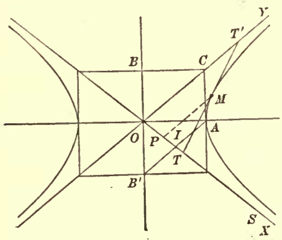

Fig. 113.

(16) $$2B'xy + H = 0, \text{ or } xy = k.$$

One deduces the value of k, on noticing that the co-ordinates of the vertex A are

$$x = y = OI = \frac{AB'}{2} = \frac{\sqrt{a^2 + b^2}}{2},$$

which satisfy the equation of the curve; whence

$$k = \frac{a^2 + b^2}{4}.$$

198. When the hyperbola is referred to its asymptotes, the tangent TT' at the point M, whose co-ordinates are x and y, has the equation

(17) $$yX + xY = 2k.$$

The abscissa of the point of intersection of the tangent with the axis OX is found by putting in this equation $Y = 0$, whence

$$X = OT = \frac{2k}{y} = 2x = 2OP;$$

one has a second proof that the point of contact M bisects the portion TT' of the tangent comprised between the asymptotes (§ 192).

THE AREA OF A HYPERBOLIC SEGMENT.

199. Next is discussed the theorem which concerns the evaluation of areas. Consider the area bounded by the axis OX, a curve, a fixed ordinate AB, and a variable ordinate MP (Fig. 114), corresponding to the abscissa x. This area, which is represented by S, is a function of the variable x, whose derivative is to be determined. Give to x an increment $\Delta x = PP'$ sufficiently small so that the ordinate of M may vary in the same sense as that of M'. Draw through the points M and M', MC, $M'D$ parallel to the axis OX. The increment ΔS of the area is greater than the parallelogram $MPP'C$, and smaller than the parallelogram $DPP'M'$. The measure of the first parallelogram is $y\Delta x \sin\theta$, θ being the angle between the axes, of the second $(y+\Delta y)\,\Delta x \sin\theta$. Therefore it follows

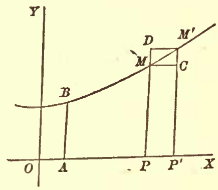

Fig. 114.

$$y\Delta x \cdot \sin\theta < \Delta S < (y + \Delta y)\,\Delta x \cdot \sin\theta,$$

and, by dividing by Δx,

$$y \sin\theta < \frac{\Delta S}{\Delta x} < (y + \Delta y) \sin\theta.$$

Let, now, Δx approach the limit zero. The ratio $\frac{\Delta S}{\Delta x}$ lies between two quantities, the one $y \sin\theta$, the other having this quantity for its limit; therefore the ratio has also the same limit $y \sin\theta$. Thus the derivative of the area considered as a function of the abscissa is $y \sin\theta$. Reciprocally, the area S is a function of $y \sin\theta$, considered as a function of x. In case the axes of co-ordinates are rectangular, the derivative of the area is equal to y.

200. Consider a hyperbola referred to its asymptotes, and determine the value of the area bounded by the asymptote OX, the hyperbola, the fixed ordinate AB corresponding to the abscissa a and the variable ordinate MP corresponding to the abscissa x (Fig. 115). It follows from equation (16) that

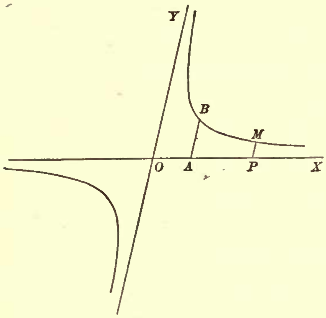

Fig. 115.

$y = \frac{k}{x}$, and, consequently,

$$S' = y \sin\theta = k \sin\theta \cdot \frac{1}{x}.$$

Since $\frac{1}{x}$ is the derivative of $\log x$; therefore $k \sin\theta \cdot \frac{1}{x}$ is the derivative of $k \sin\theta \log x$; one has, consequently,

$$S = k \sin\theta \log x + C.$$

The constant C is determined by the condition that the area be zero for $x = a$, which gives $C = -k \sin\theta \log a$. Hence, it follows

(18) $$S = k \sin\theta\,(\log x - \log a) = k \sin\theta \cdot \log\left(\frac{x}{a}\right).$$

The abscissa a being constant, if x be made to increase indefinitely, the area S increases also without limit. The same occurs when a approaches the limit zero, x remaining fixed.

In the particular case when the hyperbola is equilateral, one has $\sin\theta = 1$; if in addition k be made equal to 1, and the area be reckoned from the ordinate which corresponds to the abscissa 1, that is, from the vertex of the curve, the preceding formula reduces to

$$S = \log x.$$

It is on account of this property that Napierian logarithms have also been called hyperbolic logarithms.

If one assume $k = 1$, $a = 1$, formula (18) becomes

$$S = \sin\theta \log x.$$

The angle θ could be taken in such a way that S be the logarithm of x in any system whatever whose base is greater than e.

EXERCISES.

1. The base of a triangle is fixed; the difference of the angles at the base is $\frac{\pi}{2}$; find the locus of the third vertex of the triangle.

2. What is the locus of the centers of the circumferences which intercept given lengths on the sides of a given angle?

3. Being given two fixed straight lines and a movable straight line which intersects the first two in such a way that a triangle of constant magnitude is formed, it is required to find the locus of the centers of gravity of these triangles.

4. Two secants drawn from any point of a hyperbola to two fixed points taken on the curve intercept on one or the other asymptote constant lengths.

5. Every chord of a hyperbola bisects the portion of one or the other asymptote comprised between the tangents at its extremities.

6. If, on a chord of a hyperbola considered as a diagonal, one constructs a parallelogram whose sides are respectively parallel to the asymptotes, the other diagonal passes through the center.

7. Being given a fixed point and a fixed straight line; an angle of constant magnitude rotates about its vertex placed at the fixed point; find the locus of the center of the circle circumscribed about the triangle formed by the sides of the angle and the fixed straight line.

8. A triangle ABC is inscribed in a hyperbola; two of its sides have fixed directions; find the locus of the mid-point of the third side.

9. On one of the diagonals of a rectangle used as a chord a circle is described; find the locus of the extremities of the diameters parallel to the second diagonal.

10. Being given an angle and a fixed point, one draws through this point an arbitrary secant, and through the points in which this secant intersects the two sides of the angle, one draws straight lines respectively parallel to these sides; find the locus of the point of intersection of these parallels.

11. Find the locus of a point such that on drawing through

this point lines parallel to the asymptotes of a hyperbola, the area of the triangle formed by these parallels and the hyperbola is equal to a given constant.

12. Find the locus of a point such that one of the bisectors of the angles formed by the straight lines which join this point to two fixed points A and B has a given direction.

13. Every equilateral hyperbola circumscribed to a triangle passes through the point of intersection of the altitudes.

14. Being given an ellipse, one draws any two conjugate diameters; find the locus of the point of intersection of one of them with a straight line drawn through a fixed point perpendicular to the other, or, more generally, with a straight line making a given angle with the second diameter.

15. Being given two straight lines $A'A$ and $B'B$ and the point O; about the point O as center, with an arbitrary radius, a circle is described; at the points of intersection of the circle with the straight lines perpendiculars are erected to these lines; find the locus of the points of intersection of these perpendiculars.

CHAPTER VI

CONCERNING THE PARABOLA.

201. The second type to which the equation of the second degree may be reduced is $C'y^2 + 2D'x = 0$, or

(1) $$y^2 = 2px.$$

The case when p is negative can be treated under the case when p is positive by reversing the direction in which one measures the positive abscissas; assume therefore that p is positive. It follows immediately from the form of equation (1) that the curve is symmetrical with respect to the axis of x, and that it passes through the origin. Equation (1), solved with respect to y, gives

$$y = \pm\sqrt{2px}.$$

In order that the ordinate be real, it is necessary that the abscissa be positive; if x increase from 0 to $+\infty$, the absolute value of y increases also from 0 to ∞; thus it follows that the parabola consists of two infinite arcs AD and AD' (Fig. 116).

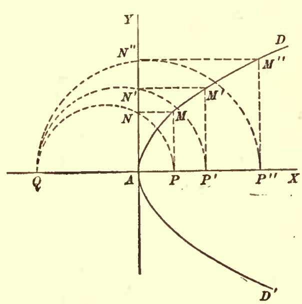

Fig. 116

The straight line AX is the axis of the parabola, the point A is the vertex, the length p, which determines the magnitude of the curve, is called the *parameter* of the parabola.

202. *Construction of the curve by points.* The ordinate MP of the point M is a mean proportional between the constant length $2p$ and the abscissa AP. Construct on AX, in the direction of negative abscissas, a length AQ equal to $2p$;

then describe diverse circumferences whose centers lie on QX, and pass through the point Q; these circumferences intersect the axis AX in the points P, P', ⋯, and the line AY in the points N, N', ⋯. Through the points P, P', ⋯, draw lines parallel to AY; through the points N, N', ⋯, lines parallel to AX; their points of intersection M, M', ⋯, belong to the parabola.

203. From the relations

$$\overline{MP}^2 = 2p \cdot AP, \qquad \overline{M'P'}^2 = 2p \cdot AP',$$

one deduces
$$\frac{\overline{MP}^2}{\overline{M'P'}^2} = \frac{AP}{AP'}.$$

The squares of the ordinates perpendicular to the axis of a parabola are proportional to the segments of the axis comprised between the vertex and the ordinates.

204. Through the point M of the curve, draw a parallel to the axis, and imagine that a movable point travels along this parallel. Replace in the function $y^2 - 2px$, x and y by the coordinates of the movable point; if the point M be situated on the side of the positive abscissas with respect to the parabola, the function will be negative, if the point M be on the other side, the function will be positive. For brevity the first region is said to be *interior* and the second *exterior* to the curve.

205. It has already been shown that the infinite branches of the hyperbola have asymptotes; the same is not true of the parabola. For, since y increases indefinitely with x, there cannot be an asymptote parallel to the axis of the parabola. In the second place, let $y = ax + b$ be the equation of any straight line oblique to the axis, the difference of the ordinates of the points of the line and of the curve which correspond to the same abscissa is equal to

$$ax + b - \sqrt{2px},$$

and can be put under the form

$$x\left(a + \frac{b}{x} - \sqrt{\frac{2p}{x}}\right).$$

When x increases indefinitely, the first factor increases indefinitely, and the second approaches the value a different from zero, the product increases indefinitely. Therefore an asymptote oblique to the axis cannot exist.

Tangent.

206. The tangent at the point M, whose co-ordinates are x and y, has the equation

$$(2) \qquad yY = p(X + x).$$

Let T be the point where the tangent intersects the axis of the parabola (Fig. 117); if in equation (2) one make $Y = 0$, then will $X = -x$; therefore $AT = AP$. This property furnishes a means for constructing the tangent to the parabola at a given point M; to construct the tangent draw MP perpendicular to the axis, take $AT = AP$, and connect the points M and P with a straight line.

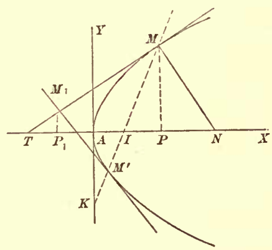

Fig. 117.

207. *To draw a tangent through an exterior point M_1.* Let x_1 and y_1 be the co-ordinates of the point M_1; the points of contact will be determined by the chord of contact

$$(3) \qquad y_1 y = p(x + x_1),$$

combined with that of the curve (1); whence it follows

$$y = y_1 \pm \sqrt{y_1^2 - 2px_1}, \quad x = \frac{y^2}{2p};$$

these values are real so long as the point M_1 is exterior to the curve.

In order to construct the line MM', one seeks the points where it intersects the co-ordinate axes; if, in equation (3), one put $y = 0$, one gets $x = -x_1$, whence AI is equal to AP_1; if one put $x=0$, one finds $y=\frac{px_1}{y_1}$; the point K may be found by a fourth proportional.

208. *To draw a tangent parallel to a given line.* If m represent the angular coefficient of the given line, the equation $\frac{p}{y}=m$, and that of the curve, determine the co-ordinates of the point of contact, $y=\frac{p}{m}$, $x=\frac{p}{2m^2}$. It follows then that the equation of the tangent will be

$$(4) \qquad Y=mX+\frac{p}{2m}.$$

209. Normal. — The normal MN at a point M of the parabola, whose co-ordinates are x and y, has the equation

$$(5) \qquad Y-y=-\frac{y}{p}(X-x).$$

On putting $Y=0$, one obtains the abscissa of the point N where it intersects the axis; one finds

$$PN=X-x=p.$$

Thus, *in the parabola the sub-normal PN is constant and equal to the parameter p.*

Diameters.

210. By applying the general equation of the diameters of a curve of the second degree to the parabola, whose equation is $y^2-2px=0$, one obtains the equation

$$(6) \qquad my-p=0, \text{ or } y=\frac{p}{m}.$$

This property has already been demonstrated in § 134; it is, *that every diameter of the parabola is parallel to the axis.*

Since the angular coefficient m of the chords can be so chosen that $\frac{p}{m}$ can take any value that one chooses, it follows that, conversely, *every straight line which is parallel to the axis is a diameter.*

Let A' be the point of intersection of the diameter with the curve (Fig. 118); since the ordinate of the point A' is equal to $\frac{p}{m}$ and the angular coefficient of the tangent at this point has the value $\frac{p}{y}$, that is m, it follows that *the tangent at the extremity of a diameter is parallel to the chords which this diameter bisects.*

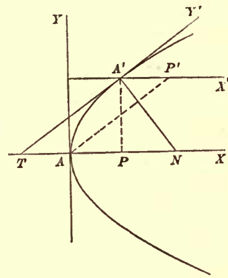

Fig. 118.

211. *Parabola referred to one of its diameters and to the tangent at its extremity.* It has been proven (§ 136) that, in case a diameter $A'X'$ and the tangent $A'Y'$ at its extremity be taken as the axes of co-ordinates, the equation of the parabola will have the form

$$(7) \qquad y^2 = 2p'x.$$

If a and b be the co-ordinates of the point A' with respect to the primitive axes, and AP' be drawn parallel to $A'T$, one knows that one has $A'P' = AT = AP$ (§ 206); the co-ordinates $A'P'$, $-A'T$ of the vertex A with respect to the new axes are therefore a and $-\sqrt{4a^2+b^2}$; since they satisfy equation (7), it follows that

$$2p' = \frac{4a^2+b^2}{a} = \frac{4a^2+2pa}{a} = 2p + 4a.$$

One has also $$p' = \frac{\overline{AP'}^2}{2A'P'} = \frac{\overline{A'T}^2}{TP} = TN.$$

If the angle $Y'A'X'$ formed by the new axes be represented by θ, it follows from the right triangles $NA'T$, $NA'P$, that

$$TN = \frac{A'N}{\sin\theta},\quad A'N = \frac{PN}{\sin\theta},$$

whence $$p' = TN = \frac{PN}{\sin^2\theta} = \frac{p}{\sin^2\theta}.$$

212. Since the parabola, referred to a diameter $A'X$ and to the tangent $A'Y$ (Fig. 119), has the equation $y^2 = 2p'x$, it is evident that the equation $yY = p'(X + x)$ represents either the tangent at the point M, if x and y be the co-ordinates of this point, or the chord of contact of the tangents drawn from an exterior point whose co-ordinates are x and y.

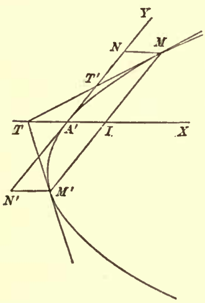

Fig. 119.

The tangents at the two extremities M and M' of a chord intersect the diameter in the same point T, such that $A'T = A'I$. It is also true that the chord of contact MM', with respect to an exterior point T, is bisected by the diameter TX which passes through this point, and for a greater reason is $A'I = A'T$.

This furnishes the means for constructing a parabola by points, in case one knows two tangents TM, TM', and the points of contact M and M'. Draw the chord MM', and join the mid-point I with the point T, the mid-point A' of the straight line TI is a point of the curve, and the tangent at this point is parallel to MM'. By means of the tangent $A'T'$, which touches the curve at A', and of each of the given tangents, one can determine two new tangents by their points of contact, and so on. This method for constructing two parallel lines by means of an arc of a parabola is frequently used, when the arc of a circle cannot be employed; that is, when the distances TM and TM' are not equal.

The Area of a Parabolic Segment.

213. It is proposed to evaluate the area S of the triangle $A'IM$ formed by the straight lines $A'I$, IM and the arc $A'M$ of the parabola (Fig. 119). If this area be regarded as a function of the abscissa of the point M, the derivative S' is given by the formula

$$S' = y \sin\theta = \sqrt{2p'x} \cdot \sin\theta = \sqrt{2p'} \cdot \sin\theta \cdot x^{\frac{1}{2}}.$$

One deduces $$S = \tfrac{2}{3}\sqrt{2p'} \cdot \sin\theta \cdot x^{\frac{3}{2}} + C.$$

The constant C is zero since the area becomes zero for $x = 0$. It follows therefore that

$$S = \frac{2}{3}\, x \cdot \sqrt{2\, p'x} \cdot \sin\theta = \frac{2}{3}\, xy \sin\theta.$$

The area S is equal to two-thirds of the parallelogram $A'IMN$, and, consequently, the area of the composite-line triangle $A'NM$ is one-third of the same parallelogram.

EXERCISES.

1. Find the locus of the vertex of an angle circumscribed about a parabola, such that the triangle formed by the sides of the angle and the arc of the parabola has a constant area.

2. Find the locus of points from which two perpendicular normals can be drawn to a parabola.

3. A secant revolves about a fixed point taken on the axis of a parabola; normals are drawn to the parabola at the points in which the secant intersects it; find the locus of the point in which these normals intersect.

4. A parabola moves parallel to itself, so that its vertex traces the parabola in its initial position; tangents are drawn from the vertex of the fixed to the movable parabola; find the locus of the points of contact.

5. Find the locus of a point from which the sum of the squares of the normals drawn to a parabola is constant.

6. Given a curve of the second degree tangent to the sides of a given angle, one draws an arbitrary tangent to this curve; find the locus of the point of intersection of the medians or the altitudes of the triangle formed by the movable tangent and the sides of the angle; find also the locus of the center of the circle circumscribed about this triangle.

7. Given an ellipse, one draws through a fixed point any two straight lines at right angles to each other, and at the points in which these lines intersect the ellipse, tangents are drawn to this ellipse; find the locus of the points of intersection of these tangents.

8. Same problem, when one replaces the perpendicular lines by lines parallel to the conjugate axes of any other given ellipse.

9. An angle of constant magnitude revolves about its vertex situated on a given curve of the second degree; at the points in which the sides of the angle meet the curve again, tangents are drawn to the curve. Find the locus of the point of intersection of these tangents.

10. Find the locus of the center of an equilateral triangle formed by three tangents or by three normals to a parabola.

11. The area of a triangle whose vertices are the points of contact of three tangents to a parabola is twice the area of the triangle formed by these tangents, and is represented by the expression

$$\pm\frac{1}{4p}(y'-y'')(y''-y''')(y'''-y'),$$

where y', y'', y''' represent the perpendiculars dropped from the vertices of the triangle to the axis.

12. An arbitrary tangent is drawn to a hyperbola, and the points in which the tangent meet the asymptotes are respectively joined to two fixed points; find the locus of the point of intersections of the two straight lines.

13. Draw to a parabola a normal so that the area comprised between this normal and the curve has a minimum value.

CHAPTER VII

FOCI AND DIRECTRICES.

215. The discussion is begun by proposing the following question: Given a point F and a straight line DE (Fig. 120), find the locus of a point whose distances from a given point and a given straight line are in a constant ratio.

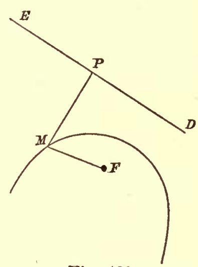

Fig. 120.

Draw in the plane any system of rectangular axes; call α and β the co-ordinates of the point F, and let $mx + ny + h = 0$ be the equation of the line DE; the distances of any point M of the plane from the point F and from the line DE are given by the formulas

$$MF = \sqrt{(x-\alpha)^2 + (y-\beta)^2}, \quad MP = \frac{\pm(mx+ny+h)}{\sqrt{m^2+n^2}};$$

if the constant ratio $\frac{MF}{MP}$ be designated by k, the locus will have the equation

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = \pm\frac{k(mx+ny+h)}{\sqrt{m^2+n^2}},$$

or

$$(x-\alpha)^2 + (y-\beta)^2 = \frac{k^2(mx+ny+h)^2}{m^2+n^2}.$$

This locus is a curve of the second degree. The quantity $AC - B^2$, which serves to distinguish the species of the curve, being equal to $1 - k^2$, the curve is an ellipse, a parabola, or a hyperbola, according as the ratio k is less, equal to, or greater than unity.

Conversely, given a curve of the second degree one proposes to seek if there exist in the plane of the curve a fixed point F

and a fixed straight line DE, such that the ratio of the distances of each of the points of the curve from the point F and the line DE is constant. If one find a point and a straight line enjoying this property, the point will be called the *focus* of the curve, and the straight line the *directrix*.

The axes of co-ordinates being arbitrary and inclosing an angle θ, suppose that one has found a point F whose co-ordinates are α and β, and a straight line DE whose equation is $mx + ny + h = 0$, such that the ratio $\frac{MF}{MP}$ is equal to a constant quantity k; since the distance MP of a point M of the curve whose co-ordinates are x and y from the directrix DE is represented by the expression

$$\frac{\pm (mx + ny + h) \sin \theta}{\sqrt{m^2 + n^2 - 2\, mn \cos \theta}},$$

one will have

$$MF = \pm \frac{k(mx + ny + h) \sin \theta}{\sqrt{m^2 + n^2 - 2\, mn \cos \theta}}.$$

Thus the distance of any point M of the curve from the focus F is expressed as an integral function of the co-ordinates x and y of the point M and is of the first degree.

Conversely, if a point F enjoy the property that its distance from any point M of the curve is expressed by an integral function of the first degree in the co-ordinates x and y of the point M, this point F is the focus; that is, that there exists a straight line DE such that the ratio of the distances of each of the points of the curve from the point F and the line DE is constant. In fact, assume that one has

$$FM = \pm (mx + ny + h),$$

where $mx + ny + h$ represents an integral function of the first degree in the co-ordinates x and y of the point M. Consider the straight line DE which has the equation

$$mx + ny + h = 0;$$

the distance of the point M from this line is given by the formula

$$MP = \frac{\pm (mx + ny + h) \sin \theta}{\sqrt{m^2 + n^2 - 2\,mn \cos \theta}};$$

one has therefore

$$\frac{MF}{MP} = \frac{\sqrt{m^2 + n^2 - 2\,mn \cos \theta}}{\sin \theta}.$$

Thus the ratio of the distances of each of the points of the curve from the fixed point F and the fixed line DE is constant; the point F is therefore a focus and the line DE the corresponding directrix of the curve. Representing the value of this constant ratio by k, one has

$$k \sin \theta = \sqrt{m^2 + n^2 - 2\,mn \cos \theta}.$$

216. Therefore the following definition can be substituted for the first. The focus is a point such that its distance from any point of the curve can be expressed by an integral function of the first degree in the variable co-ordinates of a point of the curve. It is clear, moreover, that this algebraic definition is independent of the position of the co-ordinate axes in the plane, because an integral function of the first degree preserves its character in case the axes are changed. The equation of the directrix is found by equating this function to zero.

If the y-axis be taken parallel to the directrix, the x-axis being arbitrary, the equation of the directrix will take the form $mx + h = 0$, the coefficient n will be zero and the distance of the focus from any point M of the same will be expressed by an integral function $\pm (mx + h)$ of the first degree in the abscissa x of the point M.

From what precedes it follows that the investigation of the focus and the directrix of curves of the second degree is reduced to the determination of a point F, such that its distance from any point M of the curve is expressed by an integral function of the first degree in the co-ordinates x and y of the point M. Suppose that the axes are rectangular, and let

$$(1) \qquad Ax^2 + Bxy + Cy^2 + 2\,Dx + 2\,Ey + F = 0$$

be the equation of the given curve of the second degree. Call α and β the co-ordinates of the focus sought; then will the co-ordinates of every point of the curve satisfy the equation

$$\sqrt{(x-\alpha)^2+(y-\beta)^2}=\pm(mx+nx+h),$$

or (2) $$(x-\alpha)^2+(y-\beta)^2-(mx+ny+h)^2=0.$$

Equations (1) and (2), representing at the same time the same curve, are identical, hence the coefficient of corresponding terms must be proportional; one will have, therefore, to determine the five unknown quantities α, β, m, n, h, the five equations

(3) $$\frac{1-m^2}{A}=\frac{-mn}{B}=\frac{1-n^2}{C}=\frac{-(\alpha+mh)}{D}$$
$$=\frac{-(\beta+nh)}{E}=\frac{\alpha^2+\beta^2-h^2}{F}.$$

In order to simplify the calculation, one considers separately the three curves of the second degree, referred to systems of rectangular axes which have served to simplify their equations. Later will be given another method for finding the foci, especially useful for finding the geometrical loci of the foci.

Foci and Directrices of the Ellipse.

217. Let

(4) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

be the equation of the given ellipse referred to its axes. This equation does not have any term in xy; it is necessary therefore that the coefficient $-2\,mn$ of the corresponding term in equation (2) be zero, whence it follows that one has either $n=0$, or $m=0$. Suppose that $n=0$; since the terms of the first degree are also zero, one will have $\alpha+mh=0$, $\beta=0$, and equations (3) reduce to

$$a^2(1-m^2)=b^2=h^2-\alpha^2.$$

Whence one deduces $m^2 = \frac{a^2 - b^2}{a^2}$; since m can always be supposed positive, without changing the signs of the coefficients m, n, h in equation (2), one takes $m = \frac{\sqrt{a^2 - b^2}}{a}$. If in the equation $a^2(1 - m^2) = h^2 - \alpha^2$, h be replaced by its value deduced from the equation $\alpha + mh = 0$, one gets $\alpha^2 = a^2 - b^2$, whence $\alpha = \pm\sqrt{a^2 - b^2}$, $h = \mp a$.

Thus are obtained the two foci F and F' (Fig. 121), situated on the major axis at equal distances from the center. In order to determine them, one describes with a radius equal to a about the extremity B of the minor axis as center, a circle; the points F and F' where this circle intersects the major axis, are the foci. If, for brevity, one put $a^2 - b^2 = c^2$, one has $\alpha = \pm c$, $m = \frac{c}{a}$, $h = \mp a$; the upper signs correspond to the focus F, the lower signs to the focus F'. One knows that the equation of the directrix is found by equating to zero the polynomial $mx + ny + h$; this equation reduces to $\frac{c}{a}x \mp a = 0$, or $x = \pm\frac{a^2}{c}$. Thus are obtained the two directrices; the directrix whose equation is $x = \frac{a^2}{c}$ corresponds to the focus F, and the directrix whose equation is $x = -\frac{a^2}{c}$ corresponds to the focus F'. These directrices are perpendicular to the major axis and at equal distances from the center; the determination of the point D depends upon a third proportional; one constructs it in the following manner: describe on the major axis as diameter a circle, draw through the focus F a perpendicular to this axis and, at the point N where this perpendicular meets the circle, draw a tangent to the circle; the point in which this tangent intersects the major axis is the point D.

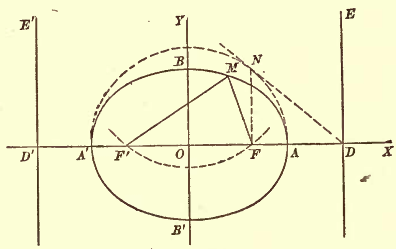

Fig. 121.

It has also been seen that the constant ratio of the distances of each of the points of the curve from the focus and from the corresponding directrix is equal to $\sqrt{m^2+n^2}$, in rectangular co-ordinates; one has therefore $k=m=\frac{c}{a}\cdot$ The ratio $\frac{c}{a}$ is called e, the eccentricity of the ellipse.

218. Suppose now $m=0$; the coefficients of the first degree should be zero; one will have $\alpha=0$, $\beta+nh=0$, and equations (3) reduce to

$$a^2=b^2(1-n^2)=h^2-\beta^2.$$

Whence may be deduced

$$n=\frac{\sqrt{b^2-a^2}}{b},\ \beta=\pm\sqrt{b^2-a^2},\ h=\mp b.$$

In order to obtain these new solutions, it suffices to permute in the first solutions the letters a and b, m and n, α and β. Since a has been supposed greater that b, these two solutions are imaginary. Thus one can assign to the constants four systems of values which render equations (2) and (4) identical; but two only of these systems of values which give the foci and the directrices are real.

219. THEOREM I. — *The sum of the distances of each of the points of an ellipse from the foci is constant.*

The distance of a focus from any point M of the curve is expressed by $\pm(mx+ny+h)$, that is $\pm\left(a-\frac{\alpha x}{a}\right)$; the sign is so chosen that the quantity will be positive. The abscissas α and x of the focus, and of a point of the ellipse being less than a in absolute value, and, consequently, the quantity within the parentheses is positive for every point of the ellipse; it will be necessary, therefore, to give the parentheses the sign $+$, and one will have

$$MF=a-ex,\ MF'=a+ex;$$

whence it follows

$$MF+MF'=2a.$$

220. Corollary I. — *The sum of the distances of a point within the ellipse from the foci is less than the major axis; the sum of the distances of a point without is greater than the major axis.*

Consider first the point N (Fig. 122), situated within the ellipse, join this point to the two foci, and prolong the straight line $F'N$ till it intersects the ellipse in M. Since the point M belongs to the ellipse, the sum of the two radii vectors $MF + MF'$ is equal to the major axes AA'; but the straight line NF is shorter than the broken line $NM + MF$; by adding to each of these expressions the same length $F'N$, it follows that the path $F'N + NF$ is shorter than $F'M + MF$, that is, less than AA'. Consider next a point P situated without the ellipse; the line PF' intersects the ellipse at a point M. The broken line $MP + PF$ is greater than the straight line MF; on adding to both the same length $F'M$, one sees that the path $F'P + PF$ is greater than $F'M + MF$, that is, greater than AA'. It is clear that the converse propositions are true. If the sum of the distances of a point of the plane from the two foci be less than the major axis, this point will lie within the ellipse. If the sum be greater than the major axis, the point will lie without. Whence it follows that one can consider the ellipse as the locus of the points of which the sum of the distances from the two foci is equal to $2a$. Thus is the ellipse constructed in elementary geometry, and it is on this property that the construction of the ellipse by points depends, or on a continuous motion, of which mention has been made at the beginning (§ 11).

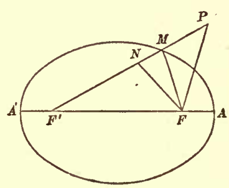

Fig. 122.

221. Corollary II. —*The ellipse in the locus of points equally distant from the focus F and the circle described about the other focus F' as center with a radius equal to the major axis.* If the foci be joined to any point M of the ellipse with straight lines (Fig. 123), and if the radius vector $F'M$ be prolonged till MH is equal to MF, one obtains a constant length $F'H$ equal to the major axis; the locus of the point H is therefore the circumference described about the focus F' as center with the major axis as radius. The portion MH of the radius being the shortest path from the point M to the circumference, the point M of the ellipse is equally distant from the focus F and the circumference. The name *director circle* has been given to this circle.

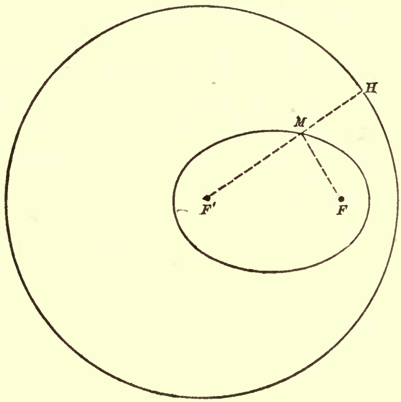

Fig. 123.

222. Theorem II. — *A tangent to the ellipse makes equal angles with the radii vectores, which are drawn from the point of contact to the foci.*

Take two points M and M' (Fig. 124) on the ellipse; about the focus F as center, with FM as radius, construct the arc of a circle which intersects the radius vector FM' at C; the length $M'C$ represents the difference of the two radii vectores FM and FM', or the increment which the radius vector FM receives when the point M has been moved to the neighboring point M'. Similarly, if one describe about the focus F' as center with the radius $F'M$ an arc of a circle which intersects in D the radius vector $F'M'$ produced,

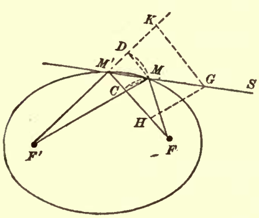

Fig. 124.

the length $M'D$ will represent the difference of the two radii vectores $F'M$ and $F'M'$, or the *negative* increment which the radius vector $F'M$ receives when the point M has been moved to the point M'. Thus, when the point M moves to the point M', the radius vector FM is increased by the increment $M'C$, while the other radius vector $F'M$ is diminished by the increment $M'D$. Since the sum of the two radii vectores $FM + F'M$ remains constant, the quantity by which the one is increased is equal to that by which the other is diminished, and, consequently, the two lengths $M'C$ and $M'D$ are equal.

Draw through the two points M and M' the secant MS; draw in the two circles previously constructed the two chords MC and MD. Lay off on the secant MS the arbitrary but invariable length MG, and through the point G draw GH parallel to MC, GK parallel to MD; from the preceding construction it follows that one has the equal ratios

$$\frac{M'C}{M'H} = \frac{M'M}{M'G} = \frac{M'D}{M'K};$$

since the two lengths $M'C$ and $M'D$ are equal, it follows that the two lengths $M'H$ and $M'K$ are also equal. Suppose now that the point M' approaches continually the point M; the secant MS will approach a limiting position MT (Fig. 125), which is a tangent to the ellipse. The points C and D will at the same time approach the point M, the chords MC and MD, prolonged, approach the tangents to the circles described about the points F and F' as centers with FM and $F'M$ as radii, and, consequently, become perpendicular to the radii FM and $F'M$; their parallels GH and GK take also directions perpendicular to the same radii, and, consequently, the angles H and K become right. The limits of the two triangles $M'GH$, $M'GK$ (Fig. 124) are two right-angled triangles MGH, MGK (Fig. 125); these two triangles, having the common hypotenuse MG and the sides MH and MK, are equal, since they are the limits of equal lengths; whence it follows that the two angles GMH, GMK are equal. Therefore the tangent MT to the ellipse bisects the angle FMK

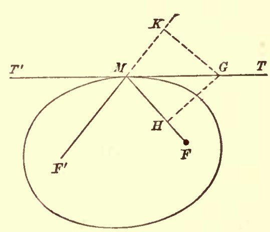

Fig. 125.

formed by the radius vector MF and the prolongation of the other $F'M$.

The vertical angles $F'MT'$ and GMK being equal, one sees that the tangent TT' makes, with the two radii vectores drawn from the point of contact, the equal angles FMT, $F'MT'$.

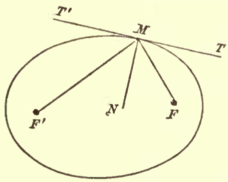

Fig. 126.

223. Corollary I. — At the point M (Fig. 126) draw to the tangent TT' a perpendicular MN; it will be a normal to the ellipse. The two angles FMN, $F'MN$ are equal, since they are the complements of the equal angles FMT, $F'MT'$; thus, *the normal to the ellipse at the point M bisects the angle FMF' formed by the radii vectores which are drawn from this point to the two foci.*

224. Corollary II. — Suppose that a light be placed at the focus F (Fig. 127) of an ellipse; the rays of light, emanating from the point F, are reflected on the ellipse, making the angle of reflection equal to the angle of incidence. Let FM be one of these rays; draw to the ellipse at this point the tangent TT'; the reflected ray, which makes with MT' an angle equal to FMT, will be reflected along MF'. Thus the reflected rays will all be concurrent at the second focus F', where they form a very brilliant image of the flame placed at the first focus F. It is on this account that the points F and F' are called *foci*.

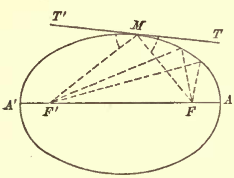

Fig. 127.

225. Corollary III. — Conversely, the ellipse is the only curve which enjoys the property that the radii vectores which are drawn from the point of contact to the two fixed points F and F' and make equal angles with the tangent. Seek, in fact, the equation of the curve in bi-polar co-ordinates (§ 4), and represent by u and v the radii vectores MF, MF' (Fig.

124). When the point M of the curve moves to the point M', the two radii vectores u and v receive the increments,

$$\Delta u = + M'C, \ \Delta v = - M'D,$$

and one has
$$\frac{\Delta v}{\Delta u} = -\frac{M'D}{M'C} = -\frac{M'K}{M'H}.$$

When the point M' approaches indefinitely the point M, the straight line MM' becomes a tangent and the two angles at H and K, as has been stated, become right. One supposes, moreover, the two angles GMH, GMK (Fig. 125) equal to each other; the two right triangles GMH, GMK are therefore equal; one has $MH = MK$, and the ratio $\frac{\Delta v}{\Delta u}$ approaches a limit equal to -1. If one consider v as a function of u, one sees that the derivative of this function is equal to -1; on returning to the primitive function, one has $v = -u + C$, and, consequently, $u + v = C$. Therefore the curve is an ellipse.

226. Corollary IV. — *The locus of the projections of the foci on the tangents to the ellipse is the circle described on the major axis as a diameter.* Prolong the radius vector $F'M$ till MH is equal to MF; the tangent bisecting the angle FMH is perpendicular to the straight line FH at its mid-point (Fig. 128); join this point to the center O of the ellipse. The straight line OI, which bisects the two sides FF', FH of the triangle $F'FH$, is parallel to the third side $F'H$, and equal to its half; the length $F'H$ being equal to the major axis AA', the distance OI is constant and equal to OA. Therefore the locus of the point I is the circumference of the circle described about the point O as center, with OA as radius.

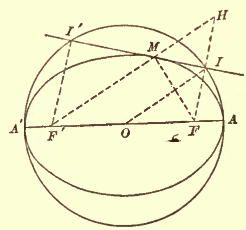

Fig. 128.

227. Problem I. — *To draw a tangent to an ellipse at a given point M on the ellipse.*

This problem has already been solved, by considering the ellipse as the projection of a circle. The same questions will be treated by another method which is applicable to the hyperbola and the parabola.

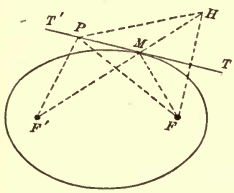

Fig. 129.

Prolong the radius vector $F'M$ (Fig. 129) till MH is equal to the other radius vector MF, and draw through the point M a straight line TT' perpendicular to FH; one will have the tangent required. Because in the isosceles triangle FMH, the line MT, drawn from the vertex perpendicular to the base FH, bisects the vertical angle. This line, being the bisector of the angle FMH formed by one of the radii vectores and the prolongation of the other, coincides with the tangent to the ellipse.

228. Remark. — One should notice that all of the points of the tangent, excepting the point of contact M, lie without the ellipse. Let P be any point of the tangent; join this point to the foci and to the point H. The tangent being perpendicular to FH at its mid-point, the distance PF is equal to PH, and, consequently, the broken line $F'P + PF$ is equal to the broken line $F'P + PH$; but the latter is greater than the line $F'H$, which is equal to the major axis of the ellipse, since the radius vector MF' was prolonged till MH is equal to MF. Since the sum of the distances of the point P from the foci is greater than the major axis, this point is situated without the ellipse.

The broken line $F'M + MF$ is the shortest path going from the point F' to a point on the tangent and then to the point F.

A broken line is said to be *convex*, in case it is situated on the same side with respect to each of its sides indefinitely prolonged. Similarly, a curve is said to be convex in case it lies entirely on the same side of every tangent to it indefinitely produced. Accordingly, it follows that the ellipse is a closed convex curve.

229. Problem II. — *To draw to an ellipse a tangent from an external point P.*

Assume that the problem is solved, and let *PM* (Fig. 130) be a tangent passing through the point *P*. If the radius vector *F'M* be prolonged till *MH* is equal to *MF*, it follows that the tangent *PM* is perpendicular to the straight line *FH* at its mid-point; it remains therefore to determine the point *H*. Since the line *F'H* is equal to the major axis *AA'*, the point *H* is on the circumference described about the focus *F'* as a center with *AA'* as a radius. On the other hand,

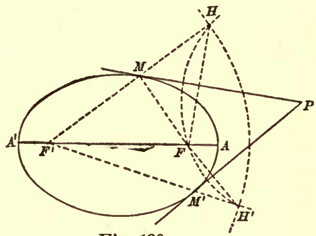

Fig. 130.

the distance *PH* being equal to *PF*, the point *H* is on the circumference described about the point *P* as a center with *PF* as a radius; the point *H* is therefore the intersection of these two circumferences. The following construction may be inferred from the preceding: Describe about the focus *F'* as center, with a radius equal to the major axis, a circle. Describe about the point *P* as center, with a radius equal to the distance *PF* of this point from the other focus, a second circle, which intersects the first in *H*. Join *F* and *H* by a straight line and draw from the point *P* a perpendicular to *FH*; the perpendicular will be the tangent required. The point of contact *M* will be determined by the intersection of the tangent with the line *F'H*. The two circles intersect in a second point *H'*; on drawing from the same point *P* a perpendicular to *FH'*, a second tangent *PM'* will be determined, whose intersection with the straight line *F'H'* will be the point of contact *M'*.

These constructions can be accomplished without drawing the ellipse. It is sufficient that the foci and the major axis be known.

230. Problem III. — *To draw to an ellipse a tangent which is parallel to a given straight line KL.*

Assume the problem to be solved, and let *ST* be a tangent parallel to *KL* (Fig. 131). If *F'M* be prolonged till *MH* is equal to MF, one knows that the tangent is perpendicular to FH at its mid-point. Whence the following construction can be inferred: Describe a circle about the focus F' as center, with a radius equal to the major axis; draw through the other focus F a straight line FH perpendicular to the given line KL; this line will intersect the circumference in a point H; draw ST perpendicular to FH at its mid-point; ST will be the tangent required. The point of contact will be determined by the intersection of the tangent with the straight line $F'H$. The straight line FH, prolonged, intersects the circumference in a second point H'; on erecting a perpendicular to FH' at its mid-point, a second tangent $S'T'$ will be found, whose point of contact M' will be determined by the intersection of $F'H'$ with $T'S'$.

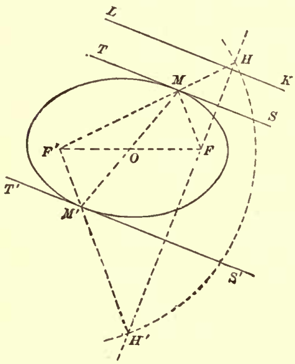

Fig. 131.

231. Problem IV.—*An ellipse is defined by its foci and its major axis. Determine the points of its intersections with a given straight line MM'.*

Let M be one of the points where the given straight line intersects the ellipse (Fig. 132); connect this point by straight lines to the two foci, and prolong the radius vector $F'M$ till MH is equal to MF; the point H belongs to the director circle described about F' as a center; if a circle be described about M as a center with a radius equal to MF, this circle will be tangent to the director circle at H; on dropping from the focus F a perpendicular upon this given line, and prolonging it till the line is double its origi-

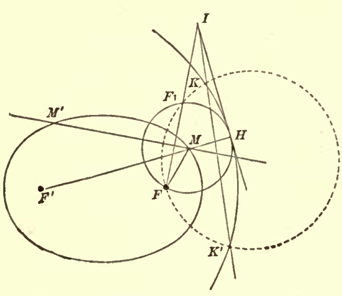

Fig. 132.

nal length, a second point F_1 is found belonging to this same circle. The problem is reduced, therefore, to finding the center M of a circle passing through two given points F and F_1 and tangent to the director circle. For this purpose one constructs through the two given points F and F_1 any circle which intersects the director circle in two points K and K'; draw from the point I, the intersection of the two lines FF_1 and KK', a tangent to the director circle; the point M, where the line $F'H$ intersects the given line, will be the point sought.

One has, in fact,

$$\overline{IH}^2 = IK \times IK' = IF \times IF_1;$$

therefore, the circle which passes through the three points F, F_1, H, is tangent to the director circle at H. Since two tangents can be drawn from the point I to the director circle, there will be two points M and M'.

When the point F_1, which is the *symmetrique* of the focus F with respect to the given straight line, is situated within the director circle, there are practically two solutions. In case the point F_1 is on the circle, the line is tangent to the ellipse. Finally, when the point F_1 is situated without the circle, the line does not intersect the ellipse.

Foci and Directrices of the Hyperbola.

232. Since the equation of the hyperbola referred to its axes is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

it is sufficient to replace b^2 by $-b^2$ in the results derived for the ellipse. One has then the two real solutions

$$\beta = 0, \quad \alpha = \pm\sqrt{a^2 + b^2} = \pm c,$$

and the polynomial of the second degree is the square of the polynomial of the first degree $a - \frac{\alpha x}{a}$. The remaining two solutions are imaginary.

The hyperbola has therefore two real foci F and F', situated on the transverse axis and at equal distances from the center (Fig. 133). They are found by drawing through the vertex A a straight line AG perpendicular to the transverse axis, meeting the asymptote in G, and laying off on the transverse axis the lengths OF and OF' equal to OG.

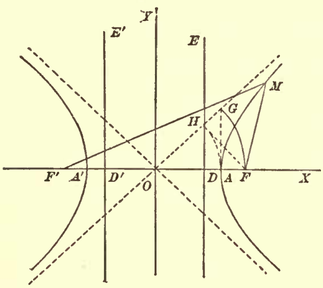

Fig. 133.

The equation of the directrix is $x = \frac{a^2}{a}$. The directrix DE corresponds to the focus F, and the directrix $D'E'$ to the focus F'. Describe about the point O as center, with OA as radius, an arc of a circle which intersects the asymptote in the point H; the point belongs to the directrix. The two triangles OAG, OHF, which have a common angle O, the sides OA and OG respectively equal to OH and OF, are equal, and the angle OHF is right; if a perpendicular HD be dropped from the point H on the transverse axis OA, one has $\overline{OH}^2 = OF \times OD$, and, consequently, $OD = \frac{a^2}{c}$. Thus the line DH is the directrix.

The constant ratio $k = \sqrt{m^2 + n^2}$ is equal to $\frac{c}{a}$; this is the *eccentricity* of the hyperbola; it is usually represented by the letter e.

233. Theorem III. — *The difference of the distances of each of the points of the hyperbola from the two foci is constant and equal to the transverse axis.*

The distance from a focus to any point M of the curve is represented by $\pm\left(a - \frac{\alpha x}{a}\right)$. The abscissas α and x of the focus and of a point of the hyperbola being in absolute value greater than a, the second term is greater in absolute value than a. It is necessary, therefore, to give the $-$ or $+$ sign to the preceding parenthesis, according as this point M is on the right or left branch, and as its distance is measured to the one or the other of the foci. In case of the right branch, one has

$$MF = -a + ex, \quad MF' = a + ex;$$

whence $$MF' - MF = 2a.$$

In case of the left branch

$$MF = a - ex, \quad MF' = -a - ex;$$

whence $$MF - MF' = 2a.$$

234. Corollary I. — *The difference of the distances of a point situated between the two branches of a hyperbola from the two foci is less than the transverse axis; in case the point is situated in either of the other two portions of the plane, the difference is greater than the transverse axis.*

Let P be a point situated between the two branches of the curve (Fig. 134); the straight line PF meets the hyperbola at the point M. One has

$$PF' - PM < MF';$$

if MF be subtracted from each member of the preceding inequality, it becomes

$$PF' - PF < MF' - MF;$$

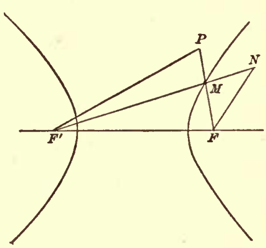

Fig. 134.

this last difference is equal to $2a$ and therefore the first is less than $2a$. Suppose now that P is situated to the right of the first branch of the hyperbola; the straight line NF' intersects this branch in M; one has

$$NF < NM + MF,$$

and on adding to each member MF',

$$NF + MF' < NF' + MF;$$

whence $$NF' - NF > MF' - MF.$$

The second difference being equal to $2a$, the first is greater than $2a$.

Whence it follows that the hyperbola may be considered as the locus of the points, such that the difference of their distances from the two foci is equal to $2a$. The construction of the hyperbola by points or by a continuous motion, given in the beginning (§ 14), depends upon this property.

235. Corollary II. — *The distance of any point M of the hyperbola from the focus F is equal to one of the normals drawn from this point to the circle described about the other focus F' as center with a radius equal to the transverse axis.* For a point M of the first branch (Fig. 135), one has

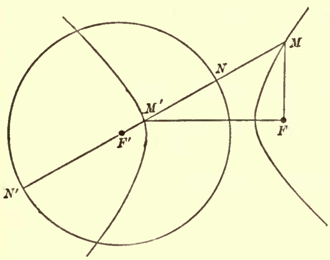

Fig. 135.

$$MF' - MF = 2a = F'N,$$

and, consequently,

$$MF = MF' - F'N = MN.$$

For a point M' of the second branch, one has

$$M'F - M'F' = 2a = F'N',$$

and, consequently,

$$M'F = M'F' + F'N = M'N'.$$

In the first case, the portion MN of the normal represents the distance of the point M from the circle, and the first branch of the hyperbola is the locus of points which are equally distant from the focus F and from the director circle.

236. Theorem IV. — *A tangent to a hyperbola bisects the angle formed by the radii vectores which are drawn from the point of contact to the foci.*

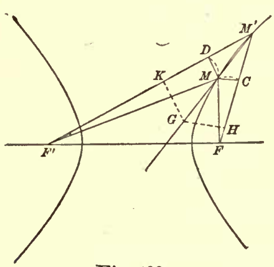

Fig. 136.

Let M and M' be two consecutive points on the hyperbola (Fig. 136). About the focus F as center describe with MF as radius the arc of a circle which intersects the radius vector FM' in C; about the focus F' as center describe an arc of a circle with a radius equal to $F'M$ which intersects the radius vector $F'M'$ in D; as the point M moves to the point M', the two radii vectores receive the increments $M'C$ and $M'D$; since the difference between the vectores is constant, these two increments are equal to one another.

Lay off on the secant MM' an arbitrary length MG, and draw through the point G, GH parallel to the chord MC and GK parallel to the chord MD. From this construction, it follows

$$\frac{M'C}{M'H}=\frac{M'M}{M'G}=\frac{M'D}{M'K};$$

since the two lengths $M'C$ and $M'D$ are equal, it follows that the two lengths $M'H$ and $M'K$ are also equal. When the point M' approaches indefinitely the point M, the secant MM' approaches a limiting position and becomes the tangent to the hyperbola at the point M; at the same time, the chords MC and MD become tangents to the circles described about the foci as centers and, consequently, perpendicular to FM and $F'M$; the lines GH and GK, which are parallel to the chords, become also perpendicular to these same radii vectores, and the angles H and K become right angles. The two triangles $M'GH$, $M'GK$, which have a common side $M'G$ and a side $M'H$ equal to $M'K$, become therefore right-angled, consequently equal to each other; whence it follows that the angles $GM'H$, $GM'K$ become equal; thus the tangent to the hyperbola at the point M is the bisector of the angle FMF'.

237. Corollary I. — The hyperbola is the only curve which possesses this property; because on calling the radii vectores u and v, and their increments Δu and Δv, one has

$$\frac{\Delta v}{\Delta u}=\frac{M'D}{M'C}=\frac{M'K}{M'H}.$$

If it be supposed that the angles $GM'H$, $GM'K$ become equal when the point M' approaches indefinitely the point M, the two triangles $GM'H$, $GM'K$ become equal and also the sides $M'H$ and $M'K$; whence

$$\lim\frac{\Delta v}{\Delta u}=1.$$

On returning to the primitive function, one has

$$v=u+C, \text{ whence } v-u=C.$$

238. Corollary II. — *An ellipse and a confocal hyperbola intersect at right angles.*

Two curves of the second degree are said to be confocal when their foci coincide; the angle at which two curves intersect is the angle formed by their tangents at the point of intersection.

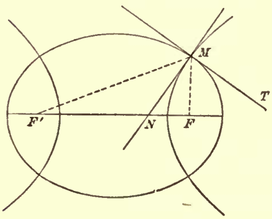

Fig. 137.

Let M be the point of intersection of an ellipse and a hyperbola which has the same foci F, F' (Fig. 137); the bisector MN of the angle FMF' is, on the one hand, perpendicular to the ellipse, on the other, tangent to the hyperbola; therefore the tangents MT, MN to the curves are perpendicular to each other.

239. Problem V. — *To draw a tangent to a hyperbola at a given point M of the hyperbola.*

Take on the radius vector MF' a length MH equal to the other radius vector MF, and draw through the point M a line MP perpendicular to FH; one has the tangent required (Fig. 138).

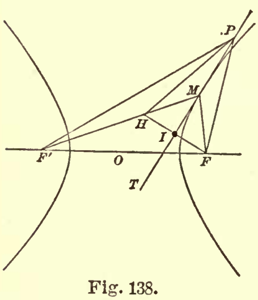

Fig. 138.

Remark. — It should be noticed that the tangent is wholly situated between the two branches of the hyperbola. Let P be any point of this tangent; one has

$$PF' - PH < F'H,$$

and, consequently,

$$PF' - PF < 2a;$$

therefore the point P lies between the two branches of the hyperbola. One branch of the hyperbola, lying always on the same side of any tangent to it, is a convex curve when viewed from any point of said tangents.

The tangent being perpendicular to FH at its mid-point I, the point I is the projection of the focus F on the tangent. The straight line OI, which is parallel to $F'H$ and equal to the half of $F'H$, is constant; whence it follows that the locus of the projections of the foci on the tangents is the circle described on the transverse axis as diameter.

240. Problem VI. — *To draw a tangent to a hyperbola from any point P situated between its branches.*

Let PM be a tangent passing through the point P (Fig. 139). If $MH = MF$ be subtracted from the radius vector MF', one knows that the tangent PM is perpendicular to FH at its mid-point. The problem is reduced to determining the position of the point H; this point will be the intersection of the circle described about the focus F' as center, with a radius equal to $2a$, and the circle described about the point P as center, with a radius equal to PF. The tangent will be formed by drawing from the point P a perpendicular to FH, and the point of contact M will be determined by prolonging the radius vector $F'H$. These two circles intersect in a second point H'; a second tangent will be found by drawing a line through P perpendicular to FH'; the point of contact of the tangent will be determined by the prolongation of the straight line $F'H'$.

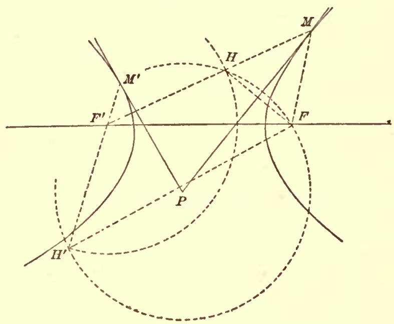

Fig. 139.

In case the point P is on one of the asymptotes, one of the tangents drawn from P coincides with this asymptote, and the point of contact is removed to infinity.

241. Problem VII. — *To draw to a hyperbola a tangent which is parallel to a given straight line OL.*

One constructs about the focus F' as center, with a radius equal to $2a$, the director circle, and draws from the focus F a straight line perpendicular to OL (Fig. 140); this straight line intersects the circle in two points H and H'; straight lines are drawn through the mid-points of the lines FH and FH' parallel to OL; these parallels will be the tangents required.

The points of contact M and M' are determined by the lines $F'H$, $F'H'$.

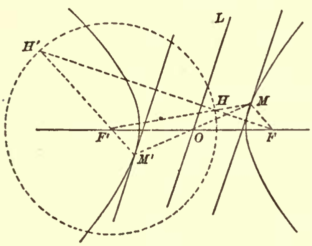

Fig. 140.

In order that the problem be possible, it is necessary that the given straight line, which can be assumed to be drawn through the center, does not intersect the hyperbola; for, then the perpendicular FH' drawn from the focus F will intersect the director circle in two points.

242. PROBLEM VIII. — *To find the points of intersection of a straight line and of a hyperbola defined by its foci and its transverse axis.*

The construction is precisely the same as for the ellipse.

THE FOCUS OF THE PARABOLA.

243. The equation of the parabola, referred to its axis and to the tangent at the vertex, is

$$y^2 - 2px = 0.$$

Since this equation contains neither a term in xy nor one in x^2, one should have, according to the general relations of § 216, $mn = 0$, $1 - m^2 = 0$; whence $n = 0$, $m = 1$. Because the coefficient of the term in y and the constant term are also zero, one has $\beta = 0$, $\alpha^2 - h^2 = 0$. Moreover, equations (3) of § 216 reduce to $1 = \frac{\alpha + h}{p}$; that is, $\alpha + h = p$. The equation $\alpha^2 - h^2 = 0$ or $(\alpha + h)(\alpha - h) = 0$ becomes $p(\alpha - h) = 0$, that is, $\alpha - h = 0$; whence it follows that $\alpha = h = \frac{p}{2}$.

Here one has a single solution. Thus the parabola possesses a single focus situated on its axis at a distance from its vertex A equal to the half of its parameter (Fig. 141). The polynomial of the second degree being the square of the polynomial of the first degree $x+\frac{p}{2}$, the distance FM is equal to $x+\frac{p}{2}$. To the focus corresponds the directrix DE, whose equation is $x=-\frac{p}{2}$; the directrix is perpendicular to the axis at a distance AD, equal to AF, from the vertex.

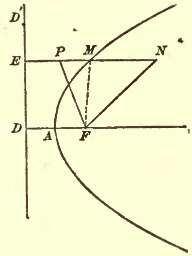

Fig. 141.

The constant ratio $l=\sqrt{m^2+n^2}$ reduces in this case to unity; whence it follows that every point of the parabola is equally distant from the focus and the directrix.

244. THEOREM V. — *The distance of any point within the parabola from the focus is less than its distance from the directrix; on the contrary, the distance of every point without from the directrix is less than its distance from the focus.*

Consider accordingly a point N which lies within the parabola; draw a perpendicular from this point to the directrix, and connect it with a straight line to the focus. The perpendicular intersects the curve in a point M which is joined to the focus. Since the point M belongs to the parabola, the distances ME and MF are equal. But the straight line NF is shorter than the broken line $NM+MF$; if MF be replaced by its equal ME, it follows that the distance NF is less than NE. Thus the internal point N is nearer to the focus than to the directrix. Consider next an external point P situated between the curve and the directrix. Connect it with the focus and draw to the directrix a perpendicular PE which is prolonged till it intersects the curve in M. Since the point M belongs to the parabola, the distances MF and ME are equal; the straight line MF, or its equal ME, is shorter than the broken line $MP+PF$; if MP be subtracted from each member of the inequality, it follows that PE is shorter than PF. In case the point P lies to the left of the directrix, it is evidently nearer to the directrix than to the focus.

It follows from the preceding discussion that the parabola may be regarded as the locus of points, each of which is equally distant from the focus and the directrix. It is in this manner that the parabola is defined in elementary geometry, and it is by means of this property that the parabola is constructed point by point, or by a continuous motion. as has already been described (§ 16).

245. Theorem VI. — *The tangent to a parabola makes equal angles with the diameter and focal vector drawn from the point of contact.*

Take on the parabola two consecutive points M and M' (Fig. 142), which we join to the focus and from which we drop perpendiculars ME and $M'E'$ upon the directrix. Construct an arc MC of a circle, about the focus F as a center, with a radius FM, and draw from M, MC' parallel to the directrix. The length $M'C$ is the difference of the two radii vectores FM' and FM; it is the increment which the radius vector FM receives when the point M moves to M'. Similarly, the length $M'C'$ is the difference of the two perpendiculars $M'E'$, ME; it is the increment which the perpendicular ME receives when the point M is removed to M'. Since the radius vector MF is always equal to the perpendicular ME, it follows that the two increments $M'C$ and $M'C'$ are equal.

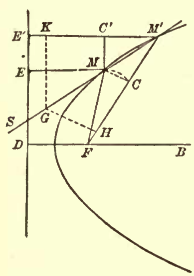

Fig. 142.

Draw through the points M and M' the secant MS and construct the chord MC in the circle described about the focus as center. Take on the secant MS an arbitrary length MG, and draw through the point G, GH parallel to MC, and GK parallel to MC'. On account of these parallels, one has the equal ratios $\frac{M'C}{M'H} = \frac{M'M}{M'G} = \frac{M'C'}{M'K}$; since the two lengths $M'C$, $M'C'$ are equal, the two lengths $M'H$ and $M'K$, which are proportional to them, are also equal.

Suppose now that the point M' approaches indefinitely the point M; the secant MS will approach a limiting position and be tangent to the parabola; the chord MC prolonged will, in a similar manner, approach a limiting position and be tangent to the circle, and consequently become perpendicular to the radius FM; the parallel GH takes also a direction perpendicular to FM. Whence it follows that the two triangles $M'GH$, $M'GK$ have as limits the two right-angled triangles MGH, MGK (Fig. 143); these two right triangles, having the common hypotenuse MG, and the two sides MH and MK equal to each other since they are the limits of equal lengths, are equal; hence the two angles GMK and GMH are equal. Therefore the tangent MT to the parabola bisects the angle FME, formed by the radius vector and the perpendicular dropped from the point of contact to the directrix. If EM be prolonged, the two vertical angles GMK and $T'ML$ will be equal and, consequently, the two angles FMT, $T'ML$, formed by the tangent with a line parallel to the axis and the radius vector FM, are equal.

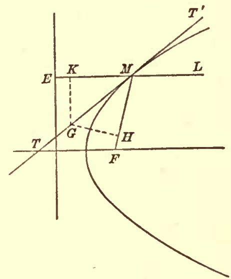

Fig. 143.

246. Corollary I. — Suppose that a light be placed at the focus F (Fig. 144) of the parabola; the rays of light, emanating from the focus F, are reflected on meeting the parabola, making the angle of reflection equal to the angle of incidence. Let FM be one of the rays; draw at the point M a tangent to the parabola; the reflected ray, making the angle LMT' equal to the angle FMT, will be parallel to the axis AB of the parabola. Similarly, every reflected ray will be parallel to the axis.

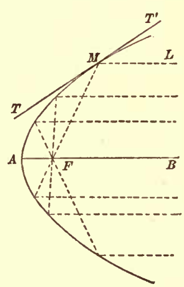

Fig. 144.

It is by means of this property of the parabola that the reflectors used in reflector telescopes and coach lamps are constructed. The interior surface, of well-polished metal, is produced by a parabola revolving about its axis; a light is placed at the focus; the luminous rays, after reflection, all become parallel to the axis; the reflector projects a pencil of parallel rays which are propagated without dispersing, and which light, therefore, at the greatest distance.

Corollary II. — Suppose that luminous rays, parallel to the axis, fall upon a parabolic mirror; after reflection, they will all converge to the focus.

Parabolic mirrors are used in the construction of telescopes. The axis is directed toward the star; the luminous rays coming from the star are reflected on the mirror, and form at the focus a very brilliant image of the star.

The parabolic form is used in the construction of speaking-trumpets and certain acoustical instruments.

247. Corollary III. — Conversely, the parabola is the only curve which enjoys this property that the tangent at any point of the curve makes equal angles with the radius vector drawn from a fixed point to the point of contact, and with a straight line drawn from the point of contact parallel to a fixed straight line. Imagine that any point M of the plane be determined by its distance MF from a fixed point F, and its distance ME from a straight line DE perpendicular to the fixed straight line FB (Fig. 142); represent these two co-ordinates by u and v (§ 17). As any point M of the curve is moved to a neighboring point M', these two co-ordinates receive the increments

$$\Delta u = M'C,\ \Delta v = M'C',$$

and one has

$$\frac{\Delta v}{\Delta u} = \frac{M'C'}{M'C} = \frac{M'K}{M'H}.$$

As the point M' approaches indefinitely the point M, the straight line MM' becomes tangent and the angle at H becomes a right angle. The two triangles GMH, GMK are at the limit right-angled and equal (Fig. 143), since they have a common hypotenuse and the angle GMH equal to GMK by hypothesis. Therefore one has

$$\lim \frac{\Delta v}{\Delta u} = 1,$$

and returning to the primitive function $v = u + C$. On removing the line DE a distance equal to the constant C, it follows that $v = u$.

248. PROBLEM IX. — *To draw a tangent at a given point of a parabola.*

FIRST METHOD. — Let T (Fig. 145) be the point in which the tangent prolonged intersects the axis, ME the perpendicular drawn from the point M to the directrix. It is known that the tangent bisects the angle FME; the angle FTM being equal to the alternate interior angle TME, and, consequently, to the angle FMT, it follows that the triangle TFM is isosceles, and the two sides FM, FT are equal. Hence, in order to construct the tangent at the point M, it is sufficient to lay off on the axis a length FT equal to the radius vector FM, and draw TM. This method is not practical in case the point M is very near the vertex A of the parabola; for then the two points M and T, being very near to each other, do not determine the tangent with sufficient precision. For this particular case the following method is used.

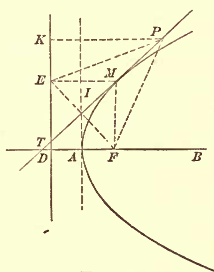

Fig. 145.

SECOND METHOD. — The tangent MT bisects the angle at the vertex M of the isosceles triangle FME, and is perpendicular to the base FE at its mid-point. Thus, in order to construct the tangent a perpendicular ME is drawn from the point M to the directrix, and a second perpendicular is drawn from the point M to the straight line FE.

It follows from this construction that the tangent at the vertex A of the parabola is perpendicular to the axis of the parabola.

REMARK. — Every point of the tangent, excepting the point of contact M, lies without the parabola. Let P be any point of the tangent; it is perpendicular to FE at its mid-point; therefore, the distances PE, PF are equal; but the oblique line PE is greater than the perpendicular PK; therefore, the distance PF is greater than PK, and, consequently, the point P is without the parabola. Whence it follows that the parabola is a convex curve when viewed from any point on a tangent.

249. Corollary. — *The locus of the projections of the focus upon tangents to the parabola is the tangent at the vertex.*

In fact, it is seen that the point *I*, mid-point of *FE*, and the projection of the focus upon the tangent, lie on a parallel to the directrix drawn through the point *A*, the mid-point of *FD*, that is, on the tangent at the vertex *A*.

250. Problem X. — *To draw a tangent from an external point P to a parabola.*

Assume that the problem is solved, and let *PM* (Fig. 146) be a tangent passing through the point *P*. If a perpendicular *ME* be drawn from the point *M* to the directrix, and the points *E* and *F* be joined, it follows that the tangent *PM* is perpendicular to *FE* at its mid-point; whence it follows that the distance *PE* is equal to *PF*, and one has the construction required: a circle is described about *P* as a center, having a radius equal to the distance *PF* of this point from the focus and intersecting the directrix in the point *E*. Join the points *F* and *E*, and draw from *P* a perpendicular to *FE*; it will be the tangent required. The point of contact *M* is determined by the intersection of the tangent with the line drawn through the point *E* parallel to the axis.

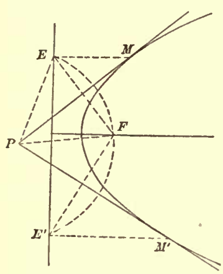

Fig. 146.

The circle intersects the directrix in a second point *E'*. In a similar manner a perpendicular is drawn from the point *P* to *FE'*, and a second tangent is constructed.

These constructions can be accomplished without tracing the parabola. It is only necessary that the focus and directrix be known.

251. Problem XI. — *To draw to a parabola a tangent which is parallel to a given straight line KL.*

Assume that the problem is solved, and let *MT* be the tangent required. If a perpendicular *ME* be drawn from the point of contact to the directrix, and the points *E* and *F* be joined, then the tangent is perpendicular to FE at its mid-point.

Whence the following construction is deducible: Draw through the focus F a straight line perpendicular to the given line KL, and produce it till it meets the directrix in E, and at the mid-point of FE erect a perpendicular TM, which will be the tangent required. The point of contact M will be determined by drawing through the point E the line ME parallel to the axis.

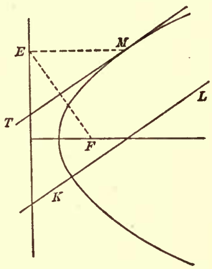

Fig. 147.

252. PROBLEM XII. — *To find the point of intersection of a given straight line and of a parabola defined by its focus and directrix.*

Let the point F_1 be a point which is symmetrical to the focus with respect to the given line (Fig. 148). The point M, being equally distant from the points F, F_1, and the directrix, is the center of a circle passing through these two points and tangent to the directrix. In order to determine the point of contact E, one lays off on the directrix, beginning at the point I in which the straight line F_1F intersects the directrix, to the one side or to the other, a length IE which is a mean proportional between the two lengths IF, IF_1; thus are the two points of intersection M and M' determined.

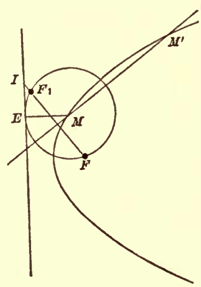

Fig. 148.

In case the point F_1, the symmetrique of the focus with respect to the given line, is situated to the right of the directrix, there are two solutions. When the point F_1 is on the directrix, the line is tangent to the parabola. Finally, when the point F_1 lies to the left of the directrix, the straight line cannot intersect the parabola.

253. THEOREM VII. — *The limiting case of an ellipse or of a hyperbola whose parameter remains finite, while the major or minor axis increases indefinitely, is a parabola.*

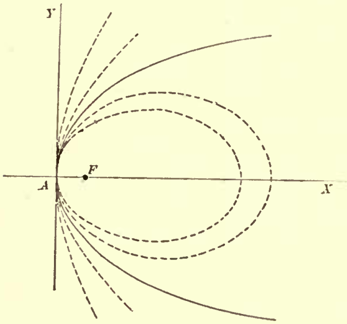

Fig. 149.

The ordinate at the focus in the parabola is equal to the parameter p; by analogy, the ordinate at the focus in the ellipse and the hyperbola is called the *parameter;* it is equal to $\frac{b^2}{a}$ and is represented by p. The ellipse, referred to its major axis and to the tangent at the left vertex (Fig. 149), has an equation of the form

$$y^2 = \frac{2b^2}{a}x - \frac{b^2}{a^2}x^2, \text{ or } y^2 = 2px - \frac{p}{a}x^2.$$

Assume now that the vertex remains fixed, and the parameter p remains finite, while the major axis $2a$ is allowed to increase indefinitely; the equation of the ellipse is reduced to the equation $y^2 = 2px$, which represents a parabola. If the points, which correspond to the same value of x, be considered, one sees that each point of the parabola is the limiting position toward which the corresponding point of the ellipse tends when a is increased indefinitely; it is this that is implied in saying that the parabola is the limit of the ellipse.

The equation of the hyperbola, referred to its major axis and to the tangent at the vertex A, is

$$y^2 = 2px + \frac{p}{a}x^2;$$

if a be allowed to increase indefinitely, the parameter p remaining finite, this equation will also reduce to

$$y^2 = 2px.$$

The parabola is the limit of the branch of the hyperbola to which the vertex A belongs; the other branch is removed indefinitely toward the left.

In the preceding discussion we have supposed that the parameter of the ellipse or the hyperbola remains finite. The same conclusion is reached, on supposing that the distance AF of the vertex A from the neighboring focus F remains finite. In fact, on calling α this distance, one has, for the ellipse,

$$p = \frac{b^2}{a} = \frac{a^2 - c^2}{a} = \frac{(a-c)(a+c)}{a} = \alpha\left(2 - \frac{\alpha}{a}\right);$$

since the parameter p has as limit the finite quantity 2α, the equation of the ellipse reduces to $y^2 = 4\alpha x$. The same will be the case for the hyperbola.

254. Remark. — This transformation of the ellipse into the parabola is important. It allows the deductions of the properties of the parabola from those of the ellipse as particular cases. Thus, in the ellipse, the diameter, or the locus of a system of parallel chords, is a straight line passing through the center; if it be supposed that the center is removed to infinity, the ellipse is transformed into the parabola, and the diameters become parallel to the axis. The ellipse is the locus of points equally distant from the focus F and from the director circle described about the focus F' as center (§ 221). If the focus F' be removed to infinity, the director circle becomes the directrix of the parabola.

The tangent to the ellipse makes equal angles with the radii vectores drawn from the point of contact to the foci (§ 222); if the focus F' be removed to infinity, the radius vector MF becomes parallel to the axis.

255. Theorem VIII. — *If two tangents be drawn to a curve of the second degree, the straight line FP, which is drawn from the focus F to the point of intersection P of the two tangents, is the bisector of the angle formed by the radii vectores FM, FM', drawn from F to the points of contact of the tangents, or the* *external angle, according as the two tangents touch the same branch of the curve or two different branches.*

Consider two tangents PM, PM' of an ellipse (Fig. 150); prolong the radius vector $F'M$ till MH is equal to MF, and similarly FM' till $M'H'$ is equal to $M'F'$; the tangents being perpendicular at the mid-points of FH and $F'H'$, it follows that

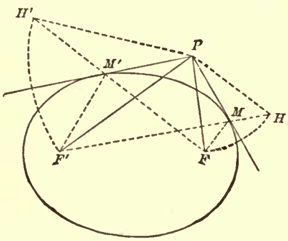

Fig. 150.

$$PH = PF, \quad PH' = PF',$$

and the two triangles $F'PH$, $H'PF$ are equal, since the three sides of the one are equal each to each to the three sides of the other, namely,

$$F'H = FH' = 2a, \quad PH = PF, \quad PF' = PH';$$

whence it follows that the angles PHM, PFM' are equal. But the angle PHM is equal to the angle PFM, therefore the angles PFM, PFM' are equal and the straight line FP is the bisector of the angle MFM'.

The same discussion holds in case the locus is a hyperbola, when the two tangents touch the same branch; but in case the tangents touch two different branches, the line FP is the bisector of the angle formed by one of the radii vectores FM and the prolongation of the other.

Consider, finally, the case when the curve is a parabola (Fig. 151). From the points of contact draw the perpendiculars MH, $M'H'$ to the directrix; since the tangents are perpendicular to FH and FH' at their mid-points, the angles PFM, PFM' are equal respectively to the angles PHM, $PH'M'$. The straight lines PH and PH', being each equal to the straight line PF, are equal to each other, and the triangle HPH' is isosceles. The angles PHM, $PH'M'$, complements of the equal angles of the isosceles triangle, are equal to each other; therefore the angles PFM, PFM' are equal. This result may otherwise be obtained immediately on regarding the parabola as the limiting case of an ellipse.

256. THEOREM IX. — *Tangents drawn from an exterior point P to an ellipse or a hyperbola, make equal angles with the straight lines drawn from this point to the foci.*

In the two equal triangles $F'PH$, $H'PF$ (Fig. 150), one has the two equal angles $F'PH$, $H'PF$; on subtracting the common part $F'PF$, one has $FPH = F'PH'$, and, on taking half of the remainders, one obtains $FPM = F'PM'$.

The same property belongs to the parabola, considered as the limiting case of an ellipse; it suffices to replace the radius vector PF' by a straight line PI parallel to the axis (Fig. 151). It is easy, moreover, to demonstrate this property directly. If about the point P as a center, a circle be described with a radius equal to PF, this circle will pass through the points H and H'; the angles MPI, FHH' are equal, since their sides are respectively perpendicular; but the inscribed angle FHH' is the half of the angle FPH' at the center, and, consequently, equal to the angle FPM'; therefore the angles MPI, $M'PF$ are equal.

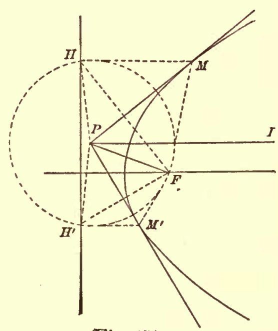

Fig. 151.

257. THEOREM X. — *The straight line FK, which joins the focus of a curve of the second degree with the point in which any secant intersects the directrix, is the bisector of the external angle formed by the radii vectores emanating from the focus to the points in which the secant cuts the curve or bisector of the angle included by the same radii vectores, according as the two points of intersection, M and M', are situated on the same branch, or different branches of the curve.*

Draw from the points M and M' perpendiculars to the directrix (Fig. 152); one has

$$\frac{MF}{ME} = \frac{M'F}{M'E'},$$

and, consequently,

$$\frac{MF}{M'F} = \frac{ME}{M'E'} = \frac{MK}{M'K}$$

In case the two points M and M' belong to the same branch of the curve, since the point K lies on the prolongation of the chord MM', the straight line FK is the bisector of the external angle of the triangle MFM'. In case the points M and M' belong to two different branches, since the point K is situated between the points M and M', the straight line FK is the bisector of the angle MFM'.

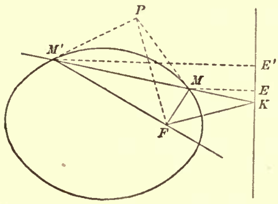

Fig. 152.

258. Theorem XI. — *If tangents be drawn from any point P on the directrix to a curve of the second degree, the chord of contact MM' passes through the corresponding focus F, and is perpendicular to the straight line FP which joins the point P to the focus* (Fig. 153).

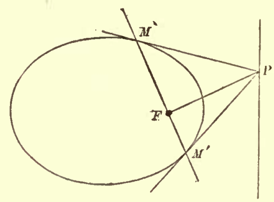

Fig. 153.

Let the tangent PM be the limiting position of a secant of the ellipse whose points of intersection with the ellipse are made to coincide; then it follows from the preceding theorem that the line FP is perpendicular to FM; it is for the same reason perpendicular to FM'; therefore the line MFM' will be a straight line perpendicular to FP.

259. Theorem XII. — *The product of the distances of the two foci from the tangent of an ellipse or a hyperbola is constant.*

Let FH, $F'H'$ be the perpendiculars dropped from the foci upon a first tangent (Fig. 154), FK, $F'K'$ the perpendiculars dropped upon a second tangent, P the point of intersection of the two tangents. Then by Theorem IX. it follows that the right triangles FPH, $F'PK'$ are similar, so also are the triangles FPK, $F'PH'$, and one has

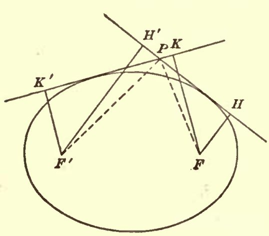

Fig. 154.

$$\frac{FH}{F'K'} = \frac{FP}{F'P} = \frac{FK}{F'H'};$$

whence $\quad FH \cdot F'H' = FK \cdot F'K'.$

If the curve be an ellipse, in drawing the tangent parallel to the major axis, it follows that the constant product is equal to b^2. When the curve is a hyperbola, if the asymptotes be regarded as the limiting position of tangents, one sees also that the product is equal to b^2.

260. PROBLEM XIII. — *To construct a curve of the second degree, given the focus F and three points A, B, C.*

Assume that the problem is solved and that the three points belong to the same branch; the point D, where the secant AB is met by the bisector of the exterior angle of the triangle AFB, is on the directrix (§ 257); the secant BC will determine in a similar manner a second point D' on the directrix. The focus F, the directrix DD', and the point A define a curve of the second degree and one only; it will be an ellipse, a parabola, or a hyperbola, according as the distance AF is less, equal to, or greater than the distance AE of the point A from the directrix. It is easily seen that this curve passes through the two points B and C; for, on account of the bisector FD, one has

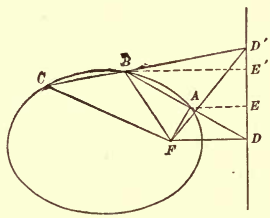

Fig. 155.

$$\frac{AF}{BF} = \frac{AD}{BD} = \frac{AE}{BE'},$$

and, consequently, $$\frac{AF}{AE} = \frac{BF}{BE'};$$

therefore the curve passes through the point B. It can be shown in a similar manner that the curve passes through C. This gives one solution.

It is possible that the three points are not on the same branch; if, for example, the two points, A and B, are on the same branch and the point C on the other branch of the hyperbola, the bisectors of the angles AFC, BFC will determine two points on the directrix. The three solutions found in this manner are hyperbolas. One has therefore, in all, four solutions; of these four curves of the second degree to which the given focus belongs and on which the three given points lie, three are always hyperbolas, the fourth is an ellipse, a hyperbola, or a parabola depending upon the disposition of the points.

261. A calculation will lead to the same result; let α and β be the co-ordinates of the focus, x' and y', x'' and y'', x''' and y''', the co-ordinates of the three given points, δ', δ'', δ''', their distances from the focus; the equation of the curve can be put under the form

$$(x-\alpha)^2+(y-\beta)^2-(mx+ny+h)^2=0,$$

where $mx+ny+h=0$, is the equation of the directrix. One can determine the three constants m, n, h by means of the three equations of the first degree:

$$\delta' = \pm(mx' + ny' + h),$$

$$\delta'' = \pm(mx'' + ny'' + h),$$

$$\delta''' = \pm(mx''' + ny''' + h).$$

Each combination of signs furnishes a system of equations; there are eight combinations; but it is to be noticed that, if the signs be changed in the three equations, the values of m, n, h change signs, and the curve is the same; therefore there are only four solutions.

The distance of a point from a straight line is expressed by a formula affected with a double sign; the same sign should be taken for any point lying on one side and the opposite sign for any point situated on the other side of the line. One knows that the ellipse lies wholly on the same side with respect to each directrix; the parabola is also situated on the same side of its directrix, but, however, the two directrices of the hyperbola lie between the two branches of the curve. When the three points lie on the same branch, their distances from either directrix have the same sign; in case, however, two of the points lie on one branch and the third on the other branch, these distances take different signs.

262. PROBLEM XIV. — *Construct a curve of the second degree, when one focus and three tangents are given.*

Assume that the problem is solved; if perpendiculars be dropped from the given focus upon the given tangents, and each prolonged a length equal to itself, three points, H, H', H'', are determined, belonging to the director circle (Fig. 156) whose center is at the second focus F'; the radius $F'H$ of this circle is equal to the axis $2a$ which passes through the two foci. The two foci F, F', along with the length $2a$, define a curve of the second degree, and one only. It is easy to see that this curve is tangent to the three given lines, for let M be the point in which the line $F'H$ intersects the straight line MT, the sum or the difference of the radii vectores MF' and MF being equal to $F'H$ or to $2a$, the point M belongs to the curve; further, the straight line MT, being perpendicular to FH at its mid-point, is tangent to the curve at the point M. The problem has thus one, and one solution only.

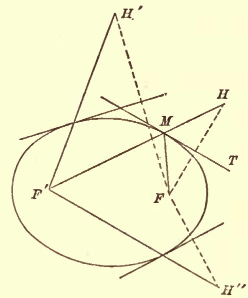

Fig. 156.

If the three points H, H', H'' should lie on the same straight line, the curve sought would be a parabola having this line for its directrix.

TRINOMIAL EQUATION COMMON TO THE THREE CURVES OF THE SECOND DEGREE.

263. If a point O of a curve of the second degree be taken for the origin, the diameter meeting the curve in this point for the x-axis, and the tangent at this point for the y-axis, the equation of the curve takes the form

$$y^2 = 2px + qx^2.$$

In fact, let

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

be the equation of the curve referred to the axes mentioned. Since the curve passes through the origin and is tangent to the axis Oy, one has

$$F = E = 0;$$

since the axis Ox is the diameter conjugate to the chords parallel to the axis Oy, the equation should contain only the second power of y, because to each value of x there should correspond two equal values of y with opposite signs; therefore $B = 0$. The coefficient C is not zero, because if it were, the conic would reduce to two straight lines parallel to the axis Oy. Hence one can solve the equation with respect to y^2 and obtain an equation of the form given above. The curve is an ellipse, a hyperbola, or a parabola, according as q is negative, positive, or zero.

Take, in particular, the point in which the focal axis meets the curve for the origin, and the direction in which one looks from this point toward the nearest focus for the positive direction of the axis Ox. Whence the coefficients p and q will have the following values:

1° *Ellipse.* On calling a and b the axes of the ellipse, one ought to have $y=0$ for $x=2a$, and $y^2=b^2$ for $x=a$. Therefore

$$pa + qa^2 = 0, \quad 2pa + qa^2 = b^2;$$

whence $$p = \frac{b^2}{a}, \quad q = -\frac{b^2}{a^2} = \frac{c^2}{a^2} - 1 = e^2 - 1.$$

2° *Hyperbola.* One should have $y = 0$ for $x = -2a$, and $y^2 = -b^2$ for $x = -a$. Therefore

$$-pa + qa^2 = 0, \quad -2pa + qa^2 = -b^2$$

whence $$p = \frac{b^2}{a}, \quad q = \frac{b^2}{a^2} = \frac{c^2}{a^2} - 1 = e^2 - 1.$$

3° *Parabola.* Here q is equal to zero, and p is the parameter.

In general, therefore, on taking the point in which the straight line drawn through the foci intersects the curve for the origin, and the straight line drawn from this point to the nearest focus for the x-axis, one can put the equation of the three curves under the form

$$y^2 = 2px + (e^2 - 1)x^2,$$

in which p is the parameter and e the eccentricity, which is greater than unity for the hyperbola, less than unity for the ellipse, and equal to unity for the parabola.

The Equations of the Curves of the Second Degree in Polar Co-ordinates.

264. A focus F is chosen as the pole, and the perpendicular drawn from this focus to the corresponding directrix DE is taken for the polar axis.

Consider now the ellipse. The ratio of the distances of any point M of the curve to the focus and to the directrix being constant and equal to the eccentricity, one has

$$\frac{MF}{ME} = e, \text{ or } MF = ME \cdot e.$$

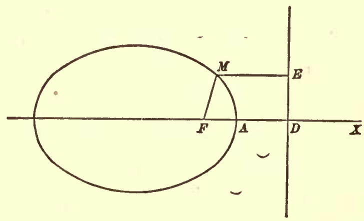

Fig. 157.

The distance FD of the focus to the directrix is equal to $\frac{b^2}{c}$. On projecting the broken line FME (Fig. 157) upon the axis, one has

$$\rho \cos \omega + ME = FD = \frac{b^2}{c}, \quad \omega = \angle XFM,$$

whence
$$ME = \frac{b^2}{c} - \rho \cos \omega;$$

on replacing ME by its value in the preceding equation, one finds

(1) $$\rho = \frac{p}{1 + e \cos \omega}.$$

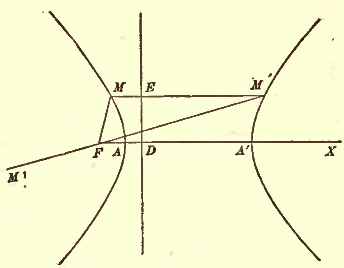

Fig. 158.

If the curve be a hyperbola (Fig. 158), the same calculation is applicable to the branch A, whose vertex is nearer the focus F taken for the pole. When the point M' is on the branch A', situated on the other side of the directrix, the projection of the broken line $FM'E$ gives

$$\rho \cos \omega - M'E = FD,$$

which leads to the equation

$$(2) \qquad \rho = \frac{-p}{1 - e \cos \omega}.$$

Moreover, if the negative radii vectores be constructed in a sense contrary to the direction indicated by the angle ω, it is easily seen that equation (1) represents the two branches of the hyperbola. Let M' be any point of the second branch, ω' the corresponding angle $A'FM'$, ρ' the radius vector FM'; owing to equation (2), one has $\rho' = \frac{-p}{1 - e \cos \omega'}$. If in equation (1), the value $\omega' + \pi$ be substituted for the angle ω, it will become

$$\rho = \frac{p}{1 - e \cos \omega'} = -\rho'.$$

Thus a negative value $-\rho'$ is obtained for ρ. But the value $\omega + \pi$ assigned to ω indicates the direction FM_1 opposite to FM'; if ρ have a positive value, it will be necessary to measure it in the direction FM_1; ρ having a negative value $-\rho'$, one measures the absolute value ρ' in the opposite direction; that is, in the direction FM', which determines the point M'. Whence it follows that equation (1) suffices to represent the two branches of the hyperbola, the first by the positive values of ρ, the second by the negative values.

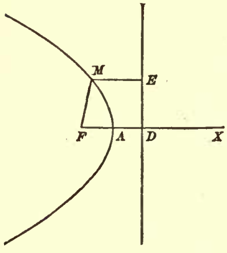

Fig. 159.

The calculation given for the ellipse is applicable to the parabola (Fig. 159); it suffices to put $e = 1$. It follows therefore that equation (1) represents the three curves of the second degree; the curve is an ellipse, a parabola, or a hyperbola according as the eccentricity e is less, equal to, or greater than unity.

EXERCISES.

1. If P be the point of intersection of the tangents drawn to a parabola at the points M and M' and F the focus, prove that

$$\frac{PM^2}{MF}=\frac{PM'^2}{M'F}.$$

2. In case of a curve of the second degree, show that the perpendicular dropped from the focus upon a chord and the diameter conjugate to this chord intersect on the directrix.

3. A semi-diameter of an ellipse or of a hyperbola is a mean proportional between the straight lines which join the foci to the extremity of the diameter conjugate to the first.

4. Show that the distance of any point of an equilateral hyperbola from its center is a mean proportional between the distances of this point from the foci.

5. Find in the plane of an ellipse a circle such that the length of the tangent drawn from every point of the ellipse to the circle is a rational, integral function of the first degree in the co-ordinates of this point.

Prove that the sum or the difference of the tangents drawn from every point of the ellipse to two circles which enjoy the preceding property is constant.

6. Find the locus of the vertex of a constant angle which is circumscribed about a parabola.

7. A chord is drawn through the focus of a parabola, and a circle is constructed on this chord as a diameter, then tangents are drawn to the circle parallel to a given straight line; find the locus of the points of contact.

8. A constant angle revolves about the focus of a curve of the second degree; tangents are drawn to the curve at the points in which the sides of the angle meet this curve; find the locus of the point of intersection of the tangents.

9. A tangent is drawn to a given ellipse at any point M and is prolonged till it intersects the tangents at the extremities of the major axis in P and Q; find the point of intersection N of the straight lines $F'P$ and FQ, and of the point of intersection N' of the straight lines FP and $F'Q$. Show that the two points N and N' are situated on the normal at the point M.

10. A curve of the second degree is given, and a secant revolves about a fixed point P; the focus F is joined to the points M and M' in which the secant intersects the curve; show that the product $\tan\frac{PFM}{2}\cdot\tan\frac{PFM'}{2}$ is constant.

11. Show that the portion of a tangent comprised between two fixed tangents to a curve of the second degree subtends a constant angle whose vertex is at a focus of this curve.

12. Prove that the point of intersection of the altitudes of a triangle circumscribed about a parabola is on the directrix, and that the circle circumscribed about the triangle passes through the focus.

13. If, at any point M of an ellipse, a normal be drawn, the portion of this normal comprised between the point M and the minor axis has for its projection on the radii vectores drawn from the point M to the two foci a length equal to the semi-major axis.

14. Prove that the portion of the normal comprised between the point M and the major axis has for its projection on the radii vectores a length equal to the parameter of the ellipse.

15. Two curves of the second degree have a common focus; if radii vectores be drawn from this focus to the extremities of any diameter of one of the curves, the sum or the difference of the ratios of these radii vectores to the radii vectores of the second curve, which have the same direction, is constant.

16. If the radii vectores which are drawn from any point M of an ellipse be prolonged till they intersect the curve at P and Q, show that the sum $\frac{MF}{FP}+\frac{MF'}{F'Q}$ is constant.

17. A mariner's compass composed of m rays revolves about its center placed at the focus of an ellipse; show that the sum of the inverse of the lengths intercepted on each ray between the focus and the point where it intersects the ellipse, is constant.

18. From any point P situated in the plane of an ellipse, tangents are drawn to this ellipse; a perpendicular PC is dropped from the point P to the chord of contact AB; the straight lines PC and AB intersect the minor axis in D and E;

show that the circle described on DE as a diameter passes through the two foci.

19. Being given two confocal ellipses, through a point P one draws to one of them tangents which intersect the second, the one in A and B, the other in C and D; demonstrate that

$$\frac{1}{PA} \pm \frac{1}{PB} = \frac{1}{PC} \pm \frac{1}{PD}.$$

20. A circle is described on the major axis of an ellipse as a diameter; the ordinate of any point M of the ellipse intersects the circle in a point N; if ω be the angle which the radius vector FM makes with the major axis, and U the angle which the radius vector ON of the circle makes with the major axis, one has the relations

$$\rho = a(1 - e\cos u), \qquad \tan\frac{\omega}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}.$$

On representing the area of the elliptic sector AFM by S, one has also

$$S = \frac{ab}{2}(u - e\sin u).$$

21. An equilateral hyperbola confocal to an ellipse intercepts, on the sides of a right angle circumscribed about an ellipse, two equal chords.

22. If one call R the radius of the circle circumscribed about a triangle which is inscribed in a parabola, c, c', c'' the chords drawn from the focus parallel to the sides of the triangle, θ, θ', θ'' the angles which the sides of the triangle make with the axis, one has

$$R\sin\theta'\cdot\sin\theta'' = p, \quad 8pR^2 = cc'c''.$$

23. Let A be the vertex, F the focus of a parabola, (ρ, ω), (ρ', ω') the co-ordinates of two points M and M' of the curve, θ the angle MFM', S the area of the sector AFM, A that of the sector MFM', l the length of the chord MM'; demonstrate the following formulas used in astronomy:

$$p = \frac{2\rho\rho'\sin^2\frac{\theta}{2}}{\rho + \rho' - 2\sqrt{\rho\rho'}\cos\frac{\theta}{2}}, \quad 2\sqrt{\rho\rho'}\cdot\cos\frac{\theta}{2} = \sqrt{(\rho+\rho')^2 - l^2},$$

$$S=\frac{1}{6}(p+\rho)\sqrt{p(2\rho-p)}=\frac{p^2}{12}\tan\frac{\omega}{2}\left(1+\tan^2\frac{\omega}{2}\right),$$

$$A=\frac{1}{3}\sqrt{\rho\rho'}\sin\frac{\theta}{2}\left(\rho+\rho'+\sqrt{\rho\rho'\cos\frac{\theta}{2}}\right)$$

$$=\frac{1}{6}\left(\rho+\rho'+\sqrt{\rho\rho'}\cos\frac{\theta}{2}\right)\sqrt{2p\left(\rho+\rho'-2\sqrt{\rho\rho'}\cos\frac{\theta}{2}\right)}$$

$$=\frac{\sqrt{2p}}{6}\left[\left(\frac{\rho+\rho'+l}{2}\right)^{\frac{3}{2}}-\left(\frac{\rho+\rho'-l}{2}\right)^{\frac{3}{2}}\right].$$

24. Consider an ellipse referred to its two axes, and an external point P whose co-ordinates are α and β; two tangents are drawn from the point P to the ellipse, and each of the points of contact are joined to the two foci. Demonstrate: 1. That the distances of the point P from any of the four lines thus constructed is equal to $\frac{1}{a}\sqrt{a^2\beta^2+b^2\alpha^2-a^2b^2}$, a and b representing the semi-axes of the curve; 2. That the sum or the difference of the tangents drawn from the two foci to the circumference of the circle described about P as a center, with a radius S, is equal to $2a$.

25. Calculate the parameter and eccentricity of a curve of the second degree, defined by its general equation (use the formulas of § 143).

CHAPTER VIII

THE CONIC SECTIONS.

265. Theorem I. — *The section of a right circular cylinder made by any plane oblique to the base is an ellipse.*

Draw through the axis OO' of the cylinder (Fig. 160) a plane perpendicular to the secant plane; this plane is taken as the plane of the figure. The plane intersects the cylinder in two diametrically opposite generatrices GG', HH', and the secant plane in the straight line AA'. Describe in the plane of the figure two circles O and O' tangent to the line AA' and the two generatrices GG', HH' of the cylinder; draw the bisectors of the angles A and A' and produce them till they intersect the axis of the cylinder in O and O'; if, about the point O as a center with the radius of the cylinder, a circle be described, this circle will touch the generatrices in G and H, and the straight line AA' at the point F; the circle described about the point O' as center will in a similar manner touch the generatrices in G' and H' and the straight line AA' at the point F'. Imagine that the figure be revolved about the axis OO'; the generatrix GG' will generate the surface of the cylinder while the two circles will generate two spheres inscribed in the cylinder and touching it internally, the first along the circumference of the great circle GLH, the second along the circumference of the great circle $G'L'H'$. Moreover, the two spheres are tangents to the

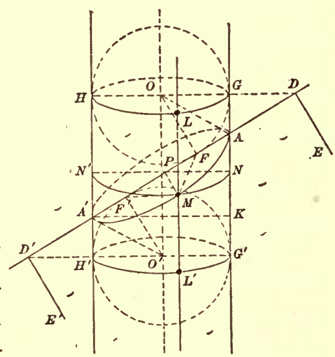

Fig. 160.

given plane, the first at the point F, the second at the point F'. In fact, the plane of the figure and the given plane are perpendicular to each other; the straight line OF, which lies in the first plane and is perpendicular to their intersection, is perpendicular to the second plane; the plane AMA' being perpendicular to OF at its extremity, is tangent to the sphere O at the point F. It may be shown in a similar manner that the given plane is tangent to the sphere O' at the point F'.

Let AMA' be the curve in which the secant plane intersects the cylinder; it will be demonstrated that this curve is an ellipse whose foci are the points F and F'. Join any point M of this curve to the two points F and F' and draw a generatrix of the cylinder through the point M; this generatrix is tangent to the upper sphere at the point L, and the lower sphere at the point L'. The two straight lines MF, ML, tangents drawn from the same point M to the sphere O, are equal; similarly, the two straight lines MF', ML', tangents drawn from the point M to the lower sphere, are equal. Hence the sum of the radii vectores $MF + MF'$ is equal to $ML + ML'$, that is, equal to the portion LL' of the generatrix comprised between the two circles of contact; this length is constant, because, by the revolution of the figure about the axis OO', the generatrix GG' is made to coincide with LL'. Hence it follows that the sum of the distances of each of the points of the curve from the two fixed points F and F' is constant and equal to GG', and consequently that the curve is an ellipse whose foci are F and F'.

Corollary. — The straight lines DE and $D'E'$, the intersections of the secant plane and the planes of the circles GH and GH', along which the inscribed spheres touch the cylinder, are the directrices of the ellipse. In fact, if a plane be drawn through the point M perpendicular to the axis of the cylinder, the section of the cylinder by this plane will be a circle NMN'. The straight line DE, the intersection of the two planes, which are perpendicular to the plane of the figure, is also perpendicular to this plane and, therefore, to the straight line AA'; in the same manner it follows that the straight line MP, the intersection of the plane of the circle and of the secant plane, is perpendicular to AA'. Since the radius vector MF is equal to

ML or to NG, and the perpendicular dropped from the point M upon the directrix DE is equal to PD, the ratio of the distances of the point M from the focus and to the directrix is $\frac{NG}{PD}$; but since PN and GD are parallel, this ratio is equal to that of AG to AD, or of AK to AA', a constant ratio, since these last two lengths are constant.

The directrix DE corresponds to the focus F and the directrix $D'E'$ to the focus F'.

266. Theorem II. — *The section of a right circular cone by a plane is a curve of the second degree.*

Draw through the axis of the cone a plane perpendicular to the secant plane; this plane intersects the cone in the two generatrices SG, SH, and the secant plane in the straight line AA'.

1° Consider now the case when the straight line AA' intersects the two generatrices SG and SH, on the same side of the vertex S (Fig. 161).

Describe two circles O and O' which are tangent to the straight line AA' and to the two elements SG', SH'. If the figure be revolved about the axis SO', so that the element SG' will generate the cone, the two circles will generate two spheres, which are tangent to the cone along the circles of contact GH, $G'H'$. The secant plane is tangent to one of the spheres at the point F, since it is perpendicular to the radius vector OF at its extremity; it is also tangent to the other sphere at the point F'.

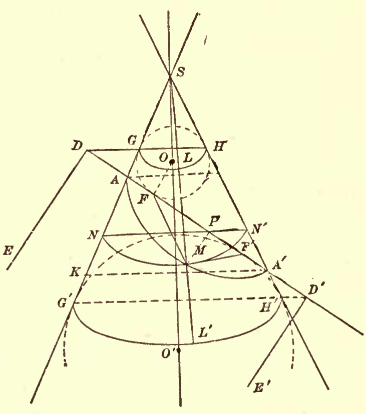

Fig. 161.

Let M be any point of the section made by the secant plane; the generatrix SM which passes through this point is tangent to the spheres at the points L and L'; draw the straight lines MF and MF'. The straight lines MF and ML are equal, since they are tangents drawn from the same point M to the sphere O; the straight lines MF' and ML' are equal, since they are tangents drawn from the point M to the sphere O'; one has, therefore,

$$MF + MF' = ML + ML' = LL'.$$

But the portion LL' of the generatrix comprised between the parallel circles GH, $G'H'$ is constant and equal to GG'; therefore the sum of the distances of each of the points of the curve from the two fixed points F and F' is constant, and, consequently, this curve is an ellipse whose foci are F and F'.

The constant sum GG' is equal to the major axis AA'. If $A'K$ be drawn through the point A' parallel to GH, one determines on the generatrix a length AK equal to the focal distance FF'; for if from the equal lengths GG', AA' one subtracts on the one hand the equal lengths AG and KG', on the other the equal lengths AF and $A'F'$, it follows that the lengths AK and FF' are equal.

Let us consider the straight lines DE and $D'E'$, the intersections of the secant plane with the planes of contact GH, $G'H'$.

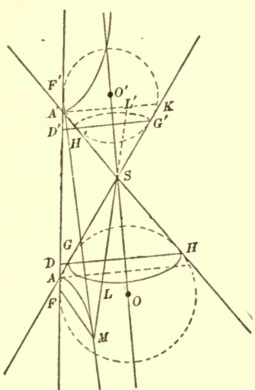

Fig. 162.

If from the point M a perpendicular MP be dropped on the major axis, the distance of the point M from the straight line DE is equal to PD. Let NMN' be the parallel circle which passes through the point M; the length MF or ML is equal to GN on account of the parallels DG, PN; one has

$$\frac{GN}{DP} = \frac{AG}{AD} = \frac{AK}{AA'}.$$

Whence the distance of each of the points of the ellipse from the focus F and from the straight line DE are to each other as the distance between the foci to the major axis. This straight line DE is a *directrix* of the ellipse; the straight line $D'E'$ is the second directrix.

2° When the straight line AA' intersects the two generatrices SG and SH on opposite sides of the vertex S (Fig. 162), one has

$$MF' - MF = ML' - ML = LL' = GG'.$$

The difference of the distances of each of the points of the curve from the two points F and F' is constant; this curve is a hyperbola whose foci are the points F and F'. The straight lines which are the intersections of the secant plane with the two planes of contact are the directrices of the hyperbola.

3° Finally, suppose that the straight line AA' is parallel to the element SH (Fig. 163). Construct a sphere tangent to the cone along the circle GH and to the secant plane at F. Let DE be the intersection of the secant plane and the plane of the circle of contact. Through the point M of the section draw the straight line ME perpendicular to DE, and the generatrix SM, which intersects the curve of contact in L; the straight line ME will be parallel to AA' and to SH; therefore the three straight lines ME, SM, SH lie in the same plane, and the three points H, L, E are on the straight line which is the intersection of the plane of contact with the plane just mentioned. The two triangles MLE, SHL are similar; since SL is equal to SH, one has also ML equal to ME; but ML equals to MF, because they are tangents drawn from the point M to the sphere; consequently MF is equal to ME. Therefore the curve is a parabola of which the point F is the focus and DE the directrix.

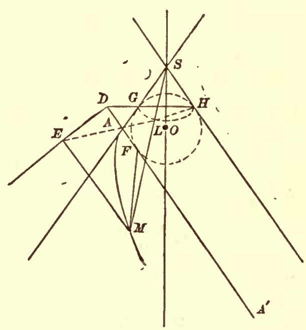

Fig. 163.

This elegant method for finding the properties of the foci and the directrices of the curves of the second degree, is due to Dandelin.

267. *To place a curve of the second degree on a given cone.*

1° The curve is an ellipse. In the triangle $AA'K$ (Fig. 161), one knows the two sides AA', AK, which are the major axis and the distance between the foci, and also the angle opposite AA', which is the complement of half the angle at the vertex of the cone. Since the major axis is greater than the focal distance, this triangle can always be constructed; the perpendicular at the mid-point of $A'K$ determines the point S, and consequently all that determines the position of the secant plane.

2° The curve is a hyperbola. In the triangle $AA'K$ (Fig. 162), one knows likewise two sides, and also an angle opposite one of them, but since the side opposite the given angle is the shortest, the construction of a triangle is not always possible. It is necessary that one has $a > c \cos \gamma$ ($2\,a$ being the transverse axis, $2\,c$ the distance between the foci of the hyperbola, $2\,\gamma$ the angle at the vertex of the cone); whence $\cos \gamma < \frac{a}{c}$ and consequently $\cos \gamma < \cos \theta$, when θ is the angle between the major axis and the asymptote; therefore the angle between the asymptotes should be less than the angle of the cone.

3° The given curve is a parabola. On joining the center O of a sphere to the point G, a right triangle OAG (Fig. 163), in which one knows the side AG which is the semi-parameter of the parabola, and the angle OAG the complement of θ. Having constructed this triangle, one constructs OS perpendicular to OA, and produces it until it intersects AG; the moment the distance SA is known, the problem is solved.

To sum up, one can place on a given cone every ellipse, every parabola, and every hyperbola in which the angle between the asymptotes is less than the angle of the cone.

268. Remark. — Suppose that the spheres used in the preceding discussion be always inscribed in the cone so that they intersect the secant plane; for this it is sufficient that the generating circles be tangent to SA, SA', and intersect AA'; the intersections of the spheres by the secant plane are circles, and one knows that, in case of the ellipse or of the hyperbola, the sum or the difference of the tangents drawn to these circles from any point of the curve is constant; that, in case of the parabola, the tangent drawn to the circle from any point of the curve is equal to the distance of this point from a certain straight line.

The Greek geometers knew the curves of the second degree as sections of a cone with circular base by a plane. APOLLONIUS (247 B.C.) wrote a treatise of eight books on conic sections, in which he gave an account of what had been discovered before his time, and gave an exposition of his discoveries concerning this subject. The treatise of APOLLONIUS contains the fundamental properties of conic sections; we may mention especially in this connection the two theorems concerning conjugate diameters (§§ 162, 163, and 191), the properties concerning the asymptotes of the hyperbola, the elementary properties of the foci.

CHAPTER IX*

THE DETERMINATION OF THE CONIC SECTIONS.

269. The general equation of the second degree

$$Ax^2 + 2\,Bxy + Cy^2 + 2\,Dx + 2\,Ey + F = 0$$

contains six coefficients; but since all of the terms may be divided by one of the coefficients, provided that this coefficient be different from zero, the equation will involve but five arbitrary parameters, which are the ratios of five coefficients to the sixth. In order to determine a curve of the second degree, it is necessary to assign values to the five parameters, or, better, that the five parameters should satisfy five relations; but in this case it is necessary to examine whether the five equations of condition have a system of real solutions, and if, moreover, the corresponding equation of the second degree represents a curve. If the five equations of condition have a system of real solutions possessing this property, there will be a curve of the second degree satisfying the proposed conditions.

In general, the relations between the parameters correspond to geometric conditions which the curve must satisfy. Thus, one can require the curve to pass through given points, to be tangent to given straight lines, etc. One will express the condition that the curve pass through a given point on requiring that the co-ordinates of the point satisfy the equation of the curve, which leads to a relation of the first degree between the coefficients. The condition that the curve is tangent to a given straight line will be found by requiring that the equation which determines the abscissas of the points of intersection of the curve and the straight line has two equal roots, which gives a relation of the second degree between the coefficients. A geometric condition which must be expressed by two relations will be regarded as double. If, for example, one should require the curve to touch a given straight line at a given point, the equation which will furnish the abscissas of the point of intersection and of the curve, should have two roots equal to a given quantity; whence there will result two relations of the first degree between the coefficients; the geometric condition stated ought, therefore, to be reckoned as two single conditions. Accordingly, it is necessary to have *five* geometric conditions in order to determine a curve of the second degree.

If one know that the curve is a parabola, the coefficients must satisfy the relation $AC - B^2 = 0$; the equation will contain but four arbitrary parameters and the parabola will be defined by four conditions.

Similarly, if one know that the curve is an equilateral hyperbola, it will be necessary that the two straight lines represented by the equation $Ax^2 + 2Bxy + Cy^2 = 0$, straight lines parallel to the asymptotes (§ 130), be perpendicular to each other, which gives a relation between the coefficients; when the axes of co-ordinates are rectangular this relation is $A + C = 0$. Four conditions are sufficient, therefore, to determine an equilateral hyperbola.

Before proceeding farther, it is best to generalize the definitions, in order to avoid the restrictions, which would introduce imaginary solutions in the statement of theorems.

Points and Imaginary Straight Lines.

270. A system of real values of x and y determine a point in a plane; in an analogous manner we call an imaginary point a system of imaginary values assigned to x and to y. If two systems of imaginary values be of the form $x = a + bi$, $y = c + di$, and $x = a - bi$, $y = c - di$, we say that the two imaginary points are conjugate.

An equation of the first degree, $Ax + By + C = 0$, with real coefficients, is satisfied by the co-ordinates of an infinite number of real points, whose locus is a straight line; but it is satisfied also by an infinity of systems of imaginary values assigned to x and y; because if any imaginary value be assigned to x, the corresponding value deduced for y will be imaginary; if two conjugate imaginary values be given to x, the two corresponding values of y will also be conjugate.

In an analogous manner, we call an imaginary straight line the *ensemble* of solutions of an equation of the first degree with imaginary coefficients. It is to be noticed that an imaginary straight line passes through one real point. Let, in fact,

$$(A' + A''i)\,x + (B' + B''i)\,y + (C' + C''i) = 0,$$

or

$$(A'x + B'y + C') + i\,(A''x + B''y + C'') = 0,$$

be an imaginary straight line. This equation is satisfied by the co-ordinates of the point of intersection of the two real straight lines

$$A'x + B'y + C' = 0, \quad A''x + B''y + C'' = 0.$$

In the case of the general equation of the first degree, involving three coefficients and consequently two arbitrary parameters, two points, real or imaginary, will determine the straight line. If x', y', x'', y'', be the co-ordinates of two given points, the straight line which passes through these points will have as its equation

$$\frac{x - x'}{x'' - x'} = \frac{y - y'}{y'' - y'}.$$

The straight line which passes through two conjugate imaginary points is real. Let, in fact, $x' = a + bi$, $y' = c + di$, $x'' = a - bi$, $y'' = c - di$; the equation of the straight line reduces to

$$\frac{x - a}{b} = \frac{y - c}{d}.$$

The point which has the co-ordinates

$$\frac{x' + x''}{2}, \quad \frac{y' + y''}{2},$$

will be called the *mid-point* of the straight line which joins the two given points; in case the two points are conjugate imaginaries, the mid-point is a real point.

An algebraic equation $f(x, y) = 0$, with real coefficients, is in general satisfied by the co-ordinates of an infinitude of real points constituting a curve; it is also satisfied by the co-ordinates of an infinitude of imaginary points, conjugate in pairs. If the coefficients be imaginary, the equation has always an infinitude of imaginary solutions, but only a limited number of real solutions; the totality of these solutions will form what we call an imaginary curve.

Two equations, the one of the first degree, the other of the second degree in x and y, have two solutions. It is said, therefore, that a straight line intersects a curve of the second degree in two points, real or imaginary. A real straight line intersects a real curve of the second degree in two points, which are real or conjugate imaginaries. This suffices to explain a fact which has already presented itself several times; when one seeks, for example, the locus of the mid-points of a series of parallel chords in an ellipse, one finds by calculation an indefinite straight line, and yet the locus which is defined geometrically is composed only of that position which lies within the ellipse; the external secants intersect the ellipse in two conjugate imaginary points; the mid-point of the chord is, moreover, a real point, and the diameter is thus prolonged without the curve.

Concerning the Intersection of Two Curves of the Second Degree.

271. We remark, first, that if the two curves coincide, that is, if the two equations are satisfied by the same systems of the variables x and y, the coefficients are proportional. In fact, the equations

(1) $$Cy^2 + 2(Bx + E)y + (Ax^2 + 2Dx + F) = 0,$$

(2) $$C'y^2 + 2(B'x + E')y + (A'x^2 + 2D'x + F') = 0,$$

have the same roots for the same value of x; one has

$$\frac{C}{C'} = \frac{Bx + E}{B'x + E'} = \frac{Ax^2 + 2Dx + F}{A'x^2 + 2D'x + F'},$$

and, since this relation should exist whatever x may be, one deduces

$$\frac{C}{C'} = \frac{B}{B'} = \frac{E}{E'} = \frac{A}{A'} = \frac{D}{D'} = \frac{F}{F'}.$$

The converse is true; in case the coefficients are proportional, the two equations will be identical and the two curves coincide.

We suppose, in what follows, that the curves are different, that is, that the coefficients are not proportional. We consider first the case when the two coefficients C and C' are different from zero; if the equations be subtracted member from member, after having been multiplied respectively by C' and C, one eliminates y^2 and obtains an equation of the form

$$(3) \qquad 2(B_1x + E_1)y + (A_1x^2 + 2D_1x + F_1) = 0,$$

which, with equation (1), forms a system equivalent to the system of the two given equations (1) and (2). The five coefficients B_1, E_1, A_1, D_1, F_1, cannot all be zero at the same time; because if that were the case the coefficients of the two equations (1) and (2) would be proportional. If the two coefficients B_1 and E_1 were zero, equation (3) would become $A_1x^2 + 2D_1x + F_1 = 0$; it would furnish two values for x; to each of which, by reason of equation (1), would correspond two values of y; in all, four solutions. Suppose that the two coefficients B_1 and E_1 were not zero at the same time; in general, the value $x = -\frac{E_1}{B_1}$, which annuls the coefficient of y in equation (1), does not reduce the polynomial $A_1x^2 + 2D_1x + F_1$ to zero; in this case, the quantity $B_1x + E_1$ being different from zero for all of the solutions of equation (3), this equation can be put under the form

$$(4) \qquad y = -\frac{A_1x^2 + 2D_1x + F_1}{2(B_1x + E_1)};$$

on substituting this value of y in equation (1), an equation of the fourth degree is found,

$$(5) \qquad a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0,$$

which, combined with equation (4), forms a system equivalent to the system of two equations (1) and (3), and, consequently, to the proposed system. The five coefficients of equation (5) cannot all be zero at the same time, because, if that were the case, equation (5) becoming an identity the two proposed equations would be satisfied by all the systems of values of x and y which would satisfy equation (4) or equation (3); the two curves would coincide with the curve represented by equation (3) and, consequently, would have proportional coefficients. Equation (5) gives four values for x; to each value of which there corresponds, owing to equation (4), one value of y, which furnishes *four* solutions of the given system.

If the value $x = -\frac{E_1}{B_1}$ annul the polynomial $A_1x^2 + 2D_1x + F_1$, equation (3) can be put under the form

$$(B_1x + E_1)(y + mx + n) = 0,$$

and decomposes into two distinct equations, $B_1x + E_1 = 0$, $y + mx + n = 0$; the first gives the value $x = -\frac{E_1}{B_1}$, to which, owing to equation (1), correspond two values of y; from the second, one gets $y = -mx - n$; and, by replacing y by this value in equation (1), one obtains an equation of the second degree in x, which furnishes two new solutions; in all, four solutions. Moreover, it can happen that this last equation of the second degree in x reduces to an identity; in this case, the co-ordinates of every point of the straight line $y + mx + n = 0$ satisfy the two proposed equations, which represent pairs of straight lines, two of which coincide.

In case one only of the coefficients C and C' is zero, one of the proposed equations will be of the form (3), and the discussion just given will be repeated.

Let us consider now the case when the two coefficients C and C' are zero at the same time. If the value $x = -\frac{E'}{B'}$, which annuls the coefficient of y in equation (2), does not reduce the polynomial $A'x^2 + 2D'x + F'$ to zero, one finds from this equation

$$y = -\frac{A'x^2 + 2D'x + F'}{2(B'x + E')},$$

and, by substituting in equation (1), an equation of the third degree in x will be obtained, which gives *three* solutions. If the value $x = -\frac{E'}{B'}$ reduces the polynomial $A'x^2 + 2D'x + F'$ to zero, equation (2) may be written

$$(B'x + E')(y + mx + n) = 0,$$

and represents two straight lines $B'x + E' = 0$, $y + mx + n = 0$, the first of which intersects the curve (1) in one point, the second in two points. It happens that one of these straight lines belongs to curve (1), and in this case the proposed equations represent pairs of straight lines, two of which coincide.

From what precedes, it follows that two curves of the second degree cannot have more than four points in common, at least that these curves consist of pairs of straight lines, two of which coincide. In case the two given equations have real coefficients, their points of intersection are real or conjugate imaginaries.

272. It is easy to form equation (5), which, in the general case, determines the abscissas of the four points of intersection of two curves of the second degree. Let $A_0y^2 + A_1y + A_2 = 0$, $A'_0y^2 + A'_1y + A'_2 = 0$, represent the two given equations, in which A_0 and A'_0 designate constants, A_1 and A'_1 polynomials of the first degree in x, A_2 and A'_2 polynomials of the second degree in x. On subtracting these equations, member from member, having multiplied them respectively by A'_0 and A_0, one has

$$(A_0A'_1 - A'_0A_1)\,y + A_0A'_2 - A'_0A_2 = 0.$$

On multiplying by A_2 and A_2', subtracting, and suppressing the factor y, one has, in a similar manner,

$$(A_0A'_2 - A'_0A_2)\,y + (A_1A'_2 - A'_1A_2) = 0.$$

The elimination of y between the last two equations leads to the equation of the fourth degree

$$(A_0A'_1 - A'_0A_1)(A_1A'_2 - A'_1A_2) - (A_0A'_2 - A'_0A_2)^2 = 0.$$

273. Corollary. — An equation of the second degree, with imaginary coefficients, cannot have more than four real solutions. In fact, the first member of the equation has the form $S + iS_1$, S and S_1 representing real polynomials of the second degree; if the equation is satisfied by real values assigned to x and to y, one will have separately $S=0$, $S_1=0$; the real points of the locus are therefore the points of intersection of the two real curves $S = 0$, $S_1 = 0$, and these points are in general four in number.

There is an exception, namely, the case cited above, when the curves consist of pairs of straight lines, two of which coincide; in this case the equation of the second degree represents two straight lines, one of which is real. Finally, if the two curves $S = 0$, $S_1 = 0$ coincide, the first member of the given equation will be divisible by a constant imaginary factor and the coefficients of the equation become real.

274. Lemma. — *Every system of n homogeneous equations of the first degree involving $n + 1$ unknown quantities, is satisfied by an infinitude of systems of values of the unknown quantities, one of which at least is different from zero.*

Let us consider first an equation involving two unknown quantities,

$$ax + by = 0.$$

If the two coefficients a and b were zero, the equation would be satisfied by arbitrary values of x and y. Suppose that one of the coefficients, for example b, be not zero; the equation can be put under the form $y = -\frac{a}{b}x$, and an arbitrary value be assigned to x; to each value of x corresponds one value of y. Thus the given equation is satisfied by an infinity of systems of values of x and y, one of which at least is different from zero.

Consider now two equations involving three unknown quantities,

$$ax + by + cz = 0,$$

$$a'x + b'y + c'z = 0.$$

If the six coefficients be all zero at the same time, the equations would be satisfied by all possible arbitrary values of x, y, z. Suppose that one at least of the coefficients, for example c, be not zero; the system of the two given equations could be replaced by the equivalent system

$$z = -\frac{ax + by}{c},$$

$$(ac' - ca')\,x + (bc' - cb')\,y = 0.$$

In accordance with what we have said, the second equation is satisfied by an infinity of systems of values of x and y, one of which at least is different from zero; to each of them corresponds one value of z given by the first equation. Thus the system of the two given equations is satisfied by an infinity of systems of x, y, z, one of which at least is different from zero.

The same reasoning can be continued indefinitely. Suppose that we have the following three equations involving four unknown quantities:

$$\begin{aligned} ax + by + cz + dt &= 0, \\ a'x + b'y + c'z + d't &= 0, \\ a''x + b''y + c''z + d''t &= 0. \end{aligned}$$

If the twelve coefficients were all zero at the same time, the equations would be satisfied by all possible arbitrary values of x, y, z, t. Suppose that one of the coefficients at least, for example d, be not zero; the system of given equations could be replaced by the equivalent system

$$t = -\frac{ax + by + cz}{d},$$

$$\begin{aligned} (ad' - a'd)\,x + (bd' - b'd)\,y + (cd' - c'd)\,z &= 0, \\ (ad'' - a''d)\,x + (bd'' - b''d)\,y + (cd'' - c''d)\,z &= 0. \end{aligned}$$

By reason of the preceding discussion, the system of the last two equations is satisfied by an infinitude of values of x, y, z, one of which at least is different from zero; to each of them corresponds one value of t given by the first equation.

275. THEOREM I. — *Through five given points, no three of which lie on a straight line, a curve of the second degree can be passed, and one only.*

Call (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) the co-ordinates of the five given points. In order that the curve of the second degree

$$(1) \qquad Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

pass through these five given points, it is necessary and sufficient that the five equations

$$(2) \qquad \begin{cases} Ax_1^2 + 2Bx_1y_1 + Cy_1^2 + 2Dx_1 + 2Ey_1 + F = 0, \\ Ax_2^2 + 2Bx_2y_2 + Cy_2^2 + 2Dx_2 + 2Ey_2 + F = 0, \\ Ax_3^2 + 2Bx_3y_3 + Cy_3^2 + 2Dx_3 + 2Ey_3 + F = 0, \\ Ax_4^2 + 2Bx_4y_4 + Cy_4^2 + 2Dx_4 + 2Ey_4 + F = 0, \\ Ax_5^2 + 2Bx_5y_5 + Cy_5^2 + 2Dx_5 + 2Ey_5 + F = 0, \end{cases}$$

be satisfied. We have thus five equations, homogeneous and of the first degree, between the six unknown quantities A, B, C, D, E, F. It follows from the preceding lemma that these equations are satisfied by an infinitude of systems of values of the unknown quantities A, B, C, D, E, F, one of which at least is different from zero. We notice that in none of these solutions, the first five coefficients are all zero at the same time, because then, by virtue of any one of equations (2), one would have $F = 0$. We remark further that if the five given points be not on a straight line, the first three coefficients A, B, C cannot be zero at the same time; because equation (1) would be reduced to the first degree, and would represent a straight line passing through the five points. On assigning to the six coefficients the values which constitute one of the preceding solutions, one obtains a curve of the second degree passing through the five given points. Thus, through five given points one curve of the second degree at least can be made to pass.

It follows that if of the five given points no three lie on a straight line, one can pass through these five points one, and only one, curve of the second degree; because if one could pass two, these two curves would have five points in common; but we have seen (§ 271) that two curves of the second degree, which are not composed of straight lines, cannot have more than four points in common.

The different solutions of the system of equations (2), in order to determine the same curve of the second degree, are formed from proportional quantities. Having learned that one of the undetermined coefficients is different from zero, one seeks the ratios of the other five to it, and will need to solve a system of five equations of the first degree in five unknown quantities.

The same conclusions are applicable to the case where three given points, and only three, are on a straight line. The locus of the second degree is composed of this straight line and the one which passes through the other two points.

If four of the points be on a straight line, the problem is indeterminate. The locus of the second degree is composed of this straight line, and of any straight line passing through the fifth point.

276. Remark. — One can, by aid of a determinant, form the equation of the second degree which passes through five given points. Consider, for this purpose, the determinant

$$\Delta = \begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x^2_1 & x_1y_1 & y^2_1 & x_1 & y_1 & 1 \\ x^2_2 & x_2y_2 & y^2_2 & x_2 & y_2 & 1 \\ x^2_3 & x_3y_3 & y^2_3 & x_3 & y_3 & 1 \\ x^2_4 & x_4y_4 & y^2_4 & x_4 & y_4 & 1 \\ x^2_5 & x_5y_5 & y^2_5 & x_5 & y_5 & 1 \end{vmatrix}.$$

This is an integral polynomial of the second degree with respect to the variables x and y. It becomes zero in case x and y are replaced by x_1 and y_1; because then the elements of the first horizontal line become equal to those of the second. The same is true if x and y be replaced by x_2 and y_2, and so on. Whence it follows that the equation $\Delta = 0$ represents a curve of the second degree passing through the five given points.

277. Corollary I.—A quadrilateral *abcd* (Fig. 164) being given, represent the equations of the two opposite sides *ab*, *cd* by $\alpha = 0$, $\beta = 0$, those of the other two opposite sides *bc*, *ad* by $\gamma = 0$, $\delta = 0$; the equation

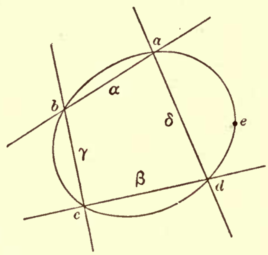

Fig. 164.

(3) $$A\alpha\beta + B\gamma\delta = 0,$$

in which the coefficients a and b are arbitrary, represents all the curves of the second degree which pass through the four points *a*, *b*, *c*, *d*. The letters α, β, γ, δ representing polynomials of the first degree in x and y, the equation is of the second degree; the co-ordinates of the point *a*, the intersection of the straight lines *ab* and *ad*, reduce the two polynomials α and δ to zero, and consequently the first member of equation (3); the same is true of the other three points *b*, *c*, *d*. Whence whatever the value of the coefficients A and B may be, the curve represented by equation (3) passes through the four points *a*, *b*, *c*, *d*. This equation represents every curve of the second degree which passes through the four points, because a fifth point *e* determines the curve and one can assign to the ratio of the coefficients such a value that the curve will pass through this fifth point taken at random in the plane.

Equation (3) has a very simple geometrical signification: the polynomials α, β, γ, δ being proportional to the distances of any point (x, y) from the sides of the quadrilateral, it follows that *the product of the distances of any point of the curve from the two opposite sides ab, cd of the inscribed quadrilateral is to the product of the distances of the same point from the two opposite sides bc, ad in a constant ratio.* The value of this ratio determines the curve.

In general, if the equations of two curves of the second degree be represented by $S=0$, $S_1=0$, the equation $S+kS_1=0$, in which k is an arbitrary parameter, represents every curve of the second degree which passes through the four points common to the first two.

278. Corollary II.—We propose to determine a parabola which passes through four given real points a, b, c, d. If these points be connected two and two by two straight lines ab, cd, which intersect and which are chosen as axes of co-ordinates, the general equation of the second degree which passes through these four points is

$$(4) \qquad \left(\frac{x}{a}+\frac{y}{c}-1\right)\left(\frac{x}{b}+\frac{y}{d}-1\right)-kxy=0;$$

a and b being the abscissas of the points a and b, c and d the ordinates of the points c and d. In order that the locus be a parabola, it is necessary that the parameter k satisfy the condition

$$(5) \qquad \left(k-\frac{1}{ad}-\frac{1}{bc}\right)^2-\frac{4}{abcd}=0.$$

In case the product $abcd$ is negative, one finds two imaginary values for k, and it is impossible to pass a real parabola through these four points. If the product be positive, one concludes that a convex quadrilateral having the four points as vertices can be formed, and obtains two real and different values for k, and, consequently, two real curves of the genus parabola passing through the four points. In case the points could be connected two and two by parallel straight lines, each pair of parallel straight lines would constitute a solution.

279. Corollary III.—It is easy to form the general equation of curves of the second degree, which pass through three given points a, b, c. If $\alpha=0$, $\beta=0$, $\gamma=0$, represent the equations of the three straight lines bc, ca, ab, the equation

$$(6) \qquad A\beta\gamma+B\gamma\alpha+C\alpha\beta=0$$

represents a curve of the second degree passing through the three given points. This equation involves two arbitrary parameters, the ratios of two of the coefficients to a third, and one could so dispose of the two parameters as to make the curve pass through two additional points chosen at random in the plane.

280. THEOREM II. — *One curve of the second degree can be drawn tangent to two given straight lines, at two given points, and made to pass through another given point, and only one.*

In order that a curve of the second degree

$$(7) \quad f(x, y) = Ax^2 + 2\,Bxy + Cy^2 + 2\,Dx + 2\,Ey + F = 0$$

be tangent to a straight line

$$(8) \qquad a(x - x_1) + b(y - y_1) = 0,$$

at a point (x_1, y_1), it is necessary and sufficient that the curve pass through the point (x_1, y_1), and that the angular coefficient of the tangent at this point be equal to that of the straight line, which furnishes two equations,

$$(9) \qquad f(x_1, y_1) = 0, \quad bf'_{x_1}(x_1, y_1) - af'_{y_1}(x_1, y_1) = 0,$$

which are homogeneous and of the first degree in the coefficients A, B, C, D, E, F.

We have thus five equations which are homogeneous and of the first degree. We have learned that such a system of equations has an infinitude of solutions in which one at least of the coefficients, say F, is different from zero; to one of these solutions there corresponds a curve of the second degree satisfying the required conditions.

There cannot exist more than one curve of the second degree satisfying these conditions, because, if there were two, the equation of the fourth degree which one obtains when one seeks their common points would have two double roots and one single root, which is impossible.

281. COROLLARY I. — The equation

$$(10) \qquad \alpha\beta - k\gamma^2 = 0,$$

in which k is an arbitrary parameter, represents every curve of the second degree tangent to two lines $\alpha = 0$, $\beta = 0$, at the points where they are intersected by the straight line $\gamma = 0$ (Fig. 165). The curve is tangent to the first straight line;

for if one makes $\alpha = 0$ in the equation of the curve, one has $\gamma^2 = 0$, and consequently the two points of intersection of the straight line and the curve coincide. The curve is in a similar manner tangent to the second straight line, and the two points of contact are situated on the straight line $\gamma = 0$. Equation (10) represents every curve possessing these properties; because the parameter k can be so determined that the curve can pass through any other point chosen at random in the plane.

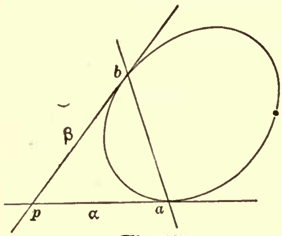

Fig. 165.

This equation signifies that *the product of the distances of any point of the curve from two tangents is to the square of the distance of this point to the chord of contact in a constant ratio.*

In general, if the equation of a curve of the second degree be represented by $S = 0$, the equation $S - k\gamma^2 = 0$ will represent every curve of the second degree tangent to the first at two points situated on the straight line $\gamma = 0$.

Corollary II. — One can determine the parameter k by the condition that the curve be a parabola. If the two tangents be chosen as axis of co-ordinates, equation (10) becomes

$$\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 - 2kxy = 0. \tag{11}$$

In order that the curve be a parabola, it is necessary that the condition

$$\left(k - \frac{1}{ab}\right)^2 - \frac{1}{a^2b^2} = 0$$

be satisfied; which furnishes the two solutions $k = 0$, $k = \frac{2}{ab}$; the straight line ab corresponds to the first; a parabola, whose equation can be put under the form

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} - 1 = 0,$$

corresponds to the second.

282. Application. — As an application, form the equation of the second degree which represents the *ensemble* of the two tangents drawn from a point p with the co-ordinates x_1, y_1 (Fig. 165) to a conic whose equation is

$$f(x, y) \equiv Ax^2 + 2\,Bxy + Cy^2 + 2\,Dx + 2\,Ey + F = 0.$$

The equation of the chord of contact ab is, as we have seen (§ 125),

$$(a) \quad \gamma \equiv (Ax_1 + By_1 + D)\,x + (Bx_1 + Cy_1 + E)\,y + Dx_1 + Ey_1 + F = 0,$$

the first member of which is designated by γ. The two tangents pa and pb represent a conic doubly tangent to the given conic $f(x, y) = 0$ at the points situated on the straight line; they are therefore represented by an equation of the form

$$(b) \qquad f(x, y) - k\gamma^2 = 0,$$

where k represents a constant coefficient which remains to be determined. For this purpose it is sufficient to express the condition that the curve (b) passes through a point taken on the conic formed by the two tangents pa and pb; we express the condition that it passes through the point p whose co-ordinates are (x_1, y_1), that is, that equation (b) is satisfied by $x = x_1$, $y = y_1$. If in γ one put $x = x_1$, $y = y_1$, γ reduces to $f(x_1, y_1)$; one has, therefore, the condition

$$f(x_1, y_1) - kf^2(x_1, y_1) = 0,$$

which gives for k the value $\dfrac{1}{f(x_1, y_1)}$, and the equation required is

$$(c) \qquad f(x, y)\,f(x_1, y_1) - \gamma^2 = 0,$$

where γ should be replaced by the expression (a). This equation is called the *quadratic equation of the tangents drawn from the point* x_1, y_1.

282. 2. We confine ourselves to stating the following results, which it is easy to verify. (See §§ 303 and 331.)

Let $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations of the three sides of a triangle. The general equation of the conics inscribed in this triangle is

$$\lambda^2\alpha^2 + \mu^2\beta^2 + \nu^2\gamma^2 - 2\,\mu\nu\beta\gamma - 2\,\lambda\nu\gamma\alpha - 2\,\lambda\mu\alpha\beta = 0,$$

λ, μ, ν representing the variable parameters. This equation can be written in the irrational form

$$(\sqrt{\lambda\alpha} + \sqrt{\mu\beta} + \sqrt{\nu\gamma})(\sqrt{\lambda\alpha} - \sqrt{\mu\beta} + \sqrt{\nu\gamma})$$
$$(\sqrt{\lambda\alpha} + \sqrt{\mu\beta} - \sqrt{\nu\gamma})(\sqrt{\lambda\alpha} - \sqrt{\mu\beta} - \sqrt{\nu\gamma}) = 0.$$

The general equation of the conic inscribed in the quadrilateral whose sides have the equations

$$\alpha + \beta + \gamma = 0,\quad \alpha - \beta + \gamma = 0,\quad \alpha + \beta - \gamma = 0,\quad \alpha - \beta - \gamma = 0,$$

is
$$\frac{\alpha^2}{\lambda} + \frac{\beta^2}{1-\lambda} - \gamma^2 = 0,$$

λ being a variable parameter. (See § 331, Examples.)

Multiple Conditions.

283. Let us examine the geometric conditions by which a curve of the second degree may be defined. Thus far we have mentioned none other than single conditions, such as points and tangents. The center is equivalent to two conditions; because if the center be taken for the origin of co-ordinates, the equation of the second degree, being deprived of the terms of the first degree, cannot contain more than three arbitrary parameters; thus the curve is defined by its center and three points.

A diameter, with the direction of the chords, is equivalent to two conditions; because if the diameter be taken for the x-axis and a line parallel to the corresponding chords for the y-axis, the equation, being deprived of the two terms of the first degree in y, does not contain more than three arbitrary parameters.

A system of conjugate diameters is equivalent to three conditions; because if they be taken for the axes of co-ordinates, the equation being reduced to the form $ax^2 + by^2 + c = 0$, contains only two arbitrary parameters. In general, let $\alpha = 0$, $\beta = 0$ be the equation of two conjugate diameters; the distances α and β of each point from the two conjugate diameters being proportional to the co-ordinates of this point with respect to these diameters, the curve will be represented by the equation

$$(12) \qquad a\alpha^2 + b\beta^2 + c = 0,$$

with two arbitrary parameters.

The equation

$$(13) \qquad \alpha^2 + k\beta = 0$$

is the general equation of the parabolas of which the straight line $\alpha = 0$ is a diameter, and the straight line $\beta = 0$ the tangent at the extremity of this diameter.

One knows that the equation of the hyperbola, referred to its asymptotes as axes, is $xy = k$. In general, let $\alpha = 0$, $\beta = 0$ be the equations of two asymptotes; the hyperbola will be represented by an equation of the form

$$(14) \qquad \alpha\beta - k = 0,$$

which contains but one arbitrary parameter k. Thus the two asymptotes are equivalent to four conditions, and the curve is determined by two asymptotes and a point or a tangent. If one were given but one asymptote whose equation is $\alpha = 0$, the equation $\beta = 0$ of the other asymptote being indeterminate, equation (14) would contain three arbitrary parameters, so that one asymptote is equivalent to two conditions.

We have seen that every equation of the second degree has one focus and one directrix; whence it follows that the equation

$$(15) \qquad (x-\alpha)^2 + (y-\beta)^2 - (mx + ny + h)^2 = 0,$$

by which the focal property is expressed in rectangular coordinates, and which involves five arbitrary parameters α, β, m, n, h, represents every curve of the second degree. A focus is equivalent to two conditions; because, if one were given a focus, its co-ordinates α and β being known, equation (15) would contain but three arbitrary parameters. Similarly, one directrix is equivalent to two conditions; because, on being given the equation of the directrix, the ratio of the three parameters m, n, h to a third is determined.

The results that we have obtained may be derived in another manner. It is clear that the two co-ordinates of a special point of a curve of the second degree, like the center, a focus, a vertex, etc., are determined when the coefficients of the equation of the second degree are known, and consequently that there exist two equations between these co-ordinates and the coefficients; if therefore such a point be given, one will have two relations between the coefficients. A similar discussion applies to the two parameters of a special straight line, such as the directrix or axis, etc.; if this straight line be given, one will have, moreover, two relations between the coefficients.

Thus, for example, if $f(x, y) = 0$ be the equation of the curve, one can express the condition that a given point is the center by requiring that its co-ordinates satisfy the two equations $f'_x = 0$, $f'_y = 0$. In order to express the condition that a given point is a vertex, it is sufficient to require that its co-ordinates satisfy the equation of the curve and that the normal at this point passes through the center.

It is to be noticed that the preceding forms under which the equation of the second degree have been put, reduce to the form $\alpha\beta - k\gamma^2 = 0$, composed of three polynomials of the first degree α, β, γ, of which the first two represent tangents drawn from an arbitrary point p of the plane, and the third represents the chord of contact. If the point p coincide with the center of the hyperbola, the tangents α and β are the asymptotes; if the chord of contact be removed to infinity, the polynomial γ reduces to a constant, and the equation $\alpha\beta - k\gamma^2 = 0$ becomes $\alpha\beta - k = 0$. Equation (12), put under the form

$$(\alpha\sqrt{a} + \beta\sqrt{-b})(\alpha\sqrt{a} - \beta\sqrt{-b}) + c = 0,$$

reduces to equation (14).

284. THE DETERMINATION OF THE FOCI OF THE CONIC. — Let α, β be the co-ordinates of a focus of a conic whose equation is $f(x, y)$. We have seen (§ 216) that the equation of the curve in rectangular co-ordinates can be written in the form

$$(15) \qquad (x-\alpha)^2 + (y-\beta)^2 - (mx + ny + h)^2 = 0.$$

This equation can be written

$$[(y-\beta) - i(x-\alpha)][(y-\beta) + i(x-\alpha)] - (mx + ny + h)^2 = 0,$$

which is of the form

$$PQ - R^2 = 0,$$

P, Q, R representing three linear functions in x and y. It follows, therefore, that the conic represented by equation (15) is tangent to the two straight lines

$$(a) \qquad y - \beta + i(x-\alpha) = 0, \quad y - \beta - i(x-\alpha) = 0,$$

the chord of contact being the *directrix*

$$mx + ny + h = 0.$$

Two straight lines (a) pass through the focus (α, β) and have the angular coefficients $+i$ and $-i$; since they are tangent to the curve, they are the two tangents drawn from the focus (α, β) to the conic. It follows, therefore, that the *focus is a point such that the two tangents drawn from this point to the conic have the angular coefficients* $\pm i$; *the directrix is the chord of contact.*

One can say also that the *ensemble* of the two tangents drawn from the focus (α, β) to the conic has the equation, in rectangular co-ordinates,

$$(x-\alpha)^2 + (y-\beta)^2 = 0,$$

an equation identical with that *of a circle of radius zero.* The two tangents drawn from the focus form, therefore, a conic whose equation has the character of the equation of a circle: *the coefficient of xy is zero, the coefficients of x^2 and y^2 are equal.*

From this follows a new method for determining the foci of a conic, a method which one can present under the one or the other of the following forms:

1° If one put

$$\gamma = (A\alpha + B\beta + D)x + (B\alpha + C\beta + E)y + D\alpha + E\beta + F,$$

the quadratic equation of the tangents drawn from the point (α, β) to the conic $f(x, y) = 0$ is (§ 282)

$$f(x, y) f(\alpha, \beta) - \gamma^2 = 0.$$

In order to express that the point (α, β) is a focus, it is sufficient to express the condition that this equation has the characteristics of the equation of a circle: in rectangular co-ordinates, the coefficient of xy is zero, the coefficients of x^2 and y^2 are equal. One has thus the two equations

$$(c) \qquad Bf(\alpha, \beta) - (A\alpha + B\beta + D)(B\alpha + C\beta + E) = 0,$$

$$Af(\alpha, \beta) - (A\alpha + B\beta + D)^2 = Cf(\alpha, \beta) - (B\alpha + C\beta + E)^2,$$

which determine α and β. If α and β be regarded in these equations as the current co-ordinates, they represent two conics whose points of intersection are the foci: thus, when the given curve is an ellipse or a hyperbola, these two conics intersect in four distinct points at finite distances, which are the two real foci and the two imaginary foci of the curve. One notices that the elimination of $f(\alpha, \beta)$ between the preceding equations furnishes an equation which can be written in the abbreviated form

$$B(f'^2_\alpha - f'^2_\beta) - (A - C) f'_\alpha f'_\beta = 0.$$

This equation, which represents a conic passing through the foci, is the equation of the *ensemble* of the axis of the conic [§ 137, eq. (24)].

2° The investigation of the foci is simplified if one form the condition

$$(d) \qquad \mathrm{a}u^2 + 2\,\mathrm{b}uv + \mathrm{c}v^2 + 2\,\mathrm{d}u + 2\,\mathrm{e}v + \mathrm{f} = 0,$$

which expresses the condition that the straight line $ux + vy + 1 = 0$ is tangent to the given conic (§ 126). Let (α, β) be a focus, the axis being rectangular; the straight line

$$y - \beta - i(x - \alpha) = 0$$

ought to be tangent to the curve. Whence, for this particular straight line one has

$$u = \frac{i}{\beta - i\alpha}, \quad v = \frac{-1}{\beta - i\alpha}.$$

Expressing the condition that these values of u and v satisfy the condition (d), one has, on developing and replacing i^2 by -1,

(e) $\mathsf{f}(\alpha^2 - \beta^2) - 2\,\mathsf{d}\alpha + 2\,\mathsf{e}\beta + \mathsf{a} - \mathsf{c} + 2\,i\,(\mathsf{f}\alpha\beta - \mathsf{e}\alpha - \mathsf{d}\beta + \mathsf{b})$.

The straight line

$$y - \beta + i\,(x - \alpha) = 0$$

being tangent to the conic, one gets a second condition which leads to the preceding by changing i into $-i$. Therefore, if the real or imaginary point (α, β) is a focus, the coefficient of i and the term independent of i ought to be zero separately in the condition (c), and one has the two equations

$$\mathsf{f}(\alpha^2 - \beta^2) - 2\,\mathsf{d}\alpha + 2\,\mathsf{e}\beta + \mathsf{a} - \mathsf{c} = 0,$$

$$\mathsf{f}\alpha\beta - \mathsf{e}\alpha - \mathsf{d}\beta + \mathsf{b} = 0,$$

already attained above [eq. (c)] in another form.

In case the axes are oblique and include an angle θ, one expresses the condition that the two straight lines

$$y - \beta = (\cos\theta \pm i\sin\theta)(x - \alpha)$$

are tangent to the conic.

INVESTIGATION OF SECANTS COMMON TO TWO CURVES OF THE SECOND DEGREE.

285. We have seen that two curves of the second degree, $S = 0$, $S_1 = 0$, have in general four points in common; through these four points, which we will suppose for the present distinct, one can pass three pairs of straight lines. In case the curves are real, the common points are real, or conjugate imaginaries taken by pairs. There are three cases to consider:

1° If the four common points a, b, c, d are real, the three couples of common secants are evidently real. 2° If the four points are imaginary and conjugate in pairs, for example a and b, c and d, the two straight lines ab and cd, which pass through two conjugate imaginary points, are real; but the other four straight lines are imaginary; because if one of them ac were real, the points a and c where the straight line ac intersects the two real straight lines ab and cd would be real. The straight line bd which passes through the two points b and d which are respectively conjugates of the points a and c, is conjugate to ac; similarly the straight line ad is the conjugate of bc; thus, in this case, one has a couple of real secants ab and cd, and two pairs ac and bd, and ad and bc, each formed by two conjugate imaginary straight lines. 3° In case two of the points of intersection a and b are real, the other two c and d conjugate imaginaries, the two straight lines ab and cd are still real, and the other four imaginary; but the two imaginary straight lines of the same pair are not conjugates; because one knows that one imaginary straight line has but one real point, which belongs also to the conjugate straight line; the two imaginary straight lines ac and bd passing through two real distinct points a and b are not conjugates.

286. The investigation of the points of intersection of the two curves depends on the solution of an equation of the fourth degree; but the question can be reduced to the solution of an equation of the third degree. The equation $S + \lambda S_1 = 0$, in which the parameter λ is arbitrary, representing every curve of the second degree which passes through the points common to the first two, one can determine the parameter λ so that this equation represents two straight lines; since the two curves have three pairs of common secants, the value of λ will be given by an equation of the third degree.

Let

$$(16)\qquad \begin{cases} Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0, \\ A'x^2 + 2B'xy + C'y^2 + 2D'x + 2E'y + F' = 0, \end{cases}$$

be the equations of the two curves. The new equation will be

$$(17)\qquad (A + \lambda A')x^2 + 2(B + \lambda B')xy \cdots = 0;$$

in order that it represent two straight lines, it is necessary and sufficient that the discriminant be zero (§ 124) and consequently that the constant λ satisfies the equation of the third degree

$$(A+\lambda A')(C+\lambda C')(F+\lambda F')-(A+\lambda A')(E+\lambda E')^2$$
$$-(C+\lambda C')(D+\lambda D')^2-(F+\lambda F')(B+\lambda B')^2$$
$$+2(B+\lambda B')(D+\lambda D')(E+\lambda E')=0.$$

On arranging this equation with respect to λ, we ~~will~~ have an equation of the form

$$(18) \qquad \Delta+\Theta\lambda+\Theta'\lambda^2+\Delta'\lambda^3=0,$$

where Δ and Δ' are the discriminants of the curves S and S_1 and where Θ and Θ' have the values

$$\Theta=A'\mathsf{a}+2B'\mathsf{b}+C'\mathsf{c}+2D'\mathsf{d}+2E'\mathsf{e}+F'\mathsf{f},$$

$$\Theta'=A\mathsf{a}'+2B\mathsf{b}'+C\mathsf{c}'+2D\mathsf{d}'+2E\mathsf{e}'+F\mathsf{f}',$$

a, **b**, **c**, **d**, **e**, **f** being the quantities already used above (§ 124) and **a**$'$, **b**$'$, **c**$'$, **d**$'$, **e**$'$, **f**$'$ the quantities analogously formed with the coefficients of the conic S_1.

One real value of λ gives two real straight lines, provided it makes the quantity

$$(A+\lambda A')(C+\lambda C')-(B+\lambda B')^2$$

negative, and two conjugate imaginary straight lines, in case it makes this quantity positive; because the first member of equation (17) has real coefficients and it decomposes into a product of two polynomials of the first degree, of which the coefficients are, in the first case, real, in the second case conjugate imaginaries (§ 123).

One imaginary value of λ gives two non-conjugate imaginary straight lines. In fact, two real straight lines or two conjugate imaginary straight lines are represented by an equation of the second degree,

$$(19) \qquad A''x^2+2B''xy+C''y^2+2D''x+E''y+F''=0,$$

with real coefficients. Equations (17) and (19), representing the same curve, have proportional coefficients; and since λ is an imaginary quantity, one deduces

$$\frac{A'}{A}=\frac{B'}{B}=\frac{C'}{C}=\frac{D'}{D}=\frac{E'}{E}=\frac{F'}{F}.$$

The two equations (16) are identical.

Suppose that the three roots of equation (18) are unequal. The three couples of straight lines being distinct, the two curves have four distinct points in common. Owing to what has been said above, in case the three roots are real, the four points are all real or all imaginary; in case one root only is real, two points are real and two imaginary. In order to distinguish the first two cases, one examines if three roots or one only make the quantity $-\mathbf{f}$ positive; in the first case, the four points are real, in the second they are imaginary.

287. We have supposed thus far that the four common points are distinct. If the two points a and b coincide, the other two being distinct, the two curves are tangent at the real point a; the couple (ab, cd) is composed of the tangent at a which is real, and the real straight line; the other two couples (ac, bd), (bc, bd) coincide. The equation of the third degree has, therefore, one single root and one double root, both real; the first gives the two real straight lines ab and cd, the second gives two straight lines, real or conjugate imaginaries, according as the two points c and d are real or imaginary.

Suppose that the two points a and c coincide, also the two points b and d; the two curves are tangents at the points a and b, which are real or conjugate imaginaries. One of the couples of straight lines consists of tangents at a and b, which are real or conjugate imaginaries; the other two coincide with the double straight line ab, which is real. The equation of the third degree has, moreover, a single and a double root, both real; the first furnishes the two tangents, the second the chord of contact. (See, for a complete discussion, Chapter XII.)

288. In order to give an application of what precedes, let us consider two ellipses having a common focus. These two ellipses cannot intersect in more than two real points; because, after what has been said in § 260, the two ellipses which have a common focus and three common points coincide; they can have therefore but two real common secants.

Let
$$(x-\alpha)^2+(y-\beta)^2-k^2\gamma^2 =0,$$
$$(x-\alpha)^2-(y-\beta)^2-k'^2\gamma'^2=0,$$

be the equations of two ellipses; since the two real common secants $k\gamma=\pm k'\gamma'$ pass through the point of intersection I of the directrices $D'I$, DI (Fig. 166), it is easy to determine them geometrically. Suppose that the two ellipses intersect in two real points A and B; one of the real common secants is the straight line AB which passes through these two points; the other IL does not intersect the curves. In order to determine this second straight line, join the point A to the focus F and drop from this point the perpendiculars AE, AE' upon the directrices; one has $k=\dfrac{AF}{AE}$, $k'=\dfrac{AF}{AE'}$, and, consequently, $\dfrac{k'}{k}=\dfrac{AE}{AE'}$. Prolong the perpendicular AE till EH is equal to AE; through the point H draw HL parallel to the first directrix, and through the point A, AL parallel to the second directrix; the point of intersection L of these two parallels will belong to the second real common secant IL.

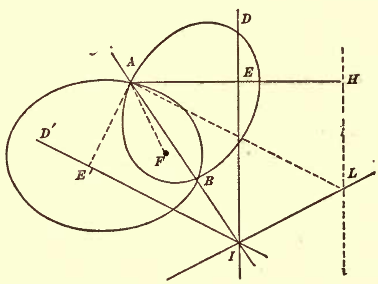

Fig. 166.

289. A circle intersects a curve of the second degree in four real or imaginary points; let

$$(x-a)^2+(y-b)^2-r^2=0$$

be the equation of the circle, $\alpha=0$, $\beta=0$ those of a pair of real common secants; the equation of the curve of the second degree can always be written in the form

$$(x-a)^2+(y-b)^2-r^2=k\alpha\beta.$$

The first member represents the square of the length of the tangent drawn from any point of the curve to the circle; whence follows the theorem: *A circle being placed in any manner whatever in the plane of a curve of the second degree, the tangent drawn from every point of the curve to the circle is to the mean proportional between the distances of this point from the two real common secants in a constant ratio.*

Suppose that the circle be tangent to the curve in two real or conjugate imaginary points, the chord of contacts will be real, and the equation of the curve will take the form

$$(x-a)^2+(y-b)^2-r^2=k\alpha^2.$$

Thus, *in case a circle is doubly tangent to a curve of the second degree, the tangent drawn from any point of the curve to the circle is to the distance of this point from the chord of contacts in a constant ratio.* The focus of a curve of the second degree can be considered as a circle with a radius zero, which has with the curve a double imaginary contact; the directrix is the chord of contact.

290. By aid of the preceding theory one determines in a very simple manner the number of normals that can be drawn from a given point to a curve of the second degree. Let, for example, an ellipse be defined by the equation

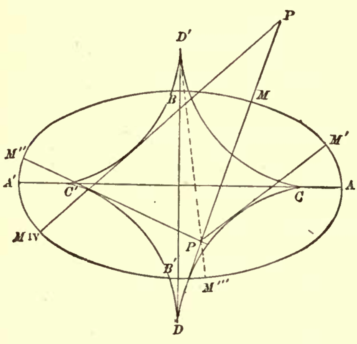

Fig. 167.

(1) $$b^2x^2+a^2y^2=a^2b^2,$$

and P a point whose co-ordinates are x_1y_1 (Fig. 167). Let x and y be the co-ordinates of the foot M of one of the normals; these unknown quantities should satisfy equation (1) and also the equation

(2) $$y_1-y=\frac{a^2y}{b^2x}(x_1-x) \text{ or } c^2xy+b^2y_1x-a^2x_1y=0,$$

which expresses the condition that the normal at the point M passes through the point P. It follows from the preceding that the point M is determined by the intersection of the ellipse (1) and of the equilateral hyperbola defined by equation (2); one of the branches of the hyperbola passes through the center of the ellipse, and the two curves have at least two real points in common. The equation of the third degree on which depends the investigation of secants common to two curves, is

$$(3) \qquad 4a^2b^2\lambda^3 + (a^2x_1^2 + b^2y_1^2 - c^4)\lambda - c^2x_1y_1 = 0.$$

If equation (3) have one real root, we have seen (§ 286) that the curves (1) and (2) cannot have more than two real points in common; if equation (3) have its three roots real, the curves (1) and (2), having at least two real common points, intersect in four real points. One ought to have, in the first case,

$$(4) \qquad (a^2x_1^2 + b^2y_1^2 - c^4)^3 + 27\,a^2b^2c^4x_1^2y_1^2 > 0,$$

and in the second case,

$$(5) \qquad (a^2x_1^2 + b^2y_1^2 - c^4)^3 + 27\,a^2b^2c^4x_1^2y_1^2 < 0.$$

If the co-ordinates x_1, y_1 satisfy the relation

$$(6) \qquad (a^2x_1^2 + b^2y_1^2 - c^4)^3 + 27\,a^2b^2c^4x_1^2y_1^2 = 0,$$

the roots of equation (3) are still real, but it has one double root, and but three distinct normals can be drawn from the point P. The points P which satisfy this condition constitute a curve $CDC'D'$ which has four cusps C, C', D, D'. Equation (6) takes the very simple form

$$a^{\frac{2}{3}}x_1^{\frac{2}{3}} + b^{\frac{2}{3}}y_1^{\frac{2}{3}} = c^{\frac{4}{3}};$$

it is plain that for every point within this curve, relation (5) is satisfied; that is, that through this point four real normals can be drawn, but no more than two real normals can be drawn through any point lying without this curve.

EXERCISES.

1. Construct a curve of the second degree, being given the directrix and three of its points.

2. Construct a parabola, the focus and two of its points being given, or a point and a tangent.

3. Construct a parabola, when its directrix and two of its points are given.

4. Construct a hyperbola, if three of its points and the directions of the asymptotes be given.

5. Construct a hyperbola, when an asymptote, a vertex, and one of its points are given.

6. Find the locus of a vertex of a parabola which has a given focus and is tangent to a given straight line.

7. Find the locus of the focus of a parabola whose vertex is at a given point and which touches a given straight line.

8. Find the locus of the foci of curves of the second degree inscribed in a given parallelogram.

9. A chord revolves about one of the foci of a curve of the second degree; find the locus of the point of intersection of the normals drawn to the curve through its two extremities.

10. Two curves of the second degree have a common focus and an angle of constant magnitude which revolves about its vertex situated at the common focus; find the locus of the point of intersection of the tangents drawn respectively to the two curves at the points where they are intersected by the sides of the angle.

11. Find the locus of the point of intersection of the straight lines drawn parallel to two fixed directions through the extremities of a chord of given length inscribed in a given circumference.

12. Find the locus of the center of an equilateral hyperbola circumscribed about a given triangle.

13. Find the locus of the foci or the vertices of a hyperbola, having an asymptote and a directrix given.

14. Find the locus of the centers of curves of the second degree which passes through the four points of intersection of two given conics. This locus does not change when each of

the conics varies, remaining similar and concentric to the other.

15. A variable circle touches a given ellipse at a given point; find the locus of the point of intersection of the tangents common to the two curves.

16. Find the locus of the center of a hyperbola which has a given focus and which intersects in a given point a given straight line parallel to one of the asymptotes.

17. Find the locus of the focus of a parabola which touches two given straight lines, one of them in a fixed point, the other in a variable point.

18. Find the locus of the point of intersection of two parabolas, which have a given point as focus, which touch a given straight line and which intersect at a given angle.

19. Being given three points A, B, C, and an indefinite straight line, a variable segment MN is taken on this straight line, and is viewed from the point A at a constant angle; find the locus of the point of intersection of the two straight lines BM and CM.

20. Two angles of constant magnitude revolve about their vertices placed at the extremities of the major axis of an ellipse; the point of intersection of two of the sides describes an ellipse; find the locus of the point of intersection of the other two sides.

21. Find the locus of the vertices of an equilateral hyperbola passing through a given point and having a given straight line as an asymptote.

22. Being given a system of conics having the foci F and F', and a fixed straight line passing through the focus F; the tangents to these various conics, at points where each of them is intersected by this straight line, are tangents to the same parabola, whose focus is the point F', and whose directrix is the secant.

The portion of each tangent comprised between the conic and the parabola is viewed from the focus F' at a constant angle.

CHAPTER X*

THEORY OF POLES AND OF POLARS.

291. Let us consider an algebraic equation of the degree m,

$$f(x, y) = 0,$$

written in an integral form. The tangent, at the point whose co-ordinates are x and y, is represented by the equation

(1) $$(X - x)f'_x + (Y - y)f'_y = 0,$$

or $$Xf'_x + Yf'_y - (xf'_x + yf'_y) = 0.$$

This equation involves, moreover, the co-ordinates of the point of contact to the degree m; but one can, owing to the relation (1), cause the terms of the mth degree to disappear. This reduction is easily accomplished by means of a special notation, which we shall presently learn. Suppose that in equation (1) x and y be replaced by $\frac{x}{z}$ and $\frac{y}{z}$, and that every term be multiplied by z^m, the polynomial $f(x, y)$ is transformed into a homogeneous polynomial of the mth degree with respect to the three letters x, y, z, a polynomial which we represent by $f(x, y, z)$. It is evident that, if one put $z = 1$ in the last polynomial, one will get the given polynomial $f(x, y)$. It is known that in case a function $f(x, y, z)$ be homogeneous and of the degree m with respect to the three letters x, y, z, one has identically

$$xf'_x + yf'_y + zf'_z = mf(x, y, z).$$

Whence one has

$$xf'_x + yf'_y = mf(x, y, z) - zf'_z.$$

The value of the second member, when one puts $z = 1$, is equal to the quantity $xf'_x + yf'_y$, which occurs in the equation of the tangent; but the point of contact being on the curve, the first term $mf(x, y, z)$ reduces to zero; the expression $xf'_x + yf'_y$ is therefore equal to the value which zf'_z takes when one puts $z = 1$. One can thus put the equation of the tangent under the form

$$Xf'_x + Yf'_y + zf'_z = 0.$$

For the sake of symmetry, one writes

(2) $$Xf'_x + Yf'_y + Zf'_z = 0.$$

When one has taken the three partial derivatives of the homogeneous function $f(x, y, z)$, one replaces in equation (2) z and Z by unity.

292. We propose now to draw from a given point p, whose co-ordinates are x_1 and y_1, tangents to the given curve. Call x and y the co-ordinates of one of the points of contact; since the tangent at this point passes through the point p, its equation (2) will be satisfied by the co-ordinates x_1 and y_1 of the point p, which furnishes the relation

$$x_1f'_x + y_1f'_y + Zf'_z = 0,$$

which, for the sake of symmetry, one writes in the form

(3) $$x_1f'_x + y_1f'_y + z_1f'_z = 0,$$

in which one may replace at will z and z_1 by unity. The points of contact will be determined by the two simultaneous equations (1) and (3). Since one of these equations is of the degree m and the other of the degree $m - 1$, the number of solutions will be at most $m(m-1)$. Hence from the point p one can draw at most $m(m - 1)$ tangents, real or imaginary, to a curve of the degree m.

In case the curve is of the second degree, equation (3) is of the first degree, and one has two solutions, which are real or conjugate imaginaries. When the two solutions are real, one can draw from the point p to the curve two real tangents. In case the two solutions are conjugate imaginaries, the two tangents are conjugate imaginaries, but the chord of contact (3) remains real. The general equation of curves of the second degree tangent to a curve of the second degree represented by the equation $f(x, y) = 0$, at points where it is intersected by the straight line (3), is (§ 281)

$$f(x, y) - \lambda (x_1 f'_x + y_1 f'_y + z_1 f'_z)^2 = 0,$$

λ designating an arbitrary parameter. If λ be determined in such a way that this curve pass through the point x_1, y_1, it will reduce necessarily to the system of two tangents emanating from this point which are represented by the equation

$$4 f(x_1, y_1) f(x, y) - (x_1 f'_x + y_1 f'_y + z_1 f'_z)^2 = 0.$$

Harmonic Proportion.

293. Being given two points A and B, one knows that there exists on the straight line AB two points C and D so situated that the ratio of their distances from the two points A and B is equal to a particular given ratio (Fig. 168). These two points C and D are called *harmonic conjugates* with respect to the two points A and B. It follows that there is an infinity of systems of harmonic conjugate points with respect to two given points; one can choose one of the points at will. In case the point C approaches the mid-point O of the straight line AB, the conjugate point D moves toward infinity, and conversely.

Fig. 168.

We represent by the symbol AB the distance of the point A from the point B, affected with + sign or − sign, according as the point B is to the right or left of the point A. In accordance with this convention it follows that $AB = -BA$, and the position of the points C and D is expressed by the relation

$$\frac{AC}{BC} = -\frac{AD}{BD}. \tag{4}$$

This relation can be written in the form

$$\frac{CA}{DA} = -\frac{CB}{DB};$$

one sees that, conversely, the two points are harmonic conjugates with respect to the two points C and D.

If the relative positions of the four points be determined by the distances of one of them from the other three, the preceding relation becomes

(5) $$\frac{2}{AB}=\frac{1}{AC}+\frac{1}{AD}.$$

On reckoning the distances from the point O, the mid-point of AB, one has

(6) $$OC \cdot OD = \overline{OB}^2.$$

294. THEOREM I. — *Being given a conic section, if through a point p of the plane one draws any secant mm'* (Fig. 169), *the locus of the point p', the harmonic conjugate of p, with respect to the two points of intersection m and m' of the secant with the curve, is a straight line.* Let

$$f(x, y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

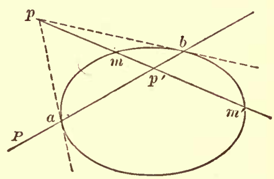

Fig. 169.

be the equation of the curve, x_1 and y_1 be the co-ordinates of the point p; any secant drawn through the point p will be represented by the equations

(7) $$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\rho,$$

in which a and b are two constants, and ρ the distance of the point p from any point m of the straight line, affected by the $+$ sign or $-$ sign, according as the point m is on the one side or the other of the point p; whence it follows that $x = x_1 + a\rho$, $y = y_1 + b\rho$. On substituting these values in the equation of the curve, an equation of the second degree in ρ is found,

$$f(x_1 + a\rho, y_1 + b\rho) = 0,$$

which determines the distances ρ' and ρ'' of the point p from the two points m and m'. The equation developed becomes

$$(Aa^2 + 2Bab + Cb^2)\rho^2 + (af'_{x_1} + bf'_{y_1})\rho + f(x_1, y_1) = 0,$$

or, if $\frac{1}{\rho}$ be regarded as unknown,

$$f(x_1, y_1)\frac{1}{\rho^2} + (af'_{x_1} + bf'_{y_1})\frac{1}{\rho} + (Aa^2 + 2\,Bab + Cb^2) = 0.$$

Call r the distance of the point p from its harmonic conjugate p'. Owing to solution (5), one should have

$$\frac{2}{r} = \frac{1}{\rho'} + \frac{1}{\rho''}.$$

But, by virtue of the last equation,

$$\frac{1}{\rho'} + \frac{1}{\rho''} = -\frac{af'_{x_1} + bf'_{y_1}}{f(x_1, y_1)};$$

whence

$$\frac{2}{r} = -\frac{af'_{x_1} + bf'_{y_1}}{f(x_1, y_1)},$$

or $$arf'_{x_1} + brf'_{y_1} + 2f(x_1, y_1) = 0.$$

The point p' belongs to the straight line pmm', and the co-ordinates x and y satisfy equations (7) of this straight line, that is, one has $x - x_1 = ar$, $y - y_1 = br$; on replacing ar and br by these values in the preceding equation, the variable parameter a and b will be eliminated, and one gets the equation of the locus

(8) $$(x - x_1)f'_{x_1} + (y - y_1)f'_{y_1} + 2f(x_1, y_1) = 0,$$

which is of the first degree. Thus the locus sought is a straight line; this straight line P is called the *polar* of the point p and the point p the *pole* of the straight line P.

By calculation it follows that the constant term

$$2f(x_1, y_1) - x_1 f'_{x_1} - y_y f'_{y_1}$$

reduces to $2\,Dx_1 + 2\,Ey_1 + 2\,F$ and the preceding equation

becomes $$xf'_{x_1} + yf'_{y_1} + (2\,Dx_1 + 2\,Ey_1 + 2\,F) = 0.$$

This reduction can be made in another manner; suppose, as above, that in the polynomial $f(x, y)$, x and y be replaced by

$\frac{x}{z}$ and $\frac{y}{z}$, and that all the terms be multiplied by z^2; this polynomial will be changed into a homogeneous polynomial of the second degree, which we will represent by $f(x, y, z)$. By reason of the theorem of homogeneous functions which has been used in § 291, one has the identity

$$xf'_x + yf'_y + zf'_z = 2f(x, y, z);$$

whence it follows

$$2f(x, y, z) - xf'_x - yf'_y = zf'_z,$$

or, on replacing x, y, z by x_1, y_1, z_1,

$$2f(x_1, y_1, z_1) - x_1f'_{x_1} - y_1f'_{y_1} = z_1f'_{z_1}.$$

Equation (8) can therefore be put under the form

$$xf'_{x_1} + yf'_{y_1} + z_1f'_{z_1} = 0,$$

or, more symmetrically,

$$xf'_{x_1} + yf'_{y_1} + zf'_{z_1} = 0, \tag{9}$$

in which, after the derivatives have been constructed, z and z_1 are replaced by unity. If this equation be developed, one sees that it is not changed when the letters x and x_1, y and y_1, z and z_1 are permuted, and thus one obtains equation (3) of the chord of contact. Hence the polar of the point p coincides with the chord of contact with respect to this point.

295. Next we examine the relative positions of the pole and of the polar. Through the point p draw a secant mm' (Fig. 170) parallel to the chords which the diameter passing through the point p bisects; the point p being the mid-point of mm', its harmonic conjugate is at infinity on this secant; whence one deduces that the polar P is parallel to the chord mm', that is, to the direction conjugate to the diameter passing through the point p.

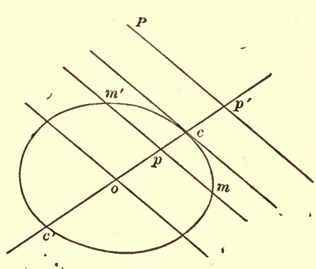

Fig. 170.

Let o be the center of the curve and p' a point of the polar situated on the diameter op, that is, the harmonic conjugate point of the point p with respect to the two extremities c and c' of the diameter, then one has $op \cdot op' = oc^2$. If the pole p be moved along the diameter oc, the polar P will be moved parallel to itself; when the pole moves from o to c, the polar, in the first place situated at infinity, moves toward the curve and becomes tangent to it at c; if the pole crosses the curve and continues its motion toward infinity, the polar intersects the curve in two real points and moves toward the center of the curve.

In case the curve is a parabola, the point c' being situated at infinity, the point c is the mid-point of pp'.

It is easily seen that the converse is true: every straight line has one pole and only one, except in the case of the parabola, when the straight line is parallel to the axis. The curve being referred to any axes, in order to determine the co-ordinates x_1 and y_1 of the pole p of a given straight line $ux + vy + w = 0$, it will be sufficient to identify this equation with equation (9), which represents the polar of p, which furnishes the two relations,

(10) $$\frac{f'_{x_1}}{u} = \frac{f'_{y_1}}{v} = \frac{f'_{z_1}}{w}.$$

On calling $\frac{2}{\lambda}$ the common value of these ratios and developing, one gets

$$A\lambda x_1 + B\lambda y_1 + D\lambda = u,$$

$$B\lambda x_1 + C\lambda y_1 + E\lambda = v,$$

$$D\lambda x_1 + E\lambda y_1 + F\lambda = w,$$

equations of the first degree in λx_1, λy_1, λ. If the conic does not reduce to two straight lines, the determinant Δ of the coefficients of the unknown quantities is not zero, and on using the general formulas for the solution of equations of the first degree, one has

$$\Delta\lambda x_1 = \mathrm{a}u + \mathrm{b}v + \mathrm{d}w,$$

$$\Delta\lambda y_1 = \mathrm{b}u + \mathrm{c}v + \mathrm{e}w,$$

$$\Delta\lambda \quad = \mathrm{d}u + \mathrm{e}v + \mathrm{f}w.$$

In case the value found for λ is not zero, these equations determine x_1 and y_1. The value of λ is zero in case the coefficients of the straight line satisfy the condition

$$\mathbf{d}u + \mathbf{e}v + \mathbf{f}w = 0,$$

that is, on supposing $\mathbf{f}$ different from zero, when the straight line passes through the center of the conic whose co-ordinates are $\frac{\mathbf{d}}{\mathbf{f}}, \frac{\mathbf{e}}{\mathbf{f}}$; or, supposing $\mathbf{f}$ to be zero, the case of the parabola, when the straight line is parallel to the axis of the parabola.

REMARK. — If the curve of the second degree reduce to two distinct straight lines whose equations are

$$\alpha = lx + my + n = 0, \quad \beta = l'x + m'y + n' = 0,$$

one will have identically

$$f(x, y) = \alpha\beta,$$

$$f'_x = l\beta + l'\alpha, \quad f'_y = m\beta + m'\alpha, \quad f'_z = n\beta + n'\alpha;$$

and on putting

$$\alpha_1 = lx_1 + my_1 + n, \quad \beta_1 = l'x_1 + m'y_1 + n',$$

the polar of the point with the co-ordinates x_1, y_1 will have the equation

$$x_1 f'_x + y_1 f'_y + f'_z = \beta\alpha_1 + \alpha\beta_1 = 0,$$

the equation of a straight line passing through the point of intersection of the two lines $\alpha = 0$, $\beta = 0$, or parallel to these two straight lines in the case where they are parallel to each other. It follows from this form of the equation of the polar that:

The polar of any point of the plane other than the point of intersection of two straight lines passes through this point of intersection. The polar of the point of intersection is indeterminate. If the pole describe a straight line $\beta - m\alpha = 0$, passing through the point of intersection of the two straight lines, the polar remains fixed, its equation being $\beta + m\alpha = 0$. Conversely, a straight line which does not pass through the point of intersection has this point for pole. A straight line $\beta + m\alpha = 0$, passing through the point of intersection of two straight lines, has an infinity of poles situated on the straight line $\beta - m\alpha = 0$. (See § 103.)

296. THEOREM II. — *The polars of all the points of a straight line pass through the pole of this straight line,* and, conversely, *the poles of all the straight lines which pass through the same point are situated on the polar of this point.*

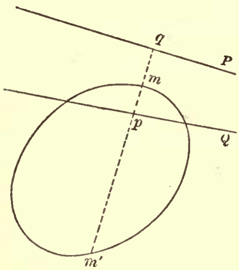

Fig. 171.

On the straight line P whose pole is p, select any point q (Fig. 171); the straight line pq intersects the conic in two points m and m'; these two points p and q being harmonic conjugates with respect to the two points m and m', the polar Q of the point q passes through the point p.

Conversely, let q be the pole of any straight line Q passing through the point p; the two points q and p being harmonic conjugates with respect to the two points m and m' in which the straight line pq intersects the conic, the point q belongs to the polar P of the point p.

CONJUGATE STRAIGHT LINES. — Two straight lines are said to be conjugates with respect to a conic in case the pole of either lies on the other. Let

$$ux + vy + w = 0, \quad u'x + v'y + w' = 0$$

be two conjugate straight lines.

On expressing the condition that the pole (x_1, y_1) of the first lies on the second, it follows that

$$u'x_1 + v'y_1 + w' = 0,$$

or, writing the preceding values for x_1 and y_1,

$$u'(\mathrm{a}u + \mathrm{b}v + \mathrm{d}w) + v'(\mathrm{b}u + \mathrm{c}v + \mathrm{e}w) + w'(\mathrm{d}u + \mathrm{e}v + \mathrm{f}w) = 0,$$

$$\mathrm{a}uu' + \mathrm{b}(uv' + vu') + \mathrm{c}vv' + \mathrm{d}(uw' + wu') + \mathrm{e}(vw' + wv') + \mathrm{f}ww' = 0.$$

This condition can, moreover, be written as follows, if one put

$$\phi(u, v, w) = \mathrm{a}u^2 + 2\,\mathrm{b}uv + \mathrm{c}v^2 + 2\,\mathrm{d}uw + 2\,\mathrm{e}vw + \mathrm{f}w^2,$$

$$u'\phi'_u + v'\phi'_v + w'\phi'_w = 0,$$

$$u\phi'_{u'} + v\phi'_{v'} + w\phi'_{w'} = 0.$$

297. Theorem III. — *Being given a conic section, if any two secants pmm', pnn', which intersect the curve in m, m', n, n', be drawn through a point p* (Fig. 172), *the points of intersection q and q' of the straight lines mm, m'n', or m'n, mn', belong to the polar of the point p.*

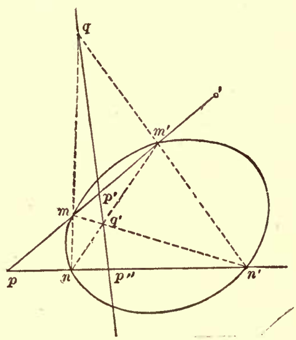

Fig. 172.

We remark in the first place that Theorem I. holds, in case the locus of the second degree reduces to a system of two straight lines; in this case the polar of the point p passes through the vertex of the angle; for, if the secant which passes through the vertex be considered, the two points m and m' coincide with this point, p and p' being harmonic conjugates.

This being established, let us consider the system of two straight lines, mn, $m'n'$, which intersect in q. The straight line pmm' intersects the conic section and the two sides of the angle mqm' in the same points, m and m'; the point p', a harmonic conjugate of the point p, is the same point on the secant pmm' whether one regard this secant as belonging to the conic section or to the angle. The point p'', a harmonic conjugate of p, will remain the same, in both cases, on the secant pnn'. The polars of the point p, with respect to the conic section and to the angle, having two common points p' and p'', coincide; but one knows that the polar with respect to the angle passes through the vertex q, therefore the point q belongs to the polar of the point p with respect to the curve. For the same reason, the point q' belongs to this same polar.

Corollary. — The curve being traced, one gets from this theorem the means for constructing the polar of the point p. Draw through the point p two secants, pmm', pnn', by means of which two points, q and q', of the polar are determined.

If the point p lie without the curve, the polar intersects the curve in two points, which are the points of contact of the tangents drawn from the point p.

Remark. — In Fig. 172 the polar of the point q is the straight line pq', and the polar of the point q' is the straight line pq. The triangle whose vertices are the points p, q, q', possesses, therefore, this remarkable property, that each of its sides is the polar of the opposite vertex with respect to the curve of the second degree. It is said that such a triangle is *autopolar* or is a *conjugate* triangle with respect to the curve of the second degree. Conversely, it is also said that the curve is **conjugate** to the triangle.

Reciprocal Polar Figures.

298. Being given a plane figure composed of the points a, b, c, ⋯ and of straight lines A, B, C, ⋯, if one construct the

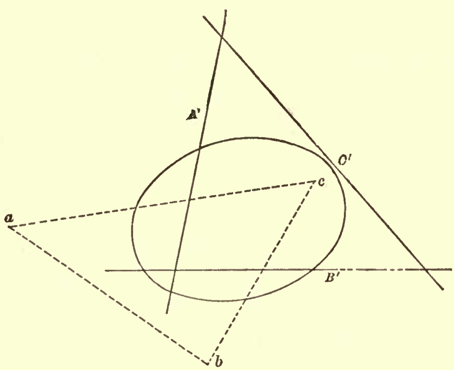

Fig. 173.

polars A', B', C', ⋯ of the points, and the poles a', b', c', ⋯ of the straight lines, with respect to a definite conic section, one forms a second figure composed, like the first, of straight lines and of points. On treating the second figure in the same manner, that is, on taking the poles of the straight lines and the polars of the points, one gets the first figure. These two figures have been called for this reason *reciprocal polar* figures (Fig. 173).

The straight line ab, which joins the two points a and b of one of the figures, has as pole the point of intersection of the two straight lines A' and B' of the other figure; and conversely, the point of intersection of the two straight lines A' and B' of one of the figures has as polar the straight line ab of the other figure. If several points $a, b, c, \cdots$ lie on a straight line in one of the figures, the straight lines $A', B', C', \cdots$ of the other figure pass through the same point, which is the pole of the straight line. Conversely, if several straight lines $A, B, C, \cdots$ pass through the same point in one of the figures, the points $a', b', c', \cdots$ of the other figure lie in a straight line.

A plane curve S being given, draw a tangent A to this curve and determine the pole a' of this tangent (Fig. 174). If the tangent A revolves about the curve S, the pole a' will describe another curve S'. Let A and B be two tangents to the curve S, a' and b' their poles; the point of intersection m of the two straight lines A and B is the pole of the straight line $a'b'$. If the tangent B approach the tangent A as its limit, the point m will approach the point of contact a of the tangent A; at the same time the secant $a'b'$ revolves about the point a' and becomes tangent to the curve S' at the point a'. Hence, conversely, the curve S is the locus of the pole a of a movable tangent A' of the curve S'. The points a and a' correspond to each other in such a way that the tangent at one of these points is the polar of the other. The two curves S and S' are for this reason called *reciprocal polars*.

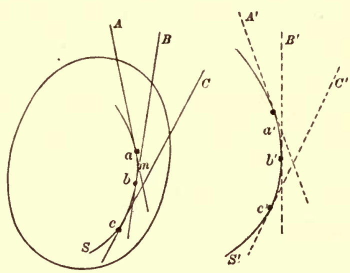

Fig. 174.

Let

(11) $$F(x, y) = 0$$

be the equation of an algebraic curve S of the degree m; the tangent A at the point a, whose co-ordinates are x and y, is represented by the equation

(12) $$XF'_x + YF'_y + ZF'_z = 0.$$

Call x_1 and y_1 the co-ordinates of the pole a' of the straight line A, with respect to a curve of reference of the second degree $f(x, y) = 0$; the equation of the polar of the point a' is

(13) $$Xf'_{x_1} + Yf'_{y_1} + Zf'_{z_1} = 0.$$

The two equations (12) and (13), which represent the same straight line, should be identical, and one has the relations

(14) $$\frac{f'_{x_1}}{F'_x} = \frac{f'_{y_1}}{F'_y} = \frac{f'_{z_1}}{F'_z}.$$

If, among the three equations (11) and (14), x and y be eliminated, one has the equation of the curve S_1, the locus of the point a'.

Seek, for example, the reciprocal polar curve of the conic section $Ax^2 + By^2 - 1 = 0$, with respect to the circle of reference

$$x^2 + y^2 - 1 = 0.$$

If x and y be replaced by $\frac{x}{z}$ and $\frac{y}{z}$, these two equations assume the homogeneous forms $Ax^2 + By^2 - z^2 = 0$, $x^2 + y^2 - z^2 = 0$, and equations (14) become $\frac{x_1}{Ax} = \frac{y_1}{By} = \frac{z_1}{z}$; whence it follows, on putting $z = z_1 = 1$, $x = \frac{x_1}{A}$, $y = \frac{y_1}{B}$; by substituting these values in the equation of the given curve, one has the equation $\frac{x_1^2}{A} + \frac{y_1^2}{B} - 1 = 0$. The polar reciprocal curve is a new conic section.

299. The *degree* or *order* of an algebraic curve has been called the degree of the equation by which it is represented in rectilinear co-ordinates, or the number of points, real or imaginary, in which the curve is intersected by any straight line. In like manner, *the class* of the curve is the number of tangents, real or imaginary, which can be drawn to the curve from any point of the plane. It is known that from any point two tangents can be drawn to a curve of the second degree; curves of the second order belong therefore to the second class.

It is very easily proven that two reciprocal polar curves S and S' (Fig. 175) are such that the order of one is equal to the class of the other. Any straight line P intersects the curve S in m points a, b, c, $\cdots$; to these m points correspond m straight lines A', B', C', $\cdots$, tangents to the curve S', and passing through the point p', the pole of the straight line P; conversely, to each tangent A' drawn from the point p' to the

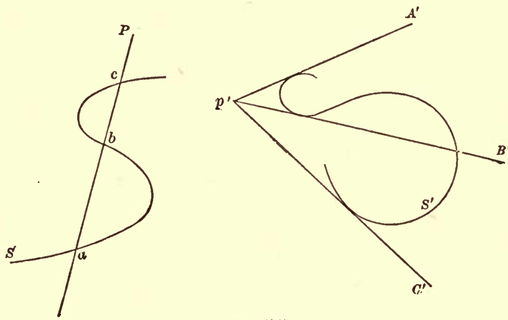

Fig. 175.

curve S' corresponds a point a belonging to the curve S and situated on the straight line P. Hence, the number of tangents which can be drawn from the point p' to the curve S' is equal to the number of points of intersection of the curve S by the straight line P, and, consequently, the class of the curve S' is equal to the order of the curve S. Similarly, the order of the curve S' is equal to the class of the curve S.

A curve of the second order being of the second class, it follows that the reciprocal polar curve of a curve of the second order is also of the second order.

In like manner, it is easy in this case to determine the species of the curve. If the center o of the curve of reference be situated without the curve S, two real tangents A and

B can be drawn from this point to the curve S (Fig. 176); the poles of these tangents lying at infinity, it follows that the curve S' has two infinite branches with different directions; it is therefore a hyperbola. Let a and b be the points of contact of the tangents A and B; the polars A' and B', of these two points are tangents to the curve S' at points situated at infinity; they are, therefore, the asymptotes. In case the center o of the curve of reference is situated on the curve S, the points a and b coincide with the point o, the polar of this point, or the asymptote is removed to infinity, and the curve S' is a parabola. Finally, if the center o of the curve of reference be situated within, the curve S' is an ellipse.

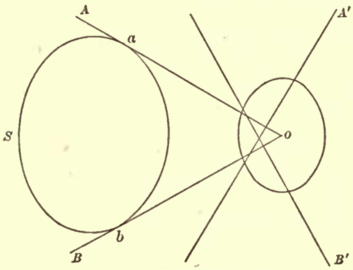

Fig. 176.

To any two points a and b of the conic S and the tangents A and B at these two points correspond two tangents A' and B' to the conic S' and their points of contact a' and b'. To the point of intersection c of the straight lines A and B corresponds the straight line $a'b'$, and to the straight line ab the point of intersection c' of the straight line A' and B'. Hence, to a point c and its polar ab in the first figure correspond in the second figure a straight line $a'b'$ and its pole c'.

300. The method of reciprocal polars plays an important rôle in the study of conic sections; it is possible when a property of these curves has been found by the method of reciprocal polars to deduce immediately a correlative property. It has been demonstrated, for example, in § 275, that through five given points one can pass a conic section, and one only; whence it follows that *a conic section can be drawn tangent to five given straight lines, and one only.* Imagine, in fact, that any conic section be drawn in a plane as the curve of reference, and that with respect to this conic section one locates the poles a', b', c', d', e' of the five given straight lines A, B,

C, D, E; a conic section S' can be drawn through the five points a', b', c', d', e'; the polar reciprocal curve of the curve S' will be a conic section S tangent to the five given straight lines. Conversely, to every conic section tangent to five straight lines corresponds a conic section passing through five points; since but one conic section can be passed through five points, it follows that but one conic section can be drawn tangent to five straight lines.

Let us consider the polars of a point p with respect to various conics which pass through four given points; if $f(x, y) = 0$, and $F(x, y) = 0$, be the equations of two of them, the equation $f + kF = 0$, in which k is an arbitrary parameter, will represent the *ensemble* of these conics. The polar of the point p, whose co-ordinates are x and y, has the equation

$$x_1(f'_x + kF'_x) + y_1(f'_y + kF'_y) + z_1(f'_z + kF'_z) = 0,$$

or $$(x_1 f'_x + y_1 f'_y + z_1 f'_z) + k\,(x_1 F'_x + y_1 F'_y + z_1 F'_z) = 0;$$

all the polars pass through the point of intersection p' of the two straight lines

$$x_1 f'_x + y_1 f'_y + z_1 f'_z = 0, \quad x_1 F'_x + y_1 F'_y + z_1 F'_z = 0.$$

It is clear that, conversely, the polars of the point p', with respect to the various conics, all pass through the point p.

If the figure be transformed by the method of reciprocal polars, it follows that the locus of the poles of a straight line P with respect to the various conics tangent to the four given straight lines is a straight line. If the straight line P be moved to infinity in an arbitrary direction, its pole with respect to each of the conics is the center of this conic; hence the locus of the centers of conics tangent to four given straight lines is a straight line. Each of the diagonals of the quadrilateral formed by four straight lines can be regarded as an ellipse or a hyperbola infinitely flattened and tangent to the four straight lines; the mid-points of the three diagonals belong to the locus and determine this straight line (§ 73).

301. THEOREM IV. — *Two conics are given in a plane:* 1° *The points of intersection of three pairs of common secants* *determine a triangle each vertex of which has as polar, with respect to each of the conics, the opposite side; 2° The points of intersection of the four common tangents to the two conics are situated two by two on the sides of this triangle.*

Let a, b, c, d (Fig. 177) be the four points common to the two conics; the points of intersection m, n, p of the three pairs

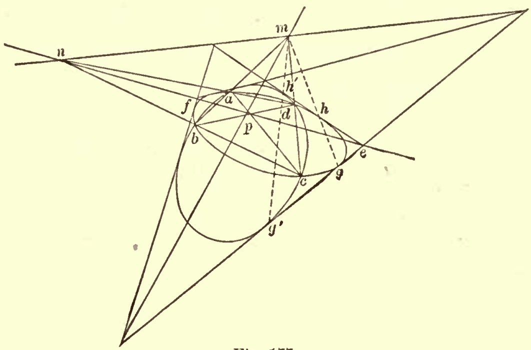

Fig. 177.

of common secants form a triangle mnp, each vertex of which, according to Theorem III., has as polar, with respect to each of the two conics, its opposite side. We notice that these three points are the only ones which enjoy the same polar property with respect to the two conics. Let m' be a point having the same polar with respect to the two conics; the straight line $m'a$ intersects this polar in a certain point q and each of the conics in a new point which is the harmonic conjugate of a with respect to the two points m' and q; these two new points ought to coincide, the straight line $m'a$ passes through one of the common points b, c, d, for example through the point b. Then the straight line $m'c$ will pass through the point d and the point m' will coincide with the point m.

Imagine now that the preceding figure be transformed by the method of reciprocal polars. To the two conics there will correspond two other conics; to the points a, b, c, d common to the first two the tangents A', B', C', D' common to the two new conics, which shows that the two conics have four common tangents.

Consider one of the tangents common to the two given conics; let g and g' be the points of contact and e the point where it intersects the straight line np; the straight lines mg, mg' intersect the conics in two other points h and h'; the point m, having the same polar np with respect to the two conics, the tangents at h and h' pass through the point e; the straight line np being also the polar of the point m with respect to the two angles geh, $g'eh'$, the two straight lines eh, eh' coincide, and the straight line ehh' is a second common tangent. For the same reason, the point of intersection f of one of the other common tangents with the straight line np belongs to the fourth tangent. Thus the six points of intersection of the four common tangents are situated two by two on the sides of the triangle mnp.

Whence it follows in addition to what precedes that the chords of contact pass four by four through the points m, n, p.

302. Let us study in particular the case where the curve of reference is a circle of radius r; the polar A' of a point a is perpendicular to oa and at a distance from the center equal to $\frac{r^2}{oa}$. The straight lines which join the center to the two points a and b inclose an angle aob equal to that included by the polars A' and B' of these points (Fig. 178).

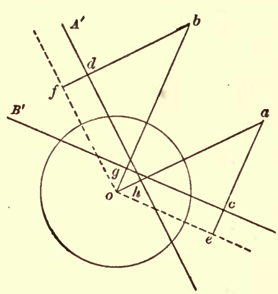

Fig. 178.

Through the center o draw lines parallel to the straight lines A' and B'; from the points a and b draw lines perpendicular to these straight lines; the right-angled triangles oae, obf are similar and give the proportions

$$\frac{oa}{ob}=\frac{ae}{bf}=\frac{ac+ce}{bd+df}=\frac{ac+og}{bd+oh};$$

whence follows $oa\,(bd+oh)=ob\,(ac+og)$; but one has

$$oa\cdot oh=ob\cdot og=r^2;$$

whence it follows that $oa \cdot bd = ob \cdot ac$, or $\frac{oa}{ob} = \frac{ac}{bd}$. Hence the distances of two points from the center are proportional to the distances of each of them from the polar of the other.

Find the reciprocal polar of a circle of radius r' with respect to a circle o. Let C' be the polar of the center c of the given circle (Fig. 179); draw to this circle any tangent A and locate the pole a' of this straight line; owing to the preceding property, one has

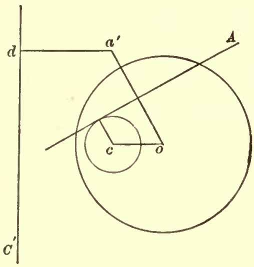

Fig. 179.

$$\frac{oa'}{oc} = \frac{a'd}{r'}, \text{ or } \frac{oa'}{a'd} = \frac{oc}{r'};$$

the ratio of the distances of each of the points a' of the locus from the point o and the fixed straight line C' is constant; therefore this locus is a curve of the second degree, of which the point o is one of the foci and the straight line C' the corresponding directrix.

By means of this transformation, most of the focal properties of curves of the second degree may be deduced from the properties of the circle. Thus, for example, two tangents A and B to the circle c form equal angles with the chord of contact ab; to two straight lines A and B correspond two points a' and b' of the conic section; to the two points a and b of the circle correspond the tangents A' and B' to this conic section at the points a' and b'; to the straight line ab or M corresponds the point of intersection m' of the straight lines A' and B'. The radii vectores drawn from the focus o to the points a', b', m' forming angles with one another equal to those of their polars A, B, M, it follows that the straight line om' is the bisector of the angle $a'ob'$ (§ 255).

The locus of the vertex m of a constant angle circumscribed about a circle is a concentric circle. To the two tangents A and B drawn from the point m to the circle correspond two points a' and b' of the conic, and to the point m the straight line $a'b'$; the angle $a'ob'$, being equal to that of the straight lines A and B, is also constant; since the point m describes a circle whose center is c, its polar $a'b'$ envelopes a conic section, of which o is one of the foci and the polar of the center c the corresponding directrix. Hence *the chord viewed from a focus of a conic section and subtending a constant angle envelopes a conic section which has the same focus and the same directrix.* The chord ab of the circles envelopes a concentric circle; therefore the point of intersection of the tangents to the conic section at a' and b' describes a conic section which has also the same focus and the same directrix.

ENVELOPE CURVES.

303. In what precedes we were led to consider curves which were tangent to a series of straight lines; in case a point describes a curve, its polar remains tangent to another curve. The *envelope* of a movable curve is the curve to which this line remains constantly tangent.

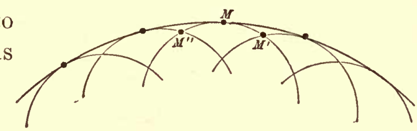

Fig. 180.

Let

(1) $f(x, y, a) = 0$

be an equation involving a variable parameter a. To each value of a corresponds a definite curve. Give to the parameter two consecutive values a and $a + h$; the curve (1) and the curve

(2) $$f(x, y, a + h) = 0$$

intersect in a point M' (Fig. 180), whose co-ordinates satisfy at the same time equations (1) and (2). The system of these two equations can be replaced by the following:

$$f(x, y, a) = 0, \quad \frac{f(x, y, a + h) - f(x, y, a)}{h} = 0,$$

which, when h approaches zero, reduces to

(3) $$f(x, y, a) = 0, \quad f'_a(x, y, a) = 0;$$

hence, when h approaches zero, the point M' is displaced on the curve (1) and approaches a limiting position M; it is this limiting point which is represented by the system (3). Each of the curves (1) contains a limiting point; the locus of these points, which is sometimes designated by the name of the locus of the *ultimate intersections* of the curve represented by equation (1), is obtained by eliminating a between equations (3).

Consider the new system of equations (1) and (2), in which a is regarded as a variable and h as a constant; this system represents the locus of the points in which each curve (a) is intersected by the curve $(a+h)$. Two of these points lie on the curve a; namely, the point of intersection M' of the curves (a) and $(a+h)$, the point of intersection M'' of the curves $(a-h)$ and (a). When h approaches zero, the points M' and M'' approach the same limiting position M, and the locus becomes tangent to the curve (a) at the point M. Hence, *the locus of the ultimate intersections of the curves represented by equation* (1) *is tangent to each of these curves.*

Remark. — When $f(x, y, a)$ is a polynomial with respect to a, to eliminate a between equations (3) is to express the condition that the equation in a

$$f(x, y, a) = 0$$

has a double root.

For example, if a enters to the second degree in $f(x, y, a)$, and if the equation of the movable curve have the form

$$Ma^2 + 2\,Na + P = 0,$$

M, N, P being polynomials in x and y, the equation of the envelope obtained by expressing the condition that the equation in a has a double root will be

$$N^2 - MP = 0.$$

According to this method it can easily be verified that the envelope of the conics whose equation is

$$\frac{\alpha^2}{\lambda} + \frac{\beta^2}{1-\lambda} - \gamma^2 = 0,$$

when α, β, γ represent given linear functions in x and y and λ a variable parameter, is composed of four straight lines (§ 282. 2).

304. Assume now that the movable curve be represented by an equation

(4) $$f(x, y, a, b) = 0,$$

containing two variable parameters a and b, connected by the relation

(5) $$\phi(a, b) = 0.$$

If b' be called the derivative of b considered as a function of a given by equation (5), one has $\phi'_a + \phi'_b b' = 0$; whence $b' = -\frac{\phi'_a}{\phi'_b}$. But if the derivative with respect to a of the function $f(x, y, a, b)$ be equated to zero when b is regarded as a function of a, one has $f'_a + f'_b b' = 0$; whence follows the relation

(6) $$\frac{\phi'_a}{f'_a} = \frac{\phi'_b}{f'_b},$$

and in order to find the equation of the envelope, the two parameters a and b are eliminated by means of the three equations (4), (5), (6).

EXAMPLE I. — Find the envelope of the normal to a parabola. The normal to the parabola $y^2 = 2px$, at the point M (Fig. 181) whose co-ordinates are x and y, has the equation $p(Y - y) + y(X - x) = 0$; if x be replaced by its value $\frac{y^2}{2p}$, this equation becomes

(7) $$pY + y(X - p) - \frac{y^3}{2p} = 0;$$

it involves an arbitrary parameter y; it is necessary to equate to zero the derivative with respect to y,

(8) $$X - p - \frac{3y^2}{2p} = 0,$$

and to eliminate y from equations (7) and (8). On replacing y^2 in

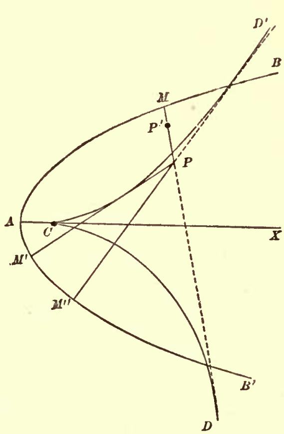

Fig. 181.

equation (7) by its value derived from equation (8), one has

$$y = -\frac{3\,pY}{2\,(X-p)};$$

substituting this value of y in equation (8), one obtains the equation of the envelope

(9) $$Y^2 = \frac{8\,(X-p)^3}{27\,p}.$$

This curve takes the form given in the figure; it has a cusp at C, because the tangent at this point being normal to the parabola at the vertex A coincides with the axis AX. When the point M describes the branch AB of the parabola, the normal revolves about the branch CD of the envelope; and, similarly, when the point M describes the branch AB' of the parabola, the normal resolves about the branch CD' of the envelope.

If one wishes to draw the normals to the parabola from a given point P, it is sufficient to regard X and Y in equation (7) as co-ordinates of the point P and the ordinate y of the foot M of the normal as unknown; three normals can be drawn from the point P to the parabola, or one only, according as this equation of the third degree in y has three real roots or one only. This problem is like that of drawing tangents from the point P to the envelope; hence the envelope, which is of the third degree, is also of the third class. In case the point P is situated between the branches of the envelope, three tangents can be drawn from this point to the envelope, and, consequently, three normals to the parabola; but, in case the point P is situated at P' without, one tangent only can be drawn to the envelope, and, consequently, one normal only to the parabola.

Example II. — Let us find the envelope of the normals to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0;$$

the equation of the normal at the point (x, y)

(10) $$\frac{a^2X}{x}-\frac{b^2Y}{y}-(a^2-b^2)=0$$

involves the variable parameters x and y connected by the relation

(11) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

Equation (6) of § 304 becomes

$$\frac{\frac{x}{a^2}}{-\frac{a^2X}{x^2}}=\frac{\frac{y}{b^2}}{\frac{b^2Y}{y^2}}=\frac{1}{b^2-a^2};$$

the third ratio has been found by adding the numerators and denominators, after having multiplied the two terms of the first by x, the two terms of the second by y, and using equations (10) and (11); whence follow

$$x^3=\frac{a^4X}{c^2},\quad y^3=-\frac{b^4Y}{c^2}.$$

Substituting these values in equation (11), one obtains the equation of the envelope

(12) $$\left(\frac{aX}{c^2}\right)^{\frac{2}{3}}+\left(\frac{bY}{c^2}\right)^{\frac{2}{3}}=1.$$

This curve has four points of inflection (Fig. 182). When the foot M describes the arc AB of the ellipse, the normal revolves about the arc CD of the envelope. If one wishes to draw the normals to the ellipse from a given point P, whose co-ordinates are X and Y, the two simultaneous equations (10) and (11) will determine the co-ordinates x and y of the foot of each of the normals; the feet of the normals are the points of intersection of the given ellipse (11) and of a hyperbola (10); hence there are four solutions. Moreover, this problem resolves itself into drawing the tangents from the point P to the envelope; it follows that the envelope which is of the sixth degree is of the fourth class. In case the point P be situated within the envelope, four tangents can be drawn from this point to the envelope, and, consequently, four real normals to the ellipse; but if the point lie without, for example at P', two real tangents only can be drawn to the envelope, and, consequently, two normals to the ellipse; thus the results obtained in § 290 are proved anew.

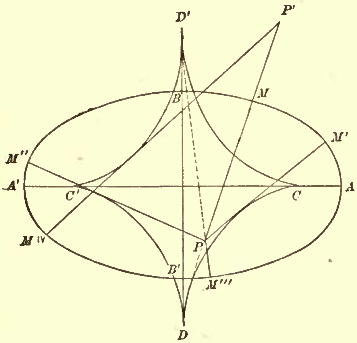

Fig. 182.

The envelope of the normals of a hyperbola has the equation

$$(13)\quad \left(\frac{aX}{c^2}\right)^{\frac{2}{3}}-\left(\frac{bY}{c^2}\right)^{\frac{2}{3}}=1.$$

305. In case a variable plane moves in a fixed plane, it can happen that a curve CD of the variable remains tangent to a curve AB of the fixed plane; this second curve is the envelope of the first. Let CD and $C'D'$ (Fig. 183) be two consecutive positions of the variable curve, M' a point of intersection of these two curves. As the curve $C'D'$ approaches continuously the curve CD, the point M' approaches a limiting position M, which is a point of the envelope (§ 303). Let M_1 be the point of the curve CD which has arrived at M', when this curve takes the position $C'D'$. We have seen (§ 31) that a variable curve can be brought from one position to another by rotating it about a fixed point I_1; the perpendicular PI_1, erected at the mid-point of the chord M_1M', passes through the point I_1. But the two points M_1 and M' have the point M as their limiting position, and the straight line PI_1 becomes the normal common to the curve CD and its envelope at the point M; this normal passes therefore through the point I, the limiting

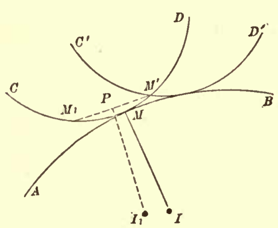

Fig. 183.

position of the point I_1, whence it follows that *the normals to the various curves situated in the variable plane, at the points where they touch their envelopes, for one position of the variable plane, pass through the same point I.* For the same position of the plane, this point is that point through which the normals to the curves described by the points of the variable plane pass.

In case a curve of the variable plane remains tangent to a given curve of the fixed plane, the common normal to the two curves can be used to determine the point I. The polar of a curve AB (Fig. 31) with respect to the point O (§ 38) is none other than the locus described by the vertex P of a right angle OPM situated in the movable plane, which so moves that one of its sides PM remains tangent to the curve AB, while the other passes through the fixed point O; that is, remains tangent to a circle of radius zero whose center is O; whence follows the construction that the point I is found by the intersection of the perpendiculars drawn from the points O and M to the two sides OP and PM of the right angle. (See construction given in § 38.)

Tangential Co-ordinates.

306. A curve may be regarded either as the locus of a point, or as the envelope of a variable straight line. From the second point of view, we represent the straight line by an equation of the form

$$ux + vy + 1 = 0, \tag{14}$$

and we say, by analogy, that the two parameters u and v, which determine its position, are the co-ordinates of the straight line.

If one be given an equation

$$\phi(u, v) = 0 \tag{15}$$

between these two parameters, and one of them be allowed to vary in a continuous manner, the other will also vary in general in a continuous manner, and the straight line will be given a motion in the plane, enveloping a curve. One can think of equation (15) as representing a curve through a series of its tangents, by means of a new system of co-ordinates u and v, which are called for convenience *tangential co-ordinates.*

In order to get the equations of this curve in linear co-ordinates, it suffices, after what has been given in § 304, to eliminate u and v from equations (14), (15), and

(16) $$\frac{x}{\phi'_u}=\frac{y}{\phi'_v}.$$

If equation (15) be algebraic, the equation in x and y, the equation which we will deduce, will also be algebraic. Make equation $\phi(u, v)=0$ homogeneous by replacing u and v by $\frac{u}{w}$, $\frac{v}{w}$ and removing the denominators. The equation $\phi=0$ can be replaced by the following

$$u\phi'_u+v\phi'_v+\phi'_w=0,$$

in which one puts, after constructing the derivatives, $w=1$. Therefore on calling λ the common value of the ratios in (16), it is necessary to eliminate u, v, λ from the four equations

$$x=\lambda\phi'_u,\ y=\lambda\phi'_v,$$

$$ux+vy+1=0,$$

$$u\phi'_u+v\phi'_v+\phi'_w=0.$$

Multiply the first of these equations by u, the second by v, and using the last two equations, we will have the equation

$$1=\lambda\phi'_w,$$

which can replace one of the last two equations, for example the last. One will have therefore to eliminate u, v, λ from the equations

(17) $$x=\lambda\phi'_u,\ y=\lambda\phi'_v,\ 1=\lambda\phi'_w,$$
$$ux+vy+1=0.$$

The degree of equation (15), written in an integral form, indicates the class of the curve. For if x_0 and y_0 be the co-ordinates of any point of the plane, each system of values of u and v, satisfying the two equations

$$ux_0 + vy_0 + 1 = 0, \quad \phi(u, v) = 0,$$

will determine a tangent, real or imaginary, passing through the point in question. In case equation (15) is of the second degree, the curve, being of the second class, is also of the second order (§ 309).

An equation of the first degree

$$Au + Bv + C = 0,$$

in tangential co-ordinates, represents a point, the point which has the linear co-ordinates $x_0 = \frac{A}{C}$, $y_0 = \frac{B}{C}$; because this equation, put under the form

$$ux_0 + vy_0 + 1 = 0,$$

indicates that the variable straight line passes always through the fixed point (x_0, y_0); the envelope reduces therefore to a point.

The properties of the equation of the first degree in linear co-ordinates, which has been studied in Book II., is here reproduced, with the modification that points are replaced by straight lines and straight lines by points. Thus the equation

$$v - v' = a(u - u'),$$

in which the parameter a is arbitrary (§ 64), is the general equation of points situated on the straight line (u', v'). The equation (§ 66)

$$(18) \qquad v - v' = \frac{v'' - v'}{u'' - u'}(u - u')$$

represents the point of intersection of the two straight lines (u', v'), (u'', v'').

Consider two consecutive tangents of the curve (15), and suppose that the second approaches continually the first; their point of intersection, represented by equation (18), will have as a limit the point of contact of the first tangent; the point of contact is therefore represented by the equation (§ 89)

$$v - v' = -\frac{\phi'_{u'}}{\phi'_{v'}}(u - u'),$$

or

(19) $$(u - u')\,\phi'_{u'} + (v - v')\,\phi'_{v'} = 0.$$

On replacing u and v by $\frac{u}{w}$ and $\frac{v}{w}$, in order to make the equation homogeneous (§ 291), this equation is simplified and written in the form

(20) $$u\phi'_{u'} + v\phi'_{v'} + w\phi'_{w'} = 0.$$

307. It is well to notice that the investigation of the envelope of a variable straight line can be reduced to the theory of reciprocal polars because this envelope curve is the reciprocal polar curve of the curve S' described by the pole of the straight line, with respect to a given conic. If one choose as curve of reference the imaginary circle $x^2 + y^2 + 1 = 0$, and if one put $x_1 = u$, $y_1 = v$, the straight line $xx_1 + yy_1 + 1 = 0$, the polar of the point $(x_1,\ y_1)$ coincides with the variable straight line (14); hence the curve S' has the equation $\phi(x_1,\ y_1) = 0$ in linear co-ordinates.

Example I. — Find the envelope of a straight line such that the product of its distances from two fixed points F and F' be equal to a given constant quantity. On choosing the straight line FF' as x-axis, and a perpendicular at the mid-point of this straight line as y-axis, calling $2c$ the distance FF', b^2 the constant product, and representing the variable straight line by the equation $ux + vy + 1 = 0$, one has the relation

$$(c^2 \pm b^2)u^2 \pm b^2v^2 - 1 = 0,$$

connecting the two variable parameters u and v; it is necessary to choose the $+$ sign or $-$ sign, according as the straight line passes to the right or left of the two points, or between them. The curve S', having the equation

$$(c^2 \pm b^2)x_1{}^2 \pm b^2y^2{}_1 - 1 = 0,$$

the equation of the curve sought S, or of the reciprocal polar (§ 298), is

$$\frac{x^2}{c^2 \pm b^2} + \frac{y^2}{\pm b^2} - 1 = 0.$$

This is an ellipse or a hyperbola whose foci are the points F and F'. This property is the converse of the theorem demonstrated in § 259.

EXAMPLE II. — Being given a quadrilateral *abcd*, find the envelope of a straight line such that the product of the distances of two opposite vertices of the quadrilateral from the straight line is to the product of the distances of the other two vertices from the straight line in a constant ratio. Call x_1 and y_1, x_2 and y_2, x_3 and y_3, x_4 and y_4, the co-ordinates of the four vertices, and represent the variable straight line by the equation $ux + vy + 1 = 0$; the two parameters u and v will be connected by the relation

$$(ux_1 + vy_1 + 1)(ux_3 + vy_3 + 1) - k^2(ux_2 + vy_2 + 1)(ux_4 + vy_4 + 1) = 0;$$

since this relation is of the second degree, it follows that the envelope is a curve of the second class or of the second order. The preceding equation is satisfied when the variable straight line coincides with one of the sides of the quadrilateral, since a factor in each term becomes zero. Hence the curve is inscribed in the quadrilateral, and one can assign to the ratio k a value such that the curve is tangent to any fifth straight line. Whence follows the general property of conic sections: *a quadrilateral being circumscribed about a conic section, the product of the distances of two opposite vertices of the quadrilateral from any tangent is to the product of the distances of the other vertices from the same tangent in a constant ratio.*

308. TANGENTIAL EQUATION OF A CONIC. — If the equation of a conic be

$$(21) \qquad Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

the tangential equation of this curve is the necessary and sufficient condition that the straight line $ux + vy + 1 = 0$ be tangent to it, that is (§ 126),

$$(22) \qquad \mathbf{a}u^2 + 2\,\mathbf{b}uv + \mathbf{c}v^2 + 2\,\mathbf{d}u + 2\,\mathbf{e}v + \mathbf{f} = 0,$$

an equation whose constant term $\mathbf{f}$ is zero if the conic be a parabola.

In case the given conic (21) consists of two distinct straight lines, one sees that condition (22) ought to require that the straight line $ux + vy + 1 = 0$ passes through the point of intersection of the two straight lines. Indeed, if a and b be the co-ordinates of this point of intersection, the first member of the tangential equation (22) is a *perfect square*, the square of

$$au + bv + 1.$$

To show this, it is sufficient to impose on this theorem the condition that the co-ordinates a and b satisfy the relations (Chapter XII.)

$$\frac{a^2}{\mathbf{a}} = \frac{ab}{\mathbf{b}} = \frac{b^2}{\mathbf{c}} = \frac{a}{\mathbf{d}} = \frac{b}{\mathbf{e}} = \frac{1}{\mathbf{f}}.$$

Equation (22) then becomes, on replacing the coefficients $\mathbf{a}$, $\mathbf{b}$, $\cdots$ by the proportional values a^2, ab, $\cdots$,

$$(au + bv + 1)^2,$$

which was to be proved.

In the particular case when equation (21) represents two parallel straight lines,

$$\mathbf{d} = 0, \ \mathbf{e} = 0, \ \mathbf{f} = 0,$$

and
$$\mathbf{b}^2 - \mathbf{ac} = - F\Delta = 0;$$

equation (22) therefore becomes

$$\mathbf{a}u^2 + 2\,\mathbf{b}uv + \mathbf{c}v^2 = 0,$$

or
$$(\mathbf{b}u + \mathbf{c}v)^2 = 0,$$

which is still a *perfect square.* It is easy to show that the condition

$$\mathbf{b}u + \mathbf{c}v = 0$$

expresses the condition that the straight line $ux + vy + 1 = 0$ is parallel to the two straight lines represented by equation (21). It further follows that this condition requires that the straight line $ux + vy + 1 = 0$ passes through the point of intersection of the two straight lines represented by equation (21).

When equation (21) represents two coincident straight lines, the first member of the tangential equation (22) is identically zero; one has in fact in this case

$$\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{d} = \mathbf{e} = \mathbf{f} = 0.$$

309. We have seen that the tangential equation of a conic is of the second degree in u and v. Conversely, an equation of the second degree between u and v,

$$(23)\quad \phi(u, v) = Au^2 + 2Buv + Cv^2 + 2Du + 2Ev + F = 0,$$

in which the discriminant $\Delta = ACF - \cdots$ is different from zero, is the tangential equation of a conic. For, in order to find the envelope of the straight line

$$ux + vy + 1 = 0,$$

whose coefficients satisfy the equation $\phi(u, v) = 0$, it is sufficient to eliminate u, v, and λ from equations (17),

$$\begin{aligned} x &= \lambda(Au + Bv + D), \\ y &= \lambda(Bu + Cv + E), \\ 1 &= \lambda(Du + Ev + F), \\ 0 &= \lambda(ux + vy + 1), \end{aligned}$$

where the last of equations (17) is multiplied by λ. The result of the elimination of λu, λv, and λ from these equations of the first degree is the condition

$$\begin{vmatrix} A & B & D & x \\ B & C & E & y \\ D & E & F & 1 \\ x & y & 1 & 0 \end{vmatrix} = 0,$$

or $\quad (24) \qquad \mathrm{a}x^2 + 2\,\mathrm{b}xy + \mathrm{c}y^2 + 2\,\mathrm{d}x + 2\,\mathrm{e}y + \mathrm{f} = 0,$

the equation of a conic. It is seen that one passes from equation (23) to (24) in the same manner as from equation (21) to equation (22). It is easy to verify that the tangential equation of the conic (24) is identical with equation (23).

It has been supposed that the discriminant Δ of equation (23) is different from zero. If this discriminant be zero without all of its minors a, b, $\cdots$, being zero, the function $\phi(u, v)$ resolves itself into a product of two factors of the first degree in u, v. In this case the equation $\phi(u, v) = 0$ will represent *two points* which could, moreover, be real or imaginary, and the first member of equation (24) will be a perfect square, the square of the first member of the equation of the straight line joining these two points. This can be demonstrated by a method identical with that given above.

Finally, if the discriminant Δ be zero, and also all of its minors, the first member of equation (23) will be a perfect square. It represents two coincident points. Equation (24) is identically zero.

CHAPTER XI*

GENERAL PROPERTIES OF CONIC SECTIONS.

THEOREMS OF PASCAL AND BRIANCHON.

310. THEOREM I.—*If three conic sections have two points in common, the three straight lines which join the other points of intersection of the curves two by two pass through a common point.*

Let $S=0$ be the equation of one of the conic sections, $\alpha=0$ the equation of the straight line which passes through the two common points; the equations of the other two conic sections will be of the form $S-k\alpha\beta=0$, $S-k'\gamma\alpha=0$. The three straight lines which pass through the other two points of intersection of the curves, considered two by two, are $\beta=0$, $\gamma=0$, $k\beta-k'\gamma=0$; the third passes through the point of intersection of the first two.

311. THEOREM II.—*If a hexagon be inscribed in a conic section, the points of intersection of the opposite sides are in a straight line.*

This theorem, which is due to PASCAL, is an application of the preceding theorem. Let *abcdef* (Fig. 184) be a hexagon inscribed in a conic section; the curve and the two pairs of straight lines *ab* and *cd*, *af* and *de* can be regarded as three conic sections having the common points *a* and *d*.

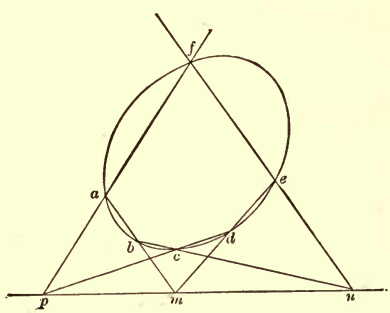

Fig. 184.

The straight line *bc* connects the other two points of intersection *b* and *c* of the curve and the two straight lines *ab* and *cd*; the straight line *ef* connects the other two points of intersection *e* and *f* of the curve and the two straight lines *af* and *de*; moreover, the two pairs of straight lines intersect each other in *m* and *p*; the three straight lines *bc*, *ef*, *mp* pass through the same point *n*; therefore the three points of intersection *m*, *n*, *p* of the opposite sides of the inscribed hexagon lie in a straight line.

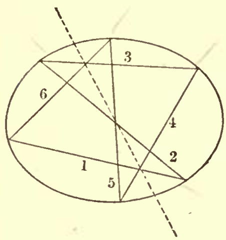

Fig. 185.

This theorem is not only applicable to a convex hexagon, but, moreover, to a hexagon formed in any manner. An inscribed hexagon is constructed by drawing six consecutive chords, in such a manner as to return finally to the point of departure. If the sides be numbered in the order in which they are constructed, the three points of intersection of the sides (1, 4), (2, 5), (3, 6) lie on a straight line (Fig. 185).

Corollary I. — If a conic section be defined by five points *a*, *b*, *c*, *d*, *e*, the preceding theorem enables one to construct as many points of the curve as one may wish. Through the point *a* draw any straight line *af* and seek the point *f* where the straight line intersects the curve (Fig. 184); one locates the point of intersection *m* of the straight lines *ab* and *de*, the point of intersection *p* of the straight line *cd* and *af*; the straight line *bc* intersects the straight line *mp* in a point *n*; the point *f*, where the straight line *ne* intersects *af*, belongs to the curve.

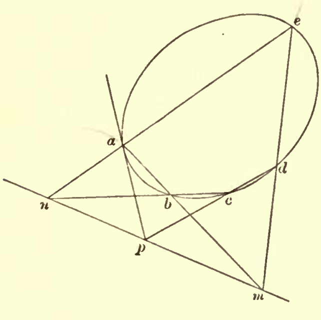

Fig. 186.

The tangent at one of these points can also be constructed. When two vertices of an inscribed hexagon, for example *a* and *f*, coincide, the corresponding side *af* becomes a tangent to the curve at the point *a*; if the theorem of the inscribed hexagon be applied, on reckoning this tangent as a side, it follows still that three points lie in a straight line. One locates therefore the point of intersection *m* of the sides *ab* and *de* (Fig. 186), the point of intersection *n* of the sides *bc* and *ae*; the straight line *cd* intersects the straight line *mn* in a point *p*; the straight line *ap* will be the tangent at *a*.

Corollary II. — *A quadrilateral abcd being inscribed in a conic section, the points of intersection of the opposite sides, and the points of intersection of the tangents at the opposite vertices lie in a straight line.* If a complete hexagon with the tangents at *a* and *c* be inscribed, one will have three points *m, n, p* in

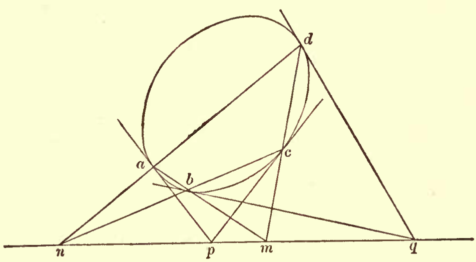

Fig. 187.

a straight line (Fig. 187). If a complete hexagon with tangents at *b* and *d* be inscribed, one will have in a similar manner three points *m, n, q* in a straight line. Therefore the four points *m, n, p, q* lie on a straight line.

Corollary III. — *A triangle being inscribed in a conic section, the points of intersection of the sides with the tangents at the opposite vertices are in a straight line.*

312. Remark. — We have seen that one conic section, and only one, can be drawn through five points *a, b, c, d, e,* no three of which lie in a straight line. The elements of this curve can be obtained in the following manner: one begins by constructing the tangents *A, B, C* at the three given points *a, b, c*. In every curve of the second degree the tangents at the extremities of a chord intersect on the diameter conjugate to this chord; consequently, the straight line which joins the point of intersection p of the straight lines A and B to the mid-point g of the straight line ab is the diameter of the chords parallel to ab; similarly, the straight line which joins the point of intersection q of the straight lines B and C to the mid-point h of bc is the diameter of the chords parallel to bc. Suppose that the two diameters pg and qh intersect in a point o. The straight line op and the straight line ok parallel to ab form a system of conjugate diameters. If a' be the length of the semi-diameter with the direction op, one has $a' = \sqrt{op \cdot og}$; in a similar manner the length b' of the semi-diameter with the direction ok may be found. It has been explained (§§ 174 and 195) how to determine the axes, in case a system of conjugate diameters a' and b' are known.

If the two diameters be parallel, the curve is a parabola. In this case, one draws the diameters which pass through a and b, then the straight lines forming with the tangents angles which are equal to those formed by the diameters with the tangents; these two straight lines intersect at the focus of the parabola. On dropping from the focus perpendiculars to the tangents A and B, and prolonging each of the perpendiculars a length equal to itself, two points of the directrix are determined.

In case three points and the tangents at two of these points are given, the tangent at the third point is determined by means of the property of the inscribed triangle; after this one proceeds as above. The construction relative to the parabola can evidently be used in case two tangents to the curve and their points of contact be known.

313. Suppose, finally, it is desired to find the elements of a parabola determined by four points a, b, c, d. If the two straight lines ab, cd be chosen as axes of co-ordinates, the equations of the parabolas passing through the given points are (§ 276)

$$\frac{x^2}{ab} \pm \frac{2xy}{\sqrt{abcd}} + \frac{y^2}{cd} + \cdots = 0.$$

Since the angular coefficients of the axes of the parabolas are $\pm\sqrt{\frac{cd}{ab}}$, it follows that these axes are parallel to the diagonals of a parallelogram constructed on the axes of co-ordinates and of which the sides will have lengths which are mean proportionals between a and b, c and d. Knowing the direction of the axis, the theorem concerning the inscribed pentagon will give, on supposing that the point e be removed indefinitely, that is, that the straight lines ae and de, for example, become parallel to the axis (Fig. 186), the tangent at one of the points. In case two tangents have been determined, the problem will be reduced to the preceding case.

314. THEOREM III. — *If a hexagon be inscribed in a conic section, the three straight lines which join the opposite vertices pass through the same point.*

This theorem, discovered by BRIANCHON, may be derived from the preceding by the method of reciprocal polars. Let $abcdef$ (Fig. 188) be a hexagon circumscribed about a conic section; the inscribed hexagon, which has as vertices the points of contact, is the corollative figure of the circumscribed hexagon, with respect to the given conic section; because the vertices $a, b, c, \cdots$ of the circumscribed hexagon are the poles of the sides $A', B', C', \cdots$ of the inscribed hexagon. The diagonal ad of the circumscribed hexagon is the polar of the point of intersection m' of the opposite sides A' and D' of the inscribed hexagon; similarly, the diagonal be is the polar of the point of intersection n' of the sides B' and E', and the diagonal cf the polar of the point of

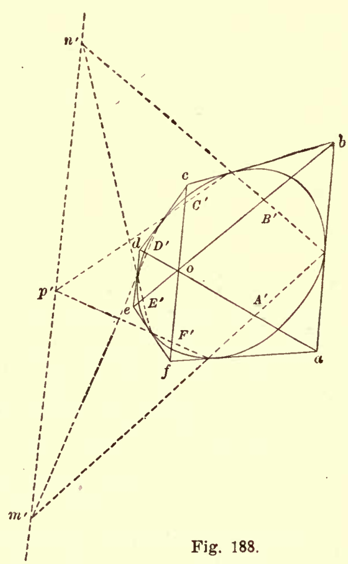

Fig. 188.

intersection p' of the sides C' and F'. Since the three points m', n', p' are in a straight line, the three straight lines ad, be, cf pass through the same point o, the pole of this straight line.

We make at this point a remark analogous to that which has been made with respect to the theorem of PASCAL. It is not necessary that the circumscribed hexagon be convex, it is sufficient that it be closed. Suppose that six tangents be drawn to a conic section; in order to construct the hexagon, beginning at the point of intersection of two tangents, one proceeds along one of them to the intersection of the next tangent; then along this second tangent, in either direction, to the intersection of a third tangent, and so on, in a similar manner, till the point of departure is reached, after having traveled along the tangents in a continuous manner. The broken line thus formed is a circumscribed hexagon. If the vertices be numbered in the order in which they are constructed, the three

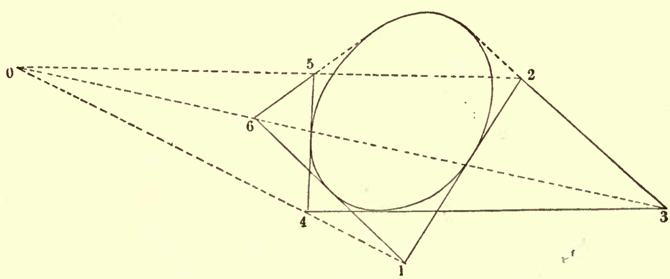

Fig. 189.

diagonals which connect the vertices (1, 4), (2, 5), (3, 6) pass through the same point (Fig. 189).

COROLLARY. — If a conic section be defined by five tangents, one can, by aid of the preceding theorem, construct as many tangents as one wishes. Let the five tangents be ab, bc, cd, de, ef (Fig. 188); determine the second tangent which passes through a point a taken arbitrarily on one of the given tangents; take the point of intersection o of the diagonals ad and be, and draw the straight line co and join the point a to the point f, where the straight line co intersects the tangent ef.

The point of contact of each of the tangents can also be determined if two sides of the circumscribed hexagon, for example the sides ab and bc, coincide; the intermediary vertex b becomes the point of contact; in order to find this point of contact, one connects the vertex e with the point of intersection o of the diagonals ad and cf (Fig. 190).

When the points of contact of the three tangents have been determined, one obtains the elements of the curve by the method which we have described in § 312.

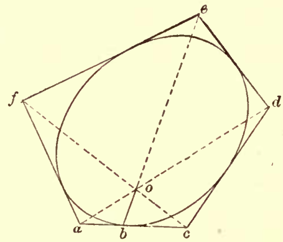

Fig. 190.

The center could also be immediately obtained by means of the theorem demonstrated in § 300.

The following corollaries may be deduced from the theorem of BRIANCHON: *If a quadrilateral be circumscribed about a conic section, the two diagonals and the two straight lines which join the points of contact of the opposite sides pass through the same point.*

If a triangle be circumscribed about a conic section, the straight lines which join the vertices to the points of contact of the opposite sides pass through the same point. It is sufficient to complete the circumscribed hexagon in the first case with the points of contact of the two opposite sides, in the second case with the three points of contact.

HOMOGRAPHIC SYSTEMS.

315. In case we are given on two given straight lines two systems of points which have a one-to-one correspondence, of the kind that if x and x' represent the distances (affected with the proper signs) of two corresponding points from two fixed points taken on the straight lines, one has the relation

$$(1) \qquad Axx' + Bx + Cx' + D = 0;$$

these two systems of points are said to be *homographic.*

This equation involves three arbitrary parameters; to three points taken at will on the first straight line there can be made to correspond three points taken at will on the second; this mode of homographic division is therefore perfectly determined.

When the point m' of the second straight line is removed to infinity, the homologous point m of the first approaches a limiting position i given by the formula $x = -\frac{C}{A}$. Similarly, in case the point m of the first straight line is removed to infinity, the point m' of the second approaches a limiting position j' given by the formula $x' = -\frac{B}{A}$. If one lay off the distances on these two straight lines, beginning with the points j' and i, the relation is simplified and becomes

$$(2) \qquad Axx' + D = 0.$$

A pencil of straight lines which passes through the same point o (Fig. 191), determines on any two secants two systems of homographic points; because, on calling x_1 and y_1 the co-ordinates of the point o with respect to the two secants, α and β the abscissa and the ordinate of the points m and m', when one of the straight lines of the pencil intersects the two secants, the variables α and β will be connected by the relation

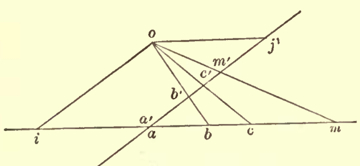

Fig. 191.

$$\frac{x_1}{\alpha} + \frac{y_1}{\beta} = 1.$$

Conversely, when one has two systems of homographic points on two straight lines, the straight lines can be so placed that one of the systems is the *perspective* of the other; it is sufficient to place one of the straight lines so that two homologous points a and a' coincide; the straight lines bb', cc', which join two pairs of homologous points, intersect in a point o; the straight line om, which joins the point o to any point m of the first straight line, will pass through the homologous point m' of the second. The straight lines oi and oj', parallel to the two straight lines, give the points i and j'. Two systems of points homographic to a third are homographic to each other; because the elimination of x from the two equations

$$(Ax' + B)x + (Cx' + D) = 0,$$

$$(A_1x_1 + B_1)x + (C_1x_1 + D_1) = 0,$$

gives an equation in x' and x_1 similar in form.

316. Consider the two pencils of straight lines which are obtained by connecting two fixed points o and o' with two systems of homographic points; these two pencils determine on any secant systems of homographic points; accordingly the two pencils of straight lines are said to be homographic.

Imagine that through a fixed point situated on the x-axis, and having the abscissa -1, one draws lines parallel to the straight lines of the two pencils; these parallels determine on the y-axis two systems of homographic points; the ordinate of each of these points being equal to the angular coefficient of the corresponding straight line, one infers that the angular coefficients m and m' of the homologous straight lines are connected by the relation

$$(3) \qquad Amm' + Bm + Cm' + D = 0.$$

Conversely, if in two pencils the angular coefficients of two homologous straight lines satisfy a relation of this form, the two pencils are homographic. Such, for example, are the two pencils which one obtains by drawing through two fixed points o and o' lines parallel to two conjugate diameters of a conic.

Two pencils homographic to a third are homographic to each other.

Two systems of homographic points are transformed in the reciprocal polar figure into two pencils of homographic straight lines. Let P and Q be the two given straight lines, p' and q' their poles with respect to a curve of the second degree whose center is o; to a point a of the first straight line corresponds a straight line A' passing through its pole p'; to a point b of the second straight line a straight line B' passing through its pole q'. The straight lines oa, ob form two homographic pencils; the pencil of straight lines A' being homographic with that of the straight lines oa and the pencil of straight lines B' with that of the straight line ob, it follows that the two pencils A' and B' are homographic.

Remark. — In the two relations (1) and (3) we have supposed that the coefficients A, B, C, D do not satisfy the condition

$$AD - CB = 0.$$

If this condition were fulfilled, the first member of relation (1), for example, would resolve into two linear factors, the one in x the other in x', and this relation (1) would take the form

$$A(x - \alpha)(x' - \alpha') = 0.$$

To a value assigned to x' would always correspond the value $x = \alpha$, and for $x' = \alpha'$, x would be indeterminate.

317. In case two systems of homographic points lie on the same straight line, there exist two double points on this line; that is, two points such that either of them, considered as belonging to one of the systems, coincides with the other, its homologous point in the other system. In fact, if one lay off the distances on the straight line, beginning with the same point, and put $x' = x$, one has, owing to relation (1), an equation of the second degree,

$$(4) \qquad Ax^2 + (B + C)x + D = 0,$$

each of whose roots gives a double point. The two double points are real or imaginary.

Suppose that there have been constructed, as already has been explained, the two points i and j' homologous with respect to infinity; if one lay off the distances beginning with the point c, mid-point of ij', equation (1) becomes

$$(5) \qquad Axx' + B(x - x') + D = 0.$$

Equation (4), which gives the double points, reduces to

$$(6) \qquad Ax^2 + D = 0.$$

Call c' the point of the second system homologous to the point c of the first; equation (5) ought to be satisfied by $x = 0$ and $x' = cc'$; one has

$$cc' = \frac{D}{B};$$

moreover,

$$cj' = -\frac{B}{A};$$

whence it follows

$$\frac{D}{A} = -cj' \times cc',$$

and equation (6) becomes

$$x^2 = cj' \times cc'. \tag{7}$$

The double points are real in case the two lengths cj' and cc' are measured in the same direction; in order to construct them, a circle is constructed on $c'j'$ as a diameter (Fig. 192); a tangent is drawn from the point c to this circle; by revolving the tangent to the straight line, the two double points e and f are determined, which are situated at equal distances from c.

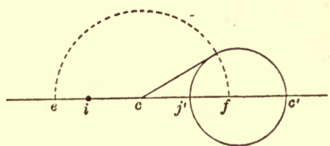

Fig. 192.

318. Two homographic pencils having a common vertex have in like manner two double straight lines, real or imaginary; their equations may be obtained by joining the vertex to the two double points of the homographic division determined by the pencil and any secant.

In case a constant angle is made to revolve about its vertex, the two sides form two homographic pencils with this point as a common vertex, the various positions of one of the sides constituting the first pencil, those of the second side the other pencil. Because if the first pencil revolve through a constant angle about the vertex, it coincides with the second. The relation between the angular coefficient of the homologous straight lines in rectangular co-ordinates is $mm' + 1 + c(m - m') = 0$; the double straight lines are imaginary and have the equation $x^2 + y^2 = 0$; they are the asymptotes of the circle $x^2 + y^2 = r^2$.

These two pencils determine on any straight line two systems of homographic points, whose double points are imaginary. Conversely, in case the double points of the two systems of homographic points on the same straight line are imaginary, these two systems of points can be obtained by the rotations of a constant angle about its vertex. The mode of division is defined by the three pairs of points (c, c'), (i, ∞), (∞, j'), the point c being the mid-point of ij'; co, the perpendicular to the straight line, intersects the circle described on $c'j'$ as diameter in a point o; the angle $c'oc$, on revolving about the point o, will give the homographic division desired.

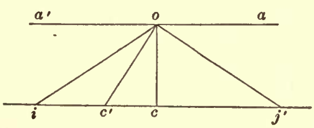

Fig. 193.

319. Involution. — Consider two systems of homographic points on the same straight line, and suppose that two homographic points a and a' be reciprocal, that is, that if to the point a of the first system there corresponds the point a' in the second, reciprocally to the point a' considered as belonging to the first system there corresponds the point a in the second. It follows that equation (1) will be satisfied when the particular values of x and x' which belong to these two points are permuted, which requires that $B = C$; but in this case all of the homologous points are reciprocal two by two, and the points are said to be in *involution*.

The equation

$$Axx' + B(x + x') + D = 0 \tag{7}$$

containing but two arbitrary parameters, two pairs of conjugate points (a, a'), (b, b') are sufficient to define the involution. The two points i and j' coincide, and if the distances beginning with this point i be laid off, equation (7) becomes

$$Axx' + D = 0; \tag{8}$$

this point is called the *center* of involution. There are two double points e and f, real or imaginary, given by the equation

$$Ax^2 + D = 0.$$

Equation (8) becomes, therefore, $xx' = \overline{ie}^2$; it follows that the two double points e and f are harmonic conjugates with respect to any two conjugate points.

The circles drawn through the two points p and q determine on a straight line an involution (Fig. 194). Let i be the point in which the straight line pq intersects the given straight line; if x and x' be the distances of this point from the two points of intersection of the secant and of one of the circumferences, one has $xx' = ip \cdot iq$; the point i is therefore the center of involution. The double points are real or imaginary, according as the point i is situated without the points p and q or between these two points. In the first case, one obtains the double points on drawing from the point i a tangent to one of the circles and revolving this tangent.

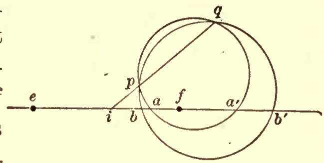

FIG. 194.

In case the involution on a straight line is defined by means of two pairs of conjugate points (a, a'), (b, b'), any two conjugate points can be easily constructed; construct a circle through the two points a and a', construct also a second circle through the two points b and b' and an arbitrary point p of the first; these two circles intersect in a second point q; the circle which passes through the two points p and q and a point m of the straight line will determine the conjugate point m'.

Let us consider in like manner two homographic pencils with a common vertex and such that two homologous straight lines are reciprocal; these straight lines determine on any secant the points of involution; all homologous straight lines are therefore reciprocal two by two and the straight lines are said to be in involution. Two double straight lines may be real or imaginary.

We have mentioned (§ 318) that, if a constant angle revolve about its vertex, its sides form two homographic pencils. When the angle is a right angle the pencils will be in involution; the double straight lines, as has been remarked, are the asymptotes of a circle.

Conversely, in case the double points of an involution on a straight line are imaginary, the pairs of conjugate points can be found by the rotation of a right angle about its vertex. The involution is defined by the two pairs of conjugate points (i, ∞), (a, a'); describe a circle on aa' as a diameter (Fig. 195); erect a perpendicular to the straight line at the point i which intersects this circle in two points p and q; a circle passing through the two points p and q will determine two conjugate points m and m', and the angle mpm' is a right angle.

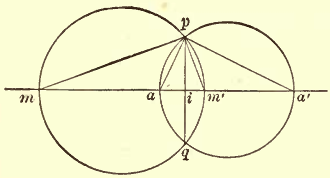

Fig. 195.

The conjugate diameters of the conic are in involution. The double straight lines are real in case of the hyperbola, imaginary in case of the ellipse.

320. Theorem I. — *If two homographic pencils be given, the locus of the point of intersection of two homologous straight lines is a conic passing through the vertices of the two pencils.*

Determine as many of the points of the locus as are situated on any straight line D; the two homographic pencils o and o' (Fig. 196) determine on this straight line two systems of homographic points (α, α'), (β, β'), (γ, γ'), $\cdots$; two homologous straight lines oe, $o'e$, which intersect on the straight line D, determine a double point e; since there cannot be on the straight line D more than two double points e and f, it follows that this straight line intersects the locus in but two points, real or imaginary; hence the locus is of the second order.

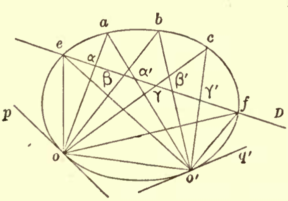

Fig. 196.

To the straight line $o'o$ of the second pencil corresponds a certain straight line op of the first; the point of intersection falls in o, and the straight line op is tangent to the curve at this point. Similarly, the curve passes through the point o' and is tangent at this point to the straight line $o'q'$ of the second pencil, the homologous line of the straight line oo' of the first.

COROLLARY. — This enables us to find the points in which a given straight line D intersects a conic defined by five points, o, o', a, b, c; if the two points o and o' be joined to the other three, one has three pairs of straight lines (oa, $o'a$), (ob, $o'b$), (oc, $o'c$), which determine the two homographic pencils o and o'; the locus of the point of intersection of the homologous straight lines is the conic passing through the five given points; the three pairs of points (α, α'), (β, β'), (γ, γ') define the homographic division on the straight line D; the two double points e and f may be found by the method described in § 317.

If the straight line pass through one of the given points, for example o, it is sufficient to construct the homologous straight line in the second pencil. Similarly, as we have already said, the tangent at o may be found by drawing the straight line op of the first pencil homologous to the straight line $o'o$ of the second. Thus may be found as many points and tangents of the conic sought as one wishes.

REMARK. — When the straight line oo', which passes through the vertices, corresponds to itself in the two pencils, it evidently constitutes a part of the locus which is then composed of two straight lines; in this case, the locus of the point of intersection of the homologous straight line is, strictly speaking, a straight line.

321. THEOREM II. — *If two systems of homographic points on two fixed straight lines A and A' be given, the straight line aa', which joins any two homologous points, envelops a conic which is tangent to the two fixed straight lines.*

Determine all the tangents to the envelope which pass through an arbitrary point p of the plane (Fig. 197); the straight lines pa, pa', which join the point p to two homologous points, form about the point p two homographic pencils; in case the variable straight line aa', in one of its positions mm', passes through the point p, it becomes a double straight line of the two pencils; since there can exist but two double straight lines pm, pn, it follows that through the point p there can be drawn to the envelope curve but two tangents, real or imaginary; this curve is therefore of the second class, and consequently of the second order.

At the point of intersection o of the two fixed straight lines A and A', considered as belonging to the second straight line,

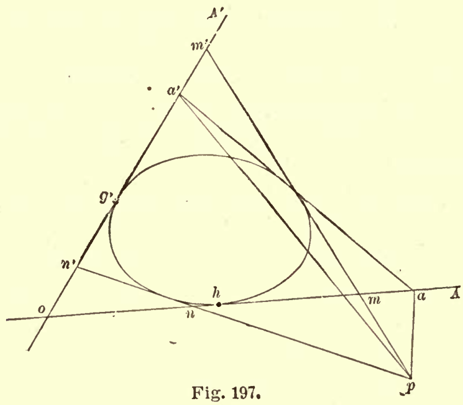

Fig. 197.

there corresponds a point h on the first; the variable straight line coincides with oh, and the curve is tangent to the straight line a at the point h. Similarly, the curve is tangent to the straight line A' at the point g' of this straight line homologous to the point o of the straight line A.

Corollary. — This theorem enables us to draw through a given point p tangents to a conic defined by five tangents; if one join to the point p the points where the two tangents A and A' are intersected by the other three, B, C, D, one obtains three pairs of straight lines determining two homographic pencils whose double straight lines are the tangents required.

If the point p be situated on one of the given tangents, A for example, the points where the tangents A and A' are intersected by the other three, B, C, D, determine on these first two tangents two systems of homographic points; one seeks on the straight line A' a point p' which is homologous to the point p on A; the straight line pp' will be tangent to the conic.

The point of contact of the tangent A is, as has been mentioned, the point of this straight line which is homologous to the point o of A'.

REMARK. — In case the point of intersection o of the two fixed straight lines corresponds to itself on the two straight lines, in the reciprocal polar figure the straight line of the vertices will correspond to itself in the two pencils; the locus becomes in this case a straight line, the envelope reduces to a point. Therefore every straight line, such as aa', passes through the same point.

322. The two preceding theorems give rise to a large number of remarkable properties. We shall now call attention to some of them.

For example, if two constant angles revolve about their vertices in such a way that the point of intersection of two sides describes a fixed straight line, the other two sides will form two homographic pencils, and, consequently, the locus of their point of intersection will be a conic passing through the two fixed vertices.

Similarly, if, on two fixed straight lines, one begin with the points where they are intersected by a variable secant drawn through a fixed point, and lay off in a definite manner two constant lengths, it is evident that the extremities of these lengths will form two systems of homographic points, and consequently that the straight line which connects them will envelop a conic tangent to two fixed straight lines.

Let us consider a triangle maa', of which the three sides revolve about the three fixed points o, o', p (Fig. 198), whilst the two vertices a and a' slide along the two fixed straight lines A and A'; the pencils o and p are homographic; similarly, p and o'. Therefore the pencils o and o' are homographic, and the point m, the third vertex of the triangle, describes a conic passing through the two points o and o'. It is easy to see that the point

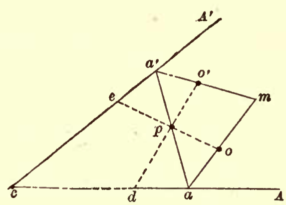

Fig. 198.

of intersection c of the straight lines A and A' and the two points d and e, where these straight lines are intersected by the straight lines po' and po, belong to the locus; thus the conic is defined by five points.

When the three fixed points o, o', p lie in a straight line, the straight line oo' corresponds to itself in the two pencils, and the locus of the vertex m is a straight line; this problem has been discussed in § 105.

Similarly, let us consider a variable triangle aba' (Fig. 199), whose three vertices slide on three fixed straight lines A, A', B, while the two sides ba, ba' revolve about the two fixed points o and o'; the pencil o determines on the straight lines A and B two systems of homographic points a and b; similarly, the pencil o' determines two systems of homographic points b and a'. Therefore the two systems a and a' are homographic, and the third side aa' of the triangle envelops a conic tangent to the two straight lines A and A'. It can be easily verified that the straight lines $o'c$ and od, which join the points o and o' to the points where the straight line B intersects the straight lines A and A', touch the conic; thus the conic will be defined by five tangents.

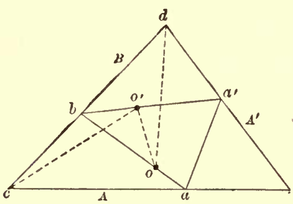

Fig. 199.

If the three straight lines A, A', B pass through the same point, the point of intersection of the straight lines A and A' corresponds to itself, and the envelope reduces to a point. Therefore the straight line aa' passes through a fixed point.

This mode of demonstration is applicable to polygons of any number of sides. Thus, if the n sides of a polygon revolve about n fixed points, and $n-1$ vertices describe straight lines, the last vertex describes a conic. In case n vertices of a polygon describe straight lines, and $n-1$ sides revolve about fixed points, the last side envelops a conic.

Theorems I. and II. make it possible, as we have seen, to construct a conic defined by five points or five tangents; but the theorems of Pascal and Brianchon furnish more simple constructions.

324. THEOREM III. — *In case, in the general equation of a pencil of conics subject to four conditions, the arbitrary parameter appears in the first degree, these conics determine on any straight line points in involution.*

If the straight line be chosen as the x-axis, and if y be made equal to zero in the given equation, one obtains an equation of the form

$$(A+kA')x^2+(B+kB')x+(C+kC')=0,$$

k being the arbitrary parameter. On calling x and x' the two roots, one has

$$\frac{x+x'}{-B-kB'}=\frac{xx'}{C+kC'}=\frac{1}{A+kA'};$$

whence it follows

$$\frac{A'(x+x')+B'}{AB'-BA'}=\frac{-A'xx'+C'}{AC'-CA'}. \qquad \text{Q.E.D.}$$

325. THEOREM IV. — *Conics which pass through four given points determine on any straight line points in involution.*

We have seen (§ 277) that the equation of conics, which pass through four given points a, b, c, d, involves one arbitrary parameter in the first degree; according to the preceding

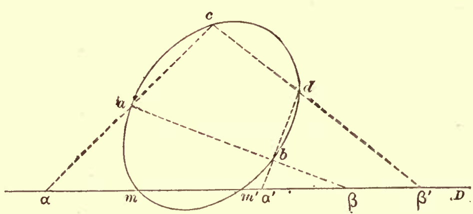

Fig. 200.

theorem these conics determine on any straight line D points in involution (Fig. 200). The pairs of straight lines (ac, bd), (ab, cd) determine two pairs of conjugate points (α, α'), (β, β'), which define the involution.

Corollary.—The double points of involution are the points of contact of the conics which pass through the four given points, and are tangent to the straight line D; since there are two double points, it follows that *there are two conics, real or imaginary, which pass through four given points and are tangent to a given straight line.* These points are determined by the construction described in § 319, and then each of the conics will be defined by five points.

326. Theorem V.—*The tangents drawn from a fixed point to conics tangent to four given straight lines are in involution.*

This theorem may be deduced from the preceding, which is due to Desargues, by the method of reciprocal polars. To the conics tangent to four given straight lines there correspond, in the reciprocal polar figure, conics which pass through four given points; to the two tangents drawn from a fixed point p to one of the first system of conics correspond the two points of intersection of the straight line P', the polar of the point p, with one of the second system of conics; these points of intersection on the straight line P' being in involution, the pencils of tangents emanating from the point p' are also in involution.

The four given straight lines (Fig. 201) form a quadrilateral; the diagonal aa' can be regarded as the limit of an ellipse tangent to the four straight lines, and of which the minor axis becomes zero; the tangents drawn from the point p to this ellipse, reduced to its major axis, are pa and pa'. A similar discussion applies to the diagonal bb'. One has therefore two pairs of conjugate straight lines (pa, pa'), (pb, pb') which define the involution.

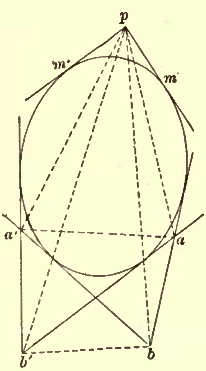

Fig. 201.

Corollary.—If the conic pass through the point p, the two tangents pm, pm' coincide and form a double straight line; since there are two double straight lines in the involution, it follows that *there are two conics, real or imaginary, which touch four given straight* *lines and pass through a given point.* On drawing a secant across the pencil, and determining the double points on the secant, one will obtain the double straight lines, and each of the two conics will be defined by five tangents.

327. THEOREM VI. — *The conics tangent to two given straight lines at two given points determine on any secant an involution of which one of the double points is situated on the chord of contact.*

This theorem is a particular case of Theorem IV. Suppose that the points a and c coincide, also that the points b and d (Fig. 200); the two straight lines ac and bd will be tangent at a and b; the two straight lines ab and cd coinciding, the two conjugate points β and β' coincide with one of the double points of the involution, to which belong the pairs of points (m, m'), (α, α').

COROLLARY. — This theorem enables one to construct a conic which passes through three given points and touches two given straight lines A and A' (Fig. 202).

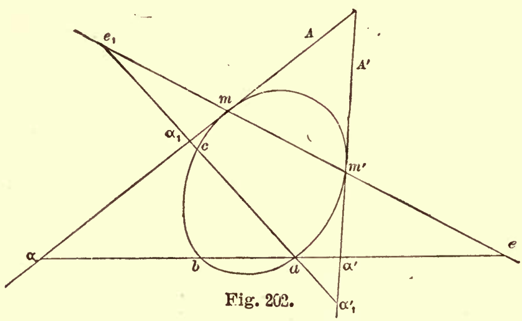

Fig. 202.

Select on the secant ab the two double points e and f of the involution defined by the two pairs of points (a, b), (α, α'). Select in a similar manner on the secant ac the two double points e_1 and f_1 of the involution defined by the two pairs of points (a, c), (α_1, α'_1).

The chord of contact, passing through one of the two points e and f, and through one of the two points e_1 and f_1, will coincide with one of the four straight lines which are found by connecting these points two by two in all possible ways. Any of these four straight lines, for example ee_1, will give a solution of the problem: the straight line ee_1 intersects the two given straight lines A and A' in two points m and m'; *one* conic can be drawn through the point a and tangent to the straight lines A and A' at the points m and m' (§ 280); this conic intersects the secant aa' in a second point conjugate to the point a in the involution defined by the double point e and the pair of points (a, a'), will pass through the point b; it can be shown in a similar manner that the conic passes through the point c. Hence there are four conics, real or imaginary, which pass through three given points and touch two given straight lines.

328. THEOREM VII. — *The tangents drawn from a fixed point to the various conics which touch two given straight lines in two given points, form an involution of which one of the double straight lines passes through the point of intersection of the two given straight lines.*

This theorem is a particular case of Theorem V. Suppose that the two tangents ab and ab' coincide (Fig. 201), also that $a'b$ and $a'b'$: the points a and a' become the points of contact of the tangents ab and $a'b'$; the two points b and b' coincide; the two straight lines pb and pb' coincide with one of the double straight lines of the involution, to which belong the two pairs of straight lines (pm, pm'), (pa, pa').

COROLLARY. — The preceding theorem enables one to construct a conic passing through two given points a and b, and touching three given straight lines mn, pm, and pn (Fig. 203). The point of intersection o of the tangents at a and b is situated on one of the two double straight lines of the involution defined by the two pairs of straight lines (pa, pb), (pm, pn), and on one of the two double straight lines of the involution defined by the two pairs of straight lines (ma, mb), (mn, mp), will coincide with one of the four points of intersection of these double straight lines taken two by two. Any one of these four points, for example the point o, will give a solution of the problem; *one* conic tangent to the two straight lines *oa* and *ob* at the points *a* and *b* and to the straight line *pm* can be determined; the second tangent which can be drawn from the point *p* to this conic will be the conjugate of the straight line *pm* in the involution defined by the double straight line *po*, and the pair of straight lines (*pa*, *pb*) will coincide with *pn*; in a similar manner it can be demonstrated that the straight line *mn* is tangent to the conic. Hence there are *four* conics, real or imaginary, passing through two given points and touching three given straight lines.

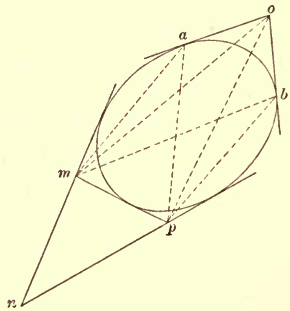

Fig. 203.

329. REMARK. — It has been stated (§ 283) that a focus can be regarded as the point of intersection of two tangents whose angular coefficients are $\pm i$, that is, tangents that are parallel to the asymptotes of a circle; to be given a focus is therefore equivalent to being given two tangents to the conic. Hence, of the conics which have a given focus in common, there is *one* tangent to three given straight lines (§ 262), *two* tangent to two given straight lines and passing through a given point, *four* (of which two are real and two imaginary) tangents to one given straight line and passing through two given points, and, finally, *four* passing through three given points (§ 260).

We have learned to construct a conic which satisfies five simple conditions, points of the curve or tangents; four conditions are sufficient for the determination of a parabola, and the discussion can be reduced to one of the preceding by a transformation with the assistance of the method of reciprocal polars. We know, in fact (§ 299), that if the center *o* of the curve of reference be situated on a conic, the polar reciprocal curve is a parabola, and that, conversely, the reciprocal polar curve of a parabola is a conic passing through the center *o* of the curve of reference. In the transformation, the condition that the curve sought is a parabola, is therefore replaced by the point *o*, the points by the straight lines, the straight lines by the points. The construction of a parabola tangent to four given straight lines is thus reduced to the construction of a conic passing through five given points; there is *one* solution, and one only. Similarly, there are *two* parabolas passing through four given points, or passing through one point and tangent to three given straight lines; *four* parabolas passing through three points and tangent to one straight line, or passing through two points and tangent to two given straight lines. On drawing the tangent to the reciprocal polar curve at o and the conjugate diameter in the curve of reference, one will have the direction of the diameters of the parabola, which makes it possible to at once apply the preceding given theorems.

M. Chasles has conceived an ingenious method for studying the properties of a system of conics which satisfy four given conditions, and he showed that these properties depend upon two integral numbers which he called the *characteristics* of the system; these represent the number of conics of the system which pass through a given point or which touch a given straight line. For example, the two characteristics of a system of conics which pass through four given points are 1 and 2; those of the system of conics which touch four given straight lines are 2 and 1; those of the system of conics which pass through three given points, and which touch one straight line, are 2 and 4; those of the system of conics which pass through one point, and which touch three straight lines, are 4 and 2; finally, those of the system of conics which pass through two points, and which touch two straight lines, are 4 and 4.

Homogeneous Co-ordinates.

330. When an algebraic curve defined by its equation $F(x, y) = 0$ is investigated, it is an advantage to consider the homogeneous integral function obtained by replacing the coordinates x and y in $F(x, y)$ by $\frac{x}{z}, \frac{y}{z}$, and multiplying the result by a suitable power of z. One has an illustration of this method when one seeks the equation of the tangent at a point of the curve (§ 291), or the co-ordinates of the point of contact of the tangents drawn from any point of the plane.

Three numbers x, y, z are called the homogeneous co-ordinates of a point, if the ratios $\frac{x}{z}$, $\frac{y}{z}$ be respectively equal to the abscissa and ordinate of this point. Thus a point whose abscissa is $\frac{3}{4}$ and ordinate $\frac{1}{6}$ has as homogeneous co-ordinates 9, 2, 12 or $9n$, $2n$, $12n$, n being any number different from zero. Let, moreover, $f(x, y, z) = 0$ be the equation of an algebraic curve rendered homogeneous by the method which we have described; this equation is called the equation of the curve in homogeneous co-ordinates.

POINTS AT INFINITY.—STRAIGHT LINE AT INFINITY.—According to the preceding definition for any point of the plane, the third of the homogeneous co-ordinates z is never zero. One considers, nevertheless, the system of values of x, y, z in which the third variable z is *zero*, and it is said that such a system $(x_1, y_1, 0)$ *corresponds to a point at infinity, and that this point is on a curve whose equation in homogeneous co-ordinates is* $f(x, y, z) = 0$, *if one have* $f(x_1, y_1, 0) = 0$.

In particular, to say that the point at infinity $(x_1, y_1, 0)$ is on the straight line $y = ax + bz$, is to say that one has the condition $y_1 = ax_1$. It is to be remarked that, in order to justify this representation of the point at infinity, one is led to the consideration of such systems of values of x, y, z as when the point is supposed to be moved continuously along a given straight line toward infinity. Thus if the homogeneous co-ordinates of the point be

$$x = x_1 - \lambda x_2,\ \ y = y_1 - \lambda y_2,\ \ z = 1 - \lambda,$$

and, consequently, its Cartesian co-ordinates

$$\frac{x_1 - \lambda x_2}{1 - \lambda},\ \ \frac{y_1 - \lambda y_2}{1 - \lambda},$$

the point is on the straight line whose homogeneous equation is

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

whose angular coefficient is $a = \frac{y_1 - y_2}{x_1 - x_2}$. If λ approach the value 1, the point approaches infinity, and for the value $\lambda = 1$, the corresponding values of x, y, z are

$$x_1 - x_2,\ y_1 - y_2,\ 0,$$

which satisfy the condition $y - ax = 0$.

Since the homogeneous equation of the first degree in x, y, z represents, in general, a straight line, and that the co-ordinates of all the points at infinity satisfy the homogeneous equation of the first degree $z = 0$, it is said that all the points at infinity are situated on a straight line (the straight line at infinity) whose equation is $z = 0$. Accordingly the two parallel straight lines

$$y = ax + bz,\ y = ax + b'z$$

are said to have a common point at infinity, or intersect on the straight line at infinity; for the equations of these two straight lines are satisfied by the same system of values of x, y, z:

$$x = 1,\ y = a,\ z = 0,$$

in which z is zero. Similarly it may be said that the two curves

$$f(x, y, z) = 0,\ \phi(x, y, z) = 0$$

have a common point $(x_1, y_1, 0)$ at infinity if one have

$$f(x_1, y_1, 0) = 0,\ \phi(x_1, y_1, 0) = 0.$$

As in § 270, an imaginary point is considered as a system of imaginary values of x, y, z, with the condition that one cannot make x, y, z real on dividing them by the same imaginary quantity. An imaginary point at infinity will be an imaginary point of which the co-ordinate z is zero.

Example. — The general equation of a circle in rectangular Cartesian co-ordinates x and y is

$$x^2 + y^2 + ax + by + c = 0,$$

therefore in homogeneous co-ordinates,

$$x^2 + y^2 + (ax + by + cz)z = 0.$$

This equation is satisfied by the two systems

$$x = 1,\ y = \sqrt{-1},\ z = 0;\quad x = 1,\ y = -\sqrt{-1},\ z = 0,$$

whatever values a, b, c may have. It can therefore be said that any circle whatever passes through the points at infinity whose homogeneous co-ordinates are $(1, \sqrt{-1}, 0)$, $(1, -\sqrt{-1}, 0)$. These two points are called the *circular points at infinity.*

Conversely, every curve of the second order which passes through the circular points at infinity is a circle. Because if the general equation of the second degree

$$Ax^2 + 2\,Bxy + Cy^2 + 2\,Dxz + 2\,Eyz + Fz^2 = 0$$

be satisfied by the co-ordinates of these two points, one has

$$A + 2\,B\sqrt{-1} - C = 0,\quad A - 2\,B\sqrt{-1} - C = 0,$$

whence, by adding and subtracting,

$$A = C,\ B = 0,$$

which shows that the conic is a circle.

Formulas of Transformation.

Suppose that one makes a change of co-ordinates, and that the formulas of transformation for the Cartesian co-ordinates are

$$(1)\qquad \begin{aligned} X' &= a + mX + nY, \\ Y' &= b + pX + qY; \end{aligned}$$

one will take, as homogeneous co-ordinates, the formulas

$$(2)\qquad \begin{aligned} x' &= az + mx + ny, \\ y' &= bz + px + qy, \\ z' &= z. \end{aligned}$$

If the point (x, y, z) be at a finite distance (z different from zero), these formulas are in fact identical with the formulas (1) according to the definition of homogeneous co-ordinates. If, on the contrary, z be zero, that is, if the point (x, y, z) be at infinity, it follows *by definition* that one will regard the values x', y', z', given by formulas (2), as the new co-ordinates of the same point; it is to be noticed that one still has $z' = 0$.

Applications.

Seek the equation in homogeneous co-ordinates of the straight line which joins the two points M_1 and M_2, which have as co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2). This equation is

$$(3) \qquad \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0;$$

in fact, the equation which has just been written represents a straight line, and this straight line passes through the two points, because the equation is evidently satisfied by the co-ordinates of the two points. One can express, as follows, the co-ordinates of any point of the straight line as a function of a parameter. The determinant (3) being zero, there exists a linear homogeneous relation between the elements of the three columns:

$$Ax + Bx_1 + Cx_2 = 0, \quad Ay + By_1 + Cy_2 = 0, \quad Az + Bz_1 + Cz_2 = 0.$$

The coefficient A is not zero, because if it were zero the co-ordinates of x_1, y_1, z_1 would be proportional to x_2, y_2, z_2, and the two points M_1, M_2 would coincide and would not determine the straight line. The coefficient A being different from zero, the relations above can be solved with respect to x, y, z, and one has for x, y, z expressions of the form

$$x = \mu x_1 + \nu x_2, \quad y = \mu y_1 + \nu y_2, \quad z = \mu z_1 + \nu z_2;$$

conversely, whatever ν and μ be, the point defined by these expressions lies on the straight line, since these expressions satisfy expression (3). Since x, y, z can be divided by the same quantity, one can divide them by μ, and on putting $\frac{\nu}{\mu} = \lambda$ one will have

$$x = x_1 + \lambda x_2, \quad y = y_1 + \lambda y_2, \quad z = z_1 + \lambda z_2,$$

excepting for the point x_2, y_2, z_2, which corresponds to $\mu = 0$.

Interpretation of λ. — If $z_1 = z_2 = 1$, one has

$$\frac{x}{z} = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad \frac{y}{z} = \frac{y_1 + \lambda y_2}{1 + \lambda};$$

one recalls the formulas already established (§ 57), and sees, on calling M the point (x, y, z), that

$$\lambda = -\frac{\overline{MM_1}}{\overline{MM_2}}.$$

If z_1 and z_2 be any quantities different from zero, one could divide x, y, z by z_1, and put $\lambda \frac{z_2}{z_1} = \lambda'$; it follows that

$$x = \frac{x_1}{z_1} + \lambda' \frac{x_2}{z_2}, \quad y = \frac{y_1}{z_1} + \lambda' \frac{y_2}{z_2}, \quad z = 1 + \lambda';$$

one returns, therefore, to the preceding case, and λ' is defined by

$$\lambda' = -\frac{\overline{MM_1}}{\overline{MM_2}};$$

moreover, λ' differs from λ only by the factor $\frac{z_2}{z_1}$, which remains constant when the co-ordinates (x_1, y_1, z_1), (x_2, y_2, z_2), remaining fixed, one imagines that the point m moves along the straight line. Whence it follows that the two points

$$(M) \quad \begin{aligned} x &= x_1 + \lambda x_2, \\ y &= y_1 + \lambda y_2, \\ z &= z_1 + \lambda z_2, \end{aligned} \qquad (M') \quad \begin{aligned} x' &= x_1 - \lambda x_2, \\ y' &= y_1 - \lambda y_2, \\ z' &= z_1 - \lambda z_2, \end{aligned}$$

are harmonic conjugates with respect to the two given points. It is easily verified that this result is true if one of the two quantities z_1 or z_2, for example z_2, approaches zero. Then the point M'_2 becomes a point at infinity, and the point M_1 is the mid-point of the segment determined by the points M and M'; one can, therefore, still say that the points M, M' are harmonic conjugates with respect to the points M_1 and M_2. If the two points M_1, M_2 be at infinity, $z_1 = z_2 = 0$, it still follows that the two points M and M', which are also at infinity, are harmonic conjugates with respect to M_1 and M_2.

PROBLEM. — Polar of a point $M_1(x_1, y_1, z_1)$ with respect to a conic

$$f(x, y, z) = Ax^2 + 2\,Bxy + Cy^2 + 2\,Dxz + 2\,Eyz + Fz^2 = 0.$$

Let $M(x, y, z)$ be a point of the polar: the co-ordinates of any point of the straight line M_1M will be

$$x_1+\lambda x,\ \ y_1+\lambda y,\ \ z_1+\lambda z,$$

and the values which it is necessary to assign to λ in order to obtain the co-ordinates of the points where the straight line M_1M intersects the conic are roots of the equation of the second degree:

$$f(x_1+\lambda x,\ y_1+\lambda y,\ z_1+\lambda z)=0,$$

$$f(x_1, y_1, z_1)+\lambda(xf'_{x_1}+yf'_{y_1}+zf'_{z_1})+\lambda^2 f(x, y, z)=0.$$

Let λ' and λ'' be the roots of this equation; the co-ordinates of the points M' and M'', where the straight line MM_1 intersects the conic, will be

$$(M') \qquad x_1+\lambda' x,\ \ y_1+\lambda' y,\ \ z_1+\lambda' z,$$

$$(M'') \qquad x_1+\lambda'' x,\ \ y_1+\lambda'' y,\ \ z_1+\lambda'' z;$$

since the point M is on the polar, the points M' and M'' should be harmonic conjugates with respect to M_1 and M. For this it is necessary and sufficient that

$$\lambda''=-\lambda',\ \ \lambda'+\lambda''=0;$$

that is,

$$(5) \qquad xf'_{x_1}+yf'_{y_1}+zf'_{z_1}=0.$$

This equation, being satisfied by the co-ordinates of any point of the polar, is the equation of the polar. Moreover, one can write it

$$(5)' \qquad x_1f'_x+y_1f'_y+z_1f'_z=0.$$

If the point M_1 be at infinity, $z_1=0$, the polar of this point is then the locus of the mid-points of the chords with the angular coefficient $\frac{y_1}{x_1}$; that is, the conjugate diameter of the direction $xy_1-yx_1=0$. The equation of this diameter is therefore

$$x_1f'_x+y_1f'_y=0,$$

as has been found above.

HOMOGENEOUS CO-ORDINATES OF A STRAIGHT LINE. — The homogeneous co-ordinates of a straight line whose equation is

$$ux + vy + wz = 0$$

are the coefficients u, v, w of this equation. Thus, the x-axis has the co-ordinates $u = 0, v \gtrless 0, w = 0$. The straight line at infinity has the co-ordinates $u = 0, v = 0, w \gtrless 0$.

Accordingly a linear, homogeneous equation in u, v, w,

$$au + bv + cw = 0,$$

expresses the condition that the straight line (u, v, w) passes through the point whose homogeneous co-ordinates are a, b, c; this equation is called the equation of this point.

The tangential equation of a curve is the condition which the co-ordinates u, v, w should satisfy in order that the straight line with the co-ordinates u, v, w be tangent to the curve; this tangential equation will be homogeneous in u, v, w. Then the tangential equation of a circle whose radius is R and whose center has the co-ordinates $a, b, 1$, may be found by expressing the condition that the distance from the center to the straight line (u, v, w) is equal to R; this gives the homogeneous equation

$$(ua + vb + w)^2 - R^2(u^2 + v^2) = 0.$$

The tangential equation of the conic

$$f(x, y, z) = Ax^2 + 2\,Bxy + Cy^2 + 2\,Dxz + 2\,Eyz + Fz^2 = 0$$

is
$$\mathsf{a}u^2 + 2\,\mathsf{b}uv + \mathsf{c}v^2 + 2\,\mathsf{d}uw + 2\,\mathsf{e}vw + \mathsf{f}w^2 = 0.$$

If $\mathsf{f} = 0$, the curve is a parabola, and the tangential equation is satisfied by the co-ordinates $u = 0, v = 0, w \gtrless 0$ of the straight line at infinity: this is what is expressed when one says that the parabola is tangent to the straight line at infinity.

Let (u_1, v_1, w_1), (u_2, v_2, w_2) be the co-ordinates of two distinct straight lines; the equation of their point of intersection will be

$$\begin{vmatrix} u & v & w \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = 0.$$

The co-ordinates u, v, w of any straight line D which passes through this point can be written

$$(D) \qquad u = u_1 + \lambda u_2, \;\; v = v_1 + \lambda v_2, \;\; w = w_1 + \lambda w_2;$$

these formulas may be verified in the same manner as formulas (4). If the sign of λ be changed, the co-ordinates of a second straight line are found to be

$$(D') \qquad u' = u_1 - \lambda u_2, \;\; v' = v_1 - \lambda v_2, \;\; w' = w_1 - \lambda w_2,$$

which is the harmonic conjugate of the first with respect to the two given straight lines; for, the equation of the straight line D is

$$ux + vy + wz = u_1x + v_1y + w_1z + \lambda(u_2x + v_2y + w_2z) = 0,$$

and that of D' is, similarly,

$$u_1x + v_1y + w_1z - \lambda(u_2x + v_2y + w_2z) = 0,$$

which proves the theorem (§ 69).

Exercise. — *Prove that the pole of the straight line* $(\mathbf{u}_1, \mathbf{v}_1, \mathbf{w}_1)$ *with respect to a curve of the second class whose tangential equation is* $\phi(\mathbf{u}, \mathbf{v}, \mathbf{w}) = 0$, *is given by the equation*

$$\mathbf{u}\phi'_{\mathbf{u}_1} + \mathbf{v}\phi'_{\mathbf{v}_1} + \mathbf{w}\phi'_{\mathbf{w}_1} = 0,$$

or

$$\mathbf{u}_1\phi'_{\mathbf{u}} + \mathbf{v}_1\phi'_{\mathbf{v}} + \mathbf{w}_1\phi'_{\mathbf{w}} = 0.$$

Trilinear Co-ordinates.

331. Definition. — Consider three linear equations

$$(6) \qquad \begin{aligned} \alpha &= ax + by + cz, \\ \beta &= a'x + b'y + c'z, \\ \gamma &= a''x + b''y + c''z, \end{aligned}$$

where the determinant

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = D$$

is different from zero. To every system of values of x, y, z in these equations there corresponds a *single* system of values of α, β, γ, and, conversely, to every system of values of α, β, γ a single system of values of x, y, z.

If x, y, z be the homogeneous co-ordinates of a point of a plane, it follows also that to every system of values of α, β, γ, all of which are not zero at the same time, there corresponds a definite point M, and to every point M of the plane there corresponds a unique system of values of α, β, γ, with the condition that systems such as α, β, γ, and $\rho\alpha$, $\rho\beta$, $\rho\gamma$, are not to be regarded as different systems.

The quantities α, β, γ, are called the trilinear co-ordinates of the point M *with respect to the triangle of reference* whose sides have the equations

$$ax + by + cz = 0, \quad a'x + b'y + c'z = 0, \quad a''x + b''y + c''z = 0.$$

GEOMETRIC INTERPRETATION. — If one take $z = 1$, the trilinear co-ordinates α, β, γ are equal to the distances of the point M from the three sides of the triangle of reference multiplied by factors which have the same sign when the point M varies. In particular, if one consider the equations

$$(7) \qquad \begin{aligned} \alpha &= -(x \cos a + y \sin a - pz), \\ \beta &= -(x \cos b + y \sin b - qz), \\ \gamma &= -(x \cos c + y \sin c - rz); \end{aligned}$$

and if one suppose the origin of Cartesian co-ordinates to be within the triangle, one sees that for $z = 1$, α, β, γ are equal to the distances of the point M from the sides affected with proper signs. This sign is $+$ for a side AB of the triangle of reference when the point M under consideration and the vertex C opposite to AB are situated on the same side of AB; it is $-$ in the contrary case.

In order to find in trilinear co-ordinates the equation of a given curve in homogeneous co-ordinates, $f(x, y, z) = 0$, we replace x, y, z by the values found by solving equations (6) with respect to x, y, z. The values obtained for x, y, z being homogeneous and linear in α, β, γ, the new equation $F(\alpha, \beta, \gamma) = 0$ will be homogeneous in α, β, γ, and of the same degree as f. Conversely, if one be given an equation $F(\alpha, \beta, \gamma) = 0$, in trilinear co-ordinates, it will be sufficient to replace α, β, γ by expressions (6) in order to have the equation of the curve in homogeneous co-ordinates.

Let, for example,

$$f(x, y, z) = ux + vy + wz = 0$$

be the equation of a straight line in homogeneous co-ordinates; if the preceding substitution be made, one will get for the equation of this same straight line

$$F = U\alpha + V\beta + W\gamma = 0.$$

One can return to equation (7) by replacing α, β, γ by their values (6). It is evident also that

$$\begin{aligned} u &= aU + a'V + a''W, \\ v &= bU + b'V + b''W, \\ w &= cU + c'V + c''W. \end{aligned} \tag{8}$$

The coefficients U, V, W of equation F are called the tangential co-ordinates of the straight line in the new system; equations (8) express the homogeneous tangential co-ordinates (u, v, w) as functions of the new (U, V, W), and, conversely, they make it possible to transform every homogeneous tangential equation in u, v, w, $\phi(u, v, w) = 0$, into another of the same degree $\Phi(U, V, W) = 0$, and conversely.

The Equation of the Straight Line at Infinity in Trilinear Co-ordinates. — The co-ordinates of a point at infinity have been defined as a system of values x, y, z, in which z is *zero*. If formulas (6) be solved with respect to z, and if the homogeneous linear expression in α, β, γ found for z be equated to zero, one obtains a condition which is called the equation of the straight line at infinity. For example, one deduces from equation (7),

$$zD = \alpha \sin(b - c) + \beta \sin(c - a) + \gamma \sin(a - b),$$

where D is a constant factor; whence one obtains for the equation of the straight line at infinity,

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0,$$

where A, B, C are the angles of the triangle of reference.

THE EQUATION OF A STRAIGHT LINE PASSING THROUGH TWO POINTS. — Let M_1 and M_2 be two points whose trilinear co-ordinates are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, then will the equation of the straight line which passes through two points be

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0.$$

The co-ordinates of a point M of the straight line may be expressed by the formulas

$$(M) \qquad \alpha = \alpha_1 + \lambda\alpha_2,\ \beta = \beta_1 + \lambda\beta_2,\ \gamma = \gamma_1 + \lambda\gamma_2,$$

where λ has the same meaning as in formulas (4). In fact, call (x_1, y_1, z_1), (x_2, y_2, z_2), (x, y, z) the homogeneous co-ordinates of the points M_1, M_2, M. One has, according to (6),

$$\alpha_1 = ax_1 + by_1 + cz_1,\ \alpha_2 = ax_2 + by_2 + cz_2,\ \alpha = ax + by + cz;$$

since

$$x = x_1 + \lambda x_2,\ \cdots,\ \cdots,$$

one has also $\alpha = \alpha_1 + \lambda\alpha_2$, and similar expressions for β and γ. Whence it follows that the point M' with the co-ordinates

$$(M') \qquad \alpha_1 - \lambda\alpha_2,\ \beta_1 - \lambda\beta_2,\ \gamma_1 - \lambda\gamma_2$$

is the harmonic conjugate of M with respect to the segment M_1M_2.

By a calculation similar to that which precedes equation (5), one can show that the polar of the point $(\alpha_1, \beta_1, \gamma_1)$, with respect to the conic

$$F(\alpha, \beta, \gamma) = A\alpha^2 + A'\beta^2 + A''\gamma^2 + 2\,B\beta\gamma + 2\,B'\gamma\alpha + 2\,B''\alpha\beta = 0,$$

has the equation

$$\alpha_1 F'_\alpha + \beta_1 F'_\beta + \gamma_1 F'_\gamma = 0, \text{ or } \alpha F'_{\alpha_1} + \beta F'_{\beta_1} + \gamma F'_{\gamma_1} = 0.$$

Tangents in Trilinear Co-ordinates. — Let $F(\alpha, \beta, \gamma)=0$ be the equation of a curve, and $\alpha_1, \beta_1, \gamma_1$ be a point situated on the curve having the homogeneous co-ordinates x_1, y_1, z_1. On replacing α, β, γ by their values (6), the equation of the curve in homogeneous co-ordinates will be obtained: by this substitution $F(\alpha, \beta, \gamma)$ is transformed identically into $f(x, y, z)$, the first member of the equation of the curve in homogeneous co-ordinates. The equation of the tangent at the point x_1, y_1, z_1 is

$$xf'_{x_1} + yf'_{y_1} + zf'_{z_1},$$

or the identity

$$f(x, y, z) = F(\alpha, \beta, \gamma)$$

gives by reason of equations (6),

$$f'_{x_1} = aF'_{\alpha_1} + a'F'_{\beta_1} + a''F'_{\gamma_1},$$
$$f'_{y_1} = bF'_{\alpha_1} + b'F'_{\beta_1} + b''F'_{\gamma_1},$$
$$f'_{z_1} = cF'_{\alpha_1} + c'F'_{\beta_1} + c''F'_{\gamma_1};$$

whence $xf'_{x_1} + yf'_{y_1} + zf'_{z_1} = \alpha F'_{\alpha_1} + \beta F'_{\beta_1} + \gamma F'_{\gamma_1}$.

The equation of the tangent at the point $\alpha_1, \beta_1, \gamma_1$ is therefore

$$\alpha F'_{\alpha_1} + \beta F'_{\beta_1} + \gamma F'_{\gamma_1} = 0.$$

The tangential co-ordinates of this tangent are given by the equations

$$\rho U = F'_{\alpha_1}, \ \rho V = F'_{\beta_1}, \ \rho W = F'_{\gamma_1},$$

ρ being a constant different from zero.

One can demonstrate in a similar manner that:

1° The equation of the point of intersection of the two straight lines D_1 and D_2, whose co-ordinates are (U_1, V_1, W_1) and (U_2, V_2, W_2), is

$$\begin{vmatrix} U & V & W \\ U_1 & V_1 & W_1 \\ U_2 & V_2 & W_2 \end{vmatrix} = 0;$$

2° The co-ordinates of any straight line D passing through this point are

$$(D) \qquad U = U_1 + \lambda U_2, \ V = V_1 + \lambda V_2, \ W = W_1 + \lambda W_2;$$

3° The straight line D', whose co-ordinates are

$$(D') \qquad U_1 - \lambda U_2, \ V_1 - \lambda V_2, \ W_1 - \lambda W_2,$$

is the harmonic conjugate of D with respect to the two straight lines D_1 and D_2;

4° If U_1, V_1, W_1 be the co-ordinates of a tangent to the curve whose tangential equation is $\Phi(U, V, W) = 0$, the equation of the point of contact is

$$U\Phi'_{U_1} + V\Phi'_{V_1} + W\Phi'_{W_1} = 0,$$

and the co-ordinates of this point are given by the formulas

$$\rho\alpha = \Phi'_{U_1}, \ \rho\beta = \Phi'_{V_1}, \ \rho\gamma = \Phi'_{W_1}.$$

Below are given some very simple applications of the preceding considerations.

Example I. — *Form the general equations of conics conjugate with respect to the triangle of reference.*

Let

$$F(\alpha, \beta, \gamma) = A\alpha^2 + A'\beta^2 + A''\gamma^2 + 2\,B\beta\gamma + 2\,B'\gamma\alpha + 2\,B''\alpha\beta = 0$$

be the equation of such a conic. The polar of each vertex of the triangle of reference with respect to this conic ought to be the opposite side. The polar of the point $(\alpha_1, \beta_1, \gamma_1)$ being

$$\alpha_1 F'_\alpha + \beta_1 F'_\beta + \gamma_1 F'_\gamma = 0,$$

that of the vertex of the triangle whose co-ordinates are $\alpha_1 = 0$, $\beta_1 = 0$, $\gamma_1 \gtrless 0$, is

$$\tfrac{1}{2} F'_\gamma = B'\alpha + B\beta + A''\gamma = 0\,;$$

this polar should coincide with the opposite side $\gamma = 0$, and hence $B = B' = 0$. Similarly it may be shown that $B'' = 0$, and the required equation will be

$$A\alpha^2 + A'\beta^2 + A''\gamma^2 = 0.$$

The tangential equation of these same conics is

$$\frac{U^2}{A} + \frac{V^2}{A'} + \frac{W^2}{A''} = 0.$$

It should be noticed in these formulas that one can suppose that one of the vertices or one of the sides of the triangle of reference is at infinity.

General equation of conics inscribed in the triangle of reference. The most general curve of the second class has in tangential co-ordinates the equation

$$AU^2 + A'V^2 + A''W^2 + 2\,BVW + 2\,B'WU + 2\,B''UV = 0;$$

on expressing the condition that this curve is tangent to the three sides of the triangle of reference whose co-ordinates are respectively

$$V = 0 \text{ and } W = 0,\ \ W = 0 \text{ and } U = 0,\ \ U = 0 \text{ and } V = 0,$$

it follows that $A = A' = A'' = 0$. The tangential equation of the conics in question is therefore

$$BVW + B'WU + B''UV = 0.$$

Moreover, the equation in trilinear co-ordinates is

$$\begin{vmatrix} 0 & B'' & B' & \alpha \\ B'' & 0 & B & \beta \\ B' & B & 0 & \gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix} = 0;$$

or $\quad B^2\alpha^2 + B'^2\beta^2 + B''^2\gamma^2 - 2\,B'B''\beta\gamma - 2\,BB''\alpha\gamma - 2\,BB'\alpha\beta = 0.$

The general equation of conics inscribed in a quadrilateral. Let, in homogeneous co-ordinates,

$$P = lx + l'y + l''z = 0, \qquad Q = mx + m'y + m''z = 0,$$

$$R = nx + n'y + n''z = 0,\ \ S = kx + k'y + k''z = 0,$$

be the equations of the four sides of the quadrilateral. Three homogeneous linear functions,

$$\alpha = ax + a'y + a''z,\ \ \beta = bx + b'y + b''z,\ \ \gamma = cx + c'y + c''z,$$

and four constants p, q, r, s, can always be found such that one has identically

(1) $$pP = \alpha + \beta + \gamma,\ \ qQ = \alpha - \beta - \gamma,$$
$$rR = \alpha - \beta + \gamma,\ \ sS = \alpha + \beta - \gamma;$$

the identification of the two members of these identities will give twelve homogeneous equations of the first degree connecting the thirteen unknown constants

$$a, a', a''; \; b, b', b''; \; c, c', c''; \; p, q, r, s.$$

The calculation can be simplified as follows. The following equation may be deduced immediately from identities (1):

(2) $$pP + qQ - rR - sS = 0,$$

which will determine p, q, r, s, or, better, the ratios of any three to the fourth. The constants p, q, r, s being thus determined, it follows that one of the identities (1) is deducible from the other three; one can write

$$2\,\alpha = pP + qQ = rR + sS,$$

$$2\,\beta = pP - rR = sS - qQ,$$

$$2\,\gamma = pP - sS = rR - qQ;$$

the functions α, β, γ are therefore known; the two expressions found for each of these linear functions are identical by reason of the identity (2). The three straight lines $\alpha = 0$, $\beta = 0$, $\gamma = 0$ are the diagonals of the given complete quadrilateral. In fact, the equation of the straight line $\alpha = 0$ can be written in either of the forms

$$pP + qQ = 0, \quad rR + sS = 0;$$

the first shows that this straight line passes through the vertex of the quadrilateral which is the intersection of the sides $P = 0$, $Q = 0$, and the second that it passes through the opposite vertex $R = 0$, $S = 0$.

Thus, on choosing as the triangle of reference the triangle formed by the three diagonals of a complete quadrilateral, the equations of the sides of the quadrilateral can be written in the form

$$\alpha + \beta + \gamma = 0, \quad \alpha + \beta - \gamma = 0, \quad \alpha - \beta + \gamma = 0, \quad \alpha - \beta - \gamma = 0.$$

The vertex $P = 0$, $Q = 0$ has the trilinear co-ordinates $\alpha = 0$, $\beta + \gamma = 0$, and the opposite vertex $R = 0$, $S = 0$ has the trilinear co-ordinates $\alpha = 0$, $\beta = \gamma$. The first of these points has the tangential equation $V + W = 0$, and the second $V - W = 0$: the *ensemble* of these two opposite vertices is therefore represented by the tangential equation $V^2 - W^2 = 0$. Similarly, the *ensemble* of the two opposite vertices $P = 0$, $S = 0$, and $Q = 0$, $R = 0$ is represented by the tangential equation $U^2 - V^2 = 0$. The general tangential equation of conics inscribed in a quadrilateral is, therefore (§ 307, Ex. II.),

$$V^2 - W^2 + \lambda (U^2 - V^2) = 0,$$

or

$$\lambda U^2 + (1 - \lambda) V^2 - W^2 = 0.$$

The equation of the same system of conics in trilinear co-ordinates is

$$\frac{\alpha^2}{\lambda} + \frac{\beta^2}{1 - \lambda} - \gamma^2 = 0.$$

REMARK. — On putting

$$\alpha = \frac{x}{c}, \quad \beta = \frac{y\sqrt{-1}}{c}, \quad \gamma = z = 1, \quad \lambda c^2 = \mu,$$

one obtains the equation

$$\frac{x^2}{\mu} + \frac{y^2}{\mu - c^2} - 1 = 0$$

of confocal conics.

EXAMPLE II. — Let us consider two reciprocal polar triangles with respect to a given conic; for simplicity take the sides of one of the triangles as lines of reference fulfilling the definition of new co-ordinates, and let

$$f(\alpha, \beta, \gamma) = \tfrac{1}{2}(A\alpha^2 + A'\beta^2 + A''\gamma^2 + 2\,B\beta\gamma + 2\,B'\gamma\alpha + 2\,B''\alpha\beta) = 0$$

be the equation of the conic. The polar of any point $(\alpha', \beta', \gamma')$ has the equation $\alpha' f'_\alpha + \beta' f'_\beta + \gamma' f'_\gamma = 0$. In particular, the polars of the three vertices $(\beta' = 0,\ \gamma' = 0)$, $(\gamma' = 0,\ \alpha' = 0)$, $(\alpha' = 0,\ \beta' = 0)$ of the triangle have the equations

$$f'_\alpha = A\alpha + B''\beta + B'\gamma, \quad f'_\beta = B''\alpha + A'\beta + B\gamma = 0,$$
$$f'_\gamma = B'\alpha + B\beta + A''\gamma = 0.$$

These polars are the sides of the second triangle. The co-ordinates of the point of intersection of two corresponding sides $\alpha = 0, f'_\alpha = 0$ satisfy the equations $\alpha = 0,\ B''\beta + B'\gamma = 0$, or $\alpha = 0,\ \frac{\beta}{B'} + \frac{\gamma}{B''} = 0$; this point is situated on the straight line $\frac{\alpha}{B} + \frac{\beta}{B'} + \frac{\gamma}{B''} = 0$, and similarly for the other two sides. Thus *the three points of intersection of the corresponding sides of two reciprocal polar triangles lie on a straight line.*

A vertex of the second triangle being given by the two equations $f'_\alpha = 0,\ f'_\beta = 0$, the straight line $Bf'_\alpha = B'f'_\beta$ passes through this point; since the equation does not contain the letter γ, this straight line passes through the vertex ($\alpha = 0$, $\beta = 0$) of the first triangle. The straight lines which join the corresponding vertices being represented by the equations $Bf'_\alpha = B'f'_\beta = B''f'_\gamma$, it follows that *these three straight lines pass through a common point.*

EXAMPLE III. — A triangle *abc* is inscribed in a conic; two of its sides *ab* and *ac* revolve about two fixed points *p* and *q* (Fig. 205); find the envelope of the third side *bc*. Let $\gamma = 0$ be the equation of the straight line *pq*, $\alpha = 0$ and $\beta = 0$ those of the tangents at the points *d* and *e* in which this straight line intersects the curve; the equation of the conic will have the form $\alpha\beta - \gamma^2 = 0$. The points *p* and *q* may be regarded as the points where the straight line $\gamma = 0$ is intersected by the two straight lines $\alpha + p\beta = 0$, $\alpha + q\beta = 0$, which pass through the point of intersection *o* of the tangents at *d* and *e*.

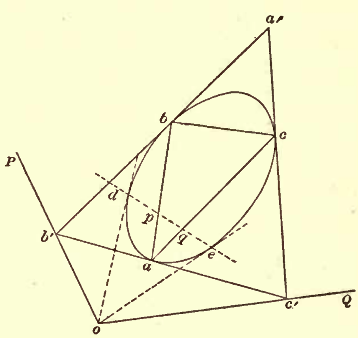

Fig. 205.

Any point *a* of the curve can be determined by the intersection of two straight lines $\alpha - a\gamma = 0$, $\beta - \frac{\gamma}{a} = 0$, which pass through the points *d* and *e*, *a* being an arbitrary parameter which defines the position of the point

a on the curve. On assigning to this parameter another value b, one obtains another point b. Any straight line passing through the point a has an equation of the form $\alpha - a\gamma + k\left(\beta - \frac{\gamma}{a}\right) = 0$; in order that this straight line pass through the point b which is represented by the two equations $\alpha - b\gamma = 0$, $\beta - \frac{\gamma}{b} = 0$, it is necessary to make $k = ab$; thus the straight line which connects any two points a and b of the curve has the equation $\alpha + ab\beta - (a+b)\gamma = 0$.

Let now a, b, c be the values of the parameter for the three vertices a, b, c of the triangle; since the side ab passes through the point p, one has $ab = p$; since the side ac passes through the point q, one has similarly $ac = q$; the side bc has the equation $\alpha + bc\beta - (b+c)\gamma = 0$; if b and c be replaced by their values $\frac{p}{a}$ and $\frac{q}{a}$, the equation becomes

$$a^2\alpha + pq\beta - (p+q)\,a\gamma = 0.$$

If the variable parameter a be eliminated between this equation and the following equation,

$$2\,a\alpha - (p+q)\,\gamma = 0,$$

which is obtained by equating to zero the derivative of the preceding equation with respect to a, one obtains the equation of the envelope of the straight line bc,

$$\alpha\beta + \frac{(p+q)^2}{4\,pq}\,\gamma^2 = 0.$$

This envelope is a conic which touches the first at the points d and e.

The tangential co-ordinates U, V, W of the variable straight line are given by the equations,

$$\rho U = a^2, \quad \rho V = pq, \quad \rho W = -(p+q)\,a;$$

the elimination of a and ρ gives the tangential equation of the envelope,

$$UV(p+q)^2 - W^2 pq = 0.$$

If tangents be drawn to the proposed conic at the points a, b, c, a circumscribed triangle $a'b'c'$ is formed, of which the two vertices b' and c' slide on the two fixed straight lines P and Q, polars of the point p and q; the curve described by the vertex a', pole of the straight line bc, is the reciprocal polar of the envelope; therefore, it is also a conic with a double contact with the first along the line de.

EXERCISES.

1. The eight points of contact of the tangents common to two given conics are situated on a conic.

2. A triangle is inscribed in a conic; two of its sides pass through two fixed points or revolve on two conics doubly tangent to the first; the envelope of the third side is a conic.— The converse theorem.

3. A polygon with n sides is inscribed in a conic; $n-1$ sides revolve about conics doubly tangent to the first; the envelope of the nth side is a conic.— The converse theorem.

4. Two conics S and S' are given, and also two tangents to the conic S'; the six straight lines which join two by two the four points in which these tangents intersect the conic S are two by two tangent to the same conic which passes through the point of intersection of the conics S and S'.— The converse theorem.

5. Being given three conics which have four given points in common, a triangle inscribed in one of them has two of its sides tangent respectively to the other two conics; the third side envelops a conic.— The converse theorem.

6. Being given n conics which have four given points in common, a polygon of n sides inscribed in one of them has $n-1$ of its sides tangent respectively to the other conics; the nth side envelops a conic.— The converse theorem.

7. A polygon in one of its positions is inscribed in a conic and circumscribed about another conic; if a vertex be made to move on the first conic, in such a way that $n-1$ sides are tangent to the second conic, the nth side will always be tangent to this second conic.

The conics which have four common points in Theorems IV., V., VI., and VII., can be replaced by homothetic conics having two common points, and in particular by circles having in pairs the same radical axis.

8. The envelope of the straight lines which intersect two given conics in four points, which are harmonically arranged, is a conic. — The converse theorem.

If the two conics have in trilinear co-ordinates the equations $\alpha^2+\beta^2+\gamma^2=0$, $A\alpha^2+B\beta^2+C\gamma^2=0$, the necessary and sufficient condition in order that the straight line $u\alpha+v\beta+w\gamma=0$ intersects the two conics in four points, which are harmonically arranged, is

$$(B+C)u^2+(C+A)v^2+(A+B)w^2=0.$$

What will happen when $A+B=0$? Apply the result to the particular case where one of the conics is a circle, and the other an equilateral concentric hyperbola.

9. We know that the polars of a point p, with respect to all of the conics which have four points in common, pass through a fixed point q; if the point p describe a straight line, the point q describes a conic. — The converse theorem.

10. If two sides of a triangle inscribed in a conic revolve about any two given curves, the third side envelops a third curve; show that the straight lines which join the vertices of the triangle to the points of contact of the opposite sides pass through a fixed point. — The converse theorem.

11. Being given a hexagon inscribed in a conic, the points of intersection of the opposite sides are located, and also the points of intersection of each of three diagonals with the two opposite sides; the nine points thus determined are situated on three straight lines which pass through a fixed point.

12. A conic S is given, and a variable conic S' is constructed, which intersects the first in two fixed points and which touches two fixed straight lines whose point of intersection is situated on the conic S; the envelope of the straight line which passes through the other two points of intersection of the conics S and S' is a conic. — The converse theorem.

13. A quadrilateral is circumscribed about a conic; if any tangent be drawn to the conic, one knows that the ratio of the product of the distances of this tangent from the two opposite vertices of a quadrilateral to the product of the distances of this same tangent from the other two opposite vertices is constant. Show that this ratio is equal to the product of the distances of the first two vertices from one of the foci divided by the product of the distances of the other two vertices from the same focus.

14. The six sides of any two triangles inscribed in the same conic are tangent to another conic. — The converse theorem.

15. Three points are said to be conjugate with respect to a conic, if the polar of one of them is the straight line which joins the other two; show that the two systems of three conjugate points with respect to a conic are situated on another conic. — The converse theorem.

16. Find the necessary and sufficient conditions in order that a conic coincide with its reciprocal polar with respect to the circle $x^2 + y^2 - 1 = 0$.

17. The tangential equation of a conic in rectangular coordinates being

$$\mathsf{a}u^2 + 2\,\mathsf{b}uv + \mathsf{c}v^2 + 2\,\mathsf{d}u + 2\,\mathsf{e}v + \mathsf{f} = 0,$$

show that the circle which is the locus of the vertices of the right angles circumscribed about the conic, has the equation

$$\mathsf{f}\,(x^2 + y^2) - 2\,\mathsf{d}x - 2\,\mathsf{e}y + \mathsf{a} + \mathsf{c} = 0.$$

18. Consider a variable conic tangent to four fixed straight lines. Show that the circle which is the locus of the vertices of the right angles circumscribed about this conic passes through two fixed points, real or imaginary.

19. Consider a variable conic tangent to three fixed straight lines. Show that the circle which is the locus of the vertices of the right angles circumscribed about this conic is the orthogonal conjugate to the triangle formed by the three straight lines.

20. Consider a conic whose equation is

$$f(x, y) - (lx + my + n)^2 = 0,$$

$f(x, y)$ being a polynomial of the second degree in x and y. Show that the tangential equation of this conic is

$$\phi(u, v) - (\lambda u + \mu v + \nu)^2 = 0,$$

ϕ being of the second degree in u and v. Apply the result to the case where

$$f(x, y) = (x - a)^2 + (y - b)^2.$$

21. Form the tangential equation of the curve generated by a point of a circumference rolling within a circumference of triple radius. (Hypocycloid with three cusps.) On writing the equation of the tangent to the curve in the form $x \sin \alpha - y \cos \alpha = p$, show that p has the form $p = a \cos(3\alpha + \alpha_0)$, a and α_0 being constants.

22. Find the envelope of the axes and the tangents at the vertex of parabolas inscribed in a triangle.

23. Find the envelope of the axes of equilateral hyperbolas circumscribed about a triangle.

24. Show that the necessary and sufficient condition in order that two straight lines whose co-ordinates are (u, v), (u', v') intersect on a conic

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$

is

$$\begin{vmatrix} A & B & D & u & u' \\ B & C & E & v & v' \\ D & E & F & 1 & 1 \\ u & v & 1 & 0 & 0 \\ u' & v' & 1 & 0 & 0 \end{vmatrix} = 0.$$

CHAPTER XII*

SECANTS COMMON TO TWO CONICS.

332. We have found above (§ 286) an equation of the third degree

(1) $$\Delta + \Theta\lambda + \Theta'\lambda^2 + \Delta'\lambda^3 = 0,$$

giving the value of λ, for which the equation

$$S + \lambda S' = 0$$

represents two straight lines. It is proposed to investigate the nature of the roots of equation (1), called equation in λ, and to study the nature of the points common to the two conics $S = 0$, $S' = 0$. We adopt for this purpose a method due to M. Darboux, which is reproduced in the excellent *Treatise on Analytic Geometry* by M. Pruvost. This method is based on the following lemmas.

(1) If the equation

$$S = Ax^2 + 2\,Bxy + Cy^2 + 2\,Dxz + 2\,Eyz + Fz^2 = 0$$

represent two parallel or concurrent straight lines, the homogeneous co-ordinates a, b, c of the point of intersection, situated at an infinite or finite distance, satisfy the conditions

(2) $$\frac{a^2}{\mathbf{a}} = \frac{ab}{\mathbf{b}} = \frac{b^2}{\mathbf{c}} = \frac{ac}{\mathbf{d}} = \frac{bc}{\mathbf{e}} = \frac{c^2}{\mathbf{f}}.$$

In fact, let

$$P = ux + vy + wz = 0,\quad Q = u'x + v'y + w'z = 0$$

be the equations of two straight lines; one will have identically

$$S = PQ;$$

whence, taking successively the partial derivatives of the two members with respect to x, y, z:

$$Ax + By + Dz = \tfrac{1}{2}(Pu' + Qu),$$

$$Bx + Cy + Ez = \tfrac{1}{2}(Pv' + Qv),$$

$$Dx + Ey + Fz = \tfrac{1}{2}(Pw' + Qw).$$

The homogeneous co-ordinates a, b, c of the point of intersection of the two straight lines reduce to zero the first members P and Q of the equation of the two straight lines; they satisfy, therefore, the three equations

$$Aa + Bb + Dc = 0,$$

$$Ba + Cb + Ec = 0,$$

$$Da + Eb + Fc = 0;$$

whence may easily be deduced

$$\frac{a}{\mathrm{a}} = \frac{b}{\mathrm{b}} = \frac{c}{\mathrm{d}},$$

$$\frac{a}{\mathrm{b}} = \frac{b}{\mathrm{c}} = \frac{c}{\mathrm{e}},$$

$$\frac{a}{\mathrm{d}} = \frac{b}{\mathrm{e}} = \frac{c}{\mathrm{f}};$$

on multiplying the first group of these relations by a, the second by b, the third by c, one obtains relations (2), which was to be proven.

Deductions. — If the equation $S = 0$ represent two straight lines, discriminant Δ is zero, and equation (1) has one of its roots equal to zero. If one have further $\Theta = 0$, this root zero is double; and then the point of intersection of the straight lines $P = 0$, $Q = 0$, which constitute the conic S, lies on S'. In fact, one has

$$\Theta = A'\mathrm{a} + 2\,B'\mathrm{b} + C'\mathrm{c} + 2\,D'\mathrm{d} + 2\,E'\mathrm{e} + F'\mathrm{f}.$$

Owing to relations (2), the condition $\Theta = 0$, in which a, b, c, etc., are replaced by the proportional quantities a^2, $2\,ab$, b^2, $\cdots$, becomes

$$A'a^2 + 2\,B'ab + C'c^2 + 2\,D'ac + 2\,E'bc + F'c^2 = 0,$$

the condition required that the point (a, b, c) lies on the conic $S' = 0$.

(2) If, besides the conditions $\Delta = 0$, $\Theta = 0$, one have further $\Theta' = 0$, one of the straight lines into which the conic $S = 0$ is decomposed is tangent to S'. In this case equation (1) has a *triple root zero.*

The identity

$$S = PQ = (ux + vy + wz)(u'x + v'y + w'z)$$

gives

$$A = uu',\ 2\,B = uv' + vu',\ C = vv',$$

$$2\,D = uw' + wu',\ 2\,E = vw' + wv',\ F = ww';$$

and, since

$$\Theta' = A\mathsf{a}' + 2\,B\mathsf{b}' + C\mathsf{c}' + 2\,D\mathsf{d}' + 2\,E\mathsf{e}' + F\mathsf{f}',$$

the condition $\Theta' = 0$ becomes

$$\mathsf{a}'uu' + \mathsf{b}'(uv' + u'v) + \mathsf{c}'vv' + \mathsf{d}'(uw' + wu')$$
$$+ \mathsf{e}'(vw' + v'w) + \mathsf{f}'ww' = 0;$$

this condition shows that the pole of one of the straight lines $P = 0$, $Q = 0$ with respect to S' lies on the other (§ 296), that is, that these two straight lines are conjugates with respect to the conic $S' = 0$. Since their point of intersection is situated on this conic, one of these straight lines is necessarily tangent to it.

333. We have to examine what happens in case the conic $S = 0$ consists of two straight lines; we occupy ourselves first with the general case. For this purpose, suppose that S and S' are any conics whatever, and call λ_1 a root of equation (1). The conic

$$S + \lambda_1 S' = 0$$

will be decomposed into two straight lines $P_1 = 0$ and $Q_1 = 0$.

Put

(3) $$S + \lambda_1 S' = S_1, \quad \alpha S + \beta S' = S'_1,$$

α and β being any two constants subject to the condition that they will not reduce the quantity $(\beta - \alpha\lambda_1)$ to zero, which determines the coefficients of S and S' in relations (3). Then $S_1 = 0$ is the equation of the pair of secants common to the conics $S = 0$, $S' = 0$, which correspond to the root $\lambda = \lambda_1$, and $S'_1 = 0$ is the equation of any conic, distinct from $S_1 = 0$, which passes through the points of intersection of the given conics $S = 0$, $S' = 0$. Since identities (3), solved with respect to S and S', give

$$(\beta - \alpha\lambda_1) S = \beta S_1 - \lambda_1 S'_1,$$

$$(\beta - \alpha\lambda_1) S' = -\alpha S_1 + S'_1,$$

the general equation $S + \lambda S' = 0$ can be written

$$S_1(\beta - \alpha\lambda) + S'_1(\lambda - \lambda_1) = 0,$$

or $$S_1 + \mu S'_1 = 0,$$

on putting

(4) $$\mu = \frac{\lambda - \lambda_1}{\beta - \lambda\alpha}.$$

Thus, μ being connected with λ by relation (4), the two equations

$$S + \lambda S' = 0, \quad S_1 + \mu S'_1 = 0$$

represent *the same conic.* If one seek the value of μ for which this conic reduces to two straight lines, one obtains an equation of the third degree in μ:

(5) $$\Delta_1 + \Theta_1\mu + \Theta'_1\mu^2 + \Delta'_1\mu^3 = 0,$$

in which $\Delta_1 = 0$, since S_1 is decomposed into two straight lines. It follows from relation (4) that to the root $\lambda = \lambda_1$ of equation (1) there corresponds the root $\mu = 0$ of equation (5) with the same degree of multiplicity.

If the root λ_1 be simple, the root $\mu = 0$ is also simple; Θ_1 is therefore different from zero, and the point of intersection of the straight lines represented by equation $S_1 = 0$ is not on the conic S'_1.

If the root λ_1 be double, the root $\mu = 0$ of equation (5) will be double; Θ_1 will be zero, and the point of intersection of the two straight lines $S_1 = 0$ will be on the conic S'_1. It can happen, as a particular case, that the minors of Δ_1 are zero; then Θ_1 is zero whatever the conic S'_1 may be; the conic $S_1 = 0$ reduces to a double straight line.

Finally, if the root λ_1 be triple, the root $\mu = 0$ of equation (5) will likewise be triple; Θ_1 and Θ'_1 will be zero; the point of intersection of the straight lines $S_1 = 0$ will be on S'_1, and one of these straight lines will be tangent to S'_1. It can happen, as a particular case, that all the minors of Δ_1 are zero; then the conic $S_1 = 0$ reduces to a *double straight line tangent* to S'_1.

Call λ_1, λ_2, λ_3 the roots of equation (1) in λ.

1° If these roots be unequal, each pair of secants consists of two distinct straight lines, whose point of intersection does not lie on any of the given conics. These conics intersect therefore in *four distinct points.*

In order to learn how many of these points are real, one has recourse to the following considerations:

(*a*) The three roots λ_1, λ_2, λ_3 are real, and the pairs of secants corresponding to two of them are real. These two pairs of real straight lines intersect in *four real and distinct points.* These four points of intersection of the conics are, therefore, real and distinct. It is plain that the third pair of secants is also real, since it passes through the points of intersection of two conics which are real.

(*b*) The roots λ_1, λ_2, λ_3 are real, but one pair only of the secants is real. Then the four points of intersection are imaginary. In fact, a pair of imaginary secants with real coefficients have in common but one real point, the point of intersection of the secants; since this point does not belong to any of the conics S or S', the points of intersection, all four of which are situated in this imaginary pair, are necessarily imaginary.

(*c*) One root λ_1 is real, the other two λ_2 and λ_3 are imaginary. Two of the points of intersection are real and two imaginary. If λ_2 be imaginary, $\lambda_2 = p + iq$, the corresponding pair

$$S + (p + iq)S' = 0$$

is composed of two *imaginary straight lines*, but not *imaginary conjugates.* In the first place, this pair is composed of two imaginary straight lines; because if one of these straight lines were real, the co-ordinates of a point of this straight line would reduce $S+pS'$ to zero on the one hand, and the coefficient S' of i on the other; that is, S and S'; the two conics would have, therefore, a common straight line, which contradicts the hypothesis. Moreover, the pair considered does not consist of conjugate imaginary straight lines; because, if that were so, the point of intersection of these straight lines would be real, and its co-ordinates would reduce to zero, simultaneously, the partial derivatives of $S+(p+iq)S'$ with respect to x, y, z, that is, the three partial derivatives of each of the polynomials S and S'; the two conics $S=0$, $S'=0$ would consist of two pairs of concurrent straight lines, which contradicts the hypothesis. The pair corresponding to the root λ_2 will be composed of two straight lines whose equations are

$$P+iQ=0, \quad R+iS=0;$$

and that corresponding to the root λ_3 of the two straight lines,

$$P-iQ=0, \quad R-iS=0.$$

These two pairs and, consequently, the two conics intersect therefore in two real points, the one situated at the intersection of the straight lines $P=0$, $Q=0$, the other at the intersection of the straight lines $R=0$, $S=0$. The other two points of intersection are imaginary, because on one imaginary straight line there can be but *one real point.*

2° If the equation in λ have a double root λ_1 and a simple root λ_3, the point of intersection O_1 of the two *assumed distinct* straight lines of the pair

$$S_1=S+\lambda_1 S'=0,$$

corresponding to the double root, lies on the conics S and S', and, more generally, on all of the conics $\alpha S+\beta S'=0$; the point of intersection O_3 of the straight lines of the couple,

$$S_3=S+\lambda_3 S'=0,$$

does not, on the contrary, lie on any of these conics other than S_3 itself. In particular, the point O_1 lies on the pair S_3, whilst O_3 does not lie on the pair S_1. The points of intersection of the two pairs of straight lines $S_1 = 0$, $S_3 = 0$, and, consequently, those of the two conics are therefore disposed in the following manner: two of these points are coincident with O_1, situated on that straight line D of the straight lines of the couple S_3, which passes through this point; the other two points A and B are at the intersection of the couple S_1 with the second straight line D' of the couple S_3. The two conics S and S' are tangent in O_1 to the straight line D, and intersect in the two points A and B situated on D'.

Since the couple $S_1 = 0$ is composed of two distinct straight lines, their point of intersection O_1, that is, the point of contact of two conics, is real; the pair S_3 containing the tangent to these conics at the point O_1 is real; the points A and B will be real or imaginary according as the couple S_1 is real or imaginary.

If the pair S_1 be a double straight line, the points of intersection of the conics are coincident, two and two, with the points where the double straight line intersects the two straight lines D and D' of the pair S_3 corresponding to the simple root. The two conics have a double contact: the double straight line is the chord of contact, and the two straight lines D and D' are the tangents at the points of contact; these points of contact are real or imaginary according as the pair S_3 is real or imaginary.

3° The equation in λ has a triple root λ_1. If the pair which corresponds to this root consist of two distinct straight lines, their point of intersection O lies on the two conics, and one of the straight lines of this pair is tangent to the two conics at O. The two conics intersect therefore in three points coincident with O and in one other point A; these points O and A are necessarily real. In this case, it is said that these two conics osculate each other in O.

If the couple which corresponds to the triple root λ be a double straight line, this straight line ought to pass through the points common to the conics and be tangent to the two conics; it will be tangent to the two conics at the same point O. These two curves intersect therefore in four points coincident with O; they are said to have a contact of the third order or to be *sub-osculatory.*

If the pair corresponding to the triple root λ_1 be indeterminate, the two conics are coincident.

REMARK. — Determine the conditions for which the equation in λ is indeterminate; it would be necessary for this that all conics represented by the equation

$$S + \lambda S' = 0$$

decompose into systems of straight lines. One will have, therefore.

$$\Delta = \Theta = \Theta' = \Delta' = 0.$$

The conditions $\Delta = 0$, $\Delta' = 0$ show that the two conics become systems of straight lines. The condition $\Theta = 0$ shows that the point of intersection O of the straight lines represented by $S = 0$ is on $S' = 0$, and $\Theta' = 0$ shows similarly that the point of intersection O' of the straight lines $S' = 0$ is on $S = 0$. If the points O and O' be distinct, it follows from the preceding that the straight line OO' ought to belong to the two conics; if O coincide with O', the two conics consist of pairs of straight lines intersecting in the same point. Conversely, if one of these conditions be fulfilled, the equation in λ is indeterminate. One has, in fact, $\Delta = \Theta = \Theta' = \Delta' = 0$. It can be verified that, in these two cases, the equation

$$S + \lambda S' = 0$$

represents straight lines whatever λ may be. In fact, if the conics S and S' be systems of straight lines having a common straight line, one has identically

$$S = PQ, \; S' = PR,$$

P, Q, R being linear functions in x, y, z, and the equation $S + \lambda S' = 0$ becomes

$$P(Q + \lambda R) = 0,$$

which represents two straight lines.

If S and S' be systems of straight lines intersecting in the same point, one has

$$S = PQ, \; S' = (aP + bQ)(a'P + b'Q);$$

$S + \lambda S'$ will then be a homogeneous polynomial of the second degree in P and Q and is consequently, whatever λ may be, resolvable into two factors of the first degree:

$$(\alpha P + \beta Q)(\alpha' P + \beta' Q).$$

EXAMPLE. — Consider a conic which, referred to rectangular axes, has the equation

$$S = Ax^2 + 2\,Bxy + Cy^2 + 2\,Ey = 0;$$

that is, a conic tangent to the x-axis at the origin; find the radius of the circle osculating this conic at the origin.

The osculating circle, being tangent to the x-axis, will have the equation

$$S' = x^2 + y^2 + 2\,Ry = 0.$$

The equation $S + \lambda S' = 0$ will therefore be

$$(A + \lambda)\,x^2 + 2\,Bxy + (C + \lambda)\,y^2 + 2\,(E + \lambda R)\,y = 0.$$

On equating the discriminant of the polynomial to zero, the following equation in λ is obtained:

$$(A + \lambda)\,(E + \lambda R)^2 = 0,$$

which has, whatever R may be, the double root $\lambda_1 = -\dfrac{E}{R}$, and the simple root $\lambda_3 = -A$. In order that the two conics osculate each other, it is necessary and sufficient that the equation λ has a triple root; that is, that $\lambda_1 = \lambda_3$, or

$$\frac{E}{R} = A, \; R = \frac{E}{A}.$$

The ordinate of the center of the osculating circle is therefore $-\dfrac{E}{A}$, and the radius of the circle is the absolute value of $\dfrac{E}{A}$. This radius is called the radius of curvature of the conic at the origin.

If we suppose

$$R = \frac{E}{A},$$

the circle is the osculating circle; in order that it be sub-osculatory, or have a contact of the third order, it is necessary and sufficient that the pair of secants corresponding to the triple root

$$\lambda = -A = -\frac{E}{R}$$

be a double straight line. This couple is

$$y^2(C - A) + 2\,Bxy = 0;$$

in order that it be a double straight line, it is necessary and sufficient that $B = 0$. Whence the equation of the conic becomes

$$Ax^2 + Cy^2 + 2\,Ey = 0,$$

and the origin is at a vertex. It is therefore only at the vertices of a conic that the osculating circle becomes sub-osculatory.

THEOREM. — *The roots of the equation in* λ *remain the same when the two conics* S *and* S' *are referred to other co-ordinate axes.*

In fact, suppose that the two conics $S = 0$, $S' = 0$ be referred to new co-ordinate axes O_1x_1, O_1y_1. The formulas for making this change of co-ordinates are

$$(6) \qquad \begin{aligned} x &= ax_1 + by_1 + cz_1, \\ y &= a'x_1 + b'y_1 + c'z_1, \\ z &= z_1, \end{aligned}$$

and, by the substitution of these values in the equations of the two conics, S and S' will become

$$S = A_1x_1^2 + 2\,B_1x_1y_1 + C_1y_1^2 + 2\,D_1x_1z_1 + 2\,E_1y_1z_1 + F_1z_1^2 = 0,$$

$$S' = A'_1x_1^2 + \cdots + F'_1z_1^2 = 0.$$

The equation $$S + \lambda S' = 0$$
will become therefore

$$(A_1 + \lambda A'_1)x_1^2 + 2(B_1 + \lambda B'_1)x_1y_1 + \cdots = 0.$$

In order that the conic $S + \lambda S' = 0$ be resolved into two straight lines, it is necessary and sufficient that the discriminant of this polynomial in x_1, y_1, z_1 be zero, which gives to determine λ a new equation of the third degree

$$\Delta_1 + \Theta_1\lambda + \Theta'_1\lambda^2 + \Delta'_1\lambda^3 = 0. \tag{7}$$

This equation has the same roots as equation (1). For it follows that both of the equations (1) and (7) give the values of λ for which the equation $S + \lambda S' = 0$ represents two straight lines. If equation (1) have three distinct roots, equation (7) will have the same roots, and consequently will

$$\frac{\Delta_1}{\Delta} = \frac{\Theta_1}{\Theta} = \frac{\Theta'_1}{\Theta'} = \frac{\Delta'_1}{\Delta'};$$

these relations will be identical if the coefficients A_1, A'_1, B_1, B'_1, $\cdots F_1$, F'_1 be replaced by their values as functions of A, A', B, B', $\cdots F$, F'; they will, therefore, still exist when *one of the equations* (1) *or* (7) *have multiple roots. Therefore, in every case these two equations have the same roots.*

EXERCISES.

1. Let $M_1(x_1, y_1)$ be a point taken on a conic whose equation is $f(x, y) = 0$. Demonstrate:

1° That the general equation of conics which osculate the conic $f = 0$ at the point M_1 is

$$f(x, y) + [\lambda(x - x_1) + \mu(y - y_1)](xf'_{x_1} + yf'_{y_1} + f'_{z_1}) = 0,$$

λ and μ being two variable parameters.

2° That the general equation of conics which are subosculatory to the conic $f = 0$ at the same point is

$$f(x, y) + \lambda[xf'_{x_1} + yf'_{y_1} + f'_{z_1}]^2 = 0,$$

λ being a variable parameter.

2. The equilateral hyperbolas which osculate a given conic $f=0$ at a point M_1 pass through a fixed point P. Find:

1° The locus of the center of these equilateral hyperbolas.

2° The locus of the point P when the point M_1 describes the conic $f=0$.

3. Place two equal parabolas whose axes are perpendicular, so that they will osculate each other.

4. Being given a parabola and a circle passing through its focus, find where the center of the circle should be in order that it have four real points of intersection with the parabola, or two real and two imaginary points, or four imaginary points. Study the form and the properties of the curve which separates these different regions (École Polytechnique, 1865).

5. Let

$$\Sigma = \mathsf{a}u^2 + 2\,\mathsf{b}uv + \mathsf{c}v^2 + 2\,\mathsf{d}uw + 2\,\mathsf{e}vw + \mathsf{f}w^2 = 0,$$

$$\Sigma' = \mathsf{a}'u^2 + 2\,\mathsf{b}'uv + \mathsf{c}'v^2 + 2\,\mathsf{d}'uw + 2\,\mathsf{e}'vw + \mathsf{f}'w^2 = 0,$$

be the tangential equations of two conics S and S'. It is required:

1° To form the equation of the third degree which gives the values of μ in order that the equation

$$\Sigma + \mu\Sigma' = 0$$

represent two points.

2° Show in what way the roots of this equation in μ are connected with the roots of the equation in λ with respect to the two conics S and S'.

Relations connecting the Roots of the Equation in λ.

Let $S = Ax^2 + 2\,Bxy + Cy^2 + 2\,Dxz + 2\,Eyz + Fz^2 = 0,$

$$S' = A'x^2 + 2\,B'xy + C'y^2 + 2\,D'xz + 2\,E'yz + F'z^2 = 0,$$

be the equations of two conics which have been made homogeneous by replacing x and y by $\frac{x}{z}$, $\frac{y}{z}$, and multiplying by z^2. The values of λ in order that the equation

$$S + \lambda S' = 0$$

represent two straight lines, are given by the equation of the third degree:

$$\Delta + \Theta\lambda + \Theta'\lambda^2 + \Delta'\lambda^3 = 0. \tag{1}$$

We have seen above that the roots of this equation remain unchanged when the conics S and S' are referred to other axes. Suppose, more generally, that α, β, γ be three linear functions in x, y, z, namely,

$$\begin{aligned} \alpha &= ax + by + cz, \\ \beta &= a'x + b'y + c'z, \\ \gamma &= a''x + b''y + c''z, \end{aligned} \tag{2}$$

which, equated to zero, represent three *non-concurrent* straight lines; one can find from these equations x, y, z as linear homogeneous functions of α, β, γ,

$$\begin{aligned} x &= l\alpha + m\beta + n\gamma, \\ y &= l'\alpha + m'\beta + n'\gamma, \\ z &= l''\alpha + m''\beta + n''\gamma; \end{aligned} \tag{3}$$

on substituting these values in the equations of the two conics, their equations will take the form

$$S = A_1\alpha^2 + 2B_1\alpha\beta + C_1\beta^2 + 2D_1\alpha\beta + 2E_1\beta\gamma + F_1\gamma^2 = 0,$$

$$S' = A_1'\alpha^2 + 2B_1'\alpha\beta + C_1'\beta^2 + 2D_1'\alpha\beta + 2E_1'\beta\gamma + F_1'\gamma^2 = 0,$$

and the equation $S + \lambda S' = 0$ will become

$$(A_1 + \lambda A_1')\alpha^2 + 2(B_1 + \lambda B_1')\alpha\beta + \cdots = 0.$$

In order that the conic $S + \lambda S' = 0$ become two straight lines, it is necessary and sufficient that this new homogeneous polynomial in α, β, γ decompose into two factors of the first degree, that is, that its discriminant

$$(A_1 + \lambda A_1')(C_1 + \lambda C_1')(F_1 + \lambda F_1') + \cdots$$

be zero; one has therefore, in order to determine λ, a new equation of the third degree:

$$\Delta_1 + \Theta_1\lambda + \Theta_1'\lambda^2 + \Delta_1'\lambda^3 = 0. \tag{4}$$

This equation has, moreover, the same roots as equation (1). Whence it follows from this result that each of equations (1) and (4) furnishes the values of λ, for which the equation $S + \lambda S' = 0$ represents two straight lines. If the first equation have three distinct roots, the second will have the same roots. Therefore,

$$\frac{\Delta_1}{\Delta} = \frac{\Theta_1}{\Theta} = \frac{\Theta'_1}{\Theta'} = \frac{\Delta'_1}{\Delta'};$$

these relations will be identities if one replace the coefficients A_1, A'_1, B_1, B_1', $\cdots$ by their values as functions of A, A', B, B', $\cdots$ etc.; they exist moreover when equation (1) has multiple roots. Therefore, in every case, equations (1) and (4) have the *same roots.*

We see finally how the roots of the equation in λ vary when all the coefficients of the equation $S = 0$ are multiplied by a constant factor K, and all of those of $S' = 0$ by a constant factor K'. Then the equations of the two conics become

$$KS = 0, \quad K'S' = 0,$$

and the general equation of the conic passing through their points of intersection becomes

$$KS + \lambda' K'S' = 0,$$

an equation which is identical with $S + \lambda S' = 0$ if one put $\lambda' = \frac{K}{K'}\lambda$. Therefore, in order to obtain the values of λ', for which the equation $KS + \lambda' K'S' = 0$ represents two straight lines, it will be sufficient to take the three roots of equation (1) in λ and multiply them by the factor $\frac{K}{K'}$.

Summing up briefly, if the equations of two conics be transformed by substituting for x, y, z expressions such as (3) (that is, on referring them to any triangle of reference), and if the equations of the two conics be multiplied by constant factors, *the roots of the equation in λ remain the same or are multiplied by a constant factor.*

Whence it follows that:

A homogeneous relation between the roots of the equation in λ expresses a property of two conics independent of the choice of the co-ordinate axes, or, more generally, of the choice of linear functions α, β, γ, that is, of the triangle of reference. Because a similar relation exists when the axes or the functions α, β, γ have been chosen in a particular way, it will exist for every other system of axes or linear functions α, β, γ.

For example, if λ_1, λ_2, λ_3 be called the three roots of the equation in λ, the relation $\lambda_1 - \lambda_2 = 0$ or, more symmetrically,

$$(\lambda_1 - \lambda_2)^2 (\lambda_2 - \lambda_3)^2 (\lambda_3 - \lambda_1)^2 = 0,$$

which is homogeneous and of the second degree in λ_1, λ_2, λ_3, expresses the condition that the equation in λ has a double root; that is, that *the two conics are tangent.*

We proceed to determine the meaning of certain other simple relations.

I. *The relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$ or $\Theta' = 0$ is the necessary and sufficient condition in order that there exist a triangle inscribed in the conic S and conjugate with respect to S'. In case one such triangle exists, there exists an infinitude of such.*

In order to prove this, suppose that there exists a triangle which is inscribed in S and conjugate to S'. Then, on calling

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0$$

the equations of the sides of this triangle, the equation of the conic S will be

$$S = 2\,B\alpha\beta + 2\,D\alpha\gamma + 2\,E\beta\gamma = 0,$$

and that of S'

$$S' = A'\alpha^2 + C'\beta^2 + F'\gamma^2.$$

It is easily seen, on equating to zero the discriminant of the polynomial $S + \lambda S'$, that one obtains the equation in λ

$$2\,BDE - \lambda (A'E^2 + C'D^2 + F'B^2) + \lambda^3 A'C'F' = 0,$$

in which the coefficient Θ' of λ^2 is zero. Therefore, the condition $\lambda_1 + \lambda_2 + \lambda_3 = 0$ is necessary. Conversely, if this condition be fulfilled in case of two conics S and S', of which the second S' is not resolvable into straight lines, there exists an infinitude of triangles which are inscribed in S and conjugate to S'. In fact, take a point P on the conic S and construct the polar of this point with respect to S' (Fig. a); this polar intersects

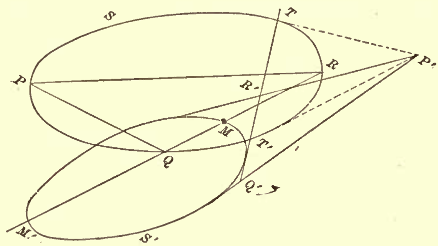

Fig. a.

the conic S' in two points M and M' and the conic S at least in one point Q; let R be the harmonic conjugate of the point Q with respect to the two points M and M'; we shall show that this point R belongs also to the conic S.

The triangle PQR being conjugate with respect to the conic S', the equation of this conic will be

$$S' = A'\alpha^2 + C'\beta^2 + F'\gamma^2 = 0,$$

on calling $\alpha = 0$, $\beta = 0$, $\gamma = 0$ the equations of the sides QR, RP, PQ of the triangle. The conic S passes through the point P the intersection of the sides $\beta = 0$, $\gamma = 0$, and the point Q the intersection of the sides $\gamma = 0$, $\alpha = 0$; its equation will therefore have the form

$$S = F\gamma^2 + 2\,B\alpha\beta + 2\,D\alpha\gamma + 2\,E\beta\gamma = 0.$$

On forming the discriminant of the polynomial $S + \lambda S'$, the coefficient of λ^2 will be

$$\Theta' = A'C'F.$$

Since this coefficient must be *zero*, and that neither A' nor C' can be zero, because the conic S' would reduce to two straight lines, therefore will $F = 0$, and the conic S is circumscribed about the triangle PQR conjugate to S'; what we wish to establish.

II. *The same relation* $\lambda_1 + \lambda_2 + \lambda_3 = 0$ *or* $\Theta' = 0$ *is the necessary and sufficient condition in order that there exist a triangle circumscribed about the conic* S' *and conjugate with respect to* S. *When one such triangle exists, there exists an infinitude.*

In fact, if there exist one such, and if α, β, γ be called the first members of the equations of the sides of this triangle, the equation of S will be

$$S = A\alpha^2 + C\beta^2 + F\gamma^2,$$

and that of S' (§ 282. 2),

$$S' = p^2\alpha^2 + q^2\beta^2 + r^2\gamma^2 - 2\,qr\beta\gamma - 2\,rp\gamma\alpha - 2\,pq\alpha\beta = 0.$$

On forming the coefficient Θ' of λ^2 in the discriminant of $S + \lambda S'$, it will be readily seen that this coefficient is zero; the condition $\lambda_1 + \lambda_2 + \lambda_3 = 0$ is, therefore, necessary. Conversely, suppose this condition fulfilled; select any tangent TT', $\alpha = 0$, to the conic S' and its pole P' with respect to S; from this pole two tangents $P'T$ and $P'T'$ can be drawn to the conic S and one tangent at least $P'Q'$ to the conic S'. Let $\gamma = 0$ be the equation of this last straight line $P'Q'$, and $\beta = 0$ the equation of its conjugate $P'R'$ with respect to the system of tangents $P'T$ and $P'T'$. Hence the triangle formed by the three straight lines $\alpha = 0$, $\beta = 0$, $\gamma = 0$ will be conjugate to S, and two of its sides $\alpha = 0$, $\gamma = 0$ will be tangent to S'; we shall show that the third side $\beta = 0$ is also tangent to S'. The equations of the two conics can be written:

$$S = A\alpha^2 + C\beta^2 + F\gamma^2 = 0,$$

$$S' = p^2\alpha^2 + q^2\beta^2 + r^2\gamma^2 - 2\,qr\beta\gamma + 2\,D'\alpha\gamma - 2\,pq\alpha\beta = 0;$$

since the first member of the equation of S' should reduce to a perfect square for $\alpha = 0$ and for $\gamma = 0$. The coefficient Θ' of λ^2 in the discriminant of $S + \lambda S'$ is $C(p^2r^2 - D'^2)$, and this coefficient should be *zero*. C cannot be zero, because if C were zero the conic S would reduce to two straight lines; one cannot have $D' = pr$, because for this value of D' one would have

$$S' = (p\alpha - q\beta + r\gamma)^2,$$

and the conic S' would be a double line. Therefore, $D' = -pr$, and the conic S' is inscribed in the triangle $P'Q'R'$ conjugate to S.

Remark. — In case the conic S is circumscribed about a triangle conjugate to S', it is said, for brevity, that *S is harmonically circumscribed* about S'; then, according to what precedes, the conic S' is also inscribed in a triangle conjugate to S, and it is said that *S' is harmonically inscribed* in S.

Example. — A triangle being given, there exists always a real or imaginary circle with respect to which the triangle is conjugate. For, on calling $\alpha = 0$, $\beta = 0$, $\gamma = 0$ the equations of the sides of the triangle, the general equation of the conics conjugate to the triangle is

$$A\alpha^2 + C\beta^2 + F\gamma^2 = 0.$$

The condition that this equation represents a circle gives two equations of the first degree, which determine the ratios of the coefficients A, C, F to any one of them. The circle thus found is called the circle conjugate to the triangle; its center is the point of intersection of the altitudes of the triangle, because if from a point a perpendicular be dropped on the polar with respect to a circle, the perpendicular passes through the center of the circle.

Having proven this, we proceed to demonstrate the following theorem:

In case a conic is inscribed in a triangle, the power of the center of the conic with respect to the circle conjugate to the triangle is equal to the algebraic sum of the squares of the axes of the conic.

Refer the conic to its axes, and let

$$S' = A'x^2 + C'y^2 - A'C' = 0$$

be the equation of the conic, A' and C' being the squares of the lengths of the axes. Let, moreover,

$$S = x^2 + y^2 + 2\,Dx + 2\,Ey + F = 0$$

be the equation of the circle conjugate to the triangle. Then the conic S' will be inscribed in a triangle conjugate to S; therefore, if the discriminant of the polynomial $S + \lambda S'$ be formed, the coefficient Θ' of λ^2 ought to be zero. This coefficient is $A'C'(F - A' - C')$; since A' and C' are different from zero, we have

$$F = A' + C',$$

which proves the theorem.

III. *The necessary and sufficient condition that there exist a triangle inscribed in a conic S and circumscribed about a conic S' is*

$$\Theta'^2 - 4\,\Theta\Delta' = 0,$$

or $$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\,\lambda_1\lambda_2 - 2\,\lambda_2\lambda_3 - 2\,\lambda_3\lambda_1 = 0,$$

or $$\sqrt{\lambda_1} \pm \sqrt{\lambda_2} \pm \sqrt{\lambda_3} = 0;$$

if there exist one triangle, there exists an infinitude.

Let $\alpha = 0$, $\beta = 0$, $\gamma = 0$ be the equations of the sides of a triangle inscribed in S and circumscribed about S'. The equations of the two conics can be written

$$S = 2\,B\alpha\beta + 2\,D\alpha\gamma + 2\,E\beta\gamma = 0,$$

$$S' = p^2\alpha^2 + q^2\beta^2 + r^2\gamma^2 - 2\,qr\beta\gamma - 2\,rp\gamma\alpha - 2\,pq\alpha\beta = 0.$$

If the discriminant of $S + \lambda S'$ be formed, it follows that the coefficients of λ^3, λ^2, λ are

$$\Delta' = -4\,p^2q^2r^2,$$

$$\Theta' = 4\,pqr\,(Ep + Dq + Br),$$

$$\Theta = -(Ep + Dq + Br)^2.$$

It follows, therefore, that $\Theta'^2 - 4\,\Theta\Delta' = 0$, a relation which can be written, owing to the relation between the coefficients and the roots,

$$(\lambda_1 + \lambda_2 + \lambda_3)^2 - 4\,(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = 0,$$

or (5) $$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\,\lambda_1\lambda_2 - 2\,\lambda_2\lambda_3 - 2\,\lambda_3\lambda_1 = 0.$$

This last relation being homogeneous with respect to the roots of the equation in λ, will hold, as has been seen, in whatever form the equations of the two conics may be written. It can be easily verified that this relation is equivalent to one of the following:

$$\sqrt{\lambda_1} \pm \sqrt{\lambda_2} \pm \sqrt{\lambda_3} = 0.$$

Conversely, if relation (5) hold, there exists an infinitude of triangles inscribed in S and circumscribed about S'. This may be proven by following the method which has been employed in the cases of Propositions I. and II.

Example I. — Consider two ellipses which have the same center and coincident axes, whose equations are

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad S' = -\frac{x^2}{a'^2} - \frac{y^2}{b'^2} + 1 = 0.$$

The values of λ for which the equation $S + \lambda S' = 0$ represents two straight lines are

$$\lambda_1 = 1, \ \lambda_2 = \frac{a'^2}{a^2}, \ \lambda_3 = \frac{b'^2}{b^2};$$

therefore the necessary and sufficient condition in order that there exist a triangle inscribed in S and circumscribed about S' is

$$1 \pm \frac{a'}{a} \pm \frac{b'}{b} = 0.$$

Example II. — Consider two circles whose equations are

$$S = x^2 + y^2 - R^2 = 0, \ S' = (x-d)^2 + y^2 - r^2 = 0;$$

the coefficients of the equation in λ are

$$\Delta' = r^2, \ \Theta' = R^2 + 2r^2 - d^2, \ \Theta = 2R^2 + r^2 - d^2;$$

the necessary and sufficient condition in order that there exist a triangle inscribed in S and circumscribed about S' is, therefore,

$$(2r^2 + R^2 - d^2)^2 - 4r^2(2R^2 + r^2 - d^2) = 0,$$

or, simplifying,

$$(d^2 - R^2)^2 - 4r^2R^2 = 0,$$

$$d^2 - R^2 = \pm 2rR,$$

a well-known relation connecting the radii of the circles and the distance between the centers of the two circles, the one circumscribed, the other inscribed or escribed to the triangle.

EXERCISES.

1. Show that the circle conjugate to a triangle is real when the triangle has an obtuse angle, and imaginary in case the angles of the triangle are acute.

2. Prove that the locus of the centers of conics inscribed in a given triangle, so that the sum of the squares of their axes is constant, is a circle whose center is the point of intersection of the altitudes.

3. In case a triangle is circumscribed about a parabola, the point of intersection of the altitudes is on the directrix.

4. When a triangle is inscribed in an equilateral hyperbola, the point of intersection of the altitudes lies on the curve.

(In these exercises, the point of intersection of the altitudes is regarded as the center of the circle conjugate to the triangle.)

5. A parabola, $$y^2 = 2px,$$

and a circle, $$x^2 + y^2 + 2ax + 2by + c = 0,$$

are given; determine the necessary and sufficient condition in order that there exist a triangle inscribed in the circle and circumscribed about the parabola. When will the circle pass through the focus of the parabola?

6. The equations of two conics being written in the form

$$S = A\alpha^2 + B\beta^2 + C\gamma^2 = 0,$$

$$S' = A'\alpha^2 + B'\beta^2 + C'\gamma^2 = 0,$$

what relation must exist among the coefficients:

1° In order that S be harmonically circumscribed about S'?

2° In order that S be circumscribed about a triangle circumscribed about S'?

7. Two conics are tangent to each other at a point M; demonstrate that the necessary and sufficient condition, in

order that there exist a triangle inscribed in S and circumscribed about S', is that the radius of curvature of S' at the point M be equal to four times that of S at the same point.

Particular Case. — If a circle which passes through the focus of a parabola be tangent to the parabola at a point M, the radius of curvature of the parabola at M is four times that of the circle.

8. Consider a triangle and the circumscribed circle; there exists a conic tangent to three sides of the triangle and tangent to the circumscribed circle at a given point M.

1° Find the center of the circle of curvature of this conic at M.

2° Find the locus of this center when the point M describes the circumscribed circle.

9. What relation should exist between the roots of the equation in λ in order that there exist a quadrilateral inscribed in the conic S, and circumscribed about the conic S'?

If there be one such quadrilateral, there will be an infinitude.

As a particular case, we apply the relation found to the case where the conics are two circles. (See § 109.)

10. Consider an ellipse E, of which the major axis and the focal distance are respectively $2a$ and $2c$. Describe a circumference of a circle C with the radius $\sqrt{2(a^2+c^2)}$ about one of the foci F of the ellipse as center. A tangent P_1P_2 is drawn from any point P_1 of the circumference C to the ellipse; from the point P_2, where it intersects the circumference C again, a second tangent P_2P_3 is drawn to the ellipse; finally, from the point P_3, where this second tangent intersects the circumference C, a third tangent P_3P_4 is drawn to the ellipse which intersects the circumference in the point P_4. It is required to prove that the second of the tangents drawn from the point P_4 to the ellipse passes through the initial point P_1. (École Normale, 1885.)

The Application of the Properties of Homogeneous Polynomials to the Theory of Curves of the Second Degree.

Let $f(x, y, z) = Ax^2 + 2\,Bxy + Cy^2 + 2\,Dxz + 2\,Eyz + Fz^2$

be a homogeneous polynomial of the second degree in x, y, z. It is known that if the discriminant Δ be different from zero, the polynomial f is resolvable into a sum of three squares linearly independent; if this discriminant be zero without all the minors being zero, the polynomial can be decomposed into a sum of two squares linearly independent; finally, if all the minors of the discriminant are zero, the polynomial is a perfect square; the converse statements are true.

1° Suppose that the discriminant

$$\Delta = ACF - AE^2 - CD^2 - FB^2 + 2\,BDE$$

is zero, and similarly all of its minors a, b, c, d, e, f. Then the polynomial f is the square of a linear function

$$f(x, y, z) = a\,(lx + my + nz)^2,$$

a being a constant which is positive or negative; and on representing the function $lx + my + nz$ by α, it follows that one has identically $f'_x = 2\,al\alpha$, $f'_y = 2\,am\alpha$, $f'_z = 2\,an\alpha$. In this case, if x, y, and z be regarded as the homogeneous co-ordinates of a point, the equation

$$f(x, y, z) = 0$$

represents two straight lines coincident with the straight line $lx + my + nz = 0$, and the three equations

$$f'_x = 0,\ f'_y = 0,\ f'_z = 0$$

represent this same straight line or are identities, since, for example, the equation $f'_z = 0$ when $n = 0$.

2° Suppose that the discriminant Δ is zero and that its minors are not all zero. Then the polynomial f can be resolved into a sum of two squares linearly independent,

(1) $$f(x, y, z) = a\alpha^2 + b\beta^2,$$

where a and b are constants and α and β are linear homogeneous functions of x, y, z:

$$\alpha = lx + my + nz,\ \beta = l'x + m'y + n'z.$$

To say that these functions are linearly independent, is to say that there do not exist two constant factors k and k', both of which are not zero, and such that $k\alpha + k'\beta$ is identically zero. The polynomial f may be written in an infinite number of different ways in form (1); we shall show how all of them can be obtained. The identity (1) can be written

$$f(x, y, z) = (\alpha\sqrt{a} + \beta\sqrt{-b})(\alpha\sqrt{a} - \beta\sqrt{-b}),$$

or
$$f(x, y, z) = PQ,$$

where P and Q designate two homogeneous linearly independent functions in x, y, z. These two linear functions are easily found; in fact, if the three coefficients A, C, F are not all zero, the polynomial f will be a trinomial of the second degree in x, y, or z, and this trinomial can be resolved into factors of the first degree, which will be P and Q; if A, C, and F be zero, the discriminant reduces to $2\,BDE$, and since it is zero, one at least of the three coefficients B, D, E is also zero, and then one of the three variables x, y, or z is a factor, and the decomposition is immediate. The polynomial being thus put under the form PQ, all possible decompositions will take the form of a sum of two squares, on noticing that one has identically

$$(2)\qquad f(x, y, z) = PQ = \frac{1}{4\,\lambda\mu}[(\lambda P + \mu Q)^2 - (\lambda P - \mu Q)^2],$$

where λ and μ designate constant coefficients different from zero. On allowing λ and μ to vary, there will be an infinitude of decompositions of f into two squares: one has all of them, because if one imagines any decomposition

$$f = a_1\alpha_1^2 + b_1\beta_1^2 = (\alpha_1\sqrt{a_1} + \beta_1\sqrt{-b_1})(\alpha_1\sqrt{a_1} - \beta_1\sqrt{-b_1}),$$

in which α_1, β_1 are linear functions, a_1 and b_1 constants, one would have identically

$$(\alpha_1\sqrt{a_1} + \beta_1\sqrt{-b_1})(\alpha_1\sqrt{a_1} - \beta_1\sqrt{-b_1}) = PQ,$$

whence, on designating a constant by k,

$$\alpha_1\sqrt{a_1}+\beta_1\sqrt{-b_1}=kP,$$

$$\alpha_1\sqrt{a_1}-\beta_1\sqrt{-b_1}=\frac{Q}{k},$$

$$\alpha_1\sqrt{a_1}=\frac{1}{2}\left(kP+\frac{Q}{k}\right),\quad \beta_1\sqrt{-b_1}=\frac{1}{2}\left(kP-\frac{Q}{k}\right),$$

and finally

$$f=a_1\alpha_1^2+b_1\beta_1^2=\frac{1}{4}\left[\left(kP+\frac{Q}{k}\right)^2-\left(kP-\frac{Q}{k}\right)^2\right];$$

an expression which becomes identical with (2) on supposing

$$\lambda=k,\ \mu=\frac{1}{k}.$$

The values of x, y, z,

$$x=x_1,\ y=y_1,\ z=z_1,$$

which reduce simultaneously to zero the linear functions P and Q, and, consequently, the functions α and β, which are equal to $\lambda P+\mu Q$ and $\lambda P-\mu Q$, reduce to zero the three partial derivatives f'_x, f'_y, f'_z.

If the coefficients of the polynomial f be real, the factors P and Q can be real or imaginary. In order to obtain, in the decomposition of (2), the squares in case of real coefficients, it will be necessary, if P and Q be real, to take λ and μ real; then one of the squares which appears in formula (2) has a negative coefficient, the other a positive; if P and Q be imaginary, one could put, since their product is real,

$$P=p+iq,\ Q=h(p-iq),$$

where h represents a real constant and p and q are real linear functions; one puts then $\lambda=h(\lambda'+i\mu')$, $\mu=\lambda'-i\mu'$, and it follows:

$$f=\frac{h^2}{\lambda'^2+\mu'^2}[(\lambda'p-\mu'q)^2+(\lambda'q+\mu'p)^2],$$

where the two squares have the same sign as that of h.

The geometric interpretation of these results is very simple. On considering x, y, z as the homogeneous co-ordinates of a point, it follows that the identity $f(x, y, z) = PQ$ shows that the equation $f = 0$ represents two distinct straight lines, real or imaginary. If z_1 be different from zero, the two straight lines intersect in the point whose Cartesian co-ordinates are $\frac{x_1}{z_1}, \frac{y_1}{z_1}$; the straight lines

$$\alpha = \lambda P + \mu Q = 0, \ \beta = \lambda P - \mu Q = 0$$

pass through this point and are harmonic conjugates with respect to the two straight lines $P = 0$ and $Q = 0$. If z_1 be zero without either of the functions P and Q reducing to the form nz, the two straight lines are parallel and have as common angular coefficient $\frac{y_1}{x_1}$. The straight lines $\alpha = 0$, $\beta = 0$ are parallel to the same direction and are harmonic conjugates with respect to the straight lines $P = 0$, $Q = 0$; one has identically, in the present case,

$$P = mQ + nz,$$

m and n being constants; if therefore one put $m\lambda = -\mu$, the straight line $\alpha = 0$ becomes the straight line at infinity and its conjugate $\beta = 0$ becomes the straight line equi-distant from the parallel straight lines $P = 0$, $Q = 0$:

$$2mQ + nz = 0;$$

since $f(x, y, z) = PQ = (mQ + nz)Q$, the two equations

$$f'_x = Q'_x(2mQ + nz) = 0,$$

$$f'_y = Q'_y(2mQ + nz) = 0$$

represent this same straight line, provided that neither of them be identically zero, which would happen, for example, if the quantity Q'_y were zero.

Finally, if one of the two functions P or Q, Q for example, were of the form nz, the straight line $Q = 0$ would be removed to infinity, the equation $f = 0$ would represent a single straight line $P = 0$ at a finite distance, and the two straight lines $\alpha = 0$ and $\beta = 0$ would be parallel to this straight line and situated at equal distances on either side of it.

3° Suppose, finally, that the discriminant Δ be different from zero. In this case, the polynomial $f(x, y, z)$ can be decomposed into three linear independent squares,

$$f(x, y, z) = a\alpha^2 + b\beta^2 + c\gamma^2, \tag{3}$$

where a, b, c represent *constants different from zero* and α, β, γ linear functions:

$$\alpha = lx + my + nz,\ \beta = l'x + m'y + n'z,\ \gamma = ux + vy + wz,$$

such that the determinant

$$\begin{vmatrix} l & m & n \\ l' & m' & m' \\ u & v & w \end{vmatrix}$$

be *different from zero.*

In order to obtain all the decompositions of f into three squares of the form (3), we notice *that one of the three linear functions α, β, γ can be chosen arbitrarily, the function γ for example, with the condition that the coefficients u, v, w of this function* DO NOT REDUCE *the following polynomial to zero,*

$$\phi(u, v, w) = \mathsf{a}u^2 + 2\,\mathsf{b}uv + \mathsf{c}v^2 + 2\,\mathsf{d}uw + 2\,\mathsf{e}vw + \mathsf{f}w^2. \tag{4}$$

In fact, u, v, w being chosen arbitrarily, let us consider the difference

$$F(x, y, z) = f(x, y, z) - \lambda\gamma^2,$$

where λ is a constant; this difference F is a homogeneous polynomial of the second degree in x, y, z. Determine λ so that the discriminant of $F(x, y, z)$ be zero; we will have the equation

$$\begin{vmatrix} A - \lambda u^2 & B - \lambda uv & D - \lambda uw \\ B - \lambda uv & C - \lambda v^2 & E - \lambda vw \\ D - \lambda uw & E - \lambda vw & F - \lambda w^2 \end{vmatrix} = 0, \tag{5}$$

whose development with respect to powers of λ is obtained by putting in the equation in λ in § 286

$$A' = -u^2,\ B' = -uv,\ C' = -v^2,\ D' = -uw,\ E' = -vw,$$

$$F' = -w^2,$$

and consequently $\Delta' = 0$, $\Theta' = 0$, $\Theta = -\phi(u, v, w)$, ϕ representing polynomial (4). Equation (5) reduces, therefore, to the equation of the *first degree*

$$\Delta - \lambda\phi(u, v, w) = 0,$$

which determines λ if $\phi(u, v, w)$ be not zero. Let c be the value of λ deduced from this equation:

$$c = \frac{\Delta}{\phi(u, v, w)}.$$

The discriminant of the function $F = f - c\gamma^2$ being zero, this function can be decomposed into a sum of two squares; it cannot be a perfect square $a\alpha^2$, because if it were such one would have

$$f(x, y, z) = c\gamma^2 + a\alpha^2;$$

the function f would be a sum of two squares and its discriminant Δ would be zero, which contradicts the hypothesis. The polynomial $F = f - c\gamma^2$ being decomposable into a sum of two squares, one can apply to it what has been said in the preceding paragraph and find all possible ways of putting it in the form $a\alpha^2 + b\beta^2$. To each of these decompositions of $F(x, y, z)$ into two squares will correspond one decomposition of f into three squares

$$f = a\alpha^2 + b\beta^2 + c\gamma^2,$$

γ having been chosen arbitrarily.

If F be decomposed into two factors

(6) $$F(x, y, z) = f - c\gamma^2 = PQ,$$

the values $x = x_1$, $y = y_1$, $z = z_1$, which reduce P and Q to zero, reduce α and β to zero, and reduce also the partial derivation F'_x, F'_y, F'_z to zero.

Since

$$\tfrac{1}{2} F'_x = \tfrac{1}{2} f'_x - cu\gamma, \quad \tfrac{1}{2} F'_y = \tfrac{1}{2} f'_y - cv\gamma, \quad \tfrac{1}{2} F'_z = \tfrac{1}{2} f'_z - cw\gamma,$$

one has, on replacing x, y, z by x_1, y_1, z_1, and representing the quantity $ux_1 + vy_1 + wz_1$ by γ_1,

(7) $$\tfrac{1}{2} f'_{x_1} = cu\gamma_1, \quad \tfrac{1}{2} f'_{y_1} = cv\gamma_1, \quad \tfrac{1}{2} f'_{z_1} = cw\gamma_1;$$

the constant γ_1 is not zero, because, if it were, $f'_{x_1}, f'_{y_1}, f'_{z_1}$ would all be zero, and the discriminant Δ would vanish. One has identically

(8) $$\tfrac{1}{2}(xf'_{x_1} + yf'_{y_1} + zf'_{z_1}) = c\gamma\gamma_1,$$

and, on putting, in this identity, $x = x_1$, $y = y_1$, $z = z_1$, and applying the theorem of homogeneous functions,

$$f(x_1, y_1, z_1) = c\gamma_1^2.$$

Whence it follows that

$$\frac{(xf'_{x_1} + yf'_{y_1} + zf'_{z_1})^2}{4f(x_1, y_1, z_1)} = c\gamma^2,$$

and, on replacing $c\gamma^2$ in the identity (6) by this expression, one gets, after removing the denominator $4f(x_1, y_1, z_1)$,

(9) $$4f(x, y, z)f(x_1, y_1, z_1) - (xf'_{x_1} + yf'_{y_1} + zf'_{z_1})^2 = 4PQf(x_1, y_1, z_1).$$

Finally, if in the relation which determines c,

$$c\phi(u, v, w) = \Delta,$$

u, v, w be replaced by their values deduced from relations (7),

$$u = \frac{1}{2}\frac{f'_{x_1}}{c\gamma_1}, \quad v = \frac{1}{2}\frac{f'_{y_1}}{c\gamma_1}, \quad w = \frac{1}{2}\frac{f'_{z_1}}{c\gamma_1},$$

it follows, since the polynomial ϕ is homogeneous,

$$\frac{1}{c\gamma_1^2}\phi(\tfrac{1}{2}f'_{x_1}, \tfrac{1}{2}f'_{y_1}, \tfrac{1}{2}f'_{z_1}) = \Delta,$$

and, on replacing $c\gamma_1^2$ by its value $f(x_1, y_1, z_1)$,

(10) $$\phi(\tfrac{1}{2}f'_{x_1}, \tfrac{1}{2}f'_{y_1}, \tfrac{1}{2}f'_{z_1}) = \Delta f(x_1, y_1, z_1);$$

which gives a remarkable identity.

Geometric Interpretation. — On considering x, y, z as the homogeneous co-ordinates of a point, it is plain that the equation $f(x, y, z)=0$ represents a conic not reducible to two straight lines, real or imaginary ellipse, hyperbola, parabola. The identity

$$f(x, y, z)=a\alpha^2+b\beta^2+c\gamma^2$$

represents the first member of the equation of this conic decomposed into a sum of three squares. If the three coefficients a, b, c have the same signs, the curve is an imaginary ellipse; if not, it is a real conic: it is known, moreover, that in all possible decompositions of f into three squares, the number of coefficients a, b, c which have *a definite sign* is *invariable.* One of the functions α, β, γ can be chosen arbitrarily, for example the function $\gamma=ux+vy+wz$, with the condition that one does not have $\phi(u, v, w)=0$, that is, with the condition that the straight line $\gamma=0$ is not tangent to the conic (§ 126). Suppose that this condition is fulfilled, the equation of the conic could be written in the form

$$f(x, y, z)=c\gamma^2+PQ=0,$$

which shows that $P=0$, $Q=0$ are the equations of the tangents at the points where the straight line $\gamma=0$ intersects the conic. We have called x_1, y_1, z_1 the values of x, y, z which reduce P and Q to zero, that is, the homogeneous co-ordinates of the point of intersection of the two straight lines $P=0$, $Q=0$; the straight line $\gamma=0$ is the chord of contact of the tangents emanating from this point, or the polar of this point; owing to the identity (8), the equation of this straight line can be written

$$xf'_{x_1}+yf'_{y_1}+zf'_{z_1}=0;$$

which is the well-known equation of the polar of the point with the co-ordinates x_1, y_1, z_1. Owing to the identity (9), the equation $PQ=0$, which represents the *ensemble* of the tangents drawn from the point (x_1, y_1, z_1) to the conic, can be written

$$4f(x, y, z)f(x_1, y_1, z_1)-(xf'_{x_1}+yf'_{y_1}+zf'_{z_1})^2=0.$$

Finally, identity (10) shows that the necessary and sufficient condition in order that $\phi(f'_{x_1}, f'_{y_1}, f'_{z_1})$ be zero is that $f(x_1, y_1, z_1)$

be zero and conversely, which means geometrically that the necessary and sufficient condition in order that the polar of the point x_1, y_1, z_1 be tangent to the curve is that this point be on the curve.

The straight lines $\alpha = 0$, $\beta = 0$ pass through the point of intersection of the tangents $P = 0$, $Q = 0$, and are harmonic conjugates with respect to these tangents. If z_1 be different from zero, the two straight lines $P = 0$, $Q = 0$ are concurrent in a point situated at a finite distance, whose Cartesian coordinates are $\frac{x_1}{z_1}, \frac{y_1}{z_1}$. If $z_1 = 0$, these two straight lines are parallel with angular coefficient $\frac{y_1}{x_1}$, or one of them is at infinity, which happens when one of the functions P or Q reduces to the form nz; in this case ($z_1 = 0$), the straight line $\gamma = 0$ has the equation

$$x_1 f'_x + y_1 f'_y = 0;$$

it coincides with the conjugate diameter of the direction of the two straight lines $P = 0$, $Q = 0$, or of that of the two which is at a finite distance; whence it is said that $\gamma = 0$ is the polar of the point at infinity $(x_1, y_1, 0)$.

Thus, if the equation be written in the form

$$f(x, y, z) = a\alpha^2 + b\beta^2 + c\gamma^2 = 0,$$

the straight line $\gamma = 0$ is the polar of the point of intersection of the other two straight lines $\alpha = 0$, $\beta = 0$; since the same is true in regard to the straight lines $\alpha = 0$, $\beta = 0$, it follows that the triangle formed by the three straight lines $\alpha = 0$, $\beta = 0$, $\gamma = 0$, is *a conjugate triangle with respect to the conic.*

Application to the reduction of the equation of the second degree.

1. Assume that $\mathbf{f}$ be different from zero; then take $\gamma = z$, that is, $u = 0$, $v = 0$, $w = 1$, and we obtain

$$c = \frac{\Delta}{\phi(0, 0, 1)} = \frac{\Delta}{\mathbf{f}}.$$

Identity (6) gives, in this case,

$$f(x, y, z) - \frac{\Delta}{\mathbf{f}} z^2 = PQ = a\alpha^2 + b\beta^2.$$

The co-ordinates of the point of intersection of the straight lines $P=0$, $Q=0$, satisfy the equations

$$f'_{x_1}=0, \quad f'_{y_1}=0,$$

which follows from relations (7), where one supposes $u=v=0$; this point is the center of the curve. The straight lines $P=0$, $Q=0$ are the asymptotes: their homogeneous equation is therefore

$$f(x, y, z)-\frac{\Delta}{\mathbf{f}}z^2=0,$$

or, in Cartesian co-ordinates,

$$f(X, Y, 1)-\frac{\Delta}{\mathbf{f}}=0.$$

These asymptotes, $P=0$, $Q=0$, can be real or imaginary. In the first case the curve is a hyperbola, the coefficients a and b have opposite signs; in the second case it is an ellipse, the coëfficients a and b have the same signs. The straight lines $\alpha=0$, $\beta=0$, whose equations have the form

$$\lambda P+\mu Q=0, \quad \lambda P-\mu Q=0,$$

are harmonic conjugates with respect to the asymptotes; they are two conjugate diameters; if the ratio $\frac{\lambda}{\mu}$ be so determined that these straight lines are perpendicular to each other, they coincide with the axes. On taking the straight lines $\alpha=0$, $\beta=0$ as axes of Cartesian co-ordinates, the equation of the conic takes the simplified form,

$$a'X'^2+b'Y'^2+\frac{\Delta}{\mathbf{f}}=0.$$

2. Suppose $\mathbf{f}=0$. Then the preceding method of reduction is no longer applicable, because $\phi(0, 0, 1)=0$. We give a second method of reduction which is applicable in all cases. It has been proven that if x_1, y_1, z_1 be the homogeneous co-ordinates of any point not situated on a conic,

$$4f(x, y, z)f(x_1, y_1, z_1)=(xf'_{x_1}+yf'_{y_1}+zf'_{z_1})^2+PQ,$$

where $P=0$, $Q=0$ are the equations of the tangents drawn from the point (x_1, y_1, z_1) to the curve. Take, in particular, $z_1 = 0$ with the condition $f(x_1, y_1, 0) \gtrless 0$, and notice that

$$xf'_{x_1} + yf'_{y_1} + zf'_{z_1} = x_1f'_x + y_1f'_y + z_1f'_z,$$

we have

$$4f(x, y, z)f(x_1, y_1, 0) = (x_1f'_x + y_1f'_y)^2 + PQ;$$

the straight lines $P = 0$, $Q = 0$ intersecting at the point at infinity $(x_1, y_1, 0)$, are parallel, or one of them is at infinity. One will have, for example,

$$Q = mP + nz,$$

where m is zero, if Q be at infinity. Then the equation $f = 0$ can be written

$$(x_1f'_x + y_1f'_y)^2 + mP^2 + nPz = 0.$$

On putting $z = 1$, one will obtain the equation of the curve in Cartesian co-ordinates; then one chooses the straight line

$$x_1f'_x + y_1f'_y = 0$$

for the new axis $O'X'$, and the straight line $P = 0$ for the axis $O'Y'$, and the equation will take the reduced form

$$Y'^2 + 2pX' + qX'^2 = 0;$$

in the particular case when the straight line $Q = 0$ is at infinity, one has $m = 0$, therefore $q = 0$, and the equation takes the very simple form

$$Y'^2 + 2pX' = 0,$$

which represents parabolas.

The ratio $\frac{y_1}{x_1}$ could be so determined that the straight line

$$x_1f'_x + y_1f'_y = 0$$

is perpendicular to $P = 0$; the first straight line will then be an axis of the conic and the second the tangent at the vertex.

BOOK IV*

THE GENERAL THEORY OF CURVES

CHAPTER I

THE CONSTRUCTION OF CURVES IN RECTILINEAR CO-ORDINATES.

334. The construction of a curve is simply the graphic representation of the *trace* of the real function of a single variable, when this variable is allowed to change in a continuous manner. If the values of y which correspond to the various values of x be calculated, a certain number of the points of the curve can be constructed, but these points are not sufficient, even for an approximate trace of the curve, because they can be connected in various different ways, and, moreover, it can happen that, between two ordinates which are very nearly equal, the curve has infinite branches. It is, therefore, indispensable first of all to know by some general method the trace of the function which represents the variations.

When the equation is solved with respect to one of the variables, y for example, one considers each of the determinations of y in particular, and examines them for the limits of x for which y remains real. Let x_0 and x_1 be the two limits; if the value of y remain finite in this interval, it furnishes a finite branch of the curve; if the value of y becomes infinite for one or more intermediate values a, b, $\cdots$ of the variable, one has various infinite branches, asymptotic to the straight lines that correspond to the values of x, which make y infinite; in such a case the interval x_0 to x_1 is subdivided into several intervals:

the first from x_0 to a, etc., in such a way that in each of them the ordinate does not become infinite. Afterwards one examines how y varies in each of the intervals, for example as x increases from x_0 to a. Sometimes one perceives immediately, from the expression for y, how this quantity varies, but more often that is not the case; in this case, however, one has recourse to the derivative. It is known, moreover, that if the function remains finite, as the variable x increases from a certain value, the function will vary in the same sense as the variable so long as the derivative preserves the same sign; the function increases if its derivative be positive and decreases if its derivative be negative. Let α, β, γ, $\cdots$ be the successive values of x comprised between x_0 and a for which the derivative changes in sign. As the variable x increases from x_0 to α, the derivative preserves the same sign, for example the sign $+$, and the function increases; from α to β, the derivative is negative and the function decreases, etc. We have demonstrated that the angular coefficient of the tangent at any point of the curve is equal to the value of the derivative at this point. Thus, the sense in which the ordinate of the curve varies is indicated by the angular coefficient of the tangent.

When the derivative changes its sign from positive to negative, the ordinate ceases to increase and then decreases; it attains therefore a *maximum* value. If, on the contrary, the derivative change from negative to positive, the ordinate ceases to decrease and then increases; it attains therefore a *minimum* value. It should be noticed that these terms *maximum* and *minimum* should not be taken with their literal meaning; they indicate only the comparison of a particular value of the ordinate with its neighboring ordinates.

In general, the derivative, remaining finite and continuous, changes in sign on becoming zero, and consequently the tangents at the points whose ordinates have the *maxima* and *minima* values are parallel to the axis OX. Every value of x which makes the derivative zero does not necessarily give a maximum or minimum value of the ordinate; one must examine if the derivative change in sign: moreover, in all the cases, the tangent is parallel to the axis OX.

335. Example I. — The strophoid defined in § 23 has the equation

$$y = \pm x\sqrt{\frac{a-x}{a+x}}.$$

When x varies from zero to $-a$, the numerical value of y increases continually from zero to infinity; whence one obtains the two infinite branches ON, ON' asymptotic to the straight line HH' (Fig. 18). If x vary from zero to a, the ordinate y begins with the value zero and returns to zero, passing always through finite values; it begins therefore by increasing, then later it decreases, and consequently it passes through a maximum value; but one does not see if the function does not experience in the interval several alternatives of increasing and decreasing. The positive value of y has the derivative

$$y' = \frac{-x^2 - ax + a^2}{\sqrt{(a+x)(a-x)}}.$$

The numerator becomes zero for two values of x, the one x_1 positive and less than a, the other negative. When x varies from zero to x_1, the derivative is positive, the function increases; from x_1 to a, the derivative is negative, the function decreases; the ordinate is a maximum for the value

$$x_1 = a\frac{\sqrt{5}-1}{2},$$

equal to the greater segment of the line a divided in a mean and extreme ratio.

336. The tangent can often be determined at certain points of the curve, or, what amounts to the same thing, certain particular values of the derivative, without recourse to the general expression for this derivative. Consider, for example, the point O of the strophoid; join this point to a neighboring point M whose co-ordinates are x and y; the angular coefficient of the secant OM is equal to the ratio $\frac{y}{x}$; the angular coefficient at the point O will be found on seeking the limit of this ratio as x approaches zero. Here one has

$$\frac{y}{x} = \pm\sqrt{\frac{a-x}{a+x}};$$

when x approaches zero, this ratio has the limit ± 1. The two branches which pass through the point O have as tangents at this point the bisectors of the angles of the axes. The tangent at the point A would be found by considering the ratio $\frac{y}{a-x}$; this ratio increasing indefinitely as x approaches a, the tangent at the point a is parallel to the axis Oy.

337. Example II. — We propose to study the curves represented by the equation $y^2 = Ax^3 + Bx^2 + Cx + D$ (it can be demonstrated that these curves, reproduced by projection, represent all the curves of the third degree). One can assume that the coefficient A is positive without changing the direction of the x-axis. There are several cases to consider: 1° The three roots of the polynomial of the third degree are real and unequal; let a, b, c be these roots arranged in order of increasing magnitude; then we may write

$$y^2 = A(x-a)(x-b)(x-c).$$

The ordinate is imaginary when x varies from $-\infty$ to a; real when x varies from a to b; imaginary when x varies from b to c; real when x

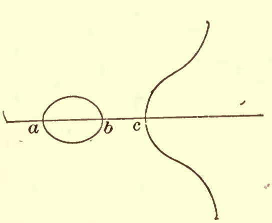

Fig. 206.

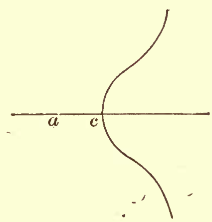

Fig. 207.

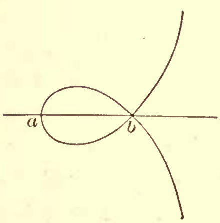

Fig. 208.

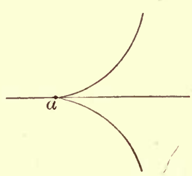

Fig. 209.

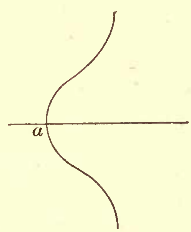

Fig. 210.

varies from c to $+\infty$. The curve is composed of a closed oval and an infinite branch (Fig. 206). 2° When the two roots a and b become equal, the oval reduces to a point a (Fig. 207). 3° When the two roots b and c are equal, the oval becomes united to the infinite branch at b (Fig. 208). 4° If the three roots a, b, c are equal, the curve has a cusp at a (Fig. 209). 5° Finally, if the polynomial of the third degree has but one real root a, the curve has the form given in Fig. 210.

The angular coefficient of the tangent is given by the formula

$$y' = \frac{3Ax^2 + 2Bx + C}{2\sqrt{Ax^3 + Bx^2 + Cx + D}} = \frac{3Ax^2 + 2Bx + C}{2y}.$$

In the first case, the numerator, which is the derivative of the polynomial of the third degree, becomes zero for a value a' comprised between a and b, and for a value b' comprised between b and c; to the first corresponds the maximum value of the ordinate in the oval. In the third case, the numerator becomes zero for the double root b; the denominator becoming zero also, the formula assumes the indeterminate form $\frac{0}{0}$ and no longer determines the tangents at the double point b; they may be found by determining the limit $\sqrt{A(b-a)}$ of the ratio $\frac{y}{x-b}$, as x approaches b.

338. When the equation, supposed algebraic, is not solved, whether this solution is possible or not, or whether it is deemed useless to solve it, we can often, by employing the theorems concerning the roots of equations, construct the curve.

Certain properties of the curve can be immediately recognized by inspection of its equation. 1° When the equation has terms all of which are of even degrees, or of odd degrees, it is clear that, if it be satisfied by $x = \alpha$, $y = \beta$, it will also be satisfied by $x = -\alpha$, $y = -\beta$; that is, the two points (α, β), $(-\alpha, -\beta)$ are situated symmetrically with respect to the origin; therefore this point is the center of the curve. 2° If the equation contains only even powers of one of the variables, y for example, the real values of y, which correspond to a particular value of x, are two by two equal and of contrary signs; if the axis be rectangular, it follows that the points of the locus are situated symmetrically with respect to the x-axis, which is an axis of the curve. 3° When the equation of the curve remains unaltered, when x is changed into y and y into x, if the equation be satisfied by $x = \alpha$, $y = \beta$, it will also be satisfied by $x = \beta$, $y = \alpha$; the two corresponding points are situated symmetrically with respect to the bisector of the angle YOX, which is an axis of the curves. It follows similarly that, if the equation does not change by substituting $-y$ for x and $-x$ for y, the bisectors of the angle YOX' is an axis.

Let $f(x, y) = 0$ be the equation of the curve; it is known that the derivative y' is given by the formula $y' = -\frac{f'_x(x, y)}{f'_y(x, y)}$. The expression for y' contains the two variables x and y; it gives the angular coefficient of the tangent at every point whose co-ordinates are known, excepting at the points where the two partial derivatives are zero at the same time.

339. EXAMPLE III. — *Construct the locus of the points such that the product of their distances from two fixed points F and F' is equal to a given number.*

Take as origin the mid-point O of the straight line FF', this straight line as the x-axis, and a perpendicular to it as the y-axis; let $2c$ be the distance FF', a^2 the constant product, the equation of the locus is

$$(1) \qquad y^4 + 2(x^2 + c^2)y^2 + (x^2 - c^2)^2 - a^4 = 0.$$

This equation involves only the even powers of each variable; each axis is therefore an axis of symmetry of the curve, and the origin is at the center. On considering y^2 as unknown, equation (1) is of the second degree; the binomial $B^2 - 4AC$ reduces in this case to $4(4c^2x^2 + a^4)$, a quantity which is always positive: the roots are therefore always real. When the last term $(x^2 - c^2)^2 - a^4$ is positive, the values of y^2 have the same sign, and since their sum $-2(x^2 + c^2)$ is negative, the two values of y^2 are negative and the four values of y are imaginary. In order that equation (1) has real roots, it is therefore necessary that we have

$$(x^2 - c^2)^2 - a^4 < 0, \text{ or } (x^2 - c^2 - a^2)(x^2 - c^2 + a^2) < 0,$$

and, consequently,

$$x^2 < a^2 + c^2 \text{ and } x^2 > c^2 - a^2.$$

Then one of the values of y^2 is positive, the other negative.

Take $OA = OA' = \sqrt{a^2 + c^2}$; the curve lies between the straight lines drawn through the point A and A' parallel to the y-axis. The second condition gives rise to the discussion of several cases.

1° $a < c$. Take $OB = OB' = \sqrt{c^2 - a^2}$, and draw at the points B and B' lines parallel to OY (Fig. 211). The curve consists of two parts, one of which is comprised between the parallel lines drawn through the points B and A, and the other between the parallels drawn through the points B' and A'. If x be given one of the values OB or OA, one of the corresponding values of y^2 is zero, and the other is negative; as x increases from OB to OA the value of y^2, which at first is zero, increases, then decreases and becomes zero again; we obtain thus a closed curve $BCAD$.

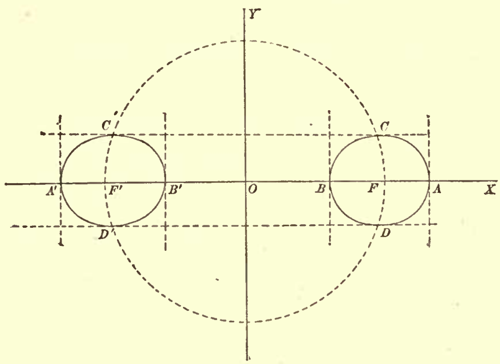

Fig. 211.

The negative values of x give a second curve $B'C'A'D'$, equal to the preceding.

The angular coefficient of the tangent is determined by the formula

$$(2) \qquad y' = -\frac{x(x^2 + y^2 - c^2)}{y(x^2 + y^2 + c^2)}.$$

At the points A and B, y is zero and y' is infinite; the tangent is therefore parallel to the y-axis. The numerator of y' becomes zero when $x^2 + y^2 = c^2$. From the point O as center, describe a circle with OF as radius. The circle intersects the curve in four points C, D, C', D', given by the formulas

$$x^2 = \frac{4\,c^4 - a^4}{4\,c^2}, \quad y^2 = \frac{a^4}{4\,c^2}.$$

Since the arc BC lies within the circle, at any of the points of this arc, the function $x^2 + y^2 - c^2$ has a negative value, and y' is positive. For points of the arc CA, the factor $x^2 + y^2 - c^2$ is positive, and y' is negative. Hence from B to C the ordinate increases, and from C to A it decreases; the ordinate at the point C is a maximum.

2° $a = c$. The second condition is satisfied whatever x may be; x may vary from $-c\sqrt{2}$ to $c\sqrt{2}$. When x varies from 0 to $c\sqrt{2}$, the positive value of y^2 begins with zero, increases, then decreases and becomes zero again; we have a closed curve $OCADO$ (Fig. 212), which passes through the origin: to the negative values of x there corresponds a curve which is the *symétrique* of the preceding with respect to the y-axis. The circle of radius OF intersects the curve in four points, whose co-ordinates have a numerical value $\frac{c}{2}$, which is a maximum; the abscissas of these points have an absolute value $\frac{c\sqrt{3}}{2}$. This curve is called the *lemniscate*.

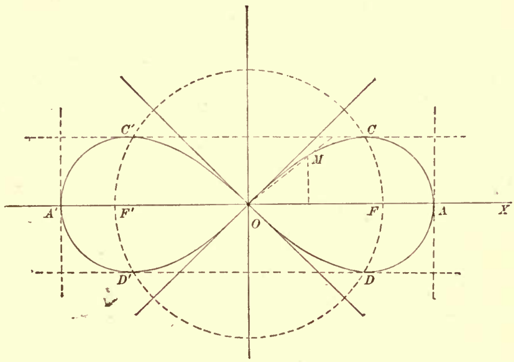

Fig. 212.

At the origin the value y' takes the form $\frac{0}{0}$; it is easy to show that this is the case at the multiple point of any algebraic curve. In fact, the value of y' is given by the formula

$$y' = -\frac{f'_x(x, y)}{f'_y(x, y)}.$$

Since $f(x, y)$ is an integral polynomial with respect to x and y, the partial derivatives $f'_x(x, y)$, $f'_y(x, y)$ are also integral polynomials with respect to the same variables. If these polynomials do not become zero, when x and y are replaced by the co-ordinates of the multiple point, y' will have at this point a unique value, whereas it should have as many different values as there are branches of the curve which pass through the multiple point. In the present case, the equation being a bi-quadratic can be solved with respect to y; to each value of y there corresponds a derivative which has a definite value when x is put equal to zero. This value of the derivative is, as has been noticed in § 336, the limit of the ratio $\frac{y}{x}$, when x approaches zero. The limit of this ratio can be found without solving the equation. Put $\frac{y}{x} = t$, or $y = tx$; on substituting in equation (1), it becomes

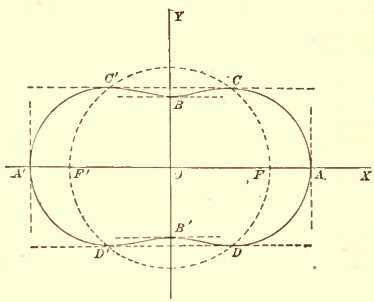

Fig. 213.

$$x^2t^4 + 2(x^2 + c^2)t^2 + x^2 - 2c^2 = 0.$$

When x is very small, one of the values of t^2 is approximately equal to unity, the other is negative and very large in absolute value; on confining ourselves to the real values of y, we have $\lim \frac{y}{x} = \pm 1$. The tangents at the point O bisect the angles formed by the axes.

3° $a > c$. The second condition is satisfied, whatever x may be; x can therefore vary from

$$-\sqrt{c^2 + a^2} \text{ to } +\sqrt{c^2 + a^2}.$$

For x equal zero, the positive value of y^2 is $a^2 - c^2$. Take on the y-axis

$$OB = OB' = \sqrt{a^2 - c^2},$$

the curve passes through the points B and B'. If x vary from 0 to $\sqrt{c^2 + a^2}$, y^2 begins with $a^2 - c^2$, decreases, and becomes zero; the locus is a closed curve whose vertices are the points A, A', B, B'. In order that the circle intersect the curve, it is necessary that $a < c\sqrt{2}$. When this condition is satisfied, the ordinate increases from B to C and diminishes from C to A: the ordinate of the point B is a minimum, that of C a maximum. If, on the other hand, one have $a > c\sqrt{2}$, the circle lies within the curve, of which the ordinate diminishes from B to A; the ordinate of the point B is a maximum. In Fig. 214 it is supposed that A is equal to $c\sqrt{2}$.

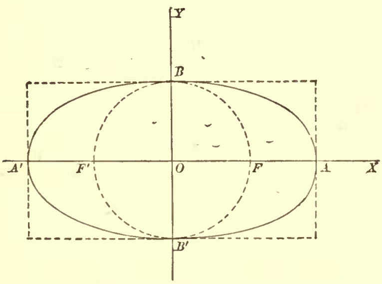

Fig. 214.

340. Example IV. —*Construct the curve*

$$2y^5 - 5xy^2 + x^5 = 0. \quad (1)$$

This equation being of the fifth degree with respect to each of the variables, it cannot be solved with respect to either of the variables; it involves only terms of odd degrees; therefore the origin is at the center of the curve. Examine how many of the roots of the equation, in which y is regarded as unknown, are real for various values of x.

Suppose in the first place that x is positive, equation (1) will have at most two real positive roots, since its first member has but two variations in signs. The derivative of the first member with respect to y is $10y(y^3 - x)$. This derivative is negative from $y = 0$ to $y = \sqrt[3]{x}$, positive from this value to infinity. The first member, which is positive for $y = 0$, decreases when x varies from 0 to $\sqrt[3]{x}$, and increases indefinitely as y becomes greater. The equation has therefore two positive roots, or it does not, according as the value $y = \sqrt[3]{x}$ renders the first member negative or positive, that is, according as one has $x^{10} < 27$, or $x^{10} > 27$. If y be changed into $-y$, the first member has but one variation; therefore the equation has one negative root and only one.

For $x = 0$, the five roots of equation (1) are zero; for values of x between 0 and $\sqrt[10]{27}$, the equation has two positive roots, and one negative; for $x = \sqrt[10]{27}$, the two positive roots are equal, because they reduce the derivative to zero. When x becomes larger than $\sqrt[10]{27}$, the equation has but one real root, which is negative. The two positive roots give an oval $OABO$ (Fig. 215), comprised within the angle YOX, and the negative root a finite branch OC situated in the angle $Y'OX$. To negative values of x there corresponds an oval $OA'B'O$ and infinite branch OC', the *symétrique* of the preceding with respect to the center. The maximum value of the abscissa for the oval $OABO$ is $\sqrt[10]{27}$. It corresponds to a point A, where the tangent is parallel to OY, since the co-ordinates of this point reduce $f'_y(x, y)$ to zero. Regarding y as an arbitrary variable, one finds that the maximum value of y for the same oval is $\sqrt[5]{4}$. This maximum value gives the point B, where the tangent is parallel to the x-axis.

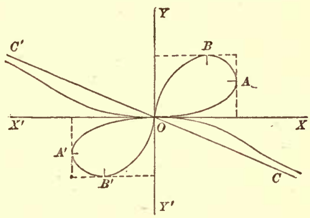

Fig. 215.

The preceding method of discussion is applicable in all cases where the equation does not contain more than three terms; because it is always possible to determine the number of real roots of a trinomial equation involving one unknown quantity.

THE INTRODUCTION OF AN AUXILIARY VARIABLE.

341. When it is impossible to solve an equation with respect to one of the variables x or y, it is possible, in certain cases, to express the two co-ordinates in terms of an auxiliary variable t, and, on following the simultaneous variation of x and y, as t varies between the limits which make these quantities real, to trace the curve.

If y be regarded as a function of x, and x as a function of t, it follows that, on taking the derivative of y with respect to t, owing to the theorem of functions,*

$$D_t y = D_x y \cdot D_t x;$$

whence it follows $$D_x y = \frac{D_t y}{D_t x},$$

* To designate the derivative of a function, the letter D is frequently employed, representing a partial *derivative*, and the variable with respect

which gives the angular coefficient of the tangent at the point which corresponds to any value of t. The values of t which reduce $D_t y$ to zero determine the points at which the tangent is parallel to the x-axis, and the values which reduce $D_t x$ to zero, the points at which the tangent is parallel to the y-axis.

342. Example V. — *Construct the curve* $y^4 - y^3x + x^3 - 2x^2y = 0$. If we put $y = tx$, it follows that

$$x = \frac{2t-1}{t^3(t-1)}, \quad y = tx = \frac{2t-1}{t^2(t-1)}.$$

The curve is constructed by allowing t to vary from $-\infty$ to $+\infty$. In order to follow the variations of x and y, construct the derivatives

$$D_t x = -\frac{6t^2 - 8t + 3}{t^4(t-1)^2}, \quad D_t y = -\frac{4t^2 - 5t + 2}{t^3(t-1)^2}.$$

The numerators do not become zero for any real value of t, and do not therefore change in sign. The values of x and y become zero for $t = \frac{1}{2}$, infinity for $t = 0$ or $t = 1$. If t vary from $-\infty$ to 0, x is negative and decreases from 0 to $-\infty$, y is positive and increases from 0 to ∞; thus the infinite branch OA is obtained (Fig. 216). As the variable t increases from 0 to $\frac{1}{2}$, x and y are positive and decrease from ∞ to 0, which gives the infinite branch BO. As the variable t increases from $\frac{1}{2}$ to 1, x and y are negative and decrease from 0 to $-\infty$, which gives the infinite branch OC. The angular coefficient of the tangent to the branch BOC at O is $\frac{1}{2}$. Finally, if t vary from 1 to ∞, x and y becoming positive and decreasing from ∞ to 0, one obtains the infinite branch DO.

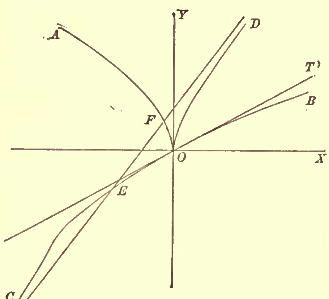

Fig. 216.

If the equation in x and y do not involve more than two groups of terms, one of the degree m, the other of the degree $m - 1$, and if the ratio $\frac{y}{x} = t$ be chosen as an auxiliary variable, the co-ordinates x and y are rational functions of this variable. If the equation contain three groups

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to which the derivative is taken is indicated by writing this variable as an index to the right and a little below the letter D. Thus, $D_t x$ and $D_t y$ indicate the derivatives of the functions x and y with respect to the variable t, $D_r y$ the derivative of y with respect to r.

of terms, the first of the degree m, the second of the degree $m-1$, the third of the degree $m-2$, on using the same auxiliary variable, the co-ordinates may be expressed by the solution of a quadratic equation, and their simultaneous variations can still be followed.

EXAMPLE VI. — *Construct the curve* $x^5y^4 - xy - x - 2 = 0$.

If one put $xy = t$, it follows that $x = \frac{t+2}{t^4-1}$, $y = \frac{t^5-t}{t+2}$. Examine how x and y vary when the auxiliary variable t varies from $-\infty$ to $+\infty$. For this purpose construct the derivatives of the two functions; one has

$$D_tx = -\frac{3t^4+8t^3+1}{(t^4-1)^2}, \quad D_ty = \frac{4t^5+10t^4-2}{(t+2)^2}.$$

The value of D_tx becomes zero for two values a and b of t, comprised, the first between $-\frac{8}{3}$ and -2, the second between -1 and 0. The value D_ty becomes zero for three values c, d, e, of t, comprised, the first between $-\frac{5}{2}$ and -2, the second between -2 and 0, and the third between 0 and 1. It follows from the preceding that $a < c$, $d < b$.

Now let us consider the following series of quantities

$$-\infty,\ a,\ c,\ -2,\ -1,\ d,\ b,\ e,\ +1,\ \infty,$$

arranged in order of magnitude. If t vary from $-\infty$ to a, x is negative, begins with 0, and decreases; y is positive, begins with infinity, decreases continually; one obtains the branch AB, asymptotic to the y-axis (Fig. 217). The variable t varying from a to c, x is negative and increases, y is positive and decreases; one obtains the branch BC. The variable t varying from c to -2, x is negative and increases to 0, y is positive and increases to ∞; whence the branch CI. The variable t varying from -2 to -1, x increases from 0 to ∞, y increases from $-\infty$ to 0; whence the doubly infinite branch DE, asymptotic to OY' and OX. The variable t varying from -1 to d, x begins with $-\infty$ and increases, y begins with 0 and increases; whence the infinite branch FG, asymptotic to OX'. As the variable t varies from d to b, x continues to increase, and y remaining positive decreases; whence the branch GH. As the variable t varies

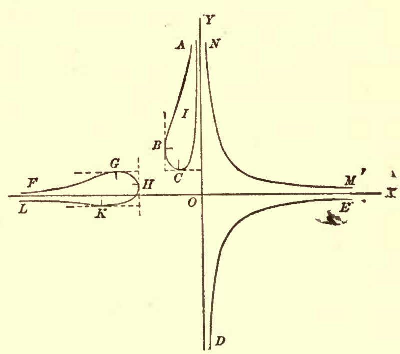

Fig. 217.

from b to e, x and y decrease; whence the branch HK, which intersects OX' in a point, whose abscissa -2 corresponds to $t=0$. As the variable t varies from e to $+1$, x decreases, y increases; whence the infinite branch KL, asymptotic to OX'. Finally, as t varies from $+1$ to $+\infty$, x decreases from ∞ to 0, and y increases from 0 to ∞; whence the double infinite branch MN, asymptotic to OX and OY.

The tangents at the points C, G, K, which correspond to the values c, d, e, of t, which reduce $D_t y$ to zero, are parallel to OX; the tangents at the points B and H are parallel to the y-axis.

343. Tangent Curves, Orthogonal Curves. — Let $f(x, y)=0$, $\phi(x, y)=0$ be the equation of two curves. Call x and y the co-ordinates of a point of intersection of the two curves; in order that they be tangent at this point, it is necessary and sufficient that the angular coefficients of the tangent to the two curves at this point *be equal:*

$$\frac{f'_x}{f'_y}=\frac{\phi'_x}{\phi'_y},$$

or

$$f'_x\phi'_y-f'_y\phi'_x=0.$$

The co-ordinates x and y ought therefore to satisfy the three equations:

$$(1) \qquad f(x, y)=0, \quad \phi(x, y)=0, \quad f'_x\phi'_y-f'_y\phi'_x=0;$$

on eliminating x and y between these three equations, that is, on expressing the condition that they have a *common solution,* we get an equation which expresses the condition that the curves are tangent at the same point.

If the curves should be tangent in k points, it would be necessary to express the condition that the equations above have k common solutions.

From a geometric point of view, to express the condition that equations (1) have k common solutions is equivalent to expressing the condition that the curve

$$f'_x\phi'_y-f'_y\phi'_x=0$$

passes through k of the points of intersection of the given curves.

In a similar manner it follows that if the given curves $f=0$, $\phi=0$ be orthogonal at a point (x, y), one ought to have

$$(2) \qquad f(x, y)=0, \quad \phi(x, y)=0, \quad f'_x\phi'_x+f'_y\phi'_y=0.$$

On eliminating x and y between these relations, we get a condition that the two curves are orthogonal *at one of their* points of intersection. In order to express the condition that they are orthogonal at k of their points of intersection, it is necessary to express the conditions that equations (2) have k common solutions.

From the geometric point of view this amounts to expressing the condition that the curve

$$f'_x\phi'_x+f'_y\phi'_y=0$$

passes through k of the points common to the proposed curves.

EXAMPLE. — Suppose that we have two conics $f=0$, $\phi=0$; in order to express the condition that they are orthogonal at their four points of intersection, it is necessary to express the condition that the curve

$$f'_x\phi'_x+f'_y\phi'_y=0,$$

which is also a conic, passes through the four points common to the two given conics; that is, that its equation can be identified with an equation of the form $f+\lambda\phi=0$.

Thus it may be easily verified that, whatever be the constants α and β, the two conics

$$2x^2+y^2-\alpha=0, \quad y^2-2\beta x=0,$$

are orthogonal at all of their points of intersection.

The same is true of the conics

$$2xy-\alpha=0, \quad x^2-y^2-\beta=0.$$

EXERCISES. — 1° *Construct the general equation of the conics which intersect at right angles the fixed conic $Ax^2+By^2-1=0$ in four points.*

These conics may be divided into several groups; to one of them belong the conics confocal to the fixed conic.

2° *Let $f(x, y)=0$, $\phi(x, y)=0$ be the equations of two algebraic curves of degrees m and n in rectangular co-ordinates; find the angle at which the curves intersect.*

Let x and y be the co-ordinates of one of the points of intersection; the tangents to the two curves at this point intersect at an angle θ, whose trigonometric tangent is given by the formula

$$\tan\theta=\frac{f'_x\phi'_y-f'_y\phi'_x}{f'_x\phi'_x+f'_y\phi'_y}.$$

By eliminating x and y between this equation and the equation of the two curves, one gets an equation of the degree mn, giving the tangents of the angle θ, at which the two curves intersect.

As an application, form the equation of the second degree which determines the tangents t of the angles at which the straight line $ux + vy + w = 0$ intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

One finds :

$$t^2(a^2v^2w^2 + b^2u^2w^2 - c^4u^2v^2) + (a^2u^2 + b^2v^2 - w^2)(2\,tc^2uv - a^2u^2 - b^2v^2) = 0.$$

CHAPTER II

CONVEXITY AND CONCAVITY.

344. Let AB be an arc of the curve corresponding to a determination of y and to the values of x comprised between a and b; we assume that the second derivative y'' of y with respect to x preserves the same sign in this interval, for example, remains positive. Draw the tangent RS at any point M of this arc, whose abscissa is x_0; let y'_0 be the value of the derivative at this point, or the angular coefficient of the tangent, and designate by Y the ordinate of any point of this straight line, the ordinate defined by the equation

$$Y - y_0 = y'_0 (x - x_0);$$

the difference $y - Y$ becomes zero for $x = x_0$, and the same is true of its derivative $y' - Y'$ or $y' - y'_0$ (Fig. 218). When the abscissa x increases from a to b, the derivative y'' of the difference $y' - y'_0$ being positive, the function $y' - y'_0$ increases; since it becomes zero for $x = x_0$, it is negative from a to x_0, positive from x_0 to b. Consider now the function $y - Y$, which has the derivative $y' - y'_0$; when x varies from a to x_0, the derivative being negative, the function decreases; since it becomes zero for $x = x_0$, it would be positive above; as x varies from x_0 to b, the derivative is positive and the function increases; since it becomes zero for $x = x_0$, it is also positive from x_0 to b, whence it follows that the difference $y - Y$ remains positive throughout the interval from a to b. We conclude from this that the arc of the curve ab is situated wholly on the same side of each of its tangents, and is said to be *convex*. Similarly, if the second-derivative were

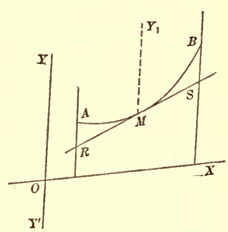

Fig. 218.

negative; the difference $y - Y$ being negative, the arc would be situated wholly on the other side of the tangent. It is easy to distinguish these two cases; draw through the point M a straight line MY_1 parallel to the axis OY, and in the direction of the positive y's; in the first case, the arc is situated on the same side of the tangent as the half-line MY_1; in the second case, the arc lies on the other side. In the first case, it is said that the arc AB is *concave* in the direction of MY_1; in the second case, in the opposite direction.

We know that the sign of y'' indicates the kind of variation of y' when x increases. If therefore one imagines that the point M travels through the arc AB, the angular coefficient will increase if y'' be positive, and, on the contrary, will decrease if y'' be negative.

345. The points of a curve at which its concavity changes its direction are called points of *inflection.* At such points therefore the second derivative changes its sign. The second derivative may change its sign on passing through *zero* or *infinity.* In general, the quantity y'', being finite and continuous, changes sign on passing through zero. Suppose that y'' changes its sign on passing through zero for $x = x_0$, it may be verified that the first derivative $y' - y'_0$ does not change sign, but that the function $y - Y$ does experience a change in sign; of the sort that at this point the curve passes from one side to the other of the tangent. If for $x = x_0$, y'' experiences a change in sign on passing through infinity, y' becoming infinite and y remaining finite, the point $x = x_0$ and $y = y_0$ is a point of inflection at which the tangent is parallel to the y-axis.

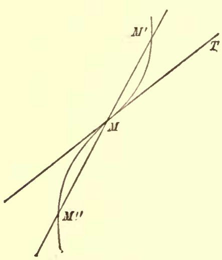

Fig. 219.

If a neighboring secant of the tangent MT be drawn through the point of inflection M, this secant will intersect the curve in two points M' and M''; the tangent MT is the limit of the secant passing through the three points M'', M, M' when the two points M'' and M' approach the point M.

346. Example I. — *Sinusoid.* Construct the curve $y = \sin x$. As x increases from 0 to π, the ordinate is positive; it begins with 0, increases to 1, and then decreases to 0, which gives the arc OAC (Fig. 220) symmetrical to the ordinate which corresponds to $x = \frac{\pi}{2}$. As x increases from π to 2π, y becomes negative, and one obtains the arc CBO', equal to the first. From 2π to 4π, the ordinate passes again through the same values which it took when x varied from 0 to 2π,

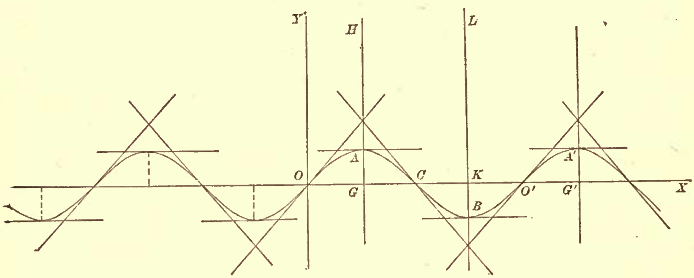

Fig. 220.

similarly from 4π to 6π, etc. Thus the curve is composed of an infinite number of equal undulations.

The angular coefficient of the tangent is $y' = \cos x$; at the origin, $y' = +1$, and the tangent is the bisector of the angle YOX. At the point C, $y' = -1$, and the tangent is parallel to the other bisector. For $x = \frac{\pi}{2}$, the derivative becomes zero and changes from positive to negative; the ordinate of the point A is a maximum. For $x = 3\frac{\pi}{2}$, the derivative becomes zero again and changes in sign from negative to positive; the ordinate of the point B is a minimum.

When x varies from 0 to π, the second derivative $y'' = -\sin x$ is negative, and the concavity of the curve is turned toward negative y's; from π to 2π, the second derivative is positive, and the concavity of the curve is reversed and turned toward positive y's; the point C is therefore a point of inflection.

It is worthy of notice that the curve has an infinity of centers, situated at equal distances along the x-axis, the points O, C, O', ⋯ whose abscissas are multiples of π. Each of them is a point of inflection.

347. Example II. — *Construct the curve* $(y - x^2)^2 - x^5 = 0$, or $y = x^2 \pm x^{\frac{5}{2}}$.

The values of y are only real when x is positive. Consider first the case when the sign before the radical is $+$; the functions $y = x^2 + x^{\frac{5}{2}}$

increases from 0 to ∞, as x varies from 0 to ∞. The derivative $y' = 2x + \frac{5}{2}x^{\frac{3}{2}}$ begins with zero and increases continually without limit; one gets therefore one infinite branch OD tangent to the x-axis at the point O, and which has its concavity turned toward positive y's (Fig. 221). Investigate now the minus sign before the radical; the value of y is positive from 0 to 1, and negative for $x > 1$. Lay off on OX a length OA equal to unity; the curve passes through A. The derivative $y' = 2x - \frac{5}{2}x^{\frac{3}{2}}$ is zero at the point O, remains positive so long as x is less than $\frac{16}{25}$, and becomes negative when x is greater than this number; the ordinate MP, which corresponds to $\frac{16}{25}$, is a maximum, and the tangent at M is parallel to the x-axis. The second derivative $2 - \frac{15}{4}x^{\frac{1}{2}}$ remains positive for $0 \leqq x < \frac{64}{225}$, but is negative for $x > \frac{64}{225}$; the point N, which corresponds to the abscissa $\frac{64}{225}$, is a point of inflection; from O to N, the concavity is turned toward the positive y's, but for $x > \frac{64}{225}$ it is turned toward the negative y's.

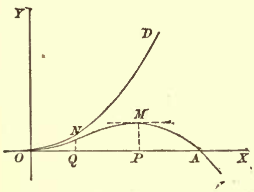

Fig. 221.

The two branches of the curve are tangent to the x-axis at the point O, without one being the continuation of the other; points which have this peculiarity are called *cusps*. In this curve, the two branches lie on the same side of the tangent. On considering the curve $(y - x^2)^2 - x^3 = 0$, there will be two branches, one situated on one side and the other on the opposite side of the tangent; the cissoid has a cusp of this kind at the vertex (Fig. 16).

348. Example III. — Let the curve be $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

It is supposed that a represents a given length; then the equation is homogeneous and defines a curve, which is called the *catenary*, for the reason that it is the curve of the form assumed by a flexible thread of which the ends are attached to two fixed points.

The equation gives equal values of y for equal values of x with contrary signs; hence the straight line OY is an axis of the curve. If x vary from 0 to ∞, the term $e^{\frac{x}{a}}$ increases, but the term $e^{-\frac{x}{a}}$ decreases; in order to know how y varies, construct the derivative of y with respect to x; namely,

$$y' = \tfrac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right).$$

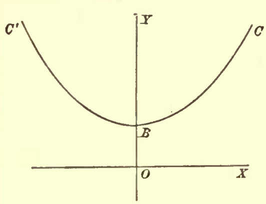

Fig. 222.

This derivative is positive for all positive values of x; therefore when x increases from 0 to ∞, the value of y increases constantly from a to ∞, which gives an infinite branch BC (Fig. 222); the branch BC', the sy-

métrique of BC with respect to OY, is obtained by assigning negative values to x.

Since the second derivative remains positive as x varies from $-\infty$ to $+\infty$, the curve turns its *concavity* toward positive y's.

349. Example IV. — *Construct the curve* $y = e^{\frac{1}{x}}$. For very small positive values of x, y is positive and very large; as x increases from 0 to $+\infty$, y decreases constantly from ∞ to 1, which gives a branch AC (Fig. 223), asymptotic on the one hand to the y-axis and on the other to the straight line $G'G$ drawn parallel to the x-axis and at a distance from this straight line equal to unity. When one gives a very small numerical negative value to x, y is positive and very small; as x varies from 0 to $-\infty$, y increases continuously from 0 to 1, whence we get a branch OD starting from the origin and asymptotic to the straight line GG'.

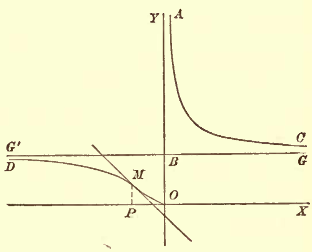

Fig. 223.

This curve presents a peculiarity which has not yet been met: the branch DO stops abruptly at the point O; points of this kind are called *points d'arrêt.*

In order to find the direction of concavity, construct first the first derivative $y' = -\frac{1}{x^2}e^{\frac{1}{x}}$. When x varies from 0 to ∞, the two factors $\frac{1}{x^2}$ and $e^{\frac{1}{x}}$ diminish, and on account of the sign $-$, y' increases; the concavity of the branch AC is turned toward positive y's; since, as x varies from $-\infty$ to 0, the factor $\frac{1}{x^2}$ increases, and the factor $e^{\frac{1}{x}}$ diminishes, it is not at first evident how y' varies and we construct the second derivative. We get $y'' = \frac{(2x+1)e^{\frac{1}{x}}}{x^4}$. As x varies from $-\infty$ to $-\frac{1}{2}$, the second derivative is negative; take $OP = \frac{1}{2}$ and let M be the corresponding point of the curve; the arc DM is concave toward the negative y's; as x increases from $-\frac{1}{2}$ to 0, y'' is positive and the arc MO turns its concavity toward the positive y's and the point M is a point of inflection.

Example V. — Let $y = e^{\sqrt[3]{x}}$.

It can be easily shown that $y'' = e^{\sqrt[3]{x}}\frac{\sqrt[3]{x} - 2}{9x^{\frac{5}{3}}}$;

the second derivative changes its sign twice, once on becoming infinite for $x = 0$, and a second time on becoming zero for $\sqrt[3]{x} = 2$, $x = 8$. The curve has therefore two points of inflection.

350. Consider the cases when the equation

$$(1) \qquad f(x, y) = 0$$

cannot be solved with respect to either of the variables x and y; the derivative y' is given by the equation

$$(2) \qquad f'_x(x, y) + f'_y(x, y) \cdot y' = 0.$$

Since y and y' are both functions of x, the first member of equation (2) is a *compound* function with respect to the independent variable x; the derivative of this function is

$$f''_{x^2} + 2f''_{xy}y' + f''_{y^2}y'^2 + f'_y y'';$$

since it is always zero, its derivative is also zero, and we have the equation

$$(3) \qquad f''_{x^2} + 2f''_{xy}y' + f''_{y^2}y'^2 + f'_y y'' = 0,$$

which determines the value of y''. If in this equation y' be replaced by its value deduced from equation (2), (3) becomes

$$y'' = -\frac{f''_{x^2}(f'_y)^2 - 2f''_{xy}.f'_x.f'_y + f''_{y^2}(f'_x)^2}{(f'_y)^3}.$$

It is by means of this formula that the direction of concavity and also the points of inflection are determined.

351. Apply this formula to the curve defined in § 339. The equation of this curve can be written in the form

$$(1) \qquad f(x, y) = \tfrac{1}{4}[(x^2 + y^2 + c^2)^2 - 4c^2x^2 - a^4] = 0.$$

From (1) we deduce

$$f'_x = x(x^2 + y^2 - c^2), \quad f'_y = y(x^2 + y^2 + c^2),$$

$$f''_{x^2} = (x^2 + y^2 - c^2) + 2x^2, \quad f''_{xy} = 2xy, \quad f''_{y^2} = (x^2 + y^2 + c^2) + 2y^2.$$

If these values be substituted in the preceding formula, we obtain, after reduction and using formula (1),

$$y'' = \frac{a^4[3c^2(y^2 - x^2) - (a^4 - c^4)]}{y^3(x^2 + y^2 + c^2)^3}.$$

Since for each portion of the curve comprised within one of the angles of the co-ordinate axes, the denominator preserves the same sign, the value of y'' cannot change sign other than when the numerator passes through zero. The co-ordinates of the points of inflection should therefore satisfy at the same time equation (1) and the equation

(2) $$y^2 - x^2 - \frac{a^4 - c^4}{3c^2} = 0;$$

whence it follows that

(3) $$y^2 + x^2 = \sqrt{\frac{a^4 - c^4}{3}}.$$

In the first case, when a is less than c, the values of x and y given by equations (2) and (3) being imaginary, the numerator of y'' has the same sign for all points of the arc BCA; it may be easily verified that this numerator is negative for the points B and A; the concavity of this arc is directed toward the negative y's (Fig. 211). In the second case, $a = c$, the numerator of y'' becomes zero at the point O only; it is negative from O to A, and the concavity of the arc OCA is directed toward the negative y's (Fig. 212); the arc $A'D'OCA$ has a point of inflection at the point O. In the third case, one has $a > c$; here the values of x and of y are real and one has at the same time $a < c\sqrt{2}$. If a be greater than $c\sqrt{2}$, the numerator is negative for all the points of the arc BA, and the concavity is directed toward the negative y's (Fig. 214); if a be less than $c\sqrt{2}$, the numerator becomes zero for a certain point G (Fig. 213) situated between B and C; from B to G, it has the same sign as at the point B; it is positive and the concavity is directed toward the positive y's; from G to A, the numerator has the same sign as at the point A; it is negative and the concavity is turned toward the side of the negative y's. The point G is a point of inflection.

Remarks concerning Algebraic Curves.

352. Let $f(x, y) = 0$ be an integral, algebraic equation of the degree m with respect to x and y, and of the degree n with respect to y; to each value of x there correspond n values of y, which, in general, are different from one another; it can be demonstrated that, when x varies in a continuous manner, each of these values varies also in a continuous manner; we assume this theorem as we have done in previous discussions. When the equation is irreducible, it cannot have multiple roots excepting for a limited number of values of x; among these values of x, consider only those which are real and suppose them arranged in order of magnitude. Let a, b, c be three consecutive values; when x varies from a to b, the number of real values of y will remain the same; because if, in the interval, an imaginary root should become real, its conjugate would also become real, and, at the moment of transition, the two roots would become equal; one obtains also, in the interval considered, a certain number of real and distinct branches which do not have a common point. When x passes through the value b, it can happen that two real roots become imaginary or conversely; at the point which corresponds to the value b and to the real double root, two branches of the curves begin or end in this case.

Among the real values of y which correspond to the same value x_0 of x, consider a value y_0 which is a simple root; if x be allowed to vary from $x_0 - h$ to $x_0 + h$, h being sufficiently small, this value of y will remain real without becoming equal to any of the others and will give rise to a real branch. Thus, *when, for a value of x_0 assigned to x, the equation has a simple real root y_0, there passes through the point whose co-ordinates are x_0 and y_0 one real branch, and only one.*

Let us consider next a value $x = b$, to which there corresponds a multiple value y_1 of the order p. Locate the point M, whose co-ordinates are $x = b$, $y = y_1$; among the p values of y which become equal to y_1 for $x = b$, there are a certain number which were real and a certain number which were imaginary; the number of the latter being even, the number of the real roots is $p - 2q$ (q can be zero). Similarly, when x varies from b to c, the number of real values of y which belongs to the value y_1 for $x = b$ is $p - 2q'$; of the sort that the total number of branches of the curve which emanate from the point M, in one direction or another, is the even number $2p - 2q - 2q'$.

353. Let us determine the tangents at the point M; transfer the origin of co-ordinates to this point, and put $y = tx$; we will have an equation $\phi(x, t) = 0$, which will determine the angular coefficients t of the secants drawn from the point M to the points where the curve is intersected by a parallel to the y-axis. Suppose that for $x = 0$ one has a real root $t = t_1$; this root will determine a straight line, and, on repeating the same reasoning of the preceding paragraph, one sees that the total number of real branches emanating from the point

M and tangent to the two directions of the straight line, is even.

It follows from what precedes that an algebraic curve cannot have a *point d'arrêt* (§ 349). It cannot have, moreover, a *point saillant* or *anguleux;* a *point anguleux* is a point at which two branches are tangent to two different straight lines.

354. When the origin is transferred to the point M, whose co-ordinates are x_0 and y_0, the equation becomes

$$(1) \qquad (xf'_{x_0} + yf'_{y_0}) + \tfrac{1}{2}(x^2 f''_{x_0^2} + 2\,xyf''_{x_0 y_0} + y^2 f''_{y_0^2}) + \cdots = 0,$$

and the equation

$$(2) \qquad x(f'_{x_0} + tf'_{y_0}) + \frac{x^2}{2}(f''_{x_0^2} + 2\,tf''_{x_0 y_0} + t^2 f''_{y_0^2}) + \cdots = 0$$

gives the points in which any straight line $y = tx$ drawn from the point M intersects the curve. When one of the first derivatives at least is different from zero, the root $x = 0$ being a simple root, the point M is called a *simple point* of the curve. For the particular value t_1 of t which reduces the first term to zero, a second root is equal to zero, and the straight line becomes a tangent to the branch of the curve. The contact is of the first order if for $t = t_1$ the coefficient of x^2 be different from zero; it is of the order p if the first coefficient different from zero is that of x^{p+1}; among the $m - 1$ other points of intersection of the straight line and of the curve, p coincides with the point M.

Suppose that the two partial derivatives of the first order be zero, without the three derivatives of the second order being zero; the root $x = 0$ being a double root of equation (2), any straight line drawn through the point M intersects the curve in two points coincident with M, and this point is called a *double point.* If all the terms be divided by x^2, equation (2) reduces to

$$(3) \ \tfrac{1}{2}(f''_{x_0^2} + 2\,tf''_{x_0 y_0} + t^2 f''_{y_0^2}) + \frac{x}{2\cdot 3}(f'''_{x_0^3} + 3\,tf'''_{x_0^2 y_0} + \cdots) + \cdots = 0.$$

When the equation of the second degree

$$(4) \qquad f''_{x_0^2} + 2\,tf''_{x_0 y_0} + t^2 f''_{y_0^2} = 0$$

has two roots t_1 and t_2 real and unequal, for a very small absolute value of x, equation (3) has two consecutive simple roots, t_1 and t_2; to these real values of t correspond two branches of the curve tangent to the straight lines $y = t_1x$, $y = t_2x$ (Fig. 208). If equation (4) have two imaginary roots, the two values of t which are consecutive are also imaginary, and the point M is an isolated point (Fig. 207). When equation (4) has its two roots equal to t_1, several cases can arise; if the two values of t consecutive to t_1 are imaginary for the positive and negative values of x, the point M is an isolated point; if they are real for the positive values of x, and imaginary for negative values or conversely, one has a cusp (Figs. 209 and 221); finally, if they are real for positive values, and also for negative values of x, one has two branches passing through the point M, from one side to the other, and tangent to the same straight line.

We notice that the equation which gives the various tangents at a multiple point is obtained by equating to zero the group of terms of lowest degree in equation (1).

354. 2. We proceed to define a case in which it is easy to find the form of an algebraic curve in the neighborhood of one of its points.

This point being taken as origin, one supposes that the equation in x, found by making $y = 0$ in the equation of the curve, has zero for a simple root. Let

$$\phi_0(x) + \phi_1(x)y + \phi_2(x)y^2 + \cdots + \phi_n(x)y^n + \cdots = 0$$

be the equation of the curve written in an integral form and arranged with respect to increasing powers of y. By hypothesis ϕ_0 contains x as a simple common factor; x can also be a factor of some of the coefficients following ϕ_0. Let $\phi_n(x)$ be the first coefficient which does not become zero for $x = 0$, then the equation can be written

$$x\{\psi_0(x) + y\psi_1(x) + \cdots + y^{n-1}\psi_{n-1}(x)\}$$
$$+ y^n\{\phi_n(x) + y\phi_{n+1}(x) + \cdots\} = 0.$$

If x be supposed very small, and if one of the very small values of y be considered, the sign of each of the parentheses is the same as the sign of its first term, and each of these terms, $\psi_0(x)$, $\phi_n(x)$, can be replaced by the value which it takes for $x = 0$.

Finally, in order to find the form of the curve in the vicinity of the origin, the equation being arranged with respect to increasing powers of y, it is sufficient to consider the term independent of y, which contains x as a simple factor, and the first of the terms following it whose coefficient does not become zero for $x = 0$. The question is reduced to the consideration of a binomial equation

$$Ax + By^n = 0$$

and one can similarly, in each of the coefficients A and B, neglect the part which becomes zero for $x = 0$.

355. One can therefore, in seeking the equation of the tangent to an algebraic curve and the points of inflection of this curve, employ the following method, which has the advantage of applicability to the curves whose equation is given in trilinear co-ordinates.

Consider a curve of the order m whose equation in homogeneous co-ordinates is $f(x, y, z) = 0$. Take on the curve a point $M_1(x_1, y_1, z_1)$ and in the plane a second point $M(x, y, z)$. Seek the points where the straight line M_1M, which joins these two points, intersects the curve. The homogeneous co-ordinates of a point of this straight line are (§ 330)

$$(5) \qquad x_1 + \lambda x, \quad y_1 + \lambda y, \quad z_1 + \lambda z;$$

in order that this point belong to the curve, it is necessary and sufficient that λ satisfy the equation

$$f(x_1 + \lambda x, \quad y_1 + \lambda y, \quad z_1 + \lambda z) = 0,$$

or (6) $$f(x_1, y_1, z_1) + \lambda(xf'_{x_1} + yf'_{y_1} + zf'_{z_1})$$
$$+ \frac{\lambda^2}{1 \cdot 2}(x^2f''_{x_1^2} + y^2f''_{y_1^2} + \cdots) + \cdots = 0.$$

On substituting successively all of the roots of this equation in λ into the expressions (5), one will obtain the co-ordinates of all of the points of intersection. The point (x_1, y_1, z_1)

being on the curve, one has $f(x_1, y_1, z_1) = 0$; equation (6) has therefore as a root $\lambda = 0$, which, substituted in expressions (5), gives no other than the point (x_1, y_1, z_1). In order that the straight line M_1M be tangent to the curve in M_1, it is necessary that it have two points of intersection coincident with M_1, that is, that equation (6) have the double root $\lambda^2 = 0$: the condition for which is

$$(7) \qquad xf'_{x_1} + yf'_{y_1} + zf'_{z_1} = 0.$$

If the co-ordinates of the point M satisfy this equation, the straight line M_1M is tangent at M_1: equation (7), in which one considers x, y, z as the current co-ordinates, is therefore the equation of the tangent at the point M_1.

In what precedes it has been assumed that the three derivatives f'_{x_1}, f'_{y_1}, f'_{z_1} are not zero at the same time. If these three derivatives were zero, the coefficient in λ in equation (6) would be zero, whatever be the position of the point M: every straight line passing through the point M_1 would intersect the curve in *two points at least coincident* with M_1. It is then said that the point M_1 is a *singular point.* The singular points are therefore characterized by the following property, that their co-ordinates reduce the three partial derivatives of f with respect to x, y, and z to zero, and, consequently, f, by virtue of the theorem of homogeneous functions.

Suppose that the point M_1 is not a singular point, and seek the condition in order that it be a point of inflection. For this purpose it is necessary and sufficient that the tangent at M_1 intersect the curve not in *two* but in *three* points coincident with M_1; in other words, if the point M be taken on the tangent (7), it is necessary that the coefficient λ^2 become zero, that is, that one has

$$(8) \qquad \phi(x, y, z) = x^2f''_{x_1^2} + y^2f''_{y_1^2} + z^2f''_{z_1^2} + 2\,yzf''_{y_1z_1} + 2\,zxf''_{x_1z_1} + 2\,xyf''_{x_1y_1} = 0.$$

If x, y, z be regarded as the current co-ordinates, equation (8) represents a conic; and since every system of values x, y, z satisfying condition (7) ought to satisfy at the same time condition (8), this conic should resolve itself into two straight lines one of which is a tangent. Therefore if M_1 be a point of inflection, the discriminant of the function of the second degree $\phi(x, y, z)$ ought to be zero.

$$H = \begin{vmatrix} f''_{x_1^2} & f''_{x_1y_1} & f''_{x_1z_1} \\ f''_{x_1y_1} & f''_{y_1^2} & f''_{y_1z_1} \\ f''_{x_1z_1} & f''_{y_1z_1} & f''_{z_1^2} \end{vmatrix} = 0.$$

Conversely, if at a *non-singular* point M_1 this function H be zero, this point is a point of inflection. In fact, we notice that the conic $\phi(x, y, z) = 0$ passes through the point (x_1, y_1, z_1) and is tangent to the curve. It passes through the point, because, owing to the theorem of homogeneous functions,

$$\phi(x_1, y_1, z_1) = x_1^2 f''_{x_1^2} + \cdots = m(m-1)f(x_1, y_1, z_1) = 0;$$

then, owing to the expression for ϕ, it may easily be shown that

$$x_1\phi'_x + y_1\phi'_y + z_1\phi'_z = 2(m-1)(xf'_{x_1} + yf'_{y_1} + zf'_{z_1}),$$

identically, which shows that the tangent to the conic at the point M_1 coincides with the tangent to the curve at this point. Therefore, if the conic ϕ decompose into two straight lines, one of them ought to pass through M_1 and be tangent to the curve; in other words, the polynomial $\phi(x, y, z)$ is resolvable into two factors of which one is the first member of (7), of the equation of the tangent. It follows that the point M_1 is indeed a point of inflection.

The determinant H becomes also zero when the point M_1 is a singular point; in fact, in this case the three partial derivatives of the polynomial $\phi(x, y, z)$, ϕ'_x, ϕ'_y, ϕ'_z become zero for $x = x_1$, $y = y_1$, $z = z_1$, since it follows from this that f'_{x_1}, f'_{y_1}, f'_{z_1} become zero at a singular point; the discriminant H of ϕ is therefore zero.

The determinant H is called the *Hessian;* if x_1, y_1, z_1 be regarded as current co-ordinates, equation $H = 0$ represents a curve of the order $3(m-2)$ called *Hessian* which passes through the points of inflection and the singular points of the curve $f = 0$. We shall see that, conversely, every point common to the two curves $f = 0$, $H = 0$, which is not a singular point of $f=0$, is a point of inflection. Whence one concludes that, if the curve $f=0$ do not have *singular points*, it has $3m(m-2)$ points of inflection real or imaginary. If the curve $f=0$ have singular points, the number of its points of inflection is diminished.

Thus a curve of the third degree without a singular point has nine points of inflection; if it have a double point, it has no more than three points of inflection; if it have a cusp, it has no more than one.

Example. — The curve of the third order which has in Cartesian co-ordinates the equation

$$X^3 + Y^3 + 1 = 0,$$

or in homogeneous co-ordinates

$$f(x, y, z) = x^3 + y^3 + z^3 = 0,$$

does not have singular points; because the three partial derivatives

$$f'_x = 3x^2, \quad f'_y = 3y^2, \quad f'_z = 3z^2$$

do not become zero for any system of values of x, y, z which are not all three zero. Here the Hessian is

$$H = 6^3 xyz = 0,$$

or $xyz = 0$; it decomposes therefore into three straight lines $x=0$, $y=0$, $z=0$, which are the two co-ordinate axes and the line at infinity. The nine points of intersection of the Hessian with the curve will be the points of inflection. One has, for the co-ordinates of these nine points, on calling ω a cubic imaginary root of unity,

$$x = 0 \text{ with } \frac{y}{z} = -1, \text{ or } \frac{y}{z} = -\omega, \text{ or } \frac{y}{z} = -\omega^2,$$

$$y = 0 \text{ with } \frac{x}{z} = -1, \text{ or } \frac{x}{z} = -\omega, \text{ or } \frac{x}{z} = -\omega^2,$$

$$z = 0 \text{ with } \frac{y}{x} = -1, \text{ or } \frac{y}{x} = -\omega, \text{ or } \frac{y}{x} = -\omega^2.$$

The last three points are at infinity. The three points

$$x = 0, \ \frac{y}{z} = -1; \ y = 0, \ \frac{x}{z} = -1; \ z = 0, \ \frac{y}{z} = -1$$

only are real. They lie on the straight line $x + y + z = 0$.

Class of a Curve. — The class of a curve $f(x, y, z) = 0$ is the number of tangents which can be drawn from a point of the plane to this curve. Let $M_2(x_2, y_2, z_2)$ be a given point;

on expressing that the tangent at the point $M_1(x_1, y_1, z_1)$ pass through M_2, one has the condition

$$(9) \qquad x_2 f'_{x_1} + y_2 f'_{y_1} + z_2 f'_{z_1} = 0,$$

which, combined with $f(x_1, y_1, z_1) = 0$, determines the points of contact of the tangents emanating from the point M_2. Every non-singular point M_1 whose co-ordinates satisfy these equations is a point such that the tangent at this point passes through M_2; equation (9) is, moreover, satisfied by the co-ordinates of all of the singular points, since these co-ordinates reduce f'_{x_1}, f'_{y_1}, f'_{z_1} to zero. If x_1, y_1, z_1 be regarded as current co-ordinates, equation (9) represents a curve of the order $(m-1)$, called the first polar *of the point M_2 with respect to the curve.*

Every point common to this curve and to the given curve $f = 0$ is a singular point or a point of contact of a tangent passing through the point M_2. These two curves have $m(m-1)$ common points; if the proposed curve does not have singular points, these common points are all of the points of contact of tangents drawn from M_2. Therefore, *a curve of the order m without singular points is of the class $m(m-1)$.*

If the curve have singular points, the number of tangents emanating from M_2 is equal to $m(m-1)$ less the number of the points of intersection of the polar (9) and of the curve, which are coincident with the singular points.

On supposing that the singular points of the curve consist of double points or cusps, and designating by d the number of double points, by r the number of points which are cusps, by i the number of points of inflection, and by c the class of the curve, the following formulas, due to Plücker, may be demonstrated:

$$c = m(m-1) - 2d - 3r,$$

$$i = 3m(m-2) - 6d - 8r.$$

Example. — A curve of the third order without a singular point is of the sixth class.

Take a curve of the third degree with a double point, for example a curve whose equation in Cartesian co-ordinates is

$$(10) \qquad Y^2 + X^2(X - a) = 0.$$

The tangents at the origin are given by the equation

$$Y^2 - aX^2 = 0;$$

if therefore a be different from zero, the origin is a double point at which there are two distinct tangents (real or imaginary, according as a is positive or negative); if a be zero, the origin is a cusp, and the tangent at this point is the x-axis. The equation rendered homogeneous is

$$f(x, y, z) = y^2z + x^2(x - az) = 0;$$

whence $$f'_x = 3x^2 - 2axz, \quad f'_y = 2yz, \quad f'_z = y^2 - ax^2.$$

The first polar of the point $M'(x', y', z')$ has the equation

(11) $$x'(3x^2 - 2axz) + 2y'yz + z'(y^2 - ax^2) = 0;$$

this polar is a conic passing through the double point situated at the origin $x = 0$, $y = 0$, and having a tangent at this point whose equation is

(12) $$axx' + yy' = 0.$$

If a be different from zero, the origin is a double point with distinct tangents, and the tangent (12) varies according to the position of the point M'. The polar conic (11) intersects therefore the curve of the third order in six points, two of which are coincident with the double point. There are therefore but four of these points of intersection which do not coincide with the singular point, and but four tangents can be drawn from the point M'. The curve is therefore of the fourth class.

If a be zero, the origin is a cusp; the polar conic (11) passes through this point, and the tangent to the conic at this point has the equation $y = 0$ as a tangent to the curve. The polar conic is therefore tangent to the curve at the cusp: it intersects it in three points coincident with this singular point, and in three other points only, which are the points of contact of the tangents drawn from the point M'. The curve is therefore of the third class.

One could, as an exercise, form the tangential equation of curve (10), that is, the condition that the straight line

$$uX + vY + w = 0$$

be tangent to this curve; it can be verified that this equation is of the fourth degree in u, v, w, and reduces to the third degree when a is equal to 0.

356. Curves in Trilinear Co-ordinates. — Let

$$F(\alpha, \beta, \gamma) = 0$$

be the equation in trilinear co-ordinates of a curve of the degree m. Take, on this curve, a point M_1, whose trilinear co-ordinates are α_1, β_1, γ_1, and, in the plane, a point $M(\alpha, \beta, \gamma)$. Find the points where the straight line M_1M, which joins these two points, intersects the curve. The trilinear co-ordinates of a point of this straight line are (§ 331)

$$\alpha_1 + \lambda\alpha, \quad \beta_1 + \gamma\beta, \quad \gamma_1 + \lambda\gamma;$$

in order that this point belong to the curve, it is necessary and sufficient that λ satisfy the equation

$$F(\alpha_1 + \lambda\alpha, \quad \beta_1 + \lambda\beta, \quad \gamma_1 + \lambda\gamma) =$$

or

$$F(\alpha_1, \beta_1, \gamma_1) + \lambda(\alpha F'_{\alpha_1} + \beta F'_{\beta_1} + \gamma F'_{\gamma_1}) + \frac{\lambda^2}{1 \cdot 2}(\alpha^2 F''_{\alpha_1^2} + \beta^2 F''_{\beta_1^2} + \cdots) \cdots = 0.$$

From this equation, which in every respect is similar to equation (6), one deduces results identical with those which one deduced from equation (6). We find thus that:

1° If the three partial derivatives F'_{α_1}, F'_{β_1}, F'_{γ_1} be not zero, the tangent at the point M_1 has the equation

$$\alpha F'_{\alpha_1} + \beta F'_{\beta_1} + \gamma F'_{\gamma_1} = 0;$$

2° If the three partial derivatives F'_{α_1}, F'_{β_1}, F'_{γ_1} be zero, the point M_1 is a *singular* point;

3° In order that the point M_1 be an inflection, it is necessary that the Hessian

$$H = \begin{vmatrix} F''_{\alpha_1^2} & F''_{\alpha_1\beta_1} & F''_{\alpha_1\gamma_1} \\ F''_{\alpha_1\beta_1} & F''_{\beta_1^2} & F''_{\beta_1\gamma_1} \\ F''_{\alpha_1\gamma_1} & F''_{\beta_1\gamma_1} & F''_{\gamma_1^2} \end{vmatrix} = 0:$$

and, conversely, if at a *non-singular* point M_1 the Hessian be zero, this point is a point of inflection.

EXAMPLE. — Consider the equation of the third degree

$$F(\alpha, \beta, \gamma) = \alpha^3 + \beta^3 + \gamma^3 + 6\,k\alpha\beta\gamma = 0;$$

it can be demonstrated that the equation of every curve of the third order without a singular point can be reduced to this form by a suitable choice of the triangle of reference.

Here we actually have

$$H = 6^3 \begin{vmatrix} \alpha & k\gamma & k\beta \\ k\gamma & \beta & k\alpha \\ k\beta & k\alpha & \gamma \end{vmatrix}$$

or, on developing,

$$H = 6^3 [- k^2 (\alpha^3 + \beta^3 + \gamma^3) + (1 + 2k^3)\, \alpha\beta\gamma].$$

The points of inflection will be the points of intersection, nine in number, of the given curve $F = 0$ with the Hessian $H = 0$. The equations $F = 0$ and $H = 0$ are homogeneous equations of the first degree with respect to the two expressions $\alpha^3 + \beta^3 + \gamma^3$, $\alpha\beta\gamma$: the determinant of the coefficients of these two expressions is $1 + 8k^3$; if therefore $1 + 8k^3$ be not zero, from the equations $F = 0$, $H = 0$ may be deduced the two following:

$$\alpha^3 + \beta^3 + \gamma^3 = 0, \quad \alpha\beta\gamma = 0,$$

which give, for the nine points of inflection,

$$\alpha = 0 \text{ with } \beta + \gamma = 0, \text{ or } \beta + \omega\gamma = 0, \text{ or } \beta + \omega^2\gamma = 0,$$

$$\beta = 0 \text{ with } \gamma + \alpha = 0, \text{ or } \gamma + \omega\alpha = 0, \text{ or } \gamma + \omega^2\alpha = 0,$$

$$\gamma = 0 \text{ with } \alpha + \beta = 0, \text{ or } \alpha + \omega\beta = 0, \text{ or } \alpha + \omega^2\beta = 0,$$

where ω designates an imaginary cubic root of unity. These nine points are the same whatever k may be. It is easily seen that the straight line which joins two of these points passes through a third.

If $1 + 8k^3 = 0$, the equations $F = 0$ and $H = 0$ represent the same curve ; then the Hessian coincides with the given curve : all points of this curve are points of inflection, which can only happen if it be composed of three straight lines. In fact, the equation $k^3 = - \frac{1}{8}$ gives for k three values:

$$-\frac{1}{2}, \quad -\frac{\omega}{2}, \quad -\frac{\omega^2}{2}.$$

For $k = - \frac{1}{2}$, the curve $F = 0$ becomes

$$F = \alpha^3 + \beta^3 + \gamma^3 - 3\,\alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma) = 0;$$

it is therefore resolved into three straight lines ; similarly if one put

$$k = -\frac{\omega}{2}, \quad k = -\frac{\omega^2}{2},$$

as is seen on substituting, in the identity above, for α, $\omega\alpha$ or $\omega^2\alpha$.

CHAPTER III

ASYMPTOTES.

357. When a curve has an infinite branch MN (Fig. 224), it can happen that the distance MH of a point M of this curve from a straight line CD approaches zero, when the point M is continuously removed toward infinity; in this case, the straight line CD is called an asymptote of the branch of the curve.

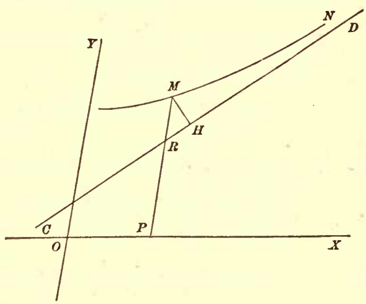

Fig. 224.

Consider the difference MR between the ordinates of the curve and of the straight line, which correspond to the same abscissa, and let β be the angle which the straight line CD makes with the y-axis; one has $MR = \frac{MH}{\sin\beta}$; if either of the quantities MH and MR approach zero, the other will also approach zero. An asymptote can therefore be defined *as a straight line such that the difference between the ordinates of the curve and of the straight line approaches the limit zero when x is indefinitely increased.*

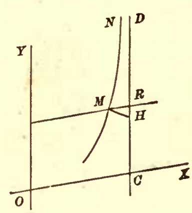

Fig. 225.

However, this definition is not applicable if the angle β be zero; that is, when the asymptote is parallel to the y-axis. In this case, if the straight line MR (Fig. 225) be drawn parallel to the x-axis, the straight line MR approaches zero, when the ordinate increases without limit. If a be the abscissa of any point of the straight line CD, the abscissa of a point M of the branch

MN approaches a when y is increased without limit, or conversely, y increases without limit, when x approaches a.

Asymptotes which are Parallel to the y-axis.

358. According to the preceding, the asymptotes of this species are obtained by seeking the finite values of x which render one of the values of y infinite. When the equation of the curve is solved with respect to y, one perceives, generally, these values at once; as examples we cite the cissoid and the strophoid discussed in §§ 20 and 23.

If the equation be algebraic, but not solvable with respect to y, we proceed in the following manner. Let m be the degree of the equation, n the largest exponent of y; the equation can be written in the form

$$\phi_0(x)y^n + \phi_1(x)y^{n-1} + \phi_2(x)y^{n-2} + \cdots = 0,$$

$\phi_0, \phi_1, \phi_2, \cdots$ representing polynomials in x, whose degrees are at most respectively equal to $m-n$, $m-n+1$, $m-n+2, \cdots$, and after dividing by y to the nth power,

$$(1) \qquad \phi_0(x) + \phi_1(x)\frac{1}{y} + \phi_2(x)\frac{1}{y^2} + \cdots + \phi_n(x)\frac{1}{y^n} = 0.$$

Suppose that a real branch *MN* be asymptotic to a straight line *CD* parallel to the axis of y and having the equation $x=a$. As the point *M* is removed to infinity on this branch, its abscissa x approaches the finite value a, while $\frac{1}{y}$ approaches zero. Since the terms of equation (1) beginning with the second approach zero, it follows that the abscissa a reduces the polynomial $\phi_0(x)$ to zero. Whence *the abscissas of asymptotes parallel to the y-axis satisfy the equation* $\phi_0(x) = 0$.

359. Conversely, let a be a real root of the equation $\phi_0(x) = 0$; it is necessary to examine, if it have real branches which approach continually the straight line $x = a$ and how many there are of such. Suppose in the first place that a be a single root of the equation $\phi_0(x) = 0$.

If x be regarded as a function of $\frac{1}{y}$ given by equation (1), when $\frac{1}{y}$ approaches zero, it being either positive or negative, one value of x, and only one, approaches a; this value of x is necessarily real; because, if it were imaginary, the conjugate root would also approach the real quantity a, which would then be a double root. One infers therefore that there exist two real branches asymptotic to the straight line $x = a$, one on the side of the positive y's, the other on the side of the negative y's.

Suppose now that a be a double root of the equation $\phi_0(x) = 0$. When $\frac{1}{y}$ approaches zero, two values of x approach a; these values can be real or conjugate imaginaries. If they be real for very small positive values of $\frac{1}{y}$, there exist two real branches asymptotic to the straight line $x = a$ on the side of the positive y's. If they be also real for very small negative values of $\frac{1}{y}$, there exist two other real branches asymptotic to the same straight line on the side of the negative y's. When the two roots are imaginary for the positive or negative values of $\frac{1}{y}$, there does not exist any real branch asymptotic to the straight line $x = a$.

In general, let p be the order of the root a; among the p values of x, which approach a when $\frac{1}{y}$ approaches zero, $p - 2q$ are real for very small positive values of $\frac{1}{y}$, $p - 2q'$ for negative values. There will be $p - 2q$ real branches asymptotic to the straight line $x = a$ on the side of the positive y's, and $p - 2q'$ real branches asymptotic to the same straight line on the side of the negative y's; in all, $2p - 2q - 2q'$ real branches asymptotic to the straight line $x = a$. It is worthy of notice that this number is even.

359. 2. In the particular case where a is a simple root of $\phi_0(x)$, it is easy to determine the nature of the curve in the neighborhood of the asymptote.

If $\phi_p(x)$ be the first of the coefficients which does not become zero for $x = a$, the equation of the curve can be written

$$(x-a)\left\{\psi_0(x)+\frac{1}{y}\psi_1(x)+\cdots+\frac{1}{y^{p-1}}\psi_{p-1}(x)\right\}$$
$$+\frac{1}{y^p}\left\{\phi_p(x)+\frac{1}{y}\phi_{p+1}(x)+\cdots\right\},$$

$\psi_0(x)$, $\psi_1(x)\cdots\psi_{p-1}$ designating integral polynomials in x, the first of which does not become zero for $x = a$.

Now, if one assign to x a neighboring value of a, and if one consider one of the very small p roots of the equation in $\frac{1}{y}$, the sign of each of the parentheses is the same as the sign of its first term; moreover, $\psi_0(x)$ and $\phi_p(x)$ have respectively the same sign as $\psi_0(a)$ and $\phi_p(a)$.

After this, it is sufficient to consider the binomial equation

$$(x-a)\psi_0(a)+\frac{1}{y^p}\phi_p(a)=0,$$

and to suppose successively x a little less, and then a little greater than a.

(We have already employed an analogous process, § 354. 2.)

Thus, when the first term of the equation, arranged with respect to the increasing powers of $\frac{1}{y}$ has a simple root a, the equation can be reduced to this term, and to the first of the following terms whose coefficient does not become zero for $x = a$. One can also replace x by a in the factors which do not become zero for $x = a$.

360. We have, thus far, in equation (1) regarded x as a function of $\frac{1}{y}$; one can, on the contrary, consider $\frac{1}{y}$ as a function of x. Suppose then that the real root a of the polynomial $\phi_0(x)$ does not reduce $\phi_1(x)$ to zero; as x approaches a, one value only of $\frac{1}{y}$ approaches zero, and this value is of necessity real. There exist, therefore, two real branches, asymptotic to the same straight line $x = a$, and one of them is given by values of x less than a, the other by values greater than a; these two branches are situated on opposite sides of the asymptote.

In order to determine their position, allow x to vary from $a-h$ to $a+h$, assuming h to be sufficiently small so that, in this interval, the equation $\phi_0(x)=0$ has only the root a, and the polynomial $\phi_1(x)$ does not become zero; it can be supposed, moreover, that h, and consequently $\frac{1}{y}$, be sufficiently small in absolute value, in order that, when the simultaneous values which correspond to a point of one of the infinite branches be assigned to x and $\frac{1}{y}$, the value of the polynomial

$$\phi_1(x)\frac{1}{y}+\phi_2(x)\frac{1}{y^2}+\cdots+\phi_n(x)\frac{1}{y^n}$$

has always the sign of its first term $\phi_1(x)\frac{1}{y}$, but from equation (1), the value of this polynomial is equal to $-\phi_0(x)$; it follows that the two quantities $\phi_1(x)\frac{1}{y}$ and $-\phi_0(x)$ have the same sign, and consequently that $\frac{1}{y}$ has the same sign as $\frac{-\phi_0(x)}{\phi_1(x)}$. When x varies from $a-h$ to $a+h$, the de-

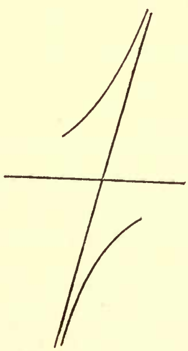

Fig. 226.

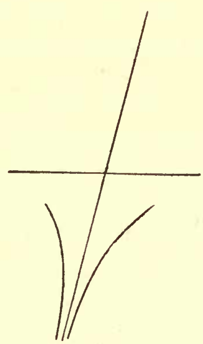

Fig. 227.

nominator $\phi_1(x)$ preserves the same sign; if a be a simple root, or more generally a root of even order, of the equation

$\phi_0(x) = 0$, the numerator $\phi_0(x)$ changes its sign for $x = a$; the value of y changes its sign also, and the two branches have opposite directions, one approaching one extremity of the asymptote and the other approaching the other extremity (Fig. 226), as in case of the hyperbola. When a is a root of even order, the numerator preserves the same sign, so also does y; the two branches are directed toward the same extremity of the asymptote (Fig. 227).

Suppose now that a reduces the successive polynomials ϕ_1, $\phi_2 \cdots \phi_{p-1}$ to zero. As x approaches a, p values of $\frac{1}{y}$ approach zero; of these values, $p - 2q$ are real for values of x less than a, $p - 2q'$ for values of x greater than a; there will be therefore $2p - 2q - 2q'$ real branches asymptotic to the straight line $x = a$.

EXAMPLE I. — Consider the curve defined by the equation

$$x^4y^4 + (x^2 - 4)(y - x)^4 = 0,$$

which, expanded, may be written

$$(x^4 + x^2 - 4)y^4 - 4x(x^2 - 4)y^3 + 6x^2(x^2 - 4)y^2 - 4x^3(x^2 - 4)y + x^4(x^2 - 4) = 0.$$

The biquadratic equation

$$\phi_0(x) = x^4 + x^2 - 4 = 0$$

has two real simple roots with contrary signs,

$$x = \pm\sqrt{\frac{\sqrt{17} - 1}{2}} = \pm a.$$

Since these values of x do not reduce $\phi_1(x)$ to zero, each of the straight lines $x = \pm a$ is asymptotic to two real branches situated on the opposite sides of the straight line and directed towards its two extremities.

EXAMPLE II. — Consider the curve $(x - 1)^2y^2 + 4 - x^2 = 0$.

The equations $\phi_0(x) = 0$ becomes $(x - 1)^2 = 0$. This equation has the double root $x = 1$. When x approaches unity, the two values of y are imaginary; the straight line $x = 1$ is not therefore asymptotic to a real branch.

Asymptotes which are not Parallel to the y-axis.

361. Let us consider an infinite branch MN of the curve (Fig. 228) which has an asymptote CD that is not parallel to the y-axis; such an asymptote has the equation

(1) $$y_1 = cx + d,$$

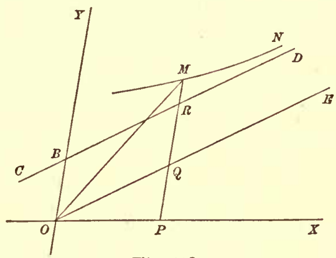

Fig. 228.

c and d being two unknown constants which are to be determined. Let y and y_1 be the ordinates of the branch of the curve and of the straight line which corresponds to the same abscissa, and δ the difference $y - y_1$, that is MR; according to the definition, δ is a function of x whose limit is zero when x is indefinitely increased. The infinite branch of the curve which we consider is therefore represented by the equation

(2) $$y = y_1 + \delta = cx + d + \delta.$$

The equation of a branch of the curve can often be easily put under the preceding form, and then the asymptote is found as follows. Let, for example, $y = \dfrac{F(x)}{f(x)}$ be the equation, in which $f(x)$ and $F(x)$ represent two integral polynomials in x, the first of the degree m, the second at most of the degree $m+1$. To each real root of the equation $f(x) = 0$ correspond two real infinite branches, asymptotic to the same straight line parallel to the y-axis, situated on opposite sides of the straight line, and directed toward its opposite extremities or toward the same extremity, according as the root a is of an odd or even order. There are, moreover, two other infinite branches which are obtained by assigning very large positive or negative values to x. If the division be effected, one obtains, on arranging the equation with respect to decreasing powers of

x, an integral quotient $cx+d$, which is at most of the first degree, whence one has

$$y = cx + d + \frac{\phi(x)}{f(x)},$$

$\phi(x)$ being an integral polynomial of a degree less than m; since this last fraction approaching the limit zero as x is indefinitely increased, it follows that the straight line $y_1 = cx + d$ is asymptotic to the two branches which we consider.

We shall cite in addition, as an example, the transcendental curve

$$y = x + \frac{1}{a^x},$$

which has an infinite branch situated in the angle YOX and asymptotic to the straight line $y = x$.

362. In general, the asymptotes cannot be found so easily. Let us return to equation (2). We find

$$c = \frac{y}{x} - \frac{d + \delta}{x}.$$

Since d has a finite value, and δ approaches zero when x increases indefinitely, one has

(3) $$c = \text{limit of } \frac{y}{x}.$$

The angular coefficient of the asymptote is equal to the limit which the ratio $\frac{y}{x}$ *approaches, when* x *increases without limit.*

The ratio $\frac{y}{x}$ being the angular coefficient of the straight line OM, the relation (3) shows that this straight line approaches as a limiting position OE parallel to the asymptote CD, when the point M is removed to infinity on the branch MN. The same equation gives $d = y - cx - \delta$, whence

(4) $$d = \text{the limit of } (y - cx).$$

The ordinate at the origin of the asymptote is equal to the limit of the difference $y - cx$, *when* x *increases without limit.*

The quantity $y - cx$ being the ordinate MQ of the curve intercepted by the straight line OE parallel to the asymptote, relation (4) shows that this ordinate approaches a limit OB, when the point M is removed to infinity on the branch MN.

The two relations (3) and (4) determine the asymptotes which are not parallel to the y-axis.

Suppose that the equation is solved with respect to y, and consider a determination of y which gives a real infinite branch, when x is increased without limit. We take, for this branch, the ratio $\frac{y}{x}$; if this ratio does not approach a finite limit, the branch does not have an asymptote. If the ratio approach a finite limit c, the difference $y - cx$ is considered; when this difference does not approach a finite limit, the branch does not have an asymptote; if, on the contrary, it approach a finite limit d, one will have $y - cx = d + \delta$, where δ approaches zero as x increases without limit; therefore the straight line $y_1 = cx + d$ will be an asymptote of the branch under consideration.

Example I. — Construct the curve

$$y = \pm x\sqrt{\frac{x-1}{x-2}},$$

referred to rectangular axes of co-ordinates. The x-axis is the axis of the curve. When x varies from o to unity, y remains finite; it is in the first place zero, then increases and becomes zero again; whence we get the oval OAO (Fig. 229). As x varies from 1 to 2, y is imaginary. When x becomes greater than 2 by a small quantity, y is real and very large; if, therefore, OB be taken equal to 2, and GG' be drawn parallel to OY, this straight line will be asymptotic to two branches of the curve.

As x increases from 2, y begins to diminish, and finally becomes very large, when x is large; thus the two branches CND and $C'N'D'$ are obtained. When x is negative, y is always real; as x varies from 0 to $-\infty$, the numerical value of y varies uniformly from 0 to ∞, and so one gets the two branches OE, OE'.

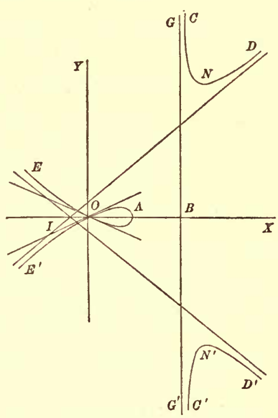

Fig. 229.

Consider, for example, ND, one of the infinite branches; this branch is found by taking the + sign in the equation, and on supposing that x is positive and very large. We have

$$\frac{y}{x}=\sqrt{\frac{x-1}{x-2}};$$

the limit of $\frac{y}{x}$ is unity. Moreover, one has

$$y-x=x\left(\sqrt{\frac{x-1}{x-2}}-1\right)=x\frac{\sqrt{x-1}-\sqrt{x-2}}{\sqrt{x-2}},$$

and, on multiplying the two terms by the sum of the radicals

$$y-x=\frac{x}{\sqrt{x-2}\,(\sqrt{x-1}+\sqrt{x-2})},$$

the limit is $\frac{1}{2}$; therefore the straight line $y=x+\frac{1}{2}$ is an asymptote of the branch considered. On dividing both terms of the fraction by x, one sees that the difference $y-x$ is greater than $\frac{1}{2}$, and, consequently, that the ordinate of the curve is greater than that of the asymptote; consequently one infers that the branch ND is situated above the asymptote. One would discover in a similar manner that the branch OE' has the same asymptote, and that it lies above the branch. The two branches $N'D'$ and OE have as asymptote a straight line which is the *symétrique* of the preceding with respect to the x-axis.

Example II. — Consider the curve $y^4-y^3x+x^3-2x^2y=0$, constructed in § 342. We have expressed the two co-ordinates x and y in terms of the auxiliary variable $t=\frac{y}{x}$. The two branches OA and OB, which are found by allowing t to approach zero, do not have an asymptote, since y becomes infinite. The two infinite branches OC and OD are obtained by making t approach unity. We have, for these branches, the limit of $\frac{y}{x}=1$; test whether the difference $y-x$ has a limit. The formulas, by means of which x and y are expressed in terms of t, give

$$y-x=(t-1)x=\frac{2t-1}{t^3};$$

this difference approaches unity, when t approaches 1. Whence it follows that the two branches under consideration have the straight line $y=x+1$ as an asymptote. The difference δ has the value

$$\delta=\frac{(1-t)(t^2+t-1)}{t^3};$$

when t varies from 1 to $+\infty$, the difference is negative and the branch OD is situated below the asymptote. The polynomial t^2+t-1 has the roots $t' = \frac{-1+\sqrt{5}}{2}$, $t'' = \frac{-1-\sqrt{5}}{2}$; when t varies from $\frac{1}{2}$ to t', δ is negative, and the arc OE is below the asymptote; as t varies from t' to 1, δ becomes positive, and the arc EC crosses to the other side of the asymptote. The other root t'' gives the point F where the branch OA intersects the asymptote.

363. Let us consider now the case when the equation supposed algebraic and integral is not solvable with respect to the variable y. Collect terms of the same degree; represent by $\phi(x, y)$ the ensemble of terms of the highest degree m, by $\psi(x, y)$ the ensemble of terms of the degree $m-1$, by $\chi(x, y)$ the terms of the degree $m-2, \ldots$; the equation may be written

$$(5) \qquad f(x, y) = \phi(x, y) + \psi(x, y) + \chi(x, y) + \cdots = 0.$$

Represent the ratio $\frac{y}{x}$ by u, and substitute ux for y in equation (5); the polynomial $\phi(x, y)$, being homogeneous and of the degree m, will contain x^m as a common factor in all of its terms, and it will follow that $\phi(x, y) = x^m\phi(1, u)$, or, for brevity, $x^m\phi(u)$. Similarly the polynomials $\phi(x, y)$, $\chi(x, y), \ldots$ will become $x^{m-1}\psi(u)$, $x^{m-2}\chi(u), \ldots$. The equation connecting x and u will therefore be

$$x^m\phi(u) + x^{m-1}\psi(u) + x^{m-2}\chi(u) + \cdots = 0,$$

and, after dividing by x^m,

$$(6) \qquad \phi(u) + \frac{1}{x}\psi(u) + \frac{1}{x^2}\chi(u) + \cdots = 0.$$

Suppose that a real branch MN (Fig. 228) be asymptotic to a straight line CD which is not parallel to the y-axis. When the point M is removed to infinity on this branch, u approaches a finite limit c, while $\frac{1}{x}$ approaches zero. Since the terms of equation (6), beginning with the second, approach zero, it follows that the value $u = c$ annuls the polynomial $\phi(u)$. Thus *the angular coefficients of the asymptotes satisfy the equation* $\phi(u) = 0$.

Take now $y - cx = v$, whence $u = \frac{y}{x} = c + \frac{v}{x}$. On substituting this value for u, and developing each term, equation (6) becomes

(7)
$$\left.\begin{aligned} &\phi(c)+\phi'(c)\frac{v}{x}+\frac{\phi''(c)}{1\cdot 2}\cdot\frac{v^2}{x^2}+\cdots \\ &+\psi(c)\cdot\frac{1}{x}+\psi'(c)\cdot\frac{v}{x^2}+\cdots \\ &+\chi(c)\cdot\frac{1}{x^2}+\cdots \end{aligned}\right\}=0.$$

Since $\phi(c)=0$, if one multiply by x, it takes the form

(8)
$$[v\phi'(c)+\psi(c)]+A\frac{1}{x}+B\frac{1}{x^2}+\cdots=0.$$

When the point M is removed to infinity on the branch MN, v approaches a finite limit d whilst $\frac{1}{x}$ approaches zero. The terms of equation (8), beginning with the second, approach zero; it follows that the value $v=d$ reduces the first term $v\phi'(c)+\psi(c)$ to zero. If c be a simple root of the equation $\phi(u)=0$, the quantity $\phi'(c)$ being different from zero, one obtains the following finite value for d:

(9)
$$d=-\frac{\psi(c)}{\phi'(c)}.$$

364. Conversely, let c be a simple real root of the equation $\phi(u)=0$; consider the corresponding finite value d given by equation (9), and construct the straight line CD, whose equation is $y=cx+d$. Owing to equation (8), when $\frac{1}{x}$ approaches zero, one value of v, and only one, approaches d; this value, necessarily real, represents the ordinate MQ, intercepted between the parallels OE and CD; it follows that there are two real branches asymptotic to the straight line CD, one on the side of the positive and the other on the side of the negative x's (Fig. 230).

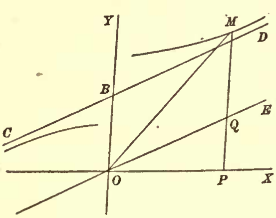

Fig. 230.

Suppose that c be a root of the pth order of $\phi(u)$ and does not reduce $\psi(u)$ to zero; according to equation (8), when $\frac{1}{x}$

approaches zero, each value of v becomes infinite; because, if one value of v preserves a finite value, the coefficients $A, B, \cdots$ would remain finite, and equation (8) would reduce to $\psi(c) = 0$, which contradicts the hypothesis. Owing to equation (6), when $\frac{1}{x}$ approaches zero, p values of u become equal to c; among these p values, $p - 2q$ are real for very small positive values of $\frac{1}{x}$, $p - 2q'$ for negative values. Draw the straight line OE, whose angular coefficient is c. To each real value of u there corresponds a straight line OM, making a very small angle with OE; the point M, in which this straight line intersects the parallel to the y-axis with the abscissa x, belongs to the curve; when $\frac{1}{x}$ approaches zero, the ordinate $v = MQ$ becomes infinitely large in absolute value, the branch described by the point M does not have an asymptote, and is similar to a branch of a parabola. There corresponds an even number $2p - 2q - 2q'$ parabolic branches to the direction c.

Discuss the case when c is a double root of the equation $\phi(u) = 0$, and annuls $\psi(c)$. Equation (7) becomes

$$(10) \qquad \left[\frac{\phi''(c)}{1.2} v^2 + \psi'(c) v + \chi(c)\right] + \frac{B}{x} + \frac{C}{x^2} + \cdots = 0.$$

When $\frac{1}{x}$ approaches zero, two values of v approach finite limits which are roots of the equation

$$(11) \qquad \frac{\phi''(c)}{1.2} v^2 + \psi'(c) v + \chi(c) = 0.$$

If the two roots d and d' of this equation be real and unequal, one value of v approaches d, when $\frac{1}{x}$ approaches zero; it is real and furnishes two real branches asymptotic to the straight line $y = cx + d$. The value of v which approaches d' furnishes in a similar manner two real branches asymptotic to the straight line $y = cx + d'$, parallel to the first. If the roots of equation (11) were equal, one could no longer make the preceding deduction; in this case one would introduce a new transformation by putting $v = d + w$.

364. 2. It is easy to find the position of the curve in the neighborhood of an asymptote, when the ordinate d of this asymptote corresponding to the origin is a simple root of the first term of the equation in $\frac{1}{x}$.

For this purpose reduce the equation in $\frac{1}{x}$ to its first term and to the first of the terms following whose coefficient does not become zero for $v = d$. One can afterwards replace v by d in the factors which do not become zero, for $v = d$ (see § 359. 2).

365. Remarks. — We have seen that a simple root c of the equation $\phi(u) = 0$ gives two real branches asymptotic to the straight line CD, which has the equation $y = cx + d$. If a certain value be assigned to v, equation (8), in which $\frac{1}{x}$ is regarded as unknown, will determine the points of intersection of the curve and of a straight line $y = cx + v$ parallel to the asymptote. Equation (7) being of the degree m with respect to $\frac{1}{x}$, equation (8) is of the degree $m - 1$; whence it follows that a parallel to the asymptote intersects the curve at most in $m - 1$ points. If the particular value d be assigned to v, the equation is depressed to the degree $m - 2$; the asymptote intersects the curve at most in $m - 2$ points.

Consider next the case when c is a double root and annuls $\psi(c)$; equation (10) being of the degree $m - 2$ with respect to $\frac{1}{x}$, a straight line parallel to $y = cx$ intersects the curve at most in $m - 2$ points. If the roots of equation (11) be real and unequal, the two asymptotes both intersect the curve at most in $m - 3$ points.

Example. — Let the curve be, $y^4 - y^3x + x^3 - 2x^2y = 0$, constructed as in § 342. One has $\phi(u) = u^4 - u^3 = u^3(u - 1)$, $\psi(u) = 1 - 2u$. The equation $\phi(u) = 0$ has a triple root zero and a simple root 1. The simple root $c = 1$, with the corresponding value for $d = 1$, gives a straight line $y = x + 1$ asymptotic to two real branches. The triple root will furnish asymptotes parallel to the x-axis; but it is plain, from the equation of the curve, that none of the values of y approach a finite limit, when x is increased indefinitely.

Equation (8) in this case becomes

$$(v-1)+(3v^2-2v)\frac{1}{x}+3v^3\frac{1}{x^2}+v^4\frac{1}{x^3}=0.$$

If we put $v=1$, we obtain an equation of the second degree,

$$1+\frac{3}{x}+\frac{1}{x^2}=0,$$

which gives the two points E and F in which the asymptote intersects the curve (Fig. 216).

366. It is an easy problem to reduce the investigation of the infinite branches of an algebraic curve to the study of the finite branches of an algebraic curve of the same degree. Let x and y be the co-ordinates of any point M of the figure for the first curve; let the point M' whose co-ordinates are x' and y' correspond to the point M, and the co-ordinates x' and y' expressed in terms of the co-ordinates of M be

$$x'=\frac{1}{x},\quad y'=\frac{y}{x},$$

from which follows, conversely,

$$x=\frac{1}{x'},\quad y=\frac{y'}{x'}.$$

If the point M describe a straight line $Ax+By+C=0$, the point M' describes the straight line $Cx'+By'+A=0$; the angular coefficient of each of the straight lines is equal to the intercept of the other on the y-axis. More generally, if the point M describe a curve of the degree m, the point M' describes a corresponding curve of the same degree; to a secant passing through two neighboring points of one of the curves corresponds a secant passing through two neighboring points of the other curve, and, consequently, to a tangent there corresponds a tangent. We can assume that the first curve is referred to axes in such a manner that the equation involves a term in y^m; then the infinite branches are obtained by making x increase without limit, and all the values of the ratio $\frac{y}{x}$ approach finite limits. Whence, if the point M describe an infinite branch of the first curve, since x' approaches zero and y' a finite value c, the point M' will describe a branch intersecting the y-axis at a point A' whose ordinate is c (Fig. 231). In this way the study of infinite branches of the first curve is reduced to the investigation of branches of the second curve in the neighborhood of points situated on a y-axis.

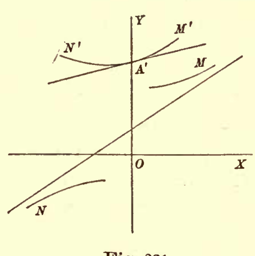

Fig. 231.

Let A' be a point in which the second curve intersects the y-axis; call d the angular coefficient of the tangent at this point;

for a point M' consecutive to A' one has $\frac{y'-c}{x'} = d + \delta$, δ approaching zero with x'; the branch $A'M'$ of the second curve is therefore represented by the equation $y' = c + dx' + \delta x'$; to this branch corresponds an infinite branch of the first curve whose equation is $y = cx + d + \delta$, δ approaching zero when x is increased without limit; to the line $y' = c + dx'$ tangent to the second curve corresponds the asymptote $y = cx + d$ of the first. We know that an even number of branches having the same tangent (§ 353) emanate always from the point A'; the first curve possesses therefore an even number of infinite branches having the same asymptote. Since the tangent at A' is the limit of the tangent at M', it follows that the asymptote is the limit of the tangent at the point M, when this point is removed to infinity.

Suppose, for example, that the point A' be an ordinary simple point, as indicated in Fig. 231; according to the sign of δ one sees that to the branch $A'M'$ there corresponds an infinite branch M situated above the asymptote to the right, and to the branch $A'N'$ a second infinite branch N situated below the asymptote to the left. If there be an inflection at A' (Fig. 232), the two infinite branches would be situated both on the same side of the asymptote, one to the right, the other to the left; in this case it is said that the curve has a point of inflection at infinity. If the point A' be a double point with distinct tangents, there will be two parallel asymptotes to each of which there corresponds two infinite branches having one of the preceding positions. If the point A' be a cusp, there will be two branches asymptotic to the same straight line, but towards the same extremity.

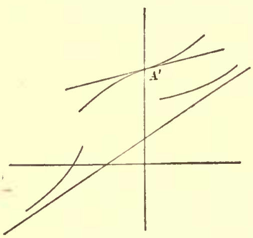

Fig. 232.

It has been assumed thus far that the tangent at A' does not coincide with the y-axis; if this were the case, to the branches which emanate from the point A' correspond, in the first figure, infinite branches without asymptotes. The direction of the tangent at the point M has as its limit the direction determined by the angular coefficient c; but it is removed to infinity. The name *parabolic branches* is given to such infinite branches. If the point A' be an ordinary simple point, the two infinite branches have the same directions as the branches of the ordinary parabola. If A' be a point of inflection, the two branches have opposite directions. The curve represented by the equation $y^3 = x$ presents this arrangement; it is composed of two infinite branches without asymptotes, and directed, one toward the positive x's, the other toward the negative x's.

367. Since to each value of x there correspond at most m real values of y, the first curve has at most m infinite branches on the side of the positive x's, and m infinite branches on the side of the negative x's; this is, moreover, a consequence of the fact that the second curve is intersected at most in m points by the y-axis. Since the number of tangents to the second curve at these points of intersection is at most equal to m, the first curve has at most m asymptotes.

The straight lines drawn from the point A' transform into straight lines parallel to the asymptote. If the point A' be a simple point (§ 354), a secant drawn from this point intersects the curve in $m-1$ additional points; therefore every straight line parallel to an asymptote intersects the first curve in $m-1$. The tangent at A' does not intersect the curve in more than $m-2$ other points; and, consequently, the asymptote does not intersect the first curve in $m-2$ points. If A' be a point of inflection, since the tangent intersects the second curve in three points, which are coincident with A' (§ 345), the asymptote will have three points of intersection at infinity; and, consequently, will not intersect the curve in more than $m-3$ points.

368. The transformation which we have made is equivalent to taking the perspective of the figure on a plane. Consider two figures situated one in the horizontal plane, the other in the vertical plane, intersecting the horizontal plane in the line LT (Fig. 233), and in such a way that

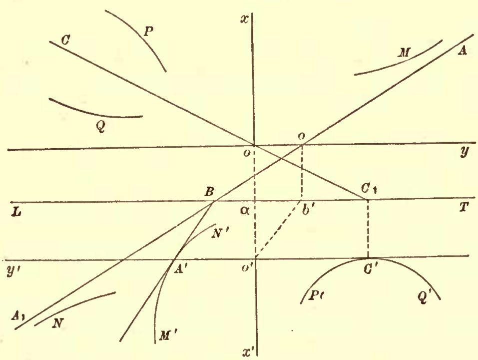

Fig. 233.

one is the perspective of the other, the eye being placed at the point whose projections are o and o'. It is evident that, when a point M is removed to infinity in the vertical plane, its perspective M' falls upon the straight line $o'y'$ parallel to *la ligne de terre* (LT); the study of the infinite branches of the curve situated in the vertical plane is thus reduced to the study of the other curves in the neighborhood of points situated on the straight line $o'y'$.

If one of the curves be referred to the axes ox and oy, the other to the axes $o'x'$ and $o'y'$, one has the formulas of transformation $x' = \frac{ab}{x}$, $y' = \frac{by}{x}$, in which a and b represent the distances ao' and ao. These formulas are identical with those which have been used above when one puts $a=b=1$.

Let A' be a point in which the second curve intersects the straight line $o'y'$; the straight line $A'B$ tangent at this point has as perspective the straight line A_1A; to the two branches M' and N', which start from the point A', correspond two infinite branches M and N, asymptotic to AA_1.

When the tangent at A' coincides with the straight line $o'y'$, as is the case at the point C', the corresponding asymptote is situated at infinity, and the two branches P' and Q' give birth to two infinite parabolic branches P and Q; the straight line drawn from the point o to a point of either of the branches P and Q approaches the limiting direction oC_1.

369. Transcendental curves can intersect their asymptotes in an infinitude of points.

For example, the curve $y = \frac{3 \sin x}{x}$ oscillates perpetually from one side to the other of the straight line OX, to which it is asymptotic, since the value of y has the limit zero (Fig. 234). The oscillations have a constant

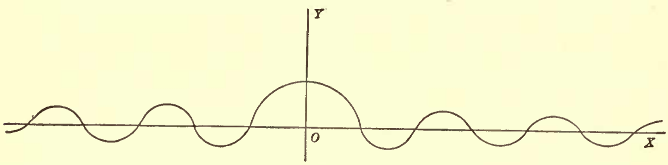

Fig. 234.

amplitude which is equal to π.

The curve $y = \frac{\sin x^2}{x}$ oscillates perpetually from one side to the other of its asymptote OX; but in this case the amplitude of oscillation diminishes continually (Fig. 233).

We have seen (§ 360) that, in algebraic curves, if an infinite branch have an asymptote, the tangent approaches a limiting position, which is

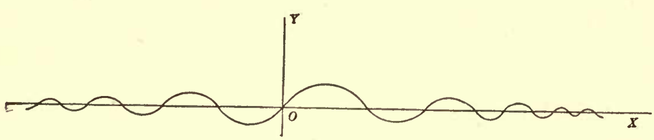

Fig. 235.

the asymptote itself, when the point of contact is removed to infinity on the branch of the curve. But this is not true in general in case of transcendental curves; thus, in the preceding example the angular coefficient of the tangent,

$$y' = 2\cos x^2 - \frac{\sin x^2}{x^2},$$

does not approach a limit; because the second term approaches zero, and the first oscillates between -2 and $+2$.

It can often happen that two infinite branches do not have a rectilinear asymptote, and notwithstanding the difference of their ordinates approaches zero; in this case, it is said that the two curves are asymptotic to each other; if one of them be a well-known simple curve, it would serve to trace the other. Consider the equation $y = \frac{F(x)}{f(x)}$, and suppose that the degree of the numerator is at least two units greater than that of the denominator; the ratio $\frac{y}{x}$ increasing without limit with x, the branches which correspond to very large values of x, positive or negative, do not have rectilinear asymptotes. If the division of the numerator by the denominator be effected, we have

$$y = ax^n + bx^{n-1} + \cdots + k + \frac{p(x)}{f(x)},$$

and the two branches considered are asymptotic to the curve

$$y = ax^n + bx^{n-1} + \cdots + k.$$

When $n = 2$, the second curve is a parabola. For example, the curve

$$y = \frac{x^3+1}{x} = x^2 + \frac{1}{x}$$

is asymptotic to the parabola $y = x^2$ (Fig. 236).

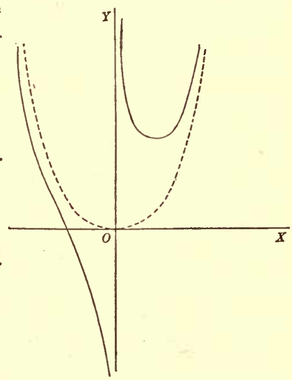

Fig. 236.

CHAPTER IV

CONSTRUCTION OF CURVES IN POLAR CO-ORDINATES.

370. Polar co-ordinates have been defined in § 3; in this system, any point, taken at random in the plane, can be determined by a value of ω comprised between 0 and 2π, and by a positive value of ρ; however each of the co-ordinates ω and ρ may vary from $-\infty$ to $+\infty$.

We have seen (§ 263) that if one of the foci of the hyperbola be taken as pole, its two branches are represented by two distinct equations, when we confine ourselves to positive radii vectores; moreover, one of the equations is sufficient, if negative radii vectores be allowed, on agreeing to measure the absolute value of each of them in the direction opposite to that indicated by the value of ω. We have also seen that this convention makes it possible to represent the limaçon of Pascal by a single equation (§ 27).

371. Spiral of Archimedes. — A point M has a uniform motion in the direction $G'G$ on an indefinite straight line $G'OG$ which revolves with a uniform motion about one of its points O. The curve described by the point M is the spiral of Archimedes (Fig. 237).

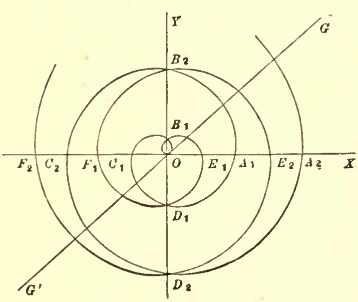

Fig. 237.

Take OX for the polar axis, the direction which the straight line OG has when the movable point M passes through O, and reckon positive polar angles in the direction of rotation of the straight line; let a be the distance which the movable point has advanced on the straight line, while it has made a complete revolution. If the variable point be considered in any of its positions after its passage through O, on calling ω the angle through which the direction OX has revolved in order to coincide with OG, and ρ the distance of the variable point from the point O, one has on putting $\frac{a}{2\pi} = b$,

$$(1) \qquad \frac{\rho}{a} = \frac{\omega}{2\pi}, \text{ or } \rho = a\,\frac{\omega}{2\pi} = b\omega.$$

Let us consider the movable point before it passes through O; call ω_1 the absolute value of the angle through which it is necessary to revolve the direction OX in order to make it coincide with OG; the point M being situated on OG', the prolongation of OG, the radius vector should be regarded as negative, and one has $\frac{-\rho}{a} = \frac{\omega_1}{2\pi}$. If one regard those angles measured in a direction opposite to the first as negative, one will have $\omega = -\omega_1$, and the preceding relation is identical with equation (1), which represents an indefinite curve. The values of ω comprised between 0 and 2π, 2π and 4π, $\cdots$, 0 and -2π, $\cdots$, give the successive helices. If we confine ourselves to the positive values of ρ and to the values of ω comprised between 0 and 2π, it would be necessary to employ a particular equation in order to represent each of the helices,

$$\rho = b\omega,\ \rho = a + b\omega, \cdots,\ \rho = a - b\omega,\ \rho = 2a - b\omega, \cdots.$$

The spiral of Archimedes is composed of two parts

$$OB_1C_1D_1A_1B_2 \text{ and } OB_1E_1D_1F_1B_2\cdots,$$

symétriques of one another with respect to OY perpendicular to the polar axis. Each portion embraces an infinitude of helices, and the portions of any straight line drawn through the pole and comprised between two consecutive helices all have the length a.

372. REMARK I.—Any point M of the plane can be defined by an infinitude of pairs of values of ρ and ω. If α represent a positive angle less than 2π which the direction OM makes with the axis OX, and a the distance OM (Fig. 238), one can select as co-ordinates of the point M the pairs of values comprised in the two formulas

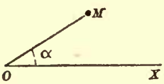

Fig. 238.

$$\rho = +a, \quad \omega = \alpha + 2k\pi,$$

$$\rho = -a, \quad \omega = \alpha + (2k+1)\pi,$$

where k is any integral number. If the point M belong to a curve defined by an equation $f(\omega, \rho) = 0$, its co-ordinates can be discovered merely by inspection of the point; it is necessary, in order to obtain them, to follow the trace of the curve.

373. Remark II. — In the formulas of transformation established in Book I., Chapter IV., we have supposed the point M determined by a positive radius vector and by a polar angle comprised between 0 and 2π. Taking in the first place the radius vector positive, one can choose as polar angle any of the angles which have the direction OM with the axis OX (Fig. 239), on agreeing to assign to the angle the + sign or − sign, according as the straight line starting with the direction OX takes the direction OM by revolving from OX toward OY, or in the opposite direction. This results in increasing or diminishing the angle represented originally by ω, by a multiple of 2π; since the sine and the cosine do not change, the formulas remain the same. Suppose now that the point M is defined by a negative radius vector; the angle ω will be one of the angles formed by the direction OM' with OX. Since the projection of OM on OX is equal to $(-\rho)\cdot\cos(\pi+\omega)$ or $\rho \cos\omega$, one has still $x = \rho\cos\omega$, and similarly $y = \rho\sin\omega$. Therefore the formulas are general.

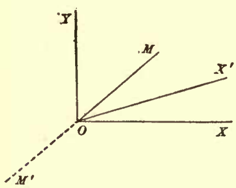

Fig. 239.

When the polar axis OX' does not coincide with OX, the position of this axis is defined by the angle α which it makes with OX. If ω in the formulas $x = \rho\cos\omega$, $y = \rho\sin\omega$, which are referred to the polar axis OX, be replaced by $\omega' + \alpha$, they become $x = \cos(\omega' + \alpha)$, $y = \rho \sin(\omega' + \alpha)$.

Suppose now that the axes of co-ordinates be oblique and take OX for the polar axis (Fig. 240); the formulas of transformation are obtained by projecting the two paths OM and OPM successively on a perpendicular to OX and on a perpendicular to OY, which gives

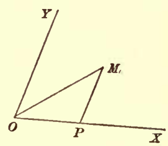

Fig. 240.

$$x = \frac{\rho \sin(\theta - \omega)}{\sin\theta}, \quad y = \frac{\rho \sin \omega}{\sin\theta}.$$

374. Remark III. — In the case where the entire curve is obtained by varying ω from 0 to 2π, one perceives the symmetry of the curve with respect to the polar axis OX, if the values α and $2\pi - \alpha$ assigned to ω give the same value for ρ, or if the values α and $\pi - \alpha$ give values for ρ, equal and contrary in signs. Similarly, the symmetry of the curve with respect to the perpendicular OY is seen, if the angles α and $\pi - \alpha$ give the same value for ρ, or the angles α and $2\pi - \alpha$ give values for ρ equal and contrary in sign. Finally, the symmetry of the curve with respect to the pole may be seen, if the angles α and $\pi + \alpha$ give the same value for ρ, or if to the same angle α there correspond two values of ρ, equal and contrary in signs.

But if, in order to obtain the entire curve it be necessary to give ω values greater than 2π, the symmetry of the curve may be discovered in another manner. For example, if it be necessary to vary ω from 0 to 4π, the symmetry with respect to the polar axis will exist, if the angles α and $2\pi - \alpha$, or α and $4\pi - \alpha$, give equal values to ρ, and moreover if the angles α and $\pi - \alpha$, or α and $3\pi - \alpha$, give values to ρ, equal and contrary in sign. If the limits of ω be farther extended, the number of comparisons is increased.

Consider, for example, the curve defined by the equation $\rho = \cos\frac{\omega}{2}$. If ω be increased two times 2π, the direction of the radius vector remains the same; moreover, if $\frac{\omega}{2}$ be increased by multiples of 2π, ρ takes the same value and one finds a point previously known; it is sufficient therefore that ω vary from 0 to 4π. If ω be increased by 2π, the radius vector returns to the same direction; but $\frac{\omega}{2}$ being increased only by π, ρ takes the same numerical value with a change in sign; it follows that the portion of the locus given by the values of ω comprised between 2π and 4π is the *symétrique* with respect to the pole of the portion given by the values of ω comprised between 0 and 2π; in other words, the pole is the center of the curve. For $\omega = \alpha$ and $\omega = 2\pi - \alpha$, the values of ρ are equal with contrary signs; here one has two points situated symmetrically with respect to OY perpendicular to the polar axes (Fig. 241); this straight line OY is an axis of the curve. The variable ω varying from 0 to π, as ρ diminishes from 1 to 0, one obtains the arc ABO tangent to the straight line OX at O. The values of ω comprised between π and 2π give the arc OBA', the *symétrique* of the first with respect to OY, and the values ω comprised between the values 2π and 4π, the curve $A'B'OB'A$, the *symétrique* of $ABOBA'$ with respect to the pole. The curve is closed and consists of four equal arcs. The polar axis is also an axis of the curve, which is at once evident on noticing that one gets equal values of ρ for the values α and $4\pi - \alpha$ of ω.

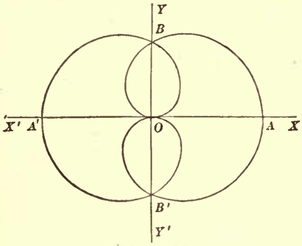

Fig. 241.

TANGENT.

375. Let M be a point of a curve referred to polar co-ordinates. Consider the tangent MT at a point M (Fig. 242), and the prolongation MA of the radius vector in the direction OM; let V be the angle through which it is necessary to revolve the prolongation MA of the radius vector about the point M, in the positive direction, in order to make it coincide with the tangent. In order to determine V, call ρ and ω the co-ordinates of the point of contact M, $\rho + \Delta\rho$, $\omega + \Delta\omega$ those of a neighboring point M' of M, and U the angle formed by MA and the

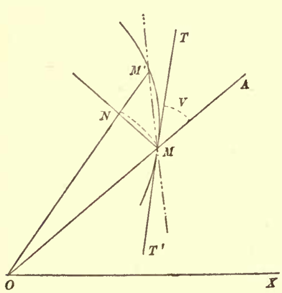

Fig. 242.

chord MM'. When the point M' approaches M indefinitely, the chord MM' approaches the tangent MT, and the angle U approaches V.

The triangle OMM' gives $\frac{OM}{OM'} = \frac{\sin OM'M}{\sin OMM'} = \frac{\sin(U - \Delta\omega)}{\sin U}$.

Since $\Delta\rho$ approaches zero when the point M' approaches M, one can suppose it to be sufficiently small that ρ and $\rho + \Delta\rho$ have the same sign: therefore one has in magnitude and in sign

$$\frac{OM}{OM'} = \frac{\rho}{\rho + \Delta\rho},$$

and the equation above gives

$$\frac{\rho}{\rho + \Delta\rho} = \frac{\sin(U - \Delta\omega)}{\sin U},$$

whence $$\frac{\rho}{\frac{\Delta\rho}{\Delta\omega}} = \frac{\sin(U - \Delta\omega)}{\frac{\sin U - \sin(U - \Delta\omega)}{\Delta\omega}} = \frac{\frac{\Delta\omega}{2}}{\sin\frac{\Delta\omega}{2}} \cdot \frac{\sin(U - \Delta\omega)}{\cos(U - \frac{1}{2}\Delta\omega)}.$$

When $\Delta\omega$ approaches 0, the ratio $\frac{\Delta\rho}{\Delta\omega}$ approaches the derivative ρ' of ρ with respect to ω, and U approaches V. Therefore one has

$$\text{tang } V = \frac{\rho}{\rho'}.$$

REMARK.—When the radius vector becomes zero for a particular value ω_0 of ω, one has a branch of the curve OC passing through the pole (Fig. 243), and the tangent to this branch at the pole is the straight line OA determined by the angle ω_0. In fact if a neighboring point M be taken and if the secant OM be revolved so that the point M approaches the point O, ρ becomes zero and the secant approaches OA as a limiting position.

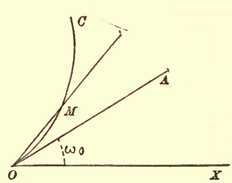

Fig. 243.

376. *Equation of the tangent.* The equation of the secant $M_1M'_1$ (Fig. 244) is (§ 83, 2)

$$\begin{vmatrix} \frac{1}{\rho} & \cos\omega & \sin\omega \\ \frac{1}{\rho_1} & \cos\omega_1 & \sin\omega_1 \\ \frac{1}{\rho_1}+\Delta\frac{1}{\rho_1} & \cos(\omega_1+\Delta\omega_1) & \sin(\omega_1+\Delta\omega_1) \end{vmatrix} = 0,$$

where ρ_1 and ω_1 are the co-ordinates of the point M_1 and $\Delta\frac{1}{\rho_1}$, $\Delta\omega_1$ the increments which $\frac{1}{\rho_1}$ and ω_1 take when one passes from the point M_1 to the neighboring point M'_1. On subtracting, in the preceding determinant, the elements of the second row from those of the third and dividing all the elements of the third row of the new determinant by $\Delta\omega_1$, one obtains, for the secant, an equation which gives, when $\Delta\omega_1$ approaches zero, the equation of the tangent at M_1.

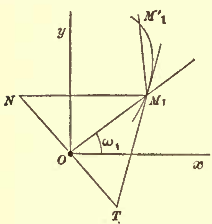

Fig. 244.

$$\begin{vmatrix} \frac{1}{\rho} & \cos\omega & \sin\omega \\ \frac{1}{\rho_1} & \cos\omega_1 & \sin\omega_1 \\ \left(\frac{1}{\rho_1}\right)' & -\sin\omega_1 & \cos\omega_1 \end{vmatrix} = 0,$$

or, on developing,

$$\frac{1}{\rho} = \frac{1}{\rho_1}\cos(\omega-\omega_1) + \left(\frac{1}{\rho_1}\right)'\sin(\omega-\omega_1).$$

Sub-tangent. The sub-tangent S_t is the radius vector of the tangent which corresponds to the polar angle $\omega = \omega_1 + \frac{\pi}{2}$. One has therefore

$$S_t = -\frac{\rho_1^2}{\rho'_1}.$$

If, in Fig. 244, ω_1 be the angle xOM_1, the sub-tangent is negative and equal to $-OT$.

Sub-normal. The sub-normal S_n is the radius vector of the normal at the point M_1 which corresponds to the polar angle

$$\omega = \omega_1 + \frac{\pi}{2}.$$

In Fig. 244, the sub-normal is positive and equal to $+ON$.

Whatever be the arrangement of the figure, the triangle TM_1N is right-angled at M_1 and the point O is the foot of the perpendicular dropped from the point M_1 upon the hypotenuse TN: this point O will lie therefore always between T and N, and, consequently, the sub-tangent and the sub-normal will have opposite signs: since the absolute value of their product is

$$\overline{OT} \cdot \overline{ON} = \overline{OM_1}^2 = \rho_1^2,$$

it follows that
$$S_t \cdot S_n = -\rho_1^2$$

and, consequently, from the value of S_t,

$$S_n = \rho_1'$$

Equation of the normal. The normal is a straight line passing through the point M_1 with the co-ordinates (ρ_1, ω_1), and through the point N, the extremity of the sub-normal with the co-ordinates $\left(\rho'_1, \omega_1 + \frac{\pi}{2}\right)$. The equation of this straight line is therefore

$$\begin{vmatrix} \frac{1}{\rho} & \cos\omega & \sin\omega \\ \frac{1}{\rho_1} & \cos\omega_1 & \sin\omega_1 \\ \frac{1}{\rho'_1} & -\sin\omega_1 & \cos\omega_1 \end{vmatrix} = 0,$$

or
$$\frac{1}{\rho} = \frac{1}{\rho_1}\cos(\omega - \omega_1) + \frac{1}{\rho'_1}\sin(\omega - \omega_1).$$

377. Example I. — *The spiral of Archimedes.* Since the equation of this curve is $\rho = b\omega$ (§ 371), it follows that $\rho' = b$, whence

$$\tan V = \frac{\rho}{\rho'} = \frac{b\omega}{b} = \omega.$$

If the point M begin with the pole and advance along the curve, the angle V, at first zero, increases constantly and approaches a right angle. The sub-normal is constant and equal to b.

EXAMPLE II. — *Logarithmic spiral.* The curve whose polar equation is $\rho = ae^{m\omega}$, a being a given length and m a given number, is called a logarithmic spiral. Suppose that the constant m is positive: if ω increase from zero to infinity, ρ will increase constantly from a to infinity, which gives an infinite branch $ABC\cdots$ consisting of an infinitude of circumvolutions about the pole (Fig. 245). If ω vary from 0 to $-\infty$, ρ constantly diminishes and approaches 0; a second infinite branch $AB'C'\cdots$ is found which makes an infinitude of circumvolutions about the pole, constantly approaching this point. If the constant m be negative, the positive values of ω would give the branch which approaches the pole, and the negative values the branch which recedes from it. In this case one has $\rho' = mae^{m\omega} = m\rho$, hence $\tan V = \frac{1}{m}$. Whence it follows that the tangent to the curve makes a constant angle with the radius vector.

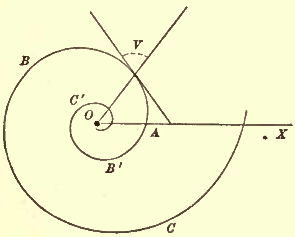

Fig. 245.

378. EXAMPLE III. — *Epicycloid.* When a circle rolls, without sliding, upon a fixed circle, a point of the rolling circle describes in the plane a curve which is called an *epicycloid.*

Let us consider the case when the two circles are equal. Let C be the fixed circle, C' the initial position of the rolling circle, and a the radius; suppose that the point of contact be A, which, by the motion of the circle C', generates the epicycloid (Fig. 246). When the rolling circle has taken the position C'', the point A is at M, and the two arcs EA, EM are equal. Take the point A as pole, and CA prolonged as polar axis. The straight line AM, perpendicular to EN the common tangent to the two circles,

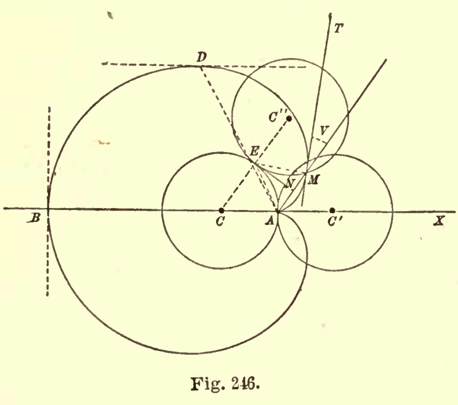

Fig. 246.

is parallel to CE; the angle AEN is a half of the angle ACE, and, consequently, a half of ω; the right triangle ANE gives $AN = AE \sin\frac{\omega}{2}$; but $AE = 2a \sin\frac{\omega}{2}$; one has, therefore,

$$\rho = 4a \sin^2\frac{\omega}{2} = 2a(1 - \cos\omega).$$

This curve is a particular case of the limaçon of Pascal (§ 26). Here one has $\rho' = 2a \sin\omega$, whence $\tan V = \tan\frac{\omega}{2}$, and, consequently, $V = \frac{\omega}{2}$. It is easy to see that the normal to the epicycloid at any point M passes through the point of contact E of the rolling circle with the fixed circle; because the angle MEN being equal to $\frac{\omega}{2}$, and, consequently, to V, the straight line EM is perpendicular to MT.

379. Example IV.— *Construct the curve* $\rho = 4 + \cos 5\omega$. The radius vector ρ is always comprised between 3 and 5; construct with radii 3 and 5 two circles about the pole as center; the curve will be wholly situated between these two circumferences (Fig. 247). When ω varies from o to $\frac{\pi}{5}$, ρ diminishes from 5 to 3, which gives the arc AB. As ω varies from $\frac{\pi}{5}$ to $\frac{2\pi}{5}$, ρ increases from 3 to 5, which gives the arc BA' the *symétrique* of the first with respect to the straight line OB. The angle 5ω has varied from o to 2π. If ω vary from $\frac{2\pi}{5}$ to $\frac{4\pi}{5}$, the angle 5ω will vary from 2π to 4π, and the same values of ρ will be reproduced in the same order; a second arc $A'B'A''$ is found equal to the first, then a third, and so on. With the fifth arc one will arrive at the point of departure. By constructing the derivative one finds $\tan V = -\frac{\rho}{5 \sin 5\omega}$. At the points A and B one has $\sin 5\omega = 0$, and, consequently, $V = \frac{\pi}{2}$.

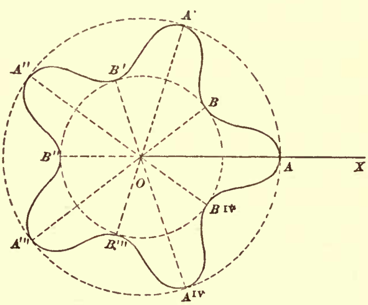

Fig. 247.

380. Example V. — The extremities of a straight line of constant length slides on two straight lines OX, OY which are perpendicular to each other; from a fixed point I on the bisector of the angle XOY one draws a straight line perpendicular to the variable straight line; find the locus of the foot of the perpendicular (Fig. 248). It is evident that the locus will be symmetrical with respect to the straight line OI. Consider first the variable straight line in the position PQ perpendicular to the bisector; a point A of the locus is determined. Make the straight line move so that the extremity Q descends along the y-axis; in some position $P'Q'$

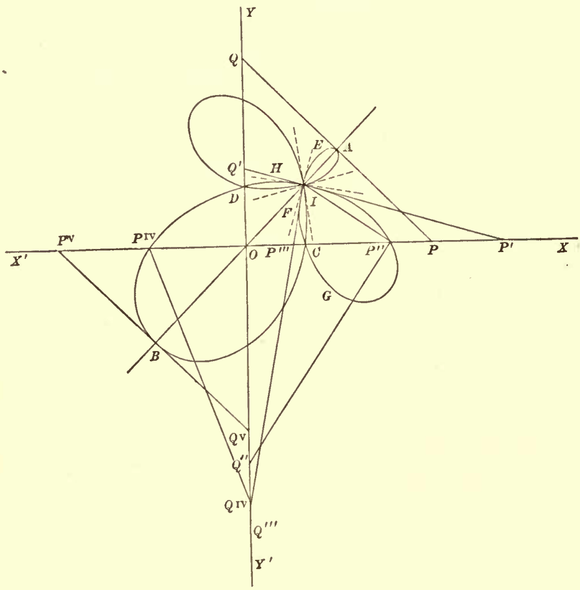

Fig. 248.

will pass through the point I, which belongs to the locus; whence one has the arc AEI, whose tangent at I is perpendicular to $P'Q'$. The extremity Q' continuing its descent will coincide with OX, and one has the arc IFC which passes through the point C, the foot of the perpendicular let fall from I to OX. As the extremity Q slides along OY', the curve passes below OX; the straight line will arrive in a certain position $P''Q''$ such that the angle $IP''Q''$ is right, which gives the point P'' of the locus; whence we have the arc CGP''. If the extremity Q'' continue its descent, the curve will return above the x-axis, and the straight line will finally assume the position $P'''Q'''$, which prolonged passes through I; we get then the arc $P''I$ whose tangent at I is perpendicular to $P'''Q'''$. If the extremity P''' continue to approach the point O, the straight line will ultimately coincide with the y-axis and we obtain the arc IHD, which passes through the point D, the foot of the perpendicular dropped from the point I upon the y-axis. If the extremity P''' slide along OX', the straight line will assume a position $P^{iv}Q^{iv}$ such that the angle $IP^{iv}Q^{iv}$ is right; the point P^{iv} belongs to the locus, and we obtain the arc DP^{iv}. Finally the straight line, in some position P^vQ^v, becomes perpendicular to the bisector OB of the angle $X'OY'$, which gives the arc $P^{iv}B$. If returning to the initial position PQ, the motion of the straight line be reversed till the final position P^vQ^v is reached, it is clear that a curve the *symétrique* of the first with respect to the straight line AB will be found.

Take as pole the point I and as the polar axis the bisector BA; call c the distance OI and $2a$ the length of the variable straight line PQ; the straight line which joins the point O to the middle of the hypotenuse PQ of the right triangle POQ is equal to a; the angle which this straight line makes with the perpendicular h dropped from the point O upon the hypotenuse is equal to 2ω; one has moreover $h = c\cos\omega + \rho$; whence follows the equation of the curve $\rho = a\cos 2\omega - c\cos\omega$.

Convexity and Concavity.

381. Consider on an arc of the curve a point M, whose co-ordinates are ω_0 and ρ_0; the tangent at this point will be represented by the equation $r = \dfrac{q}{\cos(\omega - \beta)}$, if q be the length of the perpendicular let fall from the pole upon the tangent and β be the angle which the perpendicular makes with the polar axis (§ 82). The position of the curve with respect to the tangent, in the neighborhood of the point M, depends upon the sign of the difference $r - \rho$ of the radii vectores for the same value of ω, or of the difference $\dfrac{1}{\rho} - \dfrac{1}{r}$; we assume that the radii vectores are positive. Let z be this last difference; the value of z is evidently zero at the point M; its first derivative

$$z' = \left(\frac{1}{\rho}\right)' - \left(\frac{1}{r}\right)' = \left(\frac{1}{\rho}\right)' + \frac{\sin(\omega - \beta)}{q}$$

is also zero; because one has, at the point M (§ 375),

$$\left(\frac{1}{\rho}\right)' = -\frac{\rho'}{\rho^2} = -\frac{\cot V}{\rho_0},\ \sin(\omega_0 - \beta) = \cos V,\ q = \rho_0 \sin V.$$

The second derivative,

$$z'' = \left(\frac{1}{\rho}\right)'' + \frac{\cos(\omega - \beta)}{q} = \left(\frac{1}{\rho}\right)'' + \frac{1}{r},$$

has at the point M a value equal to that of the expression $\frac{1}{\rho}+\left(\frac{1}{\rho}\right)''$. On repeating here the reasoning of § 344, it can be easily verified that, if this quantity be positive, the difference z is also positive in the neighborhood of the point M, and consequently the curve is situated on the same side of the tangent as the pole, and that, if on the contrary this quantity be negative, the difference z is negative, and the curve will lie on the other side of the tangent.

If the radius vector ρ be negative, one sees by similar consideration that the position of the curve with respect to the tangent is on the same side as the pole when the quantity $\frac{1}{\rho}+\left(\frac{1}{\rho}\right)''$ is negative and on the other side if this quantity be positive.

In general, therefore, it may be said that *at a point M of a curve it turns its convexity or its concavity toward the pole according as*

$$\frac{1}{\rho}\left[\frac{1}{\rho}+\left(\frac{1}{\rho}\right)''\right]$$

is negative or positive at this point.

There is a point of inflection at the point of the curve where the quantity $\frac{1}{\rho}+\left(\frac{1}{\rho}\right)''$ changes its sign.

Asymptotes.

382. Consider an infinite branch asymptotic to the straight line CD (Fig. 249); if the point M of the curve be joined to the pole, and if the point M be removed toward infinity on the curve, the radius vector OM will have for its limit a line OL parallel to the asymptote. Thus, *when the radius vector ρ becomes infinite for a particular value α of ω, if the branch thus determined have an asymptote, this asymptote is parallel to the direction determined by the angle α which makes ρ infinite.*

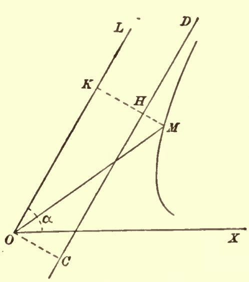

Fig. 249.

To find the distance OC of the asymptote from the straight line OL, draw from the point M, MK perpendicular to OL; the triangle MOK gives

$$MK = OM \sin KOM = \pm \rho \sin(\alpha - \omega).$$

If the product $\pm \rho \sin(\alpha - \omega)$ do not have a finite limit, the infinite branch does not have an asymptote. If, on the contrary, this product approach a finite limit, the branch of the curve does have an asymptote, the straight line CD, situated a distance OC, equal to this limit, from the line OL; because if the distance MK have the limit OC, the distance MH will have the limit zero.

A second demonstration. Suppose that the infinite branch is referred to two rectangular axes drawn through the pole, the y'-axis in the direction α, and the x'-axis in the direction $\alpha - \frac{\pi}{2}$. If Ox' be taken for a new polar axis, it follows that $\omega' = \omega - \left(\alpha - \frac{\pi}{2}\right)$, and the abscissa x' of the point M will be, in every case (§ 373),

$$x' = \rho \cos\omega' = \rho \cos\left(\omega - \alpha + \frac{\pi}{2}\right) = \rho \sin(\alpha - \omega).$$

One is thus led to seek an asymptote parallel to the y-axis. The abscissa q of the asymptote is the limit of x', when the point M is removed to infinity on the branch of the curve; one has, therefore,

$$q = \lim \rho \sin(\alpha - \omega).$$

The absolute value of q gives the distance of the asymptote from the straight line Oy'; the sign shows on which side it is situated.

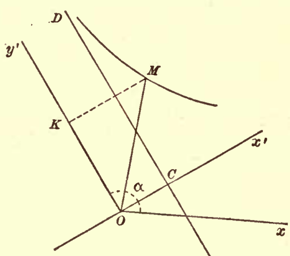

Fig. 250.

383. Example VI. — *The Hyperbola.* The polar equation of this curve is (§ 263) $\rho = \frac{p}{1 + e\cos\omega}$, in which e is greater than 1. Let α be the angle whose cosine is $-\frac{1}{e}$; when ω increases from 0 to α, ρ increases from $\frac{p}{1+e}$ to ∞, and one has the infinite branch AE; when ω varies from α to π, ρ becomes negative and varies from $-\infty$ to $-\frac{p}{e-1}$, which gives the infinite branch $E'A'$. The values of ω comprised between π and 2π give two branches which are the *symétriques* of the preceding with respect to the polar axis.

The distance MK of a point of one of the branches AE, $A'E'$ from the line OL is

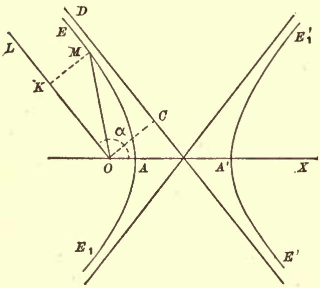

Fig. 251.

$$MK = \frac{p\sin(\alpha-\omega)}{1+e\cos\omega} = \frac{p\sin(\alpha-\omega)}{e\left(\frac{1}{e}+\cos\omega\right)} = \frac{p\sin(\alpha-\omega)}{e(\cos\omega-\cos\alpha)}.$$

Substituting a product for the difference $\cos\omega - \cos\alpha$, and $2\sin\frac{\alpha-\omega}{2}\cos\frac{\alpha-\omega}{2}$ for $\sin(\alpha-\omega)$, and suppressing the common factor $\sin\frac{\alpha-\omega}{2}$, one has

$$MK = \frac{p\cos\frac{\alpha-\omega}{2}}{e\sin\frac{\alpha+\omega}{2}}.$$

This distance has the limit $OC = \frac{p}{e\sin\alpha}$; thus is obtained the asymptote CD. The asymptote of the other two branches is situated symmetrically with respect to the polar axis.

The difference between MK and its limit is

$$\delta = \frac{p}{e}\left[\frac{\cos\frac{\alpha-\omega}{2}}{\sin\frac{\alpha+\omega}{2}} - \frac{1}{\sin\alpha}\right] = \frac{p}{e}\times\frac{2\sin\alpha\cos\frac{\alpha-\omega}{2} - 2\sin\frac{\alpha+\omega}{2}}{2\sin\alpha\sin\frac{\alpha+\omega}{2}};$$

if the product $2\sin\alpha\cos\frac{\alpha-\omega}{2}$ be replaced by the sum

$$\sin\frac{3\alpha-\omega}{2} + \sin\frac{\alpha+\omega}{2},$$

it becomes
$$\delta = \frac{p}{e}\cdot\frac{\sin\frac{3\alpha-\omega}{2} - \sin\frac{\alpha+\omega}{2}}{2\sin\alpha\sin\frac{\alpha+\omega}{2}};$$

if the numerator be transformed into a product, one has finally

$$\delta = \frac{p\sin\frac{\alpha-\omega}{2}\cos\alpha}{e\sin\alpha\sin\frac{\alpha+\omega}{2}} = \frac{p\sin\frac{\omega-\alpha}{2}}{e^2\sin\alpha\sin\frac{\omega+\alpha}{2}}.$$

When ω varies from 0 to a, the difference δ is negative; thus the branch AE is comprised between the parallels OL and CD. But, when ω varies from a to π, the difference δ is positive, and the branch $E'A'$ is situated without the parallels.

384. Example VII. — *Oblique Strophoid.* In the construction of the right strophoid, such as has been given in § 23, we suppose the straight lines OX and OY perpendicular to each other; suppose now that these straight lines include an angle θ (Fig. 252); through the fixed point A, situated on one of them, draw any secant AD, on which take, beginning with the point D, the lengths DM and DN equal to DO, and find the locus of the points M and N. When the secant revolves in the obtuse angle XOY till it becomes parallel to OY, the point M describes the arc OMB, ending at the point B on OB perpendicular to OY; the point N describes the infinite branch ON. If a distance OG be taken equal to OA, and a straight line $H'H$ be drawn through the point G parallel to OY, one obtains the asymptote of the branch ON; because NF is equal to AM, having the limit AB, the distance of the point N from the straight line has the limit zero.

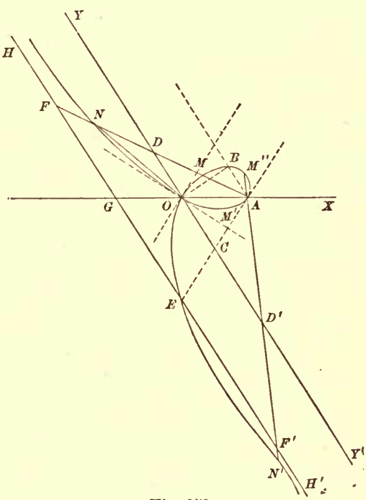

Fig. 252.

Allow the secant to revolve in the adjacent angle XOY'; the perpendicular erected at the mid-point of OA intersects OY' at a point C, so that one has $CA = CO$; when the secant occupies the position AC, one of the points arrives at A and the other at E; thus, the secant revolving from the position AO to AC, one obtains the arc $OM'A$ tangent to the straight line AC at A, and the arc OE. As the secant continues its motion, the straight line OD' becomes greater than AD' or $D'F'$, and the point N' is situated beyond the asymptote; one obtains the infinite branch EN' and the arc $AM''B$ which is a continuation of the arc OMB. It is easy to see that the tangents at the point O are the bisectors of the angles formed by the straight lines OX and OY.

If the point O be taken as pole, the straight line OX as the polar axis, and if one call the distance OA, a, and the angle YOX, θ, the angles DOM and DMO being equal to $\theta - \omega$, the angle OAD to $\theta - 2\omega$, the triangle OMA gives the relation,

$$(1) \qquad \rho = \frac{a \sin(\theta - 2\omega)}{\sin(\theta - \omega)}.$$

By aid of the equation, one may easily verify the properties which we have deduced from the geometric definition of the curve.

If the straight line OY be taken as polar axis, the equation of the curve becomes

$$(2) \qquad \rho = \frac{a \sin(2\omega + \theta)}{\sin \omega}.$$

385. Example VIII. — *To find the locus of the points of contact of tangents drawn from a given point P to the various curves of the second degree whose foci are two fixed points F and F'.*

Take FF' as x-axis (Fig. 253), and the perpendicular erected to this straight line at its mid-point as the y-axis; the general equation of conics whose foci are F and F' is

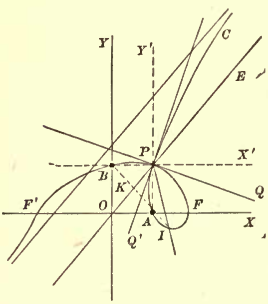

Fig. 253.

$$(1) \qquad \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1,$$

where c designates the distance OF, and a is a variable parameter; when a is greater than c, the curve is an ellipse; when a is less than c, it is a hyperbola. Let α and β be the co-ordinates of the given point P; the equation of the chord of contact of the tangents drawn from the point P to the conic (1) is

$$(2) \qquad \frac{\alpha x}{a^2} + \frac{\beta y}{a^2 - c^2} = 1.$$

The equation of the locus is found by eliminating the parameter a between equations (1) and (2). If these equations be subtracted member from member, one obtains

$$\frac{x^2 - \alpha x}{a^2} + \frac{y^2 - \beta y}{a^2 - c^2} = 0;$$

whence $$a^2 = \frac{c^2(x^2 - \alpha x)}{x^2 + y^2 - \alpha x - \beta y}, \quad a^2 - c^2 = \frac{-c^2(y^2 - \beta y)}{x^2 + y^2 - \alpha x - \beta y};$$

by substituting in equation (2), the equation of the locus is found

$$(3) \qquad (x^2 + y^2 - \alpha x - \beta y)(\beta x - \alpha y) + c^2(x - \alpha)(y - \beta) = 0.$$

The locus is of the third degree, it passes through the given point P, through the foci F and F', and through the projections of the point P upon the straight lines OX and OY.

If the axes be transferred parallel to themselves to the point P, the equation of the locus becomes

$$(4) \qquad (x^2 + y^2 + \alpha x + \beta y)(\beta x - \alpha y) + c^2 xy = 0.$$

Transforming the equation to polar co-ordinates, taking the point P as pole, and the line PX' as polar axis, one has

$$(5) \qquad \rho = \frac{(c^2 + \beta^2 - \alpha^2) \sin 2\omega + 2\alpha\beta \cos 2\omega}{2(\alpha \sin \omega - \beta \cos \omega)}.$$

By introducing auxiliary angles ϕ and ϕ_1, determined by the formulas

$$\tan \phi = \frac{\beta}{\alpha}, \quad \tan \phi_1 = \frac{2\alpha\beta}{c^2 + \beta^2 - \alpha^2},$$

this equation takes the form

$$(6) \qquad \rho = \frac{d \sin(2\omega + \phi_1)}{\sin(\omega - \phi)}$$

where the letter d designates the quantity

$$d = \frac{1}{2}\sqrt{\frac{(c^2 + \beta^2 - \alpha^2)^2 + 4\alpha^2\beta^2}{\alpha^2 + \beta^2}}.$$

If the polar axis be revolved through the angle ϕ, equation (6) becomes identical with equation (2) of the oblique strophoid (§ 384). The angle ϕ being equal to POA, the asymptote is parallel to the straight line OP. Among the confocal curves considered are a hyperbola and an ellipse which pass through the point P; one infers that the point P belongs to the locus and that the tangents at this point are the bisectors PQ, PQ' of the angles formed by the straight lines PF and PF'. The strophoid is determined by two straight lines PE, PI, and a point I on one of them. We know the straight line PE; the straight line PI is determined by the fact that the tangent PQ is the bisector of the angle EPI; the point I is determined by means of one of the points of the locus, for example, by the point A; the straight line IA should be such that $KA = KP$; the point K is therefore the mid-point of the diagonal OP.

The preceding curve is thus the locus of the feet of the normals drawn from the point P to the curves of the second degree, whose foci are the points F and F'; because in case of an ellipse and a hyperbola which intersect at right angles, the tangent to one of them is normal to the other.

386. Example IX. — *Construct the curve given by the equation*

$$\rho = a\frac{2\,\omega}{2\,\omega - 1}.$$

The value of ρ becomes zero for $\omega = 0$, and becomes infinite for $\omega = \frac{1}{2}$; draw through the pole the straight line $L'L$, which makes the angle $\frac{1}{2}$ with the polar axis (Fig. 254). When ω varies from 0 to $\frac{1}{2}$, ρ is negative, and varies from 0 to $-\infty$; one obtains an infinite branch $OA'B'$, tangent to the polar axis and comprised within the angle $X'OL'$. When ω exceeds $\frac{1}{2}$ and increases from $\frac{1}{2}$ to ∞, ρ becomes positive and decreases from ∞ to a; one obtains an infinite branch BA, which makes an infinity of circumvolutions about the circle described about the pole as center, with a radius a, continually approaching the circle. When ω varies from 0 to $-\infty$, ρ remains positive and increases from 0 to a, which gives the branch OE' within the circle. This branch makes an infinity of circumvolutions continually approaching the circle.

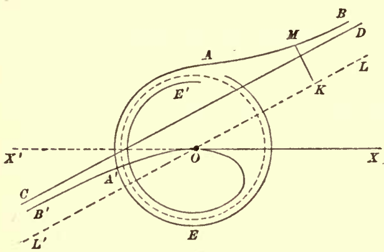

Fig. 254.

Consider the infinite branches $A'B'$, AB; the abscissa of a point of one of them, with respect to the axes according to § 382, is

$$x' = \rho \sin\left(\tfrac{1}{2} - \omega\right) = -a\omega\frac{\sin\left(\omega - \frac{1}{2}\right)}{\omega - \frac{1}{2}};$$

its limit is $\frac{-a}{2}$; the two branches have therefore as asymptote CD, whose intercept is $q = \frac{-a}{2}$.

If one put $\omega = \frac{1}{2} + \omega'$, one has

$$\delta = x' - q = \frac{a}{2\,\omega'}\left[\omega' - (1 + 2\,\omega')\sin\omega'\right].$$

The quantity placed within the parentheses becomes zero for $\omega' = 0$; its first derivative also becomes zero, but its second derivative is negative; if ω increase from zero, one infers that the first derivative commences by decreasing and, consequently, is negative and the same is true of the quantity itself; thus the difference δ is negative for the positive values of ω' made sufficiently small. One perceives in the same manner that the difference δ is positive for very small values of ω', which is evident from the formula; hence the two infinite branches with respect to the asymptote have the position indicated in the figure. It is evident, moreover, that this asymptote is intersected by the curve in an infinite number of points.

387. Example X. — *Construct the curve*

$$\rho = 1 \pm \sqrt{\frac{2 \sin \omega - 1}{\sin \omega}}.$$

Since the radius vector takes the same value when $\omega + 2\pi$ is substituted for ω, it is sufficient to vary ω from 0 to 2π. That the radius vector be real, it is necessary that the quantity under the radical be positive. The numerator changes its sign for the values $\frac{\pi}{6}$, $\frac{5\pi}{6}$ of ω and the denominator for the values 0 and π; on arranging these angles in order of magnitude

$$0, \ \frac{\pi}{6}, \ \frac{5\pi}{6}, \ \pi, \ 2\pi,$$

it is evident that the quantity under the radical is negative from 0 to $\frac{\pi}{6}$, positive from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$, negative from $\frac{5\pi}{6}$ to π, positive from π to 2π; the whole curve is therefore obtained by varying ω from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$ and from π to 2π. We notice, moreover, that the supplementary values of ω reproducing the same values of ρ, the curve is symmetrical with respect to OY perpendicular to the polar axis OX (Fig. 255).

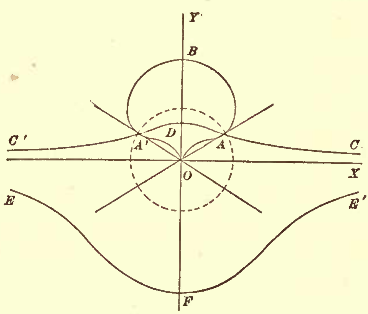

Fig. 255.

About the pole as center, describe a circle with a unit radius: this circle will bisect each of the chords which pass through the center; when ω varies from $\frac{\pi}{6}$ to $\frac{\pi}{2}$, the value of the radical increases from 0 to 1, which furnishes the two arcs AB and AO, either of which is the continuation of the other, tangent to the straight line OA; the arc AO is tangent to the straight line OY at the point O. By varying ω from $\frac{\pi}{2}$ to $\frac{5\pi}{6}$, we obtain the arc $BA'O$, the *symétrique* of BAO, with respect to the straight line OY.

When ω varies from π to $\frac{3\pi}{2}$, the value of the radical decreases from ∞ to $\sqrt{3}$, which gives the two infinite branches EF and CD; by varying ω from $\frac{3\pi}{2}$ to 2π, the two branches FE' and DC', the *symétriques* of the preceding are determined; these infinite branches are asymptotic to the polar axis.

388. Remark.—It has been shown (§ 382) that if the radius vector becomes infinite for a finite value α of ω, the position of the asymptote is determined by the formula

$$q = \lim \rho \sin(\alpha - \omega).$$

The first factor becomes infinite when the second factor approaches zero. In the preceding examples, one can without difficulty find what the product must be for values of ω consecutive to α. In case there is much difficulty, one may employ another method.

One may deduce from the preceding formula

$$\frac{1}{q} = -\lim \frac{\omega - \alpha}{\sin(\omega - \alpha)} \cdot \frac{\frac{1}{\rho}}{\omega - \alpha}.$$

The limit of the first ratio is equal to unity. If $\frac{1}{\rho}$ be regarded a function of ω, the numerator of the second ratio is the increment which this function receives when the polar angle varies from the value α to the value ω; the limit of the second fraction is therefore the derivative of $\frac{1}{\rho}$, and we have

$$\frac{1}{q} = -\left(\frac{1}{\rho}\right)' \text{ for } \omega = \alpha.$$

As a rule the value of ρ assumes the form $\rho = \frac{F(\omega)}{f(\omega)}$, the denominator becoming zero for $\omega = \alpha$, while the numerator preserves a finite value different from zero. We find for the derivative

$$\left(\frac{1}{\rho}\right)' = \frac{F(\omega)f'(\omega) - f(\omega)F'(\omega)}{F^2(\omega)},$$

which reduces to $\frac{f'(\alpha)}{F(\alpha)}$ for $\omega = \alpha$. Thus is obtained the formula

$$q = -\frac{F(\alpha)}{f'(\alpha)},$$

which is very convenient in practice.

EXERCISES.

1. Find the locus of the vertex of a variable parabola which has a fixed focus and which touches a conic which possesses the same focus (limaçon of Pascal).

2. The vertex O of a variable triangle AOB is fixed, the vertex B slides on a fixed straight line OX; find the locus described by the point of intersection of AB with the perpendicular erected to the side OA at the point O.

3. A fixed point O and a fixed straight line OP are given; find the locus of the vertex M of a variable triangle MON which fulfills the following conditions: the side ON is constant and equal to a, the side $MN=a\sqrt{2}$, finally the angles satisfy the relation $\cos(MON-2\,OMN)=\cos MOP$ (lemniscate); show that the tangent at any point M of the locus passes through the center of the circle circumscribed about the triangle which has given this point.

4. A triangle OAB right-angled at O is given, a variable conic is circumscribed about this triangle so that the normals at the three points A, O, B pass through a common point; find the locus of this point.

5. Find the locus of the foci of the parabolas which have a common chord and a common tangent parallel to this chord.

6. Find the locus of the vertices or of the foci of an equilateral hyperbola, whose center is fixed and which passes through a fixed point.

7. Find the locus of the center of a given equilateral hyperbola required to pass through two fixed points.

8. One is given a right angle YOX, and a fixed point P on the bisector of this angle; find the locus of the foot of the perpendicular drawn from the point P to a variable secant which cuts off in the angle a triangle whose area is constant.

9. At any point M of a parabola a normal is drawn which is prolonged till it intersects the axis in the point N; find the locus of the point of intersection of the tangent to the curve at M with the perpendicular to the axis at the point N.

10. Substitute in the preceding problem the hyperbola for the parabola; take the point M on one of the axes of the curve; find the locus.

11. A parabola revolves about its focus; find the locus of the points of contact of the tangents drawn parallel to a given straight line.

12. Find the locus of the mid-point of a chord normal to a given hyperbola.

13. An ellipse is given; the center of a circle with constant radius travels on a diameter of the ellipse; find the locus of the points of intersection of the secants common to the circle and the ellipse.

14. An equilateral hyperbola is given; the center of a circle, which always passes through the center of the hyperbola, touches an asymptote; find the locus of the points of intersection of the common secants.

15. Find the locus of the points of contact of tangents drawn from a given point to the circles which touch a given straight line at a given point.

16. Find the locus of the points of contact of tangents drawn from a given point to the circles which pass through a given point and touch a given straight line.

17. Find the locus of the mid-points of the chords inscribed in a given hyperbola and tangent to a circle concentric to the hyperbola.

18. Study the loci represented by the equations

$$y^4 - x^4 + 2\,ax^2y = 0, \quad x^4 + y^4 - 2\,a^3y - 2\,b^2xy = 0,$$
$$(x^2 + y^2)^2 - 6\,axy^2 - 2\,ax^3 + 2\,a^2x^2 = 0, \quad y = a + b(x - c)^m,$$
$$x^4 + y^4 - 3\,x^3 - 4\,x^2 = 0, \quad x^3y^3 + y - x = 0,$$
$$y^4 - x^4 - 2\,bxy^2 - 2\,ax^3 = 0, \quad y^4 - x^4 - 3\,x^2y^2 - 2\,x = 0,$$
$$y^4 - x^4 - 96\,a^2y^2 + 100\,a^2x^2 = 0, \quad 2\,x^3 - y^3 + (y - x)^2 = 0.$$

19. Construct the curves represented by the equations

$$2\sin y - \sin x = 0, \quad \sin x \sin y = \tfrac{1}{2}.$$
$$y = \frac{e^x}{x}, \quad y = x^x, \quad y^x = x^y.$$

20. Construct the curves represented by the equations

$$\rho = \frac{\sin\omega}{2\cos\omega - 1}, \quad \rho = 1 \pm \sqrt{\frac{\sin 3\,\omega}{\cos\omega}},$$
$$\rho^2\cos\omega - 2\,\rho\sin\omega + \cos^2\omega = 0, \quad \rho^2\cos\omega - 4\,\rho\sin\omega - \tan\omega = 0,$$
$$\rho = \cos\omega \pm \sqrt{\frac{1 - 2\cos\omega}{\sin\omega}}.$$

21. Construct the curve defined by the equation

$$y = x\sin\frac{1}{x}.$$

This curve, which passes through the origin, does not have a tangent at this point.

22. Find the envelope of straight lines such that the segments intercepted on them by two fixed conics have the same mid-point. Discuss the case when the two conics have in common a conjugate diameter with the same direction as the chords.

23. A variable conic circumscribed about a triangle ABC is such that the normals to the conic at the points A, B, C are concurrent in a point M. Find the geometric locus of the point M.

24. A variable conic inscribed in a triangle ABC is such that the normals to the conic at the points of contact are concurrent in a point M. Find the geometric locus of the point N. (This locus is the same as the preceding.)

25. Find the locus of the points of intersection of tangents common to a fixed conic S and to a variable conic passing through four fixed points A, B, C, D.

(*a*) The points A, B, C, D being arbitrary, the locus is a curve of the sixth degree, having as double points the centers of the pairs of straight lines which pass through the four points.

(*b*) If two points, A and B for example, be on the fixed conic S, the locus resolves itself into a conic and a curve of the fourth order.

(*c*) If the three points A, B, C be on S, the locus is composed of three conics.

(*d*) If the four points A, B, C, D be on S, the locus is composed of straight lines.

26. Find the locus of points of contact of tangents common to a fixed conic S and to a variable conic passing through four fixed points A, B, C, D.

Discuss, as in the preceding exercise, the reductions which occur when a certain number of the sides or diagonals of the quadrilateral $ABCD$ touch the fixed conic S.

27. The straight line whose equation in rectangular co-ordinates is

$$x \sin \alpha - y \cos \alpha = a \cos k\alpha,$$

has as envelope, when α varies, a and k remaining constant, a hypocycloid or an epicycloid (§ 378). Find the locus of the points at which perpendicular tangents can be drawn to the envelope curve. Examine the particular cases $k = 2$, $k = 3$.

CHAPTER V*

CONCERNING SIMILITUDE.

389. We recall first the definition of two homothetic figures. Let $A, B, C, \cdots$ (Fig. 256) be any system of points, situated in a plane; these points may be isolated or arranged on lines passing through O, taken arbitrarily in the plane; draw to the various points of the system the half-lines $OA, OB, OC, \cdots$, and take on these half-lines the points $A', B', C', \cdots$ so that we have

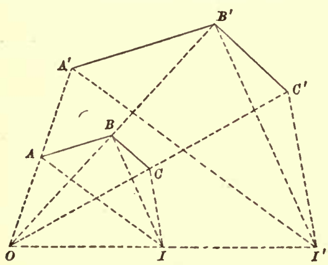

Fig. 256.

$$\frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'} = \cdots = k;$$

the system of points thus determined is said to be *similar* to the proposed system and *similarly placed.*

If the points $A', B', C', \cdots$ were taken on the prolongations of the half-lines in opposite directions, the two systems would be *similar* and *oppositely placed.* By a rotation about the point O through 180 degrees, the second system will coincide with one of the systems similar and similarly placed with respect to $A, B, C, \cdots$ (Fig. 257).

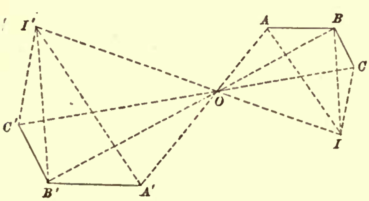

Fig. 257.

In order to abbreviate the expression, M. Chasles has called this similitude of form and of position *homothetic, direct* in the first case, and *opposite* in the second. The point O is called the *center of similitude* or *homothetic center* of the two systems, the number k is the ratio of *similitude*, and the points A and A' situated on the same half-line are called homologous points. If the ratio k vary from 0 to ∞ also the position of the center O of similitude, one obtains all the systems homothetic to the given system.

A system is *similar* to a given system, when it is equal to one of the systems homothetic to the given system.

390. We know that all the curves homothetic to a given curve S with a single center of similitude O, taken at random in its plane, may be found. Consider next some examples.

1° The curve S is a circle. If the center of the circle be taken as the center of similitude, the second curve will be a circle, whose radius can have any magnitude we wish.

2° The curve S is a parabola (Fig. 258). The curve being referred to its axis and to the tangent at its vertex, the co-ordinates x and y of any point M of this curve satisfy the equation $y^2 = 2px$. If the vertex be taken as center of similitude and x' and y' be the co-ordinates of the point M' homothetic to M, one has

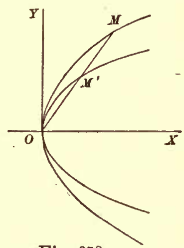

Fig. 258.

$$\frac{x}{x'} = \frac{y}{y'} = \frac{OM}{OM'} = k,$$

whence $x = kx'$, $y = ky'$; the homothetic curve will have the equation $y'^2 = \frac{2p}{k}x'$; it is a parabola whose parameter can have as large a magnitude as is desired, on account of the arbitrary ratio k; it follows that *any two parabolas are similar.*

3° The curve S is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If the center of the curve be taken as center of similitude, the homothetic curve, represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{k^2}$$

is an ellipse, whose axes can have any magnitude proportional to those of the axes of the first ellipse. It follows that *two ellipses are similar when their axes are proportional.*

The same theorem holds for the hyperbola.

4° The curve S is the logarithmic spiral $\rho = ae^{m\omega}$. If the pole be taken as the center of similitude, the homothetic curves have the equation $\rho = \frac{ae^{m\omega}}{k}\cdot$ If one put $k = e^{m\alpha}$, this equation becomes $\rho = a^{m(\omega-\alpha)}$; it represents the given spiral which has been revolved about the pole through the angle α. It follows that the only curve similar to a logarithmic spiral is this spiral itself; to a point M of the curve corresponds another point M' of the same curve, and this point M' can be taken at will, on account of the arbitrary number k.

The Equation of Homothetic Curves.

391. Let

$$f(x, y) = 0 \tag{1}$$

be the equation of the curve S. Take the origin as center of similitude and construct a curve S' homothetic to the first with the ratio k. Let (x, y) be the co-ordinates of any point M of the first curve, and (x', y') those of the homologous point M' of the second curve, the similar triangles OPM, $OP'M'$ give

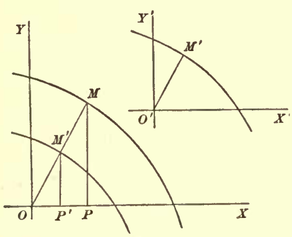

Fig. 259.

$$\frac{x}{x'} = \frac{y}{y'} = \frac{OM}{OM'} = k;$$

if $x = x'k$, $y = y'k$ be substituted in equation (1), one has equation

$$f(kx', ky') = 0, \tag{2}$$

which represents all curves homothetic to the given curve and having the origin as homothetic center. Positive values of k in this equation will correspond to a direct homothetic transformation, and negative values to indirect.

Allowing S to be fixed, transfer the curve S' in the plane, in such a manner that the origin O will fall in $O'(p, q)$, and that the axes will remain parallel to their primitive directions; the curve S' has as equation, referred to the axes $O'X'$ and $O'Y'$,

$$f(kx', ky') = 0,$$

and, with respect to the fixed axes OX and OY,

$$(3) \qquad f[k(x-p), k(y-q)] = 0.$$

In this new position, the curve S' is homothetic to the curve S; because the radii vectores drawn from the points O and O' are parallel and in the constant ratio k. Equation (3) represents therefore all the curves homothetic to the given curve, whatever be the position of the center of similitude.

392. At the same time that the origin is transferred to O', revolve the axes through the angle α; the curve S' will then have an arbitrary position in the plane, and will be simply similar to the given curve. The curve S' referred to the variable axes $O'X'$ and $O'Y'$ has the equation $f(kx', ky') = 0$; by reason of the formulas of transformation,

$$x' = (x-p)\cos\alpha + (y-q)\sin\alpha,$$
$$y' = -(x-p)\sin\alpha + (y-q)\cos\alpha,$$

the axes being assumed rectangular, the equation of the curve with respect to the two fixed axes OX and OY is obtained. This equation represents all curves similar to the given curve.

393. As an application, determine the conditions for which the two curves of the second degree,

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0,$$
$$A'x^2 + 2B'xy + C'y^2 + 2D'x + 2E'y + F' = 0,$$

are homothetic. The general equation of the curve homothetic to the first curve is (§ 391)

$$Ak^2x^2 + 2Bk^2xy + Ck^2y^2 - 2(Bk^2q + Ak^2p - Dk)x$$
$$- 2(Bk^2p + Ck^2q - Ek)y + (Ak^2p^2 + 2Bk^2pq + Ck^2q^2$$
$$- 2Dkp - 2Ekq + F) = 0.$$

In order that this equation be identical with the second, one must have

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}=\frac{-Bq-Ap+\frac{D}{k}}{D'}=\frac{-Bp-Cq+\frac{E}{k}}{E'}$$

$$=\frac{Ap^2+2Bpq+Cq^2-\frac{2D}{k}p-\frac{2E}{k}q+\frac{F}{k^2}}{F'}.$$

The elimination of the three quantities p, q, k from these five conditions will give two equations of condition; or the first two equations $\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}$, which do not involve the parameters to be eliminated, are simply the equations of condition required. Therefore, *in order that two curves of the second degree be homothetic, it is necessary that the coefficients of the terms of the second degree be proportional.*

394. It remains to inquire whether the parameters p, q, k are real or finite; we attack this question in the following manner: since the coefficients of the terms of the second degree are proportional, they can be made equal by multiplying all the terms of the second degree by a suitable factor; consider, therefore, the two equations under the form

$$(4)\qquad Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0,$$

$$(5)\qquad A'x^2+2B'xy+C'y^2+2D'x+2E'y+F'=0.$$

Several cases may arise according to the sign of the quantity $AC-B^2$.

1° $AC-B^2=0$. The two loci belong to the genus parabola; if these loci be parabolas, they are certainly similar, since all parabolas are similar; further, the axes of the two curves, having equal angular coefficients $-\frac{B}{C}$, are parallel, and, consequently, the curves are homothetic.

2° $AC-B^2>0$. The loci belong to the genus ellipse. If, in case of each curve, the axes be transferred parallel to themselves to the center of the curve, equations (4) and (5) become

(6) $$Ax^2 + 2Bxy + Cy^2 + H = 0,$$

(7) $$Ax^2 + 2Bxy + Cy^2 + H' = 0.$$

The axes of the curves, whose directions are determined by the equation $\tan 2\alpha = \frac{2B}{A - C}$ (§ 139), are parallel; if the axes of co-ordinates be revolved through the angle α, equations (6) and (7) reduce to the form

(8) $$A'x^2 + C'y^2 + H = 0,$$

(9) $$A'x^2 + C'y^2 + H' = 0.$$

The coefficients A' and C', whose values are given by the equations

$$A' + C' = A + C, \quad A' - C' = \pm\sqrt{4B^2 + (A - C)^2},$$

have the same sign; in order that equations (8) and (9) represent two real ellipses, it is necessary that the quantities H and H' have the same sign, and that, further, this sign be contrary to that of A' and C'. When this condition is fulfilled, the axes of the two ellipses having the same ratio $\sqrt{\frac{C'}{A'}}$, the ellipses are homothetic.

3° $AC - B^2 < 0$. The two loci belong to the genus hyperbola. On making the same transformation as in the preceding case, one is led to equations (8) and (9), in which A' and C' have contrary signs. When the quantities H and H' are different from zero, each of the equations represents a hyperbola. If H and H' have the same sign, the real axes of the two curves are parallel; and, since the ratio of the axes is the same, the curves are homothetic. When H and H' have contrary signs, one of the hyperbolas is similar to the conjugate of the other. In the two cases the curves have parallel asymptotes.

It follows from what precedes that, when *two equations of the second degree have coefficients of the terms of the second degree proportional, and represent real curves, these curves are homothetic; excepting, when the curves are hyperbolas, it can happen that one is homothetic to the conjugate of the other.*

General Equation of a Species of Curves.

395. *Curves of the same species* are curves comprised within the same geometric definition, and which do not differ from one another excepting in the values assigned to the parameters which are involved in the general definition. The general equation of the curves of the species considered is an equation which, in a system of co-ordinates, gives all these curves, whatever be their position in the plane, when various values are assigned to the variable parameters which it involves. Thus, when the fixed axes are rectangular, the general equation of the species *circle* is $(x-a)^2+(y-b)^2=r^2$. This equation involves three variable parameters, namely, the radius r and the two co-ordinates of the center, a and b.

Accordingly, one seeks the equation of the curve with respect to particular axes, which are chosen to simplify the calculation; then, to obtain the general equation, one refers it to the fixed axes by a transformation of co-ordinates.

The lemniscate has been defined as the locus of points such that the product of the distances of each of them from two points F and F' is equal to the square of half of the distance FF' (Fig. 260). If the mid-point O' of the straight line FF' be taken as origin, $O'F$ and a perpendicular to $O'F$ as axes of co-ordinates, and if the distance $F'F$ be designated by $2c$, the curve, referred to these particular variable axes, is represented by the equation (§ 339)

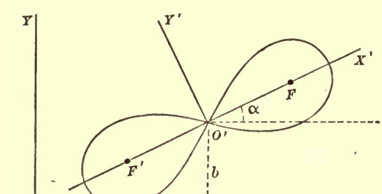

Fig. 260.

$$(x'^2+y'^2)^2+2c^2(y'^2-x'^2)=0.$$

The curve may be referred finally to the fixed rectangular axes OX, OY, by means of the formulas of transformation

$$x'=(x-a)\cos\alpha+(y-b)\sin\alpha,$$

$$y'=-(x-a)\sin\alpha+(y-b)\cos\alpha,$$

in which a and b designate the co-ordinates of the point O' with respect to the fixed axes, and α the angle formed by $O'F$ with OX. One is thus led to an equation

$$F(x, y, c, a, b, \alpha) = 0$$

involving four arbitrary parameters, and which represents all lemniscates; it is the general equation of the species.

The general equation of a species of curves, with respect to fixed axes, contains three parameters more than the equation of the same curves referred to the axes associated with the curves in a determined manner. Let n be the total number of parameters, this system of parameters could always be replaced by another system, such that the variation of three of them causes only the curve to be displaced in its plane, while the variation of the $n-3$ others gives the different curves.

396. The number of points, and, in general, the number of conditions necessary to completely determine a curve of given species, is equal to the number of arbitrary parameters which the general equation of the species involves. The remarks made concerning the curves of the second degree (§ 283), on multiple conditions, are applicable here. It is often important to previously show that the parameters which enter in the equation cannot be reduced to a smaller number.

Consider, for example, the locus of points such that the ratio of their distances from two fixed points whose co-ordinates are (a, b), (a', b') is constant and equal to k. This locus, whose equation is

$$(1) \qquad (x-a)^2 + (y-b)^2 - k^2[(x-a')^2 + (y-b')^2] = 0,$$

is a circle. Equation (1) involves five parameters; but by developing and by arranging the terms, it becomes

$$(2) \quad x^2+y^2-2\frac{a-k^2a'}{1-k^2}x-2\frac{b-k^2b'}{1-k^2}y+\frac{a^2+b^2-k^2(a'^2+b'^2)}{1-k^2}=0.$$

Three of the coefficients only involve the parameters; if they be replaced by A, B, C, equation (2) takes the form

$$(3) \qquad x^2+y^2+Ax+By+C=0.$$

Three points are sufficient to determine the coefficients A, B, C, and, consequently, the circumference. If one wish then to obtain a, b, a', b', k, one will have a system of three equations in five unknown quantities; the solution will be indeterminate, and two of the unknown quantities could be chosen at will; this signifies that, for the same circumference, one can find an infinity of pairs of two points such that the ratio of their distances from each of the points of the circumference from these two fixed points is constant.

397. The geometric definition of a species of curves embraces within it the number of arbitrary parameters involved in its general equation. The definition of the circle assumes that we know its center, whose position is determined by its two co-ordinates, and by the length of the radius, in all three constants or arbitrary parameters. The definition of the lemniscate assumes that two fixed points are given, which are equivalent to four fixed constants. In case of the ellipse, since it is necessary to know the sum of the radii vectores, the curve involves five parameters; in the definition of the spiral of Archimedes one is given a pole, which is equivalent to two constants, the position of the straight line at the instant when the movable point passes through the pole, and a ratio: in all four constants.

398. An integral polynomial of the degree m in the two variables x and y contains $\frac{(m+1)(m+2)}{2}$ terms; it follows that $\frac{(m+1)(m+2)}{2}-1$ or $\frac{m(m+3)}{2}$ points are necessary to define an algebraic curve of the mth degree. For example, nine points are necessary to determine a curve of the third degree, fourteen points to define a curve of the fourth degree.

Consequently, that every equation of the third degree can be written in the form

$$\alpha_1\alpha_2\alpha_3 - k\beta^2\gamma = 0 \tag{4}$$

where α_1, α_2, α_3, β, γ are linear functions of the first degree, and k is an arbitrary parameter because this equation contains eleven arbitrary parameters; one can choose at will one of the five linear functions, since there will still remain nine parameters. The three straight lines $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ are tangents to the curve at the points where they are intersected by the straight line $\beta = 0$; and the points where these tangents are intersected by the straight line $\gamma = 0$ belong moreover to the curve. On taking β at will, the following theorem is deduced: if a curve of the third degree be intersected by any straight line $\beta = 0$, and tangents be drawn to the curve at the three points of intersection, each of the tangents intersects the curve in one additional point, and these three points lie on a straight line. On taking γ at will, we get another theorem: if a curve of the third degree be intersected by a straight line $\gamma = 0$, and if through each of the points of intersection tangents be drawn to the curve, the points of contact of these tangents lie on a straight line.

Suppose that the straight line $\beta = 0$ be removed to infinity, the three tangents $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$ have as limits the three asymptotes of the curve; each of these asymptotes intersects the curve in one point only, and these three points lie on a straight line.

The equation

$$\alpha_1\alpha_2\alpha_3 - k\beta^3 = 0, \tag{5}$$

which involves nine arbitrary parameters, represents all the curves of the third degree. The three points where the straight line $\beta = 0$ intersects the curve are points of inflection; the tangents at these points are the straight lines

$$\alpha_1 = 0, \;\; \alpha_2 = 0, \;\; \alpha_3 = 0.$$

All the curves of the third degree can moreover be represented by the equation

$$\alpha(\alpha - a\gamma)(\alpha - b\gamma) - k\beta^2\gamma = 0, \tag{6}$$

which involves nine arbitrary parameters. The straight line $\gamma = 0$ is the tangent at the point of inflection $(\alpha = 0, \gamma = 0)$; the three tangents $\alpha = 0$, $\alpha - a\gamma = 0$, $\alpha - b\gamma = 0$ at the points where the straight line $\beta = 0$ intersects the curve pass through this point of inflection. On taking the perspective on a plane, any straight line we wish may be removed to infinity; for example, the tangent $\gamma = 0$ at the point of inflection; it is sufficient to make $\gamma = 1$ in the equation which reduces to the form $\alpha(\alpha - a)(\alpha - b) - k\beta^2 = 0$; if the straight line $\beta = 0$ be taken as the x-axis and the straight line $\alpha = 0$ as the y-axis, one obtains the equation $ky^2 = x(x-a)(x-b)$, which has been discussed in § 337.

399. Every equation of the fourth degree can be written in the form

(7) $$\alpha_1\alpha_2\alpha_3\alpha_4 - k\beta^2\phi = 0,$$

when ϕ is a polynomial of the second degree; because this equation contains sixteen parameters, one can take β at will, since there will remain fourteen parameters. Thus when a curve of the fourth degree is intersected by any straight line $\beta = 0$, and tangents be drawn to the curve at the four points of intersection, if the two other points in which each tangent intersects the curve be taken, there will be eight points of the curve situated on the conic $\phi = 0$.

A curve of the fourth degree has four asymptotes; each of them intersects the curve in two points; the eight points of intersection lie on a conic.

Condition of Similitude of Two Figures.

400. Consider a series of curves of the same species, in whose definition there enters only the linear parameter A, whose measure is represented by a, and let

(1) $$f(x, y, a) = 0$$

be the equation which represents all these curves, without reference to their position in the plane. If equation (1) were obtained without specifying the linear unit, it is necessarily homogeneous with respect to x, y, a. When the linear parameter A changes, which happens, the unit remaining the same, if the number a vary, equation (1) defines a series of homothetic curves. In fact, let A_0 be a parameter whose measure is a_0; to this parameter corresponds the particular curve

(2) $$f(x, y, a_0) = 0.$$

The curves homothetic to curve (2), the origin being the center of *homothétie* and k an arbitrary ratio, are represented by the equation (§ 391)

(3) $$f(kx, ky, a_0) = 0.$$

Let A_1 be a second parameter which is measured by a_1 such that $\frac{a_0}{a_1} = k$, equation (3) may be written

$$f(kx, ky, ka_1) = k^m f(x, y, a_1) = 0,$$

(4) or $$f(x, y, a_1) = 0,$$

which shows that the curves homothetic to curve (2) are the various curves which are obtained by varying the parameter A.

401. In general, suppose that n linear parameters $A, B, \cdots$ fail to define all the curves of the same species, without regard to their position in the plane; let $a, b, c, \cdots$ be the measures of these parameters with respect to an arbitrary unit; the equation of the curves of the species

(5) $$f(x, y, a, b, \cdots) = 0$$

will be homogeneous with respect to $x, y, a, b, \cdots$. The curves defined by equation (5) and which correspond to the two series of proportional parameters $A_0, B_0, \cdots$ and $A_1, B_1, \cdots$ are homothetic; because, if k represent the ratio of the parameters

$$\frac{a_0}{a_1} = \frac{b_0}{b_1} = \cdots = k,$$

the curves homothetic to the curve

$$f(x, y, a_0, b_0, \cdots) = 0$$

are represented by the equation

$$f(kx, ky, ka_1, kb_1, \cdots) = k^m f(x, y, a_1, b_1, \cdots) = 0,$$

or $$f(x, y, a_1, b_1, \cdots) = 0.$$

It follows from what precedes that when the curve, without regard to its position in the plane, is defined by a single magnitude, that all the curves of the species are similar. Thus, the circle being defined by its radius, the parabola by the distance of the focus from the directrix, the lemniscate by the distance between the foci, the spiral of Archimedes by the length intercepted on the radius vector between two successive helices, all circumferences are similar, and similarly all parabolas, all lemniscates, etc.

The ellipse being defined by its two axes, the condition of similarity of two ellipses is that these axes are proportional, as we have already learned in § 390. The same is true of two hyperbolas.

CHAPTER VI*

GRAPHIC SOLUTION OF EQUATIONS.

402. Consider two equations in two unknown quantities x and y

(1) $\phi(x, y) = 0,$ (2) $\psi(x, y) = 0;$

each of them defines a curve. For this system of two equations can be substituted an infinity of equivalent systems of equations; consider in particular a system

(3) $\chi(x, y) = 0,$ (4) $f(x) = 0,$

one of which only contains the variable y, a system which may be obtained by eliminating y from the two given equations. The real roots of equation (4) are the abscissas of the points common to the two curves (1) and (2). And yet, if the system of equations (3) and (4) were satisfied by a pair of values of the form $x = \alpha$, $y = \beta + \gamma i$, in which α, β, γ are real, these values would satisfy the system of equations (1) and (2); but the quantity α would not be the abscissa of a real point common to the two curves. The exception which we have pointed out never occurs when the equation $\chi(x, y) = 0$ is an algebraic equation which involves only y to the first degree.

When one wishes to solve an equation $f(x) = 0$ in a single unknown quantity, the curves determined by (1) and (2) may be selected in an infinity of different ways. The only condition to be fulfilled is that the elimination of y between equations (1) and (2) give the proposed equation. A first combination is $y = f(x)$, $y = 0$, which leads one to consider the values of the unknown quantity as the abscissas of the points of intersection of the curve $y = f(x)$ with the x-axis. This combination is rarely the most simple. It is proven in algebra that if an unknown quantity y be eliminated from two algebraic equations in two unknown quantities whose degrees are m and n, the resulting equation in x is at most of the degree mn. Consequently, if the proposed equation be algebraic and one wish to obtain its roots by the intersection of algebraic curves, the product of the degree of the equation of the two curves would be equal to the degree of the equation to be solved. We shall apply this method to the solution of the equation of the fourth degree.

403. The equation of the fourth degree may be easily reduced to the form

(5) $$x^4 + px^2 + qx + r = 0;$$

it may be regarded as the result of the elimination of y between the two equations of the second degree

(6) $x^2 - my = 0,$ (7) $m^2y^2 + pmy + qx + r = 0,$

each of which defines a parabola. Since equation (6) involves y only to the first degree, all the real roots of equation (5) are the abscissas of real points common to the two curves.

One can substitute for the parabola (7) another curve of the second degree which passes through the intersection of the curves (6) and (7). The general equation of the second degree satisfying this condition (§ 277) is

(8) $$kx^2 + m^2y^2 + qx + m(p - k)y + r = 0,$$

k being an arbitrary parameter. If one put $k = m^2$, the curve (8) would be simply a circle; the co-ordinates a and b of the center and the radius R of this circle are given by the formulas

(9) $$a = -\frac{q}{2m^2}, \quad b = \frac{m^2 - p}{2m}, \quad R^2 = a^2 + b^2 - \frac{r}{m^2}.$$

When the value of R^2 is positive, equation (8) represents a real circle, and the real roots of equation (5) are the abscissas of the points of intersection of this circle and parabola (6). When the value of R^2 is negative, equation (8) cannot have real solutions (§ 85); the same is true of the system of equations

(6) and (7), or of the equivalent system of equations (5) and (6); the four roots of equation (5) will be imaginary.

404. Consider next the equation of the third degree reduced to the form

$$x^3 + px + q = 0.$$

If this equation be multiplied by x, which introduces the root $x = 0$, an equation of the fourth degree is obtained,

$$x^4 + px^2 + qx = 0,$$

to which the preceding method is applicable. The value of R^2, being in this case equal to $a^2 + b^2$, is always positive. The circle and the parabola pass through the origin of co-ordinates; the abscissa of this point is the root $x = 0$, which one should remove.

The same parabola $x^2 - my = 0$ may serve for the solution of all equations of the third or of the fourth degree; the circle only changes depending upon the value of the coefficients of the proposed equations. This method can be employed with advantage when one would solve successively a great number of equations; then a parabola having an arbitrary parameter is traced with great care; and, in each particular example, it only remains to determine the circle.

405. When the unknown quantity x is a line, and the unit of length has not been specified, the equation $f(x) = 0$ is a homogeneous equation in the unknown quantity x and the various known lines. In case the equation is of the fourth degree, if the coefficients p, q, r be rational functions, or irrational functions of the second degree of given lengths, on taking an arbitrary length for the parameter m of the parabola, the co-ordinates of the center and the radius of the circle could be constructed with the rule and the compass.

But if the equation be a numerical equation, that is if the coefficients be given numbers, a definite value is given to m; for example, one would put $m = 1$, and construct the parabola and the circle by means of an arbitrary scale; the abscissa of one of the points of intersection measured by the same scale, will give the value of the unknown quantity x.

We know that the solution of two equations of the second degree in two unknown quantities x and y, or the determination of the points of intersection of two curves of the second degree, reduces to the solution of an equation of the fourth degree in one unknown quantity. This solution could therefore be accomplished by means of a definite parabola and a circle. Accordingly, if one of the curves of the second degree be already traced, it can be used with the circle.

406. Example I. — Draw through a given point P whose co-ordinates are x_1 and y_1 a normal to a parabola $y^2 - 2px = 0$. The co-ordinates x and y of the foot of the normal are determined by the system of equations

$$y^2 - 2px = 0, \quad xy - (x_1 - p)y - py_1 = 0.$$

If all the terms of the last equation be multiplied by y, and if $2px$ be substituted for y^2, a new parabola $x^2 - (x_1 - p)x - \frac{y_1 y}{2} = 0$ is obtained; on adding the equations of the two parabolas member to member, one obtains the circle $x^2 + y^2 - (x_1 + p)x - \frac{y_1 y}{2} = 0$. The points where this circle intersects the given parabola are the feet of the normals (§ 306).

407. Example II. — *Solve the numerical equation* $x^3 - x - 7 = 0$.

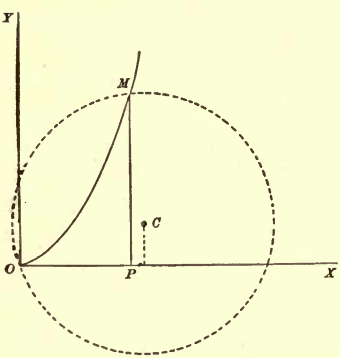

Fig. 261.

Construct by means of an accurately made scale, the parabola $x^2 = y$; describe a circle whose center C has the co-ordinates $a = \frac{7}{2}$, $b = 1$, and which passes through the origin; this circle intersects the parabola in one additional point M; therefore the proposed equation has but one real root, the abscissa OP of the point M (Fig. 261). By measuring this length by means of the scale here employed, we find $x = 2.09$.

Example III. — Solve the equation $x^3 - 5x + 1 = 0$. Construct the circle whose center C has the co-ordinates $a = -\frac{1}{2}$, $b = 3$ and which passes through the origin: this circle intersects the parabola in three points; it follows that the equation has three real roots; on measuring the abscissas, one finds that the two positive roots are 0.20 and 2.13.

408. Example IV. — Consider the transcendental equation

$$x \operatorname{tang} x = 1.$$

This equation is the result of the elimination of y between the two equations

$$y = \operatorname{tang} x, \quad xy = 1.$$

The first represents a curve composed of an infinitude of equal branches which have asymptotes perpendicular to the x-axis; the second an equilateral hyperbola (Fig. 262). It is evident that the right-hand branch of

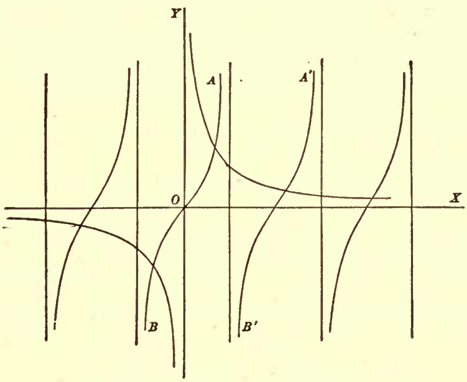

Fig. 262.

the hyperbola intersects, at least once, each of the branches OA, $B'A'$, ... of the transcendental curve; moreover there is but a single point of intersection on each branch, because, when x varies, the ordinates of the two curves vary in contrary directions; if these ordinates be equal for a certain value of x, they are necessarily unequal for every other value. The roots of the equation are equal in pairs with contrary signs; there is in the first place a root situated between 0 and $\frac{\pi}{2}$, a second root between π and $\frac{3\pi}{2}$, a third between 2π and $\frac{5\pi}{2}$, etc., ...; the number of roots is infinite. On calling x_n the nth root, the difference between x and $(n-1)\pi$ is very small when n is very large. The curve gives, for the value of the first root, 0.86.

The equations $y = \text{tang}\left(\frac{\pi}{2} - x\right)$, $y = x$ could also be discussed in this manner on putting $\frac{\pi}{2} - x = x'$, $y = \text{tang}\, x'$, $y = \frac{\pi}{2} - x'$; the hyperbola would be replaced by a straight line.

409. Remarks. — The graphic methods which we have described do not give the values of the unknown quantities with any great precision; one should not expect an approximation nearer than the hundredth part of the root.

One often attacks the problem by magnifying the traces of the two curves in order to determine the number of real roots of an equation. But, so long as the form of the two curves is not studied with care, no rigorous conclusions can be deduced from this discussion. In general, the discussion of the curves and the determination of the points of intersection offer the same difficulties as the problem proposed.

CHAPTER VII*

NOTIONS CONCERNING UNICURSAL CURVES.

410. We have learned (page 339) what facilities offer themselves for studying a curve when the co-ordinates of one of its points can be conveniently expressed as a function of a parameter. From this point of view, the most simple algebraic curves are those whose co-ordinates can be expressed as a *rational* function of a parameter. These particular curves have been called *unicursal curves.*

The theory of unicursal curves was formed by Chasles and Clebsch. The results which we give have been taken in most part from the Memoirs of Clebsch (Crelle Journal, v. LXIV. and LXXIII.) and from the lectures on geometry by Clebsch, published by Lindemann.

I. *A curve of the order* **m** *with a multiple point of the order* **m — 1** *is unicursal.*

If the multiple point be chosen as origin, the equation of the curve becomes (§ 354)

$$\phi_{m-1}(x, y) + \phi_m(x, y) = 0,$$

where ϕ_{m-1} and ϕ_m designate homogeneous polynomials in x and y of the degrees **m** and **m — 1**. A variable straight line

$$y = tx,$$

passing through the origin, intersects the curve in **m — 1** points coincident with the origin, and in *one other* variable point M whose co-ordinates are given by the equations

$$(1) \qquad x = -\frac{\phi_{m-1}(1, t)}{\phi_m(1, t)}, \quad y = -\frac{t\phi_{m-1}(1, t)}{\phi_m(1, t)},$$

that is, are rational functions of the parameter t, of the degree **m,** with the same denominator. To each value of t corresponds on the curve a single point, and conversely to each point M of the curve *distinct* from *the multiple point* $x = 0$, $y = 0$ there corresponds a single value of t, $t = \frac{y}{x}$, equal to the angular coefficient of the straight line OM; as for the multiple point $x = 0$, $y = 0$, it is obtained by $m - 1$ values of t, roots of the equation

$$\phi_{m-1}(1, t) = 0.$$

It can still be said that, if x and y be the co-ordinates of a point of the curve distinct from the multiple point, equation (1) of the degree m in t has a single common root $t = \frac{y}{x}$, and if $x = 0$, $y = 0$, these equations have $\mathbf{m - 1}$ given roots common with the equation $\phi_{m-1}(1, t) = 0$.

The following curves enter in this category of unicursal curves: the *conics* ($m = 2$); the curves of the third order with a cusp or with a double point ($m = 3$), for example, the cissoid and strophoid; then the curves of the fourth order with a triple point, for example, the curve constructed on page 422 ⋯ etc. ⋯. We shall learn later that there exist other unicursal curves of the fourth order, namely, curves having three singular points, double points, or cusps.

II. Unicursal Curves of the Third Order.

Let

(1) $$Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3 = 0$$

be the equation of a curve of the third order having a singular point at the origin. If one make $y = tx$, one obtains, for the co-ordinates of a point M of the curves, the following expressions:

(2) $$x = -\frac{A + Bt + Ct^2}{\phi(t)}, \quad y = -\frac{t(A + Bt + Ct^2)}{\phi(t)}$$

where $$\phi(t) = D + Et + Ft^2 + Gt^3.$$

The values of t, corresponding to the singular point situated at the origin, are roots of the equation of the second degree

$$A + Bt + Ct^2 = 0$$

which give the angular coefficients of the tangents at the origin; the singular point will be a double point or a cusp according as these roots are unequal or equal (§ 354).

A Curve of the Third Order with One Cusp.

Suppose first that these roots are equal: let a be their common value; one will have

$$A + Bt + Ct^2 = C(t-a)^2,$$

whence

$$x = -\frac{C(t-a)^2}{\phi(t)}, \quad y = -\frac{Ct(t-a)^2}{\phi(t)}. \tag{3}$$

Take on the curve three points corresponding to the values t_1, t_2, t_3, of the parameter, and determine the necessary and sufficient condition in order that these points be on a straight line.

Let

$$ux + vy + w = 0$$

be the equation of a straight line which does not pass through the cusp. The values t_1, t_2, t_3 of the parameter t corresponding to the points of intersection of this straight line with the curve are the roots of the equation of the third degree found by substituting in the equation of the straight line the values of x and y in expression (3). By this substitution, the trinomial $ux + vy + w$ becomes a rational function in t of which the denominator is $\phi(t)$ and of which the numerator becomes zero for the values t_1, t_2, t_3 of t. One has therefore by this substitution, identically

$$ux + vy + w = \frac{K(t-t_1)(t-t_2)(t-t_3)}{\phi(t)},$$

K being a constant depending on u, v, w. On taking the derivatives of the two members of this identity with respect to t, we will have another identity

$$ux'_t + vy'_t = \frac{k(t-t_1)(t-t_2)(t-t_3)}{\phi(t)}\left[\frac{1}{t-t_1} + \frac{1}{t-t_2} + \frac{1}{t-t_3} - \frac{\phi'(t)}{\phi(t)}\right].$$

Substitute, in this last identity in t, the value $t = a$ for t, which corresponds to the cusp. Since, for this value, x'_t and y'_t become zero and the factors

$$(a - t_1)(a - t_2)(a - t_3)$$

are not zero, the secant not passing through the cusp, it follows that

(4) $$\frac{1}{a - t_1} + \frac{1}{a - t_2} + \frac{1}{a - t_3} = \frac{\phi'(a)}{\phi(a)}.$$

Since this relation is independent of u, v, w it holds *necessarily* between the parameters t_1, t_2, t_3 of the three points on the straight line. Conversely, if this relation be satisfied, the three corresponding points are on a straight line, for on calling t'_3 the parameter of a point which is on this straight line passing through the first two, one has

$$\frac{1}{a - t_1} + \frac{1}{a - t_2} + \frac{1}{a - t'_3} = \frac{\phi'(a)}{\phi(a)},$$

whence by comparison with (4), $t'_3 = t_3$.

Point of Inflection. The tangent at a point of inflection intersects the curve in three points coincident with the point of contact: if therefore one call θ the parameter of a point of inflection, one will have, by putting

$$t_1 = t_2 = t_3 = \theta$$

in formula (4)

$$\frac{3}{a - \theta} = \frac{\phi'(a)}{\phi(a)}.$$

The curve has, therefore, a single point of inflection

Points on a conic. A conic whose equation is

$$f(x, y) = \alpha x^2 + \beta xy + \gamma y^2 + \delta x + \epsilon y + \eta = 0,$$

and which does not pass through the cusp, intersects the curve in six points, whose parameters t_1, t_2, t_3, t_4, t_5, t_6 are roots of the equation of the sixth degree obtained by replacing, in the equation of the conic, x and y by their values (3). By this substitution $f(x, y)$ becomes a rational fraction of the sixth degree in t, whose denominator is $\phi^2(t)$ and whose numerator is a polynomial which becomes zero for the values $t_1, t_2, t_3, t_4, t_5, t_6$ of t. Hence follows the identity

$$f(x, y) = \frac{H(t-t_1)(t-t_2)\cdots(t-t_6)}{\phi^2(t)},$$

where H is a certain constant independent of t; then, on taking the derivatives of both members with respect to t, one has the identity

$$f'_x x'_t + f'_y y'_t = \frac{H(t-t_1)(t-t_2)\cdots(t-t_6)}{\phi^2(t)}\left[\frac{1}{t-t_1} + \frac{1}{t-t_2} + \cdots + \frac{1}{t-t_6} - \frac{2\phi'(t)}{\phi(t)}\right];$$

and on putting in this last identity $t = a$, one has

$$(5) \quad \frac{1}{a-t_1} + \frac{1}{a-t_2} + \frac{1}{a-t_3} + \frac{1}{a-t_4} + \frac{1}{a-t_5} + \frac{1}{a-t_6} = \frac{2\phi'(a)}{\phi(a)},$$

the *necessary* condition in order that the six points corresponding to these six values of t be on a conic. It may be seen, as above, that this condition is sufficient.

These considerations are, in particular, applicable to the pedal or to the inverse curve of a parabola, with respect to a point of the parabola.

Curves of the Third Order with a Double Point.

Let a and b be the roots of the equation

$$A + Bt + Ct^2 = 0,$$

which gives the angular coefficients of the tangents at the singular point (page 435); if these roots be imaginary, the double point is isolated. The expressions for the co-ordinates x and y of a point of the curve are

$$(6) \qquad x = \frac{-C(t-a)(t-b)}{\phi(t)}, \quad y = \frac{-Ct(t-a)(t-b)}{\phi(t)}.$$

Find the necessary and sufficient condition, in order that the three points corresponding to the values t_1, t_2, t_3 of the parameter lie on a straight line. Let $ux + vy + w$ be the first member of an equation of a straight line which does not pass through a double point; if x and y be replaced by their expressions in t, this first member becomes a rational fraction, whose denominator is $\phi(t)$, and whose numerator becomes zero for the values t_1, t_2, t_3 of t. One has therefore by this substitution the identical equation

$$ux + vy + w = \frac{K(t - t_1)(t - t_2)(t - t_3)}{\phi(t)}.$$

Substitute successively for t, in this identity, the values a and b corresponding to a double point; since x and y become zero for either of these two values, it follows

$$w = \frac{K(a - t_1)(a - t_2)(a - t_3)}{\phi(a)},$$

$$w = \frac{K(b - t_1)(b - t_2)(b - t_3)}{\phi(b)},$$

whence by division

$$\frac{(a - t_1)(a - t_2)(a - t_3)}{(b - t_1)(b - t_2)(b - t_3)} = \frac{\phi(a)}{\phi(b)}. \tag{7}$$

This is the necessary condition, that three points in a straight line lie on the curve; it may be shown, as on page 507, that it is sufficient.

If the double point be real (the pedal of a parabola with respect to an external point, the inverse curve of a hyperbola with respect to a point of a curve), the constants $a, b, \phi(a), \phi(b)$ which enter in relation (7) are real. If the double point is isolated (the pedal of a parabola with respect to an interior point, the inverse curve of an ellipse with respect to a point of the curve) these constants a and b on the one hand, $\phi(a)$ and $\phi(b)$ on the other, are conjugate imaginaries. Relation (7) can then be replaced by another which involves only real elements. For this purpose suppose

$$a = p + iq,\ \ b = p - iq,\ \ \frac{\phi(a)}{\phi(b)} = \cos 2\lambda + i \sin 2\lambda$$

when 2λ designates the argument of the constant quantity $\frac{\phi(a)}{\phi(b)}$, whose modulus is 1, since $\phi(a)$ and $\phi(b)$ are conjugate imaginaries. Put

$$\frac{p-t}{q}=\cot\tau,\ \frac{a-t}{b-t}=\frac{\cos\tau+i\sin\tau}{\cos\tau-i\sin\tau}=\cos 2\tau+i\sin 2\tau;$$

to each value of τ corresponds one value of t; to each value of t correspond an infinitude of values of τ differing from each other by multiples of π. Call τ_1, τ_2, τ_3 the three values of τ which correspond to the parameters t_1, t_2, t_3 of the three points on the straight line; relation (7) gives us

$$\cos 2(\tau_1+\tau_2+\tau_3)+i\sin 2(\tau_1+\tau_2+\tau_3)=\cos 2\lambda+i\sin 2\lambda,$$

whence

(8) $$\tau_1+\tau_2+\tau_3=\lambda+k\pi,$$

where k is any integer. On making in these formulas (7) and (8) $t_1=t_2=t_3=t$, or $\tau_1=\tau_2=\tau_3=\tau$, one finds the value of the parameters t or τ which correspond to points of inflection. One can easily demonstrate that: *there are three points of inflection; these points lie on a straight line. If a and b be real, one only of these points is real; if a and b be imaginary, the three points of inflection are real.*

The necessary and sufficient condition in order that six points of the curve lie on a conic can be written

$$\frac{(a-t_1)(a-t_2)(a-t_3)(a-t_4)(a-t_5)(a-t_6)}{(b-t_1)(b-t_2)(b-t_3)(b-t_4)(b-t_5)(b-t_6)}=\left[\frac{\phi(a)}{\phi(b)}\right]^2,$$

or if a and b be imaginary,

$$\tau_1+\tau_2+\tau_3+\tau_4+\tau_5+\tau_6=2\lambda+k\pi.$$

III. The curves which we have thus far studied constitute *all unicursal curves* of the third order. In fact, let

(9) $$x=\frac{P(t)}{R(t)},\ y=\frac{Q(t)}{R(t)}$$

be the expressions of the co-ordinates of a point of the curve, where $P(t)$, $Q(t)$, $R(t)$ designate polynomials of the third degree in t which do not have a common divisor. To each value of t corresponds a single point of the curve; suppose that conversely, save for certain special points *finite in number*, to each point (x, y) of this curve corresponds one value only of t. This is equivalent to saying that the equations

$$(10) \qquad \frac{P(t)}{R(t)} = \frac{P(t')}{R(t')}, \quad \frac{Q(t)}{R(t)} = \frac{Q(t')}{R(t')}$$

do not have more than a finite number of solutions in which t is different from t', or, moreover, that there is but a finite number of values of x and y for which the equations in t

$$(9) \qquad xR(t) - P(t) = 0, \; yR(t) - Q(t) = 0$$

have two common roots. Then the curve defined by equations (9) is of the third order and has *one singular point;* equations (9) have two common roots when the point (x, y) coincides with this singular point.

Then the curve is of the third order, for on changing the values of the parameter corresponding to the points of intersection of the curve with a straight line, one gets an equation of the third degree. Whence, the equation of the tangent at the point with the parameter t is (§ 341)

$$Y - y = \frac{y'_t}{x'_t}(X - x);$$

that is, after introducing the expressions of x and y in t,

$$X(RQ' - QR') + Y(PR' - RP') + QP' - PQ' = 0.$$

In the combination such as $RQ' - QR'$, the term in t^5 disappears, and the equation of the tangent contains t to the fourth power at most. If, therefore, one seek the values of t corresponding to the points of contact of the tangents drawn from a point to the curve, one finds at most *four values* for t. The curve considered is therefore, at most, of the fourth class; it has a singular point, because a curve of the third order without a singular point is of the sixth class.

IV. The following proposition concerning curves of the fourth order will be proven:

The curve of the fourth order with three singular points (node or cusps) is unicursal. Let A, B, C be the singular points, and D a fixed point taken on the curves. A variable conic

$$S + tS_1 = 0,$$

which passes through the four points A, B, C, D, intersects the curve in eight points, seven of which are fixed; namely, two coincident with A, two with B, two with C, and one with D. The equation of the eighth degree, which gives the abscissas of the points of intersection of the conic and of the curve, has therefore seven fixed roots independent of t; one could suppress these roots by division, and there would remain an equation of the first degree in x expressing x as a *rational function* of t. Similarly, y may be obtained as a rational function of t. To each point of the curve different from the points A, B, C corresponds a value of the parameter t given by the equation

$$S + tS_1 = 0;$$

to the double points correspond two values of the parameter, which become equal when the double point becomes a cusp.

In this category of curves belong the lemniscate, the hypocycloid with three cusps, the pedals or the inverse curves of an ellipse, and of a hyperbola with respect to a point not situated on these curves.

It can be demonstrated that, conversely, every unicursal curve of the fourth order has a triple point or three singular points (node or cusp).

V. Since a curve of the third order cannot have two and a curve of the fourth order cannot have four singular points, the unicursal curves of these orders are those which have the maximum number of singular points. This theorem can be made general for unicursal curves of all degrees. In fact, it can be proven that:

A unicursal curve of the nth order, which does not have higher singularities than nodes and cusps, may have nodes or cusps $\frac{(n-1)(n-2)}{2}$ in number; and, conversely, a curve of the nth order with $\frac{(n-1)(n-2)}{2}$ nodes or cusps is unicursal.

This number, $\frac{(n-1)(n-2)}{2}$, is the *maximum* number of singular points which a curve of the nth order can have without decomposition.

If one be given the expressions of the co-ordinates of a point of an unicursal curve as a rational function of a parameter

(11) $$x = \phi(t), \quad y = \psi(t),$$

the values of t corresponding to the singular points are found by seeking the solutions common to the two equations

$$\phi(t) = \phi(t'), \quad \psi(t) = \psi(t'),$$

in which t is different from t'.

It is to be noticed that the equation of the curve is obtained by eliminating t between equations (11); that is, by expressing the conditions that these equations have a common root in t. If the point (x, y) be an ordinary point of the curve, equations (11) do not have more than *one common root;* if the point (x, y) coincide with a singular point, they have two or more roots in common, which are the values of t corresponding to the singular point. The singular points are therefore found by seeking the positions of the point (x, y) for which equations (11) in t have two or more roots in common.

EXERCISES.

1. A point M is taken on a curve of the third order with a cusp. The tangent at M intersects the curve in a point M_1, the tangent in M_1 intersects it in M_2, the tangent at M_2 intersects it in M_3, etc., $\cdots$. Show that the nth point M_n thus determined, when n is indefinitely increased, approaches the cusp.

From the point M a tangent can be drawn to the curve. Let M' be its point of contact; from the point M' a second tangent can be drawn to it; let M_2 be its point of contact, etc., $\cdots$. Show that the nth point $M^{(n)}$ thus determined approaches the point of inflection.

2. Let M be a point on a curve of the third order with a double point. The tangent at M intersects the curve at M_1, the tangent at M_1 intersects it at M_2, etc., $\cdots$; let M_n be the nth point thus determined.

If the tangents at the double point be real, the point M_n approaches, when n is increased without limit, the double point. If these tangents be imaginary, it can happen that the point M_n coincides with the point of departure M. One will then have a polygon of n sides whose vertices lie on the curve and the sides tangent to the curve. What positions is it necessary that the point M should take in order that this should happen? Study the particular cases $n = 3, 4, 5$. (Durège, *Math. Annalen*, Erster Band.)

3. From a point of inflection I, of a curve of the third order, one can draw, at the double point, a tangent IT to this curve having its point of contact at T. Show that the straight lines which join the two points I and T to the double point are harmonic conjugates with respect to the tangents at the double point.

4. The co-ordinates of a point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ can be expressed as a function of an angle ϕ by the formulas $x = a\cos\phi$, $y = b\sin\phi$. Show that the necessary and sufficient condition, in order that four points of the ellipse correspond to the values ϕ_1, ϕ_2, ϕ_3, ϕ_4 of ϕ should lie on the same circle, is

$$\phi_1 + \phi_2 + \phi_3 + \phi_4 = 2\,k\pi.$$

Through a point M taken on the ellipse three circles osculating the ellipse can be passed (without intersecting the one which has its point of contact at M); prove that the points of contact of these three circles lie on a circle which passes through M.

At the point M draw a circle osculating the ellipse; let M_1 be the point in which this circle intersects the curve; the osculating circle at M_1 intersects the ellipse at M_2; etc., $\cdots$. What should be the position of the point M in order that the point M_n obtained by repeating the construction n times should coincide with M? Study the particular cases $n = 1, 2, 3, 4$.

5. One is given an ellipse and a point P in its plane. 1° Find the number of circles osculating the ellipse so that each of the chords common to the ellipse and to the different circles passes through the point. 2° Find, for the different position of the point P, how many of these circles are real. 3° Prove that the points of contact of the circles osculating the ellipse are on the circle C. 4° Find the envelope E of these circles when the point P describes the ellipse. 5° The curve E may be regarded as the envelope of a series of circles which intersect at right angles a fixed circle and whose centers lie on a conic; determine in how many different ways the curve E may be generated in this way.

6. Express the co-ordinates of a point of a hyperbola as a rational function of a parameter t. What relation connects the parameters t_1, t_2, t_3, t_4 of the four points situated on a circle? How many circles osculating a hyperbola can be drawn through a point. The circle osculating the hyperbola at M intersects it in M_1, the osculating circle at M_1 intersects it at M_2, etc., $\cdots$; what will be the limiting position of the nth point M_n thus constructed, when n is increased without limit?

7. Consider a conic C whose co-ordinates are expressed as a rational function of a parameter t in such a way that to each point of the curve corresponds a single value t, and let A and B be two fixed points not situated on the conic.

Prove that the necessary and sufficient condition that the points t_1, t_2, t_3, t_4 of the conic C are situated on a conic passing through A and B is

$$\frac{(t_1 - a)(t_2 - a)(t_3 - a)(t_4 - a)}{(t_1 - b)(t_2 - b)(t_3 - b)(t_4 - b)} = C^{\text{th}},$$

a and b representing the values of t which correspond to the two points in which the straight line AB intersects the conic C. In what way is it necessary to modify this relation when the straight line AB is tangent to C?

8. One is given a unicursal curve of the fourth order with a triple point at which the tangents to the curve are distinct; express its co-ordinates x and y as a rational function of a parameter t and call a, b, c the three values of t which determine the triple point. Prove that the necessary and sufficient conditions that the four points t_1, t_2, t_3, t_4 situated on the curve lie on a straight line are:

$$\frac{(t_1 - a)(t_2 - a)(t_3 - a)(t_4 - a)}{(t_1 - c)(t_2 - c)(t_3 - c)(t_4 - c)} = \frac{\phi(a)}{\phi(c)},$$

$$\frac{(t_1 - b)(t_2 - b)(t_3 - b)(t_4 - b)}{(t_1 - c)(t_2 - c)(t_3 - c)(t_4 - c)} = \frac{\phi(b)}{\phi(c)},$$

$\phi(t)$ being the common denominator of the expressions in x and y Deduce the number of points of inflection and of double tangents of the curve.

In what way is it necessary to modify the preceding relations in case two of the three tangents at the triple point coincide?

Apply the preceding formulas to the curve constructed in § 342; and to the curves whose equations are

$$(x^2 + y^2)^2 - ay(x^2 - y^2) = 0,$$

$$(x^2 + y^2)^2 - a(x - y)^2 y = 0,$$

$$(x^2 + y^2)^2 - ay^3 = 0.$$

9. One is given a unicursal curve of the fourth order having three double points corresponding respectively to the values $(t = a,\ t = b)$, $(t=a',\ t=b')$, $(t = a'',\ t=b'')$. Prove that the necessary and sufficient conditions in order that the four points t_1, t_2, t_3, t_4 lie on a straight line, take the forms

$$\frac{(t_1-a)(t_2-a)(t_3-a)(t_4-a)}{(t_1-b)(t_2-b)(t_3-b)(t_4-b)}=K,$$

$$\frac{(t_1-a')(t_2-a')(t_3-a')(t_4-a')}{(t_1-b')(t_2-b')(t_3-b')(t_4-b')}=K'$$

$$\frac{(t_1-a'')(t_2-a'')(t_3-a'')(t_4-a'')}{(t_1-b'')(t_2-b'')(t_3-b'')(t_4-b'')}=K'',$$

equations which reduce to two.

Derive the number of points of inflection and double tangents to the curve.

In what way is it necessary to modify the conditions when one, two, or three double points become cusps?

Apply these formulas to the lemniscate (§ 339, 2°), to the hypocycloid with three cusps (curve generated by a point of a circumference which rolls within a circumference with a radius three times that of the rolling circle).

10. Two normals which intercept between them a portion of the axis of the parabola of constant length are drawn to a given parabola. Find the locus of their point of intersection.

11. Two parabolas whose axes include a constant angle are circumscribed about a triangle: find the locus of the fourth point of intersection of these parabolas. (École Polytechnique, 1874.)

12. a and b are the rectilinear rectangular co-ordinates of a point M; what is, for each position of this point, the nature of the roots of the equation

$$3t^4+8at^3-12bt^2+4b=0\,?$$

Construct, in particular, the locus of the positions of the point M for which the equation has a double root, and show that the co-ordinates of a point of the locus is a function of this root. (École Normale, 1884.)

13. Four tangents can be drawn from a point M of a lemniscate to this curve besides the tangent which touches the curve at M. Prove that the four points of contact of these tangents are on a straight line, and find the envelope of this straight line when the point M describes a lemniscate.

14. The envelope of the normals to a unicursal curve is a unicursal curve. What is the order and the class of this envelope in case of a conic; a cubic unicursal curve?

15. One of the foci of a conic inscribed in a given triangle describes a given conic; prove that the other focus describes a unicursal curve of the fourth order. (Astor, *Nouv. Annales*, 1885.)

16. A straight line δ intersects three sides of a triangle ABC in three points P, Q, R situated respectively on the sides BC, CA, AB. The harmonic conjugate P' of P with respect to BC, Q' of Q with respect to CA, R' of R with respect to AB are constructed; the three straight lines AP', BQ', CR' intersect in a point M.

Find the locus of this point:

1° When the straight line δ passes through a fixed point;

2° When this straight line envelops a conic.

NOTE.

We give below a collection of elementary examples for the application of the principles discussed in the text in giving an exposition of the theory of the *point* and *straight line*, the *circle*, the *parabola*, *ellipse* and *hyperbola*.

EXAMPLES I

CONCERNING THE POINT.

1. Find the points whose co-ordinates are (0, 1), (– 2, 1), (– 5, 0), (– 2, – 3).

2. Draw a triangle, the co-ordinates of whose angular points are (0, 0), (2, – 3), (– 1, 0), and find the co-ordinates of the middle points of its sides.

3. A straight line cuts the positive part of the axis of y at a distance 4, and the negative part of the axis x at a distance 3 from the origin; find the co-ordinates of the point where the part intercepted by the axes is cut in the ratio 3 : 1, the smaller segment being adjacent to the axis of x.

4. There are two points P (7, 8), and Q (4, 4); find the distance PQ, (1) with rectangular axes, and (2) with axes inclined at an angle of 60°.

5. Work Ex. 4 when P is (– 2, 0), Q (– 5, – 3).

6. The co-ordinates of P are $x = 2$, $y = 3$, and of Q, $x = 3$, $y = 4$; find the co-ordinates of R, so that $PR : RQ :: 3 : 4$.

7. The polar co-ordinates of P are $\rho = 5$, $\theta = 75°$, and of Q, $\rho = 4$, $\theta = 15°$; find the distance PQ.

8. Find the polar co-ordinates of the points whose rectangular co-ordinates are

(1) $x = \sqrt{3}$, $y = 1$; (2) $x = -\sqrt{3}$, $y = 1$; (3) $x = -1$, $y = 1$;

and draw a figure in each case.

9. Find the rectangular co-ordinates of the points whose polar co-ordinates are

(1) $\rho = 5$, $\theta = \frac{\pi}{4}$; (2) $\rho = -5$, $\theta = \frac{\pi}{3}$; (3) $\rho = 5$, $\theta = -\frac{\pi}{4}$;

and draw a figure in each case.

10. Transform the equations

$$x \cos \alpha + y \sin \alpha = a, \quad x^2 + xy + y^2 = b^2,$$

from rectangular to polar co-ordinates.

11. Transform the equation $\rho^2 = a^2 \cos 2\theta$ from polar to rectangular co-ordinates.

12. A straight line joins the points (2, 3) and (– 2, – 3); find the co-ordinates of the points which divide the line into three equal parts.

13. If ABC be a triangle, and AB, AC are taken as axes of x and y, find the co-ordinates (1) of the bisection of BC, (2) of the point where the perpendicular from A meets BC, and (3) of the point where the line bisecting the angle BAC meets BC.

14. Find the co-ordinates of the same points, when AB is the axis of x, and a straight line drawn from A perpendicular to AB the axis of y.

15. The rectangular co-ordinates of a point S are h and k, and a straight line PS is drawn, making an angle θ with the axis of x; show that the co-ordinates of P are

$$x = h + \rho \cos \theta, \quad y = k + \rho \sin \theta,$$

where $SP = \rho$.

EXAMPLES II

CONCERNING THE STRAIGHT LINE.

1. Draw the lines whose equations are

(1) $y = 5x + 2$, (2) $y - 7 = 5x + 3$, (3) $7y - 3x = 0$,

(4) $6 - x = 2y$, (5) $\frac{x}{3} + \frac{y}{11} = 2$, (6) $2x + 3 = 0$.

2. Find the equation to the straight line which passes through the points (2, 5), and (0, – 7).

3. The co-ordinates of the angular points of a triangle being given, find the equations to the three straight lines, each of which bisects two of the sides.

4. Two straight lines make each of them an angle of 45° with the axis of x, and their intercepts on the axis of y are 6 and 8; find the equation to the straight line which is equidistant from the two, the axes being rectangular.

5. Find the equation to a straight line on which the perpendicular from the origin = 6, and makes (1) an angle of 45°, and (2) an angle of 225° with the axis of x, the axes being rectangular.

6. Determine the point of intersection of the two lines ($3y - x = 0$) and ($2x + y = 1$).

7. Find the equation to the straight line which passes through the point of intersection of the straight lines

$$x - 2y - a = 0, \quad x + 3y - 2a = 0,$$

and is parallel to the line $3x + 4y = 0$.

8. Find the equation to a straight line which is equidistant from the two lines represented by the equation $y = mx + c \pm c'$.

9. Find the equation to the straight line that joins the points of intersection of the two pairs of lines

$$\left.\begin{matrix} 2x + 3y - 4a = 0, \\ 2x + y - a = 0, \end{matrix}\right\} \text{ and } \left.\begin{matrix} x + 6y - 7a = 0, \\ 3x - 2y + 2a = 0. \end{matrix}\right\}$$

10. Find the length of the perpendicular from the origin on the line $a(x - a) + b(y - b) = 0$, and the portion intercepted by the axes, which are rectangular.

11. The rectangular co-ordinates of two points are 3, 5 and 4, 4 respectively; find the equation to a straight line which bisects the distance between them and makes an angle of 45° with the axis of x.

12. Find the equation to a straight line which passes through a given point (a, b) and makes equal angles with the axes.

13. Find the equations to the diagonals of the parallelogram formed by the four lines $x = a$, $x = a'$, $y = b$, $y = b'$.

14. A straight line, inclined to the axis of x at an angle of 150°, cuts the positive axes of rectangular co-ordinates in A and B; find the equation to a straight line bisecting AB and passing through the origin.

15. Find the equations to the four sides of a square, the co-ordinates of two of its opposite angular points being (2, 3) and (3, 4), the co-ordinates being rectangular.

16. Find the distance of the origin of co-ordinates from the line $\frac{x}{2} + \frac{y}{3} = 1$, the axes being rectangular.

17. Find the equation to a straight line which passes through the intersection of the lines $x = a$, $x + y + a = 0$, and through the origin.

18. The axes of co-ordinates being inclined to each other at an angle of 60°, find the equation to a straight line parallel to the line $(x + y = 3a)$ and a distance from it equal to $\frac{1}{2}a\sqrt{3}$.

19. Show that the lines $y = 2x + 3$, $y = 3x + 4$, $y = 4x + 5$, all pass through one point.

20. Find the value of m, in order that the line $(y = mx + 3)$ may pass through the intersection of the lines $(y = x + 1)$ and $(y = 2x + 2)$.

21. A straight line cuts off intercepts on the axes, the sum of the reciprocals of which is a constant quantity; show that all straight lines which fulfill this condition pass through a fixed point.

22. A straight line slides along axes of x and y, and the difference of the intercepts is always proportional to the area it incloses; show that the line always passes through a fixed point.

23. If the distance of a point from the origin equals twice its distance from the axis of x, show that it always lies in one of two straight lines that pass through the origin; axes rectangular.

24. Find the cosine of the angle which the line $(Ax + By + C)$ makes with the axis of x, the axes being inclined at an angle of 45°.

25. If a straight line cuts the (rectangular) axes of x and y at equal distances from the origin, and a straight line be drawn from the origin, dividing it in the ratio $m : n$, find the tangent of the angle which this latter line makes with the axis of x.

26. An equilateral triangle, whose side $= a$, has its vertex at the origin, and its sides equally inclined to the positive directions of rectangular axes; find the co-ordinates of the angles, and thence of the point bisecting the base.

27. Find the polar co-ordinates of the point of intersection of the lines

$$\left\{\rho = 2a \sec\left(\theta - \frac{\pi}{2}\right)\right\} \text{ and } \left\{\rho = a \sec\left(\theta - \frac{\pi}{6}\right)\right\},$$

and the angle between them.

28. Trace the line whose polar equation is

$$\rho = 2a\cos\left(\theta + \frac{\pi}{6}\right).$$

29. Show that the polar equation to a straight line, passing through the points (ρ', θ'), (ρ'', θ''), is

$$\rho'\rho\sin(\theta' - \theta) + \rho''\rho'\sin(\theta'' - \theta') + \rho\rho''\sin(\theta - \theta'') = 0.$$

What is the geometrical interpretation of this equation?

EXAMPLES III

CONCERNING THE ANGLES FORMED BY STRAIGHT LINES AND THE STRAIGHT LINES REPRESENTED BY EQUATIONS OF SECOND AND HIGHER DEGREES.

1. Find the equation to the straight lines which pass through the point (1, 3), and make an angle of 30° with the line $(2y - x + 1 = 0)$; axes being rectangular.

2. Draw the lines represented by the equation

$$(2y - x + c)(3y + x - c) = 0,$$

and determine (1) where they intersect, and (2) at what angle; the axes being rectangular.

3. Find the equation to a straight line which passes through the point $(c, 0)$, and makes an angle of 45° with the line $(bx - ay = ab)$; axes being rectangular.

4. Find the equation to a straight line which is perpendicular to the line $(8y + 5x - 3 = 0)$ and cuts the axis of y at a distance $= 8$ from the origin; axes being rectangular.

5. Find the cosine of the angle between the lines

$$(y - 4x + 8 = 0) \text{ and } (y - 6x + 9 = 0);$$

axes being rectangular.

6. Find the angle between the lines

$$(4y + 3x + 5 = 0) \text{ and } (4x - 3y + 6 = 0);$$

axes being rectangular.

7. Find the equations to the straight lines which pass through the intersection of the lines $(y = 2x + 4)$, $(y = 3x + 6)$, and bisect the supplementary angles between them; axes being rectangular.

8. What is the geometrical signification of the equations

$$x^2 + y^2 = 0, \qquad xy = 0\,?$$

9. Find the equations to the straight lines which bisect the angles between the lines $(12x + 5y = 8)$ and $(3x - 4y = 3)$; axes being rectangular.

10. Show that the lines represented by the equation

$$6y^2 + xy - 2x + y - x^2 - 1 = 0,$$

are inclined to one another at an angle of 45°; axes being rectangular.

11. The equation $2y^2 - 3xy - 2x^2 - 3y + 6x = 0$ represents two straight lines at right angles; axes being rectangular.

12. The equation $y^2 - 2xy \sec\theta + x^2 = 0$ represents two straight lines inclined to one another at an angle θ; axes being rectangular.

13. What is the inclination of the co-ordinate axes, when the lines represented by $y^2 - x^2 = 0$, are perpendicular to one another?

14. The equations to two straight lines are

$$x + 3y - a = 0 \cdots (1), \qquad y - x + a = 0 \cdots (2);$$

find the equations to the straight lines which pass through the intersections of (1) and (2), so that the ratio of the sines of the inclination of each to (1) and (2) may be as $1 : \sqrt{5}$.

15. What must be the inclination of the axes in order that the lines $(xy - 3y - 2x + 6 = 0)$ may include an angle of 135°?

16. Find the equations to the two straight lines which pass through the origin and divide into three equal parts the distance between the points in which the axes of co-ordinates are intersected by the line $(x + y = 1)$.

7. Find the distance of the point of intersection of the lines

$$(3x + 2y + 4 = 0),\ (2x + 5y + 8 = 0),$$

from the line $(y = 5x + 6)$; the axes being rectangular.

EXAMPLES IV

CONCERNING TRANSFORMATION OF CO-ORDINATES.

1. The equation of a right line is

$$3x + 5y - 15 = 0;$$

find the equation of the same line referred to parallel axes whose origin is at (1, 2). *Ans.* $3x + 5y = 2$.

2. The equation of a locus is

$$x^2 + y^2 - 4x - 6y = 18;$$

what will this equation become if the origin be moved to the point (2, 3)? *Ans.* $x^2 + y^2 = 31$.

3. The equation of a locus is $y^2 - x^2 = 16$; what will this equation become if transformed to axes bisecting the angles between the given axes? *Ans.* $xy = 8$.

4. Transform the equation $2x^2 - 5xy + 2y^2 = 4$ from axes inclined to each other at an angle of 60°, to the axes which bisect the angles between the given axes. *Ans.* $x^2 - 27y^2 + 12 = 0$.

5. Transform the equation $y^2 + 4ay\cot\alpha - 4ax = 0$ from a rectangular system to an oblique system inclined at an angle α, the origin remaining the same, and the new axis of x coinciding with the old.
Ans. $y^2\sin^2\alpha = 4ax$.

6. The equation of a locus is $x^4 + y^4 + 6x^2y^2 = 2$; what will be the equation if the axes are turned through an angle of 45°?
Ans. $x^4 + y^4 = 1$.

7. Transform $x^2 + y^2 = 7ax$ to polar co-ordinates, the pole being at the origin, and the initial line coincident with the axis of x.
Ans. $r = 7a\cos\theta$.

8. Change the equations $r^2 = a^2\cos 2\theta$ and $r^2\cos 2\theta = a^2$ into equations between x and y. *Ans.* $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ and $x^2 - y^2 = a^2$.

EXAMPLES V

CONCERNING THE CIRCLE.

1. To find the center and radius of the circle

$$x^2 + y^2 - 6x + 4y + 4 = 0.$$

2. Investigate the line or lines represented by the equation

$$x^3 + xy^2 - x^2r - xr^2 - ry^2 + r^3 = 0.$$

3. Find the common chord of two circles

$$(x-1)^2 + (y-2)^2 = 6, \quad (x-2)^2 + (y-3)^2 = 8.$$

4. To find the equation to a straight line which passes through the centers of the two circles

$$x^2 + 2x + y^2 = 0, \quad y^2 + 2y + x^2 = 0.$$

5. To find the equation to a circle having for its diameter the straight line joining the points of intersection of the line $y = mx$ and the circle $y^2 = 2rx - x^2$.

6. Find the equation to the circle the diameter of which is the common chord of the circles

$$x^2 + y^2 = r^2, \quad (x-a)^2 + y^2 = r^2.$$

7. What is represented by the equation

$$x(x-2) + y(y-4) + 8 = 0\,?$$

8. Find a relation between the coefficients of the equation

$$A(x^2 + y^2) + Dx + Ey + F = 0,$$

in order that (1) the axis of x, and (2) the axis of y, may be tangents to the circle.

9. To find the inclination to the axis of x of the tangents drawn from any point (x', y') to the circle whose equation is

$$(x-a)^2 + (y-b)^2 - r^2 = 0.$$

10. To find the relation between the quantities a, b, r, in order that the line $\frac{x}{a} + \frac{y}{b} = 1$ may touch the circle $x^2 + y^2 = r^2$.

11. To find the equation to a circle, the center of which is at the origin of co-ordinates, and which is touched by the line

$$y = 2x + 3.$$

12. To find the intercepts on the axes of co-ordinates of the tangent to a circle $(x^2 + y^2 = r^2)$, drawn parallel to a given straight line

$$(x \cos\alpha + y \sin\alpha = p).$$

13. If $2\alpha'$, $2\alpha''$ be the inclination of two radii of a circle $x^2 + y^2 = r^2$ to the axis of x, to find the equation to the chord joining the extremities of the radii.

14. If the pole always lie on a line

$$\frac{x}{a} + \frac{y}{b} = 1,$$

and the equation to the circle is $x^2 + y^2 = r^2$, the equation to the polar is of the form

$$(ax - r^2) + k(by - r^2) = 0,$$

where k is any constant.

15. If the pole of a straight line with regard to the circle $x^2 + y^2 = r^2$ lie on the circle $x^2 + y^2 = 4r^2$, the polar will touch the circle

$$x^2 + y^2 = \frac{r^2}{4}.$$

16. Find the equation to the circle which has each of the co-ordinates of the center $= -\frac{1}{3}$, and its radius $= \frac{2}{\sqrt{3}}$, the axes being inclined at an angle of 60°.

17. Prove that the circles

$$x^2 + y^2 = (c + a)^2, \quad (x - a)^2 + y^2 = c^2$$

have only one common tangent, and find its equation.

18. Find the locus of the mid-points of chords drawn from the extremity of the diameter of any circle.

19. Show that the polar of the point (x', y') with regard to the circle $(x - a)^2 + (y - b)^2 = r^2$ is

$$(x - a)(x' - a) + (y - b)(y' - b) = r^2.$$

20. Find the locus of the vertices of all triangles which have a given base and a given vertical angle.

21. Prove Euc. III. 31, from the resulting equation.

22. Tangents are drawn to a circle $x^2 + y^2 = r^2$, at two points (x', y'), (x'', y''); to find the distance of a point (h, k) from a straight line passing through the center and the intersection of the two tangents.

23. To find the equations to straight lines touching a circle

$$x^2 + y^2 = 10,$$

at points, the common abscissa of which is unity.

24. Find the equation to a straight line touching the circle

$$(x - a)^2 + (y - b)^2 = r^2,$$

and parallel to a given line $y = mx + c$.

25. To find the equation to the straight line passing through the origin of co-ordinates, and touching the circle

$$x^2 + y^2 - 3x + 4y = 0.$$

26. To find the length of the common chord of the circles

$$(x - a)^2 + (y - b)^2 = r^2, \quad (x - b)^2 + (y - a)^2 = r^2.$$

27. Find the area between the two circles

$$x^2 + 2x + y^2 + 4y = 0, \quad x^2 + 2x + y^2 + 4y = 1.$$

28. To find the length of the chord of a circle $x^2 + y^2 = r^2$, made by the straight line $\frac{x}{a} + \frac{y}{b} = 1$.

29. If from a given point S, a perpendicular be drawn to the tangent PY at any point P of a circle, of which the center is C, and, in the line MP at right angles to CS and produced if necessary, a point Q be taken, such that $QM = SY$, to find the locus of Q.

30. Given the equation to a circle, and the chord of a circle; show that a perpendicular let fall upon the chord from the center bisects the chord.

31. Find the diameter of the circle

$$x^2 + y^2 + 2xy \cos \omega = ax + by.$$

32. In the equation $Ax + By + C = 0$, if C be constant, and A and B vary, subject to the condition $A^2 + B^2 =$ a constant, the equation represents a series of tangents to a given circle.

33. Find the equation to the circle which passes through the points $(0, 0)$, $(-8a, 0)$, $(0, 6a)$, the axes being rectangular.

34. To find the locus of mid-points of chords which pass through a given point.

35. If on any radius vector through a fixed point O, OQ be taken in a constant ratio to OP, find the locus of Q.

36. The circles represented by the equation

$$(n + 1)(x^2 + y^2) = ax + nby,$$

where n is arbitrary, have a common chord.

37. Prove algebraically that the angles in the same segment of a circle are equal, and that the angle in a semicircle is a right angle.

38. Two sides of a triangle are b and c, and they include an angle A; if these sides be taken as axes, the equation to the circumscribed circle is

$$x^2 + y^2 + 2xy \cos A - bx - cy = 0.$$

39. Given the base and vertical angle, to show that the locus of the point of intersection of the perpendiculars from the angles on the sides is a circle.

40. Given base and ratio of sides of a triangle, show that the locus of the vertex is a circle.

41. When will the locus of a point be a circle, if the square of its distance from the base of a triangle be in a constant ratio to the product of its distances from the sides?

42. When will the locus of a point be a circle, if the sum of the squares of the three perpendiculars from it on the sides of a triangle be constant?

43. Find the locus of a point, the square of whose distance from a given point is proportional to its distance from a given right line.

44. Given the base of a triangle, and m times the square of one of its sides $\pm\, n$ times the square of the other $=$ a constant; find locus of the vertex, find center and radius of resulting circle, and where it cuts base.

45. Find the equations to the circles which touch the three lines, referred to rectangular axes,

$$x = a, \quad y = 2\,b, \quad y = 2\,b'.$$

46. The locus of the centers of all circles inscribed in all right-angled triangles on the same hypotenuse is the quadrant described on the hypotenuse.

47. The equation to a circle is $y^2 + x^2 = a(y + x)$; what is the equation to that diameter which passes through the origin of co-ordinates?

48. To find the equation to a circle referred to two tangents at right angles, as axes.

49. If through any point of a quadrant whose radius is R two circles be drawn touching the bounding radii of the quadrant, and r, r' be the radii of these circles, $rr' = R^2$.

50. To find the equations to the straight lines which touch both the circles

$$x^2 + y^2 = r^2, \quad (x - a)^2 + y^2 = r'^2.$$

51. To find the equation to the circle which touches the three straight lines, referred to rectangular axes,

$$x = 0, \quad y = 0, \quad \frac{x}{a} + \frac{y}{b} = 1.$$

52. To find the equations to two circles, which touch rectangular axes of x and y, and pass through a given point $(a,\, b)$.

53. The straight lines joining the angles of a triangle with the points in which the escribed circles touch the opposite sides, meet in a point.

54. In any circle draw a chord AB; from the mid-point E of the lesser segment draw any straight line cutting AB in C, and meeting the circumference in D; join AD, and in AD take $AP = AC$; find locus of P.

55. The axes Ox, Oy cut a circle in points A, A', B, B' respectively; to compare the values of x, y at the intersection of the chords AB', $A'B$.

56. Determine the magnitude and position of the circle

$$\rho^2 - 2\rho(\cos\theta + \sqrt{3}\sin\theta) = 5.$$

EXAMPLES VI

Transform the following equation, illustrating each transformation by a figure, as at the end of § 144.

1. $1 + 2x + 3y^2 = 0$ to $y^2 = -\frac{2}{3}x$.

2. $3x^2 + 2y^2 - 2x + y - 1 = 0$ to $72x^2 + 48y^2 = 35$.

3. $3x^2 + 2xy + 3y^2 - 16y + 23 = 0$ to $4x^2 + 2y^2 = 1$.

4. $y^2 - 10xy + x^2 + y + x + 1 = 0$ to $32x^2 - 48y^2 = 9$.

5. $y^2 - 2xy + x^2 - 6y - 6x + 9 = 0$ to $y^2 = 3\sqrt{2}x$.

6. $y^2 + xy + x^2 + y + x - 5 = 0$ to $9x^2 + 3y^2 = 32$.

7. $y^2 - 2xy - x^2 + 2 = 0$ to $y^2 - x^2 + \sqrt{2} = 0$.

8. $y^2 - x^2 - y = 0$ to $4x^2 - 4y^2 + 1 = 0$.

9. Show by transformation that the equation

$$12xy + 8x - 27y - 18 = 0$$

represents two straight lines parallel to the axes.

10. Show by transformation that the equation

$$y^2 - 2xy + 3x^2 - 2y - 10x + 19 = 0$$

represents two imaginary straight lines passing through the point (3, 4).

11. Show by transformation that the equation

$$y^2 - 4xy + 5x^2 + 2y - 4x + 2 = 0$$

represents an imaginary ellipse.

12. Show that any point on the line $(y = x + 1)$ is a center of the locus

$$y^2 - 2xy + x^2 - 2y + 2x = 0.$$

13. Show by transformation that the equation

$$y^2 + 2xy + x^2 + 1 = 0$$

represents two imaginary parallel straight lines.

14. What is the equation to the axis in Ex. 5?

15. Transform $7y^2 + 16xy + 16x^2 + 32y + 64x + 28 = 0$, the axes being inclined at an angle of 60°, to $y^2 + 4x^2 = 9$, the axes being rectangular, and the axis of x remaining the same.

EXAMPLES VII

CONCERNING THE PARABOLA.

1. Find the intersections of the parabola $y^2 = 8x$ and the line $3y - 2x - 8 = 0$. *Ans.* (2, 4) and (8, 8).

2. Find the equation of the right line passing through the focus of the parabola $y^2 = 4x$, and making an angle of 45° with the axis of the curve. *Ans.* $y = x - 1$.

3. Find the points in which the focal chord $y = x - 1$ intersects the parabola $y^2 = 4x$. *Ans.* $(3 \pm 2\sqrt{2}, 2 \pm 2\sqrt{2})$.

4. Find the equation of the right line passing through the vertex of any parabola and the extremity of the focal ordinate. *Ans.* $y = 2x$.

5. Find the equation of the circle which passes through the vertex of any parabola and the extremities of the double ordinate through the focus. *Ans.* $y^2 = \frac{5}{2}px - x^2$.

6. Find the equation of the circle which passes through the vertex of the parabola $y^2 = 12x$ and the extremities of the double ordinate through the focus. *Ans.* $y^2 = 15x - x^2$.

7. Find the equations of the tangent and normal to any parabola at the extremity of the positive ordinate through the focus. *Ans.* $y = x + \frac{1}{2}p$ and $y + x = \frac{3}{2}p$.

8. Find the equations of the tangent and normal to the parabola $y^2 = 4x$, at the extremity of the positive ordinate through the focus. *Ans.* $y = x + 1$; $y + x = 3$.

9. Find the point where the normal in Ex. 7 meets the curve again, and the length of the intercepted chord. *Ans.* $(\frac{9}{2}p, -3p)$; length of chord $= 4p\sqrt{2}$.

10. Find the point where the normal in Ex. 8 meets the curve again, and the length of the intercepted chord. *Ans.* $(9, -6)$; length of chord $= 8\sqrt{2}$.

11. Find the point in a parabola where the tangent is inclined at an angle of 30° to the axis of x. *Ans.* $(\frac{3}{2}p, p\sqrt{3})$.

12. Prove that the normal at any point of a parabola bisects the angle between the focal line and the diameter passing through that point.

13. On a parabola whose latus rectum is 10, a tangent is drawn at the point whose ordinate is 6, the origin being at the principal vertex; determine where the tangent cuts the two co-ordinate axes. *Ans.* $(-3.6, 0)$ and $(0, 3)$.

14. Determine where the normal in the preceding example, at the same point, if produced, will cut the two axes. *Ans.* $(8.6, 0)$ and $(0, 10.3)$.

15. Find the angle which the tangent in Ex. 14 makes with the axis of x. *Ans.* $39° 48' 20''$.

16. In the parabola $y^2 = 12x$, find the length of the perpendicular from the focus to the tangent at the point whose abscissa is 9. *Ans.* 6.

17. In the parabola $y^2 = 8x$, find the length of the normal at the point whose abscissa is 6. *Ans.* 8.

18. The extremities of any chord of a parabola being (x', y'), (x'', y''), and the abscissa of its intersection with the axis of the curve being x, to prove that $y'x'' = x^2$, $y'y'' = -2px$.

19. Two tangents of a parabola meet the curve in (x', y') and (x'', y''), their point of intersection being (x, y); show that

$$x = \sqrt{x'x''}, \quad y = \frac{y' + y''}{2}.$$

EXAMPLES VIII

The following problems are enunciated, some for the ellipse and some for the hyperbola, though many of them are equally applicable to both curves.

1. Find the semi-axes of the ellipse $3y^2 + 2x^2 = 6$.

Comparing this equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we find

$$a = \sqrt{3}, \text{ and } b = \sqrt{2}, \textit{ Ans.}$$

2. Find the semi-axes of the ellipse $4y^2 + 3x^2 = 19$. *Ans.* $a = \sqrt{\frac{19}{3}}$, $b = \sqrt{\frac{19}{4}}$.

3. Find the points of intersection of the parabola $y^2 = 4x$ and the ellipse $3y^2 + 2x^2 = 14$. *Ans.* (1, 2) and (1, -2).

4. Find the equation of a tangent to the ellipse $3y^2 + 2x^2 = 35$, at the point whose abscissa is 2. *Ans.* $9y + 4x = 35$.

5. Find the eccentricity of the ellipse $2x^2 + 3y^2 = a^2$. *Ans.* Eccentricity $= \sqrt{\frac{1}{3}}$.

6. Find the equation of the tangent to the ellipse at the end of the latus rectum; also, find the lengths of the intercepts of this tangent on the two axes.

Ans. $y + ex = a$; the intercepts are $\frac{a}{e}$ on the axis of x, and a on the axis of y.

7. Find the axes of the hyperbola whose equation is $3y^2 - 2x^2 + 12 = 0$; also the eccentricity of the given and the conjugate hyperbola and the parameter. *Ans.* $a = \sqrt{6}$, $b = 2$; $e = \sqrt{\frac{5}{3}}$; $e' = \sqrt{\frac{5}{2}}$; $2p = \frac{8}{\sqrt{6}}$.

8. Find the intersection of the hyperbola $3y^2 - 2x^2 + 12 = 0$ and the circle $x^2 + y^2 = 16$. *Ans.* $(\pm 2\sqrt{3}, \pm 2)$.

9. Find whether the line $y = \frac{3}{4}x$ cuts the hyperbola $5y^2 - 2x^2 = -15$, or its conjugate. *Ans.* It cuts the conjugate.

10. Find the equation of an hyperbola of given transverse axis, whose vertex bisects the distance between the center and the focus.

Ans. $y^2 - 3x^2 = -3a^2$.

11. Show $\tan\frac{PSH}{2}\tan\frac{PHS}{2} = \frac{1-e}{1+e}$, where P is any point on an ellipse and H and S are its foci.

12. Find the points of intersection of an ellipse and hyperbola whose equations are $x^2 + 2y^2 = 1$, $3x^2 - 6y^2 = 1$, and show that at each of these points the tangent to the ellipse is the normal to the hyperbola.

13. If CA, CB be the semi-axes of an ellipse, show that, when SBH is a right angle, $CA^2 : CB^2 = 2 : 1$.

14. Find the condition that the line $\left(\frac{x}{m} + \frac{y}{n} = 1\right)$ should touch the hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$.

15. The tangent to an ellipse is inclined to the major axis at an angle ϕ; show that the area included by this tangent and the axes is

$$= \tfrac{1}{2}(a^2 \tan\phi + b^2 \cot\phi).$$

16. The circle described on any radius vector SP of an ellipse as diameter will touch the circle on the axis major.

17. Find where the tangents from the foot of the directrix will meet the hyperbola, and what angle they will make with the transverse axis.

18. Find the equation to the tangent at the extremity of the latus rectum of an ellipse whose equation is $\frac{x^2}{9a^2} + \frac{y^2}{4a^2} = 1$.

19. A tangent at the extremity of the latus rectum of an hyperbola meets any ordinate PM produced in R; show that $SP = MR$, where S is the focus through which the latus rectum passes.

20. Show that the equation to the normal, at the point whose eccentric angle is ϕ, is $ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2$.

21. Find the radius of a circle inscribed in a semi-ellipse, touching the axis minor.

22. From the point where the circle on the major axis is intersected by the minor axis produced, a tangent is drawn to the ellipse; find the point of contact.

23. If from the extremities of the minor axis two straight lines be drawn through any point in the ellipse, and intersect the axis major in Q and R, then $CQ \cdot CR = CA^2$.

24. If a tangent be drawn to the interior of two concentric ellipses, the axes of which are in the same straight line, meeting the exterior one in P, Q and at P, Q tangents be drawn to the latter, intersecting in R, prove that the locus of R is an ellipse.

25. Show that the locus of one end of a given straight line, whose other end and a given point in it move in straight lines at right angles to one another, is an ellipse.

26. If with the co-ordinates of any point in an elliptic quadrant as semi-axes, a concentric ellipse be described, the chord of the quadrant of the one will be a tangent to the other.

27. The locus of the center of a circle touching two circles externally is an hyperbola.

28. The locus of the center of a circle touched by one circle externally, and one internally, is an hyperbola.

29. Find the locus of the extremity of the perpendicular from the center on the tangent to the hyperbola.

30. If θ, ϕ be the eccentric angles of two points on an ellipse, the equation to the chord joining the two points is

$$\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}.$$

Hence deduce the equation to the tangent at the point whose eccentric angle is ϕ.

31. If $3\,AC = 2\,CS$ in an hyperbola, find the inclination of the asymptotes to the transverse axis.

32. If from a point P in an hyperbola, PK be drawn parallel to the transverse axis, cutting the asymptotes in I and K, then $PK \cdot PI = a^2$, or, if parallel to the conjugate, $PK \cdot PI = b^2$.

33. Is the point (2, 3) *without* or *within* the hyperbola $2\,x^2 - 3\,y^2 = 7$? Show that the straight line, joining this point with the point (6, 4), cuts the curve.

34. If A, A' be the extremities of the major axis of an ellipse, T the point where the tangent at P meets AA', QTR a line perpendicular to AA', and meeting AP, $A'P$ in Q and R respectively, then $QT = TR$.

35. Find the eccentricity and latus rectum of the conic

$$2\,y^2 + x^2 + 4\,y - 2\,x - 6 = 0,$$

the axes being rectangular.

36. Find the equations to the asymptotes of the curve

$$3x^2 - 10xy + 3y^2 - 8 = 0;$$

and find the angle between the asymptotes of

$$y^2 - 10xy + x^2 + y + x + 1 = 0,$$

the axes in the latter case being rectangular.

37. In the equilateral hyperbola, the eccentricity is the ratio of the diagonal of a square to its side.

38. A tangent at any point P of an ellipse meets the axis major produced in T, and the axis minor produced in t; to find the locus of a point Q in Tt such that $QT : Qt = m : n$.

39. To find the locus of the intersection of the ordinate of any point in an ellipse produced with the perpendicular from the center upon the tangent at that point.

40. If the normal at P meet the axis major of an ellipse in G, and GK be drawn perpendicular to SP, $GK = e \cdot PM$, where PM is the ordinate of P.

41. If SQ be drawn, always bisecting the angle PSC, in an ellipse, and equal to a mean proportional between SC and SP, find the eccentricity of the curve which is the locus of Q.

42. Two straight lines, such that the product of the tangents of their inclinations to the axis of x is constant, touch an ellipse; show that the locus of their intersection is an ellipse, or hyperbola, according as the product is negative or positive.

43. Show that the locus of the summit of a movable right angle, one side of which touches one, and the other side the other, of two confocal ellipses, is a concentric circle.

44. An ellipse and hyperbola have the same foci and coincident axes; they cut each other at right angles.

45. If P be any point in the hyperbola, S and H the foci, find the locus of the center of the circle which is inscribed in SPH.

46. If a tangent at any point of an hyperbola be intersected by the tangents at the vertices in H and K, the circle on HK as diameter passes through the foci.

EXAMPLES IX

CONCERNING THE ELLIPSE, HYPERBOLA, AND THEIR CONJUGATE DIAMETERS.

1. If CP, CQ be semi-diameters at right angles to each other,

$$\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

2. If, from the focus S of an ellipse, perpendiculars be drawn on CP, CD conjugate diameters, these perpendiculars produced backwards will intersect CD and CP in the directrix.

3. If ρ, r and ρ', r' be respectively the focal distances of two points P, D, the extremities of a pair of conjugate diameters of an ellipse, then

$$\rho r + \rho' r' = a^2 + b^2.$$

4. If a tangent to an hyperbola at P cut off CT, Ct from the axes, then $PT \cdot Pt = CD^2$, CD, being the semi-conjugate diameter.

5. In the equilateral hyperbola, the conjugate diameters make equal angles with the asymptotes.

6. From the extremities P, D of two conjugate diameters, normals are drawn to the major axis of an ellipse; the sum of the squares of these two $= \frac{b^2}{a^2}(a^2 + b^2)$.

7. If the tangent at the vertex A cut any two conjugate diameters of an ellipse produced in T and t, then $AT \cdot At = b^2$.

8. The lengths of the equal conjugate diameters of an ellipse are $\sqrt{2(a^2 + b^2)}$, and the eccentric angles of their extremities are 45° and 135°.

9. The locus of the mid-points of chords of an ellipse, which pass through a fixed point, is an ellipse with the same eccentricity; and if the fixed point be the focus, the major axis of the ellipse is SC.

10. The tangent at any point of an hyperbola is produced to meet the asymptotes; show that the triangle cut off is of constant area.

11. If the asymptotes of the hyperbola are axes, show that the equation to one directrix is $x + y - a = 0$.

12. If any two tangents be drawn to an hyperbola, and their intersections with the asymptotes be joined, the joining lines will be parallel.

13. Show that the locus of the points of quadrisection of all parallel chords in a circle is a concentric ellipse.

14. If the angle between the equal conjugate diameters of an ellipse is 60°, find the eccentricity.

15. If a be the angle between two conjugate diameters which make angles θ, θ' with the axis major,

$$\cos a = e^2 \cos\theta \cos\theta'.$$

16. CP, CD are semi-conjugate diameters of an ellipse, and PF is a perpendicular let fall from P on CD or CD produced; determine the locus of F.

17. The chords joining the extremities of the conjugate diameters of an ellipse will all touch, in their mid-points, a concentric ellipse with axes $a\sqrt{2}$, $b\sqrt{2}$ coincident with those of the original curve.

18. If a circle be described from the focus of an hyperbola, with radius equal to half the conjugate axis, it will touch the asymptotes in the points where they are cut by the directrix.

19. Trace the curve, referred to rectangular axes,

$$\frac{(4x-3y)^2}{4} + \frac{(3x+4y+6)^2}{9} = 25.$$

20. The radius of a circle, which touches an hyperbola and its asymptotes, is equal to that part of the latus rectum produced, which is intercepted between the curve and the asymptote.

21. The equation to the diameter conjugate to

$$\frac{x}{c} - \frac{y}{s} = 0 \text{ is } \frac{x}{c} + \frac{y}{s} = 0,$$

the hyperbola being referred to its asymptotes, c and s are the cos and sin of the angle formed by the line and the x-axis.

22. An ellipse being traced upon a plane, draw the axes and the directrix, and find the focus.

23. Find the angle between the asymptotes of the hyperbola $xy = bx^2 + c$, the axes being rectangular; and write the equation to the conjugate hyperbola.

24. Tangents are drawn to an hyperbola, and the portions intercepted by the asymptotes are divided in a given ratio; show that the locus of the point of division is an hyperbola.

25. Draw the asymptotes of the hyperbolas

$$xy - 2x - 3y - 2 = 0, \quad xy + 2x^2 + 3 = 0,$$

and place the curves in the proper angles.

26. Find the locus of the intersection of tangents to an ellipse, which are parallel to conjugate diameters.

27. Find the equation to the locus of the mid-points of all chords of a given length, in an ellipse.

28. If two concentric equilateral hyperbolas be described, the axes of the one being the asymptotes of the other, they will intersect at right angles.

29. If P be the mid-point of a straight line AB, which is so drawn as to cut off a constant area from the corner of a square, its locus is an equilateral hyperbola.

30. If S and H be the foci of an equilateral hyperbola, and a circle be described upon SH, then the quadrantal chord of this circle shall be a tangent to that described upon the transverse axis.

31. If α be the acute angle between the axes of co-ordinates of the ellipse $(x^2 + y^2 = c^2)$, find the lengths of the axes and the eccentricity.

32. If AA' be any diameter of a circle, PQ any ordinate to it, then the locus of the intersections of AP, $A'Q$ is an equilateral hyperbola.

33. In an equilateral hyperbola, focal chords parallel to conjugate diameters are equal.

34. If a series of straight lines have their extremities in two straight lines at right angles to one another, and all pass through a given point, the locus of their mid-points is an equilateral hyperbola.

35. PQ is an ordinate to the axis major AA' of an ellipse, meeting the curve in P and Q; draw AP, $A'Q$ intersecting in R; the locus of R is an hyperbola with the same center and axes.

36. If tangents be drawn, making a given angle with the axes of all ellipses having the same foci, the locus of the point of contact is an equilateral hyperbola.

37. If normals be drawn to an ellipse from a given point within it, the points where they meet the curve will all lie in an equilateral hyperbola which passes through the given point, and has its asymptotes parallel to the axes of the ellipse.

38. Find the locus of the mid-points of chords in a circle, which touch a concentric ellipse.

39. If normals be drawn from the extremities of conjugate diameters to an hyperbola, and the point of their intersection be joined to the center, this line produced shall be perpendicular to the straight line passing through the extremities of the conjugate diameters.

40. Given in position, a straight line AB and a point P outside it; a straight line PM is drawn, intersecting AB in C, from the extremity M of which a perpendicular MD on AB intercepts CD of a given magnitude; find the locus of M.

41. The locus of the centers of all circles, which cut off from the directions of two sides of a triangle chords equal to two given straight lines, is an equilateral hyperbola, having two conjugate diameters in the directions of these sides.

42. A straight line passes through a given point and is terminated in the sides of a given angle; find the locus of the point which divides it in a given ratio.

43. From a point P perpendiculars are dropped upon the sides of a given angle, so as to contain a quadrilateral of given area; show that the locus of P is an hyperbola whose center is the vertex of the given angle.

44. Given the base of a triangle and the difference of the tangents of the base angles; show that the locus of the vertex is an hyperbola, of which the perpendicular through the center of the base is an asymptote.

45. If about the exterior focus of an hyperbola, a circle be described with radius equal to the semi-conjugate axis, and tangents be drawn to it from any point in the hyperbola, the straight line joining the points of contact will touch the circle described on the transverse axis as diameter.

46. If, from the center of an equilateral hyperbola, a straight line be drawn through any point P, and if ϕ and ϕ' be the angles which this line and the polar of P respectively make with the transverse axis, then

$$\tan\phi \tan\phi' = 1.$$

47. Prove that the circle which passes through any three of the four points in which the equilateral hyperbola

$$x^2 + 2hxy - y^2 + 2gx + 2fy + c = 0$$

cuts the rectangular co-ordinate axes, is equal to the circle

$$x^2 + y^2 + 2gx + 2fy = 0.$$

48. Find the locus of the mid-points of a system of parallel chords, drawn between an hyperbola and the conjugate hyperbola.

49. If, in two concentric hyperbolas, whose axes are coincident, two points be taken whose abscissas are as the transverse axes of the hyperbolas, the locus of the mid-point of the straight line joining them is an hyperbola, whose axes are arithmetical means between those of the given hyperbolas.

50. If tangents be drawn from different points of an ellipse, of lengths equal to n times the semi-conjugate diameter at the point, the locus of their extremities will be a concentric ellipse with semi-axes equal to $a\sqrt{n^2+1}$, $b\sqrt{n^2+1}$.

51. If a length $PQ = CD$ be taken in the normal to an ellipse, the locus of the point Q is a circle whose radius $= a - b$ or $a + b$, according as Q is taken within or without the ellipse.

QUESTIONS PROPOSED FOR VARIOUS EXAMINATIONS

ÉCOLE POLYTECHNIQUE.

1860. A parabola P is given; let A and B be two variable points on this curve, but so chosen that the normals at A and B intersect in a point C situated on P. The tangent at C and the straight line AB intersect in a point N; find the locus described by this point and construct the curve.

1863. Two circumferences O, O' are given in a plane; from a point A situated on O, tangents are drawn to O', and the points thus determined are joined. This straight line intersects the tangent at O, at the point A, in M; find the locus described by M.

Investigate the different forms of this locus according to the relative magnitudes and position of the two circumferences O and O'; indicate the cases in which the locus is decomposable. Show that the locus of the points M is tangent to O at each of the points which are common to it and this circumference.

1864. Construct the circle whose equation is

$$x^2 + y^2 = 1,$$

and the parabola which has the equation

$$(\beta x - \alpha y)^2 + 2\,\alpha x + 2\,\beta y = \frac{3\,\alpha^2 + \beta^2 - 1}{\alpha^2},$$

where α and β are any positive parameters. It is proposed to determine:

1° The number of real points common to the two curves, for the different values of α and β.

2° The co-ordinates of the four common points when

$$\alpha^2 + \beta^2 = 1,$$

or

$$\alpha = 1, \text{ and } \beta > 0;$$

finally, when we have

$$\beta = \sqrt{(\alpha^2 - 1)(4\,\alpha^2 - 1)}.$$

1865. A parabola P is given in a plane and a circumference C passing through the focus P is considered. It is required to find the regions of the plane which the center C will occupy in order that this circumference may have successively in common with P: 1° four real points;

2° four imaginary points; 3° two real points and two imaginary points. Study the form and the properties of the curve which separate the first two regions from the third.

1866. Consider the parabola and the equilateral hyperbola which correspond, respectively, to the equations

$$y^2 - 2px = 0, \quad xy - m^2 = 0.$$

It is proposed:

1° To construct the equation whose roots are the abscissas or the ordinates of the feet of the normals common to these two curves.

2° To prove from this equation that the number of common normals is at least equal to *one*, and at most to *three*.

3° To demonstrate that if one have

$$7p^4 > 2m^4,$$

there can be but one real common normal.

1867. A triangle BOA, right-angled at O, and a straight line D situated in the plane of this triangle are given; it is proposed:

1° To construct the general equation of the equilateral hyperbolas circumscribed about the triangle BOA; 2° to calculate the equation of the locus L of the points of contact where these different hyperbolas have as tangents straight lines which are parallel to D; 3° to study the different forms of the locus L which correspond to the various directions of the straight line D.

1868. Let P_1, P_2 be two parabolas whose foci coincide with the fixed point O, and, as axis respectively, the fixed straight lines OX, OY, which are supposed to be perpendicular. A common tangent is drawn to these parabolas: let M_1 and M_2 be the points of contact; find the locus described by the mid-point of M_1M_2 when the straight line passes through a fixed point.

1869. A right-angled isosceles triangle AOB is given and it is required:

1° To find the general equation of the parabolas P which are tangent to the three sides of the triangle AOB;

2° To determine the equation of the axis of any of these parabolas;

3° To determine the equation and the form of the locus of the projections of the point O, the vertex of the right angle AOB, upon the axes of the parabolas P.

1872. Two rectangular axes of co-ordinates and two straight lines (A) and (B), respectively parallel to these axes, are given. It is required:

1° To construct the general equation of the curves of the second degree which have the origin of co-ordinates as center and which are normal to the given straight lines (A), (B);

2° To demonstrate that in general three of these curves pass through a point of the plane, namely, two ellipses and one hyperbola ;

3° To find the points of the plane for which this general rule fails.

1873. A circle and a point A are given, and it is required to find the locus of the centers of the equilateral hyperbolas which pass through the given point A and touch the given circle in two points.

Discuss the curve determined by the different positions of the point A and demonstrate that, in the general case, the points of contact of the tangents which can be drawn to the locus through the point A are situated on the circumference of the given circle.

1874. A triangle is given, and it is known that, through a point M of its plane, there pass, in general, two parabolas circumscribed about the triangle. With this condition it is required to construct and to discuss the locus of the point M for which the axes of the two corresponding parabolas inclose a constant angle.

1875. Find the geometrical locus of the intersection of the two normals drawn to the parabola at the two extremities of all the chords whose orthogonal projections upon a perpendicular to the axis have the same value.

What will happen in case this value of the projection approaches zero as a limit ?

Returning to the general case, draw through any point of the locus three normals to the parabola.

Special application to the maximum point of the locus.

1875. A conic of given form and magnitude is so displaced that each of its foci lie on a given straight line. A tangent is drawn parallel to one of the given straight lines to the conic ; find the locus of contact.

1878. 1. 1° Give a study of Newton's method, based on the consideration of successive derivatives, for determining the superior limit of the positive roots of an equation.

2° Construct the curve represented in rectangular co-ordinates by the two equations

$$x = \frac{t}{1 - t^2}, \quad y = \frac{t\,(1 - 2\,t^2)}{1 - t^2}.$$

2. A straight line D whose equation with respect to two rectangular axes ox and oy is

$$\frac{x}{p} + \frac{y}{q} = 1$$

is given.

Consider the various conics whose axes are ox and oy and which are normal to the straight line D. Each of them intersects this straight line in two points ; at these points tangents are drawn to the conic.

Find the locus of the point of contact of these tangents.

Demonstrate that this locus is a parabola and that the distance of the focus of this parabola from its vertex is the fourth of the distance of the point O from the directrix D.

Construct geometrically the axis and vertex of the parabola.

1879. 1. 1° Show how to deduce from Sturm's theorem the conditions for the reality of all the roots of an algebraic equation of a given degree.

2° Construct the curve whose equation in polar co-ordinates is

$$\rho = \frac{\sin \omega}{2\omega - 3\cos \omega}.$$

2. A conic,

$$\frac{x^2}{a} + \frac{y^2}{b} = 1,$$

referred to its axes and a point M of the conic are given. A circle is drawn through the extremities of any diameter of the conic and the point M. Prove that the locus described by the center of this circle is a conic K which passes through the origin O of the axes.

If two lines, which are perpendicular, be made to revolve about the point O, they will intersect the conic K in two points; prove that the locus of the points of intersection of the tangents drawn at these points is a straight line perpendicular to the segment OM and passes through the mid-point of this segment.

Through the point O can be drawn, independently of the normal whose foot is at O, three other straight lines normal to the conic K.

1° In the particular case where the given conic is an equilateral hyperbola and where $a = 1$, $b = -1$, show that one only of these normals is real, and calculate the co-ordinates of its foot.

2° Find the equation to the circle, in the general case, which passes through the feet of these three normals.

NOTE. — The foot of the normal is the point at which the normal is drawn to the curve.

1880. Let M and N be the points in which the x-axis intersects the circle

$$x^2 + y^2 = R^2;$$

consider any of the equilateral hyperbolas which pass through the points M and N; draw through a point Q, taken arbitrarily on the circle, tangents to the hyperbola; let A and B be the points in which the circle intersects the straight line which joins the points of contact.

Demonstrate that, of the two straight lines QA and QB, one has a fixed direction and the other passes through a fixed point P.

If the point P be fixed, the corresponding equilateral hyperbola which passes through the points M and N is determined. Construct geometrically its center, its asymptotes, and its vertices.

If the point P describe the straight line $y = x$, what is the locus described by the foci of the hyperbola? Determine its equation and construct it.

1881. 1. Consider a parabola P and a straight line AB normal to this curve at the point A (the point A having the focus as its projection upon the axis).

Find the locus of the vertices of the sections made by the plane which passes through AB in the right cylinder whose base is the parabola P.

2. An asymptote and a point P of a hyperbola are given. Suppose that one of the foci describe the perpendicular drawn from P to the given asymptote, find the locus of the point M of the intersection of the second asymptote with the directrix corresponding to the given focus.

1882. Two circles which intersect at the points A and B are given. Any conic which passes through these points and is tangent to the two circles intersects the equilateral hyperbola which has these points as vertices in two other points, C and D.

1° Demonstrate that the straight line CD passes through one of the centers of similitude of the two given circles.

2° If all the conics which pass through A and B and which are tangent to the two circles be considered, demonstrate that the locus of their centers is composed of two circles E and F.

3° Consider a conic which satisfies the conditions of the problem and which has its center on one of the circumferences E or F; demonstrate that the asymptotes of this conic intersect this circumference in two fixed points situated on the radical axis of the two given circumferences.

1883. A parabola and straight line are given. Find the locus of points such that the tangents drawn from each of them to the parabola form with the given straight line a triangle of given area.

1884. A conic

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

is given. Join the point M of this conic with the two foci F and F'.

1° It is required to express the co-ordinates of the circle inscribed within the triangle MFF', in terms of the co-ordinates of the point M.

2° In the case when the given conic is an ellipse, demonstrate that if one consider the circles inscribed within the two triangles corresponding to the two points M and M' of the conic, the radical axis of these two circles passes through the mid-point of the segment MM'.

3° For each position of the point M, the radius vector FM touches the corresponding circle at a point P. Determine in polar co-ordinates the equation of the points described by P. Take the focus F as the pole, and the axis of x as the initial line.

1885. The circumference of a variable circle is passed through the two foci of an ellipse.

1° What condition should this ellipse satisfy in order that the circumference of the circle would intersect it in four real points, and in what portion of the minor axis must the center of the circle be placed in order that the four points of intersection be real?

2° Tangents are drawn to the ellipse at each of the points of intersection; find, as the circle varies, the locus of the vertices of the quadrilateral formed by these tangents.

3° What is the locus of the points of intersection of the sides of this quadrilateral with those of another quadrilateral, the *symétrique* of the first with respect to the center of the ellipse?

4° Consider the tangents common to the ellipse and to the circle; find the locus of their points of contact with the circle.

1886. A rectangle $ABA'B'$ is given. Two equilateral hyperbolas A and B, whose asymptotes are parallel to the sides of the rectangle, pass, the one (A) through the opposite vertices A and A', the other (B) through the opposite vertices B and B' of the rectangle.

1° Demonstrate that the center of the hyperbola A has with respect to the hyperbola B the same polar P which the center of the hyperbola B has with respect to the hyperbola A.

2° The rectangle remaining fixed, allow the two hyperbolas to vary at the same time in such a manner that they are always equal without being symmetrical with respect to one of the axes of symmetry of the rectangle. Examine whether they intersect in real or imaginary points. Find the locus of the mid-point of the straight line which joins their centers, and prove that the straight line P is constantly tangent to this locus.

3° If any two of the hyperbolas (A) and (B) be considered, find the locus of the centers of the infinitude of rectangles formed with sides parallel to the asymptotes and with two opposite vertices on each of these hyperbolas.

1887. A fixed point ω and two fixed rectangular axes Ox, Oy are given in a plane. Two straight lines at right angles to each other are passed through ω which intersect Ox in B and D, Oy in A and C. Draw through A and B a parabola P tangent to the axes Ox, Oy at these points; draw through C and D a parabola P' tangent to Ox, Oy at these points.

Allow the perpendicular straight lines AB, CD to revolve about the point ω, and find:

1° The equation of the parabolas P, P', of their axes and of their directrices;

2° The equation of the locus of the point of intersection of the axes and of the directrices;

3° The equation of the locus of the point of intersection of their axes, which are composed of the two circles.

1888. A quadrilateral $OABC$ and two series of parabolas are given: the one tangent to AC at A, having OA as a diameter; the other tangent to BC at B, having OB as a diameter. It is required to find:

1° The locus of the point of contact M of a parabola of the first series with a parabola of the second series.

2° Indicate, on allowing the triangle OAB to vary, in what region of the plane must the point C be situated in order that the locus be an ellipse or in order that it be a hyperbola.

3° Demonstrate that in the hyperbola where $OABC$ is a parallelogram, the tangent common to the two parabolas at M revolves about the point of intersection K of the median of the triangle ABC.

4° Find, under the same hypothesis, the locus of the point of intersection P of the tangent to the two parabolas at M with the other common tangent DE which can be drawn to these two curves.

NOTE. — One puts $OA = a, \ OB = b.$

1889. Two rectangular axes Ox and Oy and two series of parabolas are given: the one P, with the parameter p, tangent to Oy on the side of the positive x's, and having its axis parallel to Ox; the other Q, with the parameter q, tangent to Ox on the side of the positive y's, and having its axis parallel to Oy. It is required:

1° To find the locus of the center of a conic C which is displaced without change of magnitude while constantly passing through the points common to the two series of parabolas P and Q.

2° To demonstrate that if a parabola P and a parabola Q be associated in such a manner that the straight line which joins their respective foci and remains parallel to a given direction, the sum of the angles which the tangents common to the two parabolas form with a fixed axis, Ox for example, remains constant, and to find, under these conditions, the locus of the point of intersection of the axes of the two parabolas.

3° To place a parabola P and a parabola Q in such a manner that they have three common points coincident, and to calculate, for this position of the curves, the co-ordinates of their common points and the angular coefficient of the common tangent at this point.

4° To demonstrate that every triangle circumscribed at the same time about the corresponding parabolas of the series P and Q is inscribed in a fixed conic, and to find the equation of this conic.

1890. An equilateral hyperbola H, whose equation, taken with respect to its axes, is

$$x^2 - y^2 = a^2,$$

is given in a plane.

From a point M of the plane whose co-ordinates are $x = p, \ y = q$, normals are drawn to this curve. It is required:

1° To pass through the feet of these normals a new equilateral hyperbola to which the normals at these points are concurrent, and to determine their point of intersection.

2° On representing by K an equilateral hyperbola which satisfies this condition, to determine in what region of the plane must the point M be placed in order that there be a hyperbola K corresponding to this point.

3° What line should the point M describe in order that the hyperbola K be equal to the hyperbola H.

N.B. Preserve the notation indicated.

1891. A parabola P is given and from each of its points is laid off in opposite directions, parallel to a fixed direction Δ, lengths equal to the distance of this point from the focus of the parabola.

1° Find the locus of the extremities of these lengths. It is composed of two parabolas P_1 and P_2; explain the reason for this duplicity.

2° Demonstrate that the axes of P_1 and P_2 are perpendicular to one another, that they revolve about a fixed point independent of Δ, and that, whatever this direction be, the sum of the squares of these parabolas is constant.

3° Find and construct the locus described by the parabolas P_1 and P_2 where Δ varies. Choose, as axes, the axis and the tangent at the vertex of the given parabola of parameter P. Express by θ the angle which Δ makes with the x-axis.

1892. An equilateral hyperbola H and a circumference C described on a chord DD' of this hyperbola as a diameter are given.

1° A chord of this circumference is drawn perpendicular to DD'; demonstrate that half of this chord is a mean proportional between the distances of its mid-point from the points in which it intersects the hyperbola.

2° Indicate the cases in which the points of intersection of the circumference and the hyperbola are real.

3° Find the locus of the points of intersection of the secants common to the hyperbola and circumference, when the chord DD' is displaced continually parallel to a fixed direction.

4° Let H be one of the points common to the hyperbola and the movable circle, A the point where the tangent to the circumference at H intersects the hyperbola, B the point where the tangent to the hyperbola at H intersects the circumference; prove that the straight line AB passes through a fixed point.

GENERAL ASSEMBLY

SPECIAL CLASS OF MATHEMATICS.

1833. Cut a triangle by a straight line so that the two portions of this triangle are in a constant ratio and their centers of gravity lie on the same perpendicular to the secant. Solve the same problem: 1° When two sides are equal; 2° when three sides are equal.

1837. Two equal parabolas, tangent at their vertices and having their axes lying in opposite directions, are given; one of the parabolas is supposed to revolve about the other so that, in each of the positions which it successively occupies, the revolving parabola is always tangent to the fixed parabola, and the point of contact is equally distant from the vertex of the fixed and revolving parabola; find the locus of the vertex of the revolving parabola.

1844. An ellipse and a point A on the ellipse are given, and a circle is drawn tangent to the ellipse at this point and two common tangents (exclusive of the tangent at the point A) are also drawn to the circle and ellipse; find the locus of the intersection of the two tangents when the point A travels along the ellipse.

1845. A circle and a point situated within it are given, and on every diameter of this circle an ellipse is constructed which has a diameter or major axis and which passes through this point. It is required to find:

1° The general equation of these ellipses;

2° The geometrical locus of their foci;

3° The locus of the extremities of the minor axes.

1846. A rectangle $ABCD$ being circumscribed about a given ellipse, it is known that the vertices of this rectangle lie on the same circle, which is concentric with the ellipse. This being the case, two straight lines are drawn from two opposite points of contact N and Q of every rectangle to the point of contact M of one of the other sides, and it is required to be proven:

1° That MN and MQ form equal angles with AB;

2° That $MN + MQ$ is constant;

3° That these straight lines MN, MQ envelop an ellipse which is confocal to the given ellipse.

1847. A triangle PQR is circumscribed about a circle, and a second triangle is so formed that its vertices A, B, C are the mid-points of the sides of the first. From the vertices of this second triangle, one draws to the circle the tangents Aa, Bb, Cc, which intersect respectively the sides opposite to these vertices in a, b, c. Prove that these three points lie on a straight line.

Test whether or not the theorem still holds if the circle be replaced by any conic section which is tangent to the three sides of the triangle PQR.

1848. An ellipse and a straight line TS are given in a plane. A diameter ACB is drawn through the center C of the ellipse and conjugate to the direction of the line TS, intersecting it in the point O. The straight line OC is prolonged a length OM such that $OC \cdot CM = CA^2$. The line TS is supposed to move so that it is always tangent to a given curve. It is required to find the curve described by the point M. Make an application of the method to the following case. The ellipse has the equation $\frac{x^2}{9}+\frac{y^2}{4}=1$, and the line TS remains always tangent to the curve which is represented by the equation $x^2 = ay$.

1849. An ellipse and a straight line situated without the ellipse are given. Two points N, N' are taken on the line, conjugate with respect to the ellipse (that is, two points such that the polar of one passes through the other). It is required:

1° To prove that there exists in the plane of the ellipse two points O, O', from which every segment NN' subtends a right angle; 2° Find the locus of the points O, O' when the given straight line moves parallel to itself.

1850. Two fixed axes ox, oy are given; an angle ABP of given constant magnitude is rotated about a fixed point P (A being the point in which one of the sides of the angle intersects the axis ox, and B, the point in which the other side intersects oy). It is required to prove that there exists a fixed point A' on ox, and a fixed point B' on oy such that the product $AA' \cdot BB'$ is constant for every possible position of the angle. Discuss the particular case when the lines ox, oy coincide.

1851. A straight line L is given. Two straight lines are drawn from each of its points M to two fixed points P and P'. Two other fixed points O, O' are the vertices of two angles AOB, $A'OB'$ of given constant magnitude, which are rotated about their respective vertices so that their sides OA, $O'A'$ are respectively perpendicular to the two straight lines MP, $M'P'$. Find the curve described by the point of intersection N of the two lines OA, $O'A'$ and the curve which is described by the point of intersection of the other two sides OB, $O'B'$, when the point M slides along the given line L.

1854. If the portion of the non-transverse axis comprised between the center and the normal at any point of the curve be taken as the diameter of a circle, the tangent drawn through this last point to the circle is equal to half of the real axis.

Whence discuss the following question: The two vertices and any third point of the hyperbola being given, construct the normal at this point. Give the analogous construction for the ellipse.

1857. Two conics C and C' are given, and all the possible systems of conjugate diameters are drawn in the first and through a point of the periphery of the other parabolas are drawn with the diameters of each system; show that the straight lines which join the second points of intersection of these parallels with the curve pass through a fixed point.

1862. Two parabolas with the same parameter have their axes at right angles; one of them is fixed and the other is variable. A common chord AB passes constantly through the foot D of the directrix of the fixed parabola; find the locus described by the vertex of the variable parabola.

1864. Two conics which have a common focus and proportional axes are given. Let FA, FA' be their minima radii vectores; these radii vectores rotate about the focus F, preserving their angular distance, and let FC, FC' be one position. Draw at C and C' tangents to each of the conics; find the locus of their point of intersection.

1865. Two conics tangent at a point O are given, and a common tangent OR is drawn to them, also the common external tangents AA', BB', which intersect in M. Given this construction, it is required to prove:

1° That the straight line PP', which joins the points P and P' diametrically opposite to O in the two conics, passes through the point M.

2° That the straight lines AB, $A'B'$, which join the points of contact of each conic with the common external tangents, intersect in a point R which is situated on the common tangent OR.

3° That the tangents drawn to the two conics from the point R touch the curves in the points which are situated on the straight line MC.

It will be seen, generally, that the point R does not share this property with another point, and it will be required to determine the condition which must be fulfilled in order that there exists a line such that the tangents drawn from each point of this line to the two conics furnishes four points of contact in a straight line.

1866. Demonstrate that: 1° The four points of intersection of any two conics inscribed in a given rectangle are the vertices of a parallelogram whose sides are parallel to the two fixed directions.

2° Find the locus of the points of contact of the tangents drawn from a point of the plane to all the conics inscribed in a given rectangle; or, better, the tangents parallel to a given direction.

3° Find the locus of the points of all these conics when the tangent forms a given angle with the diameter which meets it at the point of contact.

1868. It is proposed:

1° To find a geometrical locus of the points which divide in a given ratio the portion of the tangents to a fixed conic which are comprised between two fixed straight lines;

2° To classify, on considering above all the general cases, the various forms which this geometrical locus can have;

3° To find the conditions which ought to be fulfilled by the conic and the two fixed straight lines in order that the geometrical locus required should decompose into straight lines or into curves of the second order.

1869. A circle whose center is at O and a point P in the plane of this circle and lying without the circumference are given; find the locus described by the foci of an equilateral hyperbola, doubly tangent to the circle and passing through the point P.

Construct the locus on supposing the distance PO equal to three times the radius of the circle.

1870. Two ellipses have their centers at a common point and their axes lying in the directions of the same straight lines. Determine the locus of a point such that the cones which have their point as a common vertex, and the two ellipses as directrices, shall be equal.

1874. Demonstrate that the most general form of a polynomial $F(x)$ satisfying the relations:

$$F(1-x)\equiv F(x),\quad F\left(\frac{1}{x}\right)\equiv\frac{F(x)}{x^m}$$

is:

$$F(x)\equiv(x^2-x)^{2p}(x^2-x+1)^q\{A_0(x^2-x+1)^{3n}+A_1(x^2-x+1)^{3(n-1)}$$
$$(x^2-x)^2+A_2(x^2-x+1)^{3(n-2)}(x^2-x)^4+\cdots+A_n(x^2-x)^{2n}\},$$

p, q, n being integral numbers, and A_0, A_1, $A_2, \cdots$, A_n arbitrary constants.

1874. If the function e^{-x^2} of the variable x be considered and its successive derivatives be taken, it is seen that the derivative of the nth order is equal to the product of the function e^{-x^2} by an integral polynomial in x which we represent by $\phi_n(x)$.

1° Demonstrate that the polynomials $\phi(x)$ satisfy the following relations:

$$\phi_n(x)\equiv-2x\phi_{n-1}(x)-2(n-1)\phi_{n-2}(x),$$
$$\phi'_n(x)\equiv-2n\phi_{n-1}(x),$$
$$\phi''_n(x)\equiv-2x\phi'_n(x)+2n\phi_n(x).$$

2° Calculate the coefficients of the polynomial $\phi_n(x)$ arranged according to the powers of x.

1880. On a curve of the third order, which has a cusp at O, the following points are considered:

$$A_{-n},\ A_{-(n-1)},\ \cdots,\ A_{-2},\ A_{-1},\ A_0,\ A_1,\ A_2,\ \cdots,\ A_{n-1},\ A_n;$$

which are so arranged that the tangent at each of them intersects the curve in the next.

1° The co-ordinates of the point A_0 being given, find the co-ordinates of the points A_{-n}, A_n, and determine the limits which these points have when n is increased indefinitely.

2° Find the locus described by the first limiting point, when the curve of the third degree preserving its cusp at O and the tangent at this point, and passing always through three fixed points P, Q, R is deformed.

3° Study the variation of the points of intersection of this locus and the sides of the triangle PQR, when the vertices of this triangle are displaced along straight lines which pass through O.

1882. Through any point P taken in the place of a given parabola, whose vertex is at O, we draw to this parabola three normals which intersect it in the points A, B, C. Representing the lengths PA, PB, PC, PO, respectively, by a, b, c, l, it is required to form the equation whose roots are l^2-a^2, l^2-b^2, l^2-c^2, and to indicate the signs of the roots according to the position of the point P in the various regions of the plane.

1887. I. Let (x_1, y_1), (x_2, y_2), $\cdots$, (x_m, y_m) be the co-ordinates of the points of intersection of two algebraic curves whose equations, put in integral form, are $f(x, y)=0$, $F(x, y)=0$. It is assumed that the points of intersection are *simple and situated at finite distances.*

1° Show that, for each value of i, one can write:

$$f(x, y)=(x-x_i)a_i(x, y)+(y-y_i)b_i(x, y)$$
$$(i=1, 2, \cdots, m),$$
$$F(x, y)=(x-x_i)A_i(x, y)+(y-y_i)B_i(x, y),$$

the coefficients a_i, b_i, A_i, B_i being polynomials in x and y.

2° One puts

$$\phi_i(x, y)=\begin{vmatrix} a_i & b_i \\ A_i & A_i \end{vmatrix}$$
$$(i=m),$$

and

$$\Phi(x, y)=\Sigma C_i\phi_i(x, y)$$
$$(i=1),$$

and we require to determine the constants C_i so that the polynomial Φ takes, for $x=x_i$ and $y=y_i$, a given value. Prove that the polynomial ϕ thus obtained takes, as a particular case, the form of the interpolation formula of Lagrange.

3° Demonstrate that all the polynomials in x and y which, for $x=x_i$ and $y=y_i$ taking the value u_i, can be written in the form

$$\Phi+Mf+NF,$$

where M and N are polynomials in x and y.

II. Let $f=0$, $F=0$ be the equations of two conics u and U, and λ_1, λ_2, λ_3, the roots of the equation found by equating the discriminant of the function $f-\lambda F$ to zero; find the necessary and sufficient condition in order that one can inscribe, in the conic u, a quadrilateral circumscribed about the conic U.

1888. Let C be the curve which is the geometrical locus of the vertices of the angles of constant magnitude circumscribed about a given ellipse E; and D a given straight line. 1° Demonstrate that there are three conics tangent to the straight line D and touching the curve C in four points. Determine the nature of these three conics. 2° Let a_1, a_2, a_3, a_4 be the points in which the straight line D intersects the curve C; through two of these points a_1 and a_2, for example, a series of circles are passed intersecting the curve C in two variable points M and M'; and we require to find the envelopes of the straight lines MM'. 3° One supposes that the straight line D is tangent to the ellipse E, and through the points a_1, a_2, a_3, a_4, where this tangent intersects the curve C, tangents besides D are drawn to the ellipse; find the locus described by the vertices of the quadrilateral formed by these tangents, when the straight line D revolves about the ellipse E.

1889. A circle whose center is at O and a parabola P being given, consider all the conics inscribed in the quadrilateral formed by the tangents common to the circle O and to the parabola P. It is required to find: 1° (a) The envelope of the polars A, of the point O, with respect to the conics C; (b) the envelopes of the tangents T to the conics C, such that the normals at the points of contact pass through the point O; (c) the envelopes of the axes of the conics C. 2° The geometrical loci of the feet of the perpendiculars dropped from the point O, upon the polars A, upon the tangents T, and upon the axes of the conics C.

RESULTS. — (1) Find the same parabola for the envelopes (a), (b), (c); (2) a strophoid for the geometrical loci, polar of the parabola with respect to a point of its directrix.

MISCELLANEOUS QUESTIONS

ÉCOLE NORMALE SUPÉRIEURE.

1861. A secant PAB is drawn from a point P which is external to a conic. At the points A, B, in which the secant intersects the curve, we draw tangents which intersect at M and we project the point M upon the straight line AB; find the locus of these projections.

Prove: 1° that the locus passes through the point P and is tangent at this point; 2° that the locus is the same for all confocal conics; 3° that the locus may be regarded as the locus of the projections of the point P upon certain straight lines which are tangent to a curve whose equation is required.

1863. Consider the equilateral hyperbolas which are tangent to a fixed straight line AB at a given point C and which pass through a point D. From a point P which lies on AB, tangents are drawn to each of the hyperbolas and the locus of the points of contact is required.

Determine the nature of the locus according to the position of the point D.

1864. A triangle ABC and a straight line D which passes through the point A are given; there is an infinitude of conics which pass through the points A, B, C and are tangent to the straight line AD.

To each of these curves tangents are drawn parallel to AD; find the locus of the points of contact.

This locus is a conic; it is required to find the curve described by its foci when the points A, B, C remain fixed and the straight line AD revolves about the point A.

1866. A parallelogram whose diagonals are any two conjugate diameters AA', BB' is inscribed in a given ellipse. The normals are drawn to the ellipse at the vertices of this parallelogram; they form a second parallelogram $MNM'N$.

1° Demonstrate that the diagonals of each of the two parallelograms $ABA'B'$, $MNMN'$ are respectively perpendicular to the sides of the other.

2° Find the locus of the vertices of the parallelogram $MNM'N'$ when the conjugate diameters are varied.

3° Find the locus of the point of intersection of the diagonal NN and of the tangent to the preceding locus at M.

1867. Two perpendicular straight lines AB, CD are given, and the hyperbolas which have the straight line AB as asymptote and touch the straight line CD at a fixed point P are given. It is required to find:

1° The locus of the foci of all of these hyperbolas;

2° The locus of the point of intersection of the second asymptote with the perpendicular dropped from the fixed point P upon its direction;

3° The locus of the points of intersection of the asymptote with the straight line which joins the focus with the point of intersection of the two given straight lines.

1869. A triangle and a point P in its plane are given; we draw through the point P any straight line PQ, and consider the two conics which pass through the vertices of the triangles and touch the straight line PQ. Let E, E' be the two points of contact and M the mid-point of the segment EE'; find the locus described by the point M, when the straight line PQ is rotated about point P.

Construct the locus under the following hypothesis: the rectangle is reduced to a square of which a side is $2a$, and, if we choose as axes of co-ordinates the straight lines drawn through its center parallel to the sides of the square, the co-ordinates of the point P are $x = y = \frac{a}{4}$.

1873. An ellipse and a point P in its plane are given; from this point P normals are drawn to the ellipse A, and the conic B, which passes through the point P and the feet of the four normals, is considered.

1° Find the co-ordinates of the center of this conic B and those of its foci.

2° Find the locus C of the center and the locus D of the foci of the conic B, when the ellipse A varies so that its foci remain fixed.

3° Find the locus of the points of intersection of the locus D and of the straight line OP when the point P describes a circle of given radius and with its center at the center O of the ellipse A.

1874. 1° Parabolas are passed through the three vertices of a right triangle. Tangents parallel to the hypotenuse of the given triangle are drawn to these parabolas. Find the locus of the points of contact.

2° The locus sought is a conic which intersects each of the parabolas in four points. It is required to find the locus described by the center of gravity of the triangle formed by the common secants which do not pass through the origin.

1875. An infinitude of ellipses which are similar to each other and which have a fixed vertex O and a common tangent at this point are considered; it is required to find the locus of the feet of the normals drawn, from a fixed point P, to these ellipses.

Construct the locus, in the particular case where OP is inclined at 45 degrees to the fixed given tangent and on supposing, successively, that the ratio of the axes of the ellipses considered is equal to $\sqrt{3}$ or equal to 2.

1876. All the parabolas tangent to two perpendicular straight lines ox and oy are considered, such that the straight line PQ which joins their points of contact P and Q with the two straight lines, passes through a fixed given point.

1° Find the locus of the point of intersection of the normal at P to one of these parabolas with the diameter of the same curve which passes through Q.

2° It is required to determine the number of real parabolas which pass through any point of the plane.

3° Find the equation of the locus of the points of intersection which satisfy the proposed conditions and whose axes inclose a given angle.

Construct this locus in the case when the given angle is an angle of 45 degrees and where the point A is on the straight line ox.

1877. Consider all the conics circumscribed about a triangle ABC, right-angled at A and such that the tangents to these conics at B and C intersect on the altitude of the triangle. Find:

1° The locus of the point of intersections of the normals to these conics at B and C;

2° The locus of the center of these conics; determine the points of the locus which are centers of the ellipses, and of those which are centers of the hyperbolas;

3° The locus of the poles of any straight line D. This locus is a conic. Study all the straight lines D for which this conic is a parabola and find the locus of the projections of the point A upon these straight lines.

1878. A conic and two fixed points A and B on the conic are given. Any circumference which passes through the two points A and B intersects the conic in two additional variable points C and D; the straight lines AC, BD, which intersect in M, and the straight lines AD, BC, which intersect in N, are constructed.

Determine:

1° The locus of the points M and N;

2° The locus of the points of intersection of the straight line MN with the circumference to which it corresponds.

Construct both loci.

1881. Consider the curve

$$27y^2 = 4x^3.$$

1° Find the condition which the parameters m and n should satisfy in order that the straight line $y = mx + n$ should be tangent to this curve.

2° Find the locus of the points at which one can draw to the given curve two tangents parallel to two conjugate diameters of the conic represented by the equation

$$x^2 + y^2 + 2\,axy = B.$$

3° Secants are drawn through a point A in this curve, intersecting this curve in two variable points M and M'. Find the locus of the mid-point of the segment MM'. Discuss the form of this locus and indicate the axes which correspond to the secants for which the points M, M' are real.

1882. One is given a fixed point P whose co-ordinates are a and b with respect to two perpendicular axes Ox, Oy, and A and B are the feet of the perpendiculars dropped from the point P upon these two axes. One studies the curves of the second order which are tangent to the two axes at the points A and B; from the point P one draws to each of these curves two variable normals PM, PM'.

1° Determine the equation of the straight line MM' which joins the feet of the variable normals, and demonstrate that this straight line passes through a fixed point.

2° Determine the equation of the curve C locus of the points M and M'. Construct the curve C with the hypothesis $a = b$, by means of polar co-ordinates of which the pole is the point O.

1884. If a and b be the rectilinear rectangular co-ordinates of a point M, what is, for every position of this point, the nature of the roots of the equation

$$3\,t^4 + 8\,at^3 - 12\,bt^2 + 4\,b = 0\,?$$

One constructs, in particular, the locus of the positions of the point M for which the equation has a double root, on calculating the co-ordinates of a point of the locus as a function of this root.

1886. One considers the curves of the third degree C, represented by the equation

$$x^2y + a^2x = \lambda,$$

where λ represents a variable parameter.

It is required to prove that there exists two curves of this species tangent to any straight line D of the plane, whose equation is

$$y = mx + p,$$

and to calculate the co-ordinates of the two points of contact M and M'. Determine the straight lines D, for which these two points are real, and the straight lines for which they are imaginary. Determine the positions of the straight line D for which the two points M and M' coincide, and find in this case the locus described by the point of contact.

Given the co-ordinates (α, β) of a point of contact M of a conic C with a straight line D, find the co-ordinates (α', β') of the second point of contact M' situated on D. Construct the curve described by the point M' when the point M describes the straight line

$$\beta = \alpha - 2a.$$

1888. A polynomial $f(x)$ of the degree n satisfies the identity

$$nf(x) = (x-a)f'(x) + bf''(x).$$

1° Find the coefficients of $f(x)$ arranged according to the powers of $(x-a)$.

2° Find the conditions of reality of the roots.

3° Prove that if b_0 be the absolute value of b, the roots of $f(x)$ are situated between

$$a - \sqrt{\frac{n(n-1)}{2}} \cdot b_0, \quad a + \sqrt{\frac{n(n-1)}{2}} \cdot b_0.$$

Construct the curve represented by the equation:

$$x(x^2 - y^2)^2 + 4xy(x-y)^2 - 4y(2y - 3x) = 0.$$

1889. 1° Determine an integral polynomial in x of the seventh degree $f(x)$, with the condition that $f(x)+1$ and $f(x)-1$ each is divisible by $(x+1)^4$. What is the number of real roots of the equation $f(x) = 0$?

2° Consider in a plane a parabola (P) and an ellipse (E) represented respectively by the two equations

$$(P) \quad y^2 - 8x = 0, \qquad (E) \quad y^2 + 4x^2 - 4 = 0,$$

and a point $M(\alpha, \beta)$. It is required to find on the parabola P a point Q, such that the pole of the straight line MQ, with respect to the ellipse (E) is situated on the tangent to the parabola at Q.

Find the number of real solutions of the problem, according to the position of the point M in the plane.

1890. Between the co-ordinates x, y of a point A, and the co-ordinates u, v of a point B, the following relations are established:

$$x = \frac{u^3 + \lambda uv^2}{u^2 + v^2}, \quad y = \frac{v^3 + \lambda vu^2}{u^2 + v^2},$$

where λ is a given positive number.

Having found from these relations the equation which furnishes the angular coefficients α, β of the straight lines which connect the origin with the points A, B, we are required to show that, in general, to each point A there corresponds three positions of the point B. Can these points B_1, B_2, B_3 be real and distinct? What must be the position of the point A in order that this be the case? What position should A have in order that two of these points (B_2 and B_3 for example) be coincident? If A describe a locus in the preceding case, what are the loci described by the coincident points B_2, B_3 and by the point B_1?

1891. Let E be an ellipse which, referred to its axes, has the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

and let x_0, y_0 be the co-ordinates of a point M of the plane of this ellipse; consider the circle C which passes through the point M and the points of contact P, Q of the tangents drawn through the point M to the ellipse.

1° The circle (C) intersects the ellipse in two additional points P', Q'; prove that the tangents to the ellipse at these two points intersect at a point M' situated on the circle; show that a circle can be passed through M, M' and the two imaginary foci; similarly through M, M' and the two imaginary foci.

2° Let I, I', I'' be the points in which the straight lines PQ, $P'Q'$, the straight lines $P'Q$, PQ', and the straight lines PP', QQ' respectively intersect; it is assumed that the point M remains fixed and that the ellipse (E) is deformed, keeping the same foci; find the loci described by the points I, I', I'', and prove that every circle which passes through I', I'' is orthogonal to the circle described on MM' as a diameter.

1892. A circle C is represented in rectangular co-ordinates by the equation

(C) $$x^2+y^2-2x-1=0.$$

1° Find the general equation of the conics A which are doubly tangent to the circle C, so that the chord which connects the two points of contact passes through the origin of co-ordinates, and which are, besides, tangent to the straight line D which has the equation

$$y=x\sqrt{3}+\sqrt{3}.$$

2° Through any point $M(\alpha, \beta)$ of the plane there pass, in general, two conics A', A'' of this kind; where must the point M be in order that these conics be real?

3° The two conics A', A'' which pass through the point M have three other common points, M_1, M_2, M_3, of which it is required to calculate the co-ordinates as functions of the co-ordinates α, β of the point M.

4° Find the equation of the equilateral hyperbola H which passes through the four points M, M_1, M_2, M_3, and show that this hyperbola passes through four fixed points, when the point M is displaced.

5° Find the locus of the points of intersection of the two conics A', A'' and the envelope of their common secants, when the chords of contact of these two conics with the circle C are perpendicular; what is, in this case, the species of the conics A', A''?

N. B. Take as variable parameter m, the angular coefficient of the chord of contact of the conic A with the circle C.

ÉCOLE CENTRALE.

1880. Let Ox, Oy be two rectangular axes and take on Ox a point A, on Oy a point B. Draw through the point A any straight line AR with the angular coefficient m.

1° Form the equation of a hyperbola A which is tangent to the axis Ox at the point O, which passes through the point B, and which has AR as asymptote.

2° Allowing m to vary, find the locus described by the point of intersection of the tangent to the hyperbola H at B and of the asymptote AR.

3° Consider the circle circumscribed about the triangle AOB; this circle intersects the hyperbola H at the points O and B, and in two additional points P and Q. Form the equation of this straight line PQ; then, allowing m to vary, find successively the loci of the points of intersection of this straight line PQ with the parallels drawn from the point O, first to the asymptote AR, then to the second asymptote of the hyperbola H.

1881. Let $a^2y^2 + b^2x^2 = a^2b^2$ be the equation of an ellipse referred to its center O and to its axes; let α and β be the co-ordinates of a point P situated in the plane of this ellipse.

1° Demonstrate that the feet of the normals drawn through the point P to this ellipse are situated on the hyperbola represented by the equation

$$c^2xy + b^2\beta x - a^2\alpha y = 0,$$

in which $c^2 = a^2 - b^2$.

2° Consider all the conics which pass through the points A, B, C, D common to this hyperbola and the given ellipse; in each of them the diameter conjugate to the direction OP is drawn and the point O is projected upon this diameter; find the locus of this projection.

3° Two parabolas can be passed through the points A, B, C, D; find the locus of the vertex of each of them when the point P is situated on a straight line with the given angular coefficient m, drawn through the point O.

Discuss the particular cases when $m = \frac{a^3}{b^3}$ and $m = -\frac{a^3}{b^3}$.

1882. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the equation of an ellipse referred to its center and to its axes, and let α and β be the co-ordinates of a point P situated in the plane of the ellipse.

Form the general equation of the conics which pass through the points of contact M and M' of the tangents drawn from the point P to the ellipse and through the points Q and Q' where this ellipse is intersected by the straight line which corresponds to the equation

$$\frac{\alpha x}{a^2} - \frac{\beta y}{b^2} + \mu = 0.$$

Dispose of the parameter μ and of the other variable parameters which are involved in the general equation in such a manner that it represents an equilateral hyperbola passing through the point P.

The point P is allowed to move along the straight line represented by the equation $x + y = l$, and it is required to find:

1° The locus described by the projection of the center of the ellipse upon QQ';

2° The locus described by the point of intersection of the chords MM' and QQ'.

Demonstrate that this last locus passes through two fixed points, whatever l may be, and determine these points.

Find the values of l for which this locus is reduced to two straight lines, and determine these straight lines.

1884. The equation $a^2y^2 - b^2x^2 + a^2b^2 = 0$ of a hyperbola referred to its center and its axis, and the equation $y - Kx = 0$ of a straight line drawn through the center of this hyperbola are given.

I. Form the general equation of the conics which pass through the real or imaginary points common to the hyperbola and to the given straight line and which, at most, are tangent to the hyperbola at that vertex of this hyperbola which is situated on the positive portions of the x-axis. Discuss this general equation, and determine the nature of the conics which it can represent.

II. Find the locus of the centers of the conics represented by the preceding equation. This locus is a conic Δ; find the number of points and tangents which suffice to determine geometrically this conic Δ.

III. Find the locus of the points of contact of the tangents drawn to the conic Δ, parallel to the straight line whose angular coefficient is $\frac{b}{a}$, when K varies. Prove that the equation of this last locus, which is of the third degree, represents three straight lines.

1885. Two rectangular axes ox, oy, and the circle whose equation is

$$(x - a)^2 + (y - b)^2 - r^2 = 0,$$

are given.

Consider the fixed chord AB drawn through the origin and bisected by it, and a variable chord CD, with constant direction which is equal and of contrary signs to that of the fixed chord AB.

Two parabolas P, P' can be drawn through the four points A, B, C, D.

Find, the chord being displaced parallel to itself:

1° The locus of the point of intersection of the axes of the parabolas P and P';

2° The locus of the vertex and the locus of the focus of each of these parabolas.

1886. Let $OACB$ be a rectangle whose sides $OA = a$ and $OB = b$, prolonged, are taken, the first for the x-axis, the second for the y-axis. Consider all the conics which pass through the three points O, A, B and for which the polar of the point C is parallel to the straight line AB.

1° Form the general equation of the conics. Find the locus of their centers, and, on their locus, separate the portions which contain the centers of the ellipses from those which contain the centers of the hyperbolas.

2° A normal is drawn to each of these conics at the point A and at the point B; find the locus of the point of intersections of these two normals.

3° Let Δ be any one of the conics considered; if the normals be drawn through the point C to this conic, one knows that the feet of these normals are the points of intersection of the conic Δ and of a certain equilateral hyperbola. Form the equation of this equilateral hyperbola, and find the locus of the center of this hyperbola, when the conic Δ varies.

1887. Two rectangular axes Ox and Oy, and a point A on Ox and a point B on Oy, so that

$$OA = a, \quad OB = b,$$

are given.

1° Write the general equation of the parabolas which pass through the three points O, A, B. Show that, in general, there pass through each point M of the plane two of these parabolas. Find the locus of the points M for which the two parabolas are coincident, and indicate the region of the plane which contains the points where there pass only real loci.

2° Find the locus of the point M such that the axes of the two parabolas include a given angle α. Construct the locus for the case $\alpha = \frac{\pi}{2}$.

3° Find the locus of the point of each of these parabolas at which the tangent is parallel to OB, of the point where the tangent is parallel to OA, and of the point where the tangent is parallel to AB.

These loci are three conics. Construct these conics and show that no two of them have a common real point at a finite distance; locate their centers D, E, F, and compare the triangle DEF with the triangle OAB.

4° Join the origin O to the point F, center of the conic, locus of the point of contact of the tangents parallel to AB; and one erects to the straight line OF, at the point O, a perpendicular which intersects the straight line AB at P; find the locus of the point P when, the point A remaining fixed, the point B travels along the y-axis.

1888. Two rectangular axes Ox, Oy and a point A on the x-axis are given; consider the pencil of conics of which the y-axis is a directrix and the point A a vertex of the focal axis. Then pass through any point M of the plane of the axes two real or imaginary conics of this pencil.

1° Determine the portions of the plane which the point M should occupy in order that the two conics of the pencil which pass through this point be real, and those which it should occupy in order that the conics be imaginary.

2° When the position of the point is determined in order that the two conics be real, find the genus of these conics.

3° Find the locus of the points of contact of the tangents drawn from the origin of co-ordinates to all the conics of the pencil considered.

1889. Let Ox, Oy be two rectangular axes, and let LL' be a straight line, whose equation is $x - a = 0$, parallel to Oy. Consider the pencil of the parabolas which pass through the point O and which have the straight line LL' as directrix.

1° Find the locus of the focus and the locus of the vertex of these parabolas.

2° Two of the parabolas considered, real or imaginary, pass through any point of the plane xOy; determine the position of the plane in which this point must be situated in order that the two parabolas be real.

3° The co-ordinates of any point M of the plane xOy are given; form the equation which has as roots the angular coefficients of the tangents to the two parabolas of the pencil considered at the point O which pass through this point M. Whence find the equation of the curve S on which the point M must be situated in order that the tangents to the two parabolas of the pencil at the point O which pass through the point M should be perpendicular.

4° Let M be a point on the curve S, and let F, F' be the foci of the two parabolas of the pencil considered which pass through this point; demonstrate that, when the point M is displaced along the curve S, the straight line FF' revolves about a fixed point.

1890. Two rectangular axes $x'Ox$, $y'Oy$ and two points A and B, *symétriques* with respect to the point O, are given.

1° Any point P is taken on the x-axis and the parabola (P), which is tangent to the straight lines PA and PB at the points A and B, is considered. The locus of the focus and the locus of the vertex of this parabola when P travels throughout $x'Ox$ are required.

2° One selects any point Q on the y-axis, and considers the parabola (Q) which is tangent to the straight lines QA, QB at the points A and B. These two parabolas (P) and (Q), which correspond thus at a point P taken on $x'Ox$ and at a point Q on $y'Oy$, intersect at the points A, B and in two other points C, D. Form the equation of the straight line CD and find the locus described by the points C, D, when P, Q are displaced respectively on $x'Ox$ and $y'Oy$, so that the abscissa of the first is always equal to the ordinate of the second.

1891. Two rectangular axes, and a circle C which passes through the origin and whose center has the co-ordinates $x = -\frac{a}{2}$, $y = -\frac{b}{2}$, are given. Two chords, d in length, which pass through the origin are drawn in this circle. One draws from a point with the abscissa p of the x-axis straight lines perpendicular to these chords.

1° Find the equation Δ of the locus of the points such that the product of this distance from the chords be in a given ratio λ with the product of their distances from the straight lines perpendicular to these chords; find the locus of the centers of the conics represented by the equation Δ, when λ varies.

2° Discuss the nature of the conics represented by the equation Δ.

3° The ratio λ is chosen so that the conic Δ becomes a circle; find the locus of the center of the curve when the center of the circle C describes the hyperbola $x^2 + nxy = \frac{d^2}{4}$.

1892. Two circumferences, whose centers are O and C, $OC = a$, are given.

We draw through the point $A(p, q)$ of intersection of these circumferences two secants BAE and DAC having a common length $2l$. These two secants intersect the y-axis and its parallel drawn through C, in the points M, N and P, Q.

1° It is required to form the general equation of the conics which pass through the four points M, N, P, Q.

2° The conics are required to pass through a point of the plane; determine the genus of the conic as the position of the point varies.

3° Find the locus of the centers of these conics.

4° Find the locus of the point of intersection of the straight lines BC and DE, when the length $2l$ is allowed to vary.

~~MAY 29 1976~~	Due end of SPRING semester Subject to recall after —
~~JUL 10 1976~~	~~APR 15 1993~~
JAN 13 1977	
	~~MAR 24 1995~~
~~OCT 13 1977~~	DEC 14 1998
~~APR 24 1978~~	Rec'd UCB A/M/S
APR 24 1978	DEC 07 1998
~~APR 19 1982~~	
~~JUN 2 1982~~	
Due end of SPRING semester Subject to recall after —	
JAN 7 1988	
Due end of SPRING Semester Subject to recall after —	
MAY 1 1991	