A COURSE IN MATHEMATICS

WOODS AND BAILEY

VOLUME I

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A COURSE IN
MATHEMATICS

FOR STUDENTS OF ENGINEERING AND APPLIED SCIENCE

BY

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VOLUME I

ALGEBRAIC EQUATIONS
FUNCTIONS OF ONE VARIABLE, ANALYTIC GEOMETRY
DIFFERENTIAL CALCULUS

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PREFACE

This book is the first volume of a course in mathematics designed to present in a consecutive and homogeneous manner an amount of material generally given in distinct courses under the various names of algebra, analytic geometry, differential and integral calculus, and differential equations. The entire course covers the work usually required of a student in his first two years in an engineering school, the first volume containing the work of the first year. In arranging the material, however, the traditional division of mathematics into distinct subjects is disregarded, and the principles of each subject are introduced as needed and the subjects developed together. The objects are to give the student a better grasp of mathematics as a whole, and of the interdependence of its various parts, and to accustom him to use, in later applications, the method best adapted to the problem in hand. At the same time a decided advantage is gained in the introduction of the principles of analytic geometry and calculus earlier than is usual. In this way these subjects are studied longer than is otherwise possible, thus leading to greater familiarity with their methods and greater freedom and skill in their application.

In carrying out this plan in detail the subject-matter of this volume is arranged as follows:

1. An introductory chapter on elimination, including the use of determinants. This chapter may be postponed or omitted, if a teacher prefers, without seriously affecting the subsequent work.

2. Graphical representation. Here the student learns the use of a system of coördinates and the definition and plotting of a function.

3. The study of the algebraic polynomial. This includes the analytic geometry of the straight line, the more important

theorems of the theory of equations, and the definition of a derivative. Simple applications of the calculus to problems involving tangents, maxima and minima, etc., are given. In this way a student obtains an introduction to the principles of the calculus, free from the difficulties of algebraic computation.

4. The study of the algebraic function in general. The knowledge of analytic geometry and calculus is here much extended by new applications of the principles already learned. Simple applications of integration are also introduced. The study of the conics forms part of the work in this place, but other curves are also used and care is taken to avoid giving the impression that analytic geometry deals only with conic sections; in fact, the chapters which deal especially with the conics may be omitted without affecting the subsequent work.

5. The study of the elementary transcendental functions. It has been thought best to assume the knowledge of elementary trigonometry, since that subject is often presented for admission to college, — a tendency which should be encouraged. The chapter discusses the graphs, the differentiation of transcendental functions, and the solution of transcendental equations.

6. The work closes with chapters on the parametric representation of curves, polar coördinates, and curvature. In the first of these chapters the solution of locus problems, which, from some standpoints, is the most important part of analytic geometry, finds its natural place; for this problem involves, in general, the expression of the coördinates of a point on a locus in terms of an arbitrary parameter, and possibly the elimination of the parameter.

As compared with the usual first course in analytic geometry, there will be found in this volume fewer of the properties of the conic sections, except as they appear in problems set for the student. On the other hand, a greater variety of curves are given, and it is believed that greater emphasis is placed on the essential principles. All work in three dimensions is postponed to the second year, and is to be taken up in the second volume in connection with functions of two or more variables, partial differentiation, and double and triple integration.

This volume contains the matter usually given in a first course in differential calculus, with the exception of differentials, series, indeterminate forms, partial differentiation, envelopes, and some advanced applications to curves. These subjects will find their appropriate place in the further development of the course in the second volume. Integration has been sparingly used as the inverse operation of differentiation, and without employing the integral sign. Simple applications to areas and velocities are given. To do more would require the expenditure of too much time on the operation of integration, and the introduction of too many new ideas into one year's work. The integral, as a limit of a sum, with its many applications, will form an important part of the second year's work.

In the preparation of the text the needs of a student who desires to use mathematics as a tool in engineering and scientific work have been primarily considered, but it is believed that the course is also adapted to the student who studies mathematics for its own sake. Abstract discussions are avoided and frequent applications and illustrations are given. Illustrations, however, which are beyond the range of a first-year student's knowledge of physical science are omitted. The proofs are made as rigorous as the maturity of the student will admit. It is to be remembered in this connection that the earlier chapters are to be studied by students who have just entered college.

In the preparation of the book the authors have had the advice and criticism of the mathematical department of the Massachusetts Institute of Technology. In particular, they are indebted to the head of the department, Professor H. W. Tyler, at whose invitation the book has been written, and whose suggestions have been most valuable.

Massachusetts Institute of Technology
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A COURSE IN MATHEMATICS

CHAPTER I

ELIMINATION

1. Determinant notation. Elimination is the process of obtaining from a certain number of equations containing two or more unknown quantities one or more equations which do not contain all of these quantities. The quantities removed are said to have been eliminated. The solution of equations is essentially the elimination of all but one of the unknown quantities. The process of elimination leads to the formation of certain expressions in the coefficients, for which a special name and a corresponding notation have been invented. In this chapter we shall consider equations of the first degree, or *linear equations.* These are equations in which no term contains more than one unknown quantity, and that in the first degree.

Ex. 1.

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0. \end{aligned} \qquad (1)$$

To eliminate y, multiply the first equation by b_2, the second by $-b_1$, and add. To eliminate x, multiply the first equation by $-a_2$, the second by a_1, and add. There results

$$\begin{aligned} (a_1b_2 - a_2b_1)x + (c_1b_2 - c_2b_1) &= 0, \\ (a_1b_2 - a_2b_1)y + (a_1c_2 - a_2c_1) &= 0. \end{aligned} \qquad (2)$$

Unless $a_1b_2 - a_2b_1 = 0$, equations (2) give at once the solution of (1). If $a_1b_2 - a_2b_1 = 0$, the method used to eliminate y also eliminates x, and the equations need further discussion, to be given in § 6.

Ex. 2.
$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0, \\ a_2x + b_2y + c_2z + d_2 &= 0, \\ a_3x + b_3y + c_3z + d_3 &= 0. \end{aligned} \tag{1}$$

To eliminate y and z, multiply the first equation by $(b_2c_3 - b_3c_2)$, the second by $-(b_1c_3 - b_3c_1)$, the third by $(b_1c_2 - b_2c_1)$, and add. There results

$$[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)]x$$
$$+ [d_1(b_2c_3 - b_3c_2) - d_2(b_1c_3 - b_3c_1) + d_3(b_1c_2 - b_2c_1)] = 0,$$

or
$$(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1)x$$
$$+ (d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_1b_3c_2 - d_2b_1c_3 - d_3b_2c_1) = 0. \tag{2}$$

To eliminate x and z, multiply the first equation by $-(a_2c_3 - a_3c_2)$, the second by $(a_1c_3 - a_3c_1)$, the third by $-(a_1c_2 - a_2c_1)$, and add. There results

$$(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1)y$$
$$+ (a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_1d_3c_2 - a_2d_1c_3 - a_3d_2c_1) = 0. \tag{3}$$

To eliminate x and y, multiply the first equation by $(a_2b_3 - a_3b_2)$, the second by $-(a_1b_3 - a_3b_1)$, the third by $(a_1b_2 - a_2b_1)$, and add. There results

$$(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1)z$$
$$+ (a_1b_2d_3 + a_2b_3d_1 + a_3b_1d_2 - a_1b_3d_2 - a_2b_1d_3 - a_3b_2d_1) = 0. \tag{4}$$

Equations (2), (3), and (4) give the solution of (1), unless

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 = 0.$$

The exceptional case will be considered in § 6.

The binomials which occur in the solution of Ex. 1 are called *determinants of the second order*. The symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

is used to denote the determinant $a_1b_2 - a_2b_1$. Then equations (2) of Ex. 1 may be written

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x + \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y + \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = 0.$$

The polynomials which occur in the solution of Ex. 2 are called *determinants of the third order.* The symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is used to denote the determinant

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

The results of Ex. 2 may then be written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} y + \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} z + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0.$$

By the work of Ex. 2,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

which may be taken as the definition of a determinant of the third order.

Similarly a determinant of the fourth order is indicated by the symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

and is defined as equal to

$$a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

If now each of these determinants of the third order is expressed in terms of determinants of the second order, we shall have finally the determinant of the fourth order expressed as an algebraic polynomial of twenty-four terms.

2. In general a determinant of the nth order is an algebraic polynomial involving n^2 quantities, called *elements.* The symbol of the determinant is obtained by writing the elements in a square of n rows and n columns. If in such a symbol a row and a column are omitted, there is left the symbol of a determinant of the next lower order. This new determinant is said to be a *minor* of the original determinant, and is said to correspond to the element which stands at the intersection of the omitted row and column. We shall now give as definition:

A determinant is equal to the algebraic sum of the products obtained by multiplying each element of the first column by its corresponding minor, the signs of the products being alternately plus and minus.

By repeated application of the same definition to the minors obtained, we eventually make the value of the determinant depend upon determinants of the second order, and thus obtain the polynomial indicated by the original symbol.

Students who desire a more general definition and discussion of determinants are referred to treatises on the subject. We shall derive here, as simply as possible, only those properties which are of use in solving equations. Before doing so, however, we need to show that the word "column" may be changed to "row" in the above definition, thus: *A determinant is also equal to the sum of the products obtained by multiplying each element of the first row by the corresponding minor, the signs of the products being alternately plus and minus.*

For a determinant of the third order the student may verify that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

The theorem thus shown to be true for a determinant of the third order may be proved for one of the fourth order as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix}$$

$$- a_4\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \qquad \text{(by definition)}$$

$$= a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2\left\{b_1\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - c_1\begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix} + d_1\begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix}\right\}$$

$$+ a_3\left\{b_1\begin{vmatrix} c_2 & d_2 \\ c_4 & d_4 \end{vmatrix} - c_1\begin{vmatrix} b_2 & d_2 \\ b_4 & d_4 \end{vmatrix} + d_1\begin{vmatrix} b_2 & c_2 \\ b_4 & c_4 \end{vmatrix}\right\}$$

$$- a_4\left\{b_1\begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix} - c_1\begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix} + d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\right\}$$

(as already proved)

$$= a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\left\{a_2\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - a_3\begin{vmatrix} c_2 & d_2 \\ c_4 & d_4 \end{vmatrix} + a_4\begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix}\right\}$$

$$+ c_1\left\{a_2\begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix} - a_3\begin{vmatrix} b_2 & d_2 \\ b_4 & d_4 \end{vmatrix} + a_4\begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix}\right\}$$

$$- d_1\left\{a_2\begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix} - a_3\begin{vmatrix} b_2 & c_2 \\ b_4 & c_4 \end{vmatrix} + a_4\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}\right\}$$

(by a rearrangement)

$$= a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$$

(by definition)

In a similar manner the theorem may be proved successively for determinants of the fifth, the sixth, and, eventually, any order.

3. Properties of determinants.

1. *A determinant is unchanged in value if the rows and the columns are interchanged in such a manner that the first row becomes the first column, the second row the second column, and so on.*

The student may verify that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix};$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

This proves the theorem for determinants of the second and the third orders. To prove it for one of the fourth order, proceed as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix};$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}.$$

The expressions on the right of these equations are equal, and hence the determinants of the fourth order are equal. In the same manner the theorem may be proved for determinants of higher order.

It follows from this theorem that any property which is true of the rows is also true of the columns, and vice versa. The following theorems are stated for both rows and columns, but are proved for the rows only.

2. *If two consecutive rows (or columns) of a determinant are interchanged, the sign of the determinant is changed.*

The student may verify that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix};$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}; \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

The theorem is then proved for determinants of the second and the third orders. To prove it for a determinant of the fourth order, consider

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

By definition,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix};$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1\begin{vmatrix} b_3 & c_3 & d_3 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_3\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_2\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_2 & c_2 & d_2 \end{vmatrix}.$$

Comparing these two expressions, it will be noticed that the minors which multiply a_1 and a_4 (the elements of the unchanged rows) differ in the two expressions by the interchange of two consecutive rows, and that the minors which multiply a_2 and a_3 (the elements of the interchanged rows) are the same in the two expressions but are preceded by opposite signs. It is evident on reflection that these laws always hold; and hence, if the theorem is true for determinants of any order, it is true for determinants of the next higher order. The theorem is known to be true for determinants of the third order; hence it is universally true.

3. *A determinant is equal to the algebraic sum of the products obtained by multiplying each element of any row (or column) by its corresponding minor, the sign of each product being plus or minus according as the sum of the number of the row and the number of the column in which the element stands is even or odd.*

For the kth row may be made the first row of a new determinant by $k-1$ interchanges of two consecutive rows. By theorem 2, if k is odd, the new determinant is equal to the original one; and if k is even, the new determinant is equal to minus the original one. The new determinant may now be expressed by definition as the algebraic sum of the elements of its first row multiplied by their minors, which are the same as those of the kth row of the original determinant. Hence the original determinant is equal to the algebraic sum of the elements of its kth row multiplied by their minors, the products being alternately plus and minus when k is odd, and alternately minus and plus when k is even. From this the law of signs as given in the theorem at once follows.

Ex.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_4 & b_4 & c_4 & d_4 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = -\begin{vmatrix} a_4 & b_4 & c_4 & d_4 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

$$= -a_4\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} + b_4\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} - c_4\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + d_4\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

When a determinant is thus expressed it is said to be *expanded* according to the elements of the kth row. We shall call the coefficient of an element the quantity which multiplies it in the expansion.

Then *the coefficient of an element is plus or minus the corresponding minor according as the number of the row added to the number of the column is even or odd.*

The coefficient of a_1 shall be denoted by A_1, that of b_1 by B_1, and so on. Then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 + a_2A_2 + a_3A_3$$

$$= b_1B_1 + b_2B_2 + b_3B_3$$

$$= c_1C_1 + c_2C_2 + c_3C_3$$

$$= a_1A_1 + b_1B_1 + c_1C_1$$

$$= a_2A_2 + b_2B_2 + c_2C_2$$

$$= a_3A_3 + b_3B_3 + c_3C_3.$$

4. *If any two rows (or columns) of a determinant are interchanged, the sign of the determinant is changed.*

For suppose the determinant is expanded as in theorem 3, and that two rows other than that used in the expansion be interchanged. A similar interchange takes place in the minors of the expansion. Hence, if the theorem is true for each of the minors, it is true for the determinant. In other words, if the theorem is true for determinants of any order, it is true for those of the next higher order. But the theorem is certainly true for determinants of the second order. Hence it is always true.

5. *If two rows (or columns) of a determinant are the same, the determinant is equal to zero.*

Let a determinant with two rows the same be expanded according to the elements of some other row. Each minor of the expansion has two rows the same. Hence, if the theorem is true for determinants of any order, it is true for determinants of the next higher order. But the theorem is certainly true for determinants of the second order, for

$$\begin{vmatrix} a_1 & b_1 \\ a_1 & b_1 \end{vmatrix} = a_1b_1 - a_1b_1 = 0.$$

Hence it is universally true.

6. *The sum of the products obtained by multiplying the elements of any row (or column) by the coefficients of the corresponding elements of some other row (or column) is zero.*

Consider, for example,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_2A_2 + b_2B_2 + c_2C_2 + d_2D_2.$$

If we replace a_2, b_2, c_2, d_2, on the right-hand side of this equation by a_4, b_4, c_4, d_4, the same substitution must be made on the left-hand side. Then we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_4 & b_4 & c_4 & d_4 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_4A_2 + b_4B_2 + c_4C_2 + d_4D_2.$$

But the determinant is zero, by theorem 5; therefore

$$a_4A_2 + b_4B_2 + c_4C_2 + d_4D_2 = 0.$$

It is evident that the proof is general and establishes the theorem.

7. *If each element of any row (or column) is multiplied by the same quantity, the determinant is multiplied by the same quantity.*

This follows at once from theorem 3. For example,

$$\begin{vmatrix} a_1 & b_1 & kc_1 & d_1 \\ a_2 & b_2 & kc_2 & d_2 \\ a_3 & b_3 & kc_3 & d_3 \\ a_4 & b_4 & kc_4 & d_4 \end{vmatrix} = kc_1C_1 + kc_2C_2 + kc_3C_3 + kc_4C_4$$

$$= k\,[c_1C_1 + c_2C_2 + c_3C_3 + c_4C_4]$$

$$= k\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

8. *If each of the elements of any row (or column) is increased by the same multiple of the corresponding element of any other row (or column), the value of the determinant is unchanged.*

We wish to show, for example, that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \tag{1}$$

$$= \begin{vmatrix} a_1 & b_1 + kd_1 & c_1 & d_1 \\ a_2 & b_2 + kd_2 & c_2 & d_2 \\ a_3 & b_3 + kd_3 & c_3 & d_3 \\ a_4 & b_4 + kd_4 & c_4 & d_4 \end{vmatrix}. \tag{2}$$

Let the coefficients of the elements in the second column of (1) be B_1, B_2, B_3, B_4. It is evident that these are also the coefficients of the elements of the second column of (2). Hence (2) is

$$(b_1 + kd_1) B_1 + (b_2 + kd_2) B_2 + (b_3 + kd_3) B_3 + (b_4 + kd_4) B_4,$$

which equals

$$b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4 + k(d_1B_1 + d_2B_2 + d_3B_3 + d_4B_4).$$

The coefficient of k in this equation is zero, by theorem 6, and the remaining terms equal the determinant (1). Hence (2) = (1).

It is evident that the proof is general. The following are special cases: If $k = 1$, the elements of one row or column are added to the corresponding elements of another row or column; if $k = -1$, the elements of one row or column are subtracted from those of another row or column.

This theorem is often used in simplifying determinants.

Ex. 1. Consider
$$\begin{vmatrix} 1 & -2 & 1 & 2 \\ 3 & -5 & 3 & 5 \\ -1 & 2 & 3 & -4 \\ 3 & -5 & 2 & 5 \end{vmatrix}. \tag{1}$$

If the elements of the second column are added to those of the fourth column, this becomes

$$\begin{vmatrix} 1 & -2 & 1 & 0 \\ 3 & -5 & 3 & 0 \\ -1 & 2 & 3 & -2 \\ 3 & -5 & 2 & 0 \end{vmatrix}. \tag{2}$$

If twice the elements of the first column are added to those of the second column, (2) becomes

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 3 & 1 & 3 & 0 \\ -1 & 0 & 3 & -2 \\ 3 & 1 & 2 & 0 \end{vmatrix}. \tag{3}$$

If the elements of the first column are subtracted from those of the third column, (3) becomes

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 4 & -2 \\ 3 & 1 & -1 & 0 \end{vmatrix}. \tag{4}$$

Expressing (4) as the sum of the product of the elements of the first row and their coefficients, it becomes

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & -2 \\ 1 & -1 & 0 \end{vmatrix};$$

and this is equal to $\begin{vmatrix} 4 & -2 \\ -1 & 0 \end{vmatrix} = -2.$

Ex. 2. Consider $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}.$

By successive subtraction of the elements of one row from those of another we have

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \begin{vmatrix} x - x_1 & y - y_1 & 0 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \begin{vmatrix} x - x_1 & y - y_1 & 0 \\ x_1 - x_2 & y_1 - y_2 & 0 \\ x_2 & y_2 & 1 \end{vmatrix}$$

$= \begin{vmatrix} x - x_1 & y - y_1 \\ x_1 - x_2 & y_1 - y_2 \end{vmatrix}$, the last transformation being made by theorem 3.

4. Solution of *n* linear equations containing *n* unknown quantities, when the determinant of the coefficients of the unknown quantities is not zero. We are now prepared to show that the method used in § 1 to solve equations with two or three unknown quantities can be so generalized as to apply to any system of equations of the first degree in which the number of equations is equal to the number of the unknown quantities. For convenience we will take the case of four equations, but the student will readily see that the method is perfectly general.

Consider the equations

$$a_1x + b_1y + c_1z + d_1w + e_1 = 0, \tag{1}$$

$$a_2x + b_2y + c_2z + d_2w + e_2 = 0, \tag{2}$$

$$a_3x + b_3y + c_3z + d_3w + e_3 = 0, \tag{3}$$

$$a_4x + b_4y + c_4z + d_4w + e_4 = 0. \tag{4}$$

Let the determinant of the coefficients of the unknown quantities x, y, z, w be denoted by D, so that

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

and let A_1 denote the coefficient of a_1, B_1 the coefficient of b_1, and so on. We assume $D \neq 0$.

If now we multiply (1) by A_1, (2) by A_2, (3) by A_3, (4) by A_4, and add the results, we have, by theorems 3 and 6, § 3,

$$Dx + e_1A_1 + e_2A_2 + e_3A_3 + e_4A_4 = 0. \tag{5}$$

Similarly, by using B_1, B_2, B_3, B_4 as multipliers, we have

$$Dy + e_1B_1 + e_2B_2 + e_3B_3 + e_4B_4 = 0; \tag{6}$$

by using C_1, C_2, C_3, C_4 as multipliers, we have

$$Dz + e_1C_1 + e_2C_2 + e_3C_3 + e_4C_4 = 0; \tag{7}$$

and by using D_1, D_2, D_3, D_4 as multipliers, we have

$$Dw + e_1D_1 + e_2D_2 + e_3D_3 + e_4D_4 = 0. \tag{8}$$

Now it is clear that any values of x, y, z, w which satisfy (1), (2), (3), (4) satisfy also (5), (6), (7), (8). Conversely, any values which satisfy (5), (6), (7), (8) satisfy also (1), (2), (3), (4). For if we multiply (5) by a_1, (6) by b_1, (7) by c_1, (8) by d_1, and add, we obtain (1). Similarly (2), (3), (4) can be obtained from (5), (6), (7), (8). Hence (1), (2), (3), (4) and (5), (6), (7), (8) are equivalent equations.

Now
$$e_1A_1+e_2A_2+e_3A_3+e_4A_4=\begin{vmatrix} e_1 & b_1 & c_1 & d_1 \\ e_2 & b_2 & c_2 & d_2 \\ e_3 & b_3 & c_3 & d_3 \\ e_4 & b_4 & c_4 & d_4 \end{vmatrix},$$

$$e_1B_1+e_2B_2+e_3B_3+e_4B_4=\begin{vmatrix} a_1 & e_1 & c_1 & d_1 \\ a_2 & e_2 & c_2 & d_2 \\ a_3 & e_3 & c_3 & d_3 \\ a_4 & e_4 & c_4 & d_4 \end{vmatrix},$$

$$e_1C_1+e_2C_2+e_3C_3+e_4C_4=\begin{vmatrix} a_1 & b_1 & e_1 & d_1 \\ a_2 & b_2 & e_2 & d_2 \\ a_3 & b_3 & e_3 & d_3 \\ a_4 & b_4 & e_4 & d_4 \end{vmatrix},$$

and
$$e_1D_1+e_2D_2+e_3D_3+e_4D_4=\begin{vmatrix} a_1 & b_1 & c_1 & e_1 \\ a_2 & b_2 & c_2 & e_2 \\ a_3 & b_3 & c_3 & e_3 \\ a_4 & b_4 & c_4 & e_4 \end{vmatrix}.$$

Hence the solution of (5), (6), (7), and (8) is

$$x=-\frac{\begin{vmatrix} e_1 & b_1 & c_1 & d_1 \\ e_2 & b_2 & c_2 & d_2 \\ e_3 & b_3 & c_3 & d_3 \\ e_4 & b_4 & c_4 & d_4 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}}, \qquad y=-\frac{\begin{vmatrix} a_1 & e_1 & c_1 & d_1 \\ a_2 & e_2 & c_2 & d_2 \\ a_3 & e_3 & c_3 & d_3 \\ a_4 & e_4 & c_4 & d_4 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}},$$

$$z=-\frac{\begin{vmatrix} a_1 & b_1 & e_1 & d_1 \\ a_2 & b_2 & e_2 & d_2 \\ a_3 & b_3 & e_3 & d_3 \\ a_4 & b_4 & e_4 & d_4 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}}, \qquad w=-\frac{\begin{vmatrix} a_1 & b_1 & c_1 & e_1 \\ a_2 & b_2 & c_2 & e_2 \\ a_3 & b_3 & c_3 & e_3 \\ a_4 & b_4 & c_4 & e_4 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}},$$

and this is the solution, and the only solution, of (1), (2), (3), (4).

Hence we may state the following important theorem:

Any system of n linear equations containing n unknown quantities has one and only one solution when the determinant formed by the coefficients of the unknown quantities is not zero.

This solution may be written down at once, for each unknown quantity is equal to minus a fraction, of which the denominator is the determinant of the coefficients and the numerator is a similar determinant formed by replacing the coefficients of that unknown quantity by the absolute terms.

Ex. 1.

$$3x + 5y - 4 = 0,$$
$$2x - 3y + 7 = 0.$$

$$x = -\frac{\begin{vmatrix} -4 & 5 \\ 7 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & -3 \end{vmatrix}} = -\frac{23}{19}, \qquad y = -\frac{\begin{vmatrix} 3 & -4 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 3 & 5 \\ 2 & -3 \end{vmatrix}} = \frac{29}{19}.$$

Ex. 2.

$$2x - 3y + z - 1 = 0,$$
$$4x + 5y - 2z + 2 = 0,$$
$$x - 2y + 3z - 3 = 0.$$

$$x = -\frac{\begin{vmatrix} -1 & -3 & 1 \\ 2 & 5 & -2 \\ -3 & -2 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 1 \\ 4 & 5 & -2 \\ 1 & -2 & 3 \end{vmatrix}} = 0, \qquad y = -\frac{\begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & -2 \\ 1 & -3 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 1 \\ 4 & 5 & -2 \\ 1 & -2 & 3 \end{vmatrix}} = 0,$$

$$z = -\frac{\begin{vmatrix} 2 & -3 & -1 \\ 4 & 5 & 2 \\ 1 & -2 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 1 \\ 4 & 5 & -2 \\ 1 & -2 & 3 \end{vmatrix}} = 1.$$

5. Systems of n linear equations containing more than n unknown quantities. When in a set of linear equations the number of equations is less than the number of unknown quantities, the equations have usually an infinite number of solutions, but may have none. The general method of procedure in solving them is to pick out a number of the unknown quantities equal to the number of the equations and having the determinant of their coefficients not zero. These are solved by the method of § 4. We then have these unknown quantities expressed in terms of the others.

Ex. 1.
$$2x + 3y + z + 4 = 0,$$
$$x - 2y + 3z + 2 = 0.$$

If we choose x and y for the unknown quantities, we have

$$D = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7.$$

Then, solving as in § 4, we have

$$x = -\frac{\begin{vmatrix} z+4 & 3 \\ 3z+2 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}} = -\frac{11}{7}z - 2,$$

$$y = -\frac{\begin{vmatrix} 2 & z+4 \\ 1 & 3z+2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}} = \frac{5}{7}z;$$

and since z may be given any value whatever, the equations have an infinite number of solutions.

Ex. 2.
$$2x + 3y + z + 4 = 0,$$
$$2x + 3y + 2z + 3 = 0.$$

If we choose to solve for x and y, we have

$$D = \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = 0.$$

But if we choose to solve for y and z, we have

$$D = \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 3.$$

The solutions are
$$y = -\tfrac{2}{3}x - \tfrac{5}{3},$$
$$z = 1.$$

It is possible that no selection of the unknown quantities will lead to a determinant of the coefficients which is not zero. In this case the equations may have no solution. The discussion is too complex for this book, but the student will probably have no difficulty with the cases likely to occur in practice.

Ex. 3.
$$2x + 3y + z + 4 = 0,$$
$$2x + 3y + z + 3 = 0.$$

The determinant of any pair of unknown equations is zero. By subtracting the second equation from the first we have $1 = 0$, showing the equations to be contradictory.

6. Systems of n linear equations containing n unknown quantities, when the determinant of the coefficients of the unknown quantities is zero. Consider again equations (1), (2), (3), (4) of § 4, but with the assumption that $D = 0$. We may proceed exactly as in § 4, but equations (5), (6), (7), (8) do not now contain the unknown quantities. In fact, these equations are, in general, contradictory, and consequently equations (1), (2), (3), (4) have, in general, no solution.

Ex. 1.
$$x - y + z + 3 = 0,$$
$$2x + y + 3z + 1 = 0,$$
$$x + 2y + 2z + 4 = 0.$$

Here
$$D = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{vmatrix} = 0.$$

Eliminating y and z by the method of § 4, we have $0x - 24 = 0$, which is absurd. Hence the equations have no solution.

It is, of course, possible that when $D = 0$ each of the other determinants in (5), (6), (7), (8) is also zero. Each of these equations is then simply $0 = 0$, and gives no direct information about the solutions of (1), (2), (3), (4). As a matter of fact, in this case, (1), (2), (3), (4) have, in general, an infinite number of solutions, but may, under special conditions, have no solutions.

The general discussion is too complex to be given here. We shall simply state the following theorem:

A set of linear equations containing n unknown quantities has, in general, no solution when the determinant of the coefficients of the unknown quantities is zero, but may, under certain conditions, have an infinite number of solutions.

In practice, one of the n equations may be temporarily set aside, and the other $n-1$ equations, which contain n unknown quantities, may be examined by the method of § 5. If these equations can be solved, the solution can be tested in the equation which has been set aside.

Ex. 2.
$$\begin{aligned} 2x - 3y + z - 1 &= 0, \\ x - 2y + 3z + 4 &= 0, \\ 7x - 11y + 6z + 1 &= 0. \end{aligned}$$

If the method of § 4 is used, the result is $0 = 0$. Solving the first two equations for x and y, we have

$$\begin{aligned} x &= 7z + 14, \\ y &= 5z + 9, \end{aligned}$$

and these results are found on trial to satisfy the last of the given equations. Since z may have any value, the equations have an infinite number of solutions.

7. Systems of linear equations in which the number of the equations is greater than that of the unknown quantities. If there are more equations of the first degree than there are unknown quantities, there will be, in general, no values of the unknown quantities which satisfy all equations. There may be such values, however, when certain relations exist among the coefficients of the equations. To obtain these relations we may pick out a number of equations equal to the number of the unknown quantities and solve them. If the solution is substituted in the remaining equations, there will result certain expressions in the coefficients which must be zero if the equations are to be satisfied.

The most important case is that in which there are $n+1$ equations containing n unknown quantities. For example, consider

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0, \\ a_2x + b_2y + c_2z + d_2 &= 0, \\ a_3x + b_3y + c_3z + d_3 &= 0, \\ a_4x + b_4y + c_4z + d_4 &= 0. \end{aligned}$$

The solution of the first three equations, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, is (§ 4)

$$x = -\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = -\frac{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$y = -\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$z = -\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Substituting these values in the first member of the last equation, we have

$$\frac{-a_4\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} + b_4\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} - c_4\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + d_4\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

which, by theorem 3, § 3, is the same as

$$\frac{\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Hence, in order that the last equation may be satisfied, we must have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

Extending this to any number of variables, we have the theorem:

In order that a system of $n+1$ linear equations containing n unknown quantities shall have a solution, it is necessary that the determinant formed from the coefficients of the unknown quantities and the absolute terms shall be zero.

Ex. 1.
$$\begin{aligned} x + y + z - 2 &= 0, \\ 2x + y - z + 3 &= 0, \\ x - 2y - 3z + 4 &= 0, \\ 5x - 3y - 4z + 1 &= 0. \end{aligned}$$

Here
$$\begin{vmatrix} 1 & 1 & 1 & -2 \\ 2 & 1 & -1 & 3 \\ 1 & -2 & -3 & 4 \\ 5 & -3 & -4 & 1 \end{vmatrix} = 0,$$

showing that if the first three equations have a solution it will satisfy the fourth equation. In fact, the solution is $x = 1$, $y = -2$, $z = 3$.

It should be noted that the converse of the theorem stated is not necessarily true. All that has been proved is that *if* n of the equations have a solution, that solution satisfies the $(n+1)$st equation when the determinant is zero. But the determinant may be zero when the equations are contradictory.

Ex. 2.
$$\begin{aligned} 2x - 3y + z + 1 &= 0, \\ 2x - 3y + 5z + 2 &= 0, \\ 2x - 3y - 6z - 3 &= 0, \\ 2x - 3y + 2z - 8 &= 0. \end{aligned}$$

Here
$$\begin{vmatrix} 2 & -3 & 1 & 1 \\ 2 & -3 & 5 & 2 \\ 2 & -3 & -6 & -3 \\ 2 & -3 & 2 & -8 \end{vmatrix} = 0,$$

but any three of the equations may be seen to be contradictory by the method of § 6.

8. Linear homogeneous equations. An equation is homogeneous with respect to the unknown quantities when the sum of the exponents of the unknown quantities is the same in each term. In particular an equation of the first degree is homogeneous when each of the terms contains one of the unknown quantities; for example,

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0,$$

where x_1, x_2, x_3, x_4 are the unknown quantities.

This equation is, of course, satisfied by placing $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, but in practice this solution is generally unimportant. In such equations, in fact, it is usually the *ratios* of the unknown quantities which are important; for if each unknown quantity is multiplied by the same number, the equation is unaltered. In fact, if we place

$$\frac{x_1}{x_4} = x, \qquad \frac{x_2}{x_4} = y, \qquad \frac{x_3}{x_4} = z,$$

the homogeneous equation just written becomes the non-homogeneous equation

$$a_1x + a_2y + a_3z + a_4 = 0.$$

In this manner a set of homogeneous equations containing n unknown quantities may be reduced to a set of non-homogeneous equations containing $n-1$ unknown quantities by dividing each equation by one of the unknown quantities. The methods of the previous articles may then be used. But this method of procedure is open to the objection that the unknown quantity by which the equations are divided may possibly be zero when the division is invalid. It is better, therefore, to handle the homogeneous equations as they stand, slightly modifying the methods used for non-homogeneous equations in a manner which will be clear from the examples.

Ex. 1.

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 &= 0, \\ b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 &= 0, \qquad (1) \\ c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 &= 0. \end{aligned}$$

We will handle this by the method of § 4, in that we temporarily look upon x_1, x_2, x_3 as the unknown quantities. We have, in the first place,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} x_1 + \begin{vmatrix} a_4x_4 & a_2 & a_3 \\ b_4x_4 & b_2 & b_3 \\ c_4x_4 & c_2 & c_3 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} x_2 + \begin{vmatrix} a_1 & a_4x_4 & a_3 \\ b_1 & b_4x_4 & b_3 \\ c_1 & c_4x_4 & c_3 \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} x_3 + \begin{vmatrix} a_1 & a_2 & a_4x_4 \\ b_1 & b_2 & b_4x_4 \\ c_1 & c_2 & c_4x_4 \end{vmatrix} = 0,$$

which may be written as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} x_1 + \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} x_4 = 0, \qquad (2)$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} x_2 - \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} x_4 = 0, \qquad (3)$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} x_3 + \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} x_4 = 0. \qquad (4)$$

From these follow:

$$x_1 : x_2 : x_3 : x_4 = \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix} : - \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} : \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} : - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \qquad (5)$$

The result (5) holds even when one or more, but not all, of the determinants involved are equal to zero. Then the corresponding unknown quantities are equal to zero. For example, if

$$\begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix} = 0, \qquad \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix} = 0,$$

and the other determinants in (5) are not zero, (3) and (4) show that $x_2 = 0$ and $x_3 = 0$, while (2) shows that the ratio of x_1 and x_4 are correctly given by (5).

If all the determinants in (5) are zero, the values of the unknowns are not thereby determined. In this case, two of the equations (1) should be solved for two of the unknown quantities in terms of the others, and the results tested for the last equations.

It should be noted that contradictory equations cannot occur. The student should compare the contradictory equations

$$2x - 3y + 4 = 0,$$
$$2x - 3y - 2 = 0,$$

with the homogeneous equations

$$2x_1 - 3x_2 + 4x_3 = 0,$$
$$2x_1 - 3x_2 - 2x_3 = 0.$$

By subtracting one equation from the other we have

$$6x_3 = 0,$$

whence $$x_3 = 0 \text{ and } x_1 : x_2 = 3 : 2.$$

Ex. 2. The four equations

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0,$$
$$b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 = 0,$$
$$c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 = 0,$$
$$d_1x_1 + d_2x_2 + d_3x_3 + d_4x_4 = 0,$$

have, of course, the common solutions, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$. In order that they may also be satisfied by the same ratios of the unknown quantities, it is necessary that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = 0.$$

The proof is as in § 7. The condition is also sufficient, for the proof of § 7 shows that if three of the equations have a solution, that will also be a solution of the fourth equation ; and, as just noted, three homogeneous equations always have a solution.

9. Eliminants. The result of eliminating all the unknown quantities from two or more equations is an equation the left-hand member of which is called the *eliminant*, or *resultant*, of the given equations. The following cases are important:

1. $n+1$ non-homogeneous linear equations with n unknown quantities. To eliminate the unknown quantities, we may solve n of the equations and substitute the solutions in the remaining equation. The work and the result are as in § 7; that is,

The eliminant of $n+1$ non-homogeneous equations with n unknown quantities is equal to the determinant of the coefficients and the absolute terms.

2. n homogeneous linear equations with n unknown quantities. To eliminate the unknown quantities, we may solve $n-1$ equations for their ratios and substitute the results in the remaining equation. The work and the result are as in § 8; that is,

The eliminant of n homogeneous equations with n unknown quantities is equal to the determinant of the coefficients.

3. Two equations containing one unknown quantity. Let it be required to eliminate x between the equations

$$a_1x^2 + b_1x + c_1 = 0, \tag{1}$$

$$a_2x^2 + b_2x + c_2 = 0. \tag{2}$$

If we multiply each equation by x, we have

$$a_1x^3 + b_1x^2 + c_1x = 0, \tag{3}$$

and

$$a_2x^3 + b_2x^2 + c_2x = 0. \tag{4}$$

These four equations may now be considered as linear in the three unknown quantities x^3, x^2, and x. Elimination gives, by 1,

$$\begin{vmatrix} 0 & a_1 & b_1 & c_1 \\ 0 & a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \end{vmatrix} = 0. \tag{5}$$

It is clear that if equations (1) and (2) have a common solution, equation (5) must be true. Conversely, it may be shown that if (5) is true, (1) and (2) must have a common solution; but this proof is too long to be given here.

The method used in the above problem may be used for equations of any degree and is known as Sylvester's method of elimination. It consists in multiplying the given equations by successive powers of x until we have one more equation than we have powers of x. The eliminant is then found as in 1.

The method may also be used to eliminate one of the unknown quantities from two equations containing two unknown quantities.

PROBLEMS

Find the value of each of the following determinants:

1. $\begin{vmatrix} 4 & 5 \\ 3 & 6 \end{vmatrix}$.

2. $\begin{vmatrix} x & 1 \\ y & 1 \end{vmatrix}$.

3. $\begin{vmatrix} x & 1 \\ x^2 & 1 \end{vmatrix}$.

4. $\begin{vmatrix} 1 & 0 \\ 10 & 4 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & -7 \\ 2 & 3 \end{vmatrix}$.

6. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$.

7. $\begin{vmatrix} 1 & 0 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix}$.

8. $\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix}$.

9. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

10. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

11. $\begin{vmatrix} x & y & 1 \\ 1 & 2 & -1 \\ 4 & 3 & 2 \end{vmatrix}$.

12. $\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{vmatrix}$.

13. $\begin{vmatrix} 0 & a_1 & b_1 & c_1 \\ 0 & a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \end{vmatrix}$.

Prove the following relations:

14. $\begin{vmatrix} 4 & 2 & 1 \\ 3 & 4 & 2 \\ 5 & 6 & 3 \end{vmatrix} = 0.$

15. $\begin{vmatrix} 5 & 4 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0.$

16. $\begin{vmatrix} a & 0 & 0 & b & 0 \\ 0 & a & 0 & 0 & b \\ x & y & 1 & z & w \\ c & 0 & 0 & d & 0 \\ 0 & c & 0 & 0 & d \end{vmatrix} = (ad - bc)^2.$

17. $\begin{vmatrix} 1 & -2 & -3 \\ -2 & 1 & 3 \\ -3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix}.$

18. $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}.$

19. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x).$

20. $\begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ 0 & 0 & c_1 & d_1 \\ 0 & 0 & c_2 & d_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \cdot \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}.$

21. $\begin{vmatrix} 1 & 4 & -3 & 5 \\ 0 & 6 & -8 & -1 \\ 2 & -3 & 4 & 2 \\ 5 & 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 2 & 1 \\ 1 & -6 & -3 & 4 \\ 3 & 8 & 4 & -3 \\ 4 & 1 & 2 & 5 \end{vmatrix}.$

22. $\begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$

23. $\begin{vmatrix} a_0 & 3a_1 & 3a_2 & a_3 & 0 \\ 0 & a_0 & 3a_1 & 3a_2 & a_3 \\ a_0 & 2a_1 & a_2 & 0 & 0 \\ 0 & a_0 & 2a_1 & a_2 & 0 \\ 0 & 0 & a_0 & 2a_1 & a_2 \end{vmatrix} = a_0 \begin{vmatrix} a_1 & 2a_2 & a_3 & 0 \\ 0 & a_1 & 2a_2 & a_3 \\ a_0 & 2a_1 & a_2 & 0 \\ 0 & a_0 & 2a_1 & a_2 \end{vmatrix}$

$$= a_0\{a_0^2a_3^2 - 6\,a_0a_1a_2a_3 + 4\,a_0a_2^3 + 4\,a_1^3a_3 - 3\,a_1^2a_2^2\}.$$

Solve the following equations:

24. $\begin{vmatrix} 4-x & 3 \\ 3 & 9-x \end{vmatrix} = 0.$

25. $\begin{vmatrix} 1-x & 2 & 3 \\ 2 & 3-x & 5 \\ 3 & 5 & 8-x \end{vmatrix} = 0.$

Write the following equations in their expanded forms:

26. $\begin{vmatrix} x & y & 1 \\ 2 & -3 & 1 \\ 1 & 5 & 1 \end{vmatrix} = 0.$

27. $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0.$

28. $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 5 & 1 & 2 & 1 \\ 13 & 2 & -3 & 1 \\ 2 & -1 & -1 & 1 \end{vmatrix} = 0.$

29. $\begin{vmatrix} a-x & h \\ h & b-x \end{vmatrix} = 0.$

30. $\begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix} = 0.$

Solve the following equations:

31. $4x - 5y + 6 = 0,$
$7x - 9y + 11 = 0.$

32. $x + 2y - z + 3 = 0,$
$2x - y - 5 = 0,$
$x + 2z - 8 = 0.$

33. $\dfrac{1}{x} + \dfrac{1}{y} = 1,$
$\dfrac{1}{y} + \dfrac{1}{z} = 2,$
$\dfrac{1}{z} + \dfrac{1}{x} = 4.$

34. $2x + 4y + 3z - 2 = 0,$
$x - 5y + z + 1 = 0,$
$3x + 10y + 5z - 5 = 0.$

35. $2x + y + z + 2 = 0,$
$5x + 2y + 3z + 5 = 0,$
$2x + 3y - 2z + 2 = 0.$

36. $x + y + 9z - 7 = 0,$
$5x - y + 9z - 5 = 0,$
$3x - y + 3z - 2 = 0.$

37. $10x - 3y + 12z - 5 = 0,$
$4x - y + 6z - 3 = 0,$
$5x - 2y + 3z = 0.$

38. $x + y + z = a,$
$y + z + w = b,$
$z + w + x = c,$
$w + x + y = d.$

39. $10x_1 + 4x_2 + 5x_3 = 0,$
$3x_1 + x_2 + 2x_3 = 0.$

40. $x_1 + 5x_2 + 3x_3 = 0,$
$3x_1 + 3x_2 + x_3 = 0.$

41. $2x_1 + 4x_2 + x_3 = 0,$
$3x_1 + 6x_2 - x_3 = 0.$

42. $2x_1 + x_2 - 5x_3 + x_4 = 0,$
$3x_1 - 2x_2 - 4x_3 - 2x_4 = 0,$
$x_1 + x_2 + 2x_3 - x_4 = 0.$

43. $2x_1 - 3x_2 + 2x_3 - 3x_4 = 0,$
$4x_1 + 5x_2 + 4x_3 - 6x_4 = 0,$
$3x_1 - 7x_2 - 2x_3 + 3x_4 = 0.$

44. $7x_1 - 5x_2 + 3x_3 - 4x_4 = 0,$
$3x_1 + 2x_2 - 5x_3 + 9x_4 = 0,$
$5x_1 - 16x_2 + 21x_3 - 35x_4 = 0.$

Find whether or not the equations in each of the following examples have a common solution:

45. $$\begin{aligned} 2x - y + 3 &= 0, \\ 3x + y - 1 &= 0, \\ 3x - 4y + 10 &= 0. \end{aligned}$$

46. $$\begin{aligned} 5x - 2y + 7 &= 0, \\ 3x - y + 6 &= 0, \\ x + 3y - 1 &= 0. \end{aligned}$$

49. For what values of a are the following equations consistent?

$$\begin{aligned} x + a^2y + a &= 0, \\ ax + y + a^2 &= 0, \\ a^2x + ay + 1 &= 0. \end{aligned}$$

50. Eliminate x from the equations

$$\begin{aligned} xy + 3x + 1 &= 0, \\ 2xy - 4y + 2 &= 0. \end{aligned}$$

51. Eliminate x from the equations

$$\begin{aligned} xy^2 + 2y + 3 &= 0, \\ xy + 4x + 1 &= 0. \end{aligned}$$

52. Eliminate x and z from the equations

$$\begin{aligned} xy + yz - x + z + 2 &= 0, \\ xy - 2x + y + z + 2 &= 0, \\ x + 3z - 2 &= 0. \end{aligned}$$

47. $$\begin{aligned} x - 2y + 3z - 1 &= 0, \\ 2x + y - z + 1 &= 0, \\ x - 3y + 2z + 2 &= 0, \\ x - 19y + 22z - 4 &= 0. \end{aligned}$$

48. $$\begin{aligned} x - 2y + 1 &= 0, \\ y - 2z + 2 &= 0, \\ z - 2x + 3 &= 0, \\ x + y + z &= 0. \end{aligned}$$

53. Find the condition that

$$ax^2 + bx + c = 0,$$

and $$x^2 = 1,$$

have a common root.

54. Show that the condition that

$$ax^2 + bx + c = 0,$$

and $$x^3 = 1,$$

have a common root is

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0.$$

55. Show that if

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0, \\ a_3x + b_3y + c_3 &= 0, \end{aligned}$$

have a common solution, there can always be found three numbers l, k, m such that

$$\begin{aligned} a_1l + a_2k + a_3m &= 0, \\ b_1l + b_2k + b_3m &= 0, \\ c_1l + c_2k + c_3m &= 0. \end{aligned}$$

CHAPTER II

GRAPHICAL REPRESENTATION

10. Real number. The science of mathematics deals with various kinds of numbers, each of which has arisen through the desire to perform, without restriction, some one of the fundamental operations. The simplest numbers are the *positive integers*, or whole numbers. If one restricts himself to the use of these, he may add or multiply together any two of them without obtaining a new kind of number; but he may not divide one number by another not exactly contained in it, nor subtract a larger number from a smaller. In order that division may always be performed, the common *fractions*, which are the quotients of one integer divided by another, are necessary. In order that subtraction may always be possible, the idea of a *negative* number must be introduced. The integers and fractions, both positive and negative, together form the class of *rational numbers*. On these numbers the operations of addition, subtraction, multiplication, and division may always be performed without leading to a new kind of number.

The operation of evolution, however, leads to two new kinds of numbers, — the *irrational*, exemplified by $\sqrt{2}$; and the *complex*, of which $\sqrt{-2}$ is an example. The complex numbers will be noticed in § 12; we shall here speak only of the irrational numbers. An *irrational number* is defined as one which cannot be expressed exactly as an integer or a common fraction, but which may be so expressed approximately to any required degree of accuracy. The simplest examples are the roots of rational numbers; for example, $\sqrt{7}$ may be approximately expressed as $\frac{264}{100}$, $\frac{26457}{10000}$, etc., but cannot be expressed exactly. There are also irrational numbers which are not the roots of numbers and cannot be expressed by means of radical signs. A familiar example is the number $\pi = 3.14159 \cdots$. An irrational number may be either positive or negative. The rational and the irrational numbers together form the class of *real numbers.*

A rigorous investigation of the nature and properties of these numbers, especially of the irrational numbers, is too advanced for this book. An elementary discussion, however, is given in any course in algebra, and is here assumed as known.

The real numbers may be represented graphically on a number scale, constructed as follows:

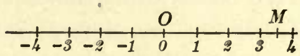

Fig. 1

On any straight line assume a fixed point O as the zero point, or *origin,* and lay off positive numbers in one direction and negative numbers in the other. If the line is horizontal, as in fig. 1, it is usual, but not necessary, to lay off the positive numbers to the right of O and the negative numbers to the left. Then any point M on the scale represents a real number, namely, the number which measures the distance of M from O; positive if M is to the right of O, and negative if M is to the left of O. Conversely, any real number is represented by one and only one real point on the scale.

11. Zero and infinity. There are two mathematical concepts usually included in the number series, for which special rules of operation are needed. These are zero, represented by the symbol 0, and infinity, represented by the symbol ∞.

Zero arises in the first place by subtracting a quantity from an equal quantity; thus, $a - a = 0$. It signifies in this sense the absence of quantity — nothing. It cannot, then, either operate upon a quantity or be operated upon; for all operations imply the existence of the quantities concerned. Literally, then, the expressions $a \times 0$, $\frac{0}{a}$, $\frac{a}{0}$, are meaningless. However, it is possible to put into these symbols conventional meanings, as follows:

Take the three expressions ax, $\frac{x}{a}$, $\frac{a}{x}$, and consider what happens when x is taken smaller and smaller, constantly nearer to zero but never equal to it. It requires only elementary arithmetic to see that ax and $\frac{x}{a}$ may each be made as small as we please by taking x sufficiently small, while $\frac{a}{x}$ becomes indefinitely great as x decreases, and may be made larger than any quantity we may choose to name. We may express the first two results concisely by the formulas

$$a \times 0 = 0, \qquad \frac{0}{a} = 0.$$

We can express the last result in a formula, however, only by introducing the concept infinity. When the value of a quantity is indefinite, but the quantity is increasing or decreasing in such a way that its numerical value is greater than any assigned quantity, however great, it is said to become infinite. It is then denoted by the symbol ∞, called infinity. We can accordingly express our third result by the formula

$$\frac{a}{0} = \infty,$$

which means that when the denominator of a fraction decreases, becoming constantly nearer to zero, the value of the fraction increases and becomes greater than any quantity which can be named.

The symbols

$$a \times \infty, \qquad \frac{a}{\infty}, \qquad \frac{\infty}{a}$$

are also literally meaningless. We can, however, give a conventional meaning to them by writing ax, $\frac{a}{x}$, $\frac{x}{a}$, and studying the effect of increasing x indefinitely. Elementary arithmetic leads to the results expressed by the formulas

$$a \times \infty = \infty, \qquad \frac{a}{\infty} = 0, \qquad \frac{\infty}{a} = \infty.$$

Two other forms also occur in practice, namely, $\frac{0}{0}$ and $\frac{\infty}{\infty}$. These arise when we have a fraction $\frac{x}{y}$ in which the numerator and the denominator either approach zero together or increase indefinitely together. The value of the fraction cannot be determined unless we know a law to govern x and y. These fractions are consequently called indeterminate forms, and will be considered later in the course.

Neither zero nor infinity can be said to have an intrinsic algebraic sign. In some cases a quantity may increase in value, remaining always positive. It is then said to be $+\infty$. At other times it may increase numerically, remaining always negative. It is then said to be $-\infty$. Often, however, the quantity is indefinitely great in such a way that the sign is ambiguous. An example is tan 90°. If an acute angle is made nearer and nearer to 90°, its tangent increases indefinitely, remaining positive. But if an obtuse angle is made nearer and nearer to 90°, its tangent increases indefinitely, remaining negative. Hence we say $\tan 90° = \infty$, and no algebraic sign can be attached to it.

Similar considerations hold for the sign of zero.

12. Complex numbers. If one restricts himself to the use of the real numbers, named in § 10, it is impossible to perform the operation of evolution without exception; for the even root of a negative number is not a real number. It is therefore necessary, if the generality of all algebraic operations is to be maintained, to introduce a new kind of number, called a *complex number.* These numbers will be used very little in this volume, and the following résumé of the matter usually contained in algebra is sufficient for our present purposes. A further discussion will be given in the second volume.

The *imaginary unit* is $\sqrt{-1}$, and is denoted by i. Then

$$i^2 = -1.$$

By multiplying this equation successively by i, we find

$$i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad i^6 = -1, \quad \cdots;$$

and, in general,

$$i^{4k} = 1, \quad i^{4k+1} = i, \quad i^{4k+2} = -1, \quad i^{4k+3} = -i,$$

where k is zero or any integer.

If b is any real number, the product bi is called a *pure imaginary number.* The square root of any negative number is pure imaginary; thus,

$$\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i, \quad \sqrt{-5} = \sqrt{5}\sqrt{-1} = i\sqrt{5}.$$

If a and b are any two real numbers, the combination $a + bi$ is called a complex imaginary number, or, more simply, a *complex number*. A complex number reduces to a pure imaginary number when $a = 0$, and to a real number when $b = 0$. If $a = 0$ and $b = 0$, the complex number $a + bi = 0$; and conversely, if $a + bi = 0$, then $a = 0$ and $b = 0$.

All operations with complex numbers are carried out by using the ordinary laws of algebra and replacing all powers of i by their values just determined.

Ex. 1. $\sqrt{-3} \times \sqrt{-2} = i\sqrt{3} \times i\sqrt{2} = i^2\sqrt{6} = -\sqrt{6}$.

Ex. 2. $\dfrac{3+\sqrt{-4}}{2-\sqrt{-4}} = \dfrac{3+2i}{2-2i} \times \dfrac{2+2i}{2+2i} = \dfrac{6+10i+4i^2}{4-4i^2} = \dfrac{2+10i}{8} = \dfrac{1+5\sqrt{-1}}{4}$.

Two complex numbers such as $a + bi$ and $a - bi$, where a and b have the same values in each, are called *conjugate complex numbers*. Their product is a real number; thus,

$$(a + bi)(a - bi) = a^2 + b^2.$$

It is clear that the complex numbers have no place on the number scale of § 10.

13. Addition of segments of a straight line. Consider any straight line connecting two points A and B. In elementary geometry only the position and the length of the line are considered, and consequently it is immaterial whether the line be called AB or BA; but in the work to follow it is often important to consider the *direction* of the line as well. Accordingly, if the direction of the line is considered as from A to B, it is called AB; but if the direction is considered from B to A, it is called BA. It will be seen later that the distinction between AB and BA is the same as that between $+a$ and $-a$ in algebra.

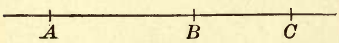

Fig. 2

Consider now two segments AB and BC on the same straight line, the point B being the end of the first segment and the beginning of the second. The segment AC is called the *sum* of AB and BC, and is expressed by the equation

$$AB + BC = AC. \quad (1)$$

This is clearly true if the points are in the position of fig. 2, but it is equally true when the points are in the position of fig. 3. Here the line BC, being opposite in direction to AB, cancels part of it, leaving AC.

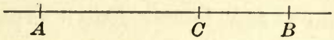

FIG. 3

If, in the last figure, the point C is moved toward A, the sum AC becomes smaller, until finally when C coincides with A we have

$$AB + BA = 0, \quad \text{or} \quad BA = -AB. \tag{2}$$

If the point C is at the left of A, as in fig. 4, we still have $AB + BC = AC$, where $AC = -CA$ by (2).

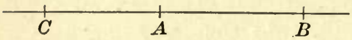

FIG. 4

It is evident that this addition is analogous to algebraic addition, and that this sum may be an arithmetical difference.

From (1) we may obtain by transposition a formula for subtraction, namely,

$$BC = AC - AB. \tag{3}$$

This is universally true since (1) is universally true.

This result is particularly important when applied to segments of the number scale of § 10. For if x is any number corresponding to the point M, we may always place $x = OM$, since both x and OM are positive when M is at the right of O, and both x and OM are negative when M is at the left of O. Now let M_1 and M_2 be any two points, and let $x_1 = OM_1$ and $x_2 = OM_2$. Then

$$M_1M_2 = OM_2 - OM_1 = x_2 - x_1.$$

On the other hand,

$$M_2M_1 = OM_1 - OM_2 = x_1 - x_2 = -M_1M_2.$$

It is clear that the segment M_1M_2 is positive when M_2 is at the right of M_1, and is negative when M_2 is at the left of M_1.

Hence, *the length and the sign of any segment of the number scale is found by subtracting the value of the x corresponding to the beginning of the segment from the value of the x corresponding to the end of the segment.*

14. Projection. Let AB and MN (figs. 5, 6) be any two straight lines in the same plane, the positive directions of which are respectively AB and MN. From A and B draw straight lines perpendicular to MN, intersecting it at points A' and B' respectively. Then $A'B'$

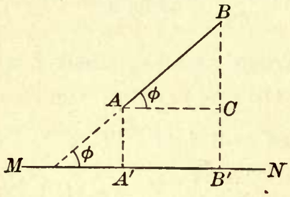

Fig. 5

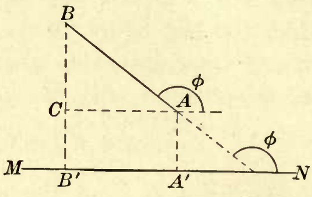

Fig. 6

is the *projection* of AB on MN, and is positive if it has the direction MN (fig. 5), and is negative if it has the direction NM (fig. 6).

Denote the angle between MN and AB by ϕ, and draw AC parallel to MN. Then in both cases, by trigonometry,

$$AC = AB \cos \phi.$$

But $AC = A'B'$, and therefore

$$A'B' = AB \cos \phi.$$

Hence, *to find the projection of one straight line upon a second, multiply the length of the first by the cosine of the angle between the positive directions of the two lines.*

Ex. It is customary in mechanics to represent a force by a straight line, the length and the direction of which denote respectively the magnitude and the direction of the force. Then the component of the force in any direction is the projection upon that direction of the line which represents the force.

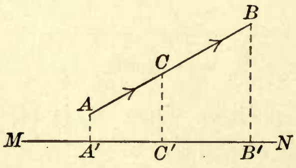

Fig. 7

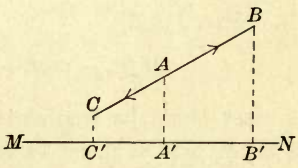

Fig. 8

In particular, let F_1 and F_2, represented respectively by AB and AC (figs. 7, 8), be two forces acting at A along the same line, and let MN be a line which makes an angle ϕ with AB.

The respective components of F_1 and F_2 are represented by $A'B'$ and $A'C'$, and the resultant component is represented by $A'B'+A'C'$.

But $A'B'=F_1\cos\phi$, and $A'C'=F_2\cos\phi$; hence, by substitution, the resultant component is $F_1\cos\phi+F_2\cos\phi$. It is to be noted that in fig. 8 F_1 and F_2 have opposite signs.

15. *The projection of a broken line upon a straight line is defined as the algebraic sum of the projections of its segments.*

Let $ABCDE$ (fig. 9) be a broken line, MN a straight line in the same plane, and AE the straight line joining the ends of the broken line.

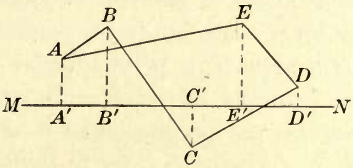

FIG. 9

Draw AA', BB', CC', DD', and EE' perpendicular to MN; then $A'B'$, $B'C'$, $C'D'$, $D'E'$, and $A'E'$ are the respective projections on MN of AB, BC, CD, DE, and AE.

But $$A'B'+B'C'+C'D'+D'E'=A'E'. \qquad \text{(by § 13)}$$

Hence, *the projection of a broken line upon a straight line is equal to the projection of the straight line joining its extremities.*

Ex. If $ABCDE$ (fig. 9) represents a polygon of forces, we have the result: the component of the resultant in any direction is the sum of the components of the forces in that direction.

16. Coördinate axes. Let $X'X$ and $Y'Y$ be two number scales at right angles to each other, with their zero points coincident at O, as in fig. 10.

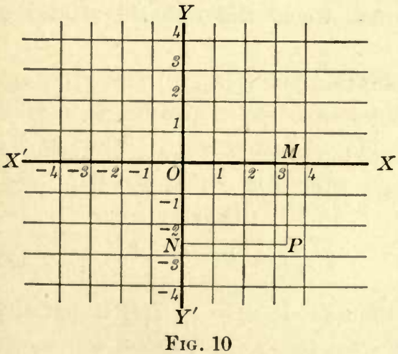

FIG. 10

Let P be any point in the plane, and through P draw straight lines perpendicular to $X'X$ and $Y'Y$ respectively, intersecting them at M and N. If now, as in § 13, we place $x=OM$, and $y=ON$, it is clear that to any point P there corresponds one and only one pair of numbers x and y, and to any pair of numbers corresponds one and only one point P.

If a point P is given, x and y may be found by drawing the two perpendiculars MP and NP as above, or by drawing only one perpendicular as MP. Then $MP = ON = y$ and $OM = x$.

On the other hand, if x and y are given, the point P may be located by finding the points M and N corresponding to the numbers x and y on the two number scales, and drawing perpendiculars to $X'X$ and $Y'Y$ respectively through M and N. These perpendiculars intersect at the required point P. Or, as is often more convenient, a point M corresponding to x may be located on its number scale, and a perpendicular to $X'X$ may be drawn through M, and on this perpendicular the value of y laid off. In fig. 10, for example, M corresponding to x may be found on the scale $X'X$, and on the perpendicular to $X'X$ at M, MP may be laid off equal to y. When the point is located in either of these ways it is said to be *plotted*. It is evident that *plotting* is most conveniently performed when the paper is ruled in squares, as in fig. 10.

These numbers x and y are called respectively the *abscissa* and the *ordinate* of the point, and together they are called its *coördinates*. It is to be noted that the abscissa and the ordinate, as defined, are respectively equal to the distances *from* $Y'Y$ and $X'X$ *to* the point, the direction as well as the magnitude of the distances being taken into account. Instead of designating a point by writing $x = a$ and $y = -b$, it is customary to write $P(a, -b)$, the abscissa always being written first in the parenthesis and separated from the ordinate by a comma. $X'X$ and $Y'Y$ are called the *axes of coördinates*, but are often referred to as the axes of x and y respectively.

17. Distance between two points. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points, and at first assume that P_1P_2 is parallel to one of the coördinate axes, as OX (fig. 11). Then $y_2 = y_1$. Now M_1M_2, the projection of P_1P_2 on OX, is evidently equal to P_1P_2. But $M_1M_2 = x_2 - x_1$ (§ 13). Hence

$$P_1P_2 = x_2 - x_1. \tag{1}$$

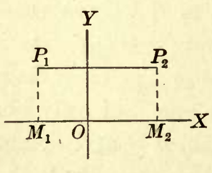

Fig. 11

In like manner, if $x_2 = x_1$, P_1P_2 is parallel to OY, and

$$P_1P_2 = y_2 - y_1. \tag{2}$$

If $x_2 \neq x_1$ and $y_2 \neq y_1$, P_1P_2 is not parallel to either axis. Let the points be situated as in fig. 12, and through P_1 and P_2 draw straight lines parallel respectively to OX and OY. They will meet at a point R, the coördinates of which are readily seen to be (x_2, y_1). By (1) and (2),

$$P_1R = x_2 - x_1, \qquad RP_2 = y_2 - y_1.$$

But in the right triangle P_1RP_2,

$$P_1P_2 = \sqrt{\overline{P_1R}^2 + \overline{RP_2}^2},$$

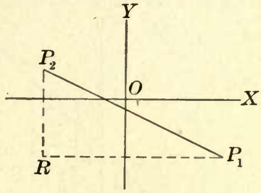

FIG. 12

whence, by substitution, we have

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \qquad (3)$$

It is to be noted that there is an ambiguity of algebraic sign on account of the radical sign. But since P_1P_2 is parallel to neither coördinate axis, the only two directions in the plane the positive directions of which have been chosen, we are at liberty to choose either direction of P_1P_2 as the positive direction, the other becoming the negative.

It is also to be noted that formulas (1) and (2) are particular cases of the more general formula (3).

Ex. Find the coördinates of a point equally distant from the three points $P_1(1, 2)$, $P_2(-1, -2)$, and $P_3(2, -5)$.

Let $P(x, y)$ be the required point. Then

$$P_1P = P_2P \text{ and } P_2P = P_3P.$$

But

$$P_1P = \sqrt{(x-1)^2 + (y-2)^2},$$

$$P_2P = \sqrt{(x+1)^2 + (y+2)^2},$$

$$P_3P = \sqrt{(x-2)^2 + (y+5)^2}.$$

$$\therefore \sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x+1)^2 + (y+2)^2},$$

$$\sqrt{(x+1)^2 + (y+2)^2} = \sqrt{(x-2)^2 + (y+5)^2},$$

whence, by solution, $x = \frac{8}{3}$ and $y = -\frac{4}{3}$. Therefore the required point is $(\frac{8}{3}, -\frac{4}{3})$.

18. Collinear points. Let $P(x, y)$ be a point on the straight line determined by $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, so situated that $P_1P = l(P_1P_2)$.

There are three cases to consider according to the position of the point P. If P is between the points P_1 and P_2 (fig. 13), the

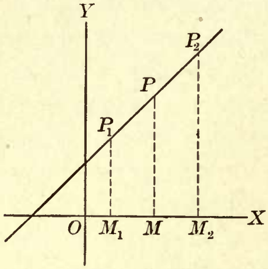

FIG. 13

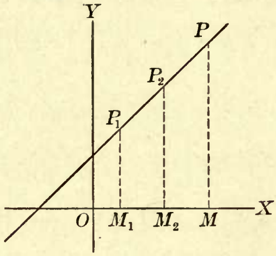

FIG. 14

segments P_1P and P_1P_2 have the same direction, and $P_1P < P_1P_2$; accordingly l is a positive number less than unity. If P is beyond P_2 from P_1 (fig. 14), P_1P and P_1P_2 still have the same direction, but $P_1P > P_1P_2$; therefore l is a positive number greater than unity.

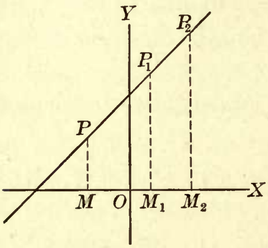

FIG. 15

Finally, if P is beyond P_1 from P_2 (fig. 15), P_1P and P_1P_2 have opposite directions, and l is a negative number, its numerical value ranging all the way from 0 to ∞.

In the first case P is called a point of internal division, and in the last two cases it is called a point of external division.

In all three figures draw P_1M_1, PM, and P_2M_2 perpendicular to OX. In each figure $OM = OM_1 + M_1M$; and since $P_1P = l(P_1P_2)$, $M_1M = l(M_1M_2)$, by geometry.

$$\therefore\ OM = OM_1 + l(M_1M_2),$$

whence, by substitution,

$$x = x_1 + l(x_2 - x_1). \qquad (1)$$

By drawing lines perpendicular to OY we can prove, in the same way,

$$y = y_1 + l(y_2 - y_1). \qquad (2)$$

In particular, if P bisects the line P_1P_2, $l = \frac{1}{2}$, and these formulas become

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}.$$

Ex. 1. Find the coördinates of a point $\frac{3}{5}$ of the distance from $P_1(2, 3)$ to $P_2(3, -3)$.

If the required point is $P(x, y)$,

$$x = 2 + \tfrac{3}{5}(3 - 2) = 2\tfrac{3}{5},$$

$$y = 3 + \tfrac{3}{5}(-3 - 3) = \tfrac{3}{5}.$$

Ex. 2. Prove analytically that the straight line dividing two sides of a triangle in the same ratio is parallel to the third side.

Let one side of the triangle coincide with OX, one vertex being at O. Then the vertices of the triangle are $O(0, 0)$, $A(x_1, 0)$, $B(x_2, y_2)$ (fig. 16). Let CD divide the sides OB and AB so that $OC = l(OB)$ and $AD = l(AB)$.

If the coördinates of C are denoted by (x_3, y_3) and those of D by (x_4, y_4), then, by the above formulas,

$$x_3 = lx_2, \qquad y_3 = ly_2,$$

and $$x_4 = x_1 + l(x_2 - x_1), \; y_4 = ly_2.$$

Since $y_3 = y_4$, CD is parallel to OA.

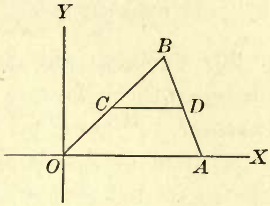

FIG. 16

19. Let us now see what happens as different real values are assigned to l. When $l = 0$, P coincides with P_1 (fig. 17). As l increases in value, the point P moves along the line toward P_2 till, when $l = 1$, it coincides with P_2. As the value of l continues to increase, the point P continues to move along the line away from P_2 and in the same direction as before.

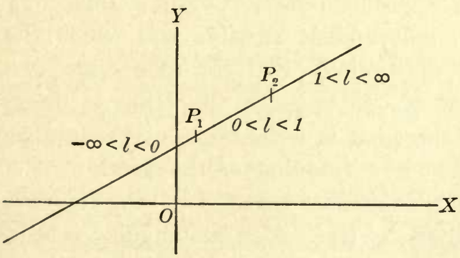

FIG. 17

If negative values are assigned to l, in ascending order of numerical magnitude, the point P moves along the line, away from P_1, in the opposite direction from P_2.

It follows that

$$x_1 + l(x_2 - x_1) \quad \text{and} \quad y_1 + l(y_2 - y_1)$$

may be made to represent the coördinates of any point of the straight line determined by the points P_1 and P_2 by assigning the appropriate value to l, the range of values for each segment of the line being indicated in fig. 17.

Ex. Consider the straight line determined by the two points $P_1(-1, -4)$ and $P_2(5, 6)$. Any other point P on this line has the coördinates

$$x = -1 + 6l, \qquad y = -4 + 10l.$$

When $l < 0$, it is clear that $x < -1$, $y < -4$; hence P lies at the left of P_1. When $0 < l < 1$, it is clear that $-1 < x < 5$, $-4 < y < 6$; hence P lies between P_1 and P_2. When $l > 1$, it is clear that $x > 5$, $y > 6$; hence P lies at the right of P_2.

20. Variable and function. A quantity which remains unchanged throughout a given problem or discussion is called a *constant.* A quantity which changes its value in the course of a problem or discussion is called a *variable.* If two quantities are so related that when the value of one is given the value of the other is determined, the second quantity is called a *function* of the first. When the two quantities are variables the first is called the *independent variable,* and the function is sometimes called the *dependent variable.* As a matter of fact, when two related quantities occur in a problem it is usually a matter of choice which is called the independent variable and which the function. Thus, the area of a circle and its radius are two related quantities such that if one is given the other is determined. We can say that the area is a function of the radius, and likewise that the radius is a function of the area.

The relation between the independent variable and the function can be graphically represented by the use of rectangular coördinates. For, if we represent the independent variable by x and the corresponding value of the function by y, x and y will determine a point in the plane, and a number of such points will outline a curve indicating the correspondence of values of variable and function. This curve is called the *graph* of the function.

Ex. 1. An important use of the graph of a function is in statistical work. The following table shows the price of standard steel rails per ton in the respective years:

1895	\$24.33	1900	\$32.29
1896	28.00	1901	27.33
1897	18.75	1902	28.00
1898	17.62	1903	28.00
1899	28.12	1904	28.00

If we plot the years as abscissas, calling 1895 the first year, 1896 the second year, etc., and plot the price of rails as ordinates, making one unit of ordinates correspond to ten dollars, we shall locate the points $P_1, P_2, \ldots, P_{10}$ in fig. 18. In order to study the variation in price, we join these points in succession by straight

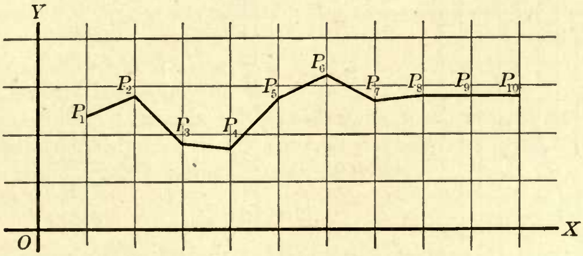

FIG. 18

lines. The resulting broken line serves merely to guide the eye from point to point, and no point of it except the vertices has any other meaning. It is to be noted that there is no law connecting the price of rails with the year. Also the nature of the function is such that it is defined only for isolated values of x.

Ex. 2. As a second example we take the law that the postage on each ounce or fraction of an ounce of first-class mail matter is two cents. The postage is then a known function of the weight. Denoting each ounce of weight by one unit of x, and each two cents of postage by one unit of y, we have the series of straight lines (fig. 19) parallel to the axis of x, representing corresponding values of weight and postage. Here the function is defined by United States law for all positive values of x, but it cannot be expressed in elementary mathematical symbols. A peculiarity of the graph is the series of breaks. The lines are not connected, but all points of each line represent corresponding values of x and y.

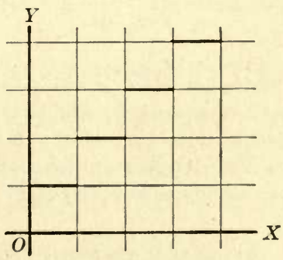

FIG. 19

Ex. 3. As a third example, differing in type from each of the preceding, let us take the following. While it is known that there is some physical law connecting the pressure of saturated steam with its temperature, so that to every temperature there is some corresponding pressure, this law has not yet been formulated mathematically. Nevertheless, knowing some corresponding values of temperature and pressure, we can construct a curve that is of considerable value. In the table* below, the temperatures are in degrees Centigrade and the pressures are in millimeters of mercury.

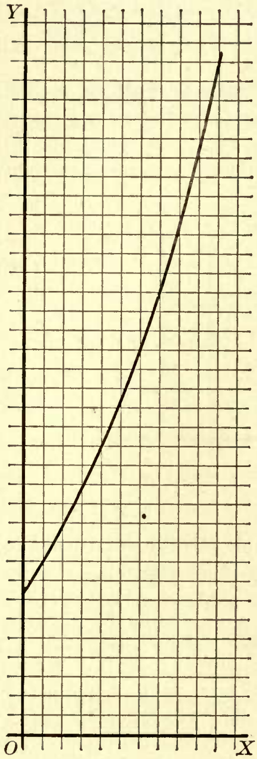

FIG. 20

TEMPERATURE	PRESSURE
100	760
105	906
110	1074.7
115	1268.7
120	1490.5
125	1743.3
130	2029.8
135	2353.7
140	2717.9
145	3126.1
150	3581.9

Let 100 represent the zero point of temperature, and let each unit of x represent 5 degrees of temperature; also let each unit of y represent 100 millimeters of pressure of mercury, and locate the points representing the corresponding values of temperature and pressure given in the above table. Through the points thus located draw a smooth curve (fig. 20) i.e. one which has no sudden changes of direction. While only the eleven points located are exact, all other points are approximately accurate, and the curve may be used for approximate computation as follows: Assume any temperature, and, laying it off as an abscissa, measure the corresponding ordinate of the curve. While not exact, it will, nevertheless, give an approximate value of the corresponding pressure. Similarly, a pressure may be assumed and the corresponding temperature determined. It may be added that the more closely together the tabulated values are taken, the better the approximation from the curve, but the curve can never be exact at all points.

* From C. H. Peabody's "Steam Tables," computed for sea level at a latitude of 45 degrees.

Ex. 4. As a final example, we will take the law of Boyle and Mariotte for perfect gases, namely, at a constant temperature the volume of a definite quantity of gas is inversely proportional to its pressure. It follows that if we represent the pressure by x and the corresponding volume by y, then $y = \frac{k}{x}$, where k is a constant and x and y are positive variables. A curve (fig. 21) in the first quadrant, the coördinates of every point of which satisfy this equation, represents the comparative changes in pressure and volume, showing that as the pressure increases by a certain amount the volume is decreased more or less, according to the amount of pressure previously exerted.

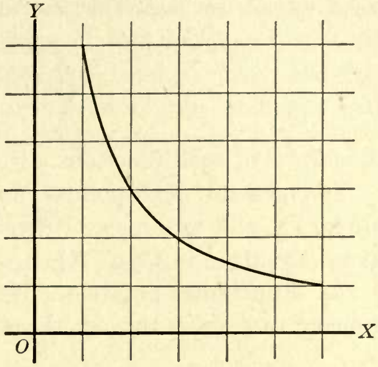

Fig. 21

This example differs from the preceding in that the law of the function is fully known and can be expressed in a mathematical formula. Consequently, we may find as many points on the curve as we please, and may therefore construct the curve to any required degree of accuracy.

21. Classes of functions. We shall consider in this book only those functions of one variable which can be expressed by means of elementary mathematical symbols. The simplest kind of such functions is the *algebraic polynomial*, expressed by

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

where all the exponents are positive integers and the coefficients $a_0, a_1, \cdots, a_{n-1}, a_n$ are real or complex numbers or zero. The number n is the degree of the polynomial. These functions are discussed in Chaps. III and IV.

The quotient of two algebraic polynomials is a *rational algebraic fraction,* expressed by

$$\frac{a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \cdots + b_{m-1}x + b_m}.$$

Examples of functions of this kind are discussed in Chap. VI.

If a function requires for its expression the use of radical signs combined with algebraic polynomials, it is an example of an *irrational algebraic function;* for example,

$$\sqrt[3]{x^3+\sqrt{\frac{x}{1+x^2}}}.$$

Examples of such functions are found in Chap. VI.

The general definition of an *algebraic function* is given in Chap. IX, and examples of non-algebraic, or *transcendental functions,* are given in Chap. XIII.

22. Functional notation. When y is a function of x it is customary to express this by the notation

$$y=f(x).$$

Then the particular value of the function obtained by giving x a definite value a is written $f(a)$. For example, if

$$f(x)=x^3+3x^2+1,$$

then

$$f(2)=2^3+3\cdot 2^2+1=21,$$

$$f(0)=0^3+3\cdot 0^2+1=1,$$

$$f(-3)=(-3)^3+3(-3)^2+1=1,$$

$$f(a)=a^3+3a^2+1.$$

If more than one function occurs in a problem, one may be expressed as $f(x)$, another as $F(x)$, another as $\phi(x)$, and so on. It is also often convenient in practice to represent different functions by the symbols $f_1(x)$, $f_2(x)$, $f_3(x)$, etc.

If $f(x)$ is any function, and we place

$$y=f(x),$$

we may, as already noted, construct a curve which is the graph of the function. *The relation between this curve and the equation* $y=f(x)$ *is such that all points the coördinates of which satisfy the equation lie on the curve; and conversely, if a point lies on the curve, its coördinates satisfy the equation.*

The curve is said to be represented by the equation, and the equation is called the equation of the curve. The curve is also called the *locus* of the equation. Its use is twofold, — on the one hand, we may study a function by means of the appearance and the properties of the curve, and, on the other hand, we may study the geometric properties of a curve by means of its equation. Both methods will be illustrated in the following pages.

PROBLEMS

1. Find the perimeter of the triangle the vertices of which are (2, 3), (– 3, 3), (1, 1).

2. Prove that the triangle the vertices of which are (– 4, – 3), (2, 1), (– 5, 5) is isosceles.

3. Prove that (6, 2), (– 2, – 4), (5, – 5), (–1, 3) are points of a circle the center of which is (2, – 1). What is its radius?

4. Prove that the quadrilateral of which the vertices are (2, 2), (4, 5), (– 1, 4), (– 3, 1) is a parallelogram.

5. Find a point equidistant from the points (– 3, 4), (5, 3), and (2, 0).

6. Find the center of a circle passing through the points (0, 0), (– 3, 3), and (5, 4).

7. Find a point on the axis of x which is equidistant from (0, 4) and (– 3, – 3).

8. A point is equally distant from the points (1, 1) and (– 2, 3), and its distance from OY is twice its distance from OX. Find its coördinates.

9. Find the points which are 4 units distant from (2, 3) and 5 units distant from the axis of y.

10. A point of the straight line joining the points (– 4, – 2) and (4, – 6) divides it into segments which are in the ratio 3 : 5. What are its coördinates?

11. Find the coördinates of a point P on the straight line determined by $P_1(2, -1)$ and $P_2(-4, 5)$, when $\frac{P_1P}{PP_2} = \frac{2}{3}$.

12. On the straight line determined by the points $P_1(2, 4)$ and $P_2(-1, -3)$ find the point three fourths of the distance from P_1 to P_2.

13. If $P(x, y)$ is a point on the straight line determined by $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, such that $\frac{P_1P}{PP_2} = \frac{l_1}{l_2}$, prove

$$x = \frac{l_1x_2 + l_2x_1}{l_1 + l_2}, \qquad y = \frac{l_1y_2 + l_2y_1}{l_1 + l_2}.$$

14. The middle point of a certain line is (1, 2) and one end is the point (– 3, 5). Find the coördinates of the other end.

15. To what point must the line drawn from (1, –1) to (– 4, 5) be extended in the same direction that its length may be trebled?

16. One end of a line is at (2, – 5) and a point one fourth of the distance to the other end is (– 1, 4). Find the coördinates of the other end of the line.

17. Find the points of trisection of the line joining $P_1(0, 3)$ and $P_2(6, -3)$.

18. Find the lengths of the medians of the triangle (2, 1), (0, – 3), (– 4, 0).

19. Given the three points $A(-3, 3)$, $B(3, 1)$, and $C(6, 0)$ upon a straight line. Find a fourth point D such that $\frac{AD}{DC} = -\frac{AB}{BC}$.

20. Given four points P_1, P_2, P_3, P_4. Find the point halfway between P_1 and P_2, then the point one third of the distance from this point to P_3, and finally the point one fourth of the distance from this point to P_4. Show that the order in which the points are taken does not affect the result.

21. Prove analytically that if in any triangle a median is drawn from the vertex to the base, the sum of the squares of the other two sides is equal to twice the square of half the base plus twice the square of the median.

22. Prove analytically that the straight line drawn between two sides of a triangle so as to cut off the same proportional parts measured from their common vertex is the same proportional part of the third side.

23. Prove analytically that if two medians of a triangle are equal the triangle is isosceles.

24. Prove analytically that in any right triangle the straight line drawn from the vertex to the middle point of the hypotenuse is equal to one half the hypotenuse.

25. Prove analytically that the lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

26. Show that the sum of the squares on the four sides of any quadrilateral is equal to the sum of the squares on the diagonals, together with four times the square on the line joining the middle points of the diagonals.

27. Prove analytically that the diagonals of a parallelogram bisect each other.

28. Prove analytically that the line joining the middle points of the non-parallel sides of a trapezoid is one half the sum of the parallel sides.

29. $OABC$ is a trapezoid of which the parallel sides OA and CB are perpendicular to OC. D is the middle point of AB. Prove analytically that $OD = CD$.

30. The following table gives the price of a bushel of wheat in the New York market from 1890 to 1904. Construct the graph.

1890	.983	1895	.669	1900	.804
1891	1.094	1896	.781	1901	.803
1892	.908	1897	.954	1902	.836
1893	.739	1898	.952	1903	.853
1894	.611	1899	.794	1904	1.107

31. The following table shows hourly barometric readings at a United States weather bureau station. Construct the graph.

1 A.M.	28.85	9 A.M.	29.04	5 P.M.	29.13
2	28.87	10	29.05	6	29.18
3	28.90	11	29.05	7	29.21
4	28.92	12 M	29.05	8	29.24
5	28.94	1 P.M.	29.05	9	29.25
6	28.97	2	29.06	10	29.29
7	28.98	3	29.08	11	29.29
8	29.02	4	29.10	12	29.29

32. The following table shows the number of inches of rainfall in Boston during the years 1880–1891. Construct the graph.

1880	38.89	1886	46.47
1881	49.22	1887	41.91
1882	48.42	1888	60.27
1883	35.56	1889	54.79
1884	53.86	1890	50.21
1885	44.07	1891	49.63

33. The following is a portion of a railway time-table. The letters indicate stations, and the adjacent number gives the distance from A to each of the other stations. The second and the third columns give the times at which two trains running in opposite directions leave each of the stations. Make a graph showing the motion of each train and thus determine the time and place of their passing.

A	10.45 A.M.	2.00	F 99	1.06 P.M.	10.48
B 21		1.30	G 126		9.53
C 44	11.50	12.56	H 151	2.59	8.56
D 64		12.11 P.M.	I 177		7.48
E 84		11.30 A.M.	K 200	4.15	7.00 A.M.

34. The following table shows the amount of $1.00 put at interest at 4% compounded annually. Construct the graph.

5 yr.	1.217	30 yr.	3.242
10	1.480	35	3.946
15	1.801	40	4.801
20	2.191	45	5.841
25	2.666	50	7.116

35. Make a graph showing the relation between the side and the area of a square.

36. Make a graph showing the relation between the radius and the area of a circle.

37. Make a graph showing the relation between the radius and the volume of a sphere.

38. The space s through which a body falls from rest in t seconds is given by the formula $s = \frac{1}{2} gt^2$. Assuming $g = 32$, construct the graph.

39. The velocity acquired by a body thrown towards the earth's surface with a velocity v_0 is given at the end of t seconds by the formula $v = v_0 + gt$. Construct the graph.

40. Two particles of mass m_1 and m_2 at a distance d from each other attract each other with a force F, given by the formula

$$F = \frac{m_1 m_2}{d^2}.$$

Assuming $m_1 = 5$ and $m_2 = 20$, construct the graph of F.

41. Ohm's law for an electric current is

$$\text{Current} = \frac{\text{Electromotive force}}{\text{Resistance}}.$$

Assuming the electromotive force to be constant, plot the curve showing the relation between the resistance and the current.

42. If $f(x) = x^4 - 3x^2 + 7x - 1$, find $f(3), f(0), f(a), f(a+h)$.

43. If $f(x) = x^3 + 1$, show that $f(2) - 4f(1) = f(0)$.

44. If $f(x) = x^4 + 2x^2 + 3$, prove that $f(-x) = f(x)$.

45. If $f(x) = x^5 + 3x^3 - 7x$, prove that $f(-x) = -f(x)$.

46. If $f(x) = x^2 - a^2$, prove that $f(a) = f(-a)$.

47. If $f_1(x) = x^2 + a^2$, and $f_2(x) = 2x$, prove that $f_1(a) - af_2(a) = 0$.

48. If $f(x) = \left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)\left(x^3 - \frac{1}{x^3}\right)$, prove that $f(a) = -f\left(\frac{1}{a}\right)$.

49. If $f(x) = \dfrac{1-x}{1+x}$, prove that $f(a) \cdot f(-a) = 1$.

50. If $f(x) = \dfrac{x^4 + 2x^3 + 2x + 1}{x^2}$, prove that $f(x) = f\left(\dfrac{1}{x}\right)$.

51. If $f(x) = \sqrt{\dfrac{x^2+1}{x}}$, find $f(3), f(0), f(-3), f(a), f\left(\dfrac{1}{a}\right)$.

52. If $f(x) = \underline{|x}$, prove that $(x+1)f(x) = f(x+1)$.

53. If $f_1(x) = \sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{a}{x}}$, and $f_2(x) = \sqrt{\dfrac{x}{a}} - \sqrt{\dfrac{a}{x}}$, prove that

$$[f_1(x)]^2 - [f_2(x)]^2 = [f_1(a)]^2.$$

54. If $f(x) = \dfrac{x+1}{x-1}$, prove that $f[f(x)] = x$.

CHAPTER III

THE POLYNOMIAL OF THE FIRST DEGREE

23. Graphical representation. An algebraic polynomial of the first degree is of the form $mx + b$, where m and b are numbers, which may be positive or negative, integral or fractional, rational or irrational. We shall restrict the values of m and b, however, to real numbers. In particular cases b may be zero, when the polynomial becomes the monomial mx.

To obtain the graph of the polynomial, we write

$$y = mx + b, \tag{1}$$

and proceed as in the examples of the previous chapter. We assign to x any number of values assumed at pleasure, say x_1, x_2, x_3, x_4, etc.; compute the corresponding values of y, namely,

$$\left.\begin{aligned} y_1 &= mx_1 + b, \\ y_2 &= mx_2 + b, \\ y_3 &= mx_3 + b, \\ y_4 &= mx_4 + b, \end{aligned}\right\} \tag{2}$$

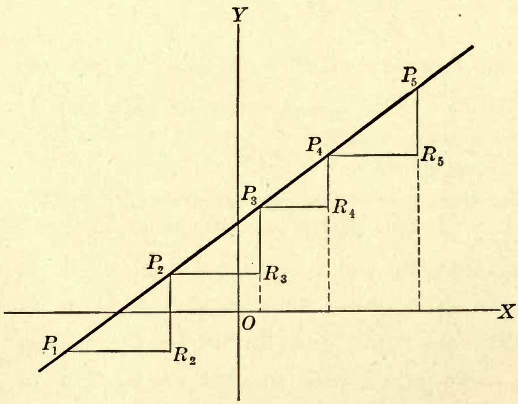

Fig. 22

and plot the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$, $P_4(x_4, y_4)$ (fig. 22). We then draw the straight lines P_1P_2, P_2P_3, P_3P_4, each connecting two successive points, and shall prove that these lines form one and the same straight line. For that purpose draw through each point lines parallel to the coördinate axes, forming the triangles shown in fig. 22. Then, by § 13,

$$\begin{aligned} P_1R_2 &= x_2 - x_1, & P_2R_3 &= x_3 - x_2, & P_3R_4 &= x_4 - x_3, \\ R_2P_2 &= y_2 - y_1, & R_3P_3 &= y_3 - y_2, & R_4P_4 &= y_4 - y_3. \end{aligned} \tag{3}$$

By subtracting each equation in (2) from the one below it, we have

$$y_2 - y_1 = m(x_2 - x_1),$$

$$y_3 - y_2 = m(x_3 - x_2),$$

$$y_4 - y_3 = m(x_4 - x_3),$$

whence
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_4 - y_3}{x_4 - x_3} = m; \tag{4}$$

or, by (3),
$$\frac{R_2P_2}{P_1R_2} = \frac{R_3P_3}{P_2R_3} = \frac{R_4P_4}{P_3R_4}.$$

Hence the triangles of the figure are similar, and the angles $R_2P_1P_2$, $R_3P_2P_3$, $R_4P_3P_4$ are equal. Therefore the line $P_1P_2P_3P_4$ is a straight line.

Again, let us take on this line any other point, such as P_5, which has not been used in constructing the graph, and draw P_4R_5 and R_5P_5 parallel to OX and OY respectively. Then, since the triangles $P_1R_2P_2$ and $P_4R_5P_5$ are similar,

$$\frac{R_5P_5}{P_4R_5} = \frac{R_2P_2}{P_1R_2};$$

that is,
$$\frac{y_5 - y_4}{x_5 - x_4} = \frac{y_2 - y_1}{x_2 - x_1} = m. \qquad \text{(by (4))}$$

Therefore
$$y_5 = mx_5 - mx_4 + y_4,$$

whence, by substituting the value of y_4 given in (2),

$$y_5 = mx_5 + b.$$

Hence the coördinates of P_5 satisfy the equation (1).

We have now shown that all points the coördinates of which satisfy equation (1) lie on a straight line, and that any point on the line has coördinates which satisfy (1). We have accordingly proved the following proposition: *The equation $y = mx + b$ always represents a straight line.*

24. The general equation of the first degree. The equation

$$Ax + By + C = 0,$$

where A, B, and C may be any numbers or zero, except that A and B cannot be zero at the same time, is called the general equation of the first degree. We shall prove: *The general equation of the first degree with real coefficients always represents a straight line.*

1. Suppose $A \neq 0$ and $B \neq 0$. If any value of x is assumed, the value of y is determined. Therefore y is a function of x, which may be expressed by solving the equation for y; thus,

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

This equation is of the form $y = mx + b$, and therefore represents a straight line by § 23.

2. Suppose $A = 0$, $B \neq 0$. The equation is then

$$By + C = 0, \quad \text{or} \quad y = -\frac{C}{B}.$$

All points the coördinates of which satisfy this equation lie on a straight line parallel to OX at a distance $-\frac{C}{B}$ units from it; and, conversely, any point on this line has coördinates which satisfy the equation. Hence the equation represents this line.

3. Suppose $A \neq 0$, $B = 0$. The equation is then

$$Ax + C = 0, \quad \text{or} \quad x = -\frac{C}{A},$$

and represents a straight line parallel to OY at a distance $-\frac{C}{A}$ units from it.

Therefore the equation $Ax + By + C = 0$ always represents a straight line.

25. In order to plot a straight line it is, in general, convenient to find the points L and K (fig. 23), in which it cuts OX and OY respectively. If the coördinates of L are $(a, 0)$ and those of K are $(0, b)$, these coördinates will satisfy the equation $Ax + By + C = 0$. By substitution we find

$$a = -\frac{C}{A}, \qquad b = -\frac{C}{B}.$$

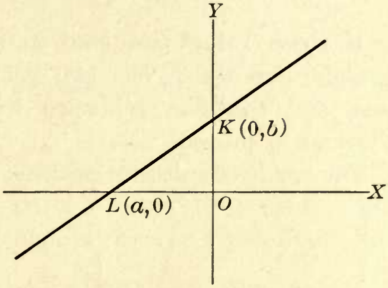

FIG. 23

The quantities a and b, which are equal in magnitude and sign to OL and OK respectively, are called the *intercepts* of the straight line. It is evident that the b found here is the same as in $y = mx + b$.

If $C = 0$, i.e. if the equation is $Ax + By = 0$, then $a = 0$ and $b = 0$, and the straight line passes through the origin. To plot the line, we must find by trial the coördinates of another point which satisfy the equation, plot this point, and draw a straight line through it and the origin.

Ex. 1. Plot the line $3x - 5y + 12 = 0$. Placing $y = 0$, we find $a = -4$. Placing $x = 0$, we find $b = 2\frac{2}{5}$. We lay off $OL = -4$, $OK = 2\frac{2}{5}$, and draw a straight line through L and K.

Ex. 2. Plot the line $3x - 5y = 0$. Here $a = 0$ and $b = 0$. If we place $x = 1$, we find $y = \frac{3}{5}$. The line is drawn through $(0, 0)$ and $(1, \frac{3}{5})$.

26. *Any straight line may be represented by an equation of the first degree.*

The proof consists in showing that the coefficients A, B, C, in the general equation of the first degree, may be so chosen that the equation may represent any straight line given in advance. Let (x_1, y_1) and (x_2, y_2) be any two points on a given straight line. The coördinates of these points will satisfy

$$Ax + By + C = 0, \tag{1}$$

provided A, B, C have such values that

$$Ax_1 + By_1 + C = 0,$$
$$Ax_2 + By_2 + C = 0.$$

Solving these equations for the ratios of A, B, C, we have (by § 8)

$$A : B : C = \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} : - \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} : \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \qquad (2)$$

If these values are used in (1), that equation represents a straight line which has two points in common with the given line, and therefore coincides with it throughout. Hence the theorem is proved.

The result of substituting from (2) in (1) is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

which is the equation of a line through two given points.

27. Slope. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ (figs. 24, 25) be two points upon a straight line. If we imagine that a point moves along the line from P_1 to P_2, the change in x caused by this motion is measured in magnitude and sign by $x_2 - x_1$, and the

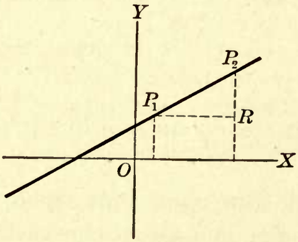

Fig. 24

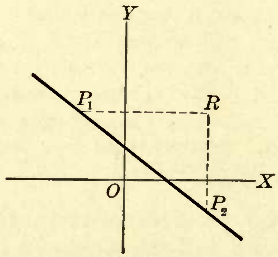

Fig. 25

change in y is measured by $y_2 - y_1$. We define the *slope* of the straight line as the ratio of the change in y to the change in x as a point moves along the line, and shall denote it by the letter m. We have then, by definition,

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

It appears from equations (4) (§ 23) that the letter m in the equation $y = mx + b$ has the meaning just defined. It follows that *if the equation of a straight line is in the form* $Ax + By + C = 0$, *its slope may be found by solving the equation for* y *and taking the coefficient of* x, thus,

$$y = -\frac{A}{B}x - \frac{C}{B}, \quad \text{whence} \quad m = -\frac{A}{B}.$$

A geometric interpretation of the slope is readily given. For if we draw through P_1 a line parallel to OX, and through P_2 a line parallel to OY, and call R the point in which these two lines intersect, then $x_2 - x_1 = P_1R$, and $y_2 - y_1 = RP_2$; and hence $m = \frac{RP_2}{P_1R}$.

It is clear from the figures, as well as from equations (4) (§ 23), that the value of m is independent of the two points P_1 and P_2 and depends only on the given line. We may therefore choose P_1 and P_2 (as in figs. 24 and 25) so that P_1R is positive. There are then two essentially different cases, according as the line runs up or down toward the right hand. In the former case RP_2 and m are positive (fig. 24); in the latter case RP_2 and m are negative (fig. 25). We may state this as follows:

The slope of a straight line is positive when an increase in x *causes an increase in* y, *and is negative when an increase in* x *causes a decrease in* y.

When the line is parallel to OX, $y_2 = y_1$, and consequently $m = 0$, as explained in § 11. If the line is parallel to OY, $x_2 = x_1$, and therefore $m = \infty$ in the sense of § 11.

28. Angles. The slope of a straight line enables us to solve many problems relating to angles, some of which we take up in this article.

1. *The angle between the axis of* x *and a known line.* Let a known line cut the axis of x at the point L. Then there are four angles formed. To avoid ambiguity, we shall agree to select that one of the four which is above the axis of x and to the right of

the line, and to consider LX as the initial line of this angle. We shall denote this angle by ϕ. Then if we take P any point on

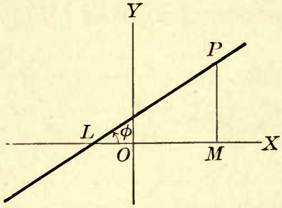

FIG. 26

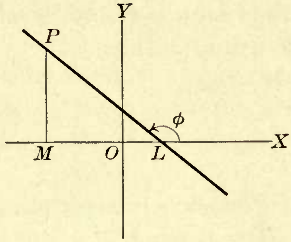

FIG. 27

the terminal line of ϕ and drop the perpendicular MP, we have, in the two cases represented by figs. 26 and 27,

$$\tan\phi = \frac{MP}{LM}\cdot$$

But $\dfrac{MP}{LM}$ is equal to the slope of the line. Therefore

$$\tan\phi = m.$$

If the straight line is parallel to OY, $\phi = 90°$ and $\tan\phi = \infty$. If the line is parallel to OX, no angle ϕ is formed; but since $m = 0$, we may say $\tan\phi = 0$, whence $\phi = 0°$ or $180°$.

2. *Parallel lines.* If two lines are parallel, they make equal angles with OX, and hence their slopes are equal. It follows that two equations which differ only in the absolute term, such as

$$Ax + By + C_1 = 0$$

and

$$Ax + By + C_2 = 0,$$

represent parallel lines.

More generally, two straight lines,

$$A_1x + B_1y + C_1 = 0$$

and

$$A_2x + B_2y + C_2 = 0,$$

are parallel if

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0.$$

3. *Perpendicular lines.* Let AB and CD (fig. 28) be two lines intersecting at right angles. Through P draw PR parallel to OX and let $RPD = \phi_1$ and $RPB = \phi_2$. Then $\tan\phi_1 = m_1$ and $\tan\phi_2 = m_2$, where m_1 and m_2 are the slopes of the lines. But by hypothesis,

$$\phi_2 = \phi_1 + 90°,$$

whence

$$\tan\phi_2 = -\cot\phi_1 = -\frac{1}{\tan\phi_1},$$

which is the same as

$$m_2 = -\frac{1}{m_1}.$$

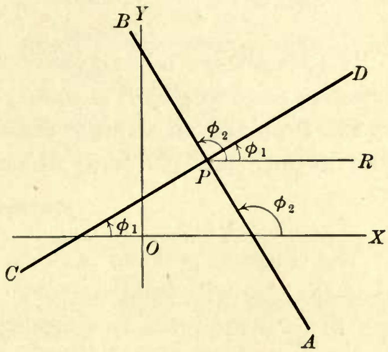

FIG. 28

That is: *Two straight lines are perpendicular when the slope of one is minus the reciprocal of the slope of the other.* This theorem may be otherwise expressed by saying that two lines are perpendicular when the product of their slopes is minus unity.

It follows that two straight lines whose equations are of the type

$$Ax + By + C_1 = 0$$

and

$$Bx - Ay + C_2 = 0$$

are perpendicular.

4. *Angle between two lines.* Let AB and CD (fig. 29) intersect at the point P, making the angle BPD, which we shall call β. Draw the line PR parallel to OX and place $RPB = \phi_1$ and $RPD = \phi_2$. Then

$$\beta = \phi_2 - \phi_1,$$

and hence

$$\tan\beta = \tan(\phi_2 - \phi_1)$$
$$= \frac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2\tan\phi_1}.$$

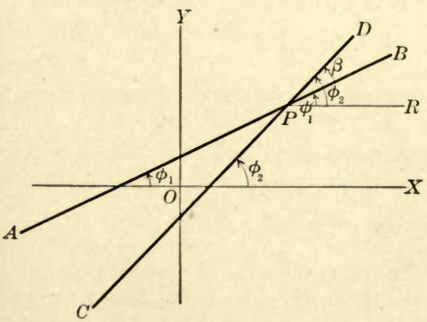

FIG. 29

But $\tan\phi_1 = m_1$ and $\tan\phi_2 = m_2$, where m_2 is the slope of CD and m_1 is the slope of AB. Therefore

$$\tan\beta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

If ϕ_2 is always taken greater than ϕ_1, $\tan\beta$ will be positive or negative according as β is acute or obtuse.

29. Problems on straight lines. We shall solve in this article certain important problems which depend on the equation

$$y = mx + b.$$

The essential problem is, in every case, to determine m and b so that the line will fulfill certain conditions. Since two quantities are to be determined, two conditions are necessary and sufficient; hence, in general, one and only one straight line can be found to satisfy two given conditions.

1. *To find the equation of a straight line which has a known slope and passes through a known point.* Let m_1 be the known slope and $P_1(x_1, y_1)$ be the known point. The equation of the line will be of the form $y = m_1 x + b$, where b, however, is unknown. But the line contains the point P_1. Therefore

$$y_1 = m_1 x_1 + b,$$

whence $$b = y_1 - m_1 x_1.$$

The required equation is, therefore,

$$y = m_1 x + y_1 - m_1 x_1;$$

or, more symmetrically,

$$y - y_1 = m_1 (x - x_1).$$

Ex. Find the equation of a straight line with the slope $-\frac{2}{3}$ passing through the point (5, 7).

First method. We have $$y = -\tfrac{2}{3}x + b;$$

then $$7 = -\tfrac{2}{3}(5) + b,$$

whence $$b = \tfrac{31}{3}.$$

Therefore the required equation is

$$y = -\tfrac{2}{3}x + \tfrac{31}{3},$$

or, finally, $$2x + 3y - 31 = 0.$$

Second method. By substituting in the formula we have

$$y - 7 = -\tfrac{2}{3}(x - 5),$$

whence $$2x + 3y - 31 = 0, \text{ as before.}$$

2. *To find the equation of a straight line passing through a known point and parallel to a known line.*

The slope of the required line is the same as that of the given line, which can be found by § 27. Hence the problem is the same as the preceding.

Ex. Find the equation of a straight line passing through $(-2, 3)$ and parallel to $3x - 5y + 6 = 0$.

First method. The slope of the given line is $\frac{3}{5}$. Therefore the required line is

$$y - 3 = \tfrac{3}{5}(x + 2), \quad \text{or} \quad 3x - 5y + 21 = 0.$$

Second method. As explained in § 28, 2, we know that the required equation is of the form

$$3x - 5y + C = 0,$$

where C is unknown. Since the line passes through $(-2, 3)$,

$$3(-2) - 5(3) + C = 0,$$

whence $C = 21$. Therefore the required equation is

$$3x - 5y + 21 = 0.$$

3. *To find the equation of a straight line passing through a known point and perpendicular to a known line.*

The slope of the required line may be found from the slope of the given line, as in § 28, 3. The problem is then the same as problem 1.

Ex. 1. Find a straight line through $(5, 3)$ perpendicular to $7x + 9y + 1 = 0$.

First method. The slope of the given line is $-\frac{7}{9}$. Therefore the slope of the required line is $\frac{9}{7}$. By problem 1, the required line is

$$y - 3 = \tfrac{9}{7}(x - 5), \quad \text{or} \quad 9x - 7y - 24 = 0.$$

Second method. As shown in § 28, 3, we know that the equation of the required line is of the form $9x - 7y + C = 0$. Substituting $(5, 3)$, we find $C = -24$. Hence the required line is $9x - 7y - 24 = 0$.

Ex. 2. Find the equation of the perpendicular bisector of the line joining (0, 5) and (5, – 11). The point midway between the given points is $(\frac{5}{2}, -3)$, by § 18. The slope of the line joining the given points is $-\frac{16}{5}$, by § 27. Hence the required line passes through $(\frac{5}{2}, -3)$, with the slope $\frac{5}{16}$. Its equation is

$$y + 3 = \tfrac{5}{16}(x - \tfrac{5}{2}), \quad \text{or} \quad 10x - 32y - 121 = 0.$$

4. *To find the equation of a straight line through two known points.*

This problem has already been solved in § 26, and the result given in the form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

which is the same as $\begin{vmatrix} x - x_1 & y - y_1 \\ x_1 - x_2 & y_1 - y_2 \end{vmatrix} = 0.$ (Ex. 2, § 3)

Or, by § 27, the slope of the required line is

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Hence, by problem 1, the equation of the required line is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Ex. Find a straight line through (1, 2) and (– 3, 5).

By the formula,

$$y - 2 = \frac{5 - 2}{-3 - 1}(x - 1), \quad \text{or} \quad 3x + 4y - 11 = 0.$$

5. *To find the condition that three known points should lie on the same straight line.* If the three points are (x_1, y_1), (x_2, y_2), and (x_3, y_3), the condition that they should lie on the same straight line is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

as is evident from 4.

30. Intersection of straight lines.

Let $$A_1x + B_1y + C_1 = 0$$
and $$A_2x + B_2y + C_2 = 0 \tag{1}$$

be the equations of two straight lines. It is required to find their point of intersection. Since the coördinates of any point on one of the lines satisfy the equation of that line, the coördinates of a point on both lines must satisfy both equations simultaneously. Hence the coördinates of the point of intersection of the lines is found by solving the two equations.

There are three cases.

1. $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \neq 0.$

The solutions are then

$$x = -\frac{\begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}, \qquad y = -\frac{\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}.$$

The two straight lines intersect in the corresponding point.

2. $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0$, but at least one of the determinants,

$$\begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix} \text{ and } \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix},$$

not equal to zero.

The equations are then contradictory and the straight lines do not intersect. In fact, § 28 shows that the straight lines are parallel.

This case may be brought into connection with case 1 as follows: In case 1 suppose that $A_1B_2 - A_2B_1$ is very small, but not zero. The values of x and y are then very large, assuming that the numerators are not small, and the point of intersection is then very remote.

Let now the lines be changed in such a manner that $A_1B_2-A_2B_1$ approaches zero. The values of x and y increase indefinitely, the point of intersection recedes indefinitely, and the lines approach parallelism.

3. $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0, \quad \begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix} = 0, \quad \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = 0.$

The equations are then not independent but represent the same straight line.

In this case the attempt to use the solutions as given in 1 leads to the indeterminate form $\frac{0}{0}$ (§ 11).

31. If the three straight lines

$$A_1x + B_1y + C_1 = 0, \tag{1}$$

$$A_2x + B_2y + C_2 = 0, \tag{2}$$

$$A_3x + B_3y + C_3 = 0, \tag{3}$$

pass through the same point, the three equations have a common solution, and therefore

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0. \tag{4}$$

Also, if the three straight lines are parallel, the determinant (4) is zero. For if (1), (2), and (3) are parallel, $A_1B_2-A_2B_1=0$, $A_2B_3-A_3B_2=0$, $A_3B_1-A_1B_3=0$, and therefore

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0.$$

Conversely, if $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0,$

the lines (1), (2), and (3) either pass through the same point or are parallel. For, by § 7, if two of the lines intersect, the coördinates of the point of intersection satisfy the other.

32. Distance of a point from a straight line.

Take the equation of any straight line, written in the form

$$y - mx - b = 0, \tag{1}$$

and consider the polynomial

$$y - mx - b, \tag{2}$$

which stands upon the left-hand side. We may substitute in (2) the coördinates (x_1, y_1) of any point P_1, and thus obtain a value of (2) which is zero when P_1 lies on the line (1), but not otherwise. We wish now to obtain the meaning of

$$y_1 - mx_1 - b$$

when P_1 is not on (1). For that purpose, let LK (fig. 30) be the line (1), and let MP_1, the ordinate of P_1, cut LK in Q. Then the abscissa of Q is x_1 and its ordinate is MQ. From (1)

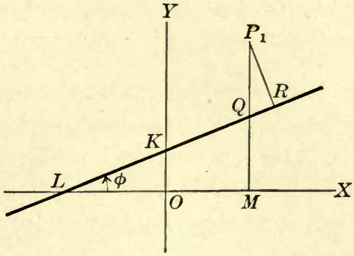

FIG. 30

$$MQ = mx_1 + b.$$

Hence
$$y_1 - mx_1 - b = y_1 - (mx_1 + b)$$
$$= MP_1 - MQ = QP_1.$$

It is clear that $y_1 - mx_1 - b$ is a positive quantity when (x_1, y_1) lies above the line LK, and is a negative quantity when (x_1, y_1) lies below LK. It is also evident from the triangle P_1QR, and from a like triangle in other cases, that the length of P_1R is numerically equal to $P_1Q \cos \phi$. But $\tan \phi = m$, and hence

$$\cos \phi = \frac{1}{\pm\sqrt{1 + m^2}}.$$

We have, then,
$$P_1R = \frac{y_1 - mx_1 - b}{\pm\sqrt{1 + m^2}}.$$

We may, if we wish, always choose the + sign in the denominator. Then P_1R is positive when P_1 is above $y = mx + b$, and negative when P_1 is below.

If the equation of the straight line is in the form

$$Ax + By + C = 0,$$

$m = -\frac{A}{B}$ and $b = -\frac{C}{B}$. Therefore

$$Ax_1 + By_1 + C = B(y_1 - mx_1 - b),$$

and

$$P_1R = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}.$$

It appears, then, that the polynomial $Ax_1 + By_1 + C$ and the perpendicular P_1R are positive for all points on one side of the line $Ax + By + C = 0$, and negative for all points on the other side. To determine which side of the line corresponds to the positive sign, it is most convenient to test some one point, preferably the origin.

33. Normal equation of a straight line. Let LK (fig. 31) be any straight line and let OD be the normal (or perpendicular) drawn from the origin. Let the length of OD be p and let the angle XOD be α. Take P any point on LK. The projection of OP on OD is equal to the sum of the projections of OM and MP (§ 15). But the projection of OP on OD is p, since ODP is a right angle. The projection of OM on OD is $x \cos\alpha$ (§ 14), and that of MP is $y \cos(\alpha - 90°) = y \sin\alpha$. Hence

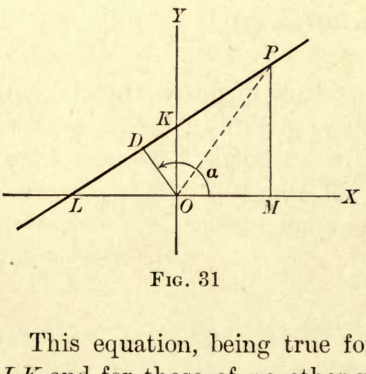

FIG. 31

$$p = x \cos\alpha + y \sin\alpha,$$

or $$x \cos\alpha + y \sin\alpha - p = 0.$$

This equation, being true for the coördinates of any point on LK and for those of no other point, is the equation of LK. It is called the *normal equation of a straight line.*

Since $\sin^2\alpha + \cos^2\alpha = 1$, it follows from § 32 that

$$x_1 \cos\alpha + y_1 \sin\alpha - p$$

is numerically equal to the distance of (x_1, y_1) from

$$x \cos\alpha + y \sin\alpha - p = 0.$$

It is sometimes desirable to change an equation

$$Ax + By + C = 0$$

into

$$x \cos\alpha + y \sin\alpha - p = 0.$$

For that purpose it is enough to notice that since any value of (x, y) which satisfies one equation must satisfy the other, the one is a multiple of the other. Hence

$$A = k \cos\alpha, \quad B = k \sin\alpha, \quad C = -kp,$$

where k is an unknown factor. But from these last equations we have

$$A^2 + B^2 = k^2.$$

Therefore

$$\cos\alpha = \frac{A}{\pm\sqrt{A^2 + B^2}},$$

$$\sin\alpha = \frac{B}{\pm\sqrt{A^2 + B^2}},$$

$$p = \frac{-C}{\pm\sqrt{A^2 + B^2}}.$$

Since p is to be positive, the sign of the radical must be opposite to that of C.

PROBLEMS

Plot the graphs of the following equations:

1. $5x - 3y + 10 = 0$.
2. $4x + 6y + 12 = 0$.
3. $x + 3y - 7 = 0$.
4. $2x - 9y = 0$.
5. $3x + 5y = 0$.
6. $4x + 7 = 0$.
7. $5y - 8 = 0$.

8. Two numbers are to be found such that one half of one plus one third of the other is equal to unity. Show how one number may be graphically found when the other is known.

9. A plane figure is in the form of a square, 3 ft. on one side, surmounted by a triangle constructed on one of its sides as a base. Express the area of the above figure in terms of the altitude of the triangle, and plot the graph of the function.

10. Express the number of inches in any length as a function of the number of centimeters, and express the same as a graph.

11. A uniform elastic string of length l is subjected to a stretching force f. If l' is the new length, $l' = l(1 + mf)$, where m is a constant. Plot the graph, showing the relation between l' and f.

12. If t represents the boiling point in degrees Centigrade at a height h in meters above sea level, then approximately $h = 295(100 - t)$. Plot the graph.

13. The pressure on a square unit of horizontal surface immersed in a liquid is equal to the weight of the column of liquid above it. Express the pressure at a depth x below the surface of a body of water, the density of the water being taken as unity. Express also the pressure x units below the surface of a body of water over which is a body of oil of density .9 and of depth 8 units. Plot the graphs.

14. A road starts at an elevation of 100 ft. above sea level and has a uniform up grade of 15 per cent; i.e. it rises 15 ft. in every 100 ft. of horizontal length. Express the distance above sea level on the road as a function of the horizontal distance from the point of departure, and construct the graph.

15. A tank of water contains 100 gal. A tap is opened, causing the water to flow out at a uniform rate of $\frac{2}{3}$ gal. per minute. Express the amount of water in the tank as a function of the time, and construct the graph.

16. Find the equation of the straight line of which the slope is 7 and the intercept on OY is -3.

17. Find the equation of the straight line passing through the point $(0, -3)$ and making an angle of 135° with OX.

18. Find the equation of a straight line making an angle of 60° with OX and cutting off an intercept -5 on OY.

19. A straight line making a zero intercept on OY makes an angle of 120° with OX. Find its equation.

20. A straight line making a zero angle with OX cuts OY at a point 5 units from the origin. Find its equation.

21. Find the acute angle between the lines $2x - 3y + 5 = 0$ and $x + 2y + 2 = 0$.

22. Find the acute angle between the lines $2x + 3y - 6 = 0$ and $2x + y + 1 = 0$.

23. Find the acute angle between the lines $4x + y - 2 = 0$ and $3x + 5y + 8 = 0$.

24. Show that $2x + 14y - 17 = 0$ bisects one of the angles between the lines $8x + 6y - 11 = 0$ and $3x - 4y + 3 = 0$.

25. Find the equation of the straight line through the point $(-4, 5)$ parallel to the line $5x - 4y + 1 = 0$.

26. Find the equation of the straight line through $(3, -1)$ parallel to the line $x - y = 8$.

27. Find the equation of the straight line through the point $(2, -11)$ perpendicular to the line $9x - 8y + 6 = 0$.

28. Find the equation of the straight line through the origin perpendicular to the line $6x + 5y - 3 = 0$.

29. Find the equation of the straight line through the points $(-2, -3)$ and $(0, 4)$.

30. Find the equation of the straight line through the points $(2, -1)$ and $(3, 2)$.

31. Find the equation of the straight line through the points $(-1, 3)$ and $(-1, 5)$.

32. Find the angle between the straight lines drawn from the origin to the points of trisection of that part of the line $6x + 4y = 24$ which is included between the coördinate axes.

33. Find the equation of the perpendicular bisector of the line joining $(-3, 5)$ and $(-4, 1)$.

34. A straight line is perpendicular to the line joining the points $(-4, -2)$ and $(2, -6)$ at a point one third of the distance from the first to the second point. What is its equation?

35. Find the equation of the straight line through $(3, 5)$ parallel to the straight line joining $(2, 5)$ and $(-5, -2)$.

36. Find the equation of the straight line parallel to the line $2x - 3y + 5 = 0$ and bisecting the straight line joining $(-1, 2)$ and $(4, 5)$.

37. Find the equation of the straight line perpendicular to $3x - 5y = 9$ and bisecting that portion of it which is included between the coördinate axes.

38. What is the equation of a straight line the intercepts of which on the axes of x and y are 2 and -5 respectively?

39. What is the equation of the straight line the intercepts of which on the axes of x and y are -4 and -7 respectively?

40. In the triangle $A\ (-2, -2)$, $B\ (1, -3)$, $C\ (0, -7)$, a straight line is drawn bisecting the adjacent sides AB and BC. Prove that it is parallel to AC and half as long.

41. Find the equation of a straight line through $(4, \frac{8}{3})$ and the point of intersection of the lines $3x - 4y - 2 = 0$ and $12x - 15y - 8 = 0$.

42. Find the equation of the straight line passing through the point of intersection of $x - 2y - 5 = 0$ and $2x - 3y - 8 = 0$ and parallel to $3x - 2y + 2 = 0$.

43. Find the equation of the straight line through the point of intersection of $6x - 2y - 11 = 0$ and $4x - 6y - 5 = 0$ and perpendicular to $4x - y + 1 = 0$.

44. Find the equation of the straight line joining the point of intersection of the lines $2x - y + 5 = 0$ and $x + y + 1 = 0$ and the point of intersection of the lines $x - y - 7 = 0$ and $2x + y - 5 = 0$.

45. Determine the value of m so that the line $y = mx + 3$ shall pass through the point of intersection of the lines $y = 2x + 1$ and $y = x + 5$.

46. Find the vertices and the angles of the triangle formed by the lines $x = 0$, $x - y + 2 = 0$, and $2x + 3y - 21 = 0$.

47. Find the distance of (3, 5) from the line $y = 4x - 8$. On which side of the line is the point?

48. How far distant from the line $2x + 3y + 8 = 0$ is the point $(7, -4)$, and on which side of the line is it?

49. Find the distance from the point $(b, -a)$ to the line $\frac{x}{a} + \frac{y}{b} = 1$.

50. The base of a triangle is the straight line joining the points $(-1, 3)$ and $(5, -1)$. How far is the third vertex $(6, -2)$ from the base?

51. The vertex of a triangle is the point $(6, -2)$ and the base is the straight line joining $(-3, 2)$ and $(4, 3)$. Find the lengths of the base and the altitude.

52. Find the distance between the two parallel lines $4x + 3y - 10 = 0$ and $4x + 3y - 8 = 0$.

53. A straight line is 7 units distant from the origin and its normal makes an angle of 30° with OX. What is its equation?

54. The normal to a straight line which is 5 units distant from the origin makes the acute angle $\tan^{-1}\frac{1}{5}$ with OX. What is the equation of the line?

55. A straight line 4 units distant from the origin makes an angle of 45° with OX. What is its equation?

56. The normal to a straight line makes an angle $\tan^{-1}\frac{3}{2}$ with OX. The line passes through the origin. What is its equation?

57. The normal to a straight line makes an angle of 90° with OX. The line is 7 units distant from the origin. What is its equation?

58. Find a point on the line $4x + 3y = 12$ equidistant from the points $(-1, -2)$ and $(1, 4)$.

59. Find the equation of the perpendicular bisector of the base of an isosceles triangle having its vertices at the points $(3, 2)$, $(-2, -3)$, and $(2, -5)$.

60. A point is equally distant from $(2, 1)$ and $(-4, 3)$, and the slope of the straight line joining it to the origin is $\frac{2}{3}$. Where is the point?

61. A point is 7 units distant from the origin, and the slope of the straight line joining it to the origin is $\frac{2}{3}$. What are its coördinates?

62. Perpendiculars are let fall from the point $(5, 0)$ upon the sides of the triangle the vertices of which are at the points $(4, 3)$, $(-4, 3)$, and $(0, -5)$. Show that the feet of the three perpendiculars lie on a straight line.

63. Find a point on the line $x + 2y - 3 = 0$, the distance of which from the axis of x equals its distance from the axis of y.

64. One diagonal of a parallelogram joins the points $(4, -2)$ and $(-4, -4)$. One end of the other diagonal is $(1, 2)$. Find its equation and length.

65. Find the equations of the straight lines through the point $(-2, 0)$ making an angle $\tan^{-1}\frac{2}{3}$ with the line $3x + 4y + 6 = 0$.

66. Find the equations of the straight lines through $(2, 2)$ making an angle of $45°$ with the line $3x - 2y = 0$.

67. Find the equations of the straight lines through the point $(2, 1)$ making an angle $\tan^{-1}\frac{1}{2}$ with the line $2x - y - 3 = 0$.

68. Derive the equation of the straight line making the intercepts a and b on the axes of x and y respectively.

69. Prove analytically that the locus of points equally distant from two points is the perpendicular bisector of the straight line joining them.

70. Prove analytically that the medians of a triangle meet in a point.

71. Prove analytically that the perpendicular bisectors of the sides of a triangle meet in a point.

72. Prove analytically that the perpendiculars from the vertices of a triangle to the opposite sides meet in a point.

73. Prove analytically that the perpendiculars from any two vertices of a triangle to the median from the third vertex are equal.

74. Prove analytically that the straight lines joining the middle points of the adjacent sides of any quadrilateral form a parallelogram.

75. Prove analytically that the straight lines drawn from a vertex of a parallelogram to the middle points of the opposite sides trisect a diagonal.

CHAPTER IV

THE POLYNOMIAL OF THE *N*TH DEGREE

34. Graph of the polynomial of the second degree.

The polynomial of the second degree is $ax^2 + bx + c$. Its graph may be plotted by equating it to y and proceeding as in §§ 20 and 23.

Ex. 1. $x^2 + 2x + 2$.

Place $y = x^2 + 2x + 2$ and assume integral values of x. The corresponding values of x and y are given in the following table:

x	y	x	y
0	2	− 1	1
1	5	− 2	2
2	10	− 3	5
3	17	− 4	10
4	26	− 5	17

As in § 20, we plot these points (fig. 32), and are then to draw a smooth curve through them.

But we notice that these points are nearer together in some places than in others. It follows that in some parts the curve would be more accurate than in others. To obviate this difficulty we assume such fractional values of x as will locate points between the more widely separated points already plotted.

We thus form the table:

x	y	x	y
1.5	7.3	− 3.5	7.3
2.5	13.3	− 4.5	13.3
3.5	21.3	− 5.5	21.3
− 2.5	3.3		

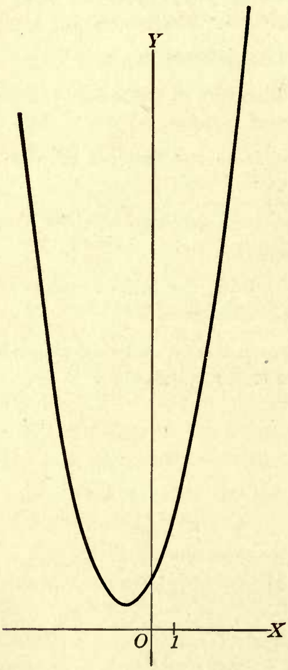

FIG. 32

Plotting these points also, and drawing the curve, we have (fig. 32) the graph of the given polynomial, $x^2 + 2x + 2$. The graph lies entirely above the axis of x, and recedes constantly from it as x increases numerically, since the polynomial is positive for all values of x, and increases in value as x increases.

Ex. 2. $2x^2 + x - 6$.

Place $y = 2x^2 + x - 6$ and assume integral values of x. Hence the table:

x	y
0	-6
1	-3
2	4
3	15

x	y
-1	-5
-2	0
-3	9

On plotting these points (fig. 33) we see that it is desirable to assume fractional values of x. Hence the table:

x	y
1.5	0
2.3	6.9
2.6	10.1
-1.5	-3
-2.5	4

x	y
-3.3	12.5
-3.7	17.7.
$-.5$	-6
$-.3$	-6.1
$-.7$	-5.7

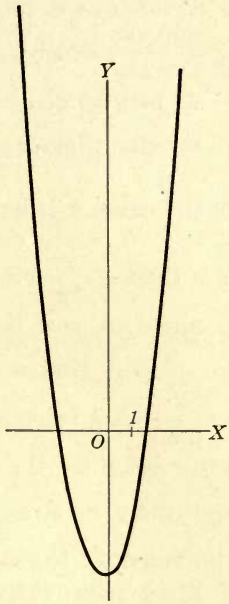

FIG. 33

In obtaining this new set of points we have assumed $-.5$, $-.3$, $-.7$ as values for x, with the aim of locating as closely as possible the turning point, or *vertex*, as it will be called, of the curve. Plotting these points also, we draw the curve (fig. 33).

It is especially to be noted that the curve cuts the axis of x when $x = -2$ and when $x = 1.5$. But these two values of x, since they make $2x^2 + x - 6$ equal to zero, are the roots of the equation $2x^2 + x - 6 = 0$.

As the graph of the polynomial in Ex. 1 did not intersect the axis of x, we conclude that the equation formed by placing it equal to zero has no real roots. Solving that equation we find that, in fact, the roots are $-1 \pm \sqrt{-1}$.

35. Let us now consider the general polynomial of the second degree, $ax^2 + bx + c$, of which the two polynomials just plotted are special cases.

If we place $y = ax^2 + bx + c$, we can write

$$y = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[-\frac{b^2 - 4ac}{4a^2} + \left(x + \frac{b}{2a}\right)^2\right].$$

The expression in brackets is the constant, $-\dfrac{b^2-4ac}{4a^2}$, plus a function of x, $\left(x+\dfrac{b}{2a}\right)^2$, which is always positive except for $x=-\dfrac{b}{2a}$, when it is zero.

At first we shall regard a as positive. It follows that y has its least value when $x=-\dfrac{b}{2a}\cdot$ Therefore the lowest point, the *vertex*, of the curve will be $\left(-\dfrac{b}{2a}, -\dfrac{b^2-4ac}{4a}\right)\cdot$ As values greater and less than $-\dfrac{b}{2a}$ are assigned to x, $x+\dfrac{b}{2a}$ increases numerically, y increases, and the corresponding point of the curve rises in the plane. Moreover, if x is assigned the values $-\dfrac{b}{2a}+k$ and $-\dfrac{b}{2a}-k$, k being any assumed constant value, the corresponding values of y are the same. Hence the curve is symmetrical with respect to the straight line $x=-\dfrac{b}{2a}$, which line passes through the vertex of the curve parallel to the axis of y.

If a is negative, it can be proved in the same way that the curve has an axis of symmetry, $x=-\dfrac{b}{2a}$, which passes through its vertex, which is in this case the highest point of the curve.

36. Now that we have proved that the graphs of all quadratic polynomials in x are alike, having a vertex and an axis of symmetry passing through it, we can plot them more easily than was possible before, as is shown by the two following examples.

Ex. 1. $4x^2-4x+1$.

$$y=4x^2-4x+1=4(x^2-x+\tfrac{1}{4})=4(x-\tfrac{1}{2})^2.$$

Therefore the vertex of the graph is $(\frac{1}{2}, 0)$, and the axis of symmetry is the line $x=\frac{1}{2}$. Beginning with the value $\frac{1}{2}$, we assign to x values greater and less than $\frac{1}{2}$, thereby locating points on both sides of the axis of symmetry, and plot the graph which is represented in fig. 34.

We see that the equation $4x^2-4x+1\doteq 0$ has two equal real roots, the graph being tangent (§ 37) to the axis of x at the point $x=\frac{1}{2}$.

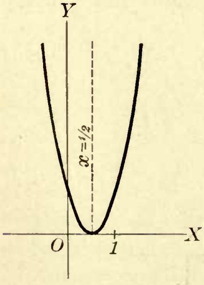

FIG. 34

Ex. 2. $-2x^2+3x$.

$$y = -2x^2 + 3x$$
$$= -2(x^2 - \tfrac{3}{2}x)$$
$$= -2[(x-\tfrac{3}{4})^2 - \tfrac{9}{16}].$$

Therefore the vertex of the graph is $(\frac{3}{4}, \frac{9}{8})$ and the axis of symmetry is the line $x = \frac{3}{4}$. The graph is represented in fig. 35. We see that it crosses the axis of x at two different points. Hence the equation $-2x^2+3x=0$ has two unequal real roots, which are found to be 0 and $\frac{3}{2}$.

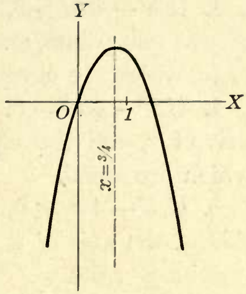

FIG. 35

37. Discriminant of the quadratic equation. Turning now to the constant $-\dfrac{b^2-4ac}{4a^2}$ in the equation

$$y = a\left[-\frac{b^2-4ac}{4a^2} + \left(x + \frac{b}{2a}\right)^2\right]$$

of § 35, we have three cases to consider.

1. If $b^2-4ac>0$, the vertex of the graph is below the axis of x when $a>0$, and above the axis of x when $a<0$, and accordingly the graph intersects the axis of x in two points.

2. If $b^2-4ac=0$, the vertex of the graph is on the axis of x, and hence the graph intersects the axis of x in a single point.

3. If $b^2-4ac<0$, the vertex of the graph is above the axis of x when $a>0$, and below the axis of x when $a<0$, and the graph does not intersect the axis of x at all.

Now let us suppose that different values are assigned to the constants a, b, and c, in such a way as to make b^2-4ac decrease, beginning with a positive value. Then the vertex of the graph rises or falls in the plane until, when $b^2-4ac=0$, it lies on the axis of x. At the same time, the points in which the graph intersects the axis of x have been approaching each other, and finally coincide, when the graph is said to be *tangent* to the axis of x.

Recalling that the abscissas of the points of the graph on the axis of x are the real roots of the equation formed by placing the expression equal to zero, we can tabulate the following results.

1. If $b^2 - 4ac > 0$, the graph of $ax^2 + bx + c$ intersects the axis of x at two points, and the equation $ax^2 + bx + c = 0$ has two real roots, which are unequal.

2. If $b^2 - 4ac = 0$, the graph of $ax^2 + bx + c$ is tangent to the axis of x, and the equation $ax^2 + bx + c = 0$ has two real roots, which are equal.

3. If $b^2 - 4ac < 0$, the graph of $ax^2 + bx + c$ is entirely on one side of the axis of x, and the equation $ax^2 + bx + c = 0$ has only imaginary roots.

The expression $b^2 - 4ac$ is called the *discriminant* of the quadratic equation, as its sign indicates the nature of the roots of the equation.

38. Graph of the polynomial of the *n*th degree. Let the polynomial be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n.$$

In general this polynomial contains $n+1$ terms. If any term is lacking, we may consider that its coefficient has become zero.

We will begin by plotting the graphs of some special numerical cases.

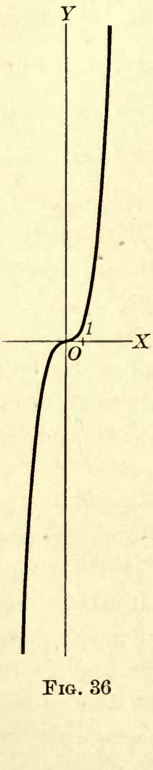

FIG. 36

Ex. 1. x^3.

Place $y = x^3$ and assume values of x. Hence the table:

x	y
0	0
1	1
2	8
− 1	− 1
− 2	− 8
.5	.1
− .5	− .1

x	y
1.5	3.4
− 1.5	− 3.4
2.3	12.2
− 2.3	− 12.2
2.7	19.7
− 2.7	− 19.7

Drawing a smooth curve through these points, we have the curve of fig. 36. It is called a cubical parabola.

Ex. 2. x^4.

Place $y = x^4$ and assume values of x. Hence the table:

x	y
0	0
1	1
2	16
– 1	1
– 2	16
.5	.1
.7	.2
.8	.4
.9	.7
1.1	1.5
1.3	2.9
1.5	5.1

x	y
1.7	8.4
1.9	13.0
– .5	.1
– .7	.2
– .8	.4
– .9	.7
– 1.1	1.5
– 1.3	2.9
– 1.5	5.1
– 1.7	8.4
– 1.9	13.0

The curve is represented in fig. 37.

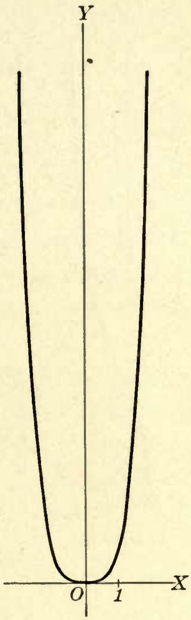

FIG. 37

Ex. 3. x^5.

Place $y = x^5$ and assume values of x. Hence the table:

x	y
0	0
1	1
2	32
– 1	– 1
– 2	– 32
.7	.2
.9	.6
1.2	2.5
1.4	5.4
1.6	10.5
1.7	14.2

x	y
1.8	18.9
1.9	24.8
– .7	– .2
– .9	– .6
– 1.2	– 2.5
– 1.4	– 5.4
– 1.6	– 10.5
– 1.7	– 14.2
– 1.8	– 18.9
– 1.9	– 24.8

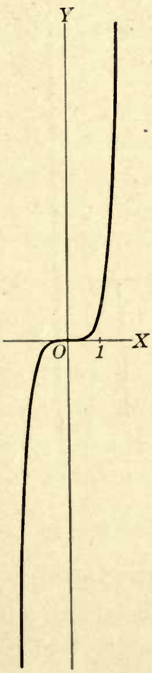

FIG. 38

The curve is represented in fig. 38.

In each of the three examples above, the curve crossed the axis of x at the origin, and the corresponding equation had the root zero.

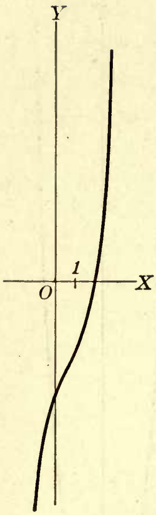

FIG. 39

Ex. 4. $x^3 - 2x^2 + 3x - 6$.

Place $y = x^3 - 2x^2 + 3x - 6$ and assume values of x. Hence the table:

x	y
0	− 6
1	− 4
2	0
3	12
− 1	− 12
− 2	− 28

x	y
1.5	− 2.6
2.5	4.6
2.7	7.2
− 1.5	− 18.4
− 1.7	− 21.8

The curve is represented in fig. 39.

This curve crosses the axis of x at the point $x = 2$, and hence the equation $x^3 - 2x^2 + 3x - 6 = 0$ has 2 for a real root. Its other roots are imaginary, i.e. $\pm\sqrt{-3}$.

Ex. 5. $4x^3 + 4x^2 - 9x - 9$.

Place $y = 4x^3 + 4x^2 - 9x - 9$ and assume values of x. Hence the table:

x	y
0	− 9
1	−10
2	21
− 1	0
− 2	− 7

x	y
1.5	0
1.3	− 5.2
1.7	6.9
− .5	− 4.0
−1.5	0
−1.3	.7

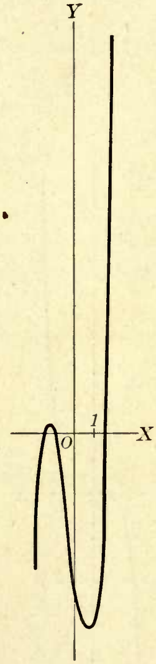

FIG. 40

This curve is represented in fig. 40. It crosses the axis of x at three points, — when $x = 1.5$, when $x = -1.5$, and when $x = -1$. Hence ± 1.5 and -1 are real roots of the equation $4x^3 + 4x^2 - 9x - 9 = 0$.

Without discussing any more numerical examples we can see that, in general, the abscissas of the points on the axis of x of the graph of the polynomial

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

are real roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0.$$

Conversely, the real roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

are the abscissas of the points at which the graph of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

intersects the axis of x, for they make $y = 0$.

Moreover, if the graph of the polynomial does not intersect the axis of x, the corresponding equation has no real roots; and conversely, if the equation has no real roots, the graph of the polynomial does not intersect the axis of x.

39. Solution of equations by factoring. Let $f(x)$ be a polynomial which can be separated into factors $f_1(x)$, $f_2(x)$, $f_3(x)$, $\cdots$, each of which is necessarily of lower degree than $f(x)$. Then the equation

$$f(x) = 0 \tag{1}$$

may be written in the form

$$f_1(x) \cdot f_2(x) \cdot f_3(x) \cdots = 0. \tag{2}$$

It is evident that any value of x which makes one of the factors $f_1(x)$, $f_2(x)$, $f_3(x)$, $\cdots$ zero, satisfies equation (2), and hence equation (1), i.e. is a root of equation (1). But such a value of x is evidently a root of some one of the equations

$$f_1(x) = 0, \qquad f_2(x) = 0, \qquad f_3(x) = 0, \qquad \cdots.$$

Conversely, any root of equation (1) must satisfy equation (2), and hence must make some one of the factors $f_1(x)$, $f_2(x)$, $f_3(x)$, $\cdots$ zero; for if no one of these factors is zero, their product cannot be zero. Hence the solution of the equation $f(x) = 0$ is reduced to the solution of the separate equations

$$f_1(x) = 0, \qquad f_2(x) = 0, \qquad f_3(x) = 0, \qquad \cdots.$$

In applying this method it is usually desirable to have no factor of higher degree than the second; but there is no advantage in carrying the factoring any further, as any quadratic equation can be readily solved.

Ex. 1. Solve the equation $x^3 = 8$.

By transposition, $$x^3 - 8 = 0;$$
whence, by factoring, $$(x-2)(x^2+2x+4) = 0.$$
$$\therefore x - 2 = 0 \quad \text{or} \quad x^2 + 2x + 4 = 0;$$
whence $$x = 2 \quad \text{or} \quad -1 \pm \sqrt{-3}.$$

Since the original equation might have been written $x = \sqrt[3]{8}$, we see that the three values of x which have been found are each a cube root of 8. In fact, every number has three cube roots, which may be found by solving the equation formed by placing x^3 equal to the number.

Ex. 2. Solve the equation $x^4 + 9 = 0$.

This equation may be written
$$(x^4 + 6x^2 + 9) - 6x^2 = 0;$$
whence, by factoring, $$(x^2 + \sqrt{6}x + 3)(x^2 - \sqrt{6}x + 3) = 0.$$
$$\therefore x^2 + \sqrt{6}x + 3 = 0, \quad \text{or} \quad x^2 - \sqrt{6}x + 3 = 0;$$
whence $$x = \frac{-\sqrt{6} \pm \sqrt{-6}}{2} \quad \text{or} \quad \frac{\sqrt{6} \pm \sqrt{-6}}{2}.$$

It is to be noted that every number has four fourth roots, which may be found by a method similar to that suggested above for finding its three cube roots.

40. Factors and roots. It follows immediately from the preceding article that *if* $x - r$ *is a factor of* $f(x)$, *then* r *is a root of the equation* $f(x) = 0$.

Conversely, *if* r *is a root of the equation* $f(x) = 0$, *then the polynomial* $f(x)$ *is divisible by* $x - r$.

Let $$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$
and let r be a root of $f(x) = 0$. Then
$$f(r) = a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0.$$
$$\begin{aligned}\therefore f(x) &= f(x) - f(r)\\ &= (a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n)\\ &\quad - (a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n)\\ &= a_0(x^n - r^n) + a_1(x^{n-1} - r^{n-1}) + \cdots + a_{n-1}(x - r).\end{aligned}$$

As $f(x)$ is expressed as a series of terms each of which, being the difference of the same positive integral powers of x and r, is divisible by $x - r$, it follows that $f(x)$ is divisible by $x - r$.

Ex. By inspection -1 is a root of the equation

$$x^4 + x^3 + 2x^2 + 3x + 1 = 0. \qquad (1)$$

Hence $x + 1$ is a factor of the left-hand member of the equation, which may accordingly be written

$$(x + 1)(x^3 + 2x + 1) = 0. \qquad (2)$$

Additional roots of equation (1) may now be found by solving the equation $x^3 + 2x + 1 = 0$ by methods given in §§ 52 and 63.

It is to be noted that the solution of the original equation has been simplified by making it depend upon the solution of a *depressed* equation, i.e. one of degree lower than the degree of the original equation.

41. By means of the second theorem we can form an equation which shall have any given quantities, $r_1, r_2, \cdots, r_n$ as roots. For if $r_1, r_2, \cdots$ are the roots of the equation, its left-hand member must contain the factors $x - r_1, x - r_2, \cdots$, the right-hand member being zero. Therefore the equation

$$(x - r_1)(x - r_2) \cdots (x - r_n) = 0$$

has the required quantities as roots. Moreover, this equation can have no other roots, since any other value of x will make no factor equal to zero, and hence the product will not be zero. Therefore the required equation is

$$(x - r_1)(x - r_2) \cdots (x - r_n) = 0.$$

Ex. 1. Form the equation having as roots $2 + 3\sqrt{-1}, 2 - 3\sqrt{-1}, -\frac{1}{3}$.

The required equation is

$$(x - 2 - 3\sqrt{-1})(x - 2 + 3\sqrt{-1})(x + \tfrac{1}{3}) = 0,$$

or

$$[(x - 2)^2 + 9][3x + 1] = 0,$$

or

$$3x^3 - 11x^2 + 35x + 13 = 0.$$

This method of forming an equation suggests a method of factoring a quadratic expression. For if r_1 and r_2 are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $ax^2 + bx + c$ is divisible by $x - r_1$ and $x - r_2$; and hence

$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Ex. 2. Factor $6x^2 + x - 1$.

The roots of the equation $6x^2 + x - 1 = 0$ are $-\frac{1}{2}$ and $\frac{1}{3}$.

$$\begin{aligned}\therefore\ 6x^2 + x - 1 &= 6(x + \tfrac{1}{2})(x - \tfrac{1}{3})\\ &= 2(x + \tfrac{1}{2}) \cdot 3(x - \tfrac{1}{3})\\ &= (2x + 1)(3x - 1).\end{aligned}$$

Ex. 3. Factor $4x^2 + 4x - 2$.

The roots of the equation $4x^2 + 4x - 2 = 0$ are $\dfrac{-1 \pm \sqrt{3}}{2}$.

$$\begin{aligned}\therefore\ 4x^2 + 4x - 2 &= 4\left(x - \frac{-1+\sqrt{3}}{2}\right)\left(x - \frac{-1-\sqrt{3}}{2}\right)\\ &= (2x + 1 - \sqrt{3})(2x + 1 + \sqrt{3}).\end{aligned}$$

Ex. 4. Factor $x^2 + 4x + 6$.

The roots of the equation $x^2 + 4x + 6 = 0$ are $-2 \pm \sqrt{-2}$.

$$\therefore\ x^2 + 4x + 6 = (x + 2 - \sqrt{-2})(x + 2 + \sqrt{-2}).$$

42. Number of roots of an equation. The fundamental proposition concerning the roots of an equation is that *every equation formed by placing a polynomial equal to zero has at least one root.* The proof of this proposition, however, depends upon methods too advanced to be used here. We shall therefore assume it as proved, and proceed to prove, as a consequence of it, that *every equation of the nth degree has n roots, and only n roots.*

Let the given equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

be denoted by

$$f(x) = 0. \qquad (1)$$

Since this equation must have at least one root, let r_1 be that root. Then $f(x)$ is divisible by $x - r_1$ (§ 40) and therefore

$$f(x) = (x - r_1)f_1(x), \qquad (2)$$

$f_1(x)$ being the other factor, and necessarily of degree $n - 1$.

Equation (1) can now be written

$$(x - r_1)f_1(x) = 0, \qquad (3)$$

and any root of

$$f_1(x) = 0 \qquad (4)$$

is a root of $f(x) = 0$ (§ 39).

But equation (4) must have at least one root; and if we let r_2 be that root, and reason as before, we may write

$$f_1(x) = (x - r_2) f_2(x), \tag{5}$$

$f_2(x)$ being of degree $n - 2$.

By substitution in (2) we shall have

$$f(x) = (x - r_1)(x - r_2) f_2(x). \tag{6}$$

After separating n linear factors in this way, the last quotient will be a_0. Therefore we shall have

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n), \tag{7}$$

the polynomial being expressed as the product of n linear factors.

Then the equation $f(x) = 0$ may be written

$$a_0(x - r_1)(x - r_2) \cdots (x - r_n) = 0, \tag{8}$$

whence it is seen to have n roots (§ 39), i.e. $r_1, r_2, \cdots, r_n$.

It can have no other roots; for if we let x have any value other than $r_1, r_2 \cdots$, or r_n, no factor of the first member of (8) is zero, and hence the product in the first member is not equal to zero. Therefore the equation of the nth degree has n, and no more than n, roots, and the polynomial of the nth degree can always be separated into n linear factors. In general, however, it is not possible to determine these factors where $n > 4$.

It is to be noted that the roots may all be different, or some of them may occur more than once. In the latter case the equation is said to have multiple roots.

43. If now the left-hand member of equation (8) of § 42 is expanded, the equation appears in the original form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

and it is evident that

$$(-r_1) + (-r_2) + (-r_3) + \cdots + (-r_n) = \frac{a_1}{a_0}, \tag{1}$$

and that
$$(-r_1)(-r_2)(-r_3) \cdots (-r_n) = \frac{a_n}{a_0}. \tag{2}$$

Equations (1) and (2) express respectively the following theorems:

1. *The sum of the roots of an equation with their signs changed is the coefficient of* x^{n-1} *divided by that of* x^n.

2. *The product of the roots of an equation with their signs changed is the constant term divided by the coefficient of* x^n.

Other theorems of this type are given in works on the theory of equations, but only these two have been stated here, since they are of special service in finding the remaining root of an equation after all the others have been determined.

Ex. 1. Three roots of the equation $2x^4 + 7x^3 + 8x^2 + 2x - 4 = 0$ are -2, $-1-\sqrt{-1}$, and $-1+\sqrt{-1}$. Find the fourth root.

The sum of all the roots is $-\frac{7}{2}$, and the sum of the three roots known is -4. Therefore the fourth root is $-\frac{7}{2}-(-4)$, or $\frac{1}{2}$.

Ex. 2. Two roots of the equation $36x^3 - 7x + 1 = 0$ are $\frac{1}{3}$ and $-\frac{1}{2}$. Find the third root.

The sum of the two roots known is $-\frac{1}{6}$, and the sum of all the roots is 0, since the coefficient of x^2 is 0; therefore the third root is $0-(-\frac{1}{6})$, or $\frac{1}{6}$.

Or the product of the roots known is $-\frac{1}{6}$, and the product of all the roots is $-\frac{1}{36}$; therefore the third root is $(-\frac{1}{36}) \div (-\frac{1}{6})$, or $\frac{1}{6}$.

44. Conjugate complex roots. Nothing was said in § 42 as to the nature of the roots $r_1, r_2, \cdots, r_n$. But if the coefficients $a_0, a_1, \cdots, a_n$ are all real, and if $a + bi$ is one of the roots, then $a - bi$ is also a root.

For if $a + bi$ is a root of $f(x) = 0$, then $f(a + bi) = 0$. When $f(a + bi)$ is expanded the terms can be separated into two sets, — those containing a alone or involving only even powers of bi as a factor, and those involving only odd powers of bi as a factor. By § 12 the terms of the first set are all real and their sum may be denoted by A; and the terms of the second set contain i to the first power as a factor, and their sum may be denoted by Bi (B, of course, being real). Then $f(a + bi) = 0$ may be written

$$A + Bi = 0,$$

whence (§ 12), $A = 0$ and $B = 0$.

If, in the above, we replace bi by $-bi$, it is evident that the terms in the first set are not affected, as they involve only even powers of bi as a factor, and those in the second set, involving only odd powers of bi as a factor, are changed in algebraic sign only. Therefore we have $f[a+(-bi)]=A-Bi$. But we have seen that $A=0$ and $B=0$; therefore $f[a+(-bi)]=0$. Since $f[a+(-bi)]=f(a-bi)$, however, it follows that $f(a-bi)=0$, and $a-bi$ is a root of the given equation $f(x)=0$.

This fact is usually stated by saying that complex roots occur in pairs.

It follows that an equation of even degree may not have any real roots, and that an equation of odd degree must have an odd number of real roots, and thus at least one real root.

45. It was proved in § 42 that every polynomial is equivalent to the product of n linear factors, i.e.

$$a_0(x-r_1)(x-r_2)\cdots(x-r_n),$$

where $r_1, r_2, \cdots, r_n$ are the roots of the corresponding equation. Now if any one of these roots is complex, there will be a corresponding conjugate complex root. Let $a+bi$ and $a-bi$ be two such roots. Then the corresponding factors are $(x-a-bi)$ and $(x-a+bi)$, which combine into $(x-a)^2+b^2$, a real quadratic factor.

Therefore every polynomial with real coefficients is equivalent to the product of real linear and quadratic factors.

46. Graphs of products of real linear and quadratic factors.

1. *All the factors linear and none repeated*, as

$$a_0(x-r_1)(x-r_2)\cdots(x-r_n).$$

Placing y equal to this expression, we have

$$y=a_0(x-r_1)(x-r_2)\cdots(x-r_n).$$

It is evident that the graph intersects the axis of x at n distinct points for which $x=r_1, x=r_2, \cdots, x=r_n$, and at no other points, as no other values of x make y zero. Now let the quantities $r_1, r_2, \cdots, r_n$ be arranged in the order of their magnitude, r_1 being the least. Then if at first $x<r_1$, all the factors are negative; and if x changes so that $r_1<x<r_2$, the first factor becomes positive while all the others remain negative. Therefore y changes

sign when x changes from being less than r_1 to being greater than r_1, and the curve crosses the axis of x at the point $x = r_1$.

Again, if x changes so that at first $r_1 < x < r_2$ and then $r_2 < x < r_3$, the second factor changes sign from minus to plus, the others retaining their original signs. Hence y again changes sign, and the curve crosses the axis of x again at the point $x = r_2$.

Continuing in this manner, we can show that the curve crosses the axis of x n times as it is traced from left to right.

2. *All the factors linear, some being repeated*, as, for example,

$$a_0(x - r_1)(x - r_2)^2(x - r_3)^3,$$

the corresponding equation being

$$y = a_0(x - r_1)(x - r_2)^2(x - r_3)^3.$$

If the r's are arranged in ascending order of magnitude, it may be proved, as in the previous case, that the graph crosses the axis of x at the points $x = r_1$, and $x = r_3$, but not at the point $x = r_2$. For if at first $r_1 < x < r_2$ and then $r_2 < x < r_3$, it is seen that no factor changes sign. But since $y = 0$ when $x = r_2$, the graph has a point on the axis of x when $x = r_2$; in fact, it is *tangent* to the axis of x. And it can be proved in general that, if a linear factor occurs an even number of times, the graph does not cross the axis of x at the corresponding point.

3. *Some of the factors quadratic*, as, for example,

$$a_0(x - r_1)(x - r_2)^2[(x - a)^2 + b^2],$$

the corresponding equation being

$$y = a_0(x - r_1)(x - r_2)^2[(x - a)^2 + b^2].$$

The only new type of factor is $(x - a)^2 + b^2$, and this is positive for all values of x. Hence there is no new point to be discussed in regard to the intersection of the graph with the axis of x.

In general, *the graph has as many points on the axis of x as the polynomial has different linear factors; it does not cross the axis at any point corresponding to a factor occurring an even number of times; and it crosses the axis of x at any point corresponding to a factor occurring an odd number of times.*

Ex. 1. $y = .5(x+2)(x+.5)(x-2)$.

1. If $x = -2$ or $-.5$ or 2, $y = 0$, and there are three points of the curve on the axis of x.

2. If $x < -2$, all three factors are negative; therefore $y < 0$, and the corresponding part of the curve lies below the axis of x. If $-2 < x < -.5$, the first factor is positive and the other two are negative; therefore $y > 0$, and the corresponding part of the curve lies above the axis of x. If $-.5 < x < 2$, the first two factors are positive and the third is negative; therefore $y < 0$, and the corresponding part of the curve lies below the axis of x. Finally, if $x > 2$, all the factors are positive; therefore $y > 0$, and the corresponding part of the curve lies above the axis of x.

3. Assuming values of x and finding the corresponding values of y, we plot the curve, as represented in fig. 41.

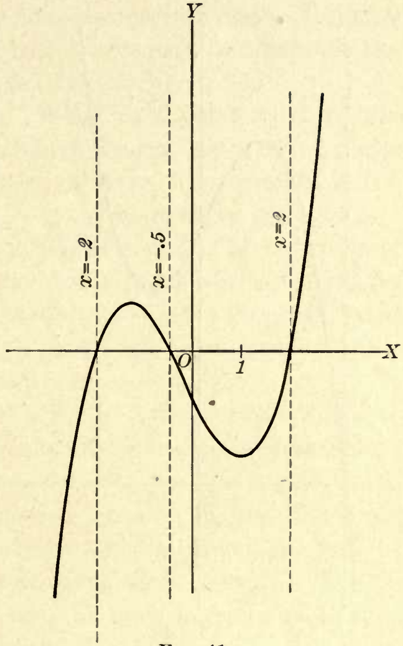

FIG. 41

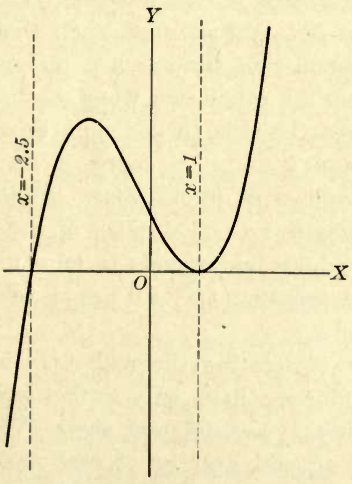

FIG. 42

Ex. 2. $y = .5(x+2.5)(x-1)^2$.

1. If $x = -2.5$ or 1, $y = 0$, and there are two points of the curve on the axis of x.

2. If $x < -2.5$, the first factor is negative and the second factor is positive; therefore $y < 0$, and the corresponding part of the curve lies below the axis of x. If $-2.5 < x < 1$, both factors are positive; therefore $y > 0$, and the corresponding part of the curve lies above the axis of x. Finally, if $x > 1$, we have the same result as when $-2.5 < x < 1$, and the curve does not cross the axis of x at the point $x = 1$, but is tangent to it.

3. Assuming values of x, and finding the corresponding values of y, we plot the curve as represented in fig. 42.

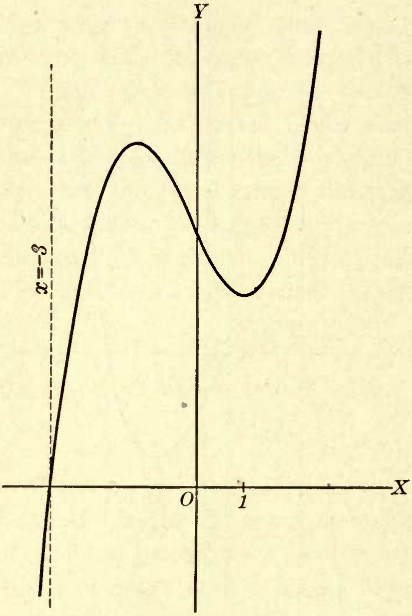

FIG. 43

Ex. 3. $y = .5(x+3)(x^2 - 2.5x + 3.5)$.

1. If $x = -3$, $y = 0$, and this curve has but one point on the axis of x.

2. If $x < -3$, the first factor is negative and the second factor is positive, as it always is, since it is equivalent to $(x - 1.25)^2 + 1.9375$; therefore $y < 0$, and the corresponding part of the curve is below the axis of x. If $x > -3$, the first factor is positive; therefore $y > 0$, and the corresponding part of the curve is above the axis of x.

3. Assuming values of x, and finding the corresponding values of y, we plot the curve as represented in fig. 43.

47. Location of roots. From the work of the last article it is evident that the real roots of the equation $f(x) = 0$ determine points on the axis of x at which the graph of $f(x)$ crosses or touches that axis. Moreover, if x_1 and x_2 $(x_1 < x_2)$ are two values of x, such that $f(x_1)$ and $f(x_2)$ are of opposite algebraic sign, the graph is on one side of the axis when $x = x_1$, and on the other side when $x = x_2$. Therefore (§ 56) it must have crossed the axis an odd number of times between the points $x = x_1$ and $x = x_2$. Of course it may have touched the axis at any number of intermediate points. Since a point of crossing corresponds to an odd number of roots of an equation, and a point of touching corresponds to an even number of roots, it follows that the equation $f(x) = 0$ has an odd number of real roots between x_1 and x_2.

The above gives a ready means of locating the real roots of an equation in the form $f(x) = 0$, for we have only to find two values of x, as x_1 and x_2, for which $f(x)$ has different signs. We then know that the equation has an odd number of real roots between these values, and the nearer together x_1 and x_2, the more nearly do we know the values of the intermediate roots. In locating the roots in this manner it is not necessary to construct the corresponding graph, though it may be helpful.

48. Descartes' rule of signs. When in a polynomial a term with one sign is immediately followed by one with the opposite sign, there is said to be a *variation* of sign. For example, in the polynomial $3x^4 + 2x^3 - 3x^2 + x - 2$ there are three variations.

The variations of sign in the left-hand member of an equation are often of value in locating the real roots of the equation, for *the number of positive roots of the equation* $f(x) = 0$ *cannot exceed the number of variations of sign in its left-hand member.* This rule is known as *Descartes' rule of signs.*

For example, the equation $3x^4 + 2x^3 - 3x^2 + x - 2 = 0$ cannot have more than three positive roots, as there are three variations of sign in its left-hand member.

To determine the greatest possible number of negative roots, replace x by $-x'$. The roots of the resulting equation will be those of the original equation with their signs changed. Accordingly the original equation can have no more negative roots than this new equation has positive roots.

If, in the equation $3x^4 + 2x^3 - 3x^2 + x - 2 = 0$, x is replaced by $-x'$, the new equation is $3x'^4 - 2x'^3 - 3x'^2 - x' - 2 = 0$. As this equation cannot have more than one positive root, the original equation cannot have more than one negative root.

Sometimes, by Descartes' rule, we can prove that an equation has imaginary roots. For example, the equation $3x^3 + x^2 + 2 = 0$ can have no positive root, and not more than one negative root. Being of odd degree, it has at least one real root (§ 44); therefore it has one negative root and two imaginary roots.

In order to prove Descartes' rule we will first prove that *if any polynomial* $f(x)$ *is multiplied by* $x - r$, *where* r *is a positive quantity, the product has at least one more variation than has* $f(x)$.

Assuming the first term of $f(x)$ to be positive, we will inclose all the terms preceding the first minus sign in a parenthesis. In a second parenthesis we will inclose all the terms with a minus sign before a positive sign occurs again, and so on. Suppose, then, that the first minus sign appears in the term containing

x^{n-k}, the next plus sign occurs in the term containing x^{n-l}, etc., and that all the terms after that containing x^{n-m} have the same sign as that term. We can now write

$$\begin{aligned} f(x) = (a_0x^n + \cdots + a_{k-1}x^{n-k+1}) \\ -(a_kx^{n-k} + \cdots + a_{l-1}x^{n-l+1}) \\ +(a_lx^{n-l} + \cdots) - \cdots \\ \pm(a_mx^{n-m} + \cdots + a_n), \end{aligned} \tag{1}$$

where all the terms within each parenthesis are of the same sign, i.e. plus. Therefore each parenthesis marks a variation.

To multiply $f(x)$ by $x - r$ we shall multiply first by x, then by $-r$, and add the partial products.

The result is an equation of the following form:

$$\begin{aligned} (x-r)f(x) = (a_0x^{n+1} \pm \cdots) - (b_kx^{n-k+1} \pm \cdots) \\ +(b_lx^{n-l+1} \pm \cdots) - \cdots \\ \pm(b_mx^{n-m+1} \pm \cdots) \mp a_nr, \end{aligned} \tag{2}$$

where $b_k = a_k + ra_{k-1}$, $b_l = a_l + ra_{l-1}$, etc., and accordingly are positive.

The signs before each parenthesis of (2) are the same as in (1), but the signs within the parenthesis are not necessarily all plus. But however the signs may occur within any parenthesis, there is at least one variation between the first term of one parenthesis and the first term of the following parenthesis. Hence, if we consider the parentheses only, the number of variations in the product is not less than the number of variations in $f(x)$.

But, in addition, we have the last term of the product, i.e. $\mp a_nr$, the sign of which differs from the sign of the first term in the last parenthesis. Hence there is at least one more variation in $(x-r)f(x)$ than in $f(x)$, as we set out to prove.

Now the equation having the roots $r_1, r_2, \cdots, r_n$ is (§ 41)

$$(x - r_1)(x - r_2)\cdots(x - r_n) = 0.$$

In expanding the left-hand member every time we multiply by a factor corresponding to a positive root, we add at least one variation of sign. Hence the number of positive roots cannot exceed the number of variations, as stated in Descartes' rule.

49. Rational roots. The real roots of any equation are either rational or irrational (§ 10), and the rational roots must be either integral or fractional. We will now derive methods of finding the rational roots, beginning with the integral roots.

An easy method of determining the integral roots depends upon the following theorem: *If the equation is written in the form*

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \tag{1}$$

where all the coefficients are integers, any integral root r must be a factor of a_n.

It has been proved in § 40 that the left-hand member of (1) is divisible by $x - r$. Since the coefficient of x is unity, and all the coefficients in the dividend are integers, all the coefficients in the quotient are integers. But the last coefficient in the quotient multiplied by r must be a_n, since there is no remainder. Hence the theorem is proved.

Accordingly, to find the integral roots of any equation with integral coefficients, we have merely to try the integral factors of a_n. When an integral root has been found, we depress the degree of the equation as in § 40, and apply the process to the new equation. In this way all the integral roots may be found. In case no integral factor of a_n proves to be a root, it follows that the equation can have no integral root.

Ex. Find the integral roots of the equation

$$4x^4 - 4x^3 - 25x^2 + x + 6 = 0.$$

The integral roots of this equation must be factors of 6, so that we have to try ± 1, ± 2, ± 3, ± 6. By trial it is found that -2 is a root, and the degree of the equation is depressed by dividing the left-hand member by $x + 2$, the depressed equation being $4x^3 - 12x^2 - x + 3 = 0$. The only possible values of integral roots of this equation are ± 1, ± 3, and 3 is found to be a root. Dividing the left-hand member by $x - 3$, we have, as the depressed equation, $4x^2 - 1 = 0$, the roots of which are $\pm \frac{1}{2}$.

Therefore the roots of the original equation are -2, 3, $\pm \frac{1}{2}$.

While all the integral roots of an equation may be found by this method, it is evident that it fails for fractional roots, as there is no way of determining what fractions ought to be tried. This difficulty is obviated by the two theorems in the next article.

50. *If a_0 is unity and all the other coefficients are integers, the equation cannot have a rational fraction in its lowest terms as a root.*

Let the equation be

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

and, if possible, let the rational fraction $\frac{p}{q}$, which is in its lowest terms, be a root. Then

$$\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + a_2\left(\frac{p}{q}\right)^{n-2} + \cdots + a_{n-1}\left(\frac{p}{q}\right) + a_n = 0.$$

Multiply through by q^{n-1}, and transpose to the second member all terms but the first. Then

$$\frac{p^n}{q} = -a_1p^{n-1} - a_2p^{n-2}q - \cdots - a_{n-1}pq^{n-2} - a_nq^{n-1}.$$

By hypothesis p and q have no common factor, and therefore $\frac{p^n}{q}$ is a rational fraction in its lowest terms, while the right-hand member of the equation is an integral expression. But two such expressions cannot be equal, and hence $\frac{p}{q}$, the rational fraction in its lowest terms, cannot be a root of the equation.

Moreover, *every equation in the form*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

in which a_0 is not unity, can be transformed into an equation with integral coefficients in which the coefficient of the highest power of the unknown quantity shall be unity.

For, dividing through by a_0, we have

$$x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_{n-1}}{a_0}x + \frac{a_n}{a_0} = 0. \tag{1}$$

If x is a root of this equation, let $x = \frac{x'}{m}$, m being an integer, and substitute in (1). Then

$$\frac{x'^n}{m^n} + \frac{a_1}{a_0}\frac{x'^{n-1}}{m^{n-1}} + \frac{a_2}{a_0}\frac{x'^{n-2}}{m^{n-2}} + \cdots + \frac{a_{n-1}}{a_0}\frac{x'}{m} + \frac{a_n}{a_0} = 0. \tag{2}$$

Multiplying (2) by m^n, we have

$$x'^n + \left(\frac{a_1}{a_0} m\right) x'^{n-1} + \left(\frac{a_2}{a_0} m^2\right) x'^{n-2} + \cdots$$
$$+ \left(\frac{a_{n-1}}{a_0} m^{n-1}\right) x' + \left(\frac{a_n}{a_0} m^n\right) = 0. \quad (3)$$

We can now determine m by inspection in such a way that all the coefficients of (3) shall be integers. The roots of this new equation are m times the roots of the original equation.

Ex. Transform equation $12x^3 + 16x^2 - 5x - 3 = 0$ to an equation having integral coefficients, the coefficient of the highest power of x being unity.

Dividing by 12, we have

$$x^3 + \tfrac{4}{3}x^2 - \tfrac{5}{12}x - \tfrac{1}{4} = 0.$$

Multiplying the roots of this equation by an integer m, we insert in each term a power of m such that the sum of its exponent and that of x' shall be equal to the degree of the equation, thus obtaining

$$x'^3 + (\tfrac{4}{3}m)x'^2 - (\tfrac{5}{12}m^2)x' - (\tfrac{1}{4}m^3) = 0.$$

For $\frac{4}{3}m$ to be an integer, m must equal $3k$ where k is an integer. Then $\frac{5}{12}m^2$ becomes $\frac{5}{12}(9k^2)$, and this is an integer only when $k = 2l$; i.e. $m = 6l$, l being an integer. Finally, $\frac{1}{4}m^3$, or $\frac{1}{4}(6l)^3$, is an integer if $l = 1$, the least value of m being the one desired.

Therefore we let $m = 6$, and our required equation is

$$x'^3 + 8x'^2 - 15x' - 54 = 0,$$

the roots of which are six times the roots of the original equation.

The roots of this equation are found by the method of § 49 to be -2, 3, and -9. Hence the roots of the original equation are $-\frac{1}{3}$, $\frac{1}{2}$, and $-\frac{3}{2}$.

We are thus in a position to determine the rational fractional roots of any equation with rational coefficients.

51. We now see that to find all the rational roots of any equation, we first find all its integral roots and then all its fractional roots, as indicated in the following example.

Ex. Find all the rational roots of the equation

$$2x^4 - 5x^3 - 2x^2 - 7x + 30 = 0. \tag{1}$$

By Descartes' rule of signs this equation cannot have more than two positive roots, and not more than two negative roots. If any of the roots are integral, they will be among the factors of 30, i.e. ± 1, ± 2, ± 3, ± 5, ± 6, ± 10, ± 15, ± 30. By trial we find $+2$ to be a root, and the depressed equation is

$$2x^3 - x^2 - 4x - 15 = 0. \tag{2}$$

By trial we find that this new equation has no integral roots, no factor of 15 being a root. Accordingly we proceed to find fractional roots.

Dividing equation (2) through by 2 and then multiplying the roots by m, we have

$$x'^3 - (\tfrac{1}{2}m)x'^2 - (2m^2)x' - (\tfrac{15}{2}m^3) = 0. \tag{3}$$

To make the coefficients of (3) integral we take $m = 2$, and the equation becomes

$$x'^3 - x'^2 - 8x' - 60 = 0. \tag{4}$$

By trial we find an integral root of this equation to be 5, and the depressed equation is

$$x^2 + 4x + 12 = 0, \tag{5}$$

the roots of which are $-2 \pm 2\sqrt{-2}$.

Therefore the three roots of the transformed equation (4) are 5 and $-2 \pm 2\sqrt{-2}$, and the roots of the first depressed equation (2) are $\frac{5}{2}$ and $-1 \pm \sqrt{-2}$, so that the roots of the given equation are 2, $\frac{5}{2}$, and $-1 \pm \sqrt{-2}$.

It is to be noted that in this example, after having found all the rational roots, we were able to find the remaining roots also, since the last depressed equation was of no higher degree than the second.

52. Irrational roots. It should be borne in mind that rational roots occur only for special values or systems of values of the coefficients. Hence, after removing the rational roots, if any, by the previous methods, we have, in general, to determine irrational roots in order to have all the real roots of the equation. But from the definition of an irrational quantity (§ 10) it is evident that we cannot find an irrational root exactly. We may, however, find an approximate value to any required degree of accuracy. There are various methods of approximation, one of which immediately follows. A more rapid method is given in § 63.*

* A method of solving algebraic equations, known as Horner's method, is found in most treatises on the theory of equations. It is convenient in arrangement of work and speedy in the hands of an expert. It may therefore be recommended to one who has often to solve equations. On the other hand, the methods of §§ 52, 63 of this book have two advantages. They may be applied to other than algebraic equations (see § 162), and depend upon principles which, if once mastered, are not easily forgotten.

Let the given equation be $f(x) = 0$, and the graph of the left-hand member be as in fig. 44, where $OM_1 = x_1$ and $OM_2 = x_2$. Then $M_1P_1 = f(x_1)$ and $M_2P_2 = f(x_2)$, and since $f(x_1)$ and $f(x_2)$ are of opposite sign, the curve crosses the axis of x between M_1 and M_2, and there is at least one real root of $f(x) = 0$ between x_1 and x_2 (§ 47).

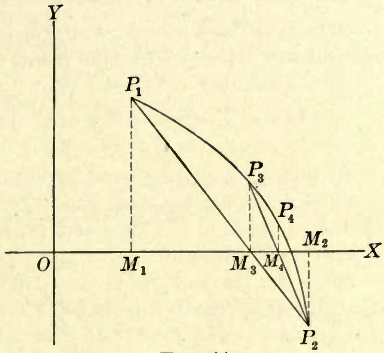

FIG. 44

Not only does the curve cross the axis of x at some point between M_1 and M_2, but it is evident from fig. 44 that the straight line P_1P_2 also intersects the axis of x at some point between M_1 and M_2, as M_3. If the points M_1 and M_2 are near together, i.e. if x_1 and x_2 differ only by a small amount, the curve in most cases differs only slightly from the straight line P_1P_2. Hence, if we replace the curve by the straight line, the abscissa of the point at which P_1P_2 intersects the axis of x will be approximately the root of the equation.

If OM_3 is denoted by x_3, it is evident (fig. 44) that there is a root of $f(x) = 0$ between x_2 and x_3, a smaller interval than that between x_1 and x_2, in which the root was first located.

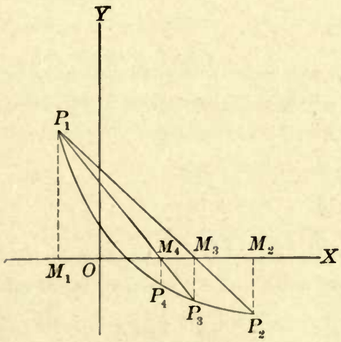

FIG. 45

If, however, the graph of $f(x)$ had been as in fig. 45, the root would have been between x_1 and x_3, an interval smaller, of course, than that between x_1 and x_2.

If $f(x_3)$ has the same sign as $f(x_1)$, we have the first case (fig. 44); and if $f(x_3)$ has the same sign as $f(x_2)$, we have the second case (fig. 45). In the first case, repeating the process, using x_3 in place of x_1, we can find an x_4 between which and x_2 the root must lie; and in the second case, using x_3 in place of x_2, we can find an x_4 between which and x_1 the root must lie.

Moreover, it is evident that the successive values of x, i.e. x_3, x_4, x_5, · · ·, found in this way are each nearer to the true value of the root of $f(x) = 0$ than the one preceding.

Ex. Find the root of the equation $x^3 + 2x - 17 = 0$ between 2 and 3.

Here $x_1 = 2$ and $x_2 = 3$; also $f(2) = -5$ and $f(3) = 16$. The equation of the straight line determined by the points $(2, -5)$ and $(3, 16)$ is (§ 29)

$$y + 5 = \frac{-5 - 16}{2 - 3}(x - 2).$$

Its intercept on OX, found by letting $y = 0$, is 2.2 +, and $f(2.2) = -1.952$.

Since $f(2.2)$ has the same sign as $f(2)$, the second straight line is determined by the points $(2.2, -1.952)$ and $(3, 16)$. Its intercept on OX is 2.28 +, and $f(2.28) = -0.587648$.

Since $f(2.28)$ and $f(2.2)$ have the same sign, the third straight line is determined by the points $(2.28, -0.587648)$ and $(3, 16)$. Its intercept on OX is 2.3 +, and $f(2.3) = -0.233$. The fourth straight line is determined by the points $(2.3, -0.233)$ and $(3, 16)$. Its intercept on OX is 2.31 +, and $f(2.31) = -0.053609$. The fifth straight line is determined by $(2.31, -0.053609)$ and $(3, 16)$. Its intercept on OX is 2.312.

Hence the irrational root of $x^3 + 2x - 17 = 0$, accurate to two places of decimals, is 2.31.

By continuing this process we can find any desired number of decimal places of the root. It is to be noted that we are obliged to find one more decimal place than the number of decimal places to which the root is to be accurate. The approximation is more rapid if the first decimal place is found by the method of § 47.

PROBLEMS

Plot the graphs of the following quadratic expressions, in each case locating the vertex of the graph and determining the nature of the roots of the corresponding equation:

1. $2x^2 + 3x - 2$.

2. $9x^2 - 3x - 2$.

3. $4x^2 + 4x + 3$.

4. $-3x^2 + 5x$.

5. $-9x^2 + 12x - 7$.

6. $4x^2 - 4x - 1$.

7. For what values of a are the roots of $ax^2 + 3x + 7 = 0$ equal? What are the roots?

8. Prove that the roots of $\left(cx + \frac{2a}{c}\right)^2 - 8ax = 0$ are equal for all values of a and c, and find them.

9. Prove that there is no real value of m for which the roots of $x^2 + (mx + 3)^2 - 16 = 0$ are equal.

For what values of k are the roots of the following quadratic equations (1) equal? (2) real and unequal? (3) imaginary?

10. $2x^2 + 3x + 2 = k.$

11. $x^2 + (2 - k)x + 1 = 0.$

12. $(k+1)x^2 + (k-1)x + (k+1) = 0.$

Plot the graphs of the following polynomials:

13. $x^3 - ax.$ $(a > 0.)$

14. $x^3 - 4x^2 + x + 1.$

15. $x^3 - 3x^2 + 1.$

16. $x^3 + x^2 + 2x + 5.$

17. $x^3 - x^2 + x - 4.$

18. $x^3 + 6x - 6.$

19. $x^3 - 12x + 3.$

20. $2x^4 + x^3 - 4x^2 - 10x - 4.$

21. $4x^4 + 12x^3 + 7x^2 - 28x - 6.$

22. $3x^4 - 10x^3 - 5x^2 + 2x.$

23. $x^4 + 6x^3 + 10x^2.$

24. $2x^5 + 2x^4 - 7x^3 - 8x^2 - 4x.$

Find all the roots of the following equations:

25. $8x^3 = 27.$

26. $8x^6 - 63x^3 - 8 = 0.$

27. $x^5 - 5x^3 + 12x = 2x^3 + 3x.$

28. $5x^6 + 27x^2 = 2x^6 - 54x^4.$

29. $(2x - a)^4 - (3x + a)^4 = 0.$

30. $x^4 - 2(a^2+1)x^2 + (a^2-1)^2 = 0.$

Form the equations having the following values for their roots:

31. $0, \frac{2}{3}, \frac{3}{2}.$

32. $a + \sqrt{b},\ a - \sqrt{b},\ -a.$

33. $0, 0, 2a \pm b, \pm\sqrt{2b}.$

34. Form a quadratic equation with real coefficients having $2 + 3i$ for one of its roots.

Factor the following quadratic expressions:

35. $4x^2 + 8x - 7.$

36. $4x^2 + 12x + 11.$

37. $4a^2x^2 + 2ax + 1.$

38. $x^2 + 2ax - a + a^2.$

39. $a^2x^2 + 2abx - a.$

40. $a^2x^2 + 2abx + b + b^2.$

If r_1 and r_2 are the roots of the equation $x^2 + px + q = 0$, find the values of the following expressions in terms of p and q without solving the equation:

41. $r_1^2 + r_2^2.$ **42.** $r_1^3 + r_2^3.$ **43.** $\frac{1}{r_1} + \frac{1}{r_2}.$ **44.** $\frac{1}{r_1^2} + \frac{1}{r_2^2}.$ **45.** $\frac{r_1}{r_2} + \frac{r_2}{r_1}.$

If r_1, r_2, r_3 are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the values of the following expressions in terms of the coefficients without solving the equation:

46. $(r_1^2 + r_2^2 + r_3^2) + 2(r_1r_2 + r_2r_3 + r_3r_1) + 3r_1r_2r_3.$

47. $r_1^2r_2r_3 + r_2^2r_3r_1 + r_3^2r_1r_2.$

48. $\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1}.$

49. Show that if $a + \sqrt{b}$ is a root of an equation with rational coefficients, then $a - \sqrt{b}$ is also a root.

Plot the graphs of the following expressions, and find all the roots of the corresponding equations:

50. $(x+1)(x-2)(x-4)$.

51. $(x-2)(x-4)(2x+3)$.

52. $(x-4)(2x+1)(3x+5)$.

53. $(x+3)(x-1)^2$.

54. $(2x-1)(x-3)^2$.

55. $(x-2)(2x+3)^2$.

56. $(2x+5)(x^2+2x+3)$.

57. $(x-5)(2x^2+3x+2)$.

58. $(x+2)(x-3)(x-2)^2$.

59. $(x-2)(x+2)(x^2+2)$.

60. $(x-2)^2(2x^2+2x+1)$.

61. $(x+1)(2x-1)(3x^2+2x+3)$.

Find all the roots of the following equations:

62. $x^3-4x^2-2x+5=0$.

63. $x^3-3x^2+4=0$.

64. $3x^3-7x^2-8x+20=0$.

65. $4x^3-8x^2-35x+75=0$.

66. $x^3+4x^2+4x+3=0$.

67. $8x^3-28x^2+30x-9=0$.

68. $12x^3-44x^2+5x+7=0$.

69. $3x^3+10x^2+10x-12=0$.

70. $3x^3+10x^2+2x-8=0$.

71. $4x^4+8x^3+3x^2-2x-1=0$.

72. $6x^4-11x^3-37x^2+36x+36=0$.

73. $3x^4-17x^3+41x^2-53x+30=0$.

74. $2x^4-9x^3-9x^2+57x-20=0$.

75. $18x^4-27x^3+10x^2+12x-8=0$.

76. $16x^4+16x^3-72x^2-20x+25=0$.

77. $x^5-2x^4-4x^3-4x^2+15x+18=0$.

78. $4x^5+12x^4+11x^3+5x^2-3x-2=0$.

79. $12x^5+44x^4-55x^3-95x^2+63x-9=0$.

80. $2x^5-5x^4-13x^3+13x^2+5x-2=0$.

Determine by Descartes' rule of signs the nature of the roots of the following equations:

81. $x^3+5x-7=0$.

82. $x^3+2x+3=0$.

83. $x^3+2x^2+5=0$.

84. $3x^4+4x^3+4x+3=0$.

85. $x^4+x^2-x-6=0$.

86. $x^4-4x^2+1=0$.

Find the real roots of the following equations, accurate to two decimal places:

87. $x^3+3x-7=0$.

88. $x^3+x+5=0$.

89. $x^4-12x+7=0$.

90. $x^4-3x^3+3=0$.

91. $x^3-x^2-6x+1=0$.

CHAPTER V

THE DERIVATIVE OF A POLYNOMIAL

53. Limits. *A variable is said to approach a constant as a limit, when, under the law which governs the change of value of the variable, the difference between the variable and the constant becomes and remains less than any quantity which can be named, no matter how small.*

If the variable is independent, it may be made to approach a limit by assigning to it arbitrarily a succession of values following some known law. Thus, if x is given in succession the values

$$x_1 = \frac{1}{2},\ x_2 = \frac{3}{4},\ x_3 = \frac{7}{8},\ \cdots,\ x_n = \frac{2^n - 1}{2^n},$$

and so on indefinitely, it approaches 1 as a limit. For we may make x differ from 1 by as little as we please by taking n sufficiently great; and for all larger values of n the difference between x and 1 is still smaller. This may be made evident graphically by marking off on a number scale the successive values of x (fig. 46), when it will be seen that the difference between x and 1 soon becomes and remains too minute to be represented.

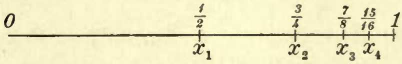

FIG. 46

Similarly, if we assign to x the succession of values

$$x_1 = \frac{1}{2},\ x_2 = -\frac{1}{3},\ x_3 = \frac{1}{4},\ x_4 = -\frac{1}{5},\ \cdots,\ x_n = (-1)^{n+1}\frac{1}{n+1},$$

x approaches 0 as a limit (fig. 47).

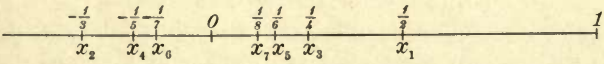

FIG. 47

If the variable is not independent but is a function of x, the values which it assumes as it approaches a limit depend upon the values arbitrarily assigned to x. For example, let $y=f(x)$, and let x be given a set of values

$$x_1,\quad x_2,\quad x_3,\quad x_4,\quad \cdots,\quad x_n,\quad \cdots,$$

approaching a limit a. Let the corresponding values of y be

$$y_1,\quad y_2,\quad y_3,\quad y_4,\quad \cdots,\quad y_n,\quad \cdots.$$

Then if there exists a number A, such that the difference between y and A becomes and remains less than any assigned quantity, y is said to approach A as a limit as x approaches a in the manner indicated. This may be seen graphically in fig. 48, where the values of x approaching a are seen on the axis of abscissas and the values of y approaching A are seen on the axis of ordinates. The curve of the function is continually nearer to the line $y=A$.

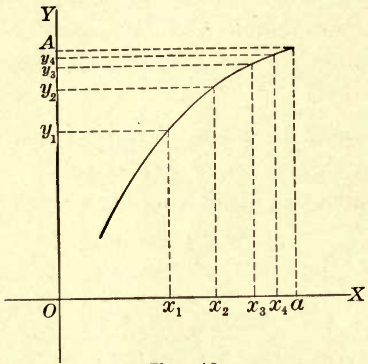

Fig. 48

In the most common cases, the limit of the function depends only upon the limit a of the independent variable and not upon the particular succession of values that x assumes in approaching a. This is clearly the case if the graph of the function is as drawn in fig. 48.

Ex. 1. Consider the function

$$y=\frac{x^2+3x-4}{x-1},$$

and let x approach 1 by passing through the succession of values

$$x=1.1,\quad x=1.01,\quad x=1.001,\quad x=1.0001,\quad \cdots.$$

Then y takes in succession the values

$$y=5.1,\quad y=5.01,\quad y=5.001,\quad y=5.0001.$$

It appears as if y were approaching the limit 5. To verify this, we place $x=1+h$, where h is not zero. By substituting and dividing by h we find $y=5+h$. From this it appears that y can be made as near 5 as we please by taking h sufficiently small, and that for smaller values of h, y is still nearer 5. Hence 5 is the limit of y as x approaches 1. Moreover, it appears that this limit is independent of the succession of values which x assumes in approaching 1.

Ex. 2. Consider $y = \dfrac{x}{1-\sqrt{1-x}}$ as x approaches zero.

Give x in succession the values .1, .01, .001, .0001, $\cdots$. Then y takes the values 1.9487, 1.9950, 1.9995, 1.9999, $\cdots$, suggesting the limit 2.

In fact, by multiplying both terms of $\dfrac{x}{1-\sqrt{1-x}}$ by $1+\sqrt{1-x}$ we find $y = 1+\sqrt{1-x}$ for all values of x except zero.

Hence it appears that y approaches 2 as x approaches 0.

We shall use the symbol $\doteq$ to mean "approaches as a limit." Then the expressions

$$\text{Lim}\, x = a$$

and

$$x \doteq a$$

have the same significance.

The expression $$\underset{x \doteq a}{\text{Lim}}\, f(x) = A$$

is read "the limit of $f(x)$, as x approaches a, is A."

54. Slope of a curve. By means of the conception of a limit we may extend the definition of "slope," given in § 27 for a straight line, so that it may be applied to any curve. For let P_1 and P_2 be any two points upon a curve (fig. 49). If P_1 and P_2 are connected by a straight line, the slope of this line is $\dfrac{y_2 - y_1}{x_2 - x_1}$. If P_2 and P_1 are close enough together, the straight line P_1P_2 will differ only a little from the arc of the curve, and its slope may be taken as approximately the slope of the curve at the point P_1. Now this approximation is closer, the nearer the point P_2 is to P_1. Hence we are led naturally to the following definition:

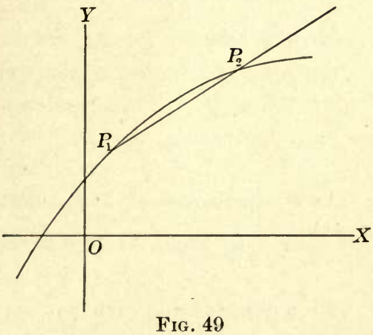

FIG. 49

The slope of a curve at a point $P_1(x_1, y_1)$ is the limit approached by the fraction $\dfrac{y_2 - y_1}{x_2 - x_1}$ where x_2 and y_2 are the coördinates of a second point P_2 on the curve, and where the limit is taken as P_2 moves toward P_1 along the curve.

Ex. 1. Consider the curve $y = x^2$ and the point (5, 25) upon it, and let $x_1 = 5,\ y_1 = 25$.

We take in succession various values for x_2 and y_2 corresponding to points on the curve which are nearer and nearer to (x_1, y_1), and arrange our results in a table as follows :

x_2	y_2	$x_2 - x_1$	$y_2 - y_1$	$\frac{y_2 - y_1}{x_2 - x_1}$
6	36	1	11	11
5.1	26.01	.1	1.01	10.1
5.01	25.1001	.01	.1001	10.01
5.001	25.010001	.001	.010001	10.0001

The arithmetical work suggests the limit 10. To verify this, place $x_2 = 5 + h$. Then $y_2 = 25 + 10h + h^2$. Consequently $\frac{y_2 - y_1}{x_2 - x_1} = 10 + h$, and as x_2 approaches x_1, h approaches 0 and $\frac{y_2 - y_1}{x_2 - x_1}$ approaches 10. Hence the slope of the curve $y = x^2$ at the point (5, 25) is 10.

Ex. 2. Find the slope of the curve $y = \frac{1}{x}$ at the point $(3, \frac{1}{3})$.

We have here $\qquad x_1 = 3, \qquad y_1 = \frac{1}{3}$.

We place $\qquad x_2 = 3 + h,\ y_2 = \frac{1}{3 + h}$.

Then $\quad x_2 - x_1 = h,\ y_2 - y_1 = \frac{-h}{9 + 3h}$, and $\frac{y_2 - y_1}{x_2 - x_1} = -\frac{1}{9 + 3h}$.

As P_2 approaches P_1 along the curve, h approaches 0, and the limit of $\frac{y_2 - y_1}{x_2 - x_1}$ is $-\frac{1}{9}$; hence the slope of the curve at the point $(3, \frac{1}{3})$ is $-\frac{1}{9}$.

In a similar manner we may find the slope of any curve the equation of which is not too complicated; but when the equation is complicated there is need of a more powerful method for finding the limit of $\frac{y_2 - y_1}{x_2 - x_1}$. This method is furnished by the operation known as differentiation, the first principles of which are explained in the following articles.

55. Increment. When a variable changes its value the quantity which is added to its first value to obtain its last value is called its increment. Thus if x changes from 5 to $5\frac{1}{2}$, its increment is $\frac{1}{2}$. If it changes from 5 to $4\frac{3}{4}$, the increment is $-\frac{1}{4}$. So, in general, if x changes from x_1 to x_2, the increment is $x_2 - x_1$. It is customary to denote an increment by the symbol Δ (Greek delta), so that

$$\Delta x = x_2 - x_1, \quad \text{and} \quad x_2 = x_1 + \Delta x.$$

If y is a function of x, any increment added to x will cause a corresponding increment of y. Thus, let $y = f(x)$, and let x change from x_1 to x_2. Then y changes from y_1 to y_2, where

$$y_1 = f(x_1) \quad \text{and} \quad y_2 = f(x_2).$$

Hence $$\Delta y = f(x_2) - f(x_1).$$

But, as shown above, $$x_2 = x_1 + \Delta x,$$

so that $$\Delta y = f(x_1 + \Delta x) - f(x_1).$$

56. Continuity. *A function y is called a continuous function of a variable x when the increment of y approaches zero as the increment of x approaches zero.*

It is clear that a continuous function cannot change its value by a sudden jump, since we can make the change in the function as small as we please by taking the increment of x sufficiently small. As a consequence of this, if a continuous function has a value A when $x = a$, and a value B when $x = b$, it will assume any value C, lying between A and B, for at least one value of x between a and b (fig. 50).

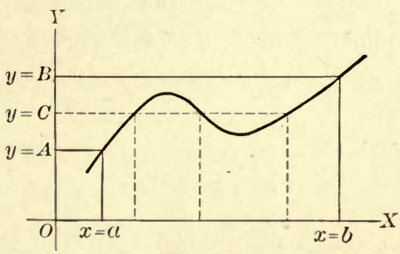

Fig. 50

In particular, if $f(a)$ is positive and $f(b)$ is negative, $f(x) = 0$ for at least one value of x between a and b.

An algebraic polynomial is a continuous function, but we shall omit the proof. The postage function (§ 20) is an example of a function which is discontinuous at certain points. Other examples are found in §§ 149, 154.

When Δx and Δy approach zero together it usually happens that $\frac{\Delta y}{\Delta x}$ approaches a limit. In this case y is said to have a derivative, defined in the next article.

57. Derivative. *When y is a continuous function of x the derivative of y with respect to x is the limit of the ratio of the increment of y to the increment of x, as the increment of x approaches zero.*

The derivative is expressed by the symbol $\frac{dy}{dx}$; or, if y is expressed by $f(x)$, the derivative may be expressed by $f'(x)$.

Thus, if $y = f(x)$,

$$\frac{dy}{dx} = f'(x) = \operatorname*{Lim}_{\Delta x \doteq 0} \frac{\Delta y}{\Delta x} = \operatorname*{Lim}_{\Delta x \doteq 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

The process of finding the derivative is called *differentiation*, and in carrying out the process we are said to differentiate y with respect to x.

The process of differentiation involves, according to the definition, the following four steps:

1. The assumption of an increment of x.
2. The computation of the corresponding increment of y.
3. The division of the increment of y by the increment of x.
4. The determination of the limit approached by this quotient as the increment of x approaches zero.

Ex. 1. Find the derivative of $y = x^3$.

(1) Assume $\Delta x = h$.

(2) Compute $\Delta y = (x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$.

(3) Find $\frac{\Delta y}{\Delta x} = 3x^2 + 3xh + h^2$.

(4) The limit is evidently $3x^2$. Hence $\frac{dy}{dx} = 3x^2$.

Ex. 2. Find the derivative of $\frac{1}{x}$.

(1) Place $y = \frac{1}{x}$ and assume $\Delta x = h$.

(2) Compute $\Delta y = \frac{1}{x+h} - \frac{1}{x} = -\frac{h}{x^2+xh}$.

(3) Find $\frac{\Delta y}{\Delta x} = -\frac{1}{x^2+xh}$.

(4) The limit is clearly $-\frac{1}{x^2}$, and therefore $\frac{dy}{dx} = -\frac{1}{x^2}$.

It appears that the operations of finding the derivative of $f(x)$ are exactly those which are used in finding the slope of the curve $y=f(x)$. Hence the derivative is a function which gives the slope of the curve at each point of it.

58. Formulas of differentiation. The obtaining of a derivative by carrying out the operations of the last article is too tedious for practical use. It is more convenient to use the definition to obtain general formulas which may be used for certain classes of functions. In this article we shall derive all formulas necessary to differentiate a polynomial.

1. $\dfrac{d(ax^n)}{dx}=nax^{n-1}$, where n is a positive integer and a any constant.

Let $$y=ax^n.$$

(1) Assume $\Delta x=h$.

(2) Then
$$\Delta y=a(x+h)^n-ax^n$$
$$=a\left(nx^{n-1}h+\frac{n(n-1)}{\underline{|2}}x^{n-2}h^2+\cdots+h^n\right).$$

(3)
$$\frac{\Delta y}{\Delta x}=a\left(nx^{n-1}+\frac{n(n-1)}{\underline{|2}}x^{n-2}h+\cdots+h^{n-1}\right).$$

(4) Taking the limit, we have $\dfrac{dy}{dx}=nax^{n-1}$.

2. $\dfrac{d(ax)}{dx}=a$, where a is a constant.

This is a special case of the preceding formula, n being here equal to 1. The student may prove it directly.

3. $\dfrac{dc}{dx}=0$, where c is a constant.

Since c is a constant, Δc is always 0, no matter what the value of x. Hence $\dfrac{\Delta c}{\Delta x}=0$, and consequently the limit $\dfrac{dc}{dx}=0$.

4. *The derivative of a polynomial is found by adding the derivatives of the terms in order.*

Let $$y = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

(1) Assume $\Delta x = h$.

(2) Then

$$\begin{aligned}\Delta y &= a_0(x+h)^n + a_1(x+h)^{n-1} + \cdots + a_{n-1}(x+h) + a_n \\ &\quad - [a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n] \\ &= h[na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1}] \\ &\quad + \frac{h^2}{2}[n(n-1)a_0x^{n-2} + (n-1)(n-2)a_1x^{n-3} + \cdots + a_{n-2}] \\ &\quad + \cdots + h^n a_0.\end{aligned}$$

(3) $$\frac{\Delta y}{\Delta x} = na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1} + \frac{h}{2}(\quad) + \cdots + h^{n-1}a_0.$$

(4) Taking the limit, we have

$$\frac{dy}{dx} = na_0x^{n-1} + (n-1)a_1x^{n-2} + \cdots + a_{n-1}.$$

Ex. Find the derivative of

$$f(x) = 6x^5 - 3x^4 + 5x^3 - 7x^2 + 8x - 2.$$

Applying formulas 1, 2, or 3 to each term in order, we have

$$f'(x) = 30x^4 - 12x^3 + 15x^2 - 14x + 8.$$

59. Tangent line. *A tangent to a curve is the straight line approached as a limit by a secant line as two points of intersection of the secant and the curve are made to approach coincidence.*

It is immaterial in what manner the two points of intersection are made to approach coincidence. In § 37 this was done by considering the curve as moved in the plane. In § 88 the secant is considered as moving parallel to itself until it becomes a tangent. In this article we are especially interested in determining a tangent at a known point of the curve. Let us call this point P_1 and a second point on the curve P_2. Then if a secant is drawn through P_1 and P_2 of a curve (fig. 51), and the point P_2 is made to move along the curve toward P_1, which is kept fixed in position, the secant will turn on P_1 as a pivot, and will approach as a limit the tangent P_1T. The point P_1 is called the *point of contact* of the tangent.

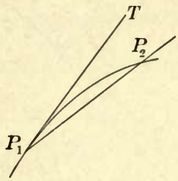

FIG. 51

From the definition it follows that the slope of the tangent is the same as the slope of the curve at the point of contact; for the slope of the tangent is evidently the limit of the slope of the secant, and this limit is the slope of the curve, by § 54.

The equation of the tangent is readily written by means of § 29, when the point of contact is known. For, let (x_1, y_1) be the point of contact, and let $\left(\frac{dy}{dx}\right)_1$ denote the value of $\frac{dy}{dx}$ when $x = x_1$ and $y = y_1$. Then (x_1, y_1) is a point on the tangent and $\left(\frac{dy}{dx}\right)_1$ is its slope. Therefore its equation is

$$y - y_1 = \left(\frac{dy}{dx}\right)_1 \left(x - x_1\right). \tag{1}$$

The equation of the tangent may also be written in terms of the abscissa of the point of contact. Let a be the abscissa of the point of contact of a tangent to a curve $y = f(x)$, and let $f'(x)$ represent as usual the derivative of $f(x)$. Then the ordinate of the point of contact is $f(a)$ and the slope of the tangent is $f'(a)$, in accordance with § 22. Hence the equation of the tangent is

$$y - f(a) = (x - a) f'(a). \tag{2}$$

Ex. 1. Find the equation of the tangent to the curve $y = x^3$ at the point (x_1, y_1) on it.

Using formula (1), we have

$$y - y_1 = 3x_1^2(x - x_1).$$

But since (x_1, y_1) is on the curve, we have $y_1 = x_1^3$. Therefore the equation can be written

$$y = 3x_1^2 x - 2x_1^3.$$

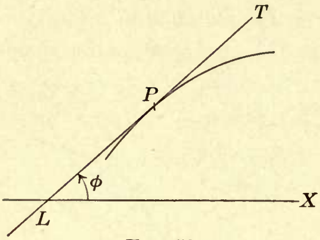

FIG. 52

EX. 2. Find the equation of the tangent to

$$y = x^2 + 3x$$

at the point the abscissa of which is 2.

We will use equation (2). Then

$$f(x) = x^2 + 3x,$$
$$f'(x) = 2x + 3.$$
$$f(2) = 10, \qquad f'(2) = 7.$$

Therefore the equation is

$$y - 10 = 7(x - 2), \quad \text{or} \quad y = 7x - 4.$$

If PT (fig. 52) is a tangent line and ϕ the angle it makes with OX, its slope equals $\tan\phi$, by § 28. Hence $\tan\phi = \frac{dy}{dx}\cdot$

60. Sign of the derivative. A function of x is called an increasing function when an increase in x causes an increase in the function. A function of x is called a decreasing function when an increase in x causes a decrease in the function. The graph of a function runs up toward the right hand when the function is increasing, and runs down toward the right hand when the function is decreasing. Thus $x^2 - x - 6$ (fig. 53) is decreasing when $x < \frac{1}{2}$, and increasing when $x > \frac{1}{2}$.

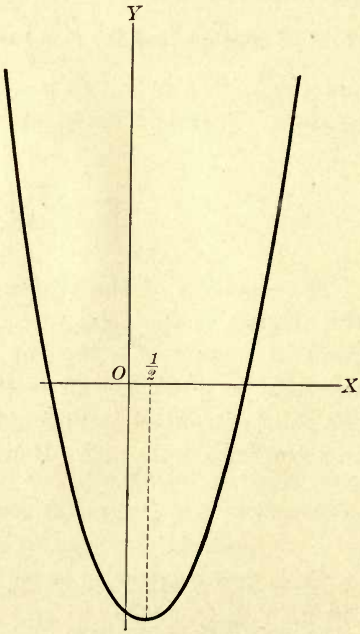

FIG. 53

The sign of the derivative enables us to determine whether a function is increasing or decreasing in accordance with the following theorem:

When the derivative of a function is positive the function is increasing; when the derivative is negative the function is decreasing.

To prove this, consider $y = f(x)$, and let us suppose that $\frac{dy}{dx}$ is positive. Then, since $\frac{dy}{dx}$ is the limit of $\frac{\Delta y}{\Delta x}$, it follows that $\frac{\Delta y}{\Delta x}$

is positive for sufficiently small values of Δx; that is, if Δx is assumed positive, Δy is also positive, and the function is increasing. Similarly, if $\frac{dy}{dx}$ is negative, Δy and Δx have opposite signs for sufficiently small values of Δx, and the function is decreasing by definition.

Ex. 1. If $y = x^2 - x - 6, \frac{dy}{dx} = 2x - 1$, which is negative when $x < \frac{1}{2}$ and positive when $x > \frac{1}{2}$. Hence the function is decreasing when $x < \frac{1}{2}$ and increasing when $x > \frac{1}{2}$, as is shown in fig. 53.

Ex. 2. If $y = \frac{1}{8}(x^3 - 3x^2 - 9x + 27)$,

$$\frac{dy}{dx} = \frac{3}{8}x^2 - \frac{3}{4}x - \frac{9}{8} = \frac{3}{8}(x+1)(x-3).$$

Now $\frac{dy}{dx}$ is positive when $x < -1$, negative when $-1 < x < 3$, and positive when $x > 3$. Hence the function is increasing when $x < -1$, decreasing when x is between -1 and 3, and increasing when $x > 3$ (fig. 54).

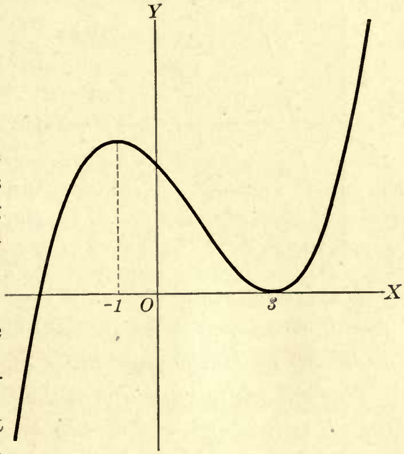

Fig. 54

It remains to examine the cases in which $\frac{dy}{dx} = 0$. Referring to the two examples just given, we see that in each the values of x which make the derivative zero separate those for which the function is increasing from those for which the function is decreasing. The points on the graph which correspond to these zero values of the derivative can be described as turning points.

Likewise, whenever $f'(x)$ is a continuous function of x, the values of x for which the derivative is positive are separated from those for which it is negative by values of x for which it is zero (§ 56). Now in most cases which occur in elementary work $f'(x)$ is a continuous function. Hence we may say,

The values of x for which a function changes from an increasing to a decreasing function are, in general, values of x which make the derivative equal to zero.

The converse proposition is, however, not always true. A value of x for which the derivative is zero is not necessarily a value of x for which the function changes from increasing to decreasing or from decreasing to increasing. For consider

$$\tfrac{1}{3}(x^3 - 9x^2 + 27x - 19).$$

Its derivative is $x^2 - 6x + 9 = (x-3)^2$, which is always positive. The function is therefore always increasing. When $x = 3$ the derivative is zero and the corresponding shape of the graph is shown in fig. 55.

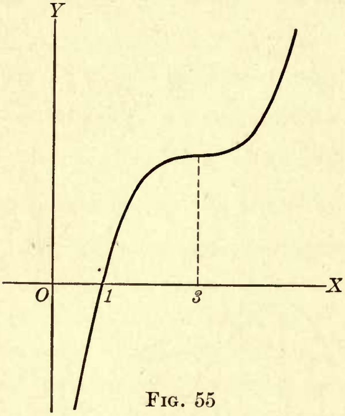

FIG. 55

61. Maxima and minima. The turning points of the graph of a function correspond to the maximum and the minimum values of the function. These terms are more precisely defined as follows:

$f(a)$ is a maximum value of the function $f(x)$ when $f(a \pm h) < f(a)$ for all values of h sufficiently small, i.e. for all values of h numerically less than some finite quantity.

$f(a)$ is a minimum value of the function $f(x)$ when $f(a \pm h) > f(a)$ for all values of h sufficiently small.

In passing through a maximum value the function changes from an increasing to a decreasing function, and in passing through a minimum value the function changes from a decreasing to an increasing function. From the work of the previous article we may accordingly frame the following rule for finding the maxima and the minima values of a function:

Find the derivative of the function, place it equal to zero, and solve the resulting equation. Take each root thus found and see if the derivative has opposite signs as x is taken first a little smaller and then a little larger than the root. If the sign of the derivative changes from plus to minus, the root substituted in the function gives a maximum value of the function. If the sign of the derivative changes from minus to plus, the root substituted in the function gives a minimum value of the function.

This rule is most readily applied when the derivative can be factored. The change of sign is then determined as in § 46. In § 62 will be given a method of distinguishing between a maximum and a minimum, which may be used when the factoring of the derivative is not convenient. In practical problems the question as to whether a value of x for which the derivative is zero corresponds to a maximum or a minimum can often be determined by the nature of the problem.

Ex. 1. Find the maximum and the minimum values of

$$f(x) = x^5 - 5x^4 + 5x^3 + 10x^2 - 20x + 5.$$

We find

$$\begin{aligned} f'(x) &= 5x^4 - 20x^3 + 15x^2 + 20x - 20 \\ &= 5(x^2-1)(x^2-4x+4) \\ &= 5(x+1)(x-1)(x-2)^2. \end{aligned}$$

The roots of $f'(x) = 0$ are -1, 1, and 2. As x passes through -1, $f'(x)$ changes from + to −. Hence $x = -1$ gives $f(x)$ a maximum value, namely 24. As x passes through $+1$, $f'(x)$ changes from − to +. Hence $x = +1$ gives $f(x)$ a minimum value, namely -4. As x passes through 2, $f'(x)$ does not change sign. Hence $x = 2$ gives $f(x)$ neither a maximum nor a minimum value.

Ex. 2. A rectangular box is to be formed by cutting a square from each corner of a rectangular piece of cardboard and bending the resulting figure. The dimensions of the piece of cardboard being 20 by 30 inches, required the largest box which can be found.

Let x be the side of the square cut out. Then if the cardboard is bent along the dotted lines of fig. 56, the dimensions of the box are $30 - 2x$, $20 - 2x$, x. Let y be the volume of the box. Then

$$\begin{aligned} y &= x(20-2x)(30-2x) \\ &= 600x - 100x^2 + 4x^3. \end{aligned}$$

$$\frac{dy}{dx} = 600 - 200x + 12x^2.$$

Equating this to zero, we have

$$3x^2 - 50x + 150 = 0,$$

$$x = \frac{25 \pm 5\sqrt{7}}{3} = 3.9 \text{ or } 12.7.$$

Hence $\dfrac{dy}{dx} = 12(x - 3.9)(x - 12.7)$.

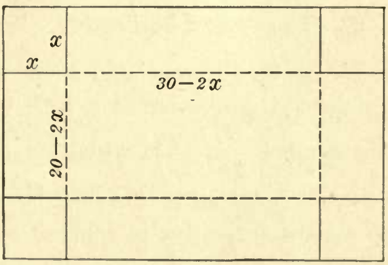

Fig. 56

$\dfrac{dy}{dx}$ changes from + to − as x passes through 3.9. Hence $x = 3.9$ gives the maximum value 1056+ for the capacity of the box. $x = 12.7$ gives a minimum value of y, but this has no meaning in the problem for which x must lie between 0 and 10.

Ex. 3. The deflection of a girder resting on three equally distant supports and loaded uniformly is given by the equation

$$v = C(-l^3x + 3lx^3 - 2x^4),$$

where C is a constant, l the distance between the supports, and x the distance from the middle support. Required the point of maximum deflection.

$$\frac{dv}{dx} = C(-l^3 + 9lx^2 - 8x^3).$$

Equating this to zero, we have

$$8x^3 - 9lx^2 + l^3 = 0.$$

It is clear that in the practical problem $x < l$. We find by trial that a root lies between $x = .4l$ and $x = .5l$. We will place

$$y = 8x^3 - 9lx^2 + l^3,$$

and apply the method of § 52. The straight line connecting $(.4l, .072l^3)$ and $(.5l, -.25l^3)$ is

$$y - .072l^3 = -3.22l^2(x - .4l)$$

and this cuts the axis of x when

$$x = \left(.4 + \frac{.072}{3.22}\right)l = .42l.$$

This is approximately the root of the equation. As a check we note that when $x = .42l$, $y = .005104l^3$; and when $x = .43l$, $y = -.028044l^3$. Hence the root lies between $.42l$ and $.43l$.

If more accuracy is required, the straight line connecting $(.42l, .005104l^3)$ and $(.43l, -.028044l^3)$ may be found. Its intercept on OX is

$$x = .4215l.$$

As shown in § 63, Ex. 2, this is correct to four decimal places.

62. The second derivative. Since $\frac{dy}{dx}$ is in general a function of x, it may be differentiated with respect to x. The result is called the second derivative of y with respect to x, and is indicated by the symbol $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, which is commonly abbreviated into $\frac{d^2y}{dx^2}$. When a function is denoted by $f(x)$ and its derivative by $f'(x)$, its second derivative is denoted by $f''(x)$; thus, if

$$y = f(x) = x^3 - 3x^2 + 6x + 7,$$

$$\frac{dy}{dx} = f'(x) = 3x^2 - 6x + 6,$$

$$\frac{d^2y}{dx^2} = f''(x) = 6x - 6.$$

Again, by differentiating $\frac{d^2y}{dx^2}$ or $f''(x)$, we may obtain an expression called the third derivative, denoted by $\frac{d^3y}{dx^3}$ or $f'''(x)$. By differentiating this we obtain the fourth derivative, and so on. To distinguish $\frac{dy}{dx}$ from these higher derivatives it is sometimes called the first derivative.

The significance of $\frac{d^2y}{dx^2}$ for the graph is obtained from the fact that $\frac{dy}{dx}$ is equal to the slope; hence $\frac{d^2y}{dx^2}$ is the derivative of the slope. Therefore, by § 60, if $\frac{d^2y}{dx^2}$ is positive, the slope is increasing; if $\frac{d^2y}{dx^2}$ is negative, the slope is decreasing. We may have, accordingly, the following four cases:

1. $\frac{dy}{dx}$ is $+$, $\frac{d^2y}{dx^2}$ is $+$.

The graph runs up toward the right with increasing slope (fig. 57).

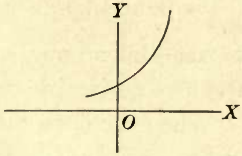

Fig. 57

2. $\frac{dy}{dx}$ is $+$, $\frac{d^2y}{dx^2}$ is $-$.

The graph runs up toward the right with decreasing slope (fig. 58).

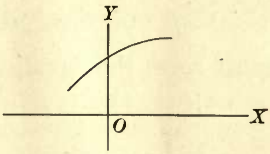

Fig. 58

3. $\frac{dy}{dx}$ is $-$, $\frac{d^2y}{dx^2}$ is $+$.

The graph runs down toward the right. The slope which is negative is increasing algebraically and hence is decreasing numerically (fig. 59).

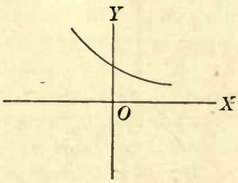

Fig. 59

4. $\frac{dy}{dx}$ is $-$, $\frac{d^2y}{dx^2}$ is $-$.

The graph runs down toward the right and the slope is decreasing algebraically (fig. 60).

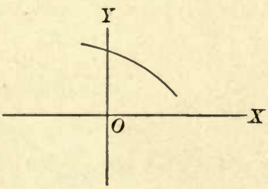

Fig. 60

The consideration of these types leads to the following conclusion: *If* $\frac{d^2y}{dx^2}$ *is positive, the graph is concave upward; if* $\frac{d^2y}{dx^2}$ *is negative, the graph is concave downward.*

From this we may deduce the following rule to distinguish maxima and minima in that we take account of the fact that the graph is concave upward when y is a minimum and concave downward when y is a maximum. *If* $\frac{dy}{dx}$ *is zero and* $\frac{d^2y}{dx^2}$ *is positive, y has a minimum value; if* $\frac{dy}{dx}$ *is zero and* $\frac{d^2y}{dx^2}$ *is negative, y has a maximum value.*

This rule cannot be applied to the case in which $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, and hence it is not so complete as the rule in § 61, but it is sometimes more convenient in application, and especially when the first derivative cannot be factored.

When the curve changes from concavity in one direction to concavity in the other, $\frac{d^2y}{dx^2} = 0$. The corresponding point is called *a point of inflection.* Hence to find the points of inflection we must solve the equation $\frac{d^2y}{dx^2} = 0$, and see if the second derivative changes sign as x passes through each root.

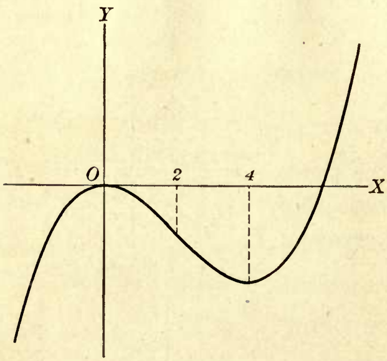

FIG. 61

Ex. 1. $y = \frac{1}{12}(x^3 - 6x^2)$,

$$\frac{dy}{dx} = \frac{1}{4}x^2 - x = \frac{1}{4}x(x-4),$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}x - 1 = \frac{1}{2}(x-2).$$

The curve (fig. 61) is concave downward when $x < 2$, is concave upward when $x > 2$, and has a point of inflection when $x = 2$. When $x = 0$, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$; the corresponding value of y is therefore a maximum. When $x = 4$, $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$; the corresponding value of y is therefore a minimum.

Ex. 2. $y = x^3 + ax = x(x^2 + a)$,

$$\frac{dy}{dx} = 3x^2 + a,$$

$$\frac{d^2y}{dx^2} = 6x.$$

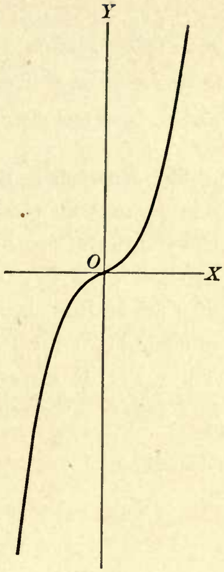

Fig. 62

The curve is concave downward when $x < 0$, is concave upward when $x > 0$, and has a point of inflection when $x = 0$. In addition we distinguish two cases:

(1) a positive. $\frac{dy}{dx}$ is always positive, and the curve cuts OX only at the origin (fig. 62).

(2) a negative. The curve has a maximum ordinate when $x = -\sqrt{-\frac{a}{3}}$, $y = -\frac{2a}{3}\sqrt{-\frac{a}{3}}$, and has a minimum ordinate when

$$x = +\sqrt{-\frac{a}{3}},$$

$$y = +\frac{2a}{3}\sqrt{-\frac{a}{3}}.$$

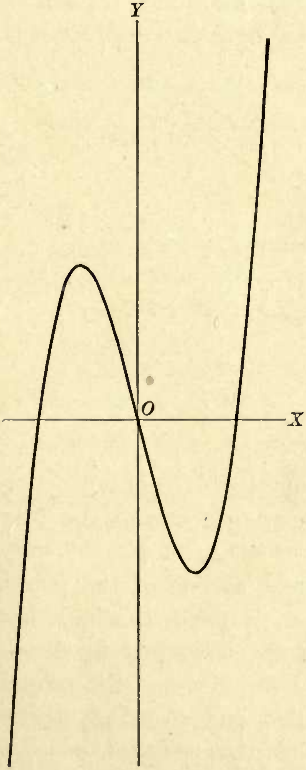

Fig. 63

It cuts OX when $x = -\sqrt{-a}$, 0, or $+\sqrt{-a}$ (fig. 63).

Ex. 3. $y = x^3 + ax + b$.

The graph of this function may be obtained by moving the graph of Ex. 2 through the distance b up or down, according to the sign of b. Our interest is especially with the intercepts on OX. The curve obtained from (1) of Ex. 2 cuts the axis of x in one and only one point. The curve obtained from (2) of Ex. 2 will intersect OX in three points, will intersect OX in one point and be tangent in another, or will intersect OX in one point only, according as the numerical value of b is less than, equal to, or greater than the distance of the turning point of the curve from OX; that is, according as

$$b^2 \lesseqgtr \left(-\frac{2a}{3}\sqrt{-\frac{a}{3}}\right)^2.$$

This condition reduces to

$$\frac{b^2}{4} + \frac{a^3}{27} \lesseqgtr 0.$$

It is to be noticed that when $a > 0$, $\frac{b^2}{4} + \frac{a^3}{27} > 0$. Hence we may cover all cases by the statement:

The equation $x^3 + ax + b = 0$ has three unequal real roots, two equal real roots and one other real root, or one real and two complex roots, according as $\frac{b^2}{4} + \frac{a^3}{27} \lesseqgtr 0$.

63. Newton's method of solving numerical equations. The results of this chapter may be applied to finding approximately the irrational roots of a numerical equation. We first find, by the method of § 47, two numbers x_1 and x_2, between which a root of $f(x) = 0$ is known to lie. It is necessary to take care that neither $f'(x)$ nor $f''(x)$ is zero for any value of x between x_1 and x_2. Then $f(x)$ is always increasing or decreasing between x_1 and x_2, and hence only one root of $f(x) = 0$ lies between x_1 and x_2. Also

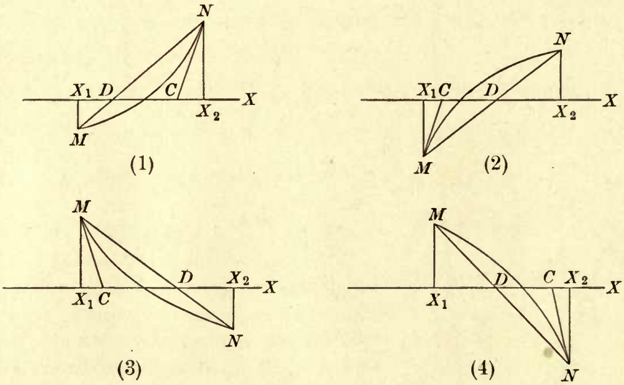

FIG. 64

the curve $y = f(x)$ is always concave upward or concave downward between x_1 and x_2. Hence the curve has one of the four shapes of fig. 64.

It appears that in each case a tangent at one of the points M or N will intersect the axis of x in a point C which lies between x_1 and x_2. In practice it is most convenient to sketch the curve with attention to the signs of the first and the second derivative, and to find the tangent at that end at which it lies between the curve and the ordinate of the point of contact. The intersection of the tangent with OX is then nearer to the intersection of the curve, i.e. to the required root of the equation, than is the abscissa of the point of contact. For example, in fig. 64, (1) and (4), the equation of the tangent is

$$y-f(x_2)=(x-x_2)\cdot f'(x_2),$$

and its point of intersection with OX is $x_2-\frac{f(x_2)}{f'(x_2)}\cdot$ Hence the root which was at first known to lie between x_1 and x_2 is now known to lie between x_1 and $x_2-\frac{f(x_2)}{f'(x_2)}\cdot$

It is well in practice to combine this method with the method of § 52. For, if we draw the secant MN, it will intersect the axis of x in a point D, and the root of the equation lies between C and D. But C and D are closer together than are x_1 and x_2, so that we have narrowed down the interval within which the root lies.

Ex. 1. Find the root of $x^3-6x-13=0$, which lies between 3 and 4.

Here

$$f(x)=x^3-6x-13,$$
$$f'(x)=3x^2-6,$$
$$f''(x)=6x.$$

When $x=3, f(x)=-4$; and when $x=4, f(x)=27$; while between $x=3$ and $x=4$, $f'(x)$ and $f''(x)$ are positive. Hence the graph is as in fig. 64, (1), where M is $(3, -4)$ and N is $(4, 27)$. The tangent at N is

$$y-27=42(x-4).$$

Hence, for C, $\qquad x=4-\frac{27}{42}=3.36.$

The equation of MN is $\qquad y-27=31(x-4).$

Hence, for D, $\qquad x=4-\frac{27}{31}=3.13.$

Therefore the root lies between 3.13 and 3.36.

As this does not fix the first decimal figure of the root, it is advisable to apply § 47 again. We find $f(3.1)=-1.809$ and $f(3.2)=+.568$. Hence the root lies between 3.1 and 3.2. Accordingly, the point M is now $(3.1, -1.809)$, and the point N is $(3.2, .568)$. The equation of the tangent at N is

$$y-.568=24.72(x-3.2),$$

and for the new point C $\qquad x=3.17702.$

The secant MN is $\qquad y-.568=23.77(x-3.2)$

and for D $\qquad x=3.176.$

The root of the equation therefore lies between 3.176 and 3.177. This result is close enough for most practical purposes, but if the operations are carried out once more it is found that the root lies between 3.1768143 and 3.1768144.

Ex. 2. In § 61, Ex. 3, the root of $8x^3 - 9lx^2 + l^3 = 0$ was found to lie between $.42\,l$ and $.43\,l$.

Placing $$f(x) = 8x^3 - 9lx^2 + l^3,$$
we have $$f'(x) = 24x^2 - 18lx,$$
$$f''(x) = 48x - 18l,$$

so that $f'(x)$ is negative and $f''(x)$ positive, when x is between $.42\,l$ and $.43\,l$. Hence the curve has the shape of fig. 64, (3). The tangent at $(.42\,l, .005104\,l^3)$ meets OX where $x = .42153\,l$. The chord connecting $(.42\,l, .005104\,l^3)$ and $(.43\,l, -.028044\,l^3)$ meets OX where $x = .42154\,l$. The root is therefore determined to four decimal places.

64. Multiple roots of an equation.

If $$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n,$$
$$f'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots$$
$$+ 2a_{n-2}x + a_{n-1},$$
$$f''(x) = n(n-1)a_0x^{n-2} + (n-1)(n-2)a_1x^{n-3}$$
$$+ (n-2)(n-3)a_2x^{n-4} + \cdots + 2a_{n-2},$$
$$f'''(x) = n(n-1)(n-2)a_0x^{n-3}$$
$$+ (n-1)(n-2)(n-3)a_1x^{n-4} + \cdots,$$

and so on. Now let $f(a), f'(a), f''(a), f'''(a)$, etc., denote the result of placing $x = a$ in these functions, and $f(a+h)$ denote the result of placing $x = a + h$ in $f(x)$. One readily computes that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \frac{h^3}{\lfloor 3}f'''(a) + \cdots + a_0h^n. \quad (1)$$

In (1) place $h = x - a$ and it becomes

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2}f''(a)$$
$$+ \frac{(x-a)^3}{\lfloor 3}f'''(a) + \cdots + a_0(x-a)^n. \quad (2)$$

If now a is a double root of $f(x) = 0$, $f(x)$ is divisible by $(x-a)^2$, by § 42, and therefore, by (2), $f(a) = 0$, $f'(a) = 0$. If a is a triple root of $f(x) = 0$, $f(x)$ is divisible by $(x-a)^3$, and therefore $f(a) = 0$, $f'(a) = 0$, $f''(a) = 0$. Similar statements may be made for multiple roots of higher order.

Conversely, if $f(a) = 0$ and $f'(a) = 0$, (2) shows that $f(x)$ is certainly divisible by $(x - a)^2$ and perhaps by a higher power of $x - a$. Therefore a is a multiple root of $f(x) = 0$. We have then the result:

A multiple root of $f(x) = 0$ is also a root of $f'(x) = 0$, and conversely.

Hence we may find the multiple roots of $f(x) = 0$ by equating to zero the highest common factor of $f(x)$ and $f'(x)$ and solving the resulting equation.

The condition that an equation $f(x) = 0$ should have multiple roots is the vanishing of the *discriminant* of the equation, which is the eliminant of the equations $f(x) = 0$ and $f'(x) = 0$, and may be found by the method of § 9.

Ex. 1. Find the discriminant of $ax^2 + bx + c = 0$.

We have to find the condition that the two equations

$$ax^2 + bx + c = 0$$

and

$$2ax + b = 0$$

should have a common root. Multiplying the last equation by x, we have

$$2ax^2 + bx = 0,$$

and the determinant of the coefficients and the absolute terms of the three equations is

$$\begin{vmatrix} a & b & c \\ 0 & 2a & b \\ 2a & b & 0 \end{vmatrix} = 0,$$

or

$$b^2 - 4ac = 0.$$

Ex. 2. Find the discriminant of $x^3 + ax + b = 0$.

We must find the eliminant of this and

$$3x^2 + a = 0.$$

Multiplying the first equation by x, and the second by x and x^2, we have the five equations

$$\begin{aligned} x^4 \quad + ax^2 + bx \quad &= 0, \\ x^3 \quad + ax + b &= 0, \\ 3x^4 \quad + ax^2 \quad &= 0, \\ 3x^3 \quad + ax \quad &= 0, \\ 3x^2 \quad + a &= 0, \end{aligned}$$

and their eliminant is

$$\begin{vmatrix} 1 & 0 & a & b & 0 \\ 0 & 1 & 0 & a & b \\ 3 & 0 & a & 0 & 0 \\ 0 & 3 & 0 & a & 0 \\ 0 & 0 & 3 & 0 & a \end{vmatrix} = 0,$$

or

$$4a^3 + 27b^2 = 0.$$ (See § 62, Ex. 3.)

PROBLEMS

Find the respective slopes of the following curves at the points noted: (1) by an approximate numerical calculation, as in § 54; (2) by placing x equal to the abscissa of the given point, plus h, and allowing h to approach zero:

1. $y = x^3$ at (2, 8).
2. $y = x^2 - 3x$ at (0, 0).
3. $y = x^3 - 3x + 1$ at $(1, -1)$.

4. Find the derivative of $x^5 - x$ by using the definition but not the formulas.

5. Find the derivative of $3x^4 + 2x$ by using the definition but not the formulas.

Find the derivative of each of the following expressions by the formulas:

6. $\frac{1}{3}x^6 - \frac{2}{5}x^5 + x$.
7. $4x^3 - 6x^2 + 5x - 8$.
8. $5x^9 - 6x^8 + 7x^6 - 4x^4 - 2x^2 + 3x - 9$.

9. By expanding and differentiating show that the derivative of $(3x + 2)^4$ is $12(3x + 2)^3$.

10. By expanding and differentiating show that the derivative of $(x + a)^n$ is $n(x + a)^{n-1}$.

11. Find the equation of the tangent to the curve $y = x^4 + 3$ at the point the abscissa of which is -2.

12. Show that the equation of the tangent to the curve $y = x^3 + ax + b$ at the point (x_1, y_1) is $y = (3x_1^2 + a)x - 2x_1^3 + b$.

13. Show that the equation of the tangent to the curve $y = ax^2 + 2bx + c$ at the point (x_1, y_1) is $y = 2(ax_1 + b)x - ax_1^2 + c$.

14. Determine the point of intersection of the tangents to the curve $y = x^3 - 5x + 7$ at the points the abscissas of which are -2 and 3 respectively.

15. Find the angle between the tangents to the curve $y = 2x^2 - 3x + 1$ at the points the abscissas of which are -1 and 2 respectively.

16. Find the area of the triangle included between the coördinate axes and the tangent to the curve $y = x^3$ at the point (2, 8).

17. Find the points on the curve $y = x^3 - 3x + 7$ at which the tangents are parallel to the line $y = 9x + 3$.

18. How many tangents has the curve $y = x^3 - 4x^2 + x - 4$ which are parallel to the line $y + 4x + 7 = 0$? Find their equations.

19. Find the points on the curve $y = x^3 + x^2 - 6$ at which it makes an angle of 45° with OX.

Find the values of x for which the following expressions are respectively increasing and decreasing:

20. $x^2 + 4x - 7$.

21. $x^3 - 2x^2 + 8$.

22. $x^4 + 8x - 10$.

23. $x^4 - 2x^2 + 6$.

24. Find the lowest point of the curve $y = 3x^2 - 8x + 7$.

25. Find the turning points of the curve $y = \frac{1}{4}x^4 - 2x^2 + \frac{1}{4}$.

Find the maximum and the minimum values of the following expressions:

26. $3x^3 - 2x^2 - 5x + 1$.

27. $3x^5 - 25x^3 + 60x - 50$.

28. Prove that the largest rectangle with a given perimeter is a square.

29. A rectangular piece of cardboard a in. long and b in. broad has a square cut out of each corner. Find the length of a side of this square when the box formed from the remainder has its greatest volume.

30. Find the dimensions of the greatest rectangle which can be inscribed in a given isosceles triangle with base b and altitude h.

31. Find the right circular cylinder of greatest volume which can be inscribed in a sphere of radius a.

32. Find the right circular cylinder of greatest volume which can be cut from a given right circular cone.

33. Find the point of the line $3x + y = 6$ such that the sum of the squares of its distances from the two points (5, 1) and (7, 3) may be a minimum.

34. Among all circular sectors with a given perimeter find the one which has the greatest area.

35. A rectangular box with a square base and open at the top is to be made out of a given amount of material. If no allowance is made for thickness of material or waste in construction, what are the dimensions of the largest box that can be made?

36. A length l of wire is to be cut into two portions, which are to be bent into the forms of a circle and a square respectively. Show that the sum of the areas of these figures will be least when the wire is cut in the ratio $\pi : 4$.

37. A piece of galvanized iron b ft. long and a ft. wide is to be bent into a U-shaped water pipe b ft. long. If we assume that the cross section of the pipe is exactly represented by a rectangle on top of a semicircle, what are the dimensions of the rectangle and the semicircle that the pipe may have the greatest capacity; (1) when the pipe is closed on top? (2) when it is open on top?

38. A stream flowing with the velocity a strikes an undershot water wheel, giving it the velocity x. Assuming that the efficiency of the wheel is proportional to the velocity x of the wheel and the loss of velocity $a - x$ of the water, what is the velocity of the wheel when it has its greatest efficiency?

39. A gardener has a certain length of wire fencing with which to fence three sides of a rectangular plot of land, the fourth side being made by a wall already constructed. Required the dimensions of the plot which contains the maximum area.

40. For a continuous girder of uniform section, uniformly loaded, and consisting of three equal spans, the deflection in the middle span is given by the equation $v = C\,(l^3x - 6\,l^2x^2 + 10\,lx^3 - 5\,x^4)$, where C is constant, l the length of the span, and x the distance from a point of support. Find the maximum deflection.

41. If ρ is the density of water and t the temperature between 0° and 30° C., $\rho = \rho_0(1 + lt + mt^2 + nt^3)$, where ρ_0 is the density when $t = 0$, and $l = .000052939$, $m = -.0000065322$, $n = .00000001445$. Show that the maximum density occurs when $t = 4.108°$.

42. Show that the curve $y = ax^2 + bx + c$ is concave upward or downward according as a is positive or negative.

43. Show that the curve $y = x^3 + ax + b$ is concave upward when x is positive and concave downward when x is negative.

Determine the values of x for which the following curves are concave upward or downward:

44. $y = x^3 - 3\,x^2 - 24.$

45. $y = x^5 - 5\,x + 6.$

Find the points of inflection of the following curves:

46. $6\,y = x^3 - 6\,x^2 + 6\,x + 1.$

47. $12\,y = x^4 - 6\,x^3 + 12\,x^2 - 2\,x + 1.$

48. $y = 3\,x^5 - 10\,x^4 + 10\,x^3 + 6\,x - 8.$

49. $y = 3\,x^5 - 5\,x^4 + 20\,x^3 - 60\,x^2 + 20\,x - 5.$

50. Prove that the curve $y = ax^3 + bx^2 + cx + d$ always has one and only one point of inflection.

Find the real roots of the following equations accurate to two decimal places:

51. $x^3 - x^2 - 2\,x + 1 = 0.$

52. $x^3 + 3\,x^2 + 4\,x + 5 = 0.$

53. $x^3 - 2\,x - 5 = 0.$

54. $x^3 - 3\,x^2 - 2\,x + 5 = 0.$

55. $x^4 - x^3 - x^2 + x - 1 = 0.$

Show that each of the following equations has equal roots and solve it:

56. $x^3 - x^2 - 8\,x + 12 = 0.$

57. $x^4 - 2\,(1 - a)\,x^3 + (1 - 3\,a)\,x^2 + a = 0.$

Find the condition that each of the following equations should have equal roots:

58. $x^3 + 3\,ax^2 + b = 0.$

59. $x^4 + 4\,ax^3 + b = 0.$

60. $x^4 + 4\,ax + b = 0.$

61. $a_0x^3 + 3\,a_1x^2 + 3\,a_2x + a_3 = 0.$

CHAPTER VI

CERTAIN ALGEBRAIC FUNCTIONS AND THEIR GRAPHS

65. Square roots of polynomials. In the previous chapters the discussion has been restricted to the polynomial. We will next study the square root of the polynomial.

At first let us assume that the polynomial can be separated into n linear real factors, as in § 42. We have, then,

$$y = \pm\sqrt{a_0(x - r_1)(x - r_2)\cdots(x - r_n)}, \tag{1}$$

and the graph of this function can readily be constructed by considering the graph of

$$y = a_0(x - r_1)(x - r_2)\cdots(x - r_n), \tag{2}$$

as given in § 46.

In the first place, the graph of (1) will intersect the axis of x in the same points as the graph of (2), i.e. in the points $x = r_1$, $x = r_2$, $\cdots$, as for these values of x the product under the radical sign is zero.

In the second place, wherever the graph of (2) is below the axis of x, the expression under the radical sign in (1) is negative, the value of the radical is imaginary, and hence there is no corresponding point of the graph. If, however, the graph of (2) is above the axis of x, there are two values of y in (1), equal in magnitude and opposite in sign, and correspondingly there are two points of the graph situated symmetrically with respect to OX. Therefore *OX is an axis of symmetry.*

As the negative values of the expression under the radical sign are separated from the positive values by zero, it follows that the values of x which make the expression zero, i.e. $r_1, r_2, \cdots, r_n$, are of the utmost importance in plotting these graphs. In fact, the lines $x = r_1$, $x = r_2$, $\cdots$, $x = r_n$ divide the plane into sections bounded by straight lines parallel to OY, in which there will be no part of the graph if the corresponding values of x make the expression negative, and in which there will be a part of the graph if the corresponding values of x make the expression positive. Hence the first step in plotting the graph is the drawing of these lines and the determination of which sections of the plane should be considered.

Ex. 1. $y = \pm\sqrt{(x+2)(x-1)(x-5)}$.

If $x = -2$, 1, or 5, $y = 0$, and the graph intersects the axis of x at three points. The lines $x = -2$, $x = 1$, $x = 5$ divide the plane (fig. 65) into four sections.

If $x < -2$, all three factors of the product are negative; hence the radical is imaginary and there can be no part of the graph in the corresponding section of the plane.

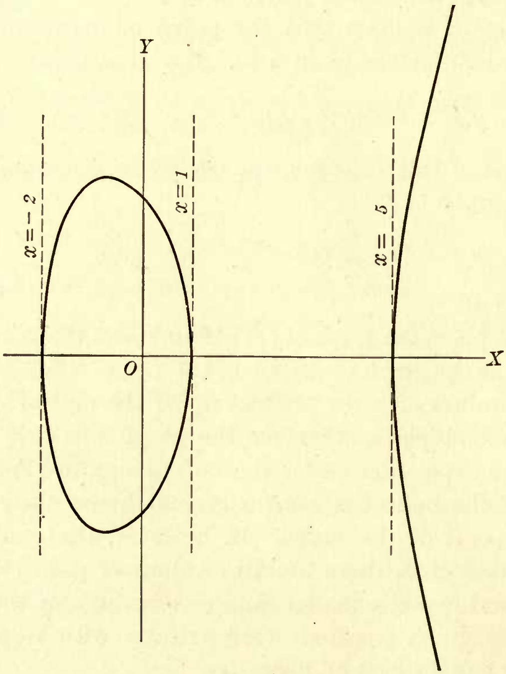

Fig. 65

If $-2 < x < 1$, the first factor is positive and the other two are negative; hence the radical is real and there is a part of the graph in the corresponding section of the plane.

If $1 < x < 5$, the first two factors are positive and the third is negative; hence the radical is imaginary and there can be no part of the graph in the corresponding section of the plane.

Finally, if $x > 5$, all three factors are positive; hence the radical is real and there is a part of the graph in the corresponding section of the plane.

Therefore the graph consists of two separate parts, and is seen (fig. 65) to consist of a closed loop and a branch of infinite length.

Ex. 2. $y = \pm\sqrt{(x+4)(x+2)(x-1)(x-4)}$.

If $x = -4$, -2, 1, or 4, $y = 0$, and the graph intersects the axis of x at four points.

The lines $x = -4$, $x = -2$, $x = 1$, and $x = 4$ divide the plane (fig. 66) into five sections.

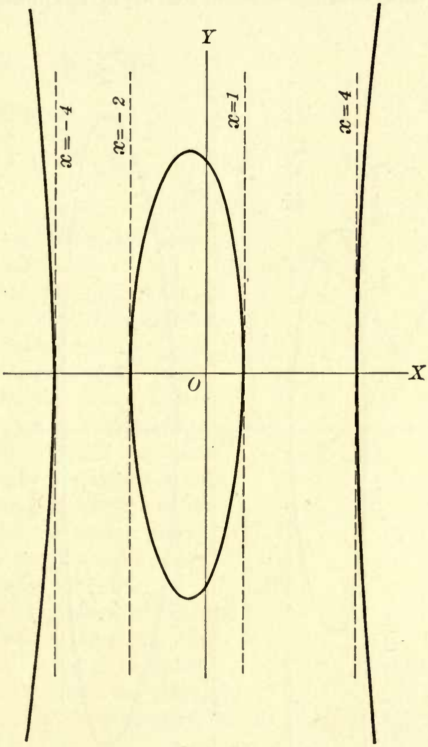

Fig. 66

If $x < -4$, all four factors are negative; hence the radical is real and there is a part of the graph in the first section.

If $-4 < x < -2$, the first factor is positive and the others are negative; hence the radical is imaginary and there can be no part of the graph in the second section.

If $-2 < x < 1$, the first two factors are positive and the other two are negative; hence the radical is real and there is a part of the graph in the third section.

If $1 < x < 4$, the first three factors are positive and the last is negative; hence the radical is imaginary and there can be no part of the graph in the fourth section.

Finally, if $x > 4$, all the factors are positive; hence the radical is real and there is a part of the graph in the fifth section.

In this example we see that the graph consists of three separate parts, and is seen (fig. 66) to consist of a closed loop and two infinite branches.

Ex. 3. $y = \pm\sqrt{-(x+4)(x+2)(x-1)(x-4)}$.

The plane is divided into five sections (fig. 67) by the lines $x = -4$, $x = -2$, $x = 1$, and $x = 4$.

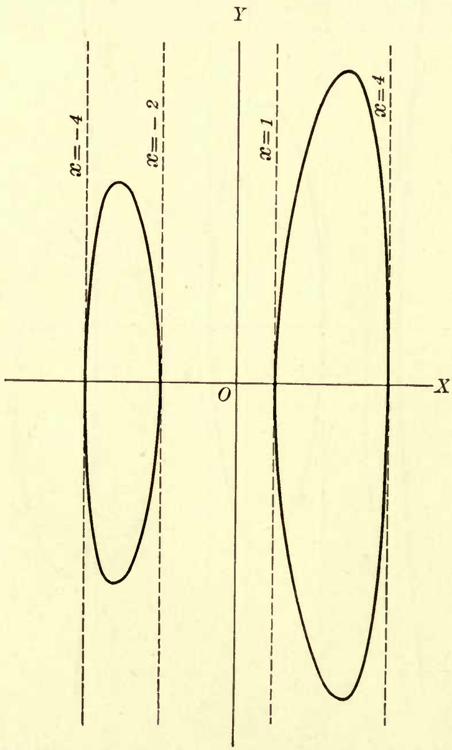

FIG. 67

Proceeding as in the previous two examples, we find y to be real if $-4 < x < -2$, or $1 < x < 4$, and to be imaginary for all other values of x. Therefore the graph consists of two separate parts, and is seen (fig. 67) to consist of two closed loops.

66. In the examples of the last article no two factors were alike, i.e. no factor occurred more than once. If any factor does occur more than once, only its first power will be left under the radical sign, or, to put it more generally, no perfect square will be left as a factor under the radical sign. As a result, there will be before the radical a factor involving x, and the presence of this factor will of necessity change the course of the reasoning to some extent, as is shown in the following examples.

Ex. 1. $y = \pm\sqrt{(x+2)(x-1)^2}$.

This will be written as

$$y = \pm(x-1)\sqrt{x+2}.$$

The line $x = -2$ divides the plane (fig. 68) into two sections.

Proceeding as in the previous examples, we find the radical to be real if $x > -2$ and imaginary if $x < -2$. Therefore there is a part of the graph to the right of the line $x = -2$, but there can be no part of the graph to the left of that line unless x can have such value as to make the coefficient of the radical zero; and this coefficient is zero only when x equals unity. Hence all of the graph lies to the right of the line $x = -2$, as shown in fig. 68.

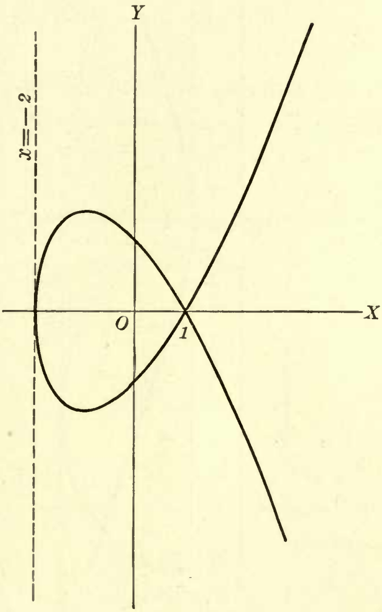

FIG. 68

Comparing this example with Ex. 1 of § 65, we see that by changing the factor $x-5$ to $x-1$ we have joined the infinite branch and the loop, making a single curve crossing itself at the point (1, 0).

Ex. 2. $y = \pm\sqrt{(x+2)^2(x-1)} = \pm(x+2)\sqrt{x-1}$.

The line $x = 1$ divides the plane (fig. 69) into two sections.

If $x > 1$, the radical is real and there is a part of the graph in the corresponding section of the plane. If $x < 1$, the radical is imaginary and there will be no points of the graph except for such values of x as make the coefficient of the radical zero. There is but one such value, i.e. -2, and therefore there is but one point of the graph, i.e. $(-2, 0)$, to the left of the line $x = 1$. The graph consists, then (fig. 69), of the isolated point A and the infinite branch.

Comparing this example also with Ex. 1 of § 65, we see that by changing the factor $x - 5$ to $x + 2$ we have reduced the loop to a single point, leaving the infinite branch as such.

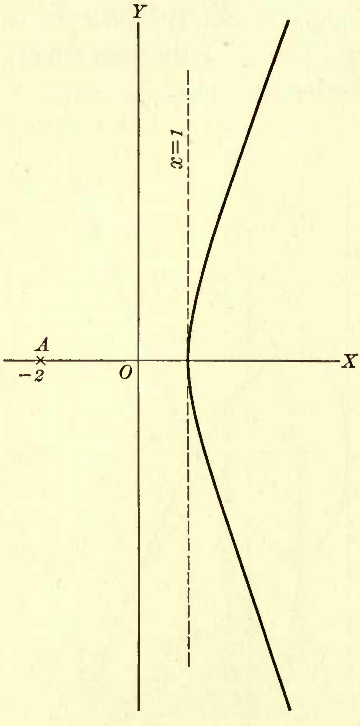

FIG. 69

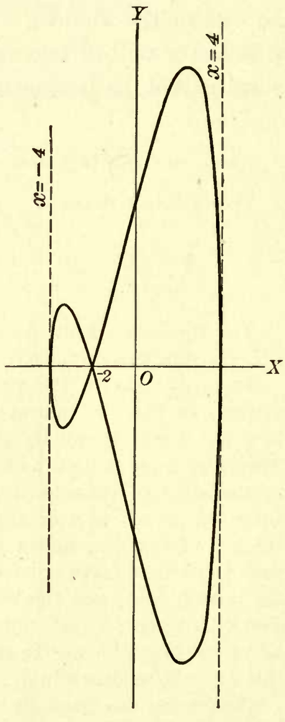

FIG. 70

Ex. 3. $y = \pm\sqrt{-(x+4)(x+2)^2(x-4)} = \pm(x+2)\sqrt{-(x+4)(x-4)}$.

The lines $x = -4$ and $x = 4$ divide the plane (fig. 70) into three sections.

If $-4 < x < 4$, the radical is real and there is a part of the graph in the corresponding portion of the plane. If $x < -4$ or $x > 4$, the radical is imaginary; and since in the corresponding sections there is no value of x which makes $x + 2$ zero, there can be no part of the graph in those sections. It is represented in fig. 70.

Comparing this example with Ex. 3 of § 65, we see that the changing of $x - 1$ to $x + 2$ has brought the two loops together, forming a single closed curve crossing itself at the point $(-2, 0)$.

67. Functions defined by equations of the second degree in y. If we have given an algebraic equation involving both y and x, y is thereby defined as a function of x. For if x is assigned any value, the corresponding values of y are determined by means of the equation. In particular, if the equation involves no power of y higher than the second, it may be readily solved for y, and the work of finding the graph is similar to that already done.

In many important cases the solution of the equation is of the form

$$y = c \pm \sqrt{(x - r_1)(x - r_2) \cdots}.$$

Comparing this case with the previous one, we see that $y = c$ is an axis of symmetry instead of $y = 0$, and that in all other respects the work is similar.

Ex. $2x^2 + y^2 + 3x - 4y - 5 = 0.$

Solving for y, we have

$$y = 2 \pm \sqrt{-2x^2 - 3x + 9},$$

or, after the expression under the radical sign has been factored,

$$y = 2 \pm \sqrt{-2(x - \tfrac{3}{2})(x + 3)}.$$

The lines $x = -3$ and $x = \frac{3}{2}$ divide the plane (fig. 71) into three sections, and, proceeding as before, we find that the curve is entirely in the middle section, i.e. when $-3 < x < \frac{3}{2}$, and that the line $y = 2$ is an axis of symmetry.

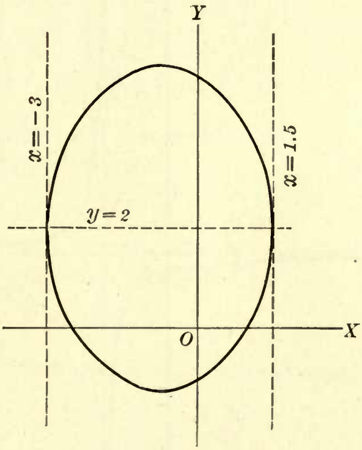

FIG. 71

In case the given equation is of higher degree in y than the second, but of the first or the second degree in x, it is evident that we can solve for x in terms of y and proceed as above, working from the y axis instead of the x axis.

It should be added that given any equation in x and y, since either may be regarded as the independent variable and the other as the function, we have perfect freedom of choice to solve for y in terms of x, or for x in terms of y, according to convenience.

68. Functions involving fractions. If the expression defining a function contains fractions, the function is not defined for a value of x which makes the denominator of any fraction zero (§ 11). But if $x = a$ is a value which makes the denominator zero, but not the numerator, and x is allowed to approach a as a limit, the value of the function increases indefinitely and is said to become infinite. The graph of a function then runs up or down indefinitely, approaching the line $x = a$ indefinitely near, but never reaching it. We have thus a graphical representation of the discussion of infinity in § 11.

When a function becomes infinite it is *discontinuous* (§ 56). In fact, this is the only kind of discontinuity which can occur in an algebraic function.

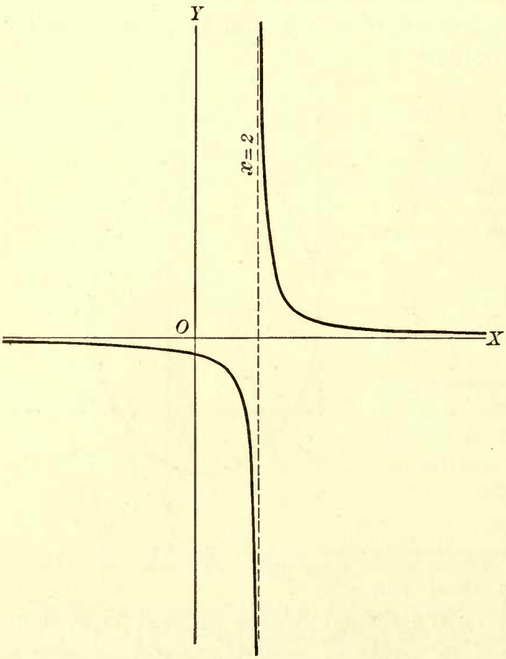

FIG. 72

Ex. 1. $y = \dfrac{1}{x-2}$.

It is evident that y is real for all values of x; also if $x < 2$, y is negative, and if $x > 2$, y is positive. Moreover, as x increases toward 2, y is negative and becomes indefinitely great; while as x decreases toward 2, y is positive and becomes indefinitely great. We can accordingly assign all values to x except 2, that value being excluded by § 11. The curve is represented in fig. 72.

It is seen that the nearer to 2 the value assigned to x, the nearer the corresponding point of the curve to the line $x = 2$. In fact, we can make this distance as small as we please by choosing an appropriate value for x. At the same time the point recedes indefinitely from OX along the curve.

Now *when a straight line has such a position with respect to a curve that as the two are indefinitely prolonged the distance between them approaches zero as a limit, the straight line is called an asymptote of the curve.*

It follows from the above definition that the line $x = 2$ and also the line $y = 0$ are asymptotes of this curve. In this example it is to be noted that the asymptote $x = 2$ is determined by the value of x which makes the function infinite.

It is clear that all equations of the type

$$y = \frac{1}{x - a}$$

represent curves of the same general shape as that plotted in fig. 72.

Ex. 2. $y = \frac{1}{x + 2} + \frac{1}{x - 2}$.

If $x = -2$ or if $x = 2$, y is infinite; hence these two values may not be assigned to x, all other values, however, being possible. The curve is represented in fig. 73.

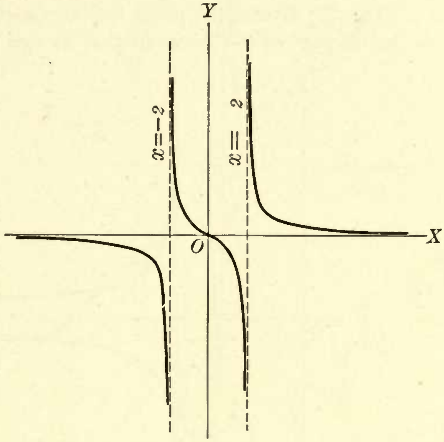

FIG. 73

By a discussion similar to that of Ex. 1, it may be proved that the lines $x = -2$ and $x = 2$, which correspond to the values of x which make the function infinite, and also the line $y = 0$, are asymptotes of the curve.

This curve is a special case of that represented by

$$y = \frac{1}{x - a} + \frac{1}{x - b};$$

and it is not difficult to see how the curve represented by

$$y = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c} + \cdots$$

will look for any number of terms.

Ex. 3. $y = \frac{1}{(x - 2)^2}$.

All values of x may be assumed except 2. The curve is represented in fig. 74. It is evident that the lines $x = 2$ and $y = 0$ are asymptotes.

This curve is a special case of that represented by

$$y = \frac{1}{(x - a)^2},$$

which is itself a special case of

$$y = \frac{1}{(x - a)^2} + \frac{1}{(x - b)^2} + \cdots.$$

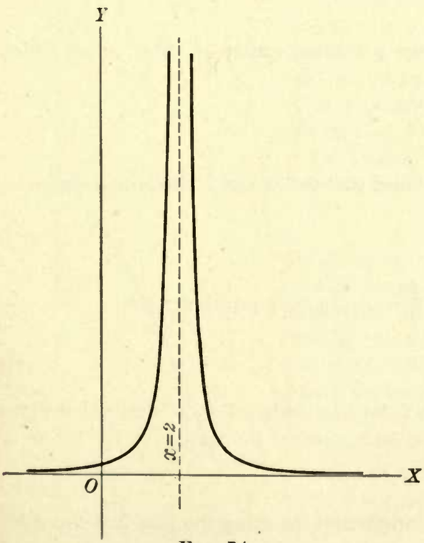

FIG. 74

Ex. 4. $y^2 = \dfrac{1}{x-3}$.

As in § 67, we solve for y, forming the equation $y = \pm\sqrt{\dfrac{1}{x-3}}$. The line $x = 3$ (fig. 75) divides the plane into two sections, and it is evident that there can be no part of the curve in that section for which $x < 3$. Moreover, this

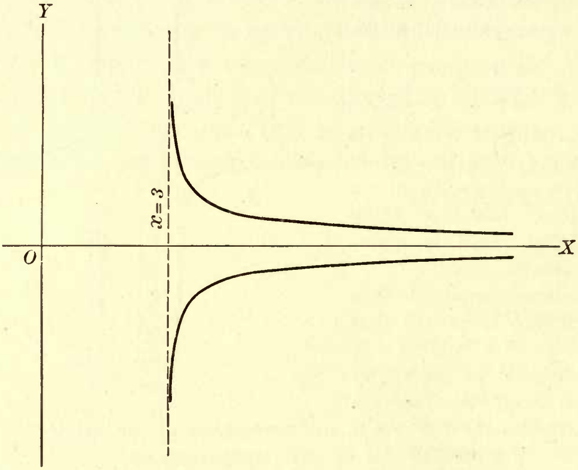

FIG. 75

line $x = 3$ is an asymptote, as in the preceding examples. The curve, which is a special case of that represented by

$$y^2 = \frac{1}{x-a},$$

is represented in fig. 75. It is to be noted that the axis of x also is an asymptote.

Ex. 5. $y = \dfrac{x^2+1}{x}$.

To plot this curve we write the equation in the equivalent form

$$y = x + \frac{1}{x}. \tag{1}$$

It is evident that all values except 0 may be assigned to x, that value being excluded as it makes y infinite. Let us also draw the line

$$y = x, \tag{2}$$

a straight line passing through the origin and bisecting the first and the third quadrants.

Comparing equations (1) and (2), we see that if any value x_1 is assigned to x, the corresponding ordinates of (1) and (2) are respectively $x_1 + \frac{1}{x_1}$ and x_1, and that they differ by $\frac{1}{x_1}$. Moreover, the numerical value of this difference decreases as greater numerical values are assigned to x_1, and it can be made less than any assigned quantity however small by taking x_1 sufficiently great. It follows that the line $y = x$ is an asymptote of the curve. It is also evident that the line $x = 0$, determined by the value of x which makes the function infinite, is an asymptote. The curve is represented in fig. 76.

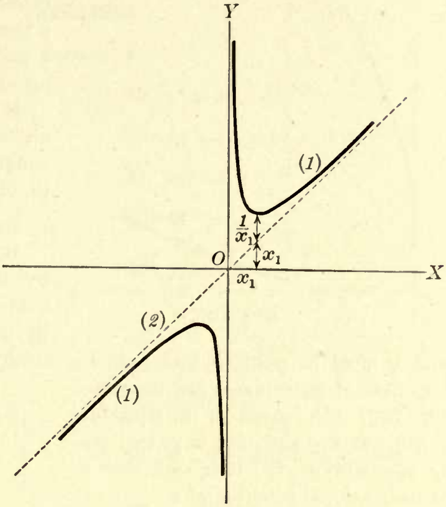

FIG. 76

69. Special irrational functions.

Ex. 1. $y^2 = x^3$.

Writing this equation in the form $y = \pm x\sqrt{x}$, we see that y is an irrational function of x, and that its graph is symmetrical with respect to OX and lies entirely to the right of the axis y. It is represented in fig. 77, and is called the semicubical parabola.

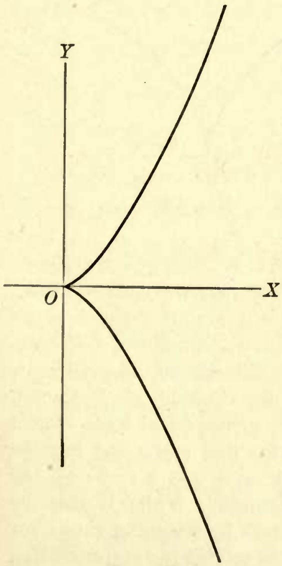

FIG. 77

In general, if the equation expressing the function is of the form

$$y = kx^n,$$

the function is *rational* or *irrational* according as n is integral or fractional. In § 38 we have plotted the graphs of some of the rational functions of this type for the special case when $k = 1$ and n has the values 3, 4, and 5 respectively. Above we have just plotted the graph of one of the irrational functions, i.e. when $n = \frac{3}{2}$.

The graphs of the irrational functions $y = x^{\frac{1}{3}}$, $y = x^{\frac{1}{4}}$, and $y = x^{\frac{1}{5}}$ may be obtained by assuming values for x and plotting as above, or by rewriting the equations in the forms $x = y^3$, $x = y^4$, and $x = y^5$, when it is immediately evident that their graphs are respectively the same in shape as those of the rational functions $y = x^3$, $y = x^4$, and $y = x^5$ already plotted, the axes of x and y, however, being changed in position.

It is to be noted that the graphs of all the functions expressed by the equation $y = x^n$ pass through the points (0, 0) and (1, 1).

Ex. 2. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

If y is defined as a function of x by the equation $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, it is evident that its graph will lie entirely in the first quadrant, since both x and y must be positive, and that its relative positions with respect to the two axes of coördinates are the same (fig. 78). The curve is a parabola (§ 79). If the equation is put in the form $y = (a^{\frac{1}{2}} - x^{\frac{1}{2}})^2$, it is seen that y is an irrational function of x.

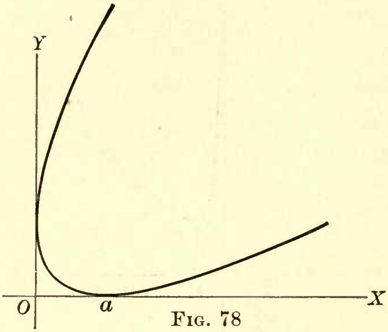

FIG. 78

Ex. 3. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Writing this equation in the form $y = \pm (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$, we see that y is an irrational function of x, and that its graph is symmetrical with respect to OX and bounded by the lines $x = -a$ and $x = a$. In the same way we may show that the graph is symmetrical with respect to OY and bounded by the lines $y = -a$ and $y = a$. It is represented in fig. 79, and is a four-cusped hypocycloid.

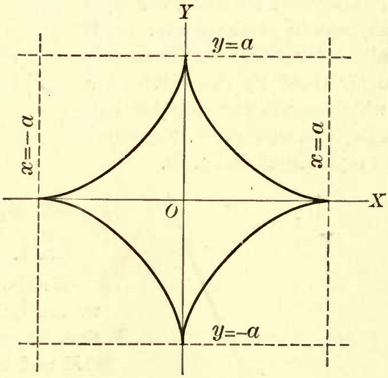

FIG. 79

Ex. 4. $x^3 + y^3 - 3\,axy = 0$.

The graph of this equation, by which y is defined as an irrational function of x, is represented in fig. 80, and is known as the Folium of Descartes. It is symmetrical with respect to the line $y = x$ and has the line $x + y + a = 0$ as an asymptote. While it may be plotted by assuming values for x and solving the corresponding cubic equations for y, it is more easily plotted when different axes of coördinates are chosen (see Ex. 38, Chap. X).

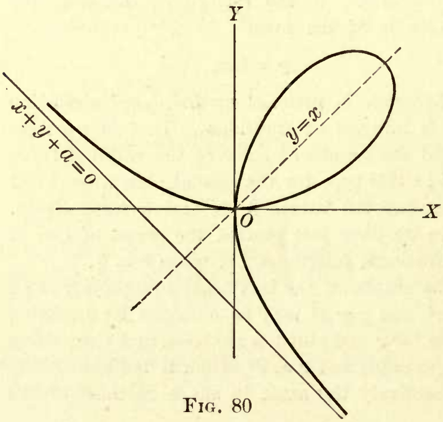

FIG. 80

PROBLEMS

Plot the graphs of the following equations:

1. $y^2 = (x-1)(x^2-4)$.
2. $y^2 = (x+2)(8x - x^2 - 15)$.
3. $4y^2 = (x+3)(2x-3)^2$.
4. $4y^2 = x^2(x+1)$.
5. $y^2 = (x-3)^2(5-2x)$.
6. $y^2 = (3x+2)(9x^2-4)$.
7. $y^2 = (x-2)^2(4x^2-4x-15)$.
8. $y^2 = (4x^2-1)(x^2-4)$.
9. $y^2 = (2x+5)^2(6+x-x^2)$.
10. $y^2 = -x^2(x+3)^2(x+1)$.
11. $y^2 = x^2(x-2)^2(x-3)$.
12. $y^2 = (1-x^2)(x^2-9)$.
13. $y^2 = (2x-5)(x^2+2)$.
14. $y^2 = (x-2)^2(x^2+2)$.
15. $y^2 = (x-2)(2x-3)^2(x^2+x+1)$.
16. $16y^2 = 4x^4 - x^6$.
17. $x^2 - y^2 - 4x + 6y - 1 = 0$.
18. $4x^2 + 9y^2 + 4x - 12y - 31 = 0$.
19. $x^2 - y^3 + 3y^2 + y - 3 = 0$.
20. $x^2 - y^4(4+y) = 0$.
21. $[x^2+3(y-1)][x^2-3(y-1)]=0$.
22. $(y-1)^2 = (x-1)^2(x-4)$.
23. $(y-x)^2 = 9 - x^2$.
24. $(x+y)^2 = y^2(y+1)$.
25. $x^2 - 4xy + 8y^2 - y^4 = 0$.
26. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$.
27. $y^3 = x^4$.
28. $y^3 = x(x^2-4)$.
29. $y^3 = x^2(x+2)$.
30. $(y+2)^3 = (x-1)(x^2-4)$.
31. $xy = 7$.
32. $xy = -7$.
33. $y = \frac{16}{1-x}$.
34. $y = \frac{1}{(x-1)^2} - \frac{1}{(x+3)^2}$.
35. $2y = 3x + \frac{1}{x}$.
36. $y - 2 = 2(x-1) + \frac{2}{x-1}$.
37. $(y-2)^2 = \frac{1}{x+1}$.
38. $y^2 = \frac{x(x+2)}{x-2}$.
39. $y^2 = \frac{x^3}{8-x}$.
40. $y^2 = \frac{x^3}{x^2-6x+8}$.
41. $x^2y^2 + 36 = 4y^2$.
42. $16a^4y^2 = b^2x^2(a^2-2ax)$.
43. $y^2 = \frac{x^2(a+x)}{a-x}$.
44. $y(x^2+a^2) = a^2(a-x)$.
45. $y^2(x^2+a^2) = a^2x^2$.
46. $a^4y^2 + b^2x^4 = a^2b^2x^2$.
47. $y^2(a^2+x^2) = x^2(a^2-x^2)$.
48. $xy^2 = 4a^2(2a-x)$.
49. $y = x^2 + \frac{1}{x}$.
50. $y = x + \frac{1}{x^2}$.

CHAPTER VII

CERTAIN CURVES AND THEIR EQUATIONS

70. The circle. When a curve has been defined by a geometric property it is often possible to find the equation of the curve by expressing the definition in algebraic symbols. This equation serves, then, as a means for plotting the curve and also as a basis for examining its other properties. In this chapter we shall derive the equation of certain important elementary curves, beginning with the circle.

A circle is the locus of a point at a constant distance from a fixed point. The fixed point is the *center* of the circle and the constant distance is the *radius*.

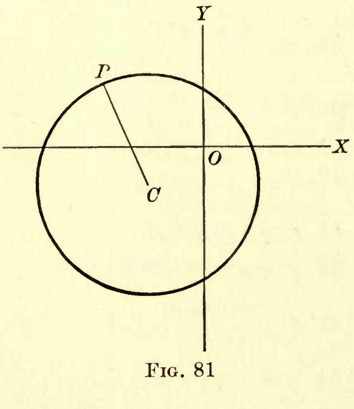

FIG. 81

Let (d, e) (fig. 81) be the coördinates of the center C, and r the radius of the circle. Then if $P(x, y)$ is a point on the circle, x and y must satisfy the equation

$$(x-d)^2+(y-e)^2=r^2, \tag{1}$$

by § 17.

Conversely, if x and y satisfy the equation (1), the point (x, y) is at a distance r from (d, e) and therefore lies on the circle.

Therefore (1) is the equation of the circle (§ 22).

Equation (1) expanded gives

$$x^2+y^2-2\,dx-2\,ey+d^2+e^2-r^2=0;$$

and if this is multiplied by any quantity A, it becomes

$$Ax^2+Ay^2+2\,Gx+2\,Fy+C=0, \tag{2}$$

where $$d=-\frac{G}{A}, \quad e=-\frac{F}{A}, \quad d^2+e^2-r^2=\frac{C}{A}.$$

Ex. The equation of a circle with the center $(\frac{1}{2}, -\frac{1}{3})$ and the radius $\frac{2}{3}$ is

$$(x-\tfrac{1}{2})^2+(y+\tfrac{1}{3})^2=\tfrac{4}{9},$$

which reduces to $\quad 36x^2+36y^2-36x+24y-3=0.$

71. Conversely, *the equation*

$$Ax^2+Ay^2+2Gx+2Fy+C=0,$$

where $A \neq 0$, represents a circle, if it represents any curve at all.

To prove this, we will transform the equation as follows:

$$x^2+2\frac{G}{A}x+y^2+2\frac{F}{A}y=-\frac{C}{A},$$

$$x^2+2\frac{G}{A}x+\frac{G^2}{A^2}+y^2+2\frac{F}{A}y+\frac{F^2}{A^2}=\frac{G^2+F^2-AC}{A^2},$$

$$\left(x+\frac{G}{A}\right)^2+\left(y+\frac{F}{A}\right)^2=\frac{G^2+F^2-AC}{A^2}.$$

There are then three possible cases:

1. $G^2+F^2-AC>0$. The equation is then of the type (1), § 70, where $d=-\frac{G}{A}$, $e=-\frac{F}{A}$, $r^2=\frac{G^2+F^2-AC}{A^2}$, and therefore represents a circle with the center $\left(-\frac{G}{A}, -\frac{F}{A}\right)$ and the radius $\sqrt{\frac{G^2+F^2-AC}{A^2}}$.

2. $G^2+F^2-AC=0$. The equation is then

$$\left(x+\frac{G}{A}\right)^2+\left(y+\frac{F}{A}\right)^2=0,$$

which can be satisfied by real values of x and y only when $x=-\frac{G}{A}$ and $y=-\frac{F}{A}$. Hence the equation represents the point $\left(-\frac{G}{A}, -\frac{F}{A}\right)$. This may be called a circle of zero radius, regarding it as the limit of a circle as the radius approaches zero.

3. $G^2+F^2-AC<0$. The equation can then be satisfied by no real values of x and y, since the sum of two positive quantities cannot be negative. Hence the equation represents no curve.

Ex. 1. The equation $x^2 + y^2 - 2x + 4y + 1 = 0$ may be written

$$(x - 1)^2 + (y + 2)^2 = 4,$$

and represents a circle with center $(1, -2)$ and radius 2.

Ex. 2. The equation $x^2 + y^2 - 2x + 4y + 5 = 0$ may be written

$$(x - 1)^2 + (y + 2)^2 = 0,$$

and is satisfied only by the point $(1, -2)$.

Ex. 3. The equation $x^2 + y^2 - 2x + 4y + 7 = 0$ may be written

$$(x - 1)^2 + (y + 2)^2 = -2,$$

and represents no curve.

72. To find the equation of a circle which will satisfy given conditions, it is necessary and sufficient to determine the three quantities d, e, r, or the ratios of the four quantities A, G, F, C. Each condition imposed upon the circle leads usually to an equation involving these quantities. In order to determine the three quantities it is necessary and in general sufficient to have three equations. Hence, in general, three conditions are necessary and sufficient to determine a circle.

It is not important to enumerate all possible conditions which may be imposed upon a circle, but the following three may be mentioned.

1. Let the condition be imposed upon the circle to pass through the known point (x_1, y_1). Then (x_1, y_1) must satisfy the equation of the circle; therefore d, e, and r must satisfy the condition

$$(x_1 - d)^2 + (y_1 - e)^2 = r^2.$$

2. Let the condition be imposed upon the circle to be tangent to the known straight line $Ax + By + C = 0$. Then the distance from the center of the circle to this line must equal the radius; therefore, by § 32, d, e, and r must satisfy the condition

$$\frac{Ad + Be + C}{\sqrt{A^2 + B^2}} = \pm r.$$

The sign will be ambiguous, unless from other conditions of the problem it is known on which side of the line the center lies.

3. Let it be required that the center of the circle should lie on the line $Ax + By + C = 0$. Then d and e must satisfy the condition

$$Ad + Be + C = 0.$$

Ex. 1. Find the equation of the circle through the three points $(2, -2)$, $(7, 3)$, and $(6, 0)$.

The quantities d, e, and r must satisfy the three conditions

$$(2 - d)^2 + (-2 - e)^2 = r^2,$$
$$(7 - d)^2 + (3 - e)^2 = r^2,$$
$$(6 - d)^2 + (0 - e)^2 = r^2.$$

Solving these we have $d = 2$, $e = 3$, and $r = 5$. Therefore the required equation is

$$(x - 2)^2 + (y - 3)^2 = 25,$$

or

$$x^2 + y^2 - 4x - 6y - 12 = 0.$$

Ex. 2. Find the equation of the circle which passes through the points $(2, -3)$ and $(-4, -1)$ and has its center on the line $3y + x - 18 = 0$.

The quantities d, e, and r must satisfy the conditions

$$(2 - d)^2 + (-3 - e)^2 = r^2,$$
$$(-4 - d)^2 + (-1 - e)^2 = r^2,$$
$$3e + d - 18 = 0.$$

Solving these equations we find $d = \frac{3}{2}$, $e = \frac{11}{2}$, $r^2 = \frac{145}{2}$. Therefore the required equation is

$$(x - \tfrac{3}{2})^2 + (y - \tfrac{11}{2})^2 = \tfrac{145}{2},$$

or

$$x^2 + y^2 - 3x - 11y - 40 = 0.$$

Ex. 3. Find the equation of a circle which is tangent to the lines

$$17x + y - 35 = 0 \quad \text{and} \quad 13x + 11y + 50 = 0,$$

and has its center on the line $88x + 70y + 15 = 0$.

The quantities d, e, and r must satisfy the conditions

$$\frac{17d + e - 35}{\sqrt{290}} = \pm r,$$
$$\frac{-13d - 11e - 50}{\sqrt{290}} = \pm r,$$
$$88d + 70e + 15 = 0.$$

These equations have the two solutions

$$d = -\tfrac{5}{6}, \qquad e = \tfrac{5}{6}, \qquad r = \frac{\sqrt{290}}{6};$$

and

$$d = 5, \qquad e = -\tfrac{13}{2}, \qquad r = \frac{3\sqrt{290}}{20}.$$

Hence each of the two circles

$$3x^2 + 3y^2 + 5x - 5y - 20 = 0$$

and

$$40x^2 + 40y^2 - 400x + 520y + 2429 = 0$$

satisfies the conditions of the problem.

Ex. 4. The equation of a circle through three given points is most readily found by means of the equation

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0.$$

If (x_1, y_1), (x_2, y_2), and (x_3, y_3) are the three given points, the quantities A, G, F, C must satisfy the equations

$$Ax_1^2 + Ay_1^2 + 2Gx_1 + 2Fy_1 + C = 0,$$
$$Ax_2^2 + Ay_2^2 + 2Gx_2 + 2Fy_2 + C = 0,$$
$$Ax_3^2 + Ay_3^2 + 2Gx_3 + 2Fy_3 + C = 0.$$

There are here four homogeneous equations in the unknowns A, G, F, C, and the result of eliminating the unknowns is, by § 9,

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0, \qquad (1)$$

which is the required equation of the circle.

It is to be noticed that the coefficient of $x^2 + y^2$ in (1) is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

When this is zero, equation (1) is of the first degree and represents a straight line. But when

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

the points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are on the same straight line (§ 29, 5) and cannot determine a circle.

73. The ellipse. *An ellipse is the locus of a point the sum of the distances of which from two fixed points is constant.*

The two fixed points are called the *foci.* Let them be denoted by F and F' (fig. 82) and let the axis of x be taken through them and the origin halfway between them. Then if P is any point on the ellipse and $2a$ represents the constant sum of its distances from the foci, we have

$$F'P + FP = 2a. \tag{1}$$

From the triangle $F'PF$ it follows that

$$F'F < 2a.$$

Hence there is a point A on the axis of x and to the right of F which satisfies the definition. We have then

$$F'A + FA = 2a,$$

or $(F'O + OA) + (OA - OF) = 2a,$

whence $\qquad OA = a.$

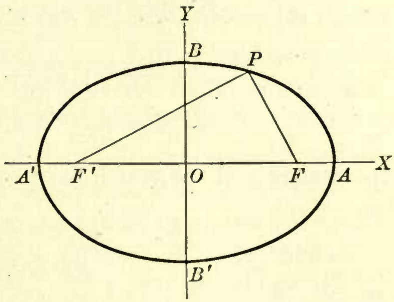

FIG. 82

Let us now place

$$\frac{OF}{OA} = e, \text{ where } e < 1.$$

Then the coördinates of F and F' are $(\pm ae, 0)$. Computing the values of $F'P$ and FP by § 17, and substituting in (1), we have

$$\sqrt{(x + ae)^2 + y^2} + \sqrt{(x - ae)^2 + y^2} = 2a. \tag{2}$$

By transposing the second radical to the right-hand side of the equation, squaring, and reducing, we have

$$a - ex = \sqrt{(x - ae)^2 + y^2} = FP. \tag{3}$$

Similarly, by transposing the first radical in (2), we have

$$a + ex = \sqrt{(x + ae)^2 + y^2} = F'P. \tag{4}$$

Either (3) or (4) leads to the equation

$$(1-e^2)x^2+y^2=a^2(1-e^2), \tag{5}$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1. \tag{6}$$

Since $e<1$, the denominator of the second fraction is positive and we place

$$a^2(1-e^2)=b^2,$$

thus obtaining

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{7}$$

We have now shown that any point which satisfies (1) has coördinates which satisfy (7).

To show, conversely, that any point whose coördinates satisfy (7) is such as to satisfy (1), let us assume (7) as given. We can then obtain (6) and (5), and (5) may be put in each of the two forms

$$x^2+2\,aex+a^2e^2+y^2=a^2+2\,aex+e^2x^2,$$
$$x^2-2\,aex+a^2e^2+y^2=a^2-2\,aex+e^2x^2,$$

the square roots of which are respectively

$$F'P=\pm(a+ex),$$
$$FP=\pm(a-ex).$$

These lead to one of the four following equations:

$$F'P+FP=2\,a,$$
$$F'P-FP=2\,a,$$
$$-F'P+FP=2\,a,$$
$$-F'P-FP=2\,a.$$

Of these, the last one is impossible, since the sum of two negative numbers cannot be positive; and the second and third are impossible, since the difference between FP and $F'P$ must be less than $F'F$, which is less than $2\,a$. Hence any point which satisfies (7) satisfies (1), and therefore (7) is the equation of the ellipse.

74. Placing $y = 0$ in (7), § 73, we find $x = \pm a$. Placing $x = 0$, we find $y = \pm b$. Hence the ellipse intersects OX in the two points $A(a, 0)$ and $A'(-a, 0)$, and intersects OY in two points $B(0, b)$ and $B'(0, -b)$. The points A and A' are called the *vertices* of the ellipse. The line AA', which is equal to $2a$, is called the *major axis*, and the line BB', which is equal to $2b$, is called the *minor axis* of the ellipse.

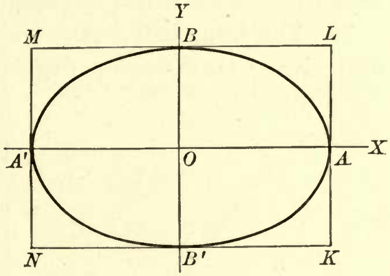

FIG. 83

Solving (5) first for y and then for x, we have

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

and $$x = \pm \frac{a}{b}\sqrt{b^2 - y^2}.$$

These equations show (1) that the ellipse is symmetrical with respect to both OX and OY, (2) that x can have no value numerically greater than a, (3) that y can have no value numerically greater than b. If we construct the rectangle $KLMN$ (fig. 83), which has O for a center and sides equal to $2a$ and $2b$ respectively, the ellipse will lie entirely within it; and if the curve is constructed in one quadrant, it can be found by symmetry in all quadrants. The form of the curve is shown in figs. 82 and 83.

75. Any equation of the form (7), § 73, in which $a > b$, represents an ellipse with the foci on OX. For if we place, as in § 73, $b^2 = a^2(1 - e^2)$, we find

$$e = \frac{\sqrt{a^2 - b^2}}{a},$$

and may fix F and F', which in § 73 were arbitrary in position, by the relation $OF = -OF' = ae$.

The foci may be found graphically by placing the point of a compass on B and describing an arc with the radius a. This arc will intersect AA' in the foci; for since $OB = b$ and $OF = \sqrt{a^2 - b^2}$, $BF = a$.

Similarly an equation of the form (7), § 73, in which $b > a$, represents an ellipse in which the foci lie on BB' at a distance $\sqrt{b^2 - a^2}$ from O. In this case $BB' = 2b$ is the major axis and $AA' = 2a$ is the minor axis.

It may be noted that the nearer the foci are taken together, the smaller is e and the more nearly $b = a$. Hence *a circle may be considered as an ellipse with coincident foci and equal axes.*

76. The hyperbola. *An hyperbola is the locus of a point the difference of the distances of which from two fixed points is constant.*

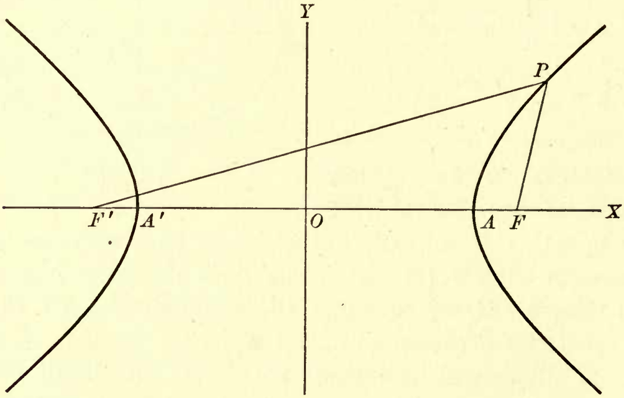

Fig. 84

The two fixed points are called the *foci*. Let them be F and F' (fig. 84) and let FF' be taken as the axis of x, the origin being halfway between F and F'. Then if P is any point on the hyperbola and $2a$ is the constant difference of its distances from F and F', we have either

$$F'P - FP = 2a, \tag{1}$$

or

$$FP - F'P = 2a. \tag{2}$$

Since in the triangle $F'PF$ the difference of the two sides FP and $F'P$ is less than $F'F$, it follows that $F'F > 2a$.

There is therefore at least one point A between O and F which satisfies the definition.

Then $$F'A - AF = 2a,$$
or $$(F'O + OA) - (OF - OA) = 2a;$$
whence $$OA = a.$$

We may therefore place

$$\frac{OF}{OA} = e, \text{ where } e > 1.$$

Then the coördinates of F and F' are $(\pm ae, 0)$ and equations (1) and (2) become

$$\sqrt{(x + ae)^2 + y^2} - \sqrt{(x - ae)^2 + y^2} = 2a \tag{3}$$

and $$\sqrt{(x - ae)^2 + y^2} - \sqrt{(x + ae)^2 + y^2} = 2a. \tag{4}$$

By transposing one of the radicals to the right-hand side of these equations, squaring, and reducing, we obtain from (3) either

$$ex + a = \sqrt{(x + ae)^2 + y^2} = F'P,$$

or $$ex - a = \sqrt{(x - ae)^2 + y^2} = FP;$$

and from (4) we obtain either

$$-(ex + a) = \sqrt{(x + ae)^2 + y^2} = F'P,$$

or $$-(ex - a) = \sqrt{(x - ae)^2 + y^2} = FP.$$

Any one of the last four equations gives

$$(1 - e^2)x^2 + y^2 = a^2(1 - e^2), \tag{5}$$

or $$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \tag{6}$$

But since $e > 1$, $a^2(1 - e^2)$ is a negative quantity and we may write $a^2(1 - e^2) = -b^2$, thus obtaining

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{7}$$

Then any point which satisfies (1) or (2) satisfies (7). Conversely, by retracing our steps, we find that if the coördinates of a point P satisfy (7), then

$$F'P = \pm (ex + a)$$

and

$$FP = \pm (ex - a).$$

Hence we must have either

$$F'P - FP = 2a,$$

$$-F'P + FP = 2a,$$

$$F'P + FP = 2a,$$

or

$$-F'P - FP = 2a.$$

The equation $F'P + FP = 2a$ is impossible, for $F'P + FP > F'F$, and $2a < F'F$. The equation $-F'P - FP = 2a$ is also clearly impossible. Hence any point which satisfies (7) satisfies either (1) or (2). Therefore (7) is the equation of the hyperbola.

77. If we place $y = 0$ in (7), § 76, we have $x = \pm a$. Hence the curve intersects OX in two points, A and A', called the *vertices*. If $x = 0$, y is imaginary. Hence the curve does not intersect OY.

Solving (7), § 76, for y and x respectively, we have

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$$

and

$$x = \pm \frac{a}{b}\sqrt{y^2 + b^2}.$$

These show (1) that the curve is symmetrical with respect to both OX and OY, (2) that x can have no value numerically less than a, and (3) that y can have all values.

Moreover, the equation for y can be written

$$y = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}.$$

As x increases the term $\frac{a^2}{x^2}$ decreases, approaching zero as a limit. Hence the more the hyperbola is prolonged, the nearer it comes to the straight lines $y = \pm\frac{b}{a}x$. Therefore the straight lines $y = \pm\frac{b}{a}x$ are the *asymptotes* of the hyperbola. They are the diagonals of the rectangle constructed as in fig. 85, and are used

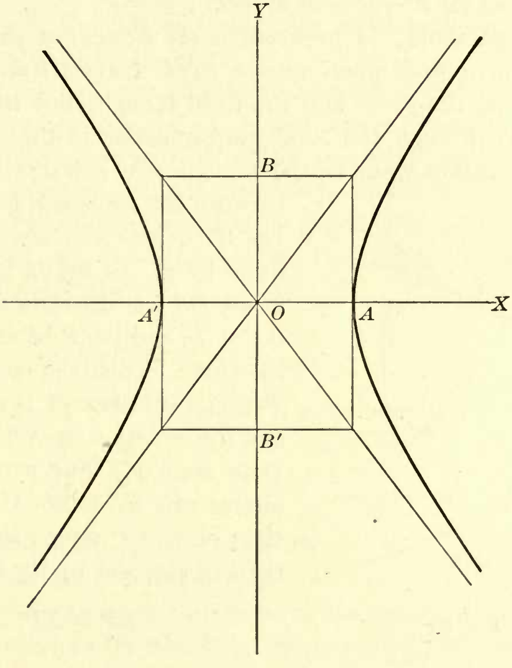

Fig. 85

conveniently as guides in drawing the curve. The line AA' is called the *transverse axis* and the line BB' the *conjugate axis* of the hyperbola. The shape of the curve is shown in figs. 84 and 85.

78. Any equation of the form (7), § 76, where a and b are any positive real values, represents an hyperbola with the foci on AA'. For if we place $-b^2 = a^2(1-e^2)$, we find $e = \frac{\sqrt{a^2+b^2}}{a}$ and may find the position of the foci from the equations $OF = -OF' = ae$. Similarly any equation of the form

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represents an hyperbola with the foci on BB'.

If $b = a$, the hyperbola is called an *equilateral hyperbola* and its equation is either $x^2 - y^2 = a^2$ or $-x^2 + y^2 = a^2$.

79. The parabola. *A parabola is the locus of a point equally distant from a fixed point and a fixed straight line.* The fixed point is called the *focus* and the fixed straight line the *directrix*. Let the line through the focus perpendicular to the directrix be taken as the axis of x, and let the origin be taken on this line halfway between the focus and the directrix. Let us denote the abscissa of the focus by p. In fig. 86 let F be the focus, RS the directrix intersecting OX at D, and let P be any point on the curve. Then the coördinates of F are $(p, 0)$, those of D are $(-p, 0)$, and the equation of RS is $x = -p$. Draw from P a line parallel to OX intersecting RS in N. If F is on the right of RS, P must also lie on the right of RS, and by the definition

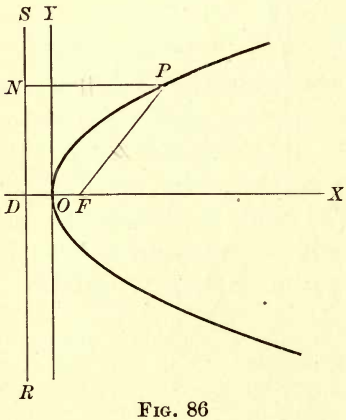

FIG. 86

$$FP = NP.$$

If, on the other hand, F is on the left of RS, P is also on the left of RS and

$$FP = PN = -NP.$$

In either case $\overline{FP}^2 = \overline{NP}^2.$

But $\overline{FP}^2 = (x - p)^2 + y^2$, and $NP = x + p$; (by § 17)

hence $(x - p)^2 + y^2 = (x + p)^2,$

which reduces to $y^2 = 4px.$ (1)

Any point on the parabola then satisfies this equation. Conversely, it is easy to show that if a point satisfies this equation, it must so lie that $FP = \pm NP$, and hence lies on the parabola.

Equation (1) shows (1) that the curve is symmetrical with respect to OX, (2) that x must have the same sign as p, and (3) that y increases as x increases numerically. The position of the curve is as shown in fig. 86 when p is positive. When p is negative F lies at the left of O and the curve extends toward the negative end of the axis of x.

Similarly the equation $x^2 = 4py$ represents a parabola for which the focus lies on the axis of y, and which extends toward the positive or the negative end of the axis of y according as p is positive or negative. In all cases O is called the *vertex* of the parabola and the line determined by O and F is called its *axis*.

80. If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two points on the parabola $y^2 = 4px$ (fig. 87), then

$$y_1^2 = 4px_1,$$

$$y_2^2 = 4px_2;$$

hence $$\frac{y_1^2}{y_2^2} = \frac{x_1}{x_2}.$$

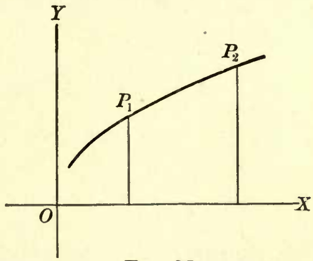

FIG. 87

That is, *the squares of the ordinates of a parabola are to each other as the abscissas.* Conversely, *if in any curve the squares of the ordinates are to each other as the abscissas, the curve is a parabola.*

For let P_1 be a known point and P any point on the curve. Then, by hypothesis,

$$\frac{y^2}{y_1^2} = \frac{x}{x_1},$$

which may be written $$y^2 = \frac{y_1^2}{x_1}x.$$

But this is the same as $y^2 = 4px$, where $p = \dfrac{y_1^2}{4x_1}$.

81. The conic. *A conic is the locus of a point the distance of which from a fixed point is in a constant ratio to its distance from a fixed straight line.*

The fixed point is called the *focus,* the fixed line the *directrix,* and the constant ratio the *eccentricity.*

We shall take the directrix as the axis of y, and a line through the focus F as the axis of x, and shall call the coördinates of the focus $(c, 0)$, where c represents OF and is positive or negative according as F lies to the right or the left of O.

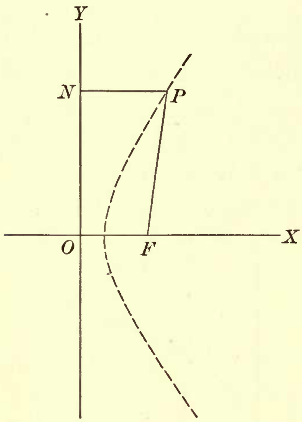

FIG. 88

Let P be any point on the conic; connect P and F, and draw PN perpendicular to OY. Then by definition

$$FP = \pm e \cdot NP, \tag{1}$$

according as P is on the right or the left of OY. In both cases

$$\overline{FP}^2 = e^2 \cdot \overline{NP}^2.$$

But $\overline{FP}^2 = (x - c)^2 + y^2$, by § 17, and $NP = x$. Therefore for any point on the conic

$$(x - c)^2 + y^2 = e^2x^2. \tag{2}$$

It is easy to show, conversely, that if the coördinates of P satisfy (2), P satisfies (1). Hence (2) is the equation of the conic.

It is clear that the parabola is a special case of a conic, for the definition of the latter becomes that of the former when $e = 1$.

It is also not difficult to show that the ellipse is a special case of a conic, where the eccentricity is e of § 73 and < 1.

For if P (fig. 89) is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we found in § 73 that

$$FP = a - ex, \qquad F'P = a + ex,$$

or

$$FP = e\left(\frac{a}{e} - x\right), \qquad F'P = e\left(\frac{a}{e} + x\right).$$

If now we take the point D so that $OD = \frac{a}{e}$, and D' so that $OD' = -\frac{a}{e}$, draw the lines DS and $D'S'$ perpendicular to OX, the line $N'PN$ perpendicular to DS, and the ordinate MP, we have

$$\frac{a}{e} - x = OD - OM = MD = PN,$$

$$\frac{a}{e} + x = D'O + OM = D'M = N'P.$$

$$\therefore \quad FP = e \cdot PN, \qquad F'P = e \cdot N'P.$$

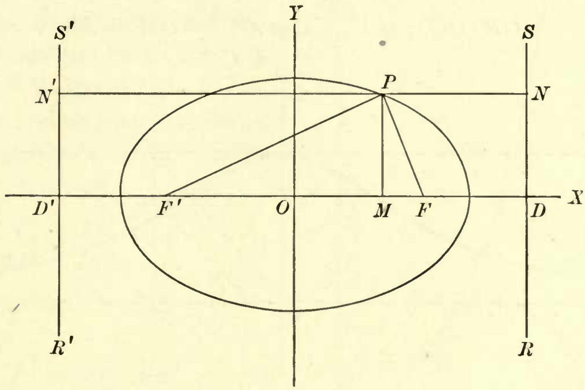

Fig. 89

The ellipse has therefore two directrices at the distances $\pm \frac{a}{e}$ from the center. When the ellipse is a circle, $e = 0$ and the directrices are at infinity.

In a similar manner we may show that the hyperbola is a special case of a conic where $e > 1$.

In § 114, Ex. 3, we shall prove that the conic is always either an ellipse, a parabola, or an hyperbola.

82. The witch. Let OBA (fig. 90) be a circle, OA a diameter, and LK the tangent to the circle at A. From O draw any line intersecting the circle at B and LK at C. From B draw a line parallel to LK and from C a line perpendicular to LK, and call the intersection of these two lines P. The locus of P is a curve called the *witch*.

To obtain its equation we will take the origin at O and the line OA as the axis of y. We will call the length of the diameter of the circle $2a$. Then by continuing CP until it meets OX at M, and calling (x, y) the coördinates of P, we have

$$OM = x, \qquad MP = y, \qquad OA = MC = 2a.$$

In the triangle OMC, $$\frac{MP}{MC} = \frac{OB}{OC} = \frac{OB \cdot OC}{\overline{OC}^2}. \tag{1}$$

If AB is drawn, OBA is a right angle and consequently

$$OB \cdot OC = \overline{OA}^2, \quad \text{also } \overline{OC}^2 = \overline{OM}^2 + \overline{MC}^2.$$

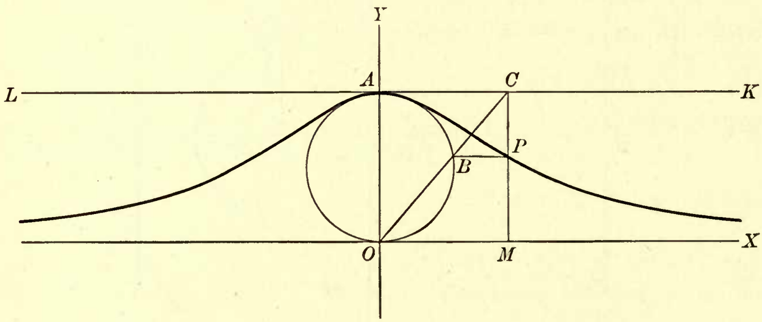

FIG. 90

Therefore $$\frac{MP}{MC} = \frac{\overline{OA}^2}{\overline{OM}^2 + \overline{MC}^2}, \tag{2}$$

that is, $$\frac{y}{2a} = \frac{4a^2}{x^2 + 4a^2}; \tag{3}$$

and finally, $$y = \frac{8a^3}{x^2 + 4a^2}. \tag{4}$$

Conversely, if equation (4) is satisfied by any point, we can deduce equations (3), (2), and (1) in order, and hence show that the point is on the witch.

Solving (4) for x, we have

$$x = \pm 2a\sqrt{\frac{2a - y}{y}}.$$

This shows (1) that the curve is symmetrical with respect to OY, (2) that y cannot be negative nor greater than $2a$, and (3) that $y = 0$ is an asymptote.

83. The cissoid. Let ODA (fig. 91) be a circle with the diameter OA, and let LK be the tangent to the circle at A. Through O draw any line intersecting the circle in D and LK in E. On OE lay off a distance OP equal to DE. Then the locus of P is a curve called the *cissoid.*

To find its equation, we will take O as the origin of coördinates and OA as the axis of x, and will call the diameter of the circle $2a$. Draw MP perpendicular to OA. Then if A and D are connected, a triangle ADE is formed similar to OMP; whence

$$\frac{OP}{MP} = \frac{AE}{DE}. \qquad (1)$$

By hypothesis $DE = OP$. Therefore

$$\overline{OP}^2 = MP \cdot AE. \qquad (2)$$

Also, in the similar triangles OAE and OPM,

$$\frac{AE}{OA} = \frac{MP}{OM}; \qquad (3)$$

whence, from (2),

$$\overline{OP}^2 = \frac{OA \cdot \overline{MP}^2}{OM}, \qquad (4)$$

or $$x^2 + y^2 = \frac{2ay^2}{x}; \qquad (5)$$

whence $$y^2 = \frac{x^3}{2a - x}. \qquad (6)$$

This equation is satisfied by the coördinates of any point upon the cissoid.

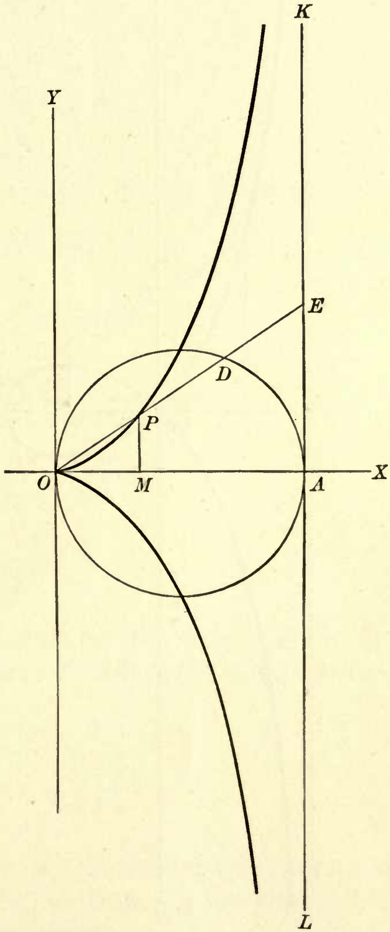

FIG. 91

Conversely, if we assume equation (6), we may deduce (5) and (4), and then by aid of (1) and (3) we have $OP = DE$.

Therefore (6) is the equation of the cissoid. It may be written

$$y = \pm x \sqrt{\frac{x}{2a - x}}.$$

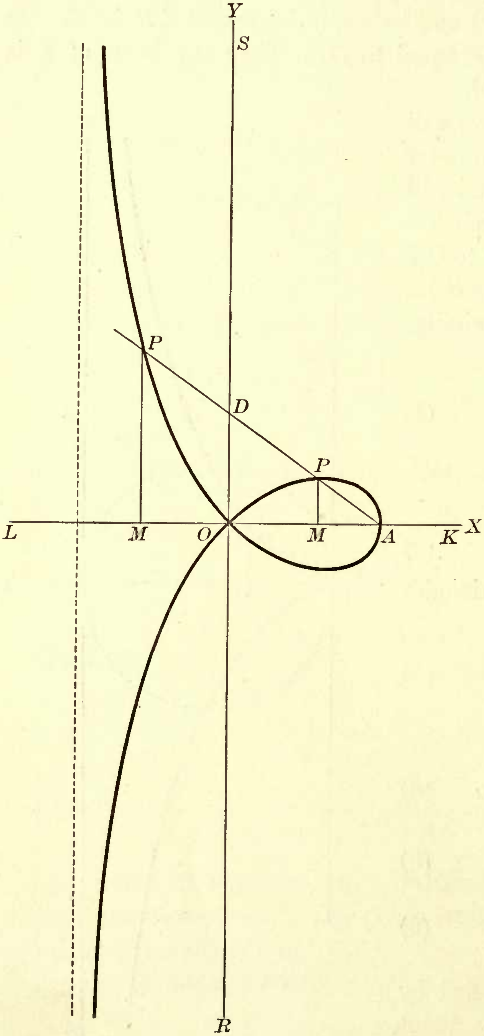

FIG. 92

From this it appears (1) that the curve is symmetrical with respect to OX, (2) that no value of x can be greater than $2a$ or less than 0, and (3) that the line $x = 2a$ is an asymptote.

84. The strophoid. Let LK and RS (fig. 92) be two straight lines intersecting at right angles at O, and let A be a fixed point on LK. Through A draw any straight line intersecting RS in D, and lay off on AD in either direction a distance DP equal to OD. The locus of P is a curve called the *strophoid*.

To find its equation, take LK as the axis of x and RS as the axis of y, and call the coördinates of A $(a, 0)$. By the definition the point P may fall in any one of the four quadrants.

If we take the positive direction on AD as measured from A towards D, we have

$$OD = PD$$

when P is in the first quadrant,

$$OD = -PD$$

when P is in the second quadrant,

$$-OD = -PD$$

when P is in the third quadrant, and

$$-OD = PD$$

when P is in the fourth quadrant.

These four equations are equivalent to the single equation

$$\overline{OD}^2 = \overline{PD}^2. \tag{1}$$

From the similar triangles OAD and APM,

$$\frac{OD}{AD} = \frac{MP}{AP} = \frac{y}{\sqrt{(x-a)^2 + y^2}},$$

$$\frac{PD}{AD} = \frac{MO}{AO} = \frac{OM}{OA} = \frac{x}{a}.$$

Hence
$$\frac{y^2}{(x-a)^2 + y^2} = \frac{x^2}{a^2} \tag{2}$$

is an equation satisfied by any point on the curve. Conversely, if (2) is given, (1) may be deduced. Therefore (2) is the equation of the strophoid.

It may be written

$$y = \pm x\sqrt{\frac{a-x}{a+x}}.$$

This shows (1) that the curve is symmetrical with respect to OX, (2) that no value of x can be less than $-a$ nor greater than $+a$, and (3) that $x = -a$ is an asymptote.

85. Examples. The use of the equation of a curve in solving problems connected with the curve will be constantly illustrated throughout the book. The following examples depend upon principles already given.

Ex. 1. Prove that in the ellipse the squares of the ordinates of any two points are to each other as the products of the segments of the major axis made by the feet of these ordinates.

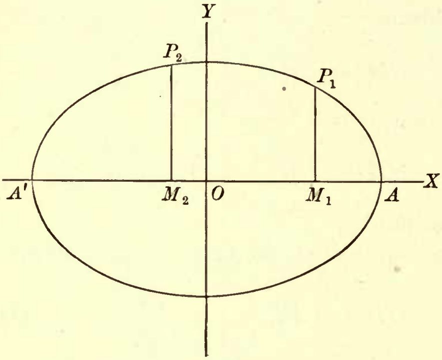

FIG. 93

We are to prove that (fig. 93)

$$\frac{\overline{M_1P_1}^2}{\overline{M_2P_2}^2} = \frac{A'M_1 \cdot M_1A}{A'M_2 \cdot M_2A}.$$

Let the coördinates of P_1 be (x_1, y_1) and let those of P_2 be (x_2, y_2). Then

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \qquad \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1,$$

whence

$$\frac{y_1^2}{y_2^2} = \frac{a^2 - x_1^2}{a^2 - x_2^2} = \frac{(a + x_1)(a - x_1)}{(a + x_2)(a - x_2)}.$$

But $y_1 = M_1P_1$, $a + x_1 = A'O + OM_1 = A'M_1$, $a - x_1 = OA - OM_1 = M_1A$, $y_2 = M_2P_2$, $a + x_2 = A'M_2$, $a - x_2 = M_2A$. Hence the proposition is proved.

Ex. 2. If M_1P_1 is the ordinate of a point P_1 of the parabola, $y^2 = 4\,px$, and a straight line drawn through the middle point of M_1P_1 parallel to the axis of x cuts the curve at Q; prove that the intercept of the line M_1Q on the axis of y equals $\frac{2}{3}\,M_1P_1$.

Let the coördinates of P_1 (fig. 94) be (x_1, y_1). Then $x_1 = \frac{y_1^2}{4\,p}$ from the equation of the parabola.

By construction, the ordinate of Q is $\frac{y_1}{2}$. Since Q is on the parabola its abscissa is found by placing $y = \frac{y_1}{2}$ in $y^2 = 4\,px$. The coördinates of Q are then $\left(\frac{y_1^2}{16\,p}, \frac{y_1}{2}\right)$. The coördinates of M are $(x_1, 0)$, which are the same as $\left(\frac{y_1^2}{4\,p}, 0\right)$. Hence the equation of MQ is, by § 29,

$$8\,px + 3\,y_1y - 2\,y_1^2 = 0.$$

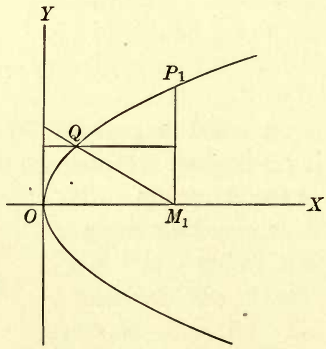

FIG. 94

The intercept of this line on OY is $\frac{2}{3}\,y_1 = \frac{2}{3}\,M_1P_1$, which was to be proved.

PROBLEMS

1. Find the equation of the circle having the center $(2, -4)$ and the radius 3.

2. Find the equation of the circle having the center $(-\frac{2}{3}, \frac{4}{3})$ and the radius 6.

3. Find the equations of the circles having the line joining $(2, 3)$ and $(-3, 1)$ as a radius.

4. Find the equation of the circle having the line joining $(a, -b)$ and $(-a, b)$ as a diameter.

5. Find the equations of the circles of radius a which are tangent to the axis of y at the origin.

6. Find the equations of the circles of radius a which are tangent to both coördinate axes.

7. Find the equation of the circle having as a diameter that part of the line $2x - 3y + 6 = 0$ which is included between the coördinate axes.

8. Find the center and the radius of the circle $x^2 + y^2 + 4x - 10y - 36 = 0$.

9. Find the center and the radius of the circle $x^2 + y^2 + 4x - 6y + 1 = 0$.

10. Find the center and the radius of the circle $3x^2 + 3y^2 - 9x + 6y - 2 = 0$.

11. Find the center and the radius of the circle $5x^2 + 5y^2 + 2x - 4y + 1 = 0$.

12. Prove that two circles are concentric if their equations differ only in the absolute term.

13. Show that the circles $x^2 + y^2 + 2Gx + 2Fy + C = 0$ and $x^2 + y^2 + 2G'x + 2F'y + C' = 0$ are tangent to each other if

$$\sqrt{(G - G')^2 + (F - F')^2} = \sqrt{G^2 + F^2 - C} \pm \sqrt{G'^2 + F'^2 - C'}.$$

14. Find the equation of the circle which passes through the points $(0, 3)$, $(3, 0)$, $(0, 0)$.

15. Find the equation of the circle circumscribing the triangle with the vertices $(0, 2)$, $(-1, 0)$, $(0, -2)$.

16. Find the equation of the circle circumscribed about the triangle the sides of which are $x + y - 2 = 0$, $9x + 5y - 2 = 0$, $y + 2x - 1 = 0$.

17. Find the equation of the circle passing through the point $(-2, 4)$ and concentric with the circle $x^2 + y^2 - 5x + 4y - 1 = 0$.

18. A circle which is tangent to both coördinate axes passes through $(4, 2)$. Find its equation.

19. The center of a circle which is tangent to the axes of x and y is on the line $2x - 3y + 6 = 0$. What is its equation?

20. A circle of radius 5 passes through the points $(2, -1)$ and $(3, -2)$. What is its equation?

21. The center of a circle which passes through the points $(1, -2)$ and $(-2, 2)$ is on the line $8x - 4y + 9 = 0$. What is its equation?

22. A circle which is tangent to OX passes through $(-3, 2)$ and $(4, 9)$. What is its equation?

23. The center of a circle which is tangent to the two parallel lines $x-3=0$ and $x-7=0$ is on the line $y=2x+4$. What is its equation?

24. The center of a circle is on the line $2x+y=0$. The circle passes through the point $(4, 2)$ and is tangent to the line $4x-3y-15=0$. What is its equation?

25. Find the equation of the circle circumscribing the isosceles triangle of which the altitude is 4 and the base is the line joining the points $(-3, 0)$ and $(3, 0)$.

26. Find the equation of the ellipse the foci of which are $(\pm 3, 0)$ and the major axis of which is 8.

27. Find the equation of the ellipse the foci of which are $(0, \pm 2)$ and the major axis of which is 6.

28. Find the equation of an ellipse when the vertices are $(\pm 6, 0)$ and one focus is $(4, 0)$.

29. Determine the semiaxes a and b in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, so that it will pass through $(1, 4)$ and $(2, -3)$.

30. If the vertices of an ellipse are $(\pm 5, 0)$ and its foci are $(\pm 3, 0)$, find its equation.

31. The center of an ellipse is at the origin and its major axis lies along OX. If its major axis is 8 and its eccentricity is $\frac{2}{3}$, find its equation.

32. Find the equation of an ellipse when its center is at the origin, one focus at the point $(-3, 0)$, and the minor axis equal to 8.

33. Find the equation of an ellipse the eccentricity of which is $\frac{4}{7}$ and the foci of which are $(0, \pm 6)$.

34. Given the ellipse $9x^2+16y^2=144$. Find its semiaxes, eccentricity, and foci.

35. Find the eccentricity and the equation of an ellipse, if the foci lie halfway between the center and the vertices, the major axis lying on OX.

36. Find the equation of an ellipse the eccentricity of which is $\frac{2}{3}$ and the ordinate at the focus is 5, the center being at the origin and the major axis lying along OX.

37. Find the equation and the eccentricity of the ellipse if the ordinate at the focus is one fourth the minor axis.

38. Find the eccentricity of an ellipse if the line connecting the positive ends of the axes is parallel to the line joining the center to the upper end of the ordinate at the left-hand focus.

39. Find the equation of an ellipse when the foci are $(\pm 2, 0)$ and the directrices are $x=\pm 5$.

40. Given the ellipse $2x^2+3y^2=1$. Find its semiaxes, foci, and directrices.

41. Find the equation of an hyperbola if the foci are $(\pm 3, 0)$ and the transverse axis is 4.

42. Find the equation of an hyperbola if the foci are $(0, \pm 4)$ and the transverse axis is 4.

43. An hyperbola has its center at the origin and its transverse axis along OX. If its eccentricity is $\frac{4}{3}$ and its transverse axis is 5, find its equation.

44. Find the equation of an hyperbola when the vertices are $(\pm 4, 0)$ and the eccentricity is $\frac{7}{5}$.

45. Show that the eccentricity of an equilateral hyperbola is equal to the ratio of a diagonal of a square to its side.

46. Find the equation of an hyperbola the vertices of which are halfway between the center and the foci, the transverse axis lying on OX.

47. Find the equation of the hyperbola with eccentricity 3 which passes through the point $(2, 4)$, its axes lying on OX and OY.

48. Find the equation of an equilateral hyperbola which passes through $(5, -2)$.

49. Find the equation of the hyperbola which has the points $(0, \pm \frac{3}{2}\sqrt{2})$ for foci and passes through the point $(2, -1)$.

50. The sum of the semiaxes of an hyperbola is 17 and its eccentricity is $\frac{13}{12}$. Find its equation, if its axes lie on OX and OY.

51. Find the equation of the hyperbola which has the asymptotes $y = \pm \frac{3}{5}x$ and passes through the point $(1, 1)$.

52. Express the angle between the asymptotes in terms of the eccentricity of the hyperbola.

53. If the vertex of an hyperbola lies two thirds of the distance from the center to the focus, find the slopes of the asymptotes.

54. Given the hyperbola $4x^2 - 25y^2 = 100$. Find its eccentricity, foci, and asymptotes.

55. Find the equation of the hyperbola which has the lines $y = \pm \frac{2}{3}x$ for its asymptotes and the points $(\pm 4, 0)$ for its foci.

56. Show that $\frac{x^2}{a^2 - k^2} + \frac{y^2}{b^2 - k^2} = 1$, where k is an arbitrary quantity, represents an ellipse confocal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when $k^2 < b^2$; and represents an hyperbola confocal to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when $k^2 > b^2$ but $< a^2$, a^2 being considered greater than b^2.

57. Find the equation of an hyperbola when the foci are $(\pm 7, 0)$ and the directrices are $x = \pm 4$.

58. Given the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Find its eccentricity, foci, directrices, and asymptotes.

59. A perpendicular is drawn from a focus of an hyperbola to an asymptote. Show that its foot is at distances a and b from the center and the focus respectively.

60. Show that in an equilateral hyperbola the distance of a point from the center is a mean proportional between its focal distances.

61. Determine p so that the parabola $y^2 = 4px$ shall pass through the point $(2, -3)$.

62. An arch in the form of a parabolic curve is 29 ft. across the bottom and the highest point is 8 ft. above the horizontal. What is the length of a beam placed horizontally across the arch 4 ft. from the top?

63. The cable of a suspension bridge hangs in the form of a parabola. The roadway, which is horizontal and 240 ft. long, is supported by vertical wires attached to the cable, the longest being 80 ft. and the shortest being 30 ft. Find the length of a supporting wire attached to the roadway 50 ft. from the middle.

64. Find the equation of a circle through the vertex and the ends of the double ordinate through the focus of the parabola $y^2 = 4px$.

65. Find the equation of the circle through the vertex, the focus, and the upper end of the ordinate at the focus, of the parabola $y^2 + 12x = 0$.

66. Find the equation of the locus of a point the distances of which from $(3, -2)$ and $(-4, 1)$ are equal.

67. Find the equations of the locus of a point the distance of which from the axis of x equals five times the distance from the axis of y.

68. Find the equation of the locus of a point the distance of which from the axis of x is one third its distance from $(0, 3)$.

69. Find the equation of the locus of a point the distance of which from the line $x = 3$ is equal to its distance from $(4, -2)$.

70. What is the locus of a point the distance of which from the line $3x + 4y - 6 = 0$ is twice its distance from $(2, 1)$?

71. A point moves so that its distance from the axis of y equals its distance from the point $(5, 0)$. Find the equation of its locus.

72. A point moves so that the square of its distance from the point $(0, 2)$ equals the cube of its distance from the axis of y. Find its locus.

73. Find the locus of the points at a constant distance 5 from the line $4x + 3y - 6 = 0$.

74. Find the locus of points equally distant from the lines $2x + 3y - 6 = 0$ and $3x - 2y + 1 = 0$.

75. Show that the locus of a point which moves so that the sum of its distances from two fixed straight lines is constant is a straight line.

76. Find the equations of the locus of a point equally distant from two fixed straight lines.

77. A point moves so that its distances from two fixed points are in a constant ratio k. Show that the locus is a circle except when $k = 1$.

78. A point moves so that the sum of the squares of its distances from the sides of an equilateral triangle is constant. Show that the locus is a circle and find its center.

79. A point moves so that the square of its distance from the base of an isosceles triangle is equal to the product of its distances from the other two sides. Show that the locus is a circle which passes through the vertices of the two base angles.

80. A point moves so that the sum of the squares of its distances from the four sides of a square is constant. Find its locus.

81. A point moves so that the sum of the squares of its distances from any number of fixed points is constant. Find its locus.

82. Find the locus of a point the square of the distance of which from a fixed point is proportional to its distance from a fixed straight line.

83. Find the locus of a point such that the lengths of the tangents from it to two concentric circles are inversely as the radii of the circles.

84. A point moves so that the length of the tangent from it to a fixed circle is equal to its distance from a fixed point. Find its locus.

85. Find the equation of the locus of a point the tangents from which to two fixed circles are of equal length.

86. Straight lines are drawn through the points $(-a, 0)$ and $(a, 0)$ so that the difference of the angles they make with the axis of x is $\tan^{-1}\frac{1}{a}$. Find the locus of their point of intersection.

87. The slope of a straight line passing through $(a, 0)$ is twice the slope of a straight line passing through $(-a, 0)$. Find the locus of the point of intersection of these lines.

88. A point moves so that the product of the slopes of the straight lines joining it to $A(-a, 0)$ and $B(a, 0)$ is constant. Prove that the locus is an ellipse or an hyperbola.

89. If in the triangle ABC $\tan A \tan \frac{1}{2}B = 2$ and AB is fixed, show that the locus of C is a parabola with its vertex at A and focus at B.

90. Given the base $2b$ of a triangle and the sum s of the tangents of the angles at the base. Find the locus of the vertex.

91. Find the locus of the center of a circle which is tangent to a fixed circle and a fixed straight line.

92. Prove that the locus of the center of a circle which passes through a fixed point and is tangent to a fixed straight line is a parabola.

93. A point moves so that its shortest distance from a fixed circle is equal to its distance from a fixed diameter of that circle. Find its locus.

94. O is a fixed point and AB is a fixed straight line. A straight line is drawn from O meeting AB at Q, and in OQ a point P is taken so that $OP \cdot OQ = k^2$. Find the locus of P.

95. If a straight line is drawn from the origin to any point Q of the line $y = a$, and if a point P is taken on this line such that its ordinate is equal to the abscissa of Q, find the locus of P.

96. AOB and COD are two straight lines which bisect each other at right angles. Find the locus of a point P such that $PA \cdot PB = PC \cdot PD$.

97. AB and CD are perpendicular diameters of a circle and M is any point on the circle. Through M, AM and BM are drawn. AM intersects CD in N, and from N a line is drawn parallel to AB meeting BM in P. Find the locus of P.

98. Given a fixed line AB and a fixed point Q. From any point R in AB a perpendicular to AB is drawn equal in length to RQ. Find the locus of the end of this perpendicular.

99. Let OA be the diameter of a fixed circle. From B, any point on the circle, draw a line perpendicular to OA and meeting it in D. Prolong the line DB to P, so that $OD : DB = OA : DP$. Find the locus of P.

100. Two straight lines are drawn through the vertex of a parabola at right angles to each other and meeting the curve at P and Q. Show that the line PQ cuts the axis of the parabola in a fixed point.

101. In the parabola $y^2 = 4px$ an equilateral triangle is so inscribed that one vertex is at the origin. What is the length of one of its sides?

102. Prove that in the ellipse half of the minor axis is a mean proportional between AF and FA'.

103. Prove that in the ellipse or the hyperbola the ordinate at the focus is an harmonic mean between AF and AF'.

104. If from any point P of an hyperbola PK is drawn parallel to the transverse axis, cutting the asymptotes in Q and R, then $PQ \cdot PR = a^2$. If PK is drawn parallel to the conjugate axis, then $PQ \cdot PR = -b^2$.

105. Show that the focal distance of any point on the hyperbola is equal to the length of the straight line drawn through the point parallel to an asymptote to meet the corresponding directrix.

106. Prove that the product of the distances of any point of the hyperbola from the asymptotes is constant.

107. Prove that in the hyperbola the squares of the ordinates of any two points are to each other as the products of the segments of the transverse axis made by the feet of these ordinates.

108. Lines are drawn through a point of an ellipse from the two ends of the minor axis. Show that the product of their intercepts on OX is constant.

109. P_1 is any point of the parabola $y^2 = 4px$, and P_1Q, which is perpendicular to OP_1, intersects the axis of the parabola in Q. Prove that the projection of P_1Q on the axis of the parabola is always $4p$.

CHAPTER VIII

INTERSECTION OF CURVES

86. General principle. If $f_m(x, y)$ is an expression involving x and y,

$$f_m(x, y) = 0 \tag{1}$$

is the equation of a curve containing all points the coördinates of which satisfy (1), and containing no other points. Similarly if $f_n(x, y)$ is any second expression in x and y,

$$f_n(x, y) = 0 \tag{2}$$

is the equation of a second curve. It follows that if we consider these two equations, any point common to the two corresponding curves will have coördinates satisfying both (1) and (2); and that, conversely, any values of x and y which satisfy both (1) and (2) are coördinates of a point common to the two curves. Hence, *to find the points of intersection of two curves, solve their equations simultaneously.*

We have already discussed in § 30 the simplest case of this problem, i.e. the intersection of two straight lines. We shall now discuss some more complex cases.

87. $f_1(x, y) = 0$ **and** $f_2(x, y) = 0$. Let

$$f_1(x, y) = 0 \tag{1}$$

be a linear equation, and $f_2(x, y) = 0$ (2)

be a quadratic equation. Since a linear equation always represents a straight line, this problem is to find the points of intersection of a straight line and a curve. Solving (1) for either x or y, and substituting the result in (2), we obtain in general a quadratic equation, as, for example,

$$ax^2 + bx + c = 0,$$

if (1) has been solved for y. We shall call this equation the *resultant* equation (§ 9). If the roots of this equation are denoted by x_1 and x_2, x_1 and x_2 are the abscissas of the required points of intersection. The corresponding ordinates are found by substituting x_1 and x_2 in succession in (1).

But according to § 37 there are three cases to be considered in the solution of the resultant equation. (1) The roots x_1 and x_2 may be real and unequal, in which case there are two points of intersection. (2) The roots x_1 and x_2 may be real and equal, in which case the corresponding ordinates are equal and the two points coincide. As in § 37, we may regard this case as a limiting case when the position of the curves is changed so as to make x_1 and x_2 approach each other, i.e. so as to make the points of intersection of the straight line and the curve approach each other along the curve. Accordingly, the straight line represented by equation (1) is *tangent* to the curve represented by equation (2). (3) Finally, the roots x_1 and x_2 may be imaginary, in which case no real points of intersection can be found, and the curves do not intersect.

Ex. 1. Find the points of intersection of

$$3x - 2y - 4 = 0 \qquad (1)$$

and

$$x^2 - 4y = 0. \qquad (2)$$

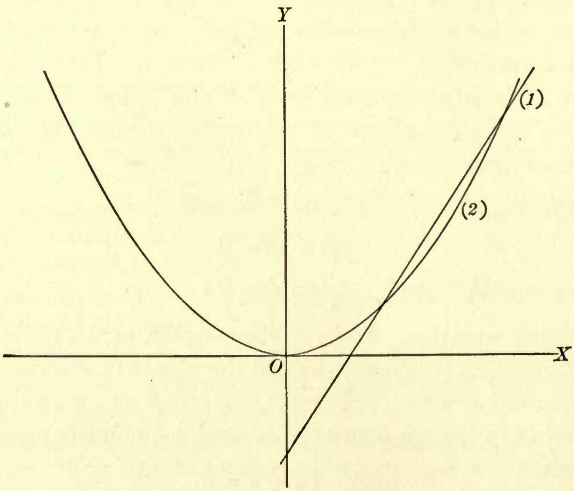

Fig. 95

Solving (1) for y and substituting the result in (2), we have $x^2 - 6x + 8 = 0$, the roots of which are 2 and 4. Substituting these values of x in (1), we find the corresponding values of y to be 1 and 4. Therefore the points of intersection are (2, 1) and (4, 4) (fig. 95).

Ex. 2. Find the points of intersection of

$$6x - 4y - 9 = 0 \quad (1)$$

and

$$x^2 - 4y = 0. \quad (2)$$

Solving (1) for y and substituting the result in (2), we have $x^2 - 6x + 9 = 0$. The roots of this equation are equal, each being 3. Hence the straight line is tangent to the curve. Substituting 3 for x in (1), we find $y = \frac{9}{4}$; hence the point of tangency is $(3, \frac{9}{4})$ (fig. 96).

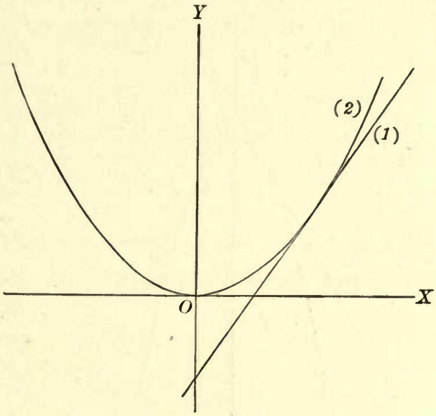

FIG. 96

Ex. 3. Find the points of intersection of

$$3x - 2y - 5 = 0 \quad (1)$$

and

$$x^2 - 4y = 0. \quad (2)$$

Proceeding as in the two previous examples, we obtain $x^2 - 6x + 10 = 0$, the roots of which are $3 \pm \sqrt{-1}$. Hence the straight line does not intersect the curve (fig. 97). The corresponding values of y are $2 \pm \frac{3}{2}\sqrt{-1}$.

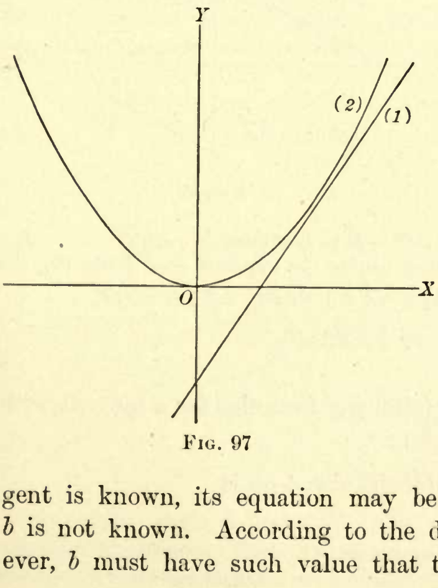

FIG. 97

It is to be noted that the straight lines of these three examples all have the same direction, differing only in the intercept on the axis of y.

88. The work of the last article suggests a method of finding the tangent to any curve represented by an equation of the second degree, the slope of the tangent being given. For if m of the required tangent is known, its equation may be written $y = mx + b$, where b is not known. According to the definition of a tangent, however, b must have such value that the points of intersection of straight line and curve shall be coincident. This condition enables us to determine b, as shown in the following examples.

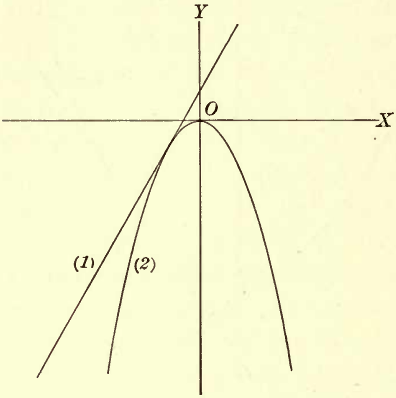

FIG. 98

Ex. 1. Find the equation of the tangent to the parabola $3x^2+2y=0$, the slope of the tangent being 2.

Since the slope of the tangent is 2, its equation may be written $y = 2x + b$. Substituting this value of y in the equation of the parabola, we have the equation

$$3x^2 + 4x + 2b = 0.$$

Since the line is to intersect the curve in two coincident points, this equation must have equal roots. The condition for equal roots, by § 37, is $(4)^2 - 4(3)(2b) = 0$, whence we find $b = \frac{2}{3}$.

Therefore the required tangent is $y = 2x + \frac{2}{3}$, or $6x - 3y + 2 = 0$ (fig. 98).

Ex. 2. Find the equation of the tangent to the ellipse $x^2 + 4y^2 = 4$, the slope of the tangent being $\frac{1}{2}$.

The equation of the tangent is $y = \frac{1}{2}x + b$. Substituting this value of y in the equation of the ellipse, we have

$$x^2 + 2bx + (2b^2 - 2) = 0.$$

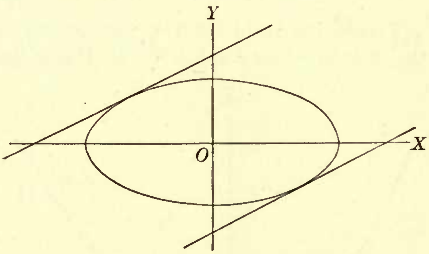

FIG. 99

The condition that this equation shall have equal roots is $(2b)^2 - 4(2b^2 - 2) = 0$, whence $b = \pm\sqrt{2}$.

In this case there are two tangents having the required slope $\frac{1}{2}$ (fig. 99), the equations of which are respectively $y = \frac{1}{2}x + \sqrt{2}$ and $y = \frac{1}{2}x - \sqrt{2}$,

or $$x - 2y \pm 2\sqrt{2} = 0.$$

By this same method the following formulas for a tangent with known slope m may be derived:

1. The tangent to the parabola $y^2 = 4px$ is

$$y = mx + \frac{p}{m}.$$

2. The tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are

$$y = mx \pm \sqrt{a^2m^2+b^2}.$$

3. The tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are

$$y = mx \pm \sqrt{a^2m^2-b^2}.$$

89. It was stated in § 87 that the result of the substitution is in general a quadratic equation. In exceptional cases, however, the resultant equation may be linear, as in the first of the following examples, or even impossible, as in the second example.

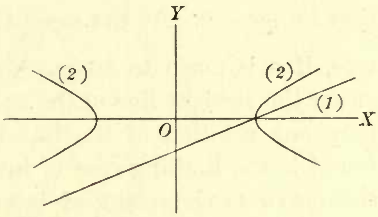

Fig. 100

Ex. 1. Consider

$$2x - 5y - 10 = 0 \qquad (1)$$

and

$$4x^2 - 25y^2 = 100. \qquad (2)$$

Substituting in (2) the value of y from (1), we have the equation $40x - 200 = 0$, whence $x = 5$. $\therefore y = 0$, and the straight line and the curve intersect in a single point (5, 0) (fig. 100).

Ex. 2. Consider

$$2x - 5y + 4 = 0 \qquad (1)$$

and

$$4x^2 - 25y^2 + 16x - 84 = 0. \qquad (2)$$

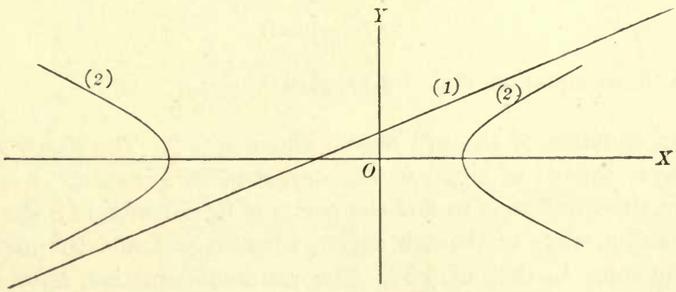

Fig. 101

Substituting in (2) the value of y from (1), we have $-100 = 0$. But this equation is impossible. Hence the given equations are contradictory, and the straight line and the curve do not intersect (fig. 101).

These exceptional cases, of which the above are illustrative examples, may be regarded as limiting cases as follows:

If x_1 and x_2 are the roots of the resultant equation $ax^2 + bx + c = 0$,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{2c}{-b - \sqrt{b^2 - 4ac}},$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{2c}{-b + \sqrt{b^2 - 4ac}}.$$

Now as $a \doteq 0$, the resultant equation approaches the linear equation $bx + c = 0$. At the same time $x_1 \doteq -\frac{c}{b}$ and $x_2 = \infty$. Therefore, if a is made to approach zero by changing the position of either the straight line or the curve in the plane, the case in which only one solution of the linear and the quadratic equations is found is the limiting case of intersection of the straight line and the curve as one point of intersection recedes indefinitely from the origin.

If $a \doteq 0$ and $b \doteq 0$, both x_1 and x_2 increase indefinitely. Hence the case in which the linear and the quadratic equations are contradictory is the limiting case of intersection, as both points of intersection recede indefinitely from the origin.

90. $f_1(x, y) = 0$ and $f_n(x, y) = 0$. Let

$$f_1(x, y) = 0 \qquad (1)$$

be a linear equation, and $f_n(x, y) = 0$ (2)

be an equation of the nth degree where $n > 2$. The *degree of a curve* is defined as equal to the degree of its equation. Accordingly, this problem is to find the points of intersection of a straight line and a curve of the nth degree where $n > 2$, and the method is the same as that of § 87. The resultant equation, after substitution from the linear equation, is, in general, of the nth degree, and its solution is found by the methods of Chaps. IV and V. The number of points of intersection will be the same as the number of real roots of the resultant equation. Hence a straight line can intersect a curve of the nth degree in n points at most. If the resultant equation has multiple roots, they correspond, in general, to points of tangency of the straight line and the curve, as in § 88; and if the resultant equation is of degree less than n, it can be shown that the straight line is the limiting position of one in which one or more points of intersection have been made to recede indefinitely.

Ex. 1. Find the points of intersection of

$$y = 2x \qquad (1)$$

and

$$y^2 = x(x-3)^2. \qquad (2)$$

The resultant equation is

$$x[(x-3)^2 - 4x] = 0,$$

or

$$x[x^2 - 10x + 9] = 0.$$

Its roots (§ 39) are the roots of $x = 0$ and $x^2 - 10x + 9 = 0$, which are 0, 1, and 9. The corresponding values of y are found from (1) to be 0, 2, and 18. Therefore the points of intersection are (0, 0), (1, 2), and (9, 18) (fig. 102).

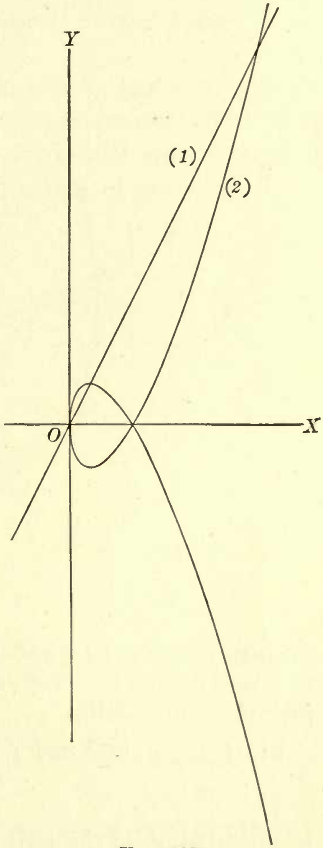

Fig. 102

Ex. 2. Find the points of intersection of

$$y = 3x + 2 \qquad (1)$$

and

$$y = x^3. \qquad (2)$$

The resultant equation is $x^3 - 3x - 2 = 0$. One root is found (§ 49) to be 2, and the depressed equation is $x^2 + 2x + 1 = 0$. Its roots are equal, both being -1. The corresponding values of y, found from (1), are 8 and -1. Therefore the points of intersection are (2, 8) and $(-1, -1)$, the latter being a point of tangency (fig. 103).

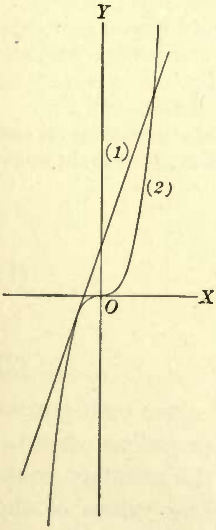

Fig. 103

Ex. 3. Find the points of intersection of

$$2x + y - 4 = 0 \tag{1}$$

and

$$y^2 = x(x^2 - 12). \tag{2}$$

The resultant equation is $x^3 - 4x^2 + 4x - 16 = 0$, or $(x - 4)(x^2 + 4) = 0$, the roots of which are 4 and $\pm 2\sqrt{-1}$. The corresponding values of y, found

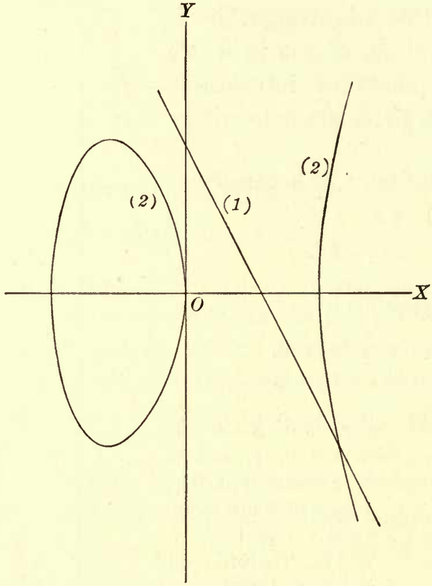

FIG. 104

from (1), are -4 and $4 \mp 4\sqrt{-1}$. The only real solution of equations (1) and (2) being $x = 4$ and $y = -4$, the straight line and the curve intersect in the single point $(4, -4)$ (fig. 104).

91. $f_m(x, y) = 0$ and $f_n(x, y) = 0$. Let

$$f_m(x, y) = 0 \tag{1}$$

be an equation of the mth degree and

$$f_n(x, y) = 0 \tag{2}$$

be an equation of the nth degree, where m and n are both greater than unity. The method is the same as in the preceding cases, i.e. the elimination of either x or y, the solution of the resultant equation, and the determination of the corresponding values of the unknown quantity eliminated. The equation resulting from the elimination is, in general, of degree mn, and the number of simultaneous solutions of the original equations is mn. If all these solutions are real, the corresponding curves intersect at mn points. If, however, any of these solutions are imaginary, or are alike, if real, the corresponding curves will intersect at a number of points less than mn. Hence, *two curves of degrees m and n respectively can intersect at mn points and no more.*

No attempt at a complete discussion will be made, on account of the unlimited number of cases which are possible. We shall merely solve a few illustrative examples, noting any interesting geometrical facts that may occur in the course of the solution.

Ex. 1. Find the points of intersection of

$$y^2 - 2x = 0 \quad (1)$$

and

$$x^2 + y^2 - 8 = 0. \quad (2)$$

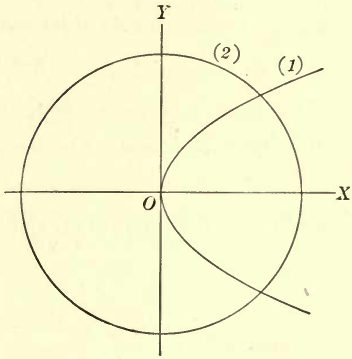

FIG. 105

Subtracting (1) from (2), we eliminate y, thereby obtaining the resultant equation $x^2 + 2x - 8 = 0$, the roots of which are -4 and 2. Substituting 2 and -4 in either (1) or (2), we find the corresponding values of y to be ± 2 and $\pm 2\sqrt{-2}$. The real solutions of the equations are accordingly $x = 2$, $y = \pm 2$, and the corresponding curves intersect at the points $(2, 2)$ and $(2, -2)$ (fig. 105).

From the figure it is also evident that the value -4 for x must make y imaginary, as both curves lie entirely to the right of the line $x = -4$.

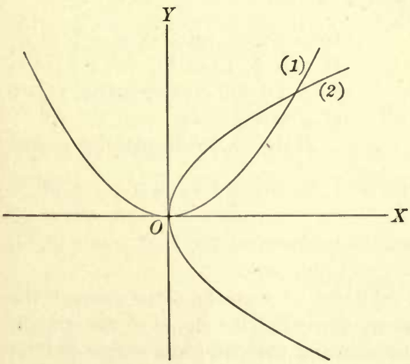

FIG. 106

Ex. 2. Find the points of intersection of

$$x^2 - 3y = 0 \quad (1)$$

and

$$y^2 - 3x = 0. \quad (2)$$

Substituting in (2) the value of y from (1), we have $x^4 - 27x = 0$. This equation may be written

$$x(x-3)(x^2 + 3x + 9) = 0,$$

the roots of which are 0, 3, and $\dfrac{-3 \pm 3\sqrt{-3}}{2}$. Substituting these values of x in (1), we find the corresponding values of y to be 0, 3, and $\dfrac{-3 \mp 3\sqrt{-3}}{2}$. Therefore the real solutions of these equations are $x = 0$, $y = 0$, and $x = 3$, $y = 3$. If we had substituted the values of x in (2), we should have at first seemed to find an additional real solution, $y = -3$ when $x = 3$. But -3 for y makes x imaginary in (1), as no part of (1) is below the axis of x. Geometrically, the line $x = 3$ intersects the curves (1) and (2) in a common point and also intersects (2) in another point. Therefore the only real solutions of these equations are the ones noted above, and the corresponding curves intersect at the two points (0, 0) and (3, 3) (fig. 106).

We see, moreover, that any results found must be tested by substitution in both of the original equations.

The remaining two solutions of these equations found by letting $x = \dfrac{-3 \pm 3\sqrt{-3}}{2}$ are imaginary.

Ex. 3. Find the points of intersection of

$$2x^2 + 3y^2 = 35 \qquad (1)$$

and

$$xy = 6. \qquad (2)$$

Since these equations are homogeneous quadratic equations we place

$$y = mx \qquad (3)$$

and substitute for y in both (1) and (2). The results are $2x^2 + 3m^2x^2 = 35$ and $mx^2 = 6$, whence

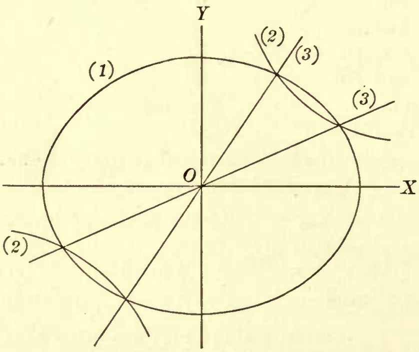

Fig. 107

$$x^2 = \frac{35}{2 + 3m^2} \qquad (4)$$

and

$$x^2 = \frac{6}{m}. \qquad (5)$$

$$\therefore \frac{35}{2 + 3m^2} = \frac{6}{m}, \qquad (6)$$

from which we find $m = \frac{3}{2}$ or $\frac{4}{9}$. If $m = \frac{3}{2}$, from (5) $x = \pm 2$; and from (3) the corresponding values of y are ± 3.

If $m = \frac{4}{9}$, in like manner we find

$$x = \pm \tfrac{3}{2}\sqrt{6} \text{ and } y = \pm \tfrac{2}{3}\sqrt{6}.$$

Therefore the ellipse and the hyperbola intersect at the four points (2, 3), $(-2, -3)$, $(\frac{3}{2}\sqrt{6}, \frac{2}{3}\sqrt{6})$, $(-\frac{3}{2}\sqrt{6}, -\frac{2}{3}\sqrt{6})$ (fig. 107).

It should be noted that (3) is the equation of a straight line through the origin, so that when we solve (6) for m we determine the slopes of the straight lines passing through the origin and intersecting the two given curves at their common points.

Ex. 4. Find the points of intersection of

$$2y^2 = x - 2 \qquad (1)$$

and $$x^2 - 4y^2 = 4. \qquad (2)$$

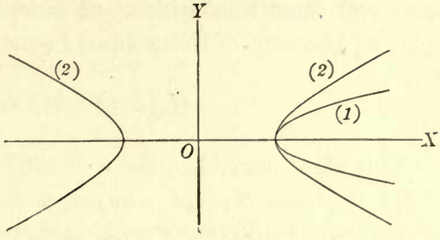

FIG. 108

Eliminating y, we have

$$x^2 - 2x = 0,$$

the roots of which are 0 and 2. When $x = 0$ we find from either (1) or (2) $y = \pm\sqrt{-1}$, and when $x = 2$ either (1) or (2) reduces to $y^2 = 0$, whence $y = 0$. Therefore these two curves are tangent at the point (2, 0) (fig. 108).

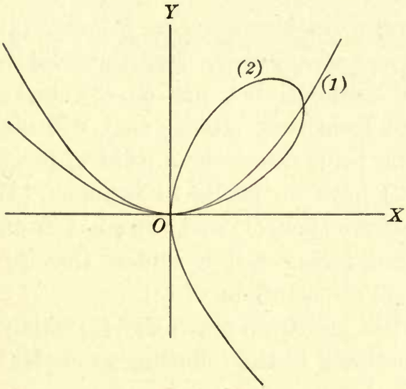

FIG. 109

Ex. 5. Find the points of intersection of

$$x^2 = 2y \qquad (1)$$

and $$x^3 - 3xy + y^3 = 0. \qquad (2)$$

Eliminating y, we have

$$x^6 - 4x^3 = 0,$$

which may be written $x^3(x^3 - 4) = 0$. The real roots of this equation are $4^{\frac{1}{3}}$ and 0, the latter being a triple root, and the two imaginary roots are $\frac{4^{\frac{1}{3}}(-1 \pm \sqrt{-3})}{2}$. Corresponding values of y are found to be $2^{\frac{1}{3}}$, 0, and $2^{-\frac{2}{3}}(-1 \mp \sqrt{-3})$. Therefore the curves intersect at the two points $(4^{\frac{1}{3}}, 2^{\frac{1}{3}})$ and (0, 0) (fig. 109).

At the point (0, 0) the parabola (1) is tangent to one part of (2) and passes through another part of (2), and for this reason the point is to be regarded as a triple point of intersection.

92. $lf_m(x, y) + kf_n(x, y) = 0$. If we have two expressions $f_m(x, y)$ and $f_n(x, y)$, we have seen in § 86 that we can form the equations of two curves by placing each of them separately equal to zero, i.e.

$$f_m(x, y) = 0, \qquad (1)$$

and $$f_n(x, y) = 0. \qquad (2)$$

Let us now form the equation of a third curve by multiplying $f_m(x, y)$ and $f_n(x, y)$ by l and k respectively, where l and k are any two quantities which are independent of both x and y, and placing the sum of the products equal to zero, i.e.

$$lf_m(x, y) + kf_n(x, y) = 0. \tag{3}$$

This third curve has the following two important properties:

1. *It passes through all points common to curves* (1) *and* (2). For the coördinates of any such points make $f_m(x, y) = 0$ and $f_n(x, y) = 0$, since they satisfy (1) and (2). Hence they will satisfy (3), i.e. be coördinates of a point of curve (3).

If either l or k is placed equal to zero, (3) reduces to either (2) or (1) as a special case.

2. *If neither l nor k is zero, it intersects curves* (1) *and* (2) *at no other points than their common points.* For the coördinates of any point on (1), for example, but not on (2), make $f_m(x, y) = 0$ and $f_n(x, y)$ different from zero. Hence they will not satisfy (3), and the corresponding point cannot be a point of (3).

It follows that if (1) and (2) have no points in common, (3) intersects neither (1) nor (2). If we treat (1) and (2) apart from possible geometrical interpretation, however, it is evident that the imaginary solutions of (1) and (2) are solutions of (3).

By assigning different values to l and k we may make (3) satisfy another condition, as will be illustrated in the following examples:

Ex. 1. Find the equation of the straight line passing through the point of intersection of the lines

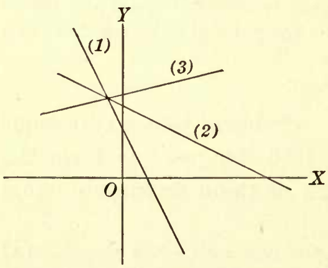

Fig. 110

$$2x + y - 1 = 0 \tag{1}$$

and

$$x + 2y - 3 = 0 \tag{2}$$

and the point (1, 2).

$$l(2x + y - 1) + k(x + 2y - 3) = 0 \tag{3}$$

passes through the point of intersection of (1) and (2), and is the equation of a straight line, since it is an equation of the first degree. Since (3) is to pass through the point (1, 2), (1, 2) must satisfy (3). Therefore $l(2 + 2 - 1) + k(1 + 4 - 3) = 0$, or $3l + 2k = 0$. Therefore, if we substitute $k = -\frac{3}{2}l$ in (3) and simplify, we shall have the equation of the required line. It is found to be $x - 4y + 7 = 0$ (fig. 110).

Ex. 2. Find the equation of a straight line passing through the point of intersection of the lines

$$x - 2y + 1 = 0 \tag{1}$$

and

$$x - 3y + 3 = 0, \tag{2}$$

and parallel to the line

$$2x + 3y + 8 = 0. \tag{3}$$

As in Ex. 1, the required line is

$$l(x - 2y + 1) + k(x - 3y + 3) = 0, \tag{4}$$

which may be written

$$(l + k)x + (-2l - 3k)y + (l + 3k) = 0.$$

Since this line is to be parallel to (3), $\frac{l+k}{2} = \frac{-2l-3k}{3}$ (§ 28, 2), whence $k = -\frac{7}{9}l$. Substituting this value of k in (4) and simplifying, we have as our required line $2x + 3y - 12 = 0$ (fig. 111).

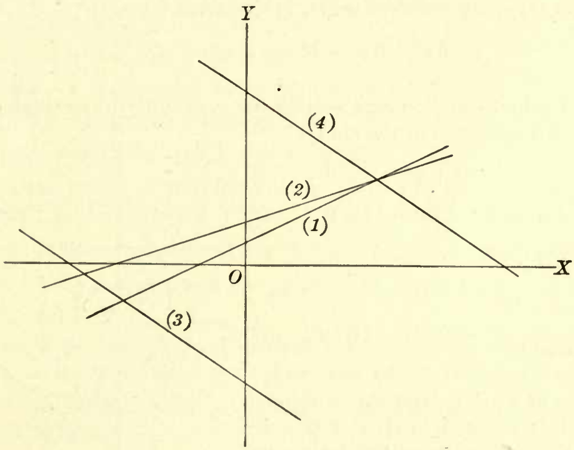

FIG. 111

Both of these examples could also have been solved by finding the point of intersection of the given lines and then, by the methods of Chap. III, passing the line through the point subject to the given condition.

93. In the two examples of the last article both equations were of the first degree. In this article we will solve some examples in which one or both equations are of the second degree.

Ex. 1. Find the equation of a circle determined by the points of intersection of the straight line

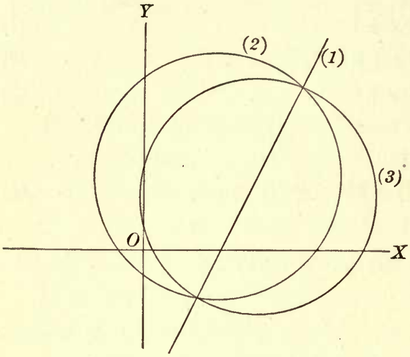

FIG. 112

$$2x - y - 6 = 0 \quad (1)$$

and the circle

$$x^2 + y^2 - 6x - 6y - 7 = 0 \quad (2)$$

and the point (1, − 1) (fig. 112).

The equation

$$l(2x - y - 6) + k(x^2 + y^2 - 6x - 6y - 7) = 0 \quad (3)$$

is the equation of a circle, since the coefficients of x^2 and y^2 are equal; and since it passes through the points of intersection of (1) and (2), it only remains to choose l and k so that it shall pass through the point (1, −1). Substituting (1, − 1) in (3), we have $3l + 5k = 0$, whence $k = -\frac{3l}{5}$. Accordingly, the equation (3) of the required circle, in simplified form, is

$$3x^2 + 3y^2 - 28x - 13y + 9 = 0.$$

Ex. 2. Find an equation representing the system of circles passing through the points of intersection of the circles

$$x^2 + y^2 - 9 = 0 \quad (1)$$

and $$x^2 + y^2 - 4x - 2y - 11 = 0 \quad (2)$$

(fig. 113).

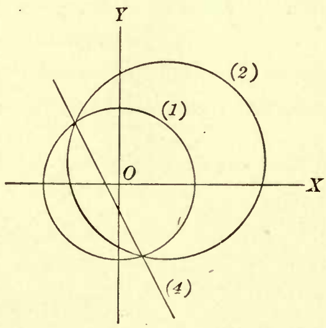

FIG. 113

The equation

$$l(x^2 + y^2 - 9) + k(x^2 + y^2 - 4x - 2y - 11) = 0 \quad (3)$$

is the required equation, for by its form it is the equation of a circle, and passes through the two points common to (1) and (2). By assigning different values to l and k we can make (3) represent any, and hence every, circle passing through the common points of (1) and (2). In other words, it represents the required system of circles.

In particular, if l and k are assigned such values as to make the coefficients of x^2 and y^2 vanish, i.e. $k = -l$ in this example, the equation reduces to

$$2x + y + 1 = 0. \quad (4)$$

But this is the equation of a straight line, and since it must, from the way in which it was formed, pass through the points common to the two circles, it must be the equation of their common chord.

In general, if
$$x^2 + y^2 + 2G_1x + 2F_1y + C_1 = 0, \quad (5)$$
$$x^2 + y^2 + 2G_2x + 2F_2y + C_2 = 0, \quad (6)$$
are the equations of any two circles, we derive the third equation
$$2(G_1 - G_2)x + 2(F_1 - F_2)y + (C_1 - C_2) = 0 \quad (7)$$
by assuming $k = -l$.

If the circles intersect, (7) is the equation of their common chord; but if they do not intersect, (7) is called their *radical axis*. It may easily be proved to be perpendicular to the line of centers and is the locus of points from which equal tangents, one to each circle, may be drawn.

PROBLEMS

Find where and how the straight line intersects the curve of the second degree in each of the following cases:

1. $2x + 3y = 5$, $4x^2 + 9y^2 + 16x - 18y - 11 = 0$.

2. $x - y + 1 = 0$, $(x + 2)^2 - 4y = 0$.

3. $x - 2y + 4 = 0$, $2x^2 - y^2 + 8x + 2y + 13 = 0$.

4. $y - 2x = 0$, $x^2 + y^2 - x + 3y = 0$.

5. $x - 2y + 4 = 0$, $5x^2 - 4y^2 + 20 = 0$.

6. $y = 8x - 5$, $2x^2 + xy - 3y^2 + 6x + 4y + 4 = 0$.

7. $2x + 3y - 6 = 0$, $x^2 + 4y^2 - 4 = 0$.

8. $x + y - 4 = 0$, $x^2 - 2xy + y^2 - 20 = 0$.

9. Find the length of the chord of the circle $x^2 + y^2 + 8x - 4y + 10 = 0$ cut from the line $2x - 3y + 3 = 0$.

10. Find the tangent to the curve $x^2 + 6x - 2y + 5 = 0$ with slope 2.

11. For what value of p will the parabola $y^2 = 4px$ be tangent to the line $y - 3x + 1 = 0$?

12. Find the tangents to the ellipse $4x^2 + 9y^2 = 36$ which are parallel to the line joining the positive ends of the axes.

13. Find the tangent to the curve $b^2x^2 + a^2y^2 + 2ab^2x = 0$ perpendicular to the line $ax + by = ab$.

14. Prove that the line $y = -mx + 2c\sqrt{m}$ is always tangent to the hyperbola $xy = c^2$, and that the point of contact is $\left(\frac{c}{\sqrt{m}},\ c\sqrt{m}\right)$.

15. Find the point of contact of the tangent to the curve
$$x^2 - 4y^2 + 2xy - 2x + 4y = 0 \text{ with slope } \tfrac{1}{2}.$$

16. Find the points of intersection of the line $8y - 25x = 0$ and the curve $x^2y^2 + 36 = 4y^2$.

17. Find the points of intersection of the line $y = 2x - 3$ and the curve $4y^2 = (x + 3)(2x - 3)^2$.

18. Find the points of intersection of the line $x - 2y + 2 = 0$ and the cissoid $x(x^2 + y^2) = 4y^2$.

19. Find the points of intersection of the line $x = 2y$ and the curve $16y^2 = 4x^4 - x^6$.

20. Find the points of intersection of the line $y = 2x - 2$ and the cissoid $x(x^2 + y^2) = 4y^2$.

21. Find the points of intersection of the line $y = mx$ and the cissoid $x(x^2 + y^2) = 2ay^2$.

22. Find the points of intersection of the line $x - y - 1 = 0$ and the witch $y = \dfrac{8}{x^2 + 4}$.

Find the points of intersection of the following pairs of curves:

23. $4y^2 = x^2(x + 1)$, $y^2 = x(x + 1)^2$.

24. $y^2 = 12x$, $y^2 = (x + 2)(x - 3)^2$.

25. $x^2 = y^2(y + 2)$, $x^2 = (y - 1)^2(y + 1)$.

26. Find the points of intersection of the parabolas $y^2 = 4ax + 4a^2$ and $y^2 = -4bx + 4b^2$.

27. Find the points of intersection of the parabola $x^2 = 4ay$ and the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$.

28. Find the points of intersection of the cissoid $y^2 = \dfrac{x^3}{2a - x}$ and the parabola $y^2 = 4ax$.

29. Find the points of intersection of the cissoid $y^2 = \dfrac{x^3}{2a - x}$ and the circle $x^2 + y^2 - 4ax = 0$.

30. Find the points of intersection of the hyperbola $xy = 2a^2$ and the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$.

31. Find the points of intersection of the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$ and the cissoid $x^2 = \dfrac{4y^3}{5a - 4y}$.

32. Find the points of intersection of the circle $x^2 + y^2 = 5a^2$ and the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$.

33. Find the equation of a straight line through the point of intersection of $7x - y - 18 = 0$ (1) and $x - 3y - 14 = 0$ (2) and the point $(-2, 1)$, without finding the point of intersection of (1) and (2).

34. Find the equation of a straight line through the point of intersection of $2x - y + 5 = 0$ (1) and $x - 4y + 13 = 0$ (2) and parallel to the line $2x + 5y + 2 = 0$, without finding the point of intersection of (1) and (2).

35. Find the equation of a straight line through the point of intersection of $4x - 6y - 5 = 0$ (1) and $6x - 4y - 5 = 0$ (2) and perpendicular to the line $x - 3y + 1 = 0$, without finding the point of intersection of (1) and (2).

36. A circle passes through the origin of coördinates and the points of intersection of the circle $x^2 + y^2 = 14$ and the line $2x + 3y + 5 = 0$. Find its equation.

37. Prove that (1, 1) is a point of the common chord of the two circles $x^2 + y^2 - 4x = 0$ and $x^2 + y^2 - 4y = 0$.

38. Find the circle passing through (1, -3) and the points of intersection of the two circles $x^2 + y^2 - 4x - 4y - 8 = 0$ and $x^2 + y^2 + x + y - 4 = 0$.

39. Find a curve of the second degree passing through (1, 1) and the points of intersection of the curves $3x^2 + 5y^2 - 15 = 0$ and $2x^2 - 3y^2 - 6 = 0$, and tell what kind of a curve it is.

40. Prove that a parabola can be passed through the points of intersection of the curves $x^2 - 2y^2 - x + 2y - 1 = 0$ and $3x^2 + 4y^2 + 2x + 2 = 0$.

41. The center of a circle is at the vertex A of a parabola $y^2 = 4px$, and its diameter is $3AF$, F being the focus of the parabola. Prove that their common chord bisects AF.

42. Show that the circle described on any focal radius of a parabola as diameter is tangent to the tangent at the vertex of the parabola.

43. Show that the circle described on any focal chord of a parabola as a diameter is tangent to the directrix of the parabola.

44. If a circle is described from a focus of an hyperbola as center, with its radius equal to half the conjugate axis, prove that it will touch the asymptotes at the points where they intersect the corresponding directrix.

CHAPTER IX

DIFFERENTIATION OF ALGEBRAIC FUNCTIONS

94. Theorems on limits. In operations with limits the following propositions are of importance.

1. *The limit of the sum of a finite number of variables is equal to the sum of the limits of the variables.*

We will prove the theorem for three variables; the proof is easily extended to any number of variables.

Let X, Y, and Z be three variables, such that $\text{Lim}\, X = A$, $\text{Lim}\, Y = B$, $\text{Lim}\, Z = C$. From the definition of limit (§ 53) we may write $X = A + a$, $Y = B + b$, $Z = C + c$, where a, b, and c are three quantities each of which becomes and remains numerically less than any assigned quantity as the variables approach their limits.

Adding, we have

$$X + Y + Z = A + B + C + a + b + c.$$

Now if ϵ is any assigned quantity, however small, we may make a, b, and c each numerically less than $\frac{\epsilon}{3}$, so that $a + b + c$ is numerically less than ϵ. Then the difference between $X + Y + Z$ and $A + B + C$ becomes and remains less than ϵ, that is,

$$\text{Lim}\,(X + Y + Z) = A + B + C = \text{Lim}\, X + \text{Lim}\, Y + \text{Lim}\, Z.$$

2. *The limit of the product of a finite number of variables is equal to the product of the limits of the variables.*

Consider first two variables X and Y such that $\text{Lim}\, X = A$ and $\text{Lim}\, Y = B$. As before, we have $X = A + a$ and $Y = B + b$. Hence

$$XY = AB + bA + aB + ab.$$

Now we may make a and b so small that bA, aB, and ab are each less than $\frac{\epsilon}{3}$, where ϵ is any assigned quantity, no matter how small. Hence

$$\text{Lim}\, XY = AB = (\text{Lim}\, X)(\text{Lim}\, Y).$$

Consider now three variables X, Y, Z. Place $XY = U$. Then, as just proved,

$$\text{Lim}\, UZ = (\text{Lim}\, U)(\text{Lim}\, Z);$$

that is,

$$\text{Lim}\, XYZ = (\text{Lim}\, XY)(\text{Lim}\, Z)$$
$$= (\text{Lim}\, X)(\text{Lim}\, Y)(\text{Lim}\, Z).$$

Similarly the theorem may be proved for any finite number of variables.

3. *The limit of a constant times a variable is equal to the constant times the limit of the variable.*

The proof is left for the student.

4. *The limit of the quotient of two variables is equal to the quotient of the limits of the variables, provided the limit of the divisor is not zero.*

Let X and Y be two variables such that $\text{Lim}\, X = A$ and $\text{Lim}\, Y = B$. Then, as before, $X = A + a$, $Y = B + b$.

Hence $\frac{X}{Y} = \frac{A+a}{B+b}$, and $\frac{X}{Y} - \frac{A}{B} = \frac{A+a}{B+b} - \frac{A}{B} = \frac{aB - bA}{B^2 + bB}$.

Now the fraction on the right of this equation may be made less than any assigned quantity by taking a and b sufficiently small.

Hence

$$\text{Lim}\, \frac{X}{Y} = \frac{A}{B} = \frac{\text{Lim}\, X}{\text{Lim}\, Y}.$$

The proof assumes that B is not zero.

95. Theorems on derivatives. The definitions of increment, continuity, and derivative given in Chap. V are perfectly general, although they are there applied only to algebraic polynomials.

In order to extend the process of differentiation to other functions, we shall need the following theorems:

1. *The derivative of a function plus a constant is equal to the derivative of the function.*

Let u be a function of x which can be differentiated, let c be a constant, and place

$$y = u + c.$$

Then if x is increased by an increment Δx, u is increased by an increment Δu, and c is unchanged. Hence the value of y becomes $u + \Delta u + c$.

Whence $$\Delta y = (u + \Delta u + c) - (u + c) = \Delta u.$$

Therefore $$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x},$$

and, taking the limit of each side of this equation,

$$\frac{dy}{dx} = \frac{du}{dx}.$$

Ex. $y = 4x^3 + 3$,

$$\frac{dy}{dx} = \frac{d}{dx}(4x^3) = 12x^2.$$

2. *The derivative of a constant times a function is equal to the constant times the derivative of the function.*

Let u be a function of x which can be differentiated, let c be a constant, and place

$$y = cu.$$

Give x an increment Δx, and let Δu and Δy be the corresponding increments of u and y. Then

$$\Delta y = c(u + \Delta u) - cu = c\Delta u.$$

Hence $$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x}$$

and $$\text{Lim}\frac{\Delta y}{\Delta x} = c\,\text{Lim}\frac{\Delta u}{\Delta x}. \qquad \text{(by theorem 2, § 94)}$$

Therefore $$\frac{dy}{dx} = c\frac{du}{dx},$$

by the definition of a derivative.

Ex. $y = 5(x^3 + 3x^2 + 1)$,

$$\frac{dy}{dx} = 5\frac{d}{dx}(x^3 + 3x^2 + 1) = 5(3x^2 + 6x) = 15(x^2 + 2x).$$

3. *The derivative of the sum of a finite number of functions is equal to the sum of the derivatives of the functions.*

Let u, v, and w be three functions of x which can be differentiated, and let

$$y = u + v + w.$$

Give x an increment Δx, and let the corresponding increments of u, v, w, and y be Δu, Δv, Δw, and Δy. Then

$$\begin{aligned}\Delta y &= (u + \Delta u + v + \Delta v + w + \Delta w) - (u + v + w) \\ &= \Delta u + \Delta v + \Delta w;\end{aligned}$$

whence

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} + \frac{\Delta w}{\Delta y}.$$

Now let Δx approach zero. By theorem 1, § 94,

$$\text{Lim}\frac{\Delta y}{\Delta x} = \text{Lim}\frac{\Delta u}{\Delta x} + \text{Lim}\frac{\Delta v}{\Delta x} + \text{Lim}\frac{\Delta w}{\Delta x};$$

that is, by the definition of a derivative,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx}.$$

The proof is evidently applicable to any finite number of functions.

Ex. $y = x^4 - 3x^3 + 2x^2 - 7x$,

$$\frac{dy}{dx} = 4x^3 - 9x^2 + 4x - 7.$$

4. *The derivative of the product of a finite number of functions is equal to the sum of the products obtained by multiplying the derivative of each factor by all the other factors.*

Let u and v be two functions of x which can be differentiated, and let

$$y = uv.$$

Give x an increment Δx, and let the corresponding increments of u, v, and y be Δu, Δv, and Δy.

Then
$$\Delta y = (u + \Delta u)(v + \Delta v) - uv$$
$$= u\,\Delta v + v\,\Delta u + \Delta u \cdot \Delta v$$

and
$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\cdot \Delta v.$$

If now Δx approaches zero, we have
$$\operatorname{Lim}\frac{\Delta y}{\Delta x} = u \operatorname{Lim}\frac{\Delta v}{\Delta x} + v \operatorname{Lim}\frac{\Delta u}{\Delta x} + \operatorname{Lim}\frac{\Delta u}{\Delta x}\cdot \operatorname{Lim}\Delta v. \quad (§ 94)$$

But $\operatorname{Lim}\frac{\Delta y}{\Delta x} = \frac{dy}{dx}$, $\operatorname{Lim}\frac{\Delta u}{\Delta x} = \frac{du}{dx}$, $\operatorname{Lim}\frac{\Delta v}{\Delta x} = \frac{dv}{dx}$, and $\operatorname{Lim}\Delta v = 0$;

therefore
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Again, let $y = uvw$.

Regarding uv as one function and applying the result already obtained, we have
$$\frac{dy}{dx} = uv\frac{dw}{dx} + w\frac{d(uv)}{dx}$$
$$= uv\frac{dw}{dx} + w\left[u\frac{dv}{dx} + v\frac{du}{dx}\right]$$
$$= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}.$$

The proof is clearly applicable to any finite number of factors.

Ex. $y = (3x-5)(x^2+1)x^3$,
$$\frac{dy}{dx} = (3x-5)(x^2+1)\frac{d(x^3)}{dx} + (3x-5)x^3\frac{d(x^2+1)}{dx} + (x^2+1)x^3\frac{d(3x-5)}{dx}$$
$$= (3x-5)(x^2+1)(3x^2) + (3x-5)x^3(2x) + (x^2+1)x^3(3)$$
$$= (18x^3 - 25x^2 + 12x - 15)x^2.$$

5. *The derivative of a fraction is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Let $y = \frac{u}{v}$ where u and v are two functions of x which can be differentiated. Let Δx, Δu, Δv, and Δy be as usual. Then

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{v\,\Delta u - u\,\Delta v}{v^2 + v\,\Delta v}$$

and
$$\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v^2 + v\,\Delta v}.$$

Now let Δx approach zero. By § 94,

$$\text{Lim}\,\frac{\Delta y}{\Delta x} = \frac{v\,\text{Lim}\,\frac{\Delta u}{\Delta x} - u\,\text{Lim}\,\frac{\Delta v}{\Delta x}}{v^2 + v\,\text{Lim}\,\Delta v};$$

whence
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Ex. $y = \frac{x^2 - 1}{x^2 + 1}$,

$$\frac{dy}{dx} = \frac{(x^2 + 1)(2x) - (x^2 - 1)2x}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}.$$

6. *If y is a function of x, then x is a function of y, and the derivative of x with respect to y is the reciprocal of the derivative of y with respect to x.*

Let Δx and Δy be corresponding increments of x and y. Then

$$\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}},$$

whence
$$\text{Lim}\,\frac{\Delta x}{\Delta y} = \frac{1}{\text{Lim}\,\frac{\Delta y}{\Delta x}};$$

that is,
$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

7. *If y is a function of u and u is a function of x, then y is a function of x, and the derivative of y with respect to x is equal to the derivative of y with respect to u times the derivative of u with respect to x.*

An increment Δx determines an increment Δu, and this in turn determines an increment Δy. Then evidently

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x},$$

whence
$$\text{Lim}\,\frac{\Delta y}{\Delta x} = \text{Lim}\,\frac{\Delta y}{\Delta u} \cdot \text{Lim}\,\frac{\Delta u}{\Delta x};$$

that is,
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Ex. $y = u^2 + 3u + 1$, where $u = \frac{1}{x^2}$,

$$\frac{dy}{dx} = (2u + 3)\left(-\frac{2}{x^3}\right) = -\frac{2 + 3x^2}{x^2} \cdot \frac{2}{x^3} = -\frac{4 + 6x^2}{x^5}.$$

The same result is obtained by substituting in the expression for y the value of u in terms of x, and then differentiating.

96. Formulas. The formulas proved in the previous article are:

$$\frac{d(u + c)}{dx} = \frac{du}{dx}, \tag{1}$$

$$\frac{d(cu)}{dx} = c\frac{du}{dx}, \tag{2}$$

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \tag{3}$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \tag{4}$$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, \tag{5}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \tag{6}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \tag{7}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}}. \tag{8}$$

Formula (8) is a combination of (6) and (7).

97. Derivative of u^n. If u is any function of x which can be differentiated and n is any real constant, then

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

To prove this formula we shall distinguish four cases:

1. When n is a positive integer.

$$\frac{d(u^n)}{dx} = \frac{d(u^n)}{du} \cdot \frac{du}{dx} \qquad \text{(by (7), § 96)}$$

$$= nu^{n-1}\frac{du}{dx}. \qquad \text{(by (1), § 58)}$$

2. When n is a positive rational fraction.

Let $n = \frac{p}{q}$ where p and q are positive integers, and place

$$y = u^{\frac{p}{q}}.$$

By raising both sides of this equation to the qth power we have

$$y^q = u^p.$$

Here we have two functions of x which are equal for all values of x. If we give x an increment Δx, we have

$$\Delta(y^q) = \Delta(u^p),$$

$$\frac{\Delta(y^q)}{\Delta x} = \frac{\Delta(u^p)}{\Delta x};$$

and therefore
$$\frac{d(y^q)}{dx} = \frac{d(u^p)}{dx},$$

whence
$$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx},$$

since p and q are positive integers. Substituting the value of y and dividing, we have

$$\frac{dy}{dx} = \frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx}.$$

Hence in this case also

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

3. When n is a negative rational number.

Let $n = -m$, where m is a positive number, and place

$$y = u^{-m} = \frac{1}{u^m}.$$

Then
$$\frac{dy}{dx} = \frac{-\dfrac{d(u^m)}{dx}}{u^{2m}} \qquad \text{(by (5), § 96)}$$
$$= -\frac{mu^{m-1}\dfrac{du}{dx}}{u^{2m}} \qquad \text{(by 1 and 2)}$$
$$= -mu^{-m-1}\frac{du}{dx}.$$

Hence in this case also

$$\frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}.$$

4. When n is an irrational number.

The formula is true in this case also, but the proof will not be given.

As a particular case of this formula, it appears that 1, § 58, is true for all real values of n.

Ex. 1. $y = (x^3 + 4x^2 - 5x + 7)^3,$

$$\frac{dy}{dx} = 3(x^3 + 4x^2 - 5x + 7)^2 \frac{d}{dx}(x^3 + 4x^2 - 5x + 7)$$
$$= 3(3x^2 + 8x - 5)(x^3 + 4x^2 - 5x + 7)^2. \qquad \text{(by § 58)}$$

Ex. 2. $y = \sqrt[3]{x^2} + \dfrac{1}{x^3} = x^{\frac{2}{3}} + x^{-3},$

$$\frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}} - 3x^{-4} \qquad \text{(by (3), § 96)}$$
$$= \frac{2}{3\sqrt[3]{x}} - \frac{3}{x^4}.$$

Ex. 3. $y = (x+1)\sqrt{x^2+1},$

$$\frac{dy}{dx} = (x+1)\frac{d(x^2+1)^{\frac{1}{2}}}{dx} + (x^2+1)^{\frac{1}{2}}\frac{d(x+1)}{dx} \qquad \text{(by (4), § 96)}$$
$$= (x+1)\left[\tfrac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x\right] + (x^2+1)^{\frac{1}{2}}$$
$$= \frac{x(x+1)}{(x^2+1)^{\frac{1}{2}}} + (x^2+1)^{\frac{1}{2}}$$
$$= \frac{2x^2 + x + 1}{\sqrt{x^2+1}}.$$

Ex. 4. $y = \sqrt[3]{\frac{x}{x^3+1}} = \left(\frac{x}{x^3+1}\right)^{\frac{1}{3}}$,

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{x}{x^3+1}\right)^{-\frac{2}{3}} \frac{d}{dx}\left(\frac{x}{x^3+1}\right)$$

$$= \frac{1}{3}\left(\frac{x^3+1}{x}\right)^{\frac{2}{3}} \frac{1-2x^3}{(x^3+1)^2} \qquad \text{(by (5), § 96)}$$

$$= \frac{1-2x^3}{3x^{\frac{2}{3}}(x^3+1)^{\frac{4}{3}}}.$$

98. Higher derivatives. It has been noted already (§ 62) that the derivative of the derivative of a function is called the second derivative of the function. Similarly the derivative of the second derivative is called the third derivative, and so on. The successive derivatives are commonly indicated by the following notation.

$y = f(x)$,	the original function,
$\frac{dy}{dx} = f'(x)$,	the first derivative,
$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$,	the second derivative,
$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = f'''(x)$,	the third derivative,
$\frac{d^ny}{dx^n} = f^{(n)}(x)$,	the nth derivative.

It is noted in § 22 that $f(a)$ denotes the value of $f(x)$ when $x = a$. Similarly $f'(a), f''(a), f'''(a)$, are used to denote the values of $f'(x)$, $f''(x)$, $f'''(x)$ respectively when $x = a$. It is to be emphasized that the differentiation is to be carried out before the substitution of the value of x.

Ex. If $f(x) = \frac{x-1}{x^2+1}$, find $f''(0)$.

$$f'(x) = \frac{-x^2+2x+1}{(x^2+1)^2};$$

$$f''(x) = \frac{2x^3-6x^2-6x+2}{(x^2+1)^3};$$

$$\therefore f''(0) = 2.$$

99. Differentiation of implicit algebraic functions. Consider any equation of the form

$$p_0y^n + p_1y^{n-1} + p_2y^{n-2} + p_3y^{n-3} + \cdots + p_{n-1}y + p_n = 0, \qquad (1)$$

where n is a positive integer, and where some or all of the coefficients $p_0, p_1, \cdots, p_n$, are polynomials in x. By means of this equation, if a value of x is given, values of y are determined. For if a numerical value is given to x, the coefficients become numerical and the equation is of the kind discussed in Chap. IV, which has been shown always to have n roots. Hence (1) defines y as a function of x. This is the most general form of an *algebraic function.* When (1) can be solved for y, so that y is expressed in terms of x by means of radical signs, y is an *explicit* algebraic function. When (1) is not solved for y, y is an *implicit* algebraic function.

For example,

$$3x^2 - 4xy + 5y^2 - 6x + 7y - 8 = 0,$$

which may be written

$$5y^2 + (7 - 4x)y + (3x^2 - 6x - 8) = 0,$$

defines y as an implicit function of x.

If the equation is solved for y, giving

$$y = \frac{-7 + 4x \pm \sqrt{209 + 64x - 44x^2}}{10},$$

y is expressed as an explicit function of x.

It may be shown by advanced methods that y defined by (1) is a continuous function of x and has a derivative with respect to x. Assuming this, it is possible to find the derivative without solving (1), for we have in (1) a function of x which is always equal to zero. Hence its derivative is zero. The derivative may be found by use of the formulas of the previous article, as shown in the examples.

Ex. 1. Given $x^2 + y^2 = 5$.

Then $$\frac{d(x^2 + y^2)}{dx} = 0,$$

that is, $$2x + 2y\frac{dy}{dx} = 0;$$

whence $$\frac{dy}{dx} = -\frac{x}{y}.$$

The derivative may also be found by solving the equation for y. Then

$$y = \pm\sqrt{5 - x^2},$$

$$\frac{dy}{dx} = \pm\frac{-x}{\sqrt{5 - x^2}} = -\frac{x}{y}.$$

Ex. 2. Given $y^3 - xy - 1 = 0$.

Then
$$\frac{d(y^3)}{dx} - \frac{d(xy)}{dx} = 0.$$

Hence
$$3y^2\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$$

and
$$\frac{dy}{dx} = \frac{y}{3y^2 - x}.$$

The second derivative may be found by differentiating the result thus obtained.

Ex. 3. If $x^2 + y^2 = 5$, we have found $\frac{dy}{dx} = -\frac{x}{y}$.

Therefore
$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= -\frac{y - x\frac{dy}{dx}}{y^2}$$

$$= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$

$$= -\frac{y^2 + x^2}{y^3}$$

$$= -\frac{5}{y^3}.$$

Ex. 4. If $y^3 - xy - 1 = 0$, we have found $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

Then
$$\frac{d^2y}{dx^2} = \frac{(3y^2 - x)\frac{dy}{dx} - y\frac{d(3y^2 - x)}{dx}}{(3y^2 - x)^2}$$

$$= \frac{(3y^2 - x)\frac{dy}{dx} - y\left(6y\frac{dy}{dx} - 1\right)}{(3y^2 - x)^2}$$

$$= \frac{(3y^2 - x)\frac{y}{3y^2 - x} - y\left(\frac{6y^2}{3y^2 - x} - 1\right)}{(3y^2 - x)^2}$$

$$= \frac{-2xy}{(3y^2 - x)^3}.$$

100. Tangents. It has been shown in § 59 that the tangent to a curve $y=f(x)$ at a point (x_1, y_1) is

$$y-y_1=\left(\frac{dy}{dx}\right)_1(x-x_1),$$

where $\left(\frac{dy}{dx}\right)_1$ denotes the value of $\frac{dy}{dx}$ at (x_1, y_1). We will apply this to some of the curves of Chap. VII, obtaining results for future reference.

Ex. 1. Consider the circle $Ax^2+Ay^2+2Gx+2Fy+C=0$.
Differentiating, we have

$$2Ax+2Ay\frac{dy}{dx}+2G+2F\frac{dy}{dx}=0;$$

whence

$$\frac{dy}{dx}=-\frac{Ax+G}{Ay+F}.$$

Hence the equation of the tangent is

$$y-y_1=-\frac{Ax_1+G}{Ay_1+F}(x-x_1),$$

that is,

$$Ax_1x-Ax_1^2+Ay_1y-Ay_1^2+Gx-Gx_1+Fy-Fy_1=0.$$

This equation may be simplified by adding to it the identity

$$Ax_1^2+Ay_1^2+2Gx_1+2Fy_1+C=0,$$

which follows from the fact that (x_1, y_1) is on the circle. There results

$$Ax_1x+Ay_1y+G(x+x_1)+F(y+y_1)+C=0.$$

This result is easily remembered from its resemblance to the equation of the circle.

The proofs of the next three examples are left to the student.

Ex. 2. The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$.

Ex. 3. The tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1$.

Ex. 4. The tangent to the parabola $y^2=4px$ is $y_1y=2p(x+x_1)$.

Ex. 5. Consider the witch $x^2y + 4a^2y - 8a^3 = 0$.

Differentiating, we have $$2xy + x^2\frac{dy}{dx} + 4a^2\frac{dy}{dx} = 0.$$

Hence the equation of the tangent is

$$y - y_1 = -\frac{2x_1y_1}{x_1^2 + 4a^2}(x - x_1);$$

that is, $$x_1^2y + 4a^2y - 4a^2y_1 + 2x_1y_1x - 3x_1^2y_1 = 0.$$

But $$x_1^2y_1 + 4a^2y_1 - 8a^3 = 0.$$

Hence the equation of the tangent may be written

$$2x_1y_1x + (x_1^2 + 4a^2)y + 8a^2y_1 - 24a^3 = 0.$$

Ex. 6. In the same manner the tangent to the cissoid $x^3 + xy^2 - 2ay^2 = 0$ at the point (x_1, y_1) is found to be

$$(3x_1^2 + y_1^2)x + (2x_1y_1 - 4ay_1)y - 2ay_1^2 = 0.$$

101. Normals. The normal to a curve at any point is the straight line perpendicular to the tangent at that point. To find its equation first find the slope of the tangent and then apply problem 3, § 29.

Ex. 1. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the slope of the tangent at (x_1, y_1) is $-\frac{b^2x_1}{a^2y_1}$. Hence the equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1),$$

which is

$$a^2y_1x - b^2x_1y - (a^2 - b^2)x_1y_1 = 0.$$

If $y = 0$,

$$x = \frac{a^2 - b^2}{a^2}x_1 = e^2x_1.$$

Hence in fig. 114

$$NF = OF - ON = ae - e^2x_1,$$

$$F'N = F'O + ON = ae + e^2x_1.$$

Then

$$\frac{F'N}{NF} = \frac{a + ex_1}{a - ex_1} = \frac{F'P_1}{FP_1}, \quad (\S\ 73)$$

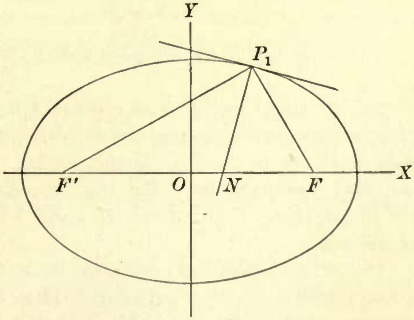

Fig. 114

and therefore, by plane geometry, the angle $F'P_1F$ is bisected by NP_1; that is, *in an ellipse the normal bisects the angle between the focal radii drawn to the point of contact.*

102. Maxima and minima. The discussion of § 61 applies here without change.

Ex. 1. A lever with the fulcrum at one end A (fig. 115) is to be used to lift a weight w applied at a distance a from the fulcrum by means of a force applied at the other end B. The lever weighing n units per unit of length, required the length of the lever that the force required may be a minimum.

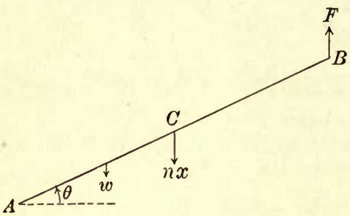

Fig. 115

Let $x = AB$, the length of the lever, θ the angle it makes with the horizontal, and F the force applied at B. Then the weight of the lever is nx, and may be considered as applied at C, the middle point of AB. By the law of the lever,

$$Fx\cos\theta = wa\cos\theta + nx\left(\frac{x}{2}\right)\cos\theta,$$

or

$$F = \frac{wa}{x} + \frac{nx}{2}.$$

Then

$$\frac{dF}{dx} = -\frac{wa}{x^2} + \frac{n}{2}$$

and

$$\frac{d^2F}{dx^2} = \frac{2\,wa}{x^3}.$$

When

$$x = \sqrt{\frac{2\,wa}{n}},\ \frac{dF}{dx} = 0 \text{ and } \frac{d^2F}{dx^2} > 0.$$

Therefore this is the required length.

Ex. 2. Light travels from a point A in one medium to a point B in another, the two media being separated by a plane surface. If the velocity in the first medium is v_1 and in the second v_2, required the path in order that the time of propagation from A to B shall be a minimum.

It is evident that the path must lie in the plane through A and B perpendicular to the plane separating the two media, and that the path will be a straight line in each medium. We have, then, fig. 116, where MN represents the intersection of the plane of the motion and the plane separating the two media, and ACB represents the path.

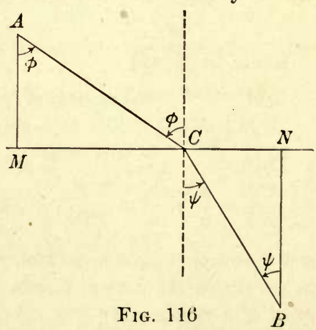

Fig. 116

Let $MA = a$, $NB = b$, $MN = c$, and $MC = x$. Then $AC = \sqrt{a^2 + x^2}$ and $CB = \sqrt{(c-x)^2 + b^2}$. The time of propagation from A to B is therefore

$$t = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{(c-x)^2 + b^2}}{v_2},$$

whence

$$\frac{dt}{dx} = \frac{x}{v_1 \sqrt{a^2 + x^2}} - \frac{c - x}{v_2 \sqrt{(c-x)^2 + b^2}},$$

and

$$\frac{d^2t}{dx^2} = \frac{a^2}{v_1 (a^2 + x^2)^{\frac{3}{2}}} + \frac{b^2}{v_2 [(c-x)^2 + b^2]^{\frac{3}{2}}}.$$

Since $\frac{d^2t}{dx^2}$ is always positive, the time is a minimum when

$$\frac{x}{v_1 \sqrt{a^2 + x^2}} - \frac{c - x}{v_2 \sqrt{(c-x)^2 + b^2}} = 0. \quad (1)$$

This equation may be solved for x, but it is more instructive to proceed as follows:

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{MC}{AC} = \sin \phi.$$

$$\frac{c - x}{\sqrt{(c-x)^2 + b^2}} = \frac{CN}{CB} = \sin \psi.$$

Then equation (1) is

$$\frac{\sin \phi}{\sin \psi} = \frac{v_1}{v_2}.$$

Now ϕ is the angle made by AC with the normal at C and is called the *angle of incidence*, and ψ is the angle made by CB with the normal at C and is called the *angle of refraction*. Hence the time of propagation is a minimum when the sine of the angle of incidence is to the sine of the angle of refraction as the velocity of the light in the first medium is to the velocity in the second medium. This is, in fact, the law according to which light is refracted.

A case of a maximum or a minimum value sometimes occurs when the derivative is infinite and consequently discontinuous. Therefore the case is not included in the previous discussions. In practice the infinite values of the derivative may be examined by the rule of § 61.

Ex. 3. $y = \sqrt[3]{(x-1)(x-2)^2} = (x-1)^{\frac{1}{3}}(x-2)^{\frac{2}{3}}$,

$$\frac{dy}{dx} = \frac{1}{3}(x-1)^{-\frac{2}{3}}(x-2)^{\frac{2}{3}} + \frac{2}{3}(x-1)^{\frac{1}{3}}(x-2)^{-\frac{1}{3}}$$

$$= \tfrac{1}{3}(x-1)^{-\frac{2}{3}}(x-2)^{-\frac{1}{3}}[(x-2) + 2(x-1)]$$

$$= \frac{3x - 4}{3\sqrt[3]{(x-1)^2(x-2)}}.$$

$\frac{dy}{dx} = 0$ when $x = \frac{4}{3}$, and changes from $+$ to $-$ as x passes through $\frac{4}{3}$. Therefore $x = \frac{4}{3}$ gives a maximum value to the function. $\frac{dy}{dx} = \infty$ when $x = 1$ or 2. When $x = 1$ $\frac{dy}{dx}$ does not change sign. When $x = 2$ $\frac{dy}{dx}$ changes from $-$ to $+$. Then $x = 2$ gives a minimum value of the function. Its graph is in fig. 117.

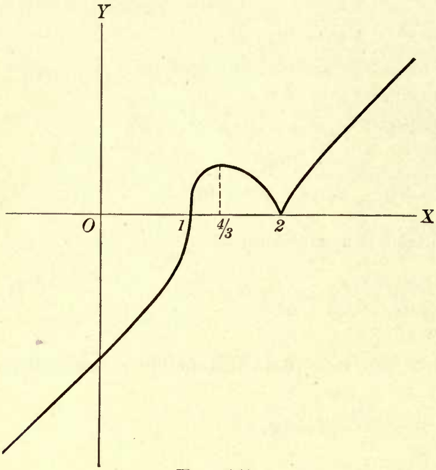

FIG. 117

103. Point of inflection. A point of inflection was defined in § 62 as a point at which the curve changes from being concave upward to concave downward, or vice versa; and the condition for it is that $\frac{d^2y}{dx^2}$ changes sign. Hence only those values of x which make $\frac{d^2y}{dx^2}$ zero or infinity need be considered in the examination of a curve for points of inflection.

Ex. 1. Find the points of inflection of the witch $y = \frac{8a^3}{x^2 + 4a^2}$.

By differentiation, $\frac{dy}{dx} = \frac{-16a^3x}{(x^2 + 4a^2)^2}$, $\frac{d^2y}{dx^2} = \frac{16a^3(3x^2 - 4a^2)}{(x^2 + 4a^2)^3}$.

It is evident that $\frac{d^2y}{dx^2} = 0$ if $x = \pm \frac{2a}{\sqrt{3}}$, and that no real finite value of x makes $\frac{d^2y}{dx^2}$ infinite.

We have, then, to consider only the points for which $x = \pm \frac{2a}{\sqrt{3}}$.

Writing $\frac{d^2y}{dx^2}$ in the form $\frac{48a^3\left(x + \frac{2a}{\sqrt{3}}\right)\left(x - \frac{2a}{\sqrt{3}}\right)}{(x^2 + 4a^2)^3}$, we see that if $x < -\frac{2a}{\sqrt{3}}$, $\frac{d^2y}{dx^2} > 0$; if $-\frac{2a}{\sqrt{3}} < x < \frac{2a}{\sqrt{3}}$, $\frac{d^2y}{dx^2} < 0$; and if $x > \frac{2a}{\sqrt{3}}$, $\frac{d^2y}{dx^2} > 0$.

Hence the curve is concave downward between the two points for which $x = -\frac{2a}{\sqrt{3}}$ and $x = \frac{2a}{\sqrt{3}}$ respectively, and concave upward at all other points. Then there are two points of inflection (fig. 90, § 82) for which $x = \pm \frac{2a}{\sqrt{3}}$. The ordinates are found from the equation to be $\frac{3a}{2}$.

Ex. 2. Examine the curve $y = (x-2)^{\frac{1}{3}}$ for points of inflection.

By differentiation, $\dfrac{dy}{dx} = \dfrac{1}{3(x-2)^{\frac{2}{3}}}$, $\quad \dfrac{d^2y}{dx^2} = -\dfrac{2}{9(x-2)^{\frac{5}{3}}}$.

It is evident that $\dfrac{d^2y}{dx^2} = \infty$ if $x = 2$, and that no value of x makes $\dfrac{d^2y}{dx^2} = 0$. If $x < 2$, $\dfrac{d^2y}{dx^2} > 0$; and if $x > 2$, $\dfrac{d^2y}{dx^2} < 0$. Hence the point for which $x = 2$ is a point of inflection, since on the left of that point the curve is concave upward and on the right of that point it is concave downward (fig. 118). The ordinate of this point is 0.

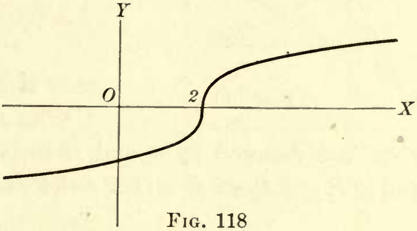

Fig. 118

104. Limit of ratio of arc to chord. The student is familiar with the determination of the length of the circumference of a circle as the limit of the length of the perimeter of an inscribed regular polygon. So, in general, if the length of an arc of any curve is required, a broken line connecting the ends of the arc is constructed by drawing a series of chords to the curve as in fig. 119. Then the length of the curve is defined as the limit of the sum of the lengths of these chords as each approaches zero, and as their number therefore increases without limit. The manner in which this limit is obtained is a question of the Integral Calculus, and will not be taken up here.

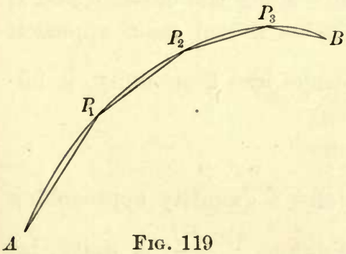

Fig. 119

We may use the definition, however, to find the limit of the ratio of the length of an arc of any curve to the length of its chord, as the length of the arc approaches zero as a limit, i.e. as the ends of the arc approach each other along the curve.

Accordingly, let P_1 and P_2 (fig. 120) be any two points of a curve, P_1P_2 the chord joining them, and P_1T and P_2T the tangents to the curve at those points respectively. We assume that the arc P_1P_2 lies entirely on one side of the chord P_1P_2, and is included between the tangents. These conditions can in general be met by taking the points P_1 and P_2 near enough together. Then it follows from the definition that

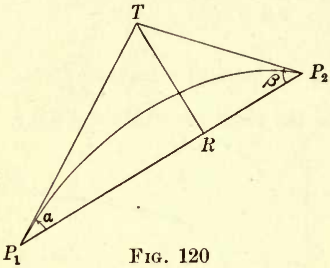

FIG. 120

$$P_1T + TP_2 > \text{arc}\, P_1P_2 > P_1P_2,$$

whence

$$\frac{P_1T + TP_2}{P_1P_2} > \frac{\text{arc}\, P_1P_2}{P_1P_2} > 1.$$

If TR is the perpendicular from T to P_1P_2, and if the angles P_2P_1T and P_1P_2T are denoted by α and β respectively, then $P_1T = P_1R \sec\alpha$, and $TP_2 = RP_2 \sec\beta = (P_1P_2 - P_1R)\sec\beta$.

$$\begin{aligned} \therefore\ P_1T + TP_2 &= P_1R \sec\alpha + (P_1P_2 - P_1R)\sec\beta \\ &= P_1P_2 \sec\beta + P_1R(\sec\alpha - \sec\beta), \end{aligned}$$

and

$$\begin{aligned} \frac{P_1T + TP_2}{P_1P_2} &= \frac{P_1P_2 \sec\beta + P_1R(\sec\alpha - \sec\beta)}{P_1P_2}, \\ &= \sec\beta + \frac{P_1R}{P_1P_2}(\sec\alpha - \sec\beta). \end{aligned}$$

Now, as P_1 and P_2 approach each other along the curve, α and β both approach zero as a limit, whence $\sec\alpha$ and $\sec\beta$ approach unity as a limit; and since $\frac{P_1R}{P_1P_2}$ is always less than unity, it follows that the limit of $\frac{P_1T + TP_2}{P_1P_2}$ is unity.

Hence $\frac{\text{arc}\, P_1P_2}{P_1P_2}$ lies between unity and a quantity approaching unity as a limit, and therefore the limit of $\frac{\text{arc}\, P_1P_2}{P_1P_2}$ is unity, i.e. *the limit of the ratio of an arc to its chord as the arc approaches zero as a limit is unity.*

105. The derivatives $\frac{dx}{ds}$ and $\frac{dy}{ds}$. On any given curve let the distance from some fixed initial point measured along the curve to any point P be denoted by s, where s is positive if P lies in one direction from the initial point, and negative if P lies in the opposite direction. The choice of the positive direction is purely arbitrary. We shall take as the positive direction of the tangent that which shows the positive direction of the curve, and shall denote the angle between the positive direction of OX and the positive direction of the tangent by ϕ.

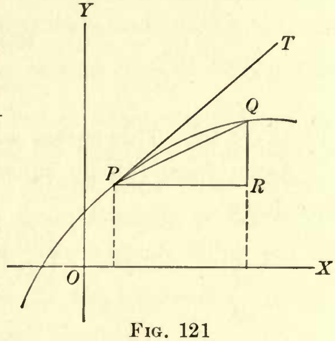

FIG. 121

Now for a fixed curve and a fixed initial point, the position of a point P is determined if s is given. Hence x and y, the coördinates of P, are functions of s, which in general are continuous and may be differentiated. We will now show that

$$\frac{dx}{ds} = \cos\phi, \qquad \frac{dy}{ds} = \sin\phi.$$

Let arc $PQ = \Delta s$ (fig. 121), where P and Q are so chosen that Δs is positive. Then $PR = \Delta x$ and $RQ = \Delta y$, and

$$\frac{\Delta x}{\Delta s} = \frac{PR}{\text{arc } PQ} = \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{PR}{\text{chord } PQ}$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \cos RPQ.$$

$$\frac{\Delta y}{\Delta s} = \frac{RQ}{\text{arc } PQ} = \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \frac{RQ}{\text{chord } PQ}$$

$$= \frac{\text{chord } PQ}{\text{arc } PQ} \cdot \sin RPQ.$$

Taking the limit, we have, since $\text{Lim}\,\frac{\text{chord } PQ}{\text{arc } PQ} = 1$ and $\text{Lim}\, RPQ = \phi$,

$$\frac{dx}{ds} = \cos\phi, \qquad \frac{dy}{ds} = \sin\phi. \tag{1}$$

From (1) we obtain by division

$$\tan\phi = \frac{\dfrac{dy}{ds}}{\dfrac{dx}{ds}} = \frac{dy}{dx}, \tag{2}$$

by (8), § 96. This agrees with § 59.

Again from (1), by squaring each equation and adding them, we have

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1. \tag{3}$$

By multiplying (3) by $\left(\dfrac{ds}{dx}\right)^2$ and again by $\left(\dfrac{ds}{dy}\right)^2$ and applying (7), § 96, we have

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{ds}{dx}\right)^2, \tag{4}$$

and

$$\left(\frac{dx}{dy}\right)^2 + 1 = \left(\frac{ds}{dy}\right)^2. \tag{5}$$

These last are the familiar trigonometric formulas

$$1 + \tan^2\phi = \sec^2\phi,$$

$$\cot^2\phi + 1 = \operatorname{cosec}^2\phi.$$

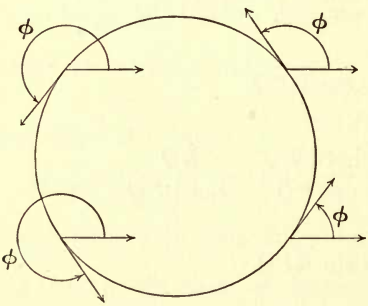

Fig. 122

For convenience we have used a figure in which ϕ is acute. But as s increases ϕ may be in any quadrant. This may be seen on the circle of fig. 122.

The student may verify that formulas (1)–(5) are true in all cases.

106. Velocity. An important application of the conception of a derivative is found in the definitions of the velocity of a moving body.

If a body moves so that the space traversed is proportional to the time, the motion is said to be *uniform*, and the velocity is the quotient of the space divided by the time, and is therefore constant. If t represents time, s the space traversed in the time t, and v the velocity, then for uniform motion $v = \frac{s}{t}\cdot$ When the space is not proportional to the time but is some other function of it, the quotient of the space divided by the time is called the mean or average velocity during the time. Thus if a railroad train goes 200 miles in 5 hours, the mean velocity is 40 miles an hour. So, in general, if a body traverses a small increment of space Δs in a small increment of time Δt, the quotient $\frac{\Delta s}{\Delta t}$ is the mean velocity in the time Δt. The mean velocity depends upon the value of Δt. To obtain a definition of the velocity at the beginning of the interval Δt, we think of Δt, and consequently of Δs, as approaching zero as a limit, and take the limit of $\frac{\Delta s}{\Delta t}$ as the velocity v; that is,

$$v = \text{Lim}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}\cdot$$

We note that if $v > 0$, an increase of time corresponds to an increase of s; while if $v < 0$, an increase of time causes a decrease of s. Consequently, the velocity is positive when the body moves in the direction in which s is measured, and negative if it moves in the opposite direction.

Ex. 1. If a body is thrown up from the earth with an initial velocity of 100 ft. per second, the space traversed, measured upward, is given by the equation

$$s = 100t - 16t^2.$$

Then
$$v = \frac{ds}{dt} = 100 - 32t.$$

When $t < 3\frac{1}{8}$, $v > 0$ and when $t > 3\frac{1}{8}$, $v < 0$. Hence the body rises for $3\frac{1}{8}$ seconds, and then falls. The highest point reached is $100(3\frac{1}{8}) - 16(3\frac{1}{8})^2 = 156\frac{1}{4}$.

Ex. 2. A man standing on a wharf 20 ft. above the water pulls in a rope attached to a boat at the uniform rate of 3 ft. per second. Required the velocity with which the boat approaches the wharf.

Let A (fig. 123) be the position of the man and C that of the boat. Let

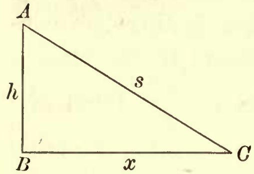

FIG. 123

$$AB = h = 20, \quad AC = s, \quad \text{and} \quad BC = x.$$

We wish to find $\dfrac{dx}{dt}$.

Now $$x = \sqrt{s^2 - 400};$$

therefore $$\frac{dx}{dt} = \frac{s}{\sqrt{s^2-400}}\frac{ds}{dt}.$$

But, by hypothesis, s is decreasing at the rate of 3 ft. per second; therefore $\dfrac{ds}{dt} = -3$, and the required expression for the velocity of the boat is

$$\frac{dx}{dt} = \frac{-3s}{\sqrt{s^2-400}}.$$

To express this in terms of the time we need to know the value of s when $t = 0$. Suppose this to be s_0; then

$$s = s_0 - 3t,$$

and $$\frac{dx}{dt} = \frac{-3s_0 + 9t}{\sqrt{s_0^2 - 400 - 6s_0t + 9t^2}}.$$

107. Components of velocity. When a body moves along its path, straight or curved, from P to Q (fig. 124), where $PQ = \Delta S$, x changes by an amount $PR = \Delta x$, and y changes by an amount $RQ = \Delta y$. We now have

$\text{Lim}\dfrac{\Delta s}{\Delta t} = \dfrac{ds}{dt} = v =$ velocity of the body in its path,

$\text{Lim}\dfrac{\Delta x}{\Delta t} = \dfrac{dx}{dt} = v_x =$ component of velocity parallel to OX.

$\text{Lim}\dfrac{\Delta y}{\Delta t} = \dfrac{dy}{dt} = v_y =$ component of velocity parallel to OY.

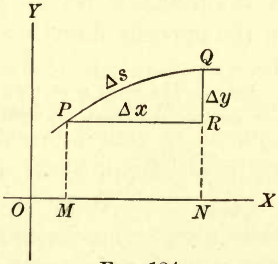

FIG. 124

Otherwise expressed, v represents the velocity of P, v_x the velocity of the projection of P upon OX, and v_y the velocity of the projection of P on OY.

By (7), § 96, and (3), § 105,

$$\frac{dx}{dt}=\frac{dx}{ds}\cdot\frac{ds}{dt}, \qquad \frac{dy}{dt}=\frac{dy}{ds}\cdot\frac{ds}{dt},$$

$$\left(\frac{ds}{dt}\right)^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2;$$

whence $$v_x = v\cos\phi, \qquad v_y = v\sin\phi,$$

$$v^2 = v_x^2 + v_y^2.$$

Ex. A man walks across the diameter of a circular courtyard at a uniform rate. A lamp, at one extremity of the diameter perpendicular to the one on which he walks, throws his shadow on the wall. Required the velocity of the shadow along the wall.

In fig. 125 let L be the lamp, M the man, and S the shadow. Let a be the radius of the courtyard and c the uniform velocity of the man. Let the variable $OM = x_1$ where $\frac{dx_1}{dt}=c$. Then the equation of the line LS is

$$ax - x_1 y - ax_1 = 0,$$

and that of the circle is

$$x^2 + y^2 = a^2.$$

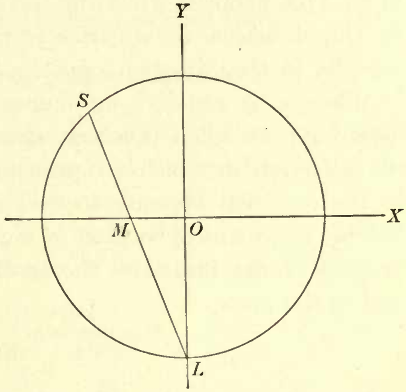

FIG. 125

Solving these equations, we have, for the coördinates of S,

$$x = \frac{2a^2x_1}{a^2+x_1^2}, \qquad y=\frac{a^3-ax_1^2}{a^2+x_1^2}.$$

Hence $$v_x = \frac{dx}{dt}=\frac{2a^4-2a^2x_1^2}{(a^2+x_1^2)^2}\frac{dx_1}{dt}=2a^2c\frac{a^2-x_1^2}{(a^2+x_1^2)^2},$$

and $$v_y=\frac{dy}{dt}=\frac{-4a^3x_1}{(a^2+x_1^2)^2}\frac{dx_1}{dt}=-2a^2c\frac{2ax_1}{(a^2+x_1^2)^2},$$

and $$v^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=4a^4c^2\frac{a^4+2a^2x_1^2+x_1^4}{(a^2+x_1^2)^4}.$$

The required velocity is $$v=\frac{2a^2c}{a^2+x_1^2}.$$

The above solution can be simplified by the use of trigonometric functions. See Ex. 2, § 153.

108. Acceleration and force. When the motion of a body is not uniform, the velocity at the end of an interval of time is not the same as at the beginning. Let v be the velocity at the beginning of the interval Δt, and $v + \Delta v$ the velocity at the end. Then the limit of the ratio of the change in the velocity to the change in time, as the latter approaches zero as a limit, is called the *acceleration;* that is, if a denotes the acceleration,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}.$$

When a is positive an increase of t corresponds to an increase of v. This happens when the body moves with increasing velocity in the direction in which s is measured, or with a decreasing velocity in the direction opposite to that in which s is measured.

When a is negative an increase of t causes a decrease of v. This happens when the body moves with decreasing velocity in the direction in which s is measured, or with increasing velocity in the direction opposite to that in which s is measured.

The *force* which acts on a moving body is measured by the product of the mass and the acceleration. Thus if F is the force and m the mass,

$$F = ma = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}.$$

From this it appears that a force is considered positive or negative according as the acceleration it produces is positive or negative. Hence a force is positive when it acts in the direction in which s is measured, and negative when it acts in the opposite direction.

Ex. Let $s = A + Bt + \frac{1}{2}Ct^2.$

Then $v = B + Ct,$

$a = C,$

and $F = mC.$

If s_0 and v_0 denote the values of s and v when $t = 0$, we have, from the last equations,

$$s_0 = A, \qquad v_0 = B,$$

and the original equation may be written

$$S = s_0 + v_0t + \tfrac{1}{2}at^2.$$

As a special case, suppose a body of mass m thrown vertically upward from a point h ft. above the surface of the earth with an initial velocity of v_0 ft. per second. Then, if s is measured upward from the surface of the earth, we have

$$s_0 = h, \qquad F = -mg, \qquad a = -g,$$

where g is the acceleration due to gravity. Then the expression becomes

$$s = h + v_0 t - \tfrac{1}{2} gt^2.$$

109. Other illustrations of the derivative.

1. *Rate of change.* If $y = f(x)$, a change of Δx units in the value of x causes a change of Δy units in the value of y. Then $\frac{\Delta y}{\Delta x}$ is the change in y per unit of change in x; that is, the change in y which would be caused by the change of a unit in x, if Δy were proportional to Δx. Passing to the limit, we have

$$\frac{dy}{dx} = \text{rate of change of } y \text{ with respect to } x.$$

For example, the velocity of a moving body is the rate of change of the space with respect to the time, and the acceleration is the rate of change of the velocity with respect to the time.

2. *Momentum.* The momentum of a moving body is the product of the mass and the velocity; that is, if M is the momentum,

$$M = mv.$$

Now, from § 108, $$F = m\frac{dv}{dt} = \frac{d(mv)}{dt} = \frac{dM}{dt}.$$

The force is therefore the derivative of the momentum with respect to the time, or, in other words, the rate of change of the momentum with respect to the time.

3. *Kinetic energy.* The kinetic energy of a moving body is equal to half the product of the mass into the square of the velocity; that is, if E is the kinetic energy,

$$E = \tfrac{1}{2} mv^2.$$

Then $$\frac{dE}{ds} = \frac{d(\frac{1}{2} mv^2)}{ds} = mv\frac{dv}{ds} = m\frac{ds}{dt}\cdot\frac{dv}{ds} = m\frac{dv}{dt} = F;$$

that is, the force is the derivative of the kinetic energy with respect to the space traversed, or, in other words, the rate of change of the kinetic energy with respect to the space.

4. *Coefficient of expansion.* Let a substance of volume v be at a temperature t. If the temperature is increased by Δt, the pressure remaining constant, the volume is increased by Δv. The change per unit of volume is then $\frac{\Delta v}{v}$, and the ratio of this change per unit of volume to the change in the temperature is $\frac{1}{v}\frac{\Delta v}{\Delta t}$. The limit of this ratio is called the coefficient of expansion; that is, the coefficient of expansion equals $\frac{1}{v}\frac{dv}{dt}$. In other words, the coefficient of expansion is the rate of change of a unit of volume with respect to the temperature.

5. *Elasticity.* Let a substance of volume v be under a pressure p. If the pressure is increased by Δp, the volume is increased by $-\Delta v$. The change in volume per unit of volume is then $-\frac{\Delta v}{v}$. The ratio of this change per unit of volume to the change in the pressure is $-\frac{1}{v}\frac{\Delta v}{\Delta p}$, and the limit of this is called the compressibility; that is, the compressibility is the rate of change of a unit volume with respect to the pressure.

The reciprocal of the compressibility is called the elasticity, which is therefore equal to $-v\frac{dp}{dv}$.

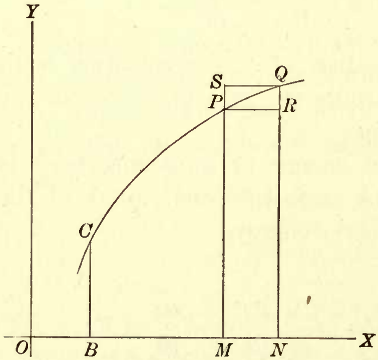

FIG. 126

Ex. For a perfect gas at constant temperature,

$$p = \frac{k}{v}.$$

Therefore the elasticity is

$$-v\frac{dp}{dv} = -v\left(-\frac{k}{v^2}\right) = \frac{k}{v} = p;$$

that is, the elasticity of a perfect gas is equal to the pressure.

6. *Areas.* Let $y = f(x)$ (fig. 126) be any curve, C a fixed point, and $P(x, y)$ a variable point upon it. We shall assume that P lies at the right of C and that the portion of the curve between C and P lies above the axis of x.

Draw the ordinates BC and MP and let A denote the area $BMPC$. Then A is a function of x, since it is determined when $OM = x$ is given. Give x an increment $\Delta x = MN$, and draw the ordinate NQ and the lines PR and QS parallel to OX. Then

$$RQ = \Delta y, \qquad MNQP = \Delta A,$$

$$MNRP = MP \cdot MN = y\,\Delta x, \qquad MNQS = NQ \cdot MN = (y + \Delta y)\,\Delta x.$$

But, from the figure,

$$MNRP < MNQP < MNQS\,*;$$

that is,

$$y\,\Delta x < \Delta A < (y + \Delta y)\,\Delta x,$$

whence

$$y < \frac{\Delta A}{\Delta x} < y + \Delta y.$$

Now as Δx approaches zero as a limit, $\frac{\Delta A}{\Delta x}$ approaches $\frac{dA}{dx}$, y is unchanged, and $y + \Delta y$ approaches y. Hence $\frac{\Delta A}{\Delta x}$, which lies between y and $y + \Delta y$, also approaches y; that is,

$$\frac{dA}{dx} = y.$$

If the curve lies below the axis of x (fig. 127), and we place, as before, $MNPR = y\Delta x$ and $MNQS = (y + \Delta y)\,\Delta x$, these areas are negative. We shall then have, as before,

$$\frac{dA}{dx} = y,$$

but the area is now considered as negative.

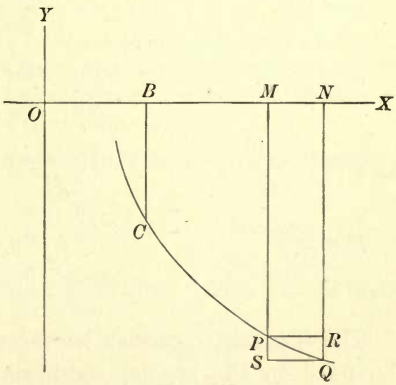

FIG. 127

110. Integration. In many applications of the calculus the derivative is known, and the problem presents itself to find

* If the curve runs down toward the right, the inequality signs will be reversed.

the function which has that derivative. For example, it may be required to find a curve when its slope is known, or to find the space traversed by a particle with known velocity or acceleration, or to find the area bounded partly by a known curve, or to find a function which has a known rate of change.

The process by which a function is found from its derivative is called *integration.* Differentiation and integration are then inverse processes, as are addition and subtraction, multiplication and division, involution and evolution. The methods of integration are in general complex and must be studied later in the integral calculus. At this time we shall give some simple examples where the integration can be carried out by reversing the formulas of differentiation.

In the first place, however, we must notice that the integration of a given function does not lead to a unique result. For, as we have seen already (§ 95),

$$\frac{d(u+c)}{dx} = \frac{du}{dx},$$

where c is any constant whatever; that is, *two functions which differ by an additive constant have the same derivative.*

Conversely, *if two functions have the same derivative, they differ by an additive constant.*

For let $$\frac{dv}{dx} = \frac{du}{dx}.$$

Then $$\frac{dv}{dx} - \frac{du}{dx} = 0,$$

or $$\frac{d(v-u)}{dx} = 0.$$

Hence* $$v - u = c, \text{ where } c = \text{constant};$$

that is, $$v = u + c.$$

The constant c cannot be determined by integration, but must be fixed by the special conditions of the problem in which it occurs.

* A proof of this conclusion will be given in the second volume.

Ex. 1. Required the curve the slope of which at any point is twice the abscissa of the point.

By hypothesis,

$$\frac{dy}{dx} = 2x.$$

Therefore $\quad y = x^2 + c. \qquad (1)$

Any curve whose equation can be derived from (1) by giving c a definite value satisfies the condition of the problem. If it is required that the curve should pass through the point (2, 3), we have, from (1),

$$3 = 4 + c; \quad \text{whence} \quad c = -1,$$

and therefore the equation of the curve is

$$y = x^2 - 1.$$

But if it is required that the curve should pass through (−3, 10), we have, from (1),

$$10 = 9 + c; \quad \text{whence} \quad c = 1,$$

and the equation is

$$y = x^2 + 1.$$

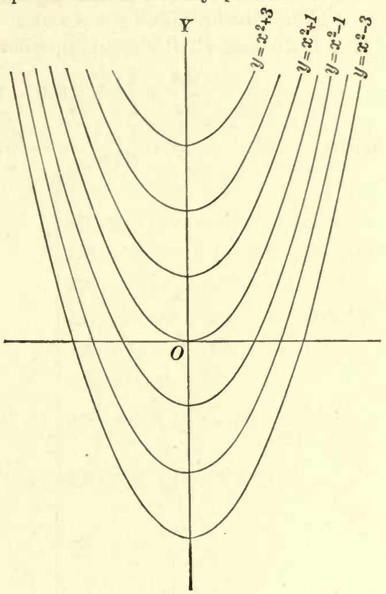

Fig. 128

Ex. 2. Required the space traversed by a particle if its velocity is equal to the square of the time.

By hypothesis, $\quad v = \frac{ds}{dt} = t^2.$

Therefore $\quad s = \frac{1}{3}t^3 + c.$

The constant c can be determined if we know the position of the particle at a given time. For instance, if when $t = 0$ the particle is at the point from which s is measured, we must have $c = 0$. On the other hand, if when $t = 0$ the particle is two units from the point at which $s = 0$, we have $c = 2$.

Ex. 3. Required the space traversed by a body if the acceleration is proportional to the time.

We have $\quad a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = kt,$

where k is a known constant. Then $\quad v = \frac{ds}{dt} = \frac{1}{2}kt^2 + c_1,$

and $\quad s = \frac{1}{6}kt^3 + c_1t + c_2.$

The constants c_1 and c_2 can be determined if we know the position and the velocity of the body at a given time. If, for example, we know that when $t = 0$, $s = 0$, and $v = 4$, we have $c_2 = 0$, $c_1 = 4$.

Ex. 4. Find the area bounded by the curve $y = \frac{1}{8}(x^3 - 3x^2 - 9x + 27)$, the axis of x, and the ordinates $x = 4$ and $x = 5$.

If A is the area $CDPM$ (fig. 129), where $OC = 4$ and $OM = x$, then (§ 109, 6)

$$\frac{dA}{dx} = \frac{1}{8}(x^3 - 3x^2 - 9x + 27),$$

whence
$$A = \frac{1}{8}\left(\frac{x^4}{4} - x^3 - \frac{9}{2}x^2 + 27x\right) + c. \tag{1}$$

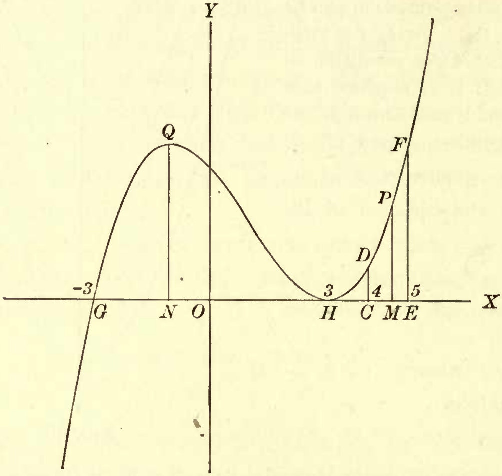

Fig. 129

If $x = 4$, MP coincides with CD and therefore $A = 0$. Substituting in (1) the corresponding values $x = 4$, $A = 0$, we find $c = -\frac{9}{2}$.

Therefore
$$A = \frac{1}{8}\left(\frac{x^4}{4} - x^3 - \frac{9}{2}x^2 + 27x\right) - \frac{9}{2}.$$

If $x = 5$, $A = CDEF$. Hence

$$CDEF = \tfrac{1}{8}\left(\tfrac{625}{4} - 125 - \tfrac{225}{2} + 135\right) - \tfrac{9}{2} = 2\tfrac{7}{32}.$$

Ex. 5. Find the area bounded by the axis of x and the portion of the curve $y = \frac{1}{8}(x^3 - 3x^2 - 9x + 27)$ between $x = -3$ and $x = 3$.

We now let $A =$ the area GNQ (fig. 129).

Then, as before, $\dfrac{dA}{dx} = \dfrac{1}{8}(x^3 - 3x^2 - 9x + 27)$.

$$A = \frac{1}{8}\left(\frac{x^4}{4} - x^3 - \frac{9}{2}x^2 + 27x\right) + c.$$

When $x = -3$, $A = 0$; therefore $c = \frac{297}{32}$,

and
$$A = \frac{1}{8}\left(\frac{x^4}{4} - x^3 - \frac{9}{2}x^2 + 27x\right) + \frac{297}{32}.$$

Placing $x = 3$, we have area $GQH = 13\frac{1}{2}$.

PROBLEMS

Find $\frac{dy}{dx}$ in each of the following cases:

1. $y = (3x + 1)(x^2 + 2x + 1)$.
2. $y = (3x^2 + 6x + 1)(5x^2 + 10x + 5)$.
3. $y = \frac{x - a}{x + a}$.
4. $y = \frac{x^3 + 1}{x^3 - 1}$.
5. $y = \frac{2x^2 - 4x + 3}{3x^2 - 6x + 1}$.
6. $y = \frac{x^3 - x^2 + x - 1}{x^4 - 1}$.
7. $y = 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}} - \frac{2}{x^{\frac{1}{2}}} + \frac{5}{x^{\frac{3}{2}}}$.
8. $y = 4x^2 - 6x + \frac{7}{x} - \frac{3}{x^2}$.
9. $y = \sqrt{x} - \frac{1}{\sqrt{x}}$.
10. $y = \sqrt[3]{x^2} - \sqrt[3]{x} + \frac{1}{\sqrt[3]{x}} - \frac{1}{\sqrt[3]{x^2}}$.
11. $y = (3x^2 - 5x + 6)^2$.
12. $y = (x^2 + 1)^3$.
13. $y = \sqrt{4x^2 + 5x - 6}$.
14. $y = \sqrt[3]{x^2 + x - 1}$.
15. $y = \frac{1}{x^2 + 1}$.
16. $y = \frac{5}{x^3 + x^2 + 1}$.
17. $y = \frac{10}{\sqrt[4]{x^2 + 1}}$.
18. $y = (2x - 3)^2 (x + 1)^3$.
19. $y = (3x - 5)^2 (x^2 - 5x + 1)$.
20. $y = (x + 1)\sqrt{x^2 + 1}$.
21. $y = (x^2 - 4x + 3)^{\frac{1}{2}} (x^3 + 1)^{\frac{2}{3}}$.
22. $y = \sqrt{x + 1} + \sqrt{x - 1}$.
23. $y = x + \sqrt{x^2 + 1}$.
24. $y = \sqrt[3]{3x^2 + 1} + \frac{1}{\sqrt[3]{3x^2 + 1}}$.
25. $y = \sqrt{\frac{x - 1}{x + 1}}$.
26. $y = \frac{\sqrt{x^2 + 1}}{x - 1}$.
27. $y = \frac{(x^2 + 1)^{\frac{1}{2}}}{(x^3 + 1)^{\frac{1}{3}}}$.
28. $y = \frac{x}{x + \sqrt{1 + x^2}}$.
29. $y = \frac{x}{\sqrt{a^2 - x^2}}$.
30. $y = \frac{1}{x + \sqrt{a^2 + x^2}}$.

Find $\frac{dy}{dx}$ from each of the following equations:

31. $x^4 - 4x^2y^2 + y^3 = 0$.
32. $x^5 - y^5 - x^3 + y = 0$.
33. $x^3y^4 + (x - y)^3 = 0$.
34. $x^5 + y^4 - x^3 - y = 0$.
35. $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = a$.
36. $y^2 = \frac{x + y}{x - y}$.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from each of the following equations:

37. $5x^2 + 2y^2 = 10$.
38. $x^7 + y^7 = a^7$.
39. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
40. $y^3 = a^2(x + y)$.
41. $y^3 + y = x^3$.
42. $y^3 - 2x^3 + 4xy = 0$.

43. Find the tangent and the normal to the parabola $y^2 - 4y - 6x - 9 = 0$ at a point the abscissa of which is -2.

44. Find the equations of the tangent and the normal to the circle

$$x^2 + y^2 - 4x + 6y - 24 = 0$$

at the point (1, 3).

45. Find the equation of the tangent to the witch $y = \dfrac{1}{x^2+1}$ at the point for which $x = 1$.

46. Find the tangent to the curve $x^5 - y^5 + x^3 - y = 0$ at the point the abscissa of which is 1.

47. Find the tangent to the curve $x^2y + x^3 - x^2 + y = 0$ at the point the abscissa of which is 1.

48. Find the equation of the tangent to the curve $y^3 - xy - a = 0$ at the point (x_1, y_1).

49. Find the equation of the tangent to the curve $x = y^3 + 1$ at the point (x_1, y_1).

50. Find the equation of the tangent to the curve $y^2 = x^3$ at the point (x_1, y_1).

51. Find the equations of the tangent and the normal to the curve $y = x + \dfrac{1}{x^2}$ at the point (x_1, y_1).

52. Find the equation of the tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point (x_1, y_1).

53. Find the equation of the tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at the point (x_1, y_1).

54. Find the tangent and the normal to the ellipse $3x^2 + 5y^2 = 15$ at the upper end of the ordinate through the right-hand focus.

55. Find the equations of the tangent and the normal to the hyperbola $4x^2 - y^2 = 12$ at a point the abscissa of which is equal to its ordinate.

56. Find in terms of x, y, and $\dfrac{dy}{dx}$ the projections upon OX of the portions of the tangent and the normal between the point of contact and OX. These are called the *subtangent* and the *subnormal.*

57. Find in terms of x, y, and $\dfrac{dy}{dx}$ the lengths of the portions of the tangent included between the point of contact and the coördinate axes.

58. Prove that a normal to an hyperbola makes equal angles with the focal radii drawn to the point where the normal intersects the hyperbola.

59. Prove that a normal to a parabola makes equal angles with the axis of the parabola and the line drawn from the focus to the point where the normal intersects the parabola.

60. Show that for an ellipse the segments of the normal between the point of the curve at which the normal is drawn and the axes are in the ratio $a^2 : b^2$.

61. Find the point at which the tangent to the curve $x^3 - xy - 1 = 0$ has the slope 2.

62. Find the coördinates of a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ such that the tangent there is parallel to the line joining the positive extremities of the major and the minor axes.

63. Find a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ such that the tangent there is equally inclined to the two axes.

64. Prove that the portion of a tangent to an hyperbola included by the asymptotes is bisected by the point of tangency.

65. If any number of hyperbolas have the same transverse axis, show that tangents to the hyperbolas at points having the same abscissa all pass through the same point on the transverse axis.

66. If a tangent to an hyperbola is intersected by the tangents at the vertices in the points Q and R, show that the circle described on QR as a diameter passes through the foci.

67. Prove that the ordinate of the point of intersection of two tangents to a parabola is the arithmetical mean between the ordinates of the points of contact of the tangents.

68. If P, Q, and R are three points on a parabola, the ordinates of which are in geometrical progression, show that the tangents at P and R meet on the ordinate of Q.

69. Show that the tangents at the extremities of the latus rectum* of a parabola are perpendicular to each other.

70. Prove that the tangents at the ends of the latus rectum of a parabola intersect on the directrix.

71. Prove analytically that if the normals at all points of an ellipse pass through the center, the ellipse is a circle.

72. Prove that the tangent at any point of the parabola $y^2=4px$ will meet the directrix and the latus rectum produced in two points equidistant from the focus.

73. Find the length of the perpendicular from the focus of the parabola $y^2=4px$ to the tangent at any point (x_1, y_1), in terms of x_1 and p.

74. If perpendiculars are let fall on any tangent to a parabola from two given points on the axis which are equidistant from the focus, prove that the difference of their squares is constant.

75. Show that the product of the perpendiculars from the foci of an ellipse upon any tangent equals the square of half the minor axis.

76. Find the equation and the length of the perpendicular from the center to any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

77. At what angles† do the loci $y^2-4x+4=0$ and $y-x+1=0$ intersect?

* The *latus rectum* of a conic is the chord through the focus perpendicular to the axis.

† The angle between two curves is the angle between their tangents at their point of intersection.

78. Find the angle between the straight line $y = 2x - 2$ and the cissoid $x(x^2 + y^2) = 4y^2$ at each of their points of intersection.

79. At what angle do the circles $x^2 + y^2 - 9 = 0$, $x^2 + y^2 - 6x - 6y + 9 = 0$ intersect?

80. Prove that the center of each of the circles

$$x^2 + y^2 = a^2 \quad \text{and} \quad x^2 + y^2 - 2ax = 0$$

is a point of the other, and find the angle at which they intersect.

81. At what angle do the circle $x^2 + y^2 = 21$ and the parabola $y^2 = 4x$ intersect each other?

82. Show that the curves $\frac{x^2}{16} + \frac{y^2}{7} = 1$ and $\frac{x^2}{4} - \frac{y^2}{5} = 1$ cut each other at right angles and are confocal.

83. Prove that an ellipse and an hyperbola with the same foci cut each other at right angles.

84. If two concentric equilateral hyperbolas are described, the axes of one being the asymptotes of the other, show that they intersect at right angles.

85. Find the angle between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ at each of their points of intersection.

86. Find the angle between the parabola $x^2 = 4ay$ and the witch $y = \frac{8a^3}{x^2 + 4a^2}$ at each of their points of intersection.

87. Prove that the cissoid $y^2 = \frac{x^3}{2a - x}$ and the parabola $y^2 = 4ax$ intersect at right angles at the origin.

88. Find the angles of intersection of the cissoid $y^2 = \frac{x^3}{2a - x}$ and the circle $x^2 + y^2 - 4ax = 0$.

89. Find the angle of intersection of the witch

$$y = \frac{8a^3}{x^2 + 4a^2} \quad \text{and the cissoid} \quad x^2 = \frac{4y^3}{5a - 4y}.$$

90. Find the angles of intersection of the circle $x^2 + y^2 = 5a^2$ and the witch

$$y = \frac{8a^3}{x^2 + 4a^2}.$$

91. Find the angle between the strophoid $y = \pm x\sqrt{\frac{a - x}{a + x}}$ and the circle $x^2 + y^2 = a^2$.

92. Find the angles of intersection of the curves

$$y^2 = 2ax \quad \text{and} \quad x^3 + y^3 - 3axy = 0.$$

93. It is required to fence off a rectangular piece of ground to contain a given area, one side to be bounded by a wall already constructed. Required the dimensions of the rectangle which will require the least amount of fencing.

94. A man on one side of a river, the banks of which are assumed to be parallel straight lines $\frac{1}{4}$ mi. apart, wishes to reach a point on the opposite side of the river and 3 mi. further along the bank. If he can walk 4 mi. an hour and swim 2 mi. an hour, find the route he should take to make the trip in the least time.

95. A rectangular piece of land to contain 96 sq. rd. is to be inclosed by a fence and divided into two equal parts by a fence parallel to one of the sides. What must be the dimensions of the rectangle that the least amount of fence may be required?

96. What are the dimensions of the rectangular beam of greatest volume that can be cut from a log a ft. in diameter and b ft. long, assuming the log to be a circular cylinder?

97. The hypotenuse of a right triangle is given. How shall the sides be chosen so that the area shall be a maximum?

98. Two towns A and B are situated respectively 2 mi. and 3 mi. back from a straight river from which they are to get their water supply, both from the same pumping station. At what point on the bank of the river should the station be placed, that the least amount of piping may be required, if the nearest points of the river to A and B respectively are 10 mi. apart?

99. AB and CD are two parallel lines distant b units apart. A transversal BF is drawn, intersecting the transversal AD at E. For what position of F is the sum of the areas of the two triangles AEB and FED a minimum?

100. A right cone is generated by revolving an isosceles triangle of constant perimeter about its altitude. Show that the cone of greatest volume will be obtained when the length of the side of the triangle is three fourths the length of the base.

101. Into a full conical wine glass whose depth is a and angle at the base is 2α there is carefully dropped a spherical ball of such size as to cause the greatest overflow. Show that the radius of the ball is

$$\frac{a \sin \alpha}{\sin \alpha + \cos 2\alpha}.$$

102. Two ships are sailing uniformly with velocities u, v along lines inclined at an angle θ. Given that at a certain time the ships are distant respectively a and b from the point of intersection of their courses, show that the least distance between the ships is

$$\frac{(av - bu)\sin \theta}{(u^2 + v^2 - 2\,uv \cos \theta)^{\frac{1}{2}}}.$$

103. Find the least ellipse which can be described about a given rectangle, assuming that the area of an ellipse with semiaxes a and b is πab.

104. Find what sector must be taken out of a given circle in order that it may form the curved surface of a cone of maximum volume.

105. The stiffness of a rectangular beam varies as the product of the breadth and the cube of the depth. Find the dimensions of the stiffest rectangular beam that can be cut from a circular cylindrical log of radius a in.

106. The strength of a rectangular beam varies as the product of its breadth and the square of its depth. Find the dimensions of the strongest rectangular beam that can be cut from a circular cylindrical log of radius a in.

107. The fuel consumed by a steamship is proportional to the cube of the velocity which would be given to the steamship in still water. If it is required to steam a fixed distance against a current flowing a mi. an hour, find the most economical rate.

108. A cistern in the form of a circular cylinder open at the top is to be constructed to contain a given amount. Required the dimensions that the amount of material expended may be the least.

109. Required the right circular cone of greatest volume which can be inscribed in a given sphere.

110. A power house stands upon one side of a river of width b mi. and a manufacturing plant stands upon the opposite side a mi. downstream. Find the most economical way to construct the connecting cable if it costs m dollars per mile on land and n dollars per mile through water.

111. Find the isosceles triangle of greatest area which can be cut from a semicircular board, assuming that the vertex of the triangle lies in the diameter.

112. Find the isosceles triangle of greatest area which can be placed in a figure bounded by a portion of a parabola and a straight line perpendicular to the axis of the parabola, assuming that the vertex of the triangle lies in the straight line.

113. Find the point of inflection of the curve $y = a + (b - x)^3$.

114. Find the points of inflection of the curve $y = \dfrac{1}{x^2 + 1}$.

115. Examine the curve $y = (x - 1)^2(x + 1)^3$ for maxima and minima and points of inflection.

116. Find the maximum and the minimum ordinates and the points of inflection of the curve $y^3 = x(x^2 - a^2)$.

117. Find the points of inflection of the curve $y = \dfrac{8}{x^2 + 4}$.

118. Show that the strophoid $y = \pm x\sqrt{\dfrac{a - x}{a + x}}$ has no point of inflection.

119. Find the points of inflection of the curve $a^4y^2 = a^2x^4 - x^6$.

120. Find the points of inflection of the curve $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1$.

121. Find where the rate of change of the ordinate of the curve

$$y = x^3 - 6x^2 + 3x + 5$$

is equal to the rate of change of the slope of the tangent.

122. A body moves in a straight line according to the law $s = \frac{1}{4}t^4 - 4t^3 + 16t^2$. Find its velocity and acceleration. When is it stationary? When is its velocity a maximum? During what interval is it moving backward?

123. A particle is moving along the curve $y^2 = 4x$, and when $x = 4$ its ordinate is increasing at the rate of 10 ft. per second. At what rate is the abscissa then changing, and how fast is the particle moving along the curve? Where will the abscissa be changing ten times as fast as the ordinate?

124. Two points, having always the same abscissa, move in such a manner that each generates one of the curves $y = x^3 - 12x^2 + 4x$ and $y = x^3 - 8x^2 - 8$. When are the points moving with equal speed in the direction of the axis of y? What will be true of the tangent lines to the curves at these points?

125. The top of a ladder a units long slides down the side of a vertical wall which rests on horizontal land. Find the ratio of the velocities of its top and bottom.

126. The altitude of a variable cylinder is constantly equal to the diameter of its base. If when the altitude is 6 ft. it is increasing at the rate of 2 ft. an hour, how fast is the volume increasing at the same instant?

127. A boat moving 8 mi. an hour is laying a submarine cable. Assuming that the water is 100 ft. deep, that the cable is attached to the bottom of the sea and stretches in a straight line to the stern of the boat, at what rate is the cable leaving the boat when 120 ft. have been paid out?

128. A ball is swung in a circle at the end of a cord 4 ft. long so as to make 100 revolutions a minute. If the cord breaks, allowing the ball to fly off at a tangent, at what rate will it be receding from the center of its previous path 10 sec. after the cord breaks, if no allowance is made for any new force acting?

129. A body slides down an inclined plane at such a rate that the distance traversed at the end of t sec. from the time it begins to move is $5t^2$. If the plane is inclined to the horizon at an angle of 30°, what is the vertical velocity of the body at the end of 3 sec.?

130. A roll of belt leather is unrolled on a horizontal surface at the rate of 5 ft. per second. If the leather is $\frac{1}{4}$ in. thick and at the start the roll was 2 ft. in diameter, at what rate is the radius decreasing at the end of 3 sec., if the roll is assumed to be a true circle?

131. An elevated car running at a constant elevation of 40 ft. above the street passes directly over a surface car, the tracks of the two cars crossing at right angles. If the speed of the elevated car is 16 mi. per hour and the speed of the surface car 8 mi. per hour, at what rate are the cars separating 5 min. after they meet?

132. Find the curve the slope of which at any point is 3 more than the square of the abscissa of that point and which passes through the point $(1, -3)$.

133. Find the curve the slope of which at any point is equal to the square of the reciprocal of the abscissa of the point and which passes through (2, 1).

134. Find the curve the slope of which at any point is equal to the square root of the abscissa of the point and which passes through (4, 9).

135. Prove that any curve the slope of which at any point is proportional to the abscissa of the point is a parabola.

136. Find the curve the slope of which at any point is proportional to the square of the ordinate of the point and which passes through (1, 1).

137. Find the area of each arch of the curve $y = 150x - 25x^2 - x^3$.

138. Find the area of the arch of the curve $y = x^3 - 3x^2 - 9x + 27$.

139. Show that the area bounded by any parabola $y^2 = 4px$, the axis of x, and the ordinate through any point of the curve is two thirds the area of a rectangle the sides of which are the coördinates of the point.

140. Express the area between the curve $y = x^n$, the axis of x, and the ordinate through the point (h, k) of the curve as a rational function of h and k.

141. Find the area of the three-sided figure bounded by the coördinate axes and the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ (§ 69).

142. Find the area between the parabola $y^2 = 8x$ and the straight line $2y - x = 0$.

143. Find the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

144. Find the area of the crescent-shaped figure between the curves $y = x^2 + 5$ and $y = 2x^2 + 1$.

145. Find the area of the closed figure bounded by the curves $y^2 = 16x$ and $y^2 = x^3$.

CHAPTER X

CHANGE OF COÖRDINATE AXES

111. Introduction. So far we have dealt with the coördinates of any point in the plane on the supposition that the axes of coördinates are fixed, and therefore to a given point corresponds one, and only one, pair of coördinates; and, conversely, to any pair of coördinates corresponds one, and only one, point. But it is sometimes advantageous to change the position of the axes, i.e. to make a *transformation of coördinates*, as it is called. In such a case we need to know the relations between the coördinates of a point with respect to one set of axes and the coördinates of the same point with respect to a second set of axes.

The equations expressing these relations are called *formulas of transformation.* It must be borne in mind that a transformation of coördinates never alters the position of the point in the plane, the coördinates alone being changed because of the new standards of reference adopted.

112. Change of origin without change of direction of axes. In this case a new origin is chosen, but the new axes are respectively parallel to the original axes.

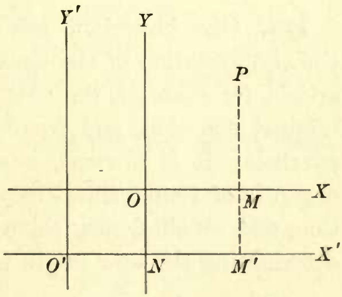

FIG. 130

Let OX and OY (fig. 130) be the original axes, and $O'X'$ and $O'Y'$ the new axes intersecting at O', the coördinates of O' with respect to the original axes being x_0 and y_0. Let P be any point in the plane, its coördinates being x and y with respect to OX and OY, and x' and y' with respect to $O'X'$ and $O'Y'$. Draw PMM' parallel to OY, intersecting OX and $O'X'$ at M and M' respectively.

Then $OM=x$, $MP=y$, $O'M'=x'$, $M'P=y'$, $ON=y_0$, and $NO'=x_0$.

But $$OM=NM'=NO'+O'M',$$

and $$MP=MM'+M'P=ON+M'P.$$

$$\therefore\ x=x_0+x',\quad y=y_0+y',$$

which are the required formulas of transformation.

Ex. 1. The coördinates of a certain point are $(3, -2)$. What will be the coördinates of this same point with respect to a new set of axes parallel respectively to the first set and intersecting at $(1, -1)$ with respect to OX and OY?

Here $x_0=1$, $y_0=-1$, $x=3$, and $y=-2$. Therefore $3=1+x'$ and $-2=-1+y'$, whence $x'=2$ and $y'=-1$.

Ex. 2. Transform the equation $y^2-2y-3x-5=0$ to a new set of axes parallel respectively to the original axes and intersecting at the point $(-2, 1)$.

The formulas of transformation are $x=-2+x'$, $y=1+y'$. Therefore the equation becomes

$$(1+y')^2-2(1+y')-3(-2+x')-5=0,$$

or $$y'^2-3x'=0.$$

As no point of the curve has been moved in the plane by this transformation, the curve has been changed in no way whatever. Its equation is different because it is referred to new axes.

After the work of transformation has been completed the primes may be dropped. Accordingly, the equation of this example may be written $y^2-3x=0$, or $y^2=3x$, the new axes being now the only ones considered.

113. One important use of transformation of coördinates is the simplification of the equation of a curve. In Ex. 2 of the last article, for example, the new equation $y^2=3x$ is simpler than the original equation, and from its form we recognize the curve as a parabola. It is obvious, however, that the position of the new origin is of fundamental importance in thus simplifying the equation, and we shall now solve an example illustrating a method of determining the new origin to advantage.

Ex. Transform the equation $y^2-4y-x^3-3x^2-3x+3=0$ to new axes parallel respectively to the original axes, so choosing the origin that there shall be no terms of the first degree in x and y in the new equation.

The formulas of transformation are

$$x = x_0 + x' \quad \text{and} \quad y = y_0 + y',$$

where suitable values of x_0 and y_0 are to be determined. The equation becomes

$$(y_0 + y')^2 - 4(y_0 + y') - (x_0 + x')^3 - 3(x_0 + x')^2 - 3(x_0 + x') + 3 = 0,$$

or, after expanding and collecting like terms,

$$y'^2 + (2y_0 - 4)y' - x'^3 - (3x_0 + 3)x'^2 - (3x_0^2 + 6x_0 + 3)x' + (y_0^2 - 4y_0 - x_0^3 - 3x_0^2 - 3x_0 + 3) = 0.$$

By the conditions of the problem we are to choose x_0 and y_0 so that

$$2y_0 - 4 = 0, \qquad 3x_0^2 + 6x_0 + 3 = 0,$$

two equations from which we find $x_0 = -1$ and $y_0 = 2$.

Therefore $(-1, 2)$ should be chosen as the new origin of axes, and the new equation is $y'^2 - x'^3 = 0$, or $y^2 = x^3$, after the primes are dropped.

114. In particular, this method of simplifying an equation is of considerable importance in studying the conics defined in Chap. VII. For consider the equation

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1. \tag{1}$$

If we place $x = x_0 + x'$, $y = y_0 + y'$, (1) becomes

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1, \tag{2}$$

which is the equation of an ellipse with its center at $x' = 0$, $y' = 0$, and its axes along $O'X'$ and $O'Y'$. Therefore (1) is an ellipse with its center at $x = x_0$, $y = y_0$, and its axes parallel to OX and OY.

Furthermore, if $a > b$, $e = \dfrac{\sqrt{a^2 - b^2}}{a}$; and the foci of the ellipse are at $(x' = \pm ae, y' = 0)$, or, what is the same thing, $(x = \pm ae + x_0, y = y_0)$. The directrices are $x' = \pm \dfrac{a}{e}$, or $x = x_0 \pm \dfrac{a}{e}\cdot$

In a similar manner

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1$$

is the equation of an hyperbola with its center at (x_0, y_0) and its axes parallel to OX and OY; and

$$(y-y_0)^2 = 4p(x-x_0)$$

represents a parabola with its vertex at (x_0, y_0) and its axis parallel to OX.

Any equation which can be reduced to a form similar to one of these can be discussed in a similar manner. A general treatment of such equations will be found in Chap. XI. We shall give here some examples.

Ex. 1. $16x^2 + 25y^2 + 64x - 150y - 111 = 0.$

Rewriting, we have

$$16(x^2 + 4x) + 25(y^2 - 6y) = 111,$$

whence
$$16(x^2 + 4x + 4) + 25(y^2 - 6y + 9) = 400,$$

or
$$\frac{(x+2)^2}{25} + \frac{(y-3)^2}{16} = 1.$$

Placing now
$$x = -2 + x', \qquad y = 3 + y',$$

we have
$$\frac{x'^2}{25} + \frac{y'^2}{16} = 1.$$

This is an ellipse with semiaxes 5 and 4, and eccentricity $\frac{3}{5}$. Its center is at $(x' = 0, y' = 0)$, its foci are at $(x' = \pm 3, y' = 0)$, and its directrices are $x' = \pm \frac{25}{3} = \pm 8\frac{1}{3}$.

Hence the original equation represents an ellipse with semiaxes 5, 4, and eccentricity $\frac{3}{5}$. Its center is at $(-2, 3)$, its foci are $(-5, 3)$ and $(1, 3)$, and its directrices are $x = -10\frac{1}{3}$ and $x = 6\frac{1}{3}$.

Ex. 2. $5y^2 - 10y - 4x - 7 = 0.$

Rewriting, we have
$$5(y^2 - 2y) = 4(x + \tfrac{7}{4}),$$
$$5(y^2 - 2y + 1) = 4(x + \tfrac{7}{4} + \tfrac{5}{4}),$$

or
$$(y-1)^2 = \tfrac{4}{5}(x+3).$$

Placing now
$$x = -3 + x',$$
$$y = 1 + y',$$

we have
$$y'^2 = \tfrac{4}{5}x',$$

which represents a parabola with vertex $(x' = 0, y' = 0)$. Its axis is along $O'X'$; its focus is $(x' = \frac{1}{5}, y' = 0)$, and its directrix is $x' = -\frac{1}{5}$.

Therefore the original equation represents a parabola with its vertex at $(-3, 1)$ and its axis parallel to OX. Its focus is $(-2\frac{4}{5}, 1)$ and its directrix is $x = -3\frac{1}{5}$.

Ex. 3. $(x-c)^2+y^2=e^2x^2$.

This is the equation of the conic, as found in § 81. We may write it as

$$(1-e^2)x^2-2cx+y^2=-c^2.$$

Then if $e\neq 1$, we may proceed as follows:

$$(1-e^2)\left(x^2-\frac{2c}{1-e^2}x+\frac{c^2}{(1-e^2)^2}\right)+y^2=-c^2+\frac{c^2}{1-e^2},$$

$$(1-e^2)\left(x-\frac{c}{1-e^2}\right)^2+y^2=\frac{c^2e^2}{1-e^2},$$

$$\frac{\left(x-\frac{c}{1-e^2}\right)^2}{\frac{c^2e^2}{(1-e^2)^2}}+\frac{y^2}{\frac{c^2e^2}{1-e^2}}=1.$$

We may now place $$\frac{c^2e^2}{(1-e^2)^2}=a^2,$$

$$\frac{c^2e^2}{1-e^2}=a^2(1-e^2)=\pm b^2,$$

and $$\frac{c}{1-e^2}=\frac{a}{e},$$

the sign of b being ± 1 according as $e \lessgtr 1$. The equation is then

$$\frac{\left(x-\frac{a}{e}\right)^2}{a^2}\pm\frac{y^2}{b^2}=1.$$

The equation accordingly represents an ellipse or an hyperbola with center at $\left(\frac{a}{e},\ 0\right)$.

If $e=1$, the equation $\qquad (x-c)^2+y^2=e^2x^2$

becomes $$y^2=2cx-c^2=2c\left(x-\frac{c}{2}\right),$$

which represents a parabola with the vertex at $\left(\frac{c}{2},\ 0\right)$.

115. Change of direction of axes without change of origin.

Case I. *Rotation of axes.* Let OX and OY (fig. 131) be the original axes, and OX' and OY' be the new axes, making $\angle\phi$ with OX and OY respectively. Then $\angle XOY'=90°+\phi$, and $\angle YOX'=90°-\phi$.

Let P be any point in the plane, its coördinates being x and y with respect to OX and OY, and x' and y' with respect to OX' and OY'. Then by construction $OM = x$, $ON = y$, $OM' = x'$, and $M'P = y'$. Draw OP.

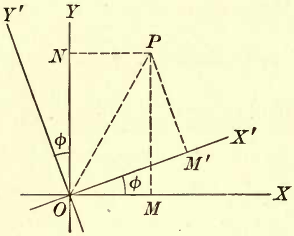

FIG. 131

The projection of OP on OX is OM, and the projection of the broken line $OM'P$ on OX is $OM' \cos\phi + M'P \cos(90° + \phi)$ or $OM' \cos\phi - M'P \sin\phi$.

$$\therefore\ OM = OM' \cos\phi - M'P \sin\phi, \quad (1)$$

by § 15.

In like manner the projection of OP on OY is ON, and the projection of the broken line $OM'P$ on OY is $OM' \cos(90° - \phi) + M'P \cos\phi$.

$$\therefore\ ON = OM' \sin\phi + M'P \cos\phi, \quad (2)$$

by § 15.

Replacing OM, ON, OM', $\cdots$ by their values, we have

$$x = x' \cos\phi - y' \sin\phi,$$
$$y = x' \sin\phi + y' \cos\phi.$$

Ex. 1. Transform the equation $xy = 5$ to new axes, having the same origin and making an angle of 45° with the original axes.

Here $\phi = 45°$, and the formulas of transformation are $x = \dfrac{x' - y'}{\sqrt{2}}$, $y = \dfrac{x' + y'}{\sqrt{2}}$.

Substituting and simplifying, we have as the new equation $x^2 - y^2 = 10$, from which we recognize the curve to be an equilateral hyperbola.

Ex. 2. Transform the equation $34x^2 + 41y^2 - 24xy = 100$ to new axes with the same origin, so choosing the angle ϕ that the new equation shall have no term in xy.

The formulas of transformation are

$$x = x' \cos\phi - y' \sin\phi,$$
$$y = x' \sin\phi + y' \cos\phi,$$

where ϕ is to be determined.

Substituting in the equation and collecting like terms, we have

$$(34 \cos^2\phi + 41 \sin^2\phi - 24 \sin\phi \cos\phi)x^2$$
$$+ (34 \sin^2\phi + 41 \cos^2\phi + 24 \sin\phi \cos\phi)y^2$$
$$+ (24 \sin^2\phi + 14 \sin\phi \cos\phi - 24 \cos^2\phi)xy = 100.$$

By the conditions of the problem we are to choose ϕ so that

$$24 \sin^2\phi + 14 \sin\phi \cos\phi - 24 \cos^2\phi = 0.$$

One value of ϕ satisfying this equation is $\tan^{-1}\frac{3}{4}$. Accordingly we substitute $\sin\phi = \frac{3}{5}$ and $\cos\phi = \frac{4}{5}$, when the equation reduces to $x^2 + 2y^2 = 4$, which is the equation of an ellipse.

CASE II. *Interchange of axes.* If the axes of x and y are simply interchanged, their directions are changed, and hence such a transformation is of the type under consideration in this article. The formulas for such a transformation are evidently $x = y'$, $y = x'$.

CASE III. *Rotation and interchange of axes.* Finally, if the axes are rotated through an angle ϕ and then interchanged, the formulas, being merely a combination of the two already found, are

$$x = y' \cos\phi - x' \sin\phi, \qquad y = y' \sin\phi + x' \cos\phi.$$

A special case of some importance occurs when $\phi = 270°$. We have then $x = x'$, $y = -y'$.

Cases II and III, it should be added, occur much less frequently than Case I.

In case both the origin and the direction of the axes are to be changed, the processes may evidently be performed successively, preferably in this order: (1) change of origin; (2) change of direction.

116. Oblique coördinates. Up to the present time we have always constructed the coördinate axes at right angles to each other. This is not necessary, however, and in some problems, indeed, it is of advantage to make the axes intersect at some other angle. Accordingly, in fig. 132, let OX and OY intersect at some angle ω other than 90°.

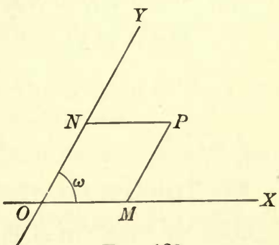

FIG. 132

We now define x for any point in the plane as the distance from OY to the point, measured parallel to OX; and y as the distance from OX to the point, measured parallel to OY. The algebraic signs are determined according to the same rules as were adopted in § 16.

It is immediately evident that the rectangular coördinates are but a special case of this new type of coördinates, called *oblique* coördinates, since the new definitions of x and y include those previously given. In fact, the term Cartesian or rectilinear coördinates includes both the rectangular and the oblique.

Oblique coördinates are usually less convenient than the rectangular, and are very little used in this book. If necessary, the formulas obtained by using rectangular coördinates can be transformed into similar ones in oblique coördinates by the formulas of the following article. When no angle is specified the angle between the axes is understood to be a right angle.

117. Change from rectangular to oblique axes without change of origin. Let OX and OY (fig. 133) be the original axes at right angles to each other, and OX' and OY' the new axes, making angles ϕ and ϕ' respectively with OX. Then $\omega = \phi' - \phi$. Let P be any point in the plane, its rectangular coördinates being x and y, and its oblique coördinates being x' and y'. Draw PM parallel to OY, PM' parallel to OY', $M'N$ parallel to OY, and $RM'N'$ parallel to OX. Then $\angle RM'P = \phi'$.

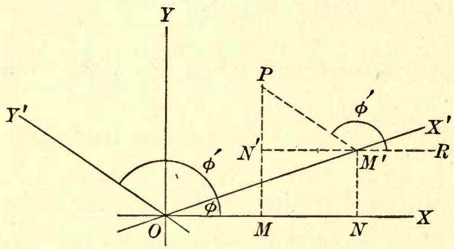

FIG. 133

$$\text{But } OM = ON + NM = ON + M'N' = OM' \cos\phi + M'P \cos\phi',$$

$$MP = MN' + N'P = NM' + N'P = OM' \sin\phi + M'P \sin\phi'.$$

$$\therefore\ x = x' \cos\phi + y' \cos\phi',$$

$$y = x' \sin\phi + y' \sin\phi'.$$

Ex. Transform the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes as axes.

Since the equations of the asymptotes are $y = \pm\frac{b}{a}x$, $\phi = \tan^{-1}\left(-\frac{b}{a}\right)$, and $\phi' = \tan^{-1}\frac{b}{a}$, if we choose to have the hyperbola lie in the first and the third quadrants with respect to the new axes. The formulas of transformation become

$$x = \frac{a}{\sqrt{a^2 + b^2}}(x' + y'), \qquad y = \frac{b}{\sqrt{a^2 + b^2}}(-x' + y').$$

Substituting and simplifying, we have as the new equation $xy = \frac{a^2 + b^2}{4}\cdot$ Unless $b = a$, the axes are oblique and $\omega = 2\tan^{-1}\frac{b}{a}\cdot$

118. Degree of the transformed equation. In reviewing this chapter we see that the expressions for the original coördinates in terms of the new are all of the first degree. Hence the result of any transformation cannot be of higher degree than that of the original equation. On the other hand, the result cannot be of lower degree than that of the original equation; for it is evident that if any equation is transformed to new axes and then back to the original axes, it must resume its original form exactly. Hence if the degree had been lowered by the first transformation, it must be increased to its original value by the second transformation. But this is impossible, as we have just noted.

It follows that the degree of an equation is unchanged by any single transformation of coördinates, or by any number of successive transformations. In particular, the proposition that any equation of the first degree represents a straight line is true for oblique as for rectangular coördinates.

PROBLEMS

1. What are the new coördinates of the points $(2, 3)$, $(-4, 5)$, and $(5, -7)$ if the origin is transferred to the point $(3, -2)$, the new axes being parallel to the old?

2. Transform the equation $x^2 + 4y^2 - 2x + 8y + 1 = 0$ to new axes parallel to the old axes and meeting at the point $(1, -1)$ with respect to the old axes.

3. Transform the equation $y^3 - 6y^2 + 3x^2 + 12y - 18x + 35 = 0$ to new axes parallel to the original axes and meeting at $(2, -3)$ with respect to the original axes.

4. Find the equation of the ellipse when the origin is taken at the lower extremity of the minor axis, and the minor axis is the axis of y.

5. Find the equation of the ellipse when the origin is at the left-hand vertex, the major axis lying along OX.

6. Find the equation of the hyperbola when the origin is at the left-hand vertex, the transverse axis lying along OX.

7. Find the equation of the strophoid when the origin is at A (fig. 92), the axes being parallel to those of § 84.

8. Find the equation of the strophoid when the asymptote is the axis of y, the axis of x being as in § 84.

9. Find the equation of the witch (fig. 90) when LK is the axis of x and OA the axis of y.

10. Find the equation of the witch when the origin is taken at the center of the circle used in constructing it, the axes being parallel to those of § 82.

11. Find the equation of the cissoid when its asymptote is the axis of y and its axis is the axis of x.

12. Find the equation of the cissoid when the origin is at the center of the circle used in its definition, the direction of the axes being as in § 83.

13. Find the equation of the parabola when the origin is at the focus and the axis of x is the axis of the curve.

14. Find the equation of the parabola when the axis of the curve and the directrix are taken as the axes of x and y respectively.

15. Transform $y^2 - 8x - 10y + 1 = 0$ to new axes parallel to the old, so choosing the origin that the new equation shall contain only terms in y^2 and x.

16. Transform the equation $12x^2 + 18y^2 - 12x + 12y - 31 = 0$ to new axes parallel to the old, so choosing the origin that there shall be no terms of the first degree in the new equation.

17. Show that any equation of the form $xy + ax + by + c = 0$ can always be reduced to the form $xy = k$ by choosing new axes parallel to the old, and determine the value of k.

18. Show that the equation $ax^2 + by^2 + cx + dy + e = 0$ $(a \neq 0,\ b \neq 0)$ can always be put in the form $ax^2 + by^2 = k$ by choosing new axes parallel to the old, and determine the value of k.

19. Show that the equation $y^2 + ay + bx + c = 0$ $(b \neq 0)$ can always be reduced to the form $y^2 + bx = 0$ by choosing new axes parallel to the given ones.

20. Find the equation of an ellipse if its axes are 6 and 2, its center is at $(-3, 2)$, and its major axis is parallel to OX.

21. Find the equation of an ellipse if its axes are $\frac{3}{7}$ and $\frac{1}{5}$, its center is at $(-2, -3)$, and its major axis is parallel to OX.

22. Find the equation of an hyperbola if its transverse axis is 4, its conjugate axis 2, its center at $(1, -2)$, and its transverse axis parallel to OX.

23. Find the equation of an hyperbola if its transverse axis is $\sqrt{2}$, its conjugate axis $\sqrt{\frac{2}{3}}$, its center at $(2, 3)$, and its transverse axis parallel to OX.

24. The vertex of a parabola is at $(3, -2)$ and its focus is at $(5, -2)$. Find its equation.

25. The vertex of a parabola is at $(4, 5)$ and its focus is at $(4, 1)$. Find its equation.

26. The center of an ellipse is at the point $(2, 3)$, its eccentricity is $\frac{1}{2}$, and the length of its major axis, which is parallel to the axis of x, is 10. What is the equation of the ellipse?

27. Find the equation of an ellipse when the vertices are $(-2, 0)$, $(4, 0)$, and one focus is at the origin.

28. The center of an hyperbola is at $(-1, -2)$, its eccentricity is $1\frac{1}{2}$, and its transverse axis, which is parallel to OX, is 4. Find its equation.

29. The vertex of a parabola is at the point $(-4, -2)$, and it passes through the origin of coördinates. Find its equation, its axis being parallel to OX.

30. Given the ellipse $4x^2 + 9y^2 + 8x - 36y + 4 = 0$; find its eccentricity, center, vertices, foci, and directrices.

31. Given the ellipse $3x^2 + 5y^2 + 18x - 20y + 32 = 0$; find its eccentricity, center, vertices, foci, and directrices.

32. Given the hyperbola $9x^2 - 4y^2 - 54x - 32y - 19 = 0$; find its eccentricity, center, vertices, foci, directrices, and asymptotes.

33. Given the hyperbola $3x^2 - 2y^2 + 6x + 8y - 11 = 0$; find its eccentricity, center, vertices, foci, directrices, and asymptotes.

34. Given the parabola $72x^2 + 48x + 180y - 37 = 0$; find its vertex, focus, axis, and directrix.

35. Given the parabola $y^2 - 5x + 6y - 1 = 0$; find its vertex, focus, axis, and directrix.

36. What are the coördinates of the points (0, 1), (1, 0), (1, 1) if the axes are rotated through an angle of 60°?

37. Transform the equation $3x^2 + 3y^2 - 10xy + 8 = 0$ to a new set of axes by rotating the original axes through an angle of 45°, the origin not being changed.

38. Find the equation of the folium $x^3 + y^3 - 3axy = 0$ after the axes have been rotated through an angle of 45°.

39. By rotating the axes through an angle of 45° and changing the origin, prove that the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ is a parabola.

40. Transform $5x^2 - 12xy + 10y^2 - 14 = 0$ to a new set of axes, making an angle $\tan^{-1}\frac{2}{3}$ with the original set.

41. Show that the equation $x^2 + y^2 = a^2$ will be unchanged by transformation to any pair of rectangular axes, if the origin is unchanged.

42. Transform the equation $x^2 - y^2 = 36$ to new axes bisecting the angles between the original axes.

43. Transform the equation $4x^2 - 3xy + 8y^2 = 1$ to one which has no xy-term, by rotating the axes through the proper angle.

44. By rotating the axes through the proper angle transform the equation $3x^2 + 2\sqrt{3}xy + y^2 + 2x - 2\sqrt{3}y = 0$ to another which shall have no term in xy.

45. Transform the equation

$$x^2 - 5y^2 - 6\sqrt{3}xy + [2 + 12\sqrt{3}]x + [20 - 6\sqrt{3}]y - 15 + 12\sqrt{3} = 0$$

to a new set of rectangular axes making an angle of 60° with the original axes and intersecting at the point $(-1, 2)$ with respect to the original axes.

46. Transform the equation $4x^2 + 9y^2 = 36$ from rectangular axes to oblique axes with the same origin, making angles $\tan^{-1}\frac{1}{3}$ and $\tan^{-1}(-\frac{4}{3})$ respectively with OX.

47. Find the equation of the hyperbola $3x^2 - 4y^2 = 12$ referred to its asymptotes as coördinate axes.

48. Show that the lines $y = \pm x$ intersect the strophoid at the origin only, and find the equation of the curve referred to these lines as axes.

49. Transform the equation $2x^2 - 3y^2 = 6$ from rectangular axes to oblique axes having the same origin and making the angles $\tan^{-1}\frac{1}{2}$ and $\tan^{-1}\frac{4}{3}$ respectively with OX.

50. Prove that the formulas for transposing from a set of rectangular axes to a set of oblique axes having the same origin and the same axis of x are

$$x = x' + y'\cos\omega,$$
$$y = y'\sin\omega,$$

where ω is the angle between the oblique axes.

51. By transforming the equation $y = mx + b$ by the formulas of example 50, show that the equation of a straight line in oblique coördinates is

$$y = \frac{\sin\phi}{\sin(\omega - \phi)}x + c,$$

where ω is the angle between OX and OY, ϕ the angle between the line and OX, and c the intercept on OY.

52. Derive the result of example 51 directly by use of the trigonometric formulas connecting the sides and the angles of an oblique triangle.

53. By use of the transformation of example 50, prove that the equation of a circle in oblique coördinates is

$$(x - d)^2 + (y - e)^2 + 2(x - d)(y - e)\cos\omega = r^2,$$

where ω is the angle between the axes, and (d, e) is the center.

54. Obtain the result of example 53 directly by use of the trigonometric relations connecting the sides and the angles of an oblique triangle.

CHAPTER XI

THE GENERAL EQUATION OF THE SECOND DEGREE

119. Introduction. The most general equation of the second degree is of the form

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

where the coefficients may have any values, including zero, except that A, B, and H cannot be zero together.

We shall proceed to show that this equation always represents an ellipse, an hyperbola, a parabola, or a limiting case of one of these, if it represents any curve, and shall derive criteria by which the nature of the curve can be readily determined.

120. Removal of the *xy*-term. Let us make a transformation of coördinates to new rectangular axes, making an angle ϕ with the original ones, the origin being unchanged. The formulas of transformation are (§ 115)

$$x = x' \cos\phi - y' \sin\phi,$$
$$y = x' \sin\phi + y' \cos\phi.$$

Substituting, we have

$$A'x'^2 + 2H'x'y' + B'y'^2 + 2G'x' + 2F'y' + C' = 0,$$

where $$A' = A\cos^2\phi + 2H\sin\phi\cos\phi + B\sin^2\phi,$$
$$H' = (B - A)\sin\phi\cos\phi + H(\cos^2\phi - \sin^2\phi),$$
$$B' = A\sin^2\phi - 2H\sin\phi\cos\phi + B\cos^2\phi,$$
$$G' = G\cos\phi + F\sin\phi,$$
$$F' = F\cos\phi - G\sin\phi,$$
and $$C' = C.$$

We may now determine ϕ so that H' shall vanish; that is, so that

$$2(B-A)\cos\phi\sin\phi + 2H(\cos^2\phi - \sin^2\phi) = 0.$$

This equation is equivalent to

$$2H\cos 2\phi + (B-A)\sin 2\phi = 0,$$

whence
$$\tan 2\phi = \frac{2H}{A-B},$$

or
$$\phi = \tfrac{1}{2}\tan^{-1}\frac{2H}{A-B}.$$

To compute the values of A' and B', we have

$$\begin{aligned} A' &= A\cos^2\phi + 2H\sin\phi\cos\phi + B\sin^2\phi \\ &= A\frac{1+\cos 2\phi}{2} + H\sin 2\phi + B\frac{1-\cos 2\phi}{2} \\ &= \tfrac{1}{2}[A+B+(A-B)\cos 2\phi + 2H\sin 2\phi]. \end{aligned}$$

But, since $\tan 2\phi = \dfrac{2H}{A-B}$,

$$\sin 2\phi = \pm\frac{2H}{\sqrt{(A-B)^2+4H^2}}, \quad \cos 2\phi = \pm\frac{A-B}{\sqrt{(A-B)^2+4H^2}},$$

and therefore
$$\begin{aligned} A' &= \frac{1}{2}\left[A+B\pm\frac{(A-B)^2+4H^2}{\sqrt{(A-B)^2+4H^2}}\right] \\ &= \tfrac{1}{2}[A+B\pm\sqrt{(A-B)^2+4H^2}]. \end{aligned}$$

Similarly,
$$B' = \tfrac{1}{2}[A+B\mp\sqrt{(A-B)^2+4H^2}].$$

From these results it follows that

$$A'B' = AB - H^2.$$

Hence if $AB-H^2$ is positive, A' and B' have the same sign; if $AB-H^2$ is negative, A' and B' have opposite signs; if $AB-H^2$ is zero, either A' or B' is zero.

The discussion of the general equation is then reduced to that of the simpler equation

$$A'x'^2 + B'y'^2 + 2\,G'x' + 2\,F'y' + C' = 0.$$

This equation we will consider in the next two articles, dropping the primes for convenience.

121. The equation $Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0$. We shall prove the theorem: *The equation*

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0,$$

where the coefficients are such that

$$AF^2 + BG^2 - ABC \neq 0,$$

represents a conic, if it represents any curve at all. In particular,

(1) *when A and B have the same sign, it represents an ellipse* * *or no curve;*

(2) *when A and B have opposite signs, it represents an hyperbola;*

(3) *when either A or B is zero, it represents a parabola.*

Suppose first that neither A nor B is zero. Then the equation may be rearranged as follows:

$$A\left(x^2 + 2\frac{G}{A}x\right) + B\left(y^2 + 2\frac{F}{B}y\right) = -C.$$

We may then complete the squares of the expressions in the parentheses; thus,

$$A\left(x^2 + 2\frac{G}{A}x + \frac{G^2}{A^2}\right) + B\left(y^2 + 2\frac{F}{B}y + \frac{F^2}{B^2}\right) = \frac{G^2}{A} + \frac{F^2}{B} - C,$$

that is,
$$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = \frac{AF^2 + BG^2 - ABC}{AB}.$$

* The circle is considered a special case of an ellipse (see § 75).

Since $AF^2 + BG^2 - ABC$ is not zero, we may divide by the right-hand member of the equation, obtaining

$$\frac{\left(x+\frac{G}{A}\right)^2}{M} + \frac{\left(y+\frac{F}{B}\right)^2}{N} = 1,$$

where, for convenience, we place

$$M = \frac{AF^2 + BG^2 - ABC}{A^2B},$$

$$N = \frac{AF^2 + BG^2 - ABC}{AB^2}.$$

We may now transfer the origin of coördinates to the point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$, the new axes remaining parallel to the old, by the formulas

$$x = -\frac{G}{A} + x', \qquad y = -\frac{F}{B} + y'.$$

The equation is then $\qquad \frac{x'^2}{M} + \frac{y'^2}{N} = 1.$

Now if A and B have the same sign, M and N will have the same sign. If this sign is positive, we may place $M = a^2$, $N = b^2$, and the equation is

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1,$$

which represents an ellipse.

The axes of the ellipse are parallel to the original coördinate axes, and its center is at the point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$ referred to the original axes. If $A = B$, the ellipse is a circle.

If M and N are both negative, the equation

$$\frac{x'^2}{M} + \frac{y'^2}{N} = 1$$

can be satisfied by no real values of x and y.

If A and B have opposite signs, M and N have opposite signs, and we may place either $M=a^2$, $N=-b^2$, or $M=-a^2$, $N=b^2$, thus obtaining either

$$\frac{x'^2}{a^2}-\frac{y'^2}{b^2}=1,$$

or

$$-\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1,$$

either of which represents an hyperbola.

The axes of the hyperbola are parallel to the original coördinate axes, and its center is at the point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$ referred to the original axes.

The first and the second parts of the theorem are therefore proved.

Consider now the case in which either A or B is zero. If, for example, $A=0$, $B\neq 0$, the equation is

$$By^2+2\,Gx+2\,Fy+C=0,$$

and the condition to be fulfilled by the coefficients is $BG^2\neq 0$, which is equivalent to $G\neq 0$, since B cannot be zero.

We may arrange the equation as follows:

$$y^2+2\frac{F}{B}y=-2\frac{G}{B}x-\frac{C}{B}.$$

Completing the square, we have

$$\left(y+\frac{F}{B}\right)^2=-\frac{2\,G}{B}\left(x+\frac{C}{2\,G}-\frac{F^2}{2\,GB}\right).$$

If now we transform to a new origin by placing

$$x=-\frac{C}{2\,G}+\frac{F^2}{2\,GB}+x',\qquad y=-\frac{F}{B}+y',$$

we have

$$y'^2=-\frac{2\,G}{B}x',$$

which is the equation of a parabola.

Similarly, if $B=0$ but $A\neq 0$, the equation may be reduced to the form

$$x'^2=-\frac{2\,F}{A}y',$$

which is also a parabola.

In each case the axis of the parabola is parallel to one of the original coördinate axes.

Hence the third part of the theorem is proved.

122. The limiting cases. We shall consider now the equation

$$Ax^2 + By^2 + 2\,Gx + 2\,Fy + C = 0$$

when the coefficients are such that

$$AF^2 + BG^2 - ABC = 0.$$

The figures represented are limiting cases of a conic, since the equation of this article may be obtained from that of the previous article by allowing the coefficients to change in such a way that $AF^2 + BG^2 - ABC$ approaches zero. We have three cases:

1. A and B have the same sign.

By proceeding as in § 121, we may put the equation in the form

$$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = 0;$$

and if, as before, we place

$$x = -\frac{G}{A} + x', \qquad y = -\frac{F}{B} + x',$$

we have

$$Ax'^2 + By'^2 = 0.$$

Since A and B have the same sign, we may consider them as positive, and factor the equation as follows:

$$(\sqrt{A}x' + i\sqrt{B}y')(\sqrt{A}x' - i\sqrt{B}y') = 0,$$

which is satisfied by real values of x' and y' only when $x' = 0$, $y' = 0$, or in the old coördinates $x = -\frac{G}{A}$, $y = -\frac{F}{B}$.

Hence in this case the equation represents a point. This may be considered the limiting case of the ellipse.

2. A and B have opposite signs. We may put the equation in the form

$$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = 0,$$

or

$$Ax'^2 + By'^2 = 0.$$

Since A and B have opposite signs, we will consider A as positive and B as negative. The equation can then be separated into two real factors

$$(\sqrt{A}x'+\sqrt{-B}y')(\sqrt{A}x'-\sqrt{-B}y')=0.$$

Consequently the equation represents the two straight lines intersecting in the point $x'=0$, $y'=0$, or $x=-\frac{G}{A}$, $y=-\frac{F}{B}$.

This may be considered the limiting case of the hyperbola.

3. One of the coefficients A or B is zero. For example, let $A=0$, $B\neq 0$. Then the condition $AF^2+BG^2-ABC=0$ becomes $G=0$. Hence the equation is

$$By^2+2Fy+C=0.$$

This may be factored into

$$B(y-y_1)(y-y_2)=0,$$

and accordingly represents either two parallel straight lines, two coincident straight lines, or no real locus, according as y_1 and y_2 are real and unequal, real and equal, or imaginary.

This is considered a limiting case of the parabola.

123. The determinant $AB-H^2$. Returning now to the general equation of the second degree,

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0,$$

and remembering that if it is reduced to the form

$$A'x'^2+B'y'^2+2G'x'+2F'y'+C'=0,$$

we have

$$AB-H^2=A'B',$$

we may state the following theorem:

The equation

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$

always represents a conic or one of the limiting cases, if it represents any curve at all.

1. If $AB - H^2 > 0$, *the equation represents an ellipse, a point, or no curve.*

2. If $AB - H^2 < 0$, *the equation represents an hyperbola or two intersecting straight lines.*

3. If $AB - H^2 = 0$, *the equation represents a parabola, two parallel lines, two coincident lines, or no curve.*

124. The discriminant of the general equation. We have seen in § 122 that

$$A'x'^2 + B'y'^2 + 2G'x' + 2F'y' + C' = 0 \tag{1}$$

represents one of the limiting cases of the conic sections when

$$A'F'^2 + B'G'^2 - A'B'C' = 0.$$

It is useful to have this condition in terms of the coefficients of the general equation

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0. \tag{2}$$

This might be done by substituting for A', B', G', F', and C' the values given in § 120, but this method is tedious. We may obtain the result by noticing that the first member of (1) can be factored rationally in x and y when it represents a limiting case, and not otherwise. The same must be true of equation (2). We shall proceed then to find the condition under which (2) can be factored.

1. Assume $A \neq 0$. (2) may now be considered as a quadratic equation in x, and factored by the method of § 41. Solving (2) for x, we have

$$x = \frac{-(Hy + G) \pm \sqrt{(H^2 - AB)y^2 + 2y(HG - AF) + (G^2 - CA)}}{A}.$$

It is necessary, however, that y should not appear under the radical sign, and for this it is necessary and sufficient that the quantity under the radical sign must be a perfect square. The necessary and sufficient condition for this is (§ 37)

$$(HG - AF)^2 - (H^2 - AB)(G^2 - CA) = 0,$$

that is,

$$ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0. \tag{3}$$

2. Assume $A = 0$, but $B \neq 0$. The equation may then be considered as a quadratic equation in y, and handled in the same manner as before with the same result.

3. Assume $A = 0$, $B = 0$. Then H cannot equal zero. The equation can consequently be written

$$xy + \frac{G}{H}x + \frac{F}{H}y + \frac{C}{2H} = 0.$$

The factors of this, if they exist at all, are clearly of the form

$$(x + a)(y + b) = 0,$$

whence $$a = \frac{F}{H}, \qquad b = \frac{G}{H}, \qquad ab = \frac{C}{2H}.$$

The necessary and sufficient condition that two quantities a and b can be found satisfying these equations is

$$2FG - CH = 0.$$

But this is just what (3) becomes when $A = 0$, $B = 0$. Hence, *the necessary and sufficient condition that*

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

represents a limiting case of a conic is

$$ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0.$$

The expression (3) is called the discriminant of (1) and is denoted by Δ. In determinant form

$$\Delta = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}.$$

125. Classification of curves of the second degree. The results of the previous articles are exhibited in the table on the following page, which gives the simplest forms to which the general equation

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

can be reduced under the various hypotheses, where

$$D = AB - H^2,$$
$$\Delta = ABC + 2FGH - AF^2 - BG^2 - CH^2.$$

	$\Delta \neq 0$	$\Delta = 0$
$D > 0$	Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, or no curve	Point $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$
$D < 0$	Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, or $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Two intersecting straight lines $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$
$D = 0$	Parabola $y^2 = 4px$, or $x^2 = 4py$	Two parallel straight lines $(y - y_1)(y - y_2) = 0$, or $(x - x_1)(x - x_2) = 0$, or no locus

126. Center of a conic. It is frequently desirable to find the center of a conic represented by the general equation. Now, if the origin of coördinates is taken at the center of the curve, the equation can contain no terms of the first degree in x and y; for if it is satisfied by any point (x_1, y_1), it must also be satisfied by the symmetrically placed point $(-x_1, -y_1)$. We will accordingly take the center as (x_0, y_0) and make the transformation

$$x = x_0 + x',$$
$$y = y_0 + y'.$$

The general equation then becomes

$$Ax'^2 + 2Hx'y' + By'^2 + 2(Ax_0 + Hy_0 + G)x' + 2(Hx_0 + By_0 + F)y' + Ax_0^2 + 2Hx_0y_0 + By_0^2 + 2Gx_0 + 2Fy_0 + C = 0,$$

where, by the condition for the center,

$$\begin{aligned} Ax_0 + Hy_0 + G &= 0, \\ Hx_0 + By_0 + F &= 0. \end{aligned} \tag{1}$$

By multiplying each of these by a properly chosen factor and adding, we obtain the equivalent equations

$$\begin{aligned} (AB - H^2)x &= HF - BG, \\ (AB - H^2)y &= HG - AF. \end{aligned} \tag{2}$$

Three cases then occur:

1. $AB - H^2 \neq 0$. Equations (2) have then a single solution and the curve has a center. This occurs for the ellipse, the hyperbola, and their limiting cases.

2. $AB - H^2 = 0$, but not each of the expressions $HF - BG$ and $HG - AF$ equal to zero. At least one of equations (2) expresses an absurdity, and hence equations (1) have no solution and the curve has no center. This occurs in the case of the parabola.

3. $AB - H^2 = 0$, $HF - BG = 0$, $HG - AF = 0$. Equations (2) are each $0 = 0$. Equations (1) are identical, and any point on the line expressed by each of them is a center of the curve. In this case one easily calculates that $\Delta = 0$. The curve then consists of two parallel straight lines (§ 125), and the line of centers is the line halfway between the two parallel lines.

127. If for the equation

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

the origin is transferred to the center of the curve, when such exists, the equation becomes

$$Ax'^2 + 2Hx'y' + By'^2 + C' = 0,$$

where $$C' = Ax_0^2 + 2Hx_0y_0 + By_0^2 + 2Gx_0 + 2Fy_0 + C.$$

This quantity C' may be expressed in terms of the original coefficients as follows. Take the equations (1) of § 126, multiply the first one by x_0, the second by y_0, and add them. There results

$$Ax_0^2 + 2Hx_0y_0 + By_0^2 + Gx_0 + Fy_0 = 0.$$

Subtracting this from the value of C', as given above, we have

$$C' = Gx_0 + Fy_0 + C,$$

whence, by substituting the values of x_0 and y_0, as given by (2) (§ 126), we have

$$C' = \frac{ABC + 2FGH - AF^2 - BG^2 - CH^2}{AB - H^2} = \frac{\Delta}{D}.$$

128. Directions for handling numerical equations. In case it is necessary to reduce a numerical equation to its simplest form, the procedure, based on the foregoing discussion, is as follows:

First compute $AB - H^2$ and determine the type of the curve (§ 123). Δ may also be computed if wished, but it is not necessary.

If $AB - H^2 \neq 0$, find the center, as in § 126, and transfer the origin to it. Then, as in § 120, turn the axes through an angle

$$\phi = \frac{1}{2}\tan^{-1}\frac{2H}{A-B} = \tan^{-1}\frac{\pm 2H}{\sqrt{(A-B)^2+4H^2} \pm (A-B)},$$

computing A' and B' by the formulas of § 120. The two values of $\tan\phi$ are the slopes of the axes of the curve.

If $AB - H^2 = 0$, write the equation in the form

$$(\sqrt{A}x + \sqrt{B}y)^2 + 2Gx + 2Fy + C = 0,$$

$\sqrt{B}$ being taken with the same sign as H, and let

$$y' = \frac{\sqrt{A}x + \sqrt{B}y}{\sqrt{A+B}}, \qquad x' = \frac{\sqrt{B}x - \sqrt{A}y}{\sqrt{A+B}}.$$

Solve these equations for x and y and substitute in the given equation.

The equation is now in the form $y'^2 + 2G'x' + 2F'y' + C' = 0$, and the further reduction is made by the method of § 121.

Ex. 1. $8x^2 - 4xy + 5y^2 - 36x + 18y + 9 = 0.$

Here $AB - H^2 = 36$, and the curve is an ellipse or a limiting case of an ellipse.

The center, found by § 126, is $(2, -1)$, and the equation transferred to the center as origin becomes

$$8x'^2 - 4x'y' + 5y'^2 - 36 = 0.$$

We now turn the axes through $\phi = \frac{1}{2}\tan^{-1}(-\frac{4}{3}) = \tan^{-1}2$ or $\tan^{-1}(-\frac{1}{2})$, and find, from § 120, $A' = 9$ or 4, $B' = 4$ or 9.

The ambiguity is removed by noticing that if we take $\tan\phi = 2$, the formulas of transformation (§ 115) are

$$x' = \frac{x'' - 2y''}{\sqrt{5}}, \qquad y' = \frac{2x'' + y''}{\sqrt{5}},$$

which give $A' = 4$, $B' = 9$.

The simplest equation is then

$$4x''^2 + 9y''^2 = 36.$$

The slopes of the axes are 2 and $-\frac{1}{2}$.

Ex. 2. $36x^2 - 48xy + 16y^2 + 52x - 260y - 39 = 0.$

Here $$AB - H^2 = 0.$$

We write $$(6x - 4y)^2 + 52x - 260y - 39 = 0,$$

and place $$y' = \frac{6x - 4y}{\sqrt{52}} = \frac{3x - 2y}{\sqrt{13}},$$

$$x' = \frac{-4x - 6y}{\sqrt{52}} = \frac{-2x - 3y}{\sqrt{13}}.$$

Solving for x and y and substituting, we have

$$y'^2 + \sqrt{13}\,y' + \sqrt{13}\,x' - \tfrac{3}{4} = 0,$$

or $$y''^2 = -\sqrt{13}\,x'',$$

where $$x'' = -\frac{4}{\sqrt{13}} + x', \qquad y'' = \frac{\sqrt{13}}{2} + y'.$$

The curve is a parabola, the axis of which is $y'' = 0$ or $6x - 4y + 13 = 0$.

129. Equation of a conic through five points. The general equation of the second degree

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

contains six constants, the ratios of which are alone essential. Five independent equations are sufficient to determine these ratios. Therefore a conic is, in general, determined by five conditions. The simplest conditions are that the conic should pass through the five points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), and (x_5, y_5). The five equations to determine the ratios of A, H, B, G, F, and C are then

$$Ax_1^2 + 2Hx_1y_1 + By_1^2 + 2Gx_1 + 2Fy_1 + C = 0,$$
$$Ax_2^2 + 2Hx_2y_2 + By_2^2 + 2Gx_2 + 2Fy_2 + C = 0,$$
$$Ax_3^2 + 2Hx_3y_3 + By_3^2 + 2Gx_3 + 2Fy_3 + C = 0,$$
$$Ax_4^2 + 2Hx_4y_4 + By_4^2 + 2Gx_4 + 2Fy_4 + C = 0,$$
$$Ax_5^2 + 2Hx_5y_5 + By_5^2 + 2Gx_5 + 2Fy_5 + C = 0.$$

Eliminating the coefficients between these and the general equation, we have

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0,$$

which is the required equation of a conic through five given points.

The equation of a conic through five points may also be found in the following manner:

Let us take any four of the given points and connect them by straight lines so as to form a quadrilateral (fig. 134).

Let the equation of P_1P_2 be $A_1x + B_1y + C_1 = 0$, or, more shortly, $f_1(x, y) = 0$. Similarly, let the equation of P_2P_3 be $f_2(x, y) = 0$, that of P_3P_4 be $f_3(x, y) = 0$, and that of P_4P_1 be $f_4(x, y) = 0$.

Form now the equation

$$lf_1(x, y) \cdot f_3(x, y) + kf_2(x, y) \cdot f_4(x, y) = 0, \tag{1}$$

where l and k are undetermined factors. This equation is of the second degree in x and y; therefore it represents a conic section. Moreover, this conic section passes through P_1; for the coördinates of P_1 make $f_1(x, y) = 0$ and $f_4(x, y) = 0$, and therefore satisfy equation (1). Similarly, this conic passes through P_2, P_3, and P_4. If now we substitute in (1) the coördinates of P_5, we determine values of l and k, which we must assume in order that the conic may pass through P_5. We thus determine the equation of a conic through the five given points.

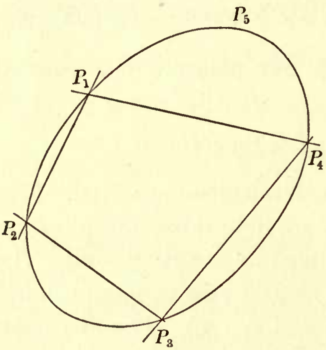

FIG. 134

Ex. Let it be required to pass a conic through the points $P_1(2, 3)$, $P_2(-1, 2)$, $P_3(-3, -1)$, $P_4(0, -4)$, $P_5(1, 1)$.

The equation of P_1P_2 is $x - 3y + 7 = 0$, that of P_2P_3 is $3x - 2y + 7 = 0$, that of P_3P_4 is $x + y + 4 = 0$, and that of P_4P_1 is $7x - 2y - 8 = 0$.

We form the equation

$$l(x - 3y + 7)(x + y + 4) + k(3x - 2y + 7)(7x - 2y - 8) = 0,$$

and, substituting the coördinates of P_5, find $k = \frac{5}{4}l$.

Hence the required conic is

$$(x - 3y + 7)(x + y + 4) + \tfrac{5}{4}(3x - 2y + 7)(7x - 2y - 8) = 0,$$

or

$$109x^2 - 108xy + 8y^2 + 169x - 10y - 168 = 0.$$

If three of the points lie in a straight line, the method is applicable, but it is evident that the conic must be one of the limiting cases, for it must consist of the straight line in which the three points lie, and the straight line connecting the other two points.

If four or five of the points lie in a straight line, the method is not applicable. It is geometrically evident that in this case the problem is indeterminate; for the conic may consist of the straight line in which the four points lie, together with any line through the fifth point, if that is not on the line with the four, or any line whatever if the fifth point lies on a straight line with the four others.

If it is required to determine a parabola, only four points are necessary. This follows from the fact that one relation connecting the coefficients is always given, namely, $AB - H^2 = 0$. We form, as before, the equation

$$lf_1(x, y) \cdot f_3(x, y) + kf_2(x, y) \cdot f_4(x, y) = 0.$$

We form, then, the equation $AB - H^2 = 0$ out of the coefficients of this equation. The result is a quadratic equation in $\frac{l}{k}$, and hence we will have two, one, or no real parabolas, according as the values of $\frac{l}{k}$ are real, equal, or imaginary. It should be noticed that in this connection "parabola" may mean two parallel straight lines.

Ex. Let it be required to pass a parabola through the points $P_1(1, -1)$, $P_2(2, 3)$, $P_3(2, -5)$, $P_4(5, 7)$.

We find the equations of the following lines: P_1P_2, $4x - y - 5 = 0$; P_2P_3, $x - 2 = 0$; P_3P_4, $4x - y - 13 = 0$; P_4P_1, $2x - y - 3 = 0$. The equation of the conic is then

$$l(4x - y - 5)(4x - y - 13) + k(x - 2)(2x - y - 3) = 0,$$

or

$$(16l + 2k)x^2 + (-8l - k)xy + ly^2$$
$$+ (-72l - 7k)x + (18l + 2k)y + 65l + 6k = 0,$$

and the condition $AB - H^2 = 0$ is

$$(16l + 2k)l - (4l + \tfrac{1}{2}k)^2 = 0,$$

whence

$$k = 0 \text{ or } -8l.$$

There are accordingly two parabolas,

$$16x^2 - 8xy + y^2 - 72x + 18y + 65 = 0,$$

and

$$y^2 - 16x + 2y + 17 = 0.$$

The first equation, however, represents a limiting case of a parabola, since it factors into

$$4x - y - 5 = 0 \quad \text{and} \quad 4x - y - 13 = 0,$$

which represent two parallel straight lines.

130. Oblique coördinates. We have assumed, thus far, that the general equation is referred to rectangular coördinates. If, however, the equation

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0$$

has reference to oblique coördinates, it may be transformed to any conveniently chosen pair of rectangular coördinates. Formulas for this purpose are given in § 117, and it has been proved in § 118 that such a transformation does not alter the degree of the equation. Therefore the new equation is of the form

$$A'x'^2 + 2\,H'x'y' + B'y'^2 + 2\,G'x' + 2\,F'y' + C' = 0.$$

This equation may now be investigated by the methods of this chapter.

Hence we have the result:

Any equation of the second degree, whether referred to rectangular or to oblique coördinates, represents a conic.

PROBLEMS

Determine the nature and the position of the following conics:

1. $4\,xy + 3\,y^2 - 8\,x + 16\,y + 19 = 0.$

2. $x^2 - 6\,xy + 9\,y^2 - 280\,x - 20 = 0.$

3. $11\,x^2 - 4\,xy + 14\,y^2 - 26\,x + 32\,y + 59 = 0.$

4. $5\,x^2 - 26\,xy + 5\,y^2 + 10\,x - 26\,y + 71 = 0.$

5. $4\,xy + 6\,x - 8\,y + 1 = 0.$

6. $x^2 - 2\,xy + y^2 + 2\,x - 2\,y + 1 = 0.$

7. $13\,x^2 + 10\,xy + 13\,y^2 + 6\,x - 42\,y - 27 = 0.$

8. $x^2 - 4\,xy - 2\,y^2 - 14\,x + 4\,y + 25 = 0.$

9. $6\,x^2 - 5\,xy - 6\,y^2 - 46\,x - 9\,y + 60 = 0.$

10. $4\,x^2 - 8\,xy + 4\,y^2 + 6\,x - 8\,y + 1 = 0.$

11. $x^2 + 6\,xy + 9\,y^2 - 6\,x - 18\,y + 5 = 0.$

12. $41\,x^2 - 24\,xy + 34\,y^2 - 188\,x + 116\,y + 196 = 0.$

13. $31\,x^2 - 24\,xy + 21\,y^2 + 48\,x - 84\,y + 84 = 0.$

14. Show that, if A and B in the general equation have opposite signs, the conic is an hyperbola.

15. Show that the conic represented by the general equation is an equilateral hyperbola when $A = -B$.

16. Prove that the necessary and sufficient conditions that the general equation should represent a circle are $A = B$, $H = 0$, provided the axes are rectangular.

17. Show that, if the general equation contains the term in xy and not more than one of the terms containing x^2 or y^2, the conic is an hyperbola.

18. Show that $xy + ax + by + c = 0$ is the general equation of the hyperbola when the axes of coördinates are parallel to the asymptotes.

19. Prove that any homogeneous equation in x and y represents a system of straight lines passing through the origin.

20. Find the angle between the two straight lines represented by the equation $Ax^2 + 2\,Hxy + By^2 = 0$.

21. Show that the asymptotes of the hyperbola are parallel to the two straight lines $Ax^2 + 2\,Hxy + By^2 = 0$.

22. Show that, if the focus is taken upon the directrix, the conic becomes one of the limiting cases.

Find the equations of the conics through the following points:

23. $(3, 2)$, $(-2, -3)$, $(\frac{1}{2}, -3)$, $(2, -2)$, $(\frac{3}{2}, -\frac{5}{2})$.

24. $(1, 2)$, $(6, 3)$, $(3, 2)$, $(2, 1)$, $(9, 2)$.

25. $(0, a)$, $(a, 0)$, $(0, -a)$, $(-a, 0)$, (a, a).

26. $(1, 1)$, $(-1, 5)$, $(2, 4)$, $(0, 3)$, $(3, 1)$.

27. Find the equation of a parabola through the four points $(4, -4)$, $(9, 4)$, $(6, -1)$, $(5, -2)$.

28. A point moves so that the sum of the squares of its distances from two intersecting straight lines is constant. Prove that the locus is an ellipse, and find its eccentricity in terms of the angle between the lines.

CHAPTER XII

TANGENT, POLAR, AND DIAMETER FOR CURVES OF THE SECOND DEGREE

131. Equation of a tangent. It has been shown in § 59 that the tangent to a curve at a point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_1 (x - x_1),$$

where $\left(\frac{dy}{dx}\right)_1$ denotes the value of $\frac{dy}{dx}$ at (x_1, y_1).

Applying this theorem to the conic

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

we first find, by differentiation,

$$2Ax + 2Hy + 2Hx\frac{dy}{dx} + 2By\frac{dy}{dx} + 2G + 2Fy\frac{dy}{dx} = 0,$$

whence
$$\frac{dy}{dx} = -\frac{Ax + Hy + G}{Hx + By + F}.$$

Therefore the equation of the tangent at the point (x_1, y_1) is

$$y - y_1 = -\frac{Ax_1 + Hy_1 + G}{Hx_1 + By_1 + F}(x - x_1),$$

that is,
$$Ax_1x - Ax_1^2 + Hxy_1 + Hx_1y - 2Hx_1y_1 + By_1y - By_1^2$$
$$+ Gx - Gx_1 + Fy - Fy_1 = 0.$$

This equation may be simplified by adding to it the identity

$$Ax_1^2 + 2Hx_1y_1 + By_1^2 + 2Gx_1 + 2Fy_1 + C = 0,$$

which follows from the fact that (x_1, y_1) is on the conic. There results

$$Ax_1x + H(x_1y + xy_1) + By_1y + G(x + x_1) + F(y + y_1) + C = 0.$$

This result is easily remembered from its resemblance to the equation of the conic.

132. Definition and equation of a polar. We have just seen in § 131 that the equation

$$Ax_1x + H(x_1y + xy_1) + By_1y + G(x + x_1) + F(y + y_1) + C = 0 \quad (1)$$

represents the tangent line to the conic

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0, \quad (2)$$

provided the point (x_1, y_1) is on the conic. But no matter what is the position of the point (x_1, y_1), (1), being of the first degree, represents some straight line which from the form of the equation must in some way be related to the conic (2) and the point (x_1, y_1).

This line is called the polar of the point (x_1, y_1) with respect to the conic, and the point is called the pole of the line.

The tangent line now appears as only the special case of the polar which occurs when the pole is on the conic.

Ex. 1. The polar of the point (3, − 2) with respect to the ellipse

$$4x^2 + 5y^2 - 2x + 3y - 1 = 0$$

is

$$12x - 10y - (x + 3) + \tfrac{3}{2}(y - 2) - 1 = 0,$$

or

$$22x - 17y - 14 = 0.$$

Ex. 2. Find the pole of the line $2x - 3y + 6 = 0$ with respect to the hyperbola $4x^2 - 5y^2 + 4x - 2y + 3 = 0$.

The polar of (x_1, y_1) is

$$4x_1x - 5y_1y + 2(x + x_1) - (y + y_1) + 3 = 0,$$

or

$$(4x_1 + 2)x + (-5y_1 - 1)y + 2x_1 - y_1 + 3 = 0.$$

This will be the same as the given line if

$$\frac{4x_1 + 2}{2} = \frac{5y_1 + 1}{3} = \frac{2x_1 - y_1 + 3}{6}.$$

These reduce to the two equations for x_1 and y_1,

$$12x_1 - 10y_1 + 4 = 0,$$

$$2x_1 - 11y_1 + 1 = 0;$$

whence

$$x_1 = -\tfrac{17}{56}, \quad y_1 = \tfrac{1}{28}.$$

133. Fundamental theorem on polars. When the equation

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0 \quad (1)$$

represents one of the limiting cases of the conics, the polar has little importance. We shall therefore assume that the conic is either an ellipse (including the circle), a parabola, or an hyperbola. The properties of its poles and polars are then conveniently found by use of the proposition:

If P_2 is any point on the polar of another point P_1, the polar of P_2 passes through P_1.

For the polar of $P_1(x_1, y_1)$ with respect to (1) is

$$Ax_1x + H(x_1y + xy_1) + By_1y + G(x + x_1) + F(y + y_1) + C = 0, \quad (2)$$

and if $P_2(x_2, y_2)$ is on (2), we must have

$$Ax_1x_2 + H(x_1y_2 + x_2y_1) + By_1y_2 + G(x_2 + x_1) + F(y_2 + y_1) + C = 0. \quad (3)$$

Again, the polar of P_2 with respect to (1) is

$$Ax_2x + H(x_2y + xy_2) + By_2y + G(x + x_2) + F(y + y_2) + C = 0, \quad (4)$$

and this passes through (x_1, y_1) because of (3).

134. Chord of contact. An inspection of the figures of the conics shows that a point not on a conic must lie so that in general either two tangents or no tangent can be drawn from it to the conic. In the former case the point is said to be outside the conic; in the latter case, inside. Let us take now a point P_1 outside the conic, and let the two tangents drawn from it to the conic touch the conic in L and K (fig. 135). Now the polar of a point on a conic is the tangent to the conic at that point (§ 132). Hence P_1L is the polar of L, and P_1K is the polar of K. Therefore, by the fundamental theorem (§ 133), the polar of P_1 must pass through L and K. Hence *the polar is the straight line LK, which is called the chord of contact of tangents from P_1.*

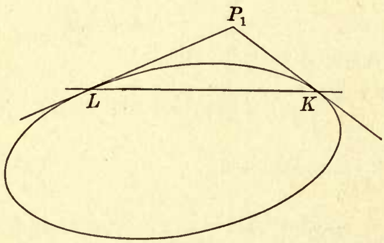

FIG. 135

Conversely, *if a straight line intersects a conic, its pole is the point of intersection of the tangents at the points of intersection.* The proof of this is left to the student.

The chord of contact may be used to find the equations of the tangents through a point not on the conic.

Ex. Find the tangents to the conic $x^2 + 2xy + y^2 + 2x + 6y + 1 = 0$ which pass through the point $(4, -2)$.

Since this point is not on the conic, its coördinates not satisfying the equation of the conic, we form the equation of its polar, i.e. $3x + 5y - 1 = 0$, which will be the chord of contact of the tangents drawn from the point to the conic, provided any can be drawn. Solving the equations of the polar and the conic simultaneously, we find that they intersect at the points $(7, -4)$ and $(2, -1)$. Hence there are two tangents which are respectively $2x + 3y - 2 = 0$ and $x + 2y = 0$.

135. Construction of a polar. Whether a point lies inside or outside a conic, the polar may be obtained by the following construction. Draw through P_1 (fig. 136) two straight lines, one intersecting the conic in L and K, and the other intersecting the conic in M and N. Let the tangents at L and K intersect in R and the tangents at M and N intersect in S. Then R is the pole of LK and S is the pole of MN, by § 134. Since P_1 lies on both LK and MN, its polar passes through R and S by the fundamental theorem. Therefore RS is the required polar.

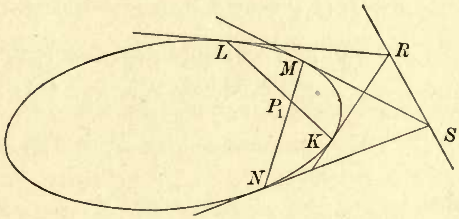

FIG. 136

This construction may also be used when P_1 is outside the conic.

136. The harmonic property of polars. An important property of poles and polars is stated in the theorem: *Any secant passing through P_1 is divided harmonically by the conic and the polar of P_1.*

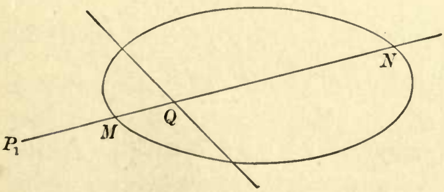

FIG. 137

Let P_1N (fig. 137) be any secant through P_1, M and N be the points in which P_1N cuts the conic, and Q the point in which it cuts the polar of P_1. We are to prove that the line MN is divided harmonically, i.e. that it is divided externally and internally in the same ratio. We are to prove, then, that

$$\frac{P_1M}{P_1N} = \frac{MQ}{QN},$$

whence, by placing $MQ = P_1Q - P_1M$, $QN = P_1N - P_1Q$, and solving for P_1Q, we have

$$P_1Q = \frac{2\,P_1M \cdot P_1N}{P_1M + P_1N}.$$

Let the point P_1 be (x_1, y_1), the equation of the conic be

$$Ax^2 + 2\,Hxy + By^2 + 2\,Gx + 2\,Fy + C = 0, \tag{1}$$

and that of the polar of P_1 be

$$Ax_1x + H(x_1y + xy_1) + By_1y + G(x + x_1) + F(y + y_1) + C = 0. \tag{2}$$

Let (x, y) be a variable point on P_1N, r the variable distance P_1P, and θ the angle made by P_1N and OX. Then

$$\cos\theta = \frac{x - x_1}{r}, \qquad \sin\theta = \frac{y - y_1}{r},$$

that is, $$x = r\cos\theta + x_1, \qquad y = r\sin\theta + y_1. \tag{3}$$

Now if P coincides with either M or N, the values of x and y given by (3) satisfy (1). Substitution gives

$$\begin{aligned} r^2[A\cos^2\theta + 2\,H\sin\theta\cos\theta + B\sin^2\theta] \\ + 2\,r[Ax_1\cos\theta + H(x_1\sin\theta + y_1\cos\theta) \\ + By_1\sin\theta + G\cos\theta + F\sin\theta] + C' = 0, \end{aligned}$$

where $$C' = Ax_1^2 + 2\,Hx_1y_1 + By_1^2 + 2\,Gx_1 + 2\,Fy_1 + C.$$

The roots of this equation are P_1M and P_1N. Hence, by § 43,

$$P_1M + P_1N = -\frac{2[Ax_1\cos\theta + H(x_1\sin\theta + y_1\cos\theta) + By_1\sin\theta + G\cos\theta + F\sin\theta]}{A\cos^2\theta + 2\,H\sin\theta\cos\theta + B\sin^2\theta},$$

$$P_1M \cdot P_1N = \frac{C'}{A\cos^2\theta + 2\,H\sin\theta\cos\theta + B\sin^2\theta},$$

whence $$\frac{2\,P_1M \cdot P_1N}{P_1M + P_1N} = -\frac{C'}{Ax_1\cos\theta + H(x_1\sin\theta + y_1\cos\theta) + By_1\sin\theta + G\cos\theta + F\sin\theta}. \tag{4}$$

Also, if the point P coincides with Q, the values of x and y given by (3) satisfy (2). Substitution gives

$$r[Ax_1\cos\theta + H(x_1\sin\theta + y_1\cos\theta) + By_1\sin\theta + G\cos\theta + F\sin\theta] + C' = 0.$$

The root of this is P_1Q. Therefore P_1Q

$$= -\frac{C'}{Ax_1\cos\theta + H(x_1\sin\theta + y_1\cos\theta) + By_1\sin\theta + G\cos\theta + F\sin\theta}. \quad (5)$$

Comparing (4) and (5), we have

$$P_1Q = \frac{2\,P_1M\cdot P_1N}{P_1M + P_1N},$$

which was to be proved.

The theorem of this article is often made the basis of the definition of the polar.

137. Reciprocal polars. Consider a given conic and a rectilinear figure, such as the triangle ABC with sides a, b, c (fig. 138). Construct the lines a', b', c', the polars of A, B, C, respectively with respect to the conic. The lines a', b', c' form a new triangle $A'B'C'$. The fundamental theorem shows that A', B', C' are the poles of

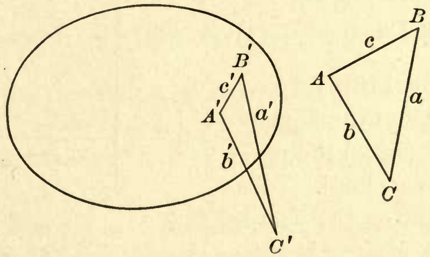

FIG. 138

a, b, c respectively. Hence the two triangles are so related that the vertices of one are the poles of the sides of the other. They are called *reciprocal polars.* A similar construction holds for any figure composed of straight lines.

Consider next any curve K and a tangent line a (fig. 139). Let A be the pole of a with respect to a conic C. As the tangent rolls around the curve K, the point A describes another curve k. Let a and b be two tangents to K, and M their point of intersection, and let A and B be the two corresponding points of k, and m the chord AB. Then, by the fundamental theorem, m is the polar of M.

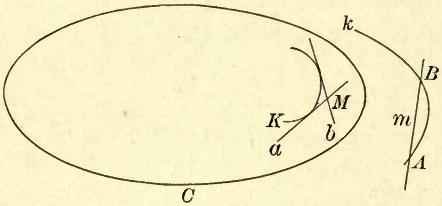

FIG. 139

Now let a and b approach coincidence. Then M approaches a point on K, B and A approach coincidence, and m approaches a tangent to k. Hence the points of K are the poles of the tangents to k.

We have then two curves such that the points of either are the poles of the tangents of the other. These curves are called *reciprocal polars.*

The study of reciprocal polars forms an important part of geometry, but lies outside the limits of this work.

138. Definition and equation of a diameter. *A diameter of a conic is the locus of the middle points of a system of parallel chords.*

Let

$$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0 \qquad (1)$$

be any conic (fig. 140), RS any chord which makes the angle θ with OX, and $P_1(x_1, y_1)$ the middle point of this chord. Take $P(x, y)$ any point on the chord, and let $P_1P = r$, where r is positive if P_1P has the direction of RS, and negative if P_1P has the direction SR. Then for any position of P we have

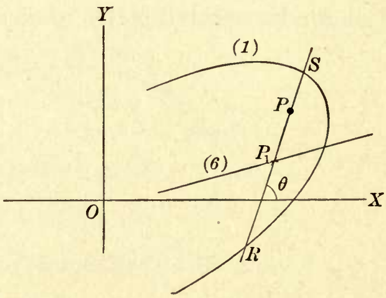

FIG. 140

$$\frac{x - x_1}{r} = \cos\theta, \qquad \frac{y - y_1}{r} = \sin\theta;$$

whence
$$x = x_1 + r\cos\theta, \qquad y = y_1 + r\sin\theta. \qquad (2)$$

Now if P coincides with either R or S, the values of x and y in (2) satisfy (1). Substituting, we have

$$\begin{aligned} r^2[A\cos^2\theta + 2H\sin\theta\cos\theta + B\sin^2\theta] \\ + 2r[Ax_1\cos\theta + Hx_1\sin\theta + Hy_1\cos\theta \\ + By_1\sin\theta + G\cos\theta + F\sin\theta] \\ + [Ax_1^2 + 2Hx_1y_1 + By_1^2 + 2Gx_1 + 2Fy_1 + C] = 0, \quad (3) \end{aligned}$$

the roots of which are P_1S and P_1R. But, by hypothesis, $P_1R = -P_1S$. Hence the roots of equation (3) are equal in magnitude and opposite in sign. Therefore the coefficient of r in (3) must be zero, that is,

$$Ax_1\cos\theta + Hx_1\sin\theta + Hy_1\cos\theta + By_1\sin\theta + G\cos\theta + F\sin\theta = 0. \quad (4)$$

If, for convenience, we assume that $\cos\theta \neq 0$, and this will generally be the case, we may divide by $\cos\theta$ and replace $\tan\theta$ by the usual symbol for the slope m, thus obtaining

$$Ax_1 + Hy_1 + G + m(Hx_1 + By_1 + F) = 0. \quad (5)$$

If we allow RS to move parallel to itself, so that m remains fixed but P_1 changes, (5) always holds true, and in fact shows that P_1 is always a point of the straight line

$$Ax + Hy + G + m(Hx + By + F) = 0. \quad (6)$$

Conversely, any point $P_1(x_1, y_1)$ on line (6) makes the values of r in (3) equal in magnitude but opposite in sign, and if P_1 lies so that these roots are real, it will be the middle point of a chord with slope m.

The straight line (6) is of infinite length, and it is customary to regard the entire line as the diameter, though it is evident that not all of its points correspond to chords of the system which intersect the conic in real points.

139. The last statement of the previous article may be explained as follows:

The equation $y = mx + b$ may be made to represent any line of slope m by assigning an appropriate value to b. For some values of b the corresponding line intersects the conic (1) of § 138 in real points, and is one of the chords bisected by the diameter (6).

For other values of b, however, the line does not intersect the conic in real points, the simultaneous values of x and y satisfying their equations being imaginary. But if these imaginary values of x and y are substituted for x_1, x_2 and y_1, y_2 respectively in the formulas $x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$ of § 18, the resulting values of x and y are real, and furthermore they satisfy the equation of the diameter.

This fact is sometimes expressed by saying that the line is a chord of the conic which intersects it in imaginary points, and that its middle point is a real point of the diameter. It is from this point of view that the entire line is regarded as the diameter, since every point of it is the middle point of some chord of the system.

140. *If the conic has a center, every diameter passes through the center.* For, by § 126, the center satisfies the equations

$$Ax + Hy + G = 0, \qquad Hx + By + F = 0,$$

and hence satisfies (6) of § 138 for any and all values of m.

In the parabola, however, all diameters are parallel to each other and to the axis; for the slope of the diameter is, from (6), § 138, $-\frac{A + Hm}{H + Bm}$. But for the parabola $H = \sqrt{AB}$, so that the slope of the diameter becomes $-\frac{A + \sqrt{AB}m}{\sqrt{AB} + Bm}$, which reduces to $-\frac{\sqrt{A}}{\sqrt{B}}$. This is independent of m, and equal to the slope of the axis (§ 128).

It is evident that the axes of a conic are diameters, for from the symmetry of the curves they contain the middle points of all chords which are perpendicular to them. In fact, they are the only diameters which are perpendicular to the chords which they bisect, as will be proved later on.

141. Diameter of a parabola. If the equation of the parabola is written in its simplest form, $y^2 = 4px$, the equation of the diameter becomes $y = \frac{2p}{m}$.

From this equation it is evident that the only diameter perpendicular to the chords which it bisects is the axis of the parabola.

Ex. 1. Find the equation of the diameter of the parabola $2y^2 + 3x = 0$ bisecting chords with slope 2.

Since $m = 2$ and $p = -\frac{3}{8}$, the equation of the diameter is $y = \frac{2(-\frac{3}{8})}{2}$, or $2y + 3 = 0$.

Ex. 2. A diameter of the parabola $y^2 = 2x$ passes through the point $(2, -1)$. What is its equation, and what is the slope of the chords bisected by it?

If m is the slope of the chords bisected, the equation of the diameter is $y = \frac{1}{m}$. But $(2, -1)$ is a point of this diameter.

$\therefore -1 = \frac{1}{m}$, whence $m = -1$; also the diameter is $y = \frac{1}{-1}$, or $y = -1$.

This equation of the diameter could have been written down immediately, for the diameter is parallel to OX, so that if one of its points is distant -1 from OX, all its points are distant -1 from OX, and its equation is $y = -1$.

If we solve the equations of the diameter and the parabola simultaneously, we find the coördinates of O' (fig. 141), their point of intersection, to be $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$.

The equation of the tangent at O' is found to be $y = mx + \frac{p}{m}$, whence it is seen that its slope is m.

Calling O' the end of the diameter, we express the above theorem as follows: *The tangent at the end of a diameter is parallel to the chords bisected by the diameter.*

If we consider the tangent as the limiting position of a chord which is moved, yet retains its original slope, the above theorem seems almost immediately evident.

142. Parabola referred to a diameter and a tangent as axes. Let $O'X'$ (fig. 141) be a diameter of parabola

$$y^2 = 4px, \tag{1}$$

bisecting chords of slope m, and $O'Y'$ be the tangent at O'. Then the coördinates of O' are $\left(\frac{p}{m^2}, \frac{2p}{m}\right)$, and the slope of $O'Y'$ is m.

First transposing (1) to $O'X'$ and $O'Y''$, where $O'Y''$ is parallel to OY, we have the formulas of transformation

$$x = \frac{p}{m^2} + x'', \qquad y = \frac{2p}{m} + y''.$$

The new form of the equation is

$$y''^2 + \frac{4p}{m}y'' = 4px''.$$

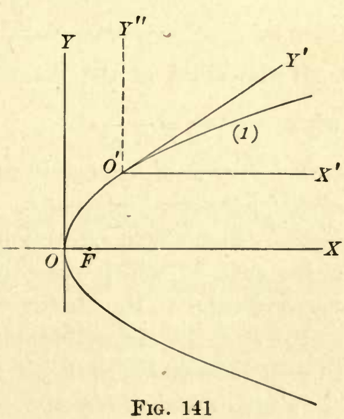

FIG. 141

Using now the formulas of transformation of § 117, which become

$$x'' = x' + \frac{y'}{\sqrt{1+m^2}}, \qquad y'' = \frac{my'}{\sqrt{1+m^2}},$$

since $\phi = 0$ and $\phi' = \tan^{-1} m$, we have, finally,

$$y'^2 = 4\left[\frac{p(1+m^2)}{m^2}\right]x'.$$

By § 17, however, $\quad FO' = \dfrac{p(1+m^2)}{m^2}.$

Therefore if we denote FO' by p', after dropping the primes from x and y, the equation becomes

$$y^2 = 4p'x.$$

It is to be noted that an equation in the form $y^2 = 4px$ always represents a parabola, the x axis being a diameter, the y axis a tangent, and the distance of the focus from the origin being one fourth the coefficient of x.

143. Diameters of an ellipse and an hyperbola. If the equation of the ellipse is written in its simplest form, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the common slope of the chords is denoted by m_1, the equation of the diameter becomes

$$y = -\frac{b^2}{a^2 m_1}x.$$

If the slope of the diameter is denoted by m_2, $m_2 = -\dfrac{b^2}{a^2 m_1}$, whence $m_1 m_2 = -\dfrac{b^2}{a^2}$.

If $b \neq a$, $m_1 m_2$ cannot in general be -1, *and the diameter of an ellipse cannot in general be perpendicular to the chords which it bisects.* The single exception is when the chords are parallel to either axis, in which case the diameter is the other axis and is perpendicular to the chords which it bisects, as noted above.

If $b = a$, the ellipse becomes a circle, and $m_1 m_2$ is always equal to -1. Hence *the diameter of a circle is always perpendicular to the chords which it bisects.*

Ex. 1. Find the equation of a diameter of the ellipse $4x^2 + 9y^2 = 36$ bisecting chords parallel to the line $x + 2y + 1 = 0$.

Here $a^2 = 9$, $b^2 = 4$, and $m_1 = -\frac{1}{2}$. $\therefore$ the diameter is $y = -\frac{4}{9(-\frac{1}{2})}x$, or $9y - 8x = 0$.

Ex. 2. $2y + 3x = 0$ is a diameter of the ellipse $4x^2 + 9y^2 = 36$. What is the slope of the chords which it bisects?

The slope of the diameter is $-\frac{3}{2}$, and by the formula is $-\frac{b^2}{a^2m_1}$, m_1 being the slope of the chords bisected. As $a^2 = 9$ and $b^2 = 4$, $-\frac{b^2}{a^2m_1}$ becomes $-\frac{4}{9m_1}$. $\therefore -\frac{3}{2} = -\frac{4}{9m_1}$, whence $m_1 = \frac{8}{27}$.

Ex. 3. Find the diameter of the circle $4x^2 + 4y^2 + 4x - 8y - 11 = 0$ bisecting chords of slope 2.

The center of the circle is $(-\frac{1}{2}, 1)$, so that the required diameter will be $y - 1 = -\frac{1}{2}(x + \frac{1}{2})$, or $2x + 4y - 3 = 0$.

Ex. 4. Find the diameter of the circle $4x^2 + 4y^2 + 4x - 8y - 11 = 0$, which passes through the point $(2, -1)$.

The center of the circle is $(-\frac{1}{2}, 1)$, and the straight line determined by the two points $(2, -1)$ and $(-\frac{1}{2}, 1)$, i.e. $4x + 5y - 3 = 0$, is the required diameter.

In the case of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ it is to be noticed that the parallel chords may be drawn in two ways. They may join points on the same branch of the hyperbola, or points of one branch to points of the other branch, as represented in fig. 142.

In whichever way the chords are drawn, if their common slope is denoted by m_1, the equation of the diameter is

$$y = \frac{b^2}{a^2m_1}x.$$

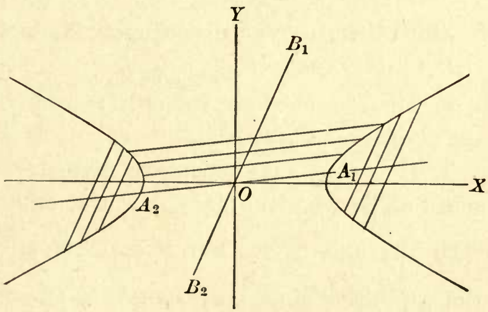

FIG. 142

This equation differs from that for the diameter of the ellipse only in the sign of the right-hand member.

If m_2 is the slope of the diameter, $m_1m_2 = \frac{b^2}{a^2}$, and, as in the case of the ellipse, a diameter of an hyperbola cannot be perpendicular to the chords it bisects, except in the two special cases of the transverse axis and the conjugate axis.

144. Conjugate diameters. In § 143 we have seen that if the slope of the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is denoted by m_1, and the slope of the diameter is denoted by m_2,

$$m_2 = -\frac{b^2}{a^2 m_1}, \quad \text{whence} \quad m_1 m_2 = -\frac{b^2}{a^2}. \tag{1}$$

Similarly, if the slope of the chords is m_2, the slope of the diameter bisecting them must be $-\frac{b^2}{a^2 m_2}$, which, by (1), must be m_1.

Hence the proposition: *If m_1 and m_2 are the slopes of two diameters of an ellipse, and*

$$m_1 m_2 = -\frac{b^2}{a^2},$$

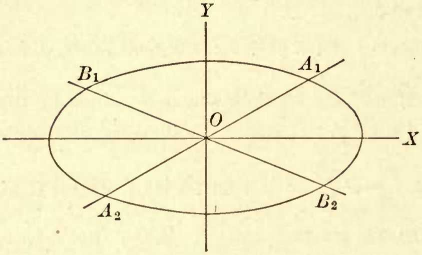

FIG. 143

then each diameter bisects all chords parallel to the other. Such diameters are called conjugate diameters. As the major and the minor axis each bisects chords parallel to the other, they are conjugate diameters.

It follows that:

1. The two axes are the only pair of conjugate diameters which are perpendicular to each other.

2. If one of two conjugate diameters of an ellipse makes an acute angle with the axis of x, the other makes an obtuse angle with the axis of x. For if $m_1 > 0$, $m_2 < 0$, since $m_1 m_2 = -\frac{b^2}{a^2}$. But a positive slope corresponds to an acute angle, and a negative slope to an obtuse angle. Hence the upper portions of conjugate diameters always lie on opposite sides of the minor axis, as OA_1 and OB_1 in fig. 143, A_1A_2 and B_1B_2 being conjugate diameters.

In similar manner for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *if the slopes of two diameters m_1 and m_2 are such that*

$$m_1 m_2 = \frac{b^2}{a^2},$$

the corresponding diameters are conjugate, and each bisects all chords parallel to the other. The transverse and the conjugate axes are conjugate diameters, each of which bisects chords parallel to the other.

It follows that:

1. The two axes are the only pair of conjugate diameters that are perpendicular to each other.

2. Two conjugate diameters make either both acute or both obtuse angles with the transverse axis; for m_1m_2 being always positive, m_1 and m_2 have the same sign.

3. Two conjugate diameters lie on opposite sides of either asymptote; for since $m_1m_2 = \frac{b^2}{a^2}$, if $m_1 < \frac{b}{a}$, then $m_2 > \frac{b}{a}$, and the corresponding conjugate diameters are on opposite sides of the asymptote $y = \frac{b}{a}x$ (fig. 146).

145. Ellipse and hyperbola referred to conjugate diameters as axes. Let the conjugate diameters OA_1 and OB_1 of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

(fig. 144) be chosen as new axes OX' and OY', and let them make angles ϕ and ϕ' respectively with OX.

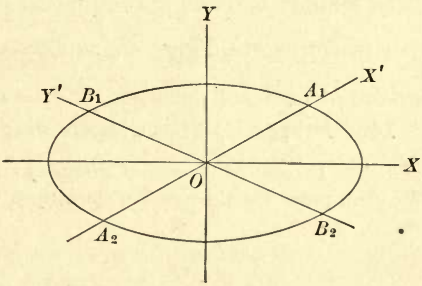

FIG. 144

Then the formulas of transformation are

$$\begin{aligned} x &= x'\cos\phi + y'\cos\phi', \\ y &= x'\sin\phi + y'\sin\phi', \end{aligned} \qquad (2)$$

where
$$\tan\phi\tan\phi' = -\frac{b^2}{a^2},$$

i.e.
$$\frac{\sin\phi\sin\phi'}{b^2} + \frac{\cos\phi\cos\phi'}{a^2} = 0, \qquad (3)$$

since OX' and OY' are conjugate diameters.

Substituting in (1) and collecting like terms, we have

$$\left(\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}\right)x'^2+2\left(\frac{\cos\phi\cos\phi'}{a^2}+\frac{\sin\phi\sin\phi'}{b^2}\right)x'y'$$
$$+\left(\frac{\cos^2\phi'}{a^2}+\frac{\sin^2\phi'}{b^2}\right)y'^2=1. \qquad (4)$$

But the coefficient of $x'y'$ is zero, by virtue of (3); and if the intercepts on OX' and OY' are denoted by a' and b' respectively, i.e. $OA_1=a'$ and $OB_1=b'$, (4) becomes

$$\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}=1, \qquad (5)$$

where $a'=\dfrac{1}{\sqrt{\dfrac{\cos^2\phi}{a^2}+\dfrac{\sin^2\phi}{b^2}}}$, and $b'=\dfrac{1}{\sqrt{\dfrac{\cos^2\phi'}{a^2}+\dfrac{\sin^2\phi'}{b^2}}}$.

The equation of an ellipse can assume the form (5) only when the axes chosen are a pair of conjugate diameters, as only then will the coefficient of xy be zero. Conversely, any equation in form (5) is an ellipse referred to a pair of conjugate diameters as axes.

In similar manner, the equation of the hyperbola referred to a pair of conjugate diameters as axes is $\dfrac{x^2}{a'^2}-\dfrac{y^2}{b'^2}=1$, where at present no geometrical meaning will be assigned to b'.

146. Properties of conjugate diameters.

1. *The tangent at the end of a diameter is parallel to the conjugate diameter.* We shall prove the theorem for the ellipse, the same form of proof being applicable to the hyperbola.

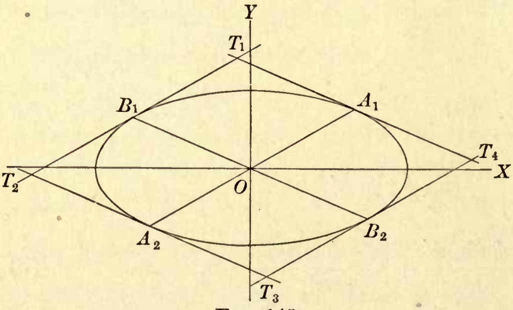

FIG. 145

In fig. 145 let A_1 have coördinates (x_1, y_1). Then the slope of OA_1 is $\dfrac{y_1}{x_1}$, and if the slope of OB_1 is m_2, $m_2\dfrac{y_1}{x_1}=-\dfrac{b^2}{a^2}$, whence $m_2=-\dfrac{b^2x_1}{a^2y_1}$. The equation of the tangent A_1T_1 at A_1 is $\dfrac{x_1x}{a^2}+\dfrac{y_1y}{b^2}=1$, the slope of which is $-\dfrac{b^2x_1}{a^2y_1}$. Hence this tangent is parallel to the conjugate diameter OB_1.

2. *The sum of the squares of the halves of two conjugate diameters of an ellipse is constant and equal to the sum of the squares of the halves of the major and the minor axes*, i.e. $a'^2 + b'^2 = a^2 + b^2$.

We have just seen that the slope of OB_1 (fig. 145) is $-\frac{b^2x_1}{a^2y_1}$, so that its equation is

$$y = -\frac{b^2x_1}{a^2y_1}x. \tag{1}$$

Solving this equation simultaneously with the equation of the ellipse, in order to find coördinates of B_1, we substitute the value of y from (1) in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. As a result $x^2 = \frac{a^4y_1^2}{a^2y_1^2 + b^2x_1^2}$. But $A_1(x_1, y_1)$ is a point of the ellipse, so that $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, or $b^2x_1^2 + a^2y_1^2 = a^2b^2$.

$$\therefore\ x^2 = \frac{a^4y_1^2}{a^2b^2} = \frac{a^2y_1^2}{b^2},$$

and
$$x = \pm\frac{ay_1}{b}.$$

By substitution in (1),
$$y = \mp\frac{bx_1}{a}.$$

Therefore the coördinates of B_1 are $\left(-\frac{ay_1}{b}, \frac{bx_1}{a}\right)$.

If, as in § 145, we denote OA_1 by a' and OB_1 by b', by § 17,

$$a'^2 = x_1^2 + y_1^2,$$

and
$$b'^2 = \frac{a^2y_1^2}{b^2} + \frac{b^2x_1^2}{a^2},$$

and hence
$$\begin{aligned} a'^2 + b'^2 &= \frac{a^2 + b^2}{a^2}x_1^2 + \frac{a^2 + b^2}{b^2}y_1^2 \\ &= (a^2 + b^2)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) \\ &= a^2 + b^2, \end{aligned}$$

since $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, as noted above.

3. *The area of the parallelogram formed by drawing tangents to an ellipse at the ends of conjugate diameters is constant and equal to* $4ab$. Let $T_1T_2T_3T_4$ (fig. 145) be a parallelogram formed by the tangents at the ends of the conjugate diameters A_1A_2 and B_1B_2. Now the area of this parallelogram is evidently four times the area of the parallelogram $A_1OB_1T_1$. But $A_1T_1 = OB_1 = b' = \frac{\sqrt{a^4y_1^2 + b^4x_1^2}}{ab}$, from work above; and since the equation of A_1T_1 is $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$, the perpendicular distance from O to A_1T_1 is, by § 32, $\frac{a^2b^2}{\sqrt{a^4y_1^2 + b^4x_1^2}}$. Hence the area of $A_1OB_1T_1 = \left(\frac{\sqrt{a^4y_1^2 + b^4x_1^2}}{ab}\right)\left(\frac{a^2b^2}{\sqrt{a^4y_1^2 + b^4x_1^2}}\right) = ab$, and the area of the large parallelogram is $4\,ab$, as was to be proved.

147. It was noted in § 144 that conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie on opposite sides of the asymptotes, whence it follows that if one of two conjugate diameters intersects the hyperbola, the other cannot intersect it. In order, then, to state for the hyperbola propositions analogous to 2 and 3 of the last article, it is customary to consider, in connection with the above hyperbola, the hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. These two hyperbolas are called *conjugate* hyperbolas, either one being considered the *primary* and the other being called the *conjugate.*

It may readily be proved that if the slopes of two diameters are such that $m_1 m_2 = \frac{b^2}{a^2}$, they are conjugate diameters of both the above hyperbolas. Moreover it is evident (fig. 146) that if one diameter intersects one hyperbola, the other intersects the conjugate hyperbola.

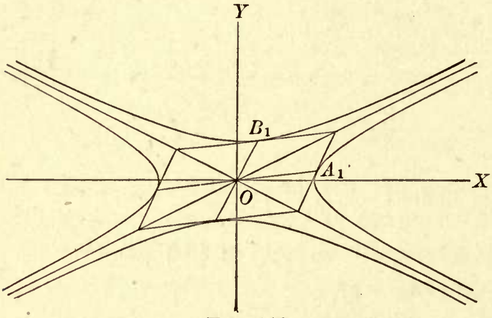

FIG. 146

Now if OA_1 and OB_1 are conjugate diameters, and OA_1 is called a', as in § 145, and we apply the same method as was applied to the ellipse, we shall find $OB_1 = b'$ of § 145.

With this value of b', theorem 2, § 146, becomes for the hyperbola $a'^2 - b'^2 = a^2 - b^2$, while theorem 3 is the same for the hyperbola as for the ellipse.

The proofs of these last statements are left to the student, the work being exactly like that for the ellipse.

PROBLEMS

Find the polars of each of the following points with respect to the given conic, and find the points in which the polar intersects the conic:

1. $(1, 2)$, $23x^2 - 11xy + 2y^2 + 36x - 9y + 9 = 0$.

2. $(-1, -2)$, $3x^2 - 3xy + 4x + y - 3 = 0$.

3. $(0, 0)$, $2x^2 - 2y^2 - 2x + 2y - 1 = 0$.

4. $(4, -2)$, $5y^2 + 18y + 4x + 5 = 0$.

Find the poles of each of the following polars with respect to the given conic:

5. $2x - y = 0$, $x^2 + 8xy - 2y^2 - 12x + 6y - 9 = 0$.

6. $x - 3y + 2 = 0$, $x^2 + y^2 - 2x + 4y = 0$.

7. $x + 2y - 13 = 0$, $3x^2 + 8y^2 - 26x - 76y + 231 = 0$.

8. $3x - 2y - 9 = 0$, $3x^2 - 4y^2 + 6x - 24y - 45 = 0$.

Find the equations of the tangents from each of the following points to the given conic:

9. $(2, 3)$, $4x^2 - 5xy + 2y^2 + 3x - 2y = 0$.

10. $(0, 1)$, $3x^2 - 4y^2 + 12x = 0$.

11. $(1, -2)$, $2x^2 - 2y^2 - 6x - 6y - 1 = 0$.

12. $(2, 4)$, $x^2 + y^2 - 6x - 2y + 5 = 0$.

13. $(2, 0)$, $5y^2 + 4x - 2y - 3 = 0$.

14. $(-1, -1)$, $3x^2 + 8y^2 - 8x - 12y + 4 = 0$.

15. Prove that the polar of a given point with respect to any one of the circles $x^2 + y^2 - 2kx + c^2 = 0$, when k is variable, always passes through a fixed point whatever the value of k.

16. T is the pole of a chord PQ of the parabola $y^2 = 4px$. Prove that the perpendiculars from P, T, and Q upon any tangent to the parabola are in geometric progression.

17. If P is any point, LM its polar with respect to any central conic, C the center of the conic, R the point in which the perpendicular from C to LM meets LM, and S the point in which the perpendicular from P to LM meets the axis of the conic, prove $CR \cdot PS = b^2$.

18. Prove that the perpendicular from any point (x_1, y_1) to its polar with respect to any central conic intersects the axis of the conic at a distance e^2x_1 from the center of the conic.

19. Prove that if in any conic the pole of the normal at P lies on the normal at Q, then the pole of the normal at Q lies on the normal at P.

20. If P_1 and P_2 are any two points, and C the center of a conic, show that the perpendiculars from P_1 and C to the polar of P_2 are to each other as the perpendiculars from P_2 and C to the polar of P_1.

21. If m_1 is the slope of the polar of a point P_1 with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and m_2 is the slope of the line joining P_1 to the center, show that $m_1m_2 = -\frac{b^2}{a^2}$. Find the similar relation for the hyperbola.

22. Prove that the portion of the axis included between the polars of two points with respect to a parabola equals the projection on the axis of the line joining the points.

23. Show that for any conic section the polar of the focus is the directrix.

24. Where is the polar of the center of an ellipse or hyperbola with respect to that curve?

25. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ find the equations of two conjugate diameters, one of which bisects the chord determined by the upper end of the minor axis and the right-hand focus.

26. If P_1 and P_2 are the extremities of any two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that the sum of the squares of the perpendiculars drawn from P_1 and P_2 to the major axis of the ellipse is equal to b^2.

27. Show that there can be only one pair of equal conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, namely $y = \frac{b}{a}x$, $y = -\frac{b}{a}x$.

28. Show that the equation of any ellipse referred to its equal conjugate diameters as axes is $x^2 + y^2 = \frac{a^2 + b^2}{2}$.

29. In any ellipse show that the diameters parallel to the lines joining the extremities of the axes are conjugate.

30. One diameter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the upper end of the right-hand latus rectum. What is the slope of the conjugate diameter?

31. What must be the relation between the semiaxes a and b of an ellipse when the diameters passing through the upper extremities of the left-hand latus rectum and the right-hand latus rectum are conjugate?

32. Show that the polar of any point on a diameter of a central conic is parallel to the conjugate diameter.

33. Show that if an ellipse and an hyperbola have the same axes in magnitude and position, then the asymptotes of the hyperbola coincide with the equal conjugate diameters of the ellipse.

34. Prove that tangents at the ends of conjugate diameters of an hyperbola intersect on the asymptotes.

35. Prove that the straight line joining the ends of a pair of conjugate diameters of an hyperbola is parallel to one asymptote and bisected by the other.

36. If an hyperbola has a pair of equal conjugate diameters, prove that it is an equilateral hyperbola.

37. Show that in an equilateral hyperbola conjugate diameters are equally inclined to the asymptotes.

38. Show that in an equilateral hyperbola all diameters at right angles to each other are equal.

39. Show that every diameter of an equilateral hyperbola is equal to its conjugate.

40. Prove that the tangents at the ends of any chord of a conic intersect on the diameter which bisects the chord.

41. The chords which join the ends of any diameter to any point of the curve are called *supplemental chords.* Prove that two diameters which are parallel to any pair of supplemental chords are conjugate.

42. If the tangent at the vertex A of an ellipse cuts any two conjugate diameters produced in T and t, show that $AT \cdot At = -b^2$.

43. Show that if any tangent meets any two conjugate diameters, the product of its segments is equal to the square of the half of the parallel diameter.

44. If from the focus of an ellipse a perpendicular is drawn to a diameter, show that it will meet the conjugate diameter on the corresponding directrix.

45. The tangent at any point P_1 of an ellipse cuts the equal conjugate diameters in T and T_1. Show that the triangles TCP_1 and T_1CP_1 are in the ratio $\overline{CT}^2 : \overline{CT_1}^2$.

46. Show that the product of the focal distances of any point of a central conic is equal to the square of half the corresponding conjugate diameter.

47. Find where the tangents from the foot of the directrix will meet the hyperbola, and what angles they will make with the transverse axis.

48. Show that the perpendicular from the focus upon a polar with respect to an ellipse or an hyperbola meets the line drawn from the center to the pole on the corresponding directrix.

CHAPTER XIII

ELEMENTARY TRANSCENDENTAL FUNCTIONS

148. Definition. *Any function of x which is not algebraic is called transcendental.* The elementary transcendental functions are the *trigonometric*, the *inverse trigonometric*, the *exponential*, and the *logarithmic* functions, the definitions and the simplest properties of which are supposed to be known to the student. In this chapter we shall discuss the graphs and the derivatives of these functions.

149. Graphs of trigonometric functions.

Ex. 1. $y = \sin x$.

The values of y are found from a table of trigonometric functions. In plotting it is desirable to express x in circular measure; e.g. for the angle 180° we lay off $x = \pi = 3.1416$. When x is a multiple of π, $y = 0$; when x is an odd multiple of $\frac{\pi}{2}$, $y = \pm 1$; for other values of x, y is numerically less than 1. The graph consists of an indefinite number of congruent arches alternately above and below the axis of x, the width of each arch being π and the height being 1 (fig. 147).

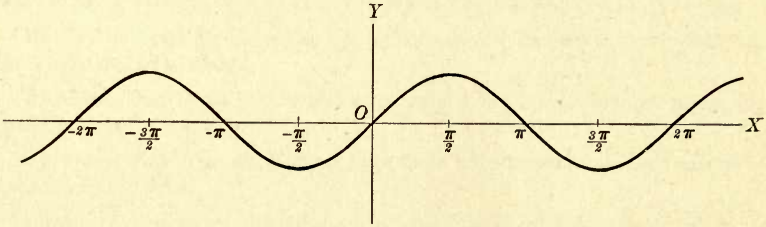

FIG. 147

The curve $y = \sin x$ may be constructed without the use of tables by a method illustrated in fig. 148.

Let P_1 be any point on the circumference of a circle of radius 1 with its center at C, and let AO be a diameter of the circle extended indefinitely. With a pair of dividers lay off on AO produced a distance ON_1 equal to the arc OP_1. This may be done by considering the arc OP_1 as composed of a number of straight lines each of which differs unappreciably from its arc. From N_1 draw a line perpendicular to AO, and from P_1 draw a line parallel to AO. Let these lines intersect in Q_1. Then $N_1Q_1 = M_1P_1 = CP_1 \sin OCP_1$. But $CP_1 = 1$, and the circular measure of OCP_1 is $OP_1 = ON_1$. If, then, we take $ON_1 = x$, $N_1Q_1 = y$,

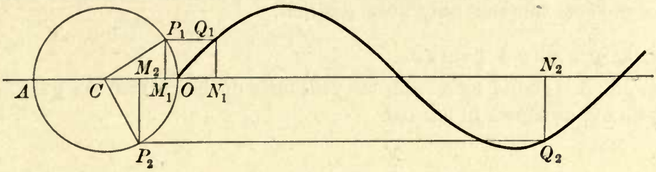

FIG. 148

Q_1 is a point of the curve $y = \sin x$. By varying the position of the point P_1 we may construct as many points of the curve as we wish. The figure shows the construction of another point Q_2.

Ex. 2. $y = a \sin bx$.

When x is a multiple of $\frac{\pi}{b}$, $y = 0$; when x is an odd multiple of $\frac{\pi}{2b}$, $y = \pm a$; for all other values of x, y is numerically less than a. The curve is similar in its general shape to that of Ex. 1, but the width of each arch is now $\frac{\pi}{b}$, and its height is a. Fig. 149 shows the curve when $a = 3$ and $b = 2$.

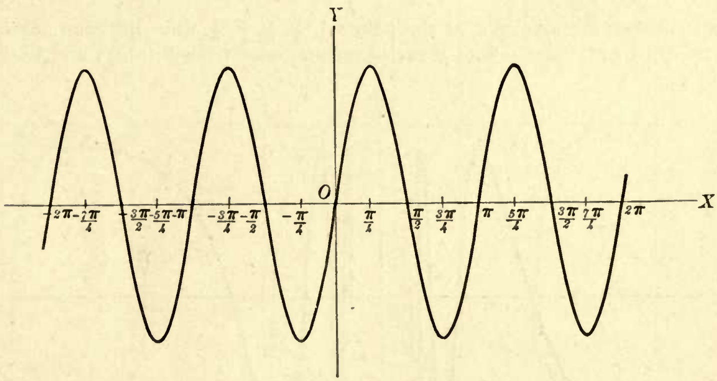

FIG. 149

Ex. 3. $y = a \sin (bx + c)$.

Place $x = -\frac{c}{b} + x'$, $y = y'$.

The equation then becomes $y' = a \sin bx'$.

The graph is consequently the same as in Ex. 1, the effect of the term $+ c$ being merely to shift the origin.

Ex. 4. $y = a\cos bx$.

This may be written $$y = a\sin\left(bx + \frac{\pi}{2}\right),$$

which is a curve of Ex. 3. Hence the graph of the cosine function differs from that of the sine function only in its position.

Ex. 5. $y = \sin x + \frac{1}{2}\sin 2x$.

The graph is found by adding the ordinates of the two curves $y = \sin x$ and $y = \frac{1}{2}\sin 2x$, as shown in fig. 150.

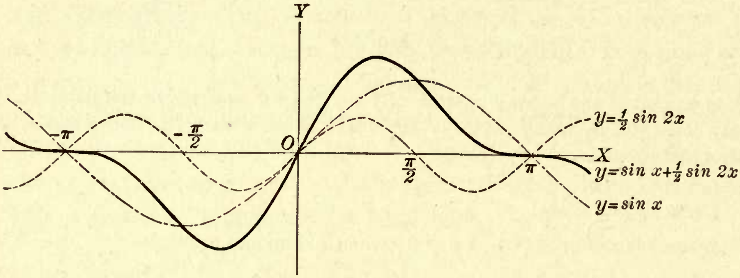

FIG. 150

Ex. 6. $y = \sin\frac{\pi}{x}$.

$y = 0$ when $\frac{\pi}{x} = k\pi$, i.e. when $x = \frac{1}{k}$, where k is any integer. Hence the graph crosses the axis of x at the points 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc. Between any consecutive two of these points y varies continuously from 0 to ± 1 and back to

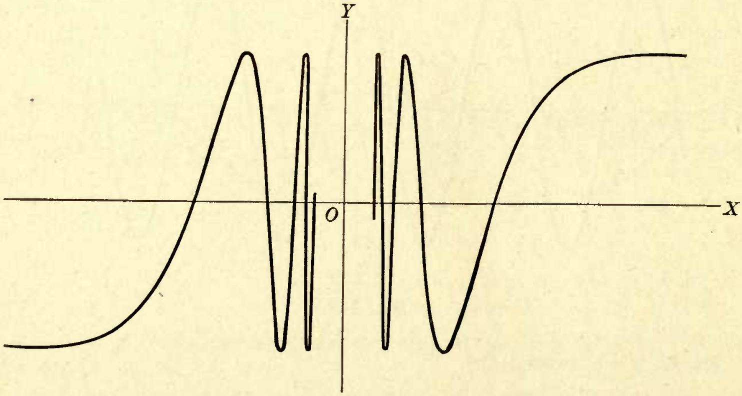

FIG. 151

zero. It follows that as x approaches 0, the corresponding point on the graph oscillates an infinite number of times back and forth between the straight lines

$y = \pm 1$. It is therefore physically impossible to construct the graph in the neighborhood of the origin. This is shown in fig. 151 by the break in the curve.

It should be borne in mind, however, that the value of y can be calculated for any value of x no matter how small. E.g. let $x = \frac{12}{125}$; then $\frac{\pi}{x} = \frac{125\pi}{12} = 10\pi + \frac{5}{12}\pi$, and $y = \sin\frac{5\pi}{12} = \sin 75° = .9659$.

The value of y is not defined for $x = 0$, and the function is discontinuous at that point.

Ex. 7. $y = \tan x$.

When x is a multiple of π, $y = 0$; when x is an odd multiple of $\frac{\pi}{2}$, y is infinite, in the sense of §§ 11 and 68. The curve has therefore an unlimited number of asymptotes perpendicular to OX, namely $x = \pm\frac{\pi}{2}$, $x = \pm\frac{3\pi}{2}$, $\cdots$, which divide the plane into an infinite number of sections, in each of which is a distinct branch of the curve, as shown in fig. 152.

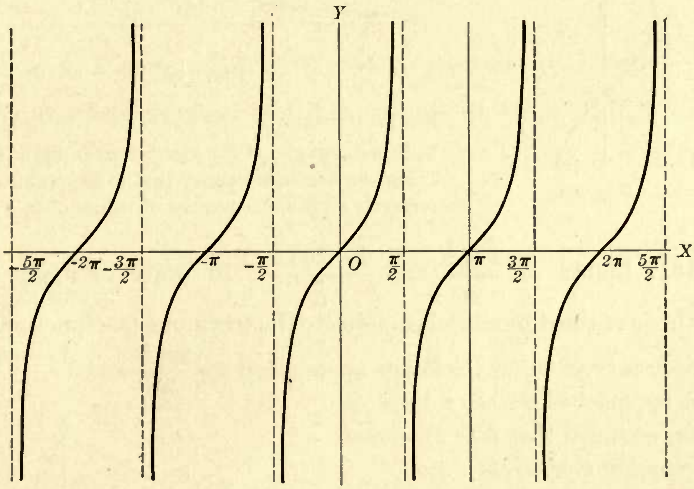

Fig. 152

150. Graphs of inverse trigonometric functions. The graphs of the inverse trigonometric functions are evidently the same as those of the direct functions, but differently placed with reference to the coördinate axes. It is to be noticed particularly that to any value of x corresponds an infinite number of values of y.

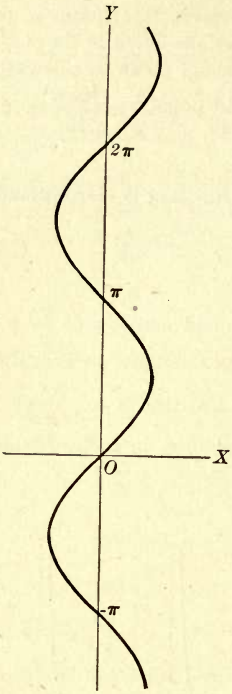

FIG. 153

Ex. 1. $y = \sin^{-1}x$.

From this $x = \sin y$, and we may plot the graph by assuming values of y and computing those of x (fig. 153).

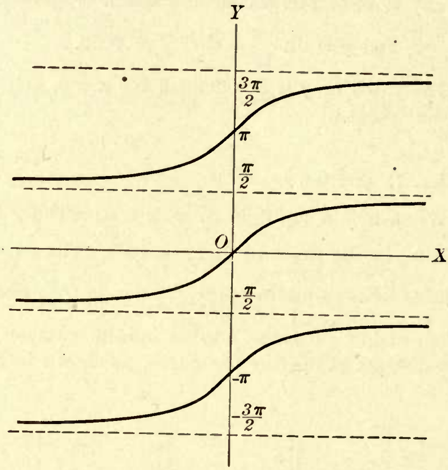

FIG. 154

Ex. 2. $y = \tan^{-1}x$.

Then $x = \tan y$, and the graph is as in fig. 154.

These curves show clearly that to any value of x corresponds an infinite number of values of y.

151. Limits of $\frac{\sin h}{h}$ and $\frac{1-\cos h}{h}$. In order to apply the methods of the differential calculus to the trigonometric functions, it is necessary to know the limits approached by $\frac{\sin h}{h}$ and $\frac{1-\cos h}{h}$, as h approaches zero as a limit, it being assumed that *h is expressed in circular measure.*

1. Let AOB (fig. 155) be the angle h, r the radius of the arc AB described from O as a center, a the length of AB, p the length of the perpendicular BC from B to OA, and t the length of the tangent drawn from B to meet OA produced in D.

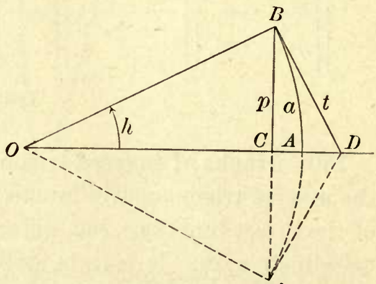

FIG. 155

Revolve the figure on OA as an axis until B takes the position B'. Then $BCB' = 2p$, $BAB' = 2a$, $B'D = BD$. Evidently

$$BD + DB' > BAB' > BCB',$$

whence

$$t > a > p.$$

Dividing through by r, we have

$$\frac{t}{r} > \frac{a}{r} > \frac{p}{r},$$

that is,

$$\tan h > h > \sin h.$$

Dividing by $\sin h$, we have

$$\frac{1}{\cos h} > \frac{h}{\sin h} > 1,$$

or, by inverting,

$$\cos h < \frac{\sin h}{h} < 1.$$

Now as h approaches zero, $\cos h$ approaches 1. Hence $\frac{\sin h}{h}$, which lies between $\cos h$ and 1, must also approach 1; that is,

$$\operatorname*{Lim}_{h \doteq 0} \frac{\sin h}{h} = 1.$$

2. To find the limit of $\frac{1 - \cos h}{h}$, as h approaches 0, we proceed as follows:

$$\frac{1 - \cos h}{h} = \frac{2\sin^2 \frac{h}{2}}{h} = \frac{\sin^2 \frac{h}{2}}{\frac{h}{2}} = \frac{h}{2}\left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2.$$

Now as h approaches zero as a limit, $\frac{\sin \frac{h}{2}}{\frac{h}{2}}$ approaches 1, as just shown, and therefore $\frac{h}{2}\left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2$ approaches zero, by 2, § 94.

Therefore

$$\operatorname*{Lim}_{h \doteq 0} \frac{1 - \cos h}{h} = 0.$$

152. Differentiation of trigonometric functions. The formulas for the differentiation of trigonometric functions are as follows, where u represents any function of x which can be differentiated:

$$\frac{d}{dx}\sin u = \cos u \frac{du}{dx}, \tag{1}$$

$$\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}, \tag{2}$$

$$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx}, \tag{3}$$

$$\frac{d}{dx}\cot u = -\csc^2 u \frac{du}{dx}, \tag{4}$$

$$\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx}, \tag{5}$$

$$\frac{d}{dx}\csc u = -\csc u \cot u \frac{du}{dx}. \tag{6}$$

1. By (7), § 96, $\quad \frac{d}{dx}\sin u = \frac{d}{du}\sin u \cdot \frac{du}{dx}.$

To find $\frac{d}{du}\sin u$, we place $y = \sin u$.

Then if u receives an increment Δu, y receives an increment Δy, where

$$\Delta y = \sin(u + \Delta u) - \sin u = 2\cos\left(u + \frac{\Delta u}{2}\right)\sin\frac{\Delta u}{2},$$

the last reduction being made by the trigonometric formula

$$\sin a - \sin b = 2\cos\frac{a+b}{2}\sin\frac{a-b}{2}.$$

Then we have

$$\frac{\Delta y}{\Delta u} = 2\cos\left(u + \frac{\Delta u}{2}\right)\frac{\sin\frac{\Delta u}{2}}{\Delta u} = \cos\left(u + \frac{\Delta u}{2}\right)\frac{\sin\frac{\Delta u}{2}}{\frac{\Delta u}{2}}.$$

Let Δu approach zero. By 2, § 94,

$$\text{Lim}\frac{\Delta y}{\Delta u} = \text{Lim}\cos\left(u + \frac{\Delta u}{2}\right)\text{Lim}\frac{\sin\frac{\Delta u}{2}}{\frac{\Delta u}{2}}.$$

But $\operatorname{Lim}\dfrac{\Delta y}{\Delta u}=\dfrac{dy}{du}$, $\operatorname{Lim}\cos\left(u+\dfrac{\Delta u}{2}\right)=\cos u$, and $\operatorname{Lim}\dfrac{\sin\dfrac{\Delta u}{2}}{\dfrac{\Delta u}{2}}$ $= 1$ (§ 151).

Hence
$$\frac{d}{du}\sin u=\cos u$$

and
$$\frac{d}{dx}\sin u=\cos u\,\frac{du}{dx}.$$

2. To find $\dfrac{d}{dx}\cos u$, we write

$$\cos u=\sin\left(\frac{\pi}{2}-u\right).$$

Then
$$\frac{d}{dx}\cos u=\frac{d}{dx}\sin\left(\frac{\pi}{2}-u\right)$$
$$=\cos\left(\frac{\pi}{2}-u\right)\frac{d}{dx}\left(\frac{\pi}{2}-u\right) \qquad \text{(by (1))}$$
$$=-\cos\left(\frac{\pi}{2}-u\right)\frac{du}{dx}$$
$$=-\sin u\,\frac{du}{dx}.$$

3. To find $\dfrac{d}{dx}\tan u$, we write

$$\tan u=\frac{\sin u}{\cos u}.$$

Then
$$\frac{d}{dx}\tan u=\frac{d}{dx}\frac{\sin u}{\cos u}$$
$$=\frac{\cos u\dfrac{d}{dx}\sin u-\sin u\dfrac{d}{dx}\cos u}{\cos^2 u} \qquad \text{(by (5), § 96)}$$
$$=\frac{(\cos^2 u+\sin^2 u)\dfrac{du}{dx}}{\cos^2 u} \qquad \text{(by (1) and (2))}$$
$$=\sec^2 u\,\frac{du}{dx}.$$

4. To find $\frac{d}{dx}\cot u$, we write

$$\cot u = \frac{\cos u}{\sin u}.$$

Then
$$\frac{d}{dx}\cot u = \frac{d}{dx}\frac{\cos u}{\sin u}$$

$$= \frac{\sin u \frac{d}{dx}\cos u - \cos u \frac{d}{dx}\sin u}{\sin^2 u} \qquad \text{(by (5), § 96)}$$

$$= \frac{-\sin^2 u - \cos^2 u}{\sin^2 u}\frac{du}{dx} \qquad \text{(by (1) and (2))}$$

$$= -\csc^2 u \cdot \frac{du}{dx}.$$

5. To find $\frac{d}{dx}\sec u$, we write

$$\sec u = \frac{1}{\cos u} = (\cos u)^{-1}.$$

Then
$$\frac{d}{dx}\sec u = -(\cos u)^{-2}\frac{d}{dx}\cos u \qquad \text{(by § 97)}$$

$$= \frac{\sin u}{\cos^2 u}\frac{du}{dx} \qquad \text{(by (2))}$$

$$= \sec u \tan u \frac{du}{dx}.$$

6. To find $\frac{d}{dx}\csc u$, we write

$$\csc u = \frac{1}{\sin u} = (\sin u)^{-1}.$$

Then
$$\frac{d}{dx}\csc u = -(\sin u)^{-2}\frac{d}{dx}\sin u \qquad \text{(by § 97)}$$

$$= -\csc u \cot u \frac{du}{dx}. \qquad \text{(by (1))}$$

Ex. 1. $y = \tan 2x - \tan^2 x = \tan 2x - (\tan x)^2.$

$$\frac{dy}{dx} = \sec^2 2x \frac{d}{dx}(2x) - 2(\tan x)\frac{d}{dx}\tan x$$
$$= 2\sec^2 2x - 2\tan x \sec^2 x.$$

Ex. 2. A particle moves in a straight line so that

$$s = k \sin bt,$$

where $t =$ time, $s =$ space, and b and k are constants. Then

$$\text{velocity} = v = \frac{ds}{dt} = bk \cos bt,$$

$$\text{acceleration} = a = \frac{d^2s}{dt^2} = -b^2k \sin bt = -b^2 s,$$

$$\text{force} = F = ma = -mb^2s.$$

Let O be the position of the particle when $t = 0$, and let $OA = +k$ and $OB = -k$. Then it appears from the formulas for s and v that the particle oscillates forward and backward between B and A. It describes the distance OA in the time $\frac{\pi}{2b}$, and moves from B to A and back to B in the time $\frac{2\pi}{b}$.

The formula $F = -mb^2s$ shows that the particle is acted on by a force directed toward O and proportional to the distance of the particle from O.

The motion of the particle is called *simple harmonic motion.*

Ex. 3. A wall is to be braced by means of a beam which must pass over a lower wall b units high and standing a units in front of the first wall. Required the shortest beam which can be used.

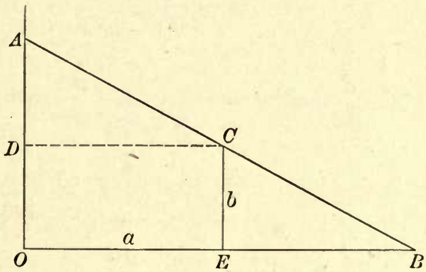

FIG. 156

Let $AB = l$ (fig. 156) be the beam, and C the top of the lower wall. Draw the line CD parallel to OB and let $EBC = \theta$. Then

$$\begin{aligned} l &= BC + CA \\ &= EC \csc\theta + DC \sec\theta \\ &= b \csc\theta + a \sec\theta \end{aligned}$$

$$\frac{dl}{d\theta} = -b \csc\theta \cot\theta + a \sec\theta \tan\theta$$

$$= \frac{a \sin^3\theta - b \cos^3\theta}{\sin^2\theta \cos^2\theta}.$$

$$\frac{dl}{d\theta} = 0, \quad \text{when} \quad a \sin^3\theta = b \cos^3\theta,$$

that is, when $$\tan\theta = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}.$$

When θ has a smaller value than this, $a \sin^3\theta < b \cos^3\theta$, and when θ has a larger value, $a \sin^3\theta > b \cos^3\theta$. Hence l is a minimum when $\tan\theta = \frac{b^{\frac{1}{3}}}{a^{\frac{1}{3}}}$. Then

$$\begin{aligned} l &= b \csc\theta + a \sec\theta \\ &= \frac{b\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{b^{\frac{1}{3}}} + \frac{a\sqrt{a^{\frac{2}{3}} + b^{\frac{2}{3}}}}{a^{\frac{1}{3}}} \\ &= (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}. \end{aligned}$$

153. Differentiation of inverse trigonometric functions. The formulas for the differentiation of the inverse trigonometric functions are as follows:

1. $\dfrac{d}{dx}\sin^{-1}u = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, when $\sin^{-1}u$ is in the first or the fourth quadrant;

   $= -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, when $\sin^{-1}u$ is in the second or the third quadrant.

2. $\dfrac{d}{dx}\cos^{-1}u = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, when $\cos^{-1}u$ is in the first or the second quadrant;

   $= \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$, when $\cos^{-1}u$ is in the third or the fourth quadrant.

3. $\dfrac{d}{dx}\tan^{-1}u = \dfrac{1}{1+u^2}\dfrac{du}{dx}$.

4. $\dfrac{d}{dx}\cot^{-1}u = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$.

5. $\dfrac{d}{dx}\sec^{-1}u = \dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$, when $\sec^{-1}u$ is in the first or the third quadrant;

   $= -\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$, when u is in the second or the fourth quadrant.

6. $\dfrac{d}{dx}\csc^{-1}u = -\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$, when u is in the first or the third quadrant;

   $= \dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$, when u is in the second or the fourth quadrant.

The proofs of these formulas are as follows:

1. If $$y = \sin^{-1} u,$$
then $$\sin y = u.$$
Hence $$\cos y \frac{dy}{dx} = \frac{du}{dx}, \qquad \text{(by § 152)}$$
or $$\frac{dy}{dx} = \frac{1}{\cos y} \frac{du}{dx}.$$

But $\cos y = \sqrt{1 - u^2}$, when y is in the first or the fourth quadrant, and $\cos y = -\sqrt{1 - u^2}$ when y is in the second or the third quadrant.

2. If $$y = \cos^{-1} u,$$
then $$\cos y = u;$$
whence $$-\sin y \frac{dy}{dx} = \frac{du}{dx},$$
or $$\frac{dy}{dx} = -\frac{1}{\sin y} \frac{du}{dx}.$$

But $\sin y = +\sqrt{1 - u^2}$ when y is in the first or the second quadrant, and $\sin y = -\sqrt{1 - u^2}$ when y is in the third or the fourth quadrant.

3. If $$y = \tan^{-1} u,$$
then $$\tan y = u.$$
Hence $$\sec^2 y \frac{dy}{dx} = \frac{du}{dx};$$
whence $$\frac{dy}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}.$$

4. If $$y = \cot^{-1} u,$$
then $$\cot y = u.$$
Hence $$-\csc^2 y \frac{dy}{dx} = \frac{du}{dx},$$
whence $$\frac{dy}{dx} = -\frac{1}{1 + u^2} \frac{du}{dx}.$$

5. If $$y = \sec^{-1} u,$$
then $$\sec y = u.$$
Hence $$\sec y \tan y \frac{dy}{dx} = \frac{du}{dx},$$
whence $$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \frac{du}{dx}.$$

But $\sec y = u$ and $\tan y = +\sqrt{u^2 - 1}$ when y is in the first or the third quadrant, and $\tan y = -\sqrt{u^2 - 1}$ when y is in the second or the fourth quadrant.

6. If $$y = \csc^{-1} u,$$
$$\csc y = u.$$
Hence $$-\csc y \cot y \frac{dy}{dx} = \frac{du}{dx},$$
whence $$\frac{dy}{dx} = -\frac{1}{\csc y \cot y} \frac{du}{dx}.$$

But $\csc y = u$ and $\cot y = +\sqrt{u^2 - 1}$ when y is in the first or the third quadrant, and $\cot y = -\sqrt{u^2 - 1}$ when y is in the second or the fourth quadrant.

Ex. 1. $y = \sin^{-1} \sqrt{1 - x^2}$, where y is an acute angle.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (1 - x^2)}} \cdot \frac{d}{dx}(1 - x^2)^{\frac{1}{2}} = -\frac{1}{\sqrt{1 - x^2}}.$$

This may also be done by noticing that $\sin^{-1} \sqrt{1 - x^2} = \cos^{-1} x$.

Ex. 2. The example of § 107 may be solved by drawing a straight line from S to O (fig. 125), denoting the angle YOS by θ and the subtended arc by s.

Then $$s = a\theta,$$
and $$\theta = 2 \cdot YLS = 2 \tan^{-1} \frac{OM}{OL} = 2 \tan^{-1} \frac{x_1}{a}.$$
Hence $$s = 2 a \tan^{-1} \frac{x_1}{a},$$
and $$v = \frac{ds}{dt} = 2 a \cdot \frac{\frac{1}{a}}{1 + \frac{x_1^2}{a^2}} \cdot \frac{dx_1}{dt} = \frac{2 a^2 c}{a^2 + x_1^2}.$$

154. The exponential and the logarithmic functions. The equation

$$y = a^x$$

defines y as a continuous function of x, called the *exponential function,* such that to any real value of x corresponds one and only one real positive value of y. A proof of this statement depends upon higher mathematics, but the student is already familiar with the methods by which the value of y may be computed for simple values of x. If $x = n$, an integer, y is determined by raising a to the nth power by multiplication. If x is a positive fraction $\frac{p}{q}$, y is the qth root of the pth power of a. If x is a positive irrational number, the approximate value of y may be obtained by expressing x approximately as a rational number. If $x = 0$, $y = a^0 = 1$. Finally, if $x = -m$, where m is any positive number, $y = a^{-m} = \frac{1}{a^m}$.

Practically, however, the value of a^x is most readily obtained by means of the inverse function, the *logarithm ;* for if

$$y = a^x,$$

then $$x = \log_a y.$$

When $a = 10$, tables of logarithms are readily accessible. Suppose a is not 10, and let b be such a number that

$$10^b = a,$$

i.e. $$b = \log_{10} a.$$

Then we have

$$y = a^x = (10^b)^x = 10^{bx}.$$

Hence $$bx = \log_{10} y,$$

and $$x = \frac{\log_{10} y}{b} = \frac{\log_{10} y}{\log_{10} a}.$$

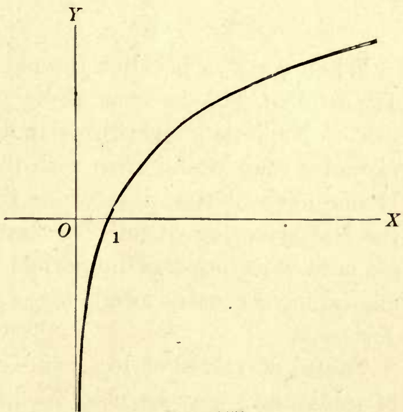

FIG. 157

Ex. 1. The graph of $y = \log_{(1.5)} x$ is shown in fig. 157.

It is to be noticed that the curve has the negative portion of the y axis for an asymptote, and has no points corresponding to negative values of x.

Ex. 2. The graph of $y=(1.5)^x$ is shown in fig. 158.

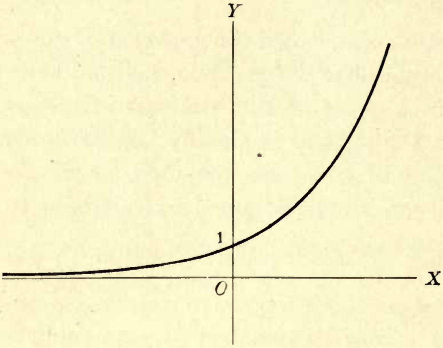

Fig. 158

155. The number *e*. In the theory and the use of the exponential and the logarithmic functions, an important part is played by a certain irrational number, commonly denoted by the letter e. This number is defined by an infinite series, thus:

$$e=1+\frac{1}{1}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\cdots.$$

It will be shown in the second volume that this series converges; i.e. that the greater the number of terms taken the more nearly does their sum approach a certain number as a limit. Assuming this, we may compute e to seven decimal places by taking the first eleven terms. There results

$$e=2.7182818\cdots.$$

When $y=e^x$, x is called the natural or Napierian logarithm of y. The student will discover as he proceeds with his study that the use of Napierian logarithms in theoretical work causes simpler formulas than would arise with the use of the common logarithm. Hence, in theoretical discussions, the expression $\log x$ usually means the Napierian logarithm. On the other hand, when the chief interest is in calculation of numerical values, as in the solution of triangles, $\log x$ usually means $\log_{10} x$. *In this book we shall use* $\log x$ *for* $\log_e x$.

Tables of values of $\log_e x$ and e^x are found in many collections of tables, and may be used in finding the graphs. It is evident, however, that the graphs will not differ in general shape from those in Exs. 1 and 2 of § 154.

We give the graphs of certain other functions which involve e and present other points of interest.

Ex. 1. $y = e^{-x^2}$.

The curve (fig. 159) is symmetrical with respect to OY and is always above OX. When $x = 0$, $y = 1$. As x increases numerically y decreases, approaching zero. Hence OX is an asymptote.

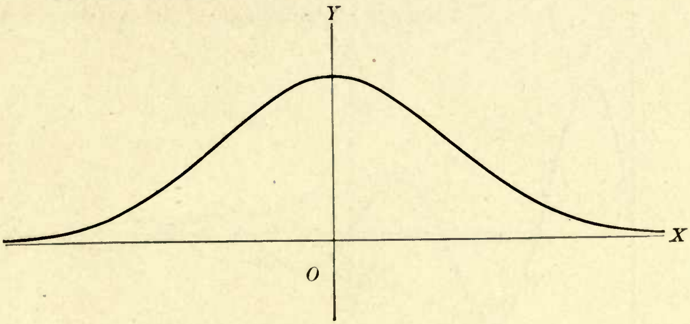

Fig. 159

Ex. 2. $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$.

This is the curve (fig. 160) made by a string held at the ends and allowed to hang freely. It is called the *catenary*.

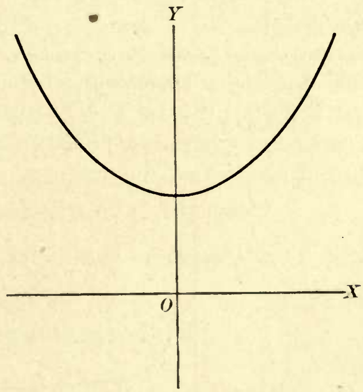

Fig. 160

Ex. 3. $y = e^{-ax} \sin bx$.

The values of y may be computed by multiplying the ordinates of the curve $y = e^{-ax}$ by the value of $\sin bx$ for the corresponding abscissas. Since the values of $\sin bx$ oscillate between ± 1, the value of $e^{-ax} \sin bx$ cannot exceed those of e^{-ax}. Hence the graph lies in the portion of the plane between the curves $y = e^{-ax}$ and $y = -e^{-ax}$. When x is a multiple of $\frac{\pi}{b}$, y is zero. The graph therefore crosses the axis of x an infinite number of times. Fig. 161 shows the graph when $a = 1$, $b = 2\pi$.

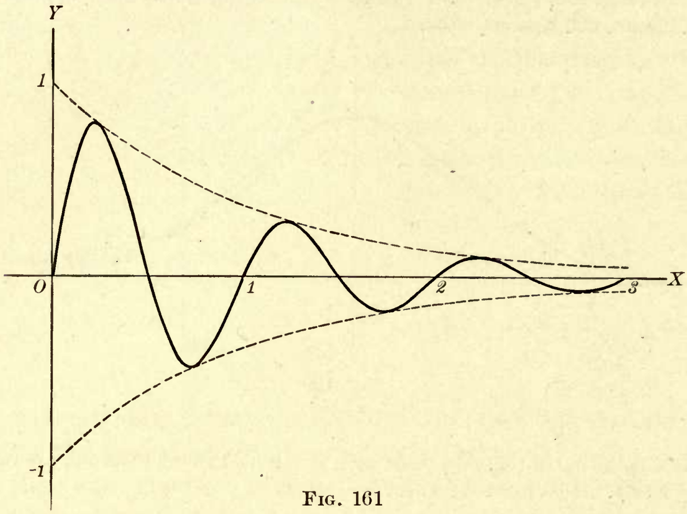

FIG. 161

Ex. 4. $y = e^{\frac{1}{x}}$.

When x approaches zero, being positive, y increases without limit. When x approaches zero, being negative, y approaches zero; e.g. when $x = \frac{1}{1000}$, $y = e^{1000}$, and when $x = -\frac{1}{1000}$, $y = e^{-1000} = \frac{1}{e^{1000}}$. The function is therefore discontinuous for $x = 0$.

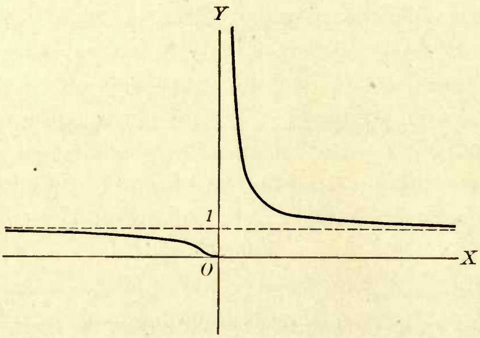

FIG. 162

The line $y = 1$ is an asymptote (fig. 162), for as x increases without limit, being positive or negative, $\frac{1}{x}$ approaches 0 and y approaches 1.

Ex. 5. $y = \dfrac{10}{1 + e^{\frac{1}{x}}}$.

As x approaches zero positively, y approaches zero. As x approaches zero negatively, y approaches 10. As x increases indefinitely, y approaches 5. The curve (fig. 163) is discontinuous when $x = 0$.

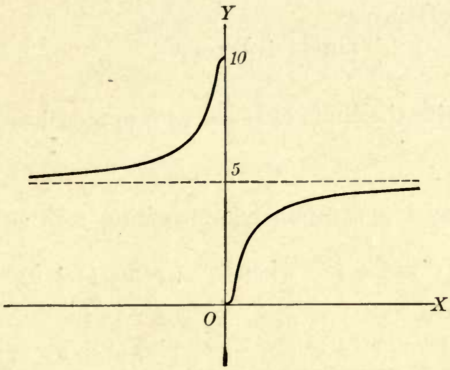

FIG. 163

156. Limits of $(1 + h)^{\frac{1}{h}}$ and $\dfrac{e^h - 1}{h}$. In obtaining the formulas for the differentiation of the exponential and the logarithmic functions it is necessary to know certain limits, the rigorous derivation of which requires methods which are too advanced for this book. We must content ourselves, therefore, with indicating somewhat roughly the general nature of the proof.

1. We require the limit of $(1 + h)^{\frac{1}{h}}$ as h approaches zero. We begin by expanding $(1 + h)^{\frac{1}{h}}$ by the binomial theorem and making certain simple transformations; thus:

$$(1 + h)^{\frac{1}{h}} = 1 + \frac{1}{h} \cdot h + \frac{\frac{1}{h}\left(\frac{1}{h} - 1\right)}{\underline{|2}} h^2 + \frac{\frac{1}{h}\left(\frac{1}{h} - 1\right)\left(\frac{1}{h} - 2\right)}{\underline{|3}} h^3 + \cdots$$

$$= 1 + \frac{1}{1} + \frac{(1 - h)}{\underline{|2}} + \frac{(1 - h)(1 - 2h)}{\underline{|3}} + \cdots$$

$$= 1 + \frac{1}{1} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots + R,$$

where R represents the sum of all terms involving h, h^2, h^3, etc. Now it may be shown by advanced methods that as h approaches zero R also approaches zero, and at the same time

$$1 + \frac{1}{1} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots$$

approaches e. Hence

$$\operatorname*{Lim}_{h \doteq 0} (1+h)^{\frac{1}{h}} = e.$$

2. We require the limit of $\frac{e^h - 1}{h}$ as h approaches zero.

Let us place $$e^h - 1 = k,$$

where evidently k is a number approaching zero as h approaches zero. Then

$$e^h = 1 + k, \quad \text{whence} \quad h = \log(1+k).$$

Then we have

$$\frac{e^h - 1}{h} = \frac{k}{\log(1+k)} = \frac{1}{\frac{1}{k}\log(1+k)} = \frac{1}{\log(1+k)^{\frac{1}{k}}}.$$

Now as h approaches zero k approaches 0, and $(1+k)^{\frac{1}{k}}$ approaches e by the previous proof. Hence $\log(1+k)^{\frac{1}{k}}$ approaches $\log e$, which is 1. Therefore

$$\operatorname*{Lim}_{h \doteq 0} \frac{e^h - 1}{h} = 1.$$

157. Differentiation of exponential and logarithmic functions. The formulas for the differentiation of the exponential and logarithmic functions are as follows, where, as usual, u represents any function which can be differentiated with respect to x, log means the Napierian logarithm, and a is any constant:

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}. \tag{1}$$

$$\frac{d}{dx} \log u = \frac{1}{u} \frac{du}{dx}. \tag{2}$$

$$\frac{d}{dx} a^u = a^u \log a \frac{du}{dx}. \tag{3}$$

$$\frac{d}{dx} \log_a u = \frac{\log_a e}{u} \frac{du}{dx}. \tag{4}$$

1. By (7), § 96, $\frac{d}{dx}e^u = \frac{d}{du}e^u \cdot \frac{du}{dx}.$

To find $\frac{d}{du}e^u$, place $y = e^u$. Then if u receives an increment Δu, y receives an increment Δy, where

$$\Delta y = e^{u+\Delta u} - e^u = e^u(e^{\Delta u} - 1).$$

Then $$\frac{\Delta y}{\Delta u} = e^u \frac{e^{\Delta u} - 1}{\Delta u}.$$

Now let Δu approach zero.

By § 94, $$\text{Lim}\,\frac{\Delta y}{\Delta u} = e^u\,\text{Lim}\,\frac{e^{\Delta u} - 1}{\Delta u}.$$

But $$\text{Lim}\,\frac{\Delta y}{\Delta u} = \frac{dy}{du} = \frac{d}{du}e^u,$$

and $$\text{Lim}\,\frac{e^{\Delta u} - 1}{\Delta u} = 1. \qquad \text{by 2, § 156}$$

Therefore $$\frac{d}{du}e^u = e^u,$$

and $$\frac{d}{dx}e^u = e^u \frac{du}{dx}.$$

2. If $$y = \log u,$$

then $$e^y = u.$$

Hence $$e^y \frac{dy}{dx} = \frac{du}{dx}, \qquad \text{by (1)}$$

whence $$\frac{dy}{dx} = \frac{1}{e^y}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}.$$

3. Let $$y = a^u.$$

Then it is always possible to find a quantity b such that

$$a = e^b,$$

whence $$b = \log a.$$

Then $$y = (e^b)^u = e^{bu},$$

and $$\frac{dy}{dx} = e^{bu}\frac{d}{dx}(bu) \qquad \text{by (1)}$$

$$= be^{bu}\frac{du}{dx}$$

$$= (\log a)\, a^u \frac{du}{dx}.$$

4. If $$y = \log_a u,$$

then $$a^y = u,$$

and $$a^y \log a \frac{dy}{dx} = \frac{du}{dx}; \qquad \text{by (3)}$$

whence $$\frac{dy}{dx} = \frac{1}{\log a}\cdot\frac{1}{u}\cdot\frac{du}{dx}.$$

But if $$\log a = b,$$

$$a = e^b,$$

whence $$a^{\frac{1}{b}} = e,$$

and therefore $$\frac{1}{b} = \log_a e,$$

or $$\frac{1}{\log a} = \log_a e.$$

Hence $$\frac{dy}{dx} = \frac{\log_a e}{u}\frac{du}{dx}.$$

Ex. 1. $y = \log(x^2 - 4x + 5)$.

$$\frac{dy}{dx} = \frac{2x - 4}{x^2 - 4x + 5}.$$

Ex. 2. $y = e^{-x^2}$.

$$\frac{dy}{dx} = -2xe^{-x^2}.$$

Ex. 3. $y = e^{-ax}\cos bx$.

$$\frac{dy}{dx} = \cos bx \frac{d}{dx}(e^{-ax}) + e^{-ax}\frac{d}{dx}(\cos bx) = -ae^{-ax}\cos bx - be^{-ax}\sin bx.$$

158. An important property of the exponential functions is expressed in the following theorem: *If the rate of change of a function is proportional to the value of the function, the function is an exponential function.*

Let $$\frac{dy}{dx} = ay.$$

Then $$\frac{1}{y}\frac{dy}{dx} = a.$$

Hence $$\log y = ax + c_1,$$

or $$y = e^{ax + c_1} = e^{c_1}e^{ax} = ce^{ax}.$$

Ex. Let p be the atmospheric pressure at the distance h above the surface of the earth and ρ the density of the air. We will assume that the density is proportional to the pressure. Then if ρ_0 and p_0 are the density and the pressure respectively at the surface of the earth,

$$\frac{\rho}{\rho_0} = \frac{p}{p_0},$$

whence $$\rho = \frac{\rho_0}{p_0}\cdot p.$$

Let now the height h be increased by a distance Δh. The pressure will be decreased by an amount Δp, where $-\Delta p$ is equal to the weight of a column of air standing on a base of unit area and having a height Δh. If ρ is the density at the height h and $\rho - \Delta\rho$ the density at the height $h + \Delta h$, it is evident that the weight of this column of air lies between $(\rho - \Delta\rho)\Delta h$ and $\rho\Delta h$; that is,

$$(\rho - \Delta\rho)\Delta h < -\Delta p < \rho\Delta h,$$

whence $$\rho - \Delta\rho < -\frac{\Delta p}{\Delta h} < \rho.$$

Taking the limit, we have

$$\frac{dp}{dh} = \text{Limit}\frac{\Delta p}{\Delta h} = -\rho = -\frac{\rho_0}{p_0}p.$$

Therefore $$p = ce^{-\frac{\rho_0}{p_0}h}.$$

Since when $h = 0$, $e^{-\frac{\rho_0}{p_0}h} = 1$ and $p = p_0$, it follows that $c = p_0$.

Hence $$p = p_0 e^{-\frac{\rho_0}{p_0}h},$$

or $$h = \frac{p_0}{\rho_0}\log\frac{p_0}{p}.$$

159. Sometimes the work of differentiating a function is simplified by first taking the logarithm of the function and then applying the formulas of this article.

Ex. 1. Let $$y = \sqrt{\frac{1-x^2}{1+x^2}}.$$

Then $$\log y = \log \sqrt{\frac{1-x^2}{1+x^2}}$$
$$= \tfrac{1}{2}\log(1-x^2) - \tfrac{1}{2}\log(1+x^2).$$

Hence $$\frac{1}{y}\frac{dy}{dx} = -\frac{x}{1-x^2} - \frac{x}{1+x^2}$$
$$= \frac{-2x}{(1-x^2)(1+x^2)}$$

and $$\frac{dy}{dx} = \frac{-2xy}{(1-x^2)(1+x^2)}$$
$$= \frac{-2x}{(1-x^2)(1+x^2)}\sqrt{\frac{1-x^2}{1+x^2}}$$
$$= \frac{-2x}{(1+x^2)\sqrt{1-x^4}}.$$

This method is especially useful for functions of the form u^v, where u and v are both functions of x. Such functions occur rarely in practice, and cannot be differentiated by any of the formulas so far given. By taking the logarithm of the function, however, a form is obtained which may be differentiated.

Ex. 2. Let $$y = x^{\sin x}.$$

Then $$\log y = \log(x^{\sin x})$$
$$= \sin x \cdot \log x.$$

Therefore $$\frac{1}{y}\frac{dy}{dx} = (\sin x)\frac{1}{x} + \cos x \cdot \log x,$$

and $$\frac{dy}{dx} = x^{\sin x - 1} \cdot \sin x + x^{\sin x}\cos x \cdot \log x.$$

160. Hyperbolic functions. Certain combinations of exponential functions are called *hyperbolic functions.* In their names and properties they are analogous to the trigonometric functions, but the reason for this cannot be shown at present. The fundamental hyperbolic functions are the *hyperbolic sine* (sinh), the *hyperbolic cosine* (cosh), the *hyperbolic tangent* (tanh), the *hyperbolic* *cotangent* (coth), the *hyperbolic secant* (sech), and the *hyperbolic cosecant* (cosech), defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$

$$\text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}.$$

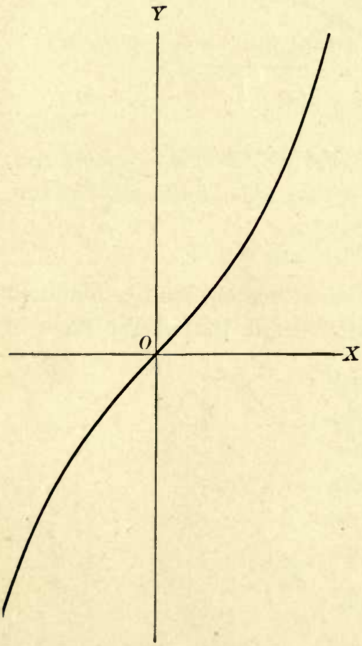

FIG. 164

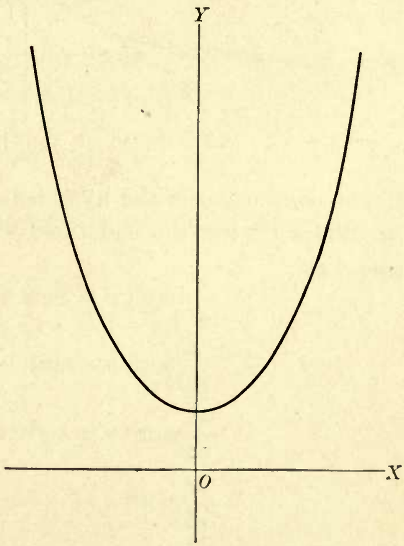

FIG. 165

The graph of sinh x is given in fig. 164, that of cosh x in fig. 165, and that of tanh x in fig. 166.

Relations between hyperbolic functions may be derived by expressing each in terms of the exponential functions. The student may in this way prove the following relations:

$$\cosh^2 x - \sinh^2 x = 1,$$
$$\tanh^2 x + \text{sech}^2 x = 1,$$
$$\coth^2 x - \text{cosech}^2 x = 1,$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y,$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y,$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$$

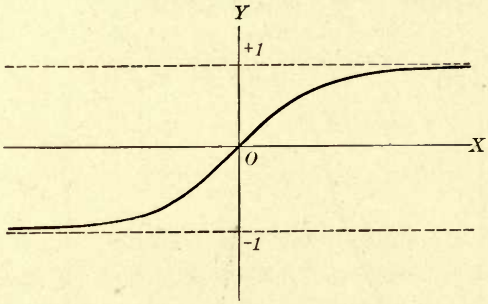

FIG. 166

The derivatives of the hyperbolic functions are readily obtained by differentiating the equations which define them. We have in this way:

$$\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx},$$
$$\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx},$$
$$\frac{d}{dx}\tanh u = \text{sech}^2 u \frac{du}{dx},$$
$$\frac{d}{dx}\coth u = -\text{cosech}^2 u \frac{du}{dx},$$
$$\frac{d}{dx}\text{sech}\, u = -\text{sech}\, u \tanh u \frac{du}{dx},$$
$$\frac{d}{dx}\text{cosech}\, u = -\text{cosech}\, u \coth u \frac{du}{dx}.$$

161. Inverse hyperbolic functions. If

$$x = \sinh y,$$

then

$$y = \sinh^{-1} x,$$

called the *inverse hyperbolic sine of x.*

This function may be expressed as a logarithm as follows:

We have $$y = \sinh^{-1} x,$$

and $$x = \sinh y = \frac{e^y - e^{-y}}{2}.$$

Placing $e^{-y} = \frac{1}{e^y}$ and clearing of fractions, we have

$$e^{2y} - 2\,xe^y = 1.$$

Treating this as a quadratic equation in e^y, we have

$$e^y = x \pm \sqrt{x^2 + 1};$$

but since we know that for any real value of y, e^y is positive, we discard the minus sign before the radical and have

$$y = \sinh^{-1} x = \log (x + \sqrt{x^2 + 1}).$$

In the same manner, the student may prove the following:

$$\cosh^{-1} x = \log (x \pm \sqrt{x^2 - 1})$$
$$= \pm \log (x + \sqrt{x^2 - 1}),$$

$$\tanh^{-1} x = \tfrac{1}{2} \log \frac{1 + x}{1 - x},$$

$$\coth^{-1} x = \tfrac{1}{2} \log \frac{x + 1}{x - 1},$$

$$\text{sech}^{-1} x = \log \frac{1 \pm \sqrt{1 - x^2}}{x} = \pm \log \frac{1 + \sqrt{1 - x^2}}{x},$$

$$\text{cosech}^{-1} x = \log \frac{1 + \sqrt{1 + x^2}}{x}.$$

The derivative of the inverse hyperbolic functions can be obtained by differentiating the expressions just obtained, or by proceeding in the same manner as in § 153. In either way we find:

$$\frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx},$$

$$\frac{d}{dx}\cosh^{-1}u = \pm\frac{1}{\sqrt{u^2-1}}\frac{du}{dx},$$

$$\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx},$$

$$\frac{d}{dx}\coth^{-1}u = \frac{1}{1-u^2}\frac{du}{dx},$$

$$\frac{d}{dx}\operatorname{sech}^{-1}u = \mp\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx},$$

$$\frac{d}{dx}\operatorname{cosech}^{-1}u = -\frac{1}{u\sqrt{1+u^2}}\frac{du}{dx}.$$

Ex. Consider the motion of a particle of unit mass falling from rest, and impeded by a force proportional to the square of its velocity. The total force acting on the particle is then $g - kv^2$, where g is the acceleration due to gravity, and k is a constant. Hence

$$\frac{dv}{dt} = g - kv^2;$$

whence

$$\frac{1}{g-kv^2}\frac{dv}{dt} = 1,$$

or

$$\frac{1}{g}\cdot\frac{1}{1-\frac{k}{g}v^2}\cdot\frac{dv}{dt} = 1.$$

To bring this under one of the known formulas of differentiation we will place

$$\sqrt{\frac{k}{g}}\,v = u;$$

whence

$$\frac{dv}{dt} = \sqrt{\frac{g}{k}}\frac{du}{dt}.$$

We have, therefore,

$$\frac{1}{\sqrt{kg}}\frac{1}{1-u^2}\frac{du}{dt} = 1;$$

whence

$$\frac{1}{\sqrt{kg}}\tanh^{-1}u = t + c,$$

or

$$\frac{1}{\sqrt{kg}}\tanh^{-1}\sqrt{\frac{g}{k}}\,v = t + c.$$

But since the body falls from rest, when $t = 0$, $v = 0$; therefore $c = 0$. The equation may be written

$$v = \sqrt{\frac{k}{g}} \tanh t\sqrt{kg},$$

that is,

$$\frac{ds}{dt} = \sqrt{\frac{k}{g}} \frac{\sinh t\sqrt{kg}}{\cosh t\sqrt{kg}}.$$

Hence

$$s = \frac{1}{g} \log \cosh t\sqrt{kg} + c.$$

162. Transcendental equations. Equations involving transcendental functions can often be solved by methods similar to those used for algebraic equations. Graphical methods can often be used to advantage.

Ex. 1. $\sin x = a$.

The solutions of this equation are the abscissas of the points of intersection of the curve $y = \sin x$ and the straight line $y = a$ (fig. 167). If $a > 1$ or $a < -1$, there are no real solutions; otherwise there are an infinite number of solutions. Let us call the smallest positive root x_1, where $0 < x_1 < \frac{\pi}{2}$ if a is positive, and

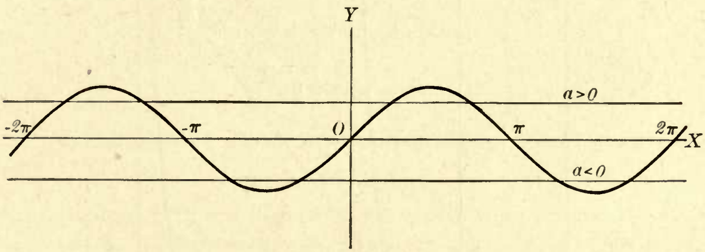

Fig. 167

$\pi < x_1 < 2\pi$ if a is negative. The value of x_1 must be found from a table or approximately from the graph. The next largest positive root is then $\pi - x_1$ when a is positive, and $3\pi - x_1$ when a is negative; and all other roots, positive or negative, are found by adding or subtracting multiples of 2π. Hence the general solution is $2k\pi + x_1$ and $(2k+1)\pi - x_1$, or, more compactly written,

$$k\pi + (-1)^k x_1,$$

where k is any positive or negative integer or zero.

Ex. 2. $\cos x = a$.

The general solution is $2k\pi \pm x_1$, where x_1 is the smallest positive solution and k is an integer or zero. The proof is left to the student.

Ex. 3. $\tan x = a$.

The general solution is $k\pi + x_1$. The proof is left to the student.

Ex. 4. $\cos 2x = 2\cos x$.

When an equation involves two or more trigonometric functions it is well to write it in terms of one. The above equation may be written

$$2\cos^2 x - 1 = 2\cos x,$$

which is a quadratic equation in $\cos x$. Solving, we have in the first place

$$\cos x = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{3},$$

but the plus sign may be disregarded, since for real angles $\cos x$ is not greater than 1 numerically. The equation

$$\cos x = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}$$

is now to be solved as in Ex. 2. There results $x = 2k\pi \pm 1.946$.

Ex. 5. $\tan x = kx$.

The roots of this equation are the abscissas of the points of intersection of the curve $y = \tan x$ and the straight line $y = kx$ (fig. 168).

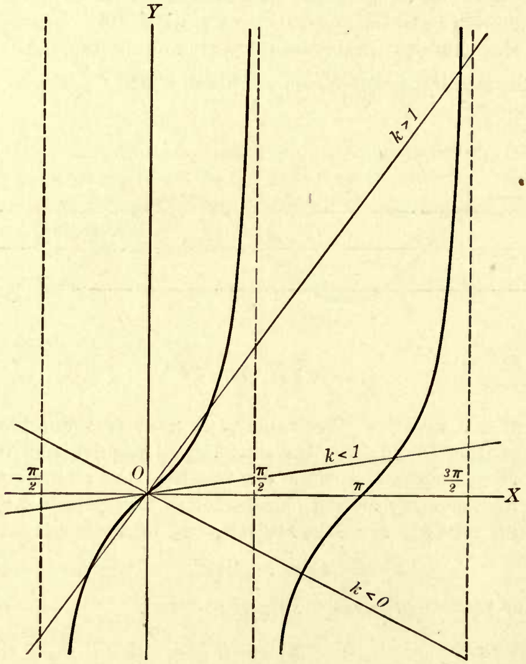

FIG. 168

The two intersect at the origin, but the other intersections depend upon the value of k. Since the slope of the curve $y = \tan x$ is 1 when $x = 0$, and >1 when $0 < x < \frac{\pi}{2}$, we need to distinguish three cases, according as $k > 1$, $0 < k \overline{\gtrless} 1$, or $k < 0$.

The graph shows that if $k > 1$, the smallest positive root lies between 0 and $\frac{\pi}{2}$; if $0 < k \leq 1$, the smallest positive root lies between π and $\frac{3\pi}{2}$; and if $k < 0$, the smallest positive root lies between $\frac{\pi}{2}$ and π.

We shall now find the smallest positive root in the special case

$$\tan x = 2x.$$

We must first locate the root (§ 47), either by the graph or by means of a table. If a table is used, it must be one in which angles are given in radians. We shall use the table on page 132 of Professor B. O. Peirce's "Short Table of Integrals." We find, by looking for a place in the tables where the tangent of an angle is approximately equal to twice the angle, that when $x = 1.1636$ ($66° 40'$), $\tan x = 2.3183$, and when $x = 1.1665$ ($66° 50'$), $\tan x = 2.3369$. Consider now the curve

$$y = \tan x - 2x.$$

When $x_1 = 1.1636$, $y_1 = -.0089$, and when $x_2 = 1.1665$, $y_2 = .0039$.

Hence the curve intersects OX between x_1 and x_2, and a root of the equation

$$\tan x - 2x = 0$$

is therefore located to two decimal places. To locate the root more closely we will use the method of § 63. We have

$$\frac{dy}{dx} = \sec^2 x - 2,$$

and
$$\frac{d^2y}{dx^2} = 2 \tan x \sec x,$$

both of which are positive when x is between x_1 and x_2. Hence that portion of the curve

$$y = \tan x - 2x$$

appears as in fig. 64, (1), and its intersection with OX lies between the tangent at (x_2, y_2) and the chord connecting (x_1, y_1) and (x_2, y_2). The tangent at (x_2, y_2) is

$$y - .0039 = 4.461(x - 1.1665);$$

the chord is
$$y - .0039 = \frac{.0128}{.0029}(x - 1.1665),$$

and the point of intersection of each with OX is found to be

$$x = 1.1656$$

to four places of decimals. This is therefore the root of the equation to four decimal places.

Ex. 6. $e^x - 4x^2 - 2x + 3 = 0$.

The roots of this equation are the abscissas of the points of intersections of the curves $y = e^x$ and $y = 4x^2 + 2x - 3$, and may be found graphically or by means of tables to lie between -1 and -2 and between 0 and 1. To determine the root between 0 and 1, we place $y = e^x - 4x^2 - 2x + 3$. When $x_1 = 0$, $y_1 = 4$, and when $x_2 = 1$, $y_2 = -.282$.

Also
$$\frac{dy}{dx} = e^x - 8x - 2,$$
and
$$\frac{d^2y}{dx^2} = e^x - 8,$$

which are both negative when x is between 0 and 1. Hence the portion of the curve in question has the shape of fig. 64, (4), and its intersection with OX lies between that of the tangent at (x_2, y_2) and that of the chord connecting (x_1, y_1) and (x_2, y_2). The tangent is

$$y + .282 = -7.282(x - 1),$$

which intersects OX when $x = .97-$. The chord is

$$y + .282 = -4.282(x - 1),$$

which intersects OX when $x = .93+$.

If we now place $x_1 = .93$, $y_1 = .2149$, and if $x_2 = .97$, $y_2 = -.0657$, the tangent at (x_2, y_2) is

$$y + .0657 = -7.1221(x - .97),$$

which intersects OX where $x = .9608-$; and the chord between (x_1, y_1) and (x_2, y_2) is

$$y + .0657 = \frac{-.2806}{.04}(x - .97),$$

which intersects OX where $x = .9606+$.

Hence a root of the equation lies between .9606 and .9608.

PROBLEMS

Plot the graphs of the following equations:

1. $y = \operatorname{ctn} x$.
2. $y = \sec x$.
3. $y = \csc x$.
4. $y = \operatorname{vers} x$.
5. $y = \frac{1}{3}\sin 3x$.
6. $y = \sin x + \frac{1}{3}\sin 3x$.
7. $y = \sin x + \sin 2x$.
8. $y = 2\sin x - \sin 2x$.
9. $y = \cos x + \frac{1}{9}\cos 3x$.
10. $y = 1 - \frac{1}{2}\cos x - \frac{2}{3}\cos 2x$.
11. $y = x\sin\frac{1}{x}$.
12. $y = x^2\sin\frac{1}{x}$.
13. $y = e^{\frac{1+x}{1-x}}$.
14. $y = xe^{\frac{1-x}{x}}$.
15. $y = xe^{\frac{1}{x}}$.
16. $y = \log(\sin x)$.
17. $y = \tan^{-1}(ax + b)$.
18. $y = \log\frac{x-1}{x+1}$.

19. Plot the graph of the equation $y = \frac{1}{x}\sin x$, and determine what relation it has to the hyperbolas $xy = \pm 1$.

20. Plot the graph of the equation $y = \sin x^2$, and show that the distance between two consecutive intercepts on OX approaches zero as a limit.

Find $\frac{dy}{dx}$ in each of the following cases:

21. $y = \sin(ax+b)\cos(ax-b)$.

22. $y = \tan(ax+b)\operatorname{ctn}(ax+c)$.

23. $y = \dfrac{\tan 2x + \operatorname{ctn} 2x}{\sec 4x}$.

24. $y = \dfrac{\operatorname{ctn} 2x + 2}{\csc 2x}$.

25. $y = \dfrac{\sec 3x}{\tan 3x + 1}$.

26. $y = \csc 2x - \operatorname{ctn} 2x$.

27. $y = \sec^m nx \csc^n mx$.

28. $y = \sec^2 2x + \tan 2x$.

29. $y = \operatorname{ctn} 4x \csc 2x$.

30. $y = \sin(x\cos x)$.

31. $y = (\cos^2 x + \frac{2}{3})\sin^3 x$.

32. $y = (2\sec^4 x + 3\sec^2 x)\sin x$.

33. $y = \dfrac{\sin^2 x}{\sqrt{1-\cos 2x}}$.

34. $y = \cos\sqrt{1-x^2}$.

35. $y = \dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x}$.

36. $y = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$.

37. $y = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$.

38. $y = \cos^{-1}\dfrac{2\sqrt{x}}{1+x}$.

39. $y = \sec^{-1}\dfrac{a}{\sqrt{a^2-x^2}}$.

40. $y = \sin^{-1}\dfrac{a-x}{a+x}$.

41. $y = \sec^{-1}\dfrac{1}{2}\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)$.

42. $y = \csc^{-1}(x^2+2x)$.

43. $y = \operatorname{ctn}^{-1}\dfrac{2ax}{x^2-a^2} - 2\tan^{-1}\dfrac{x}{a}$.

44. $y = \dfrac{1}{\sqrt{a^2-b^2}}\cos^{-1}\dfrac{b+a\cos x}{a+b\cos x}$.

45. $y = \sin^{-1}(2x\sqrt{1-x^2})$.

46. $y = \csc^{-1}\dfrac{2x-1}{2\sqrt{x^2-x}}$.

47. $y = \sec^{-1}\sqrt{4x^2+4x+2}$.

48. $y = \csc^{-1}\dfrac{1}{2}\left(x^2+\dfrac{1}{x^2}\right)$.

49. $y = e^{x^2+2x}$.

50. $y = \log\sqrt{\dfrac{x^2-a^2}{x^2+a^2}}$.

51. $y = a^{\frac{x}{\sqrt{1-x^2}}} e^{\sqrt{1-x^2}}$.

52. $y = \log(2x+1+2\sqrt{x^2+x})$.

53. $y = e^x\sqrt[x]{a}$.

54. $y = a^{\tan x}$.

55. $y = x^2\log x^2$.

56. $y = a^{\tan x \sec^2 x}$.

57. $y = \tan 2x\, a^{\sec 2x}$.

58. $y = e^{(a+x)^2}\sin mx$.

59. $y = \csc^{-1}(\sec 2x)$.

60. $y = (x+a)e^{\tan^{-1}\sqrt{\frac{x}{a}}}$.

61. $y = \tan^{-1}\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

62. $y = \log\dfrac{3\tan x + 1}{\tan x + 3}$.

63. $y = \sec^{-1}\dfrac{e^x+e^{-x}}{2}$.

64. $y = e^{x\cos a}\cos(x\sin a)$.

65. $y = \log\tan(x^2+a^2)$.

66. $y = x\cos^{-1}x - \sqrt{1-x^2}$.

67. $y = \dfrac{1}{a}\log(\sec ax + \tan ax)$.

68. $y = (x + a\sqrt{1-x^2})e^{a\sin^{-1}x}$.

69. $y = \log\sqrt{1+x^2} + x\operatorname{ctn}^{-1}x$.

70. $y = \dfrac{x\cos^{-1}x}{\sqrt{1-x^2}} - \log\sqrt{1-x^2}$.

71. $y = \dfrac{e^{ax}(a\sin mx - m\cos mx)}{m^2+a^2}$.

72. $y = x^2 \operatorname{ctn}^{-1}\frac{a}{x} + a^2 \tan^{-1}\frac{x}{a} - ax.$

73. $y = \log\left(\frac{x}{1+\sqrt{1-x^2}}\right) - \frac{\sin^{-1}x}{x}.$

74. $y = \log(x + \sqrt{x^2 - a^2}) + \cos^{-1}\frac{x}{a}.$

75. $y = \log(x + \sqrt{x^2 - a^2}) - \csc^{-1}\frac{x}{a}.$

76. $y = \tan^{-1}\sqrt{x^2 - 2x} - \frac{\log(x-1)}{\sqrt{x^2 - 2x}}.$

77. $y = \frac{2}{\sqrt{a^2 - b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right).$

78. $y = \log\tan(2x+1) + \csc(4x+2).$

79. $y = \sqrt{2ax - x^2} + a\cos^{-1}\frac{\sqrt{2ax - x^2}}{a}.$

80. $y = x\sqrt{a^2 + x^2} + a^2\log(x + \sqrt{a^2 + x^2}).$

81. $y = \log\sqrt{\frac{2\sqrt{x} - 1}{2\sqrt{x} + 1}} - \frac{2\sqrt{x}\sec^{-1}2\sqrt{x}}{\sqrt{4x - 1}}.$

82. $y = \frac{x-a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}.$

83. $y = \tan^{-1}(x - \sqrt{x^2 - 1}) + \log\frac{\sqrt{x^2 - 1}}{x}.$

84. $y = \frac{\log(e^x + 2)}{\sqrt{e^{2x} + 2e^x - 1}} + \csc^{-1}\sqrt{e^{2x} + 2e^x}.$

85. $y = \frac{1}{2}\sqrt{1 - x^2} - \left(1 - \frac{x^2}{2}\right)\tan^{-1}\sqrt{1 - x^2}.$

86. $y = \frac{1}{2\sqrt{2}}\log\frac{\sqrt{2 + 2x^2} - x}{\sqrt{2 + 2x^2} + x} + \log(x + \sqrt{1 + x^2}).$

87. $y = (\sin\sqrt{x})^{\tan\sqrt{x}}.$

88. $y = \sqrt[x]{\sin x}.$

89. $y = x^{(x^x)}.$

90. $y = (x)^{e^x}.$

91. $y = (e)^{x^x}.$

92. $y = (a + x)^{\tan^{-1}(a+x)}.$

Find $\frac{dy}{dx}$ in each of the following cases:

93. $x^y + \sec xy = 0.$

94. $y\tan^{-1}x - y^2 + x^2 = 0.$

95. $y\sin x - \cos(x - y) = 0.$

96. $ye^{ny} = ax^m.$

97. $e^x\sin y - e^y\cos x = 0.$

98. $y\sin x + x\cos y = xy.$

99. $y\log x = x\sin y.$

100. $xy = \tan^{-1}\frac{x}{y}.$

Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ in each of the following cases:

101. $\log(x^2 + y^2) - 2\tan^{-1}\frac{y}{x} = 0.$

102. $e^x + e^y = 1.$

103. $\log\frac{1-x}{1+y} - \log\frac{1+x}{1-y} = 1.$

104. $x - y = \log(x + y).$

105. $e^{x+y} = xy.$

106. At what points is the curve $y = \sin x + \sin 2x$ parallel to the axis of x?

107. What value must be assigned to m that the curve $y = \frac{1}{5mx} + \tan^{-1}(x + m)$ may be parallel to OX at the point the abscissa of which is 1?

108. Find the angle of intersection of the curves $y = \sin x$ and $y = \cos x$.

109. Find the angle of intersection of the curves $y = \sin x$ and $y = \sin(x + a)$.

110. Find the angle of intersection of the curves $y = \sin x$ and $y = \sin 2x$.

111. Show that the portion of the tangent to the curve

$$y = \frac{a}{2}\log\frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}$$

included between the point of contact and the axis of y is constant. (From this property the curve is called the *tractrix*.)

112. Find the points of inflection of the curve $y = 2\sin x - \frac{1}{2}\sin 2x$.

113. Find the points of inflection of the curve $xy = a^2 \log\frac{x}{a}$.

114. Find the points of inflection of the curve $y = e^{-x^2}$

115. Prove that the curve

$$y = \tfrac{1}{2}x - \tfrac{2}{3}\sin x + \tfrac{1}{12}\sin 2x$$

has an indefinite number of points of inflection, and that two of them lie between the points for which $x = 6$ and $x = 10$ respectively.

116. Plot the curve $y = \sin^2 x$, finding maxima and minima, and points of inflection.

117. Plot the curve $y = e^{-ax}\cos bx$, and prove that it is tangent to the curve $y = e^{-ax}$ wherever they have a point in common. Find maxima and minima and points of inflection of this curve when $a = b = 1$.

118. Plot the curve $y = x^n e^{-x}$ $(n > 0)$, finding maxima and minima and points of inflection.

119. A body moves in a plane so that $x = a\cos t + b$, $y = a\sin t + c$, where t denotes time and a, b, and c are constants. Find the path of the body, and show that its velocity is constant.

120. A rectilinear motion is expressed by the equation $s = 5 - 2\cos^2 t$. Show that the motion is a simple harmonic motion, and express the velocity and the acceleration at any point in terms of s.

121. A, the center of one circle, is on a second circle with center at B. A moving straight line, AMN, intersecting the two circles at M and N respectively, has constant angular velocity about A. Prove that BN has constant angular velocity about B.

122. Two particles are moving on the same straight line, and their distances from the fixed point O on the line at any time t are respectively $x = a \cos \omega t$ and $x' = a \cos\left(\omega t + \frac{\pi}{3}\right)$, ω and a being constants. Find the greatest distance between them.

123. A ladder b ft. long leans against a side of a house. Its foot is drawn away in the horizontal direction at the rate of a ft. per second. How fast is its center moving?

124. If a particle moves so that

$$s = e^{-\frac{1}{2}ct}(a \sin ht + b \cos ht),$$

find expressions for the velocity and the acceleration. Hence show that the particle is acted on by two forces, one proportional to the distance from the origin and the other proportional to the velocity. Describe the motion of the particle.

125. If $s = ae^{kt} + be^{-kt}$, show that the particle is acted on by a repulsive force which is proportional to the distance from the point from which s is measured.

126. BC is a rod a ft. long, connected with a piston rod at C, and at B with a crank AB b ft. long, revolving about A. Find C's velocity in terms of AB's angular velocity.

127. A man walks along the diameter, 200 ft. in length, of a semicircular courtyard at a uniform rate of 5 ft. per second. How fast will his shadow move along the wall when the rays of the sun are at right angles to the diameter?

128. How fast is the shadow in the preceding problem moving if the sun's rays make an angle α with the diameter?

129. Given that two sides and the included angle of a triangle have at a certain moment the values 6 ft., 10 ft., and 30° respectively, and that these quantities are changing at the rates of 3 ft., − 2 ft., and 10° per second respectively, what is the area of the triangle at the given moment, and how fast is it changing?

130. One side of a triangle is l ft., and the opposite angle is α. Find the other angles of the triangle when its area is a maximum.

131. A tablet 8 ft. high is placed on a wall so that the bottom of the tablet is 20 ft. from the ground. How far from the wall should a person stand in order that he may see the tablet at the best advantage, i.e. in order that the angle between the lines from the observer's standpoint to the top and the bottom of the tablet may be the greatest?

132. A weight P is dragged along the ground by a force F. If the coefficient of friction is K, in what direction should the force be applied to produce the best result?

133. An open gutter is to be constructed of boards in such a way that the bottom and the sides, measured on the inside, are to be each 4 in. wide, and both sides are to have the same slope. How wide should the gutter be across the top in order that its capacity may be as great as possible?

134. Above the center of a round table is a hanging lamp. What must be the ratio of the height of the lamp above the table to the radius of the table that the edge of the table may be most brilliantly lighted, given that the illumination varies inversely as the square of the distance and directly as the cosine of the angle of incidence?

135. A steel girder 27 ft. long is to be moved on rollers along a passageway and into a corridor 8 ft. in width at right angles to the passageway. If the horizontal width of the girder is neglected, how wide must the passageway be in order that the girder may go around the corner?

136. Find the area of an arch of the curve $y = \sin x$.

137. Find the area bounded by the axis of y and the portion of the curves $y = \sin x$, $y = \cos x$, lying between $x = 0$ and $x = \pi$.

138. Find the area bounded by the portions of the curves $y = \frac{1}{2}\sin 2x$ and $y = \sin x + \frac{1}{2}\sin 2x$ that extend between $x = 0$ and $x = \pi$.

139. Find the area between the curve $y = e^x$, the axis of x, and the ordinates $x = 0$ and $x = 1$.

140. Find the area bounded by the axis of x, the catenary, and the ordinates $x = \pm a$.

141. Find the area bounded by the axis of x, the curve $y = \frac{1}{x}$, and the ordinates $x = 1$ and $x = 2$.

142. Find where the ordinate of the witch should be drawn in order that the area between that ordinate, the witch, the axis of y and the axis of x should be equal to the area of the circle used in the definition.

143. Show that for the catenary $\frac{ds}{dx} = \frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$, and thence find an expression for the length of s.

144. Find the curve the slope of which at any point is k times the reciprocal of the abscissa of the point, and which passes through (2, 3).

145. Find the curve the slope of which at any point is k times the ordinate of the point, and which passes through the point (a, b).

146. Find the space traversed by a moving body in the time t if its velocity is proportional to the distance traveled.

Solve the following equations:

147. $\tan x = \cos x$.

148. $\cos 2x = \frac{1}{2}\cos x$.

149. $\sin 2\theta \cos 2\theta + 2\sin\theta = 0$.

150. $\sin 4x - 2\sin x \cos 2x = 0$.

151. $\sin^4 x + 3\cos^4 x - 4\sin^2 x \cos^2 x = 0$.

152. $\tan x = x$.

153. $\tan x = \frac{1}{2}x$.

154. $x - \frac{1}{2}\sin x = \frac{1}{10}$.

155. $e^x = x^2$.

156. $\log x = \frac{1}{3}x$.

CHAPTER XIV

PARAMETRIC REPRESENTATION OF CURVES

163. Definition. Thus far we have considered a curve as represented by a single equation connecting x and y. Another useful method is to express x and y each as a function of a third independent variable; thus:

$$x = f_1(t), \qquad y = f_2(t),$$

where t is an independent variable and $f_1(t)$ and $f_2(t)$ are continuous functions of t. As t varies, x and y also vary, and the point (x, y) traces out a curve. By eliminating t between the two equations the curve may often be expressed by a single equation between x and y.

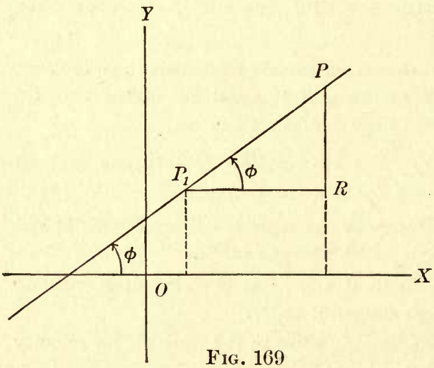

Fig. 169

164. The straight line. Let $P_1(x_1, y_1)$ (fig. 169) be a fixed point on a straight line and ϕ be the angle which the line makes with a line P_1R parallel to OX. Let $P(x, y)$ be any point on the line, and r the distance from P_1 to P, where r is positive when P is on the terminal line of ϕ, and negative when P_1 is on the backward extension of the terminal line. Then, for all possible positions of P

$$\frac{x - x_1}{r} = \cos\phi, \qquad \frac{y - y_1}{r} = \sin\phi;$$

whence $$x = x_1 + r\cos\phi, \qquad y = y_1 + r\sin\phi.$$

This is a parametric representation of the straight line, where r is the arbitrary parameter. Illustrations of the use of these equations have been given in §§ 136 and 138.

Another parametric representation of a straight line is furnished by the equations of § 19,

$$x = x_1 + l(x_2 - x_1),$$
$$y = y_1 + l(y_2 - y_1),$$

where l is the parameter and (x_1, y_1) and (x_2, y_2) are fixed points.

More generally, the equations

$$x = a + bt, \qquad y = f + gt,$$

where a, b, f, g, are constants, and t is an arbitrary parameter, represent a straight line. For these equations are equivalent to

$$y - f = \frac{g}{b}(x - a).$$

165. The circle. Let $P(x, y)$ (fig. 170) be any point on a circle with its center at the origin O, and its radius equal to a. Let ϕ be the angle made by OP and OX. Then from the definition of the sine and cosine

$$x = a \cos\phi,$$
$$y = a \sin\phi,$$

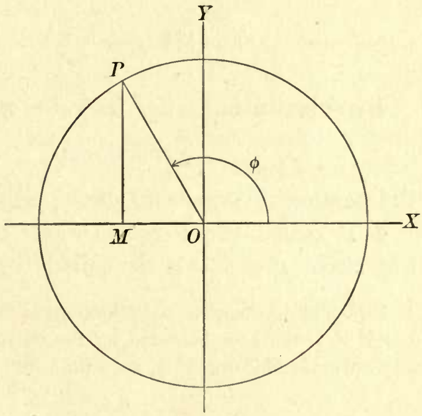

FIG. 170

the parametric equations of the circle with ϕ as the arbitrary parameter.

166. The ellipse. Take the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (a > b)$$

and on its major axis as a diameter construct a circle. Take $P(x, y)$ (fig. 171) any point on the ellipse, draw the ordinate MP

and prolong it until it meets the circle in Q. Call the coördinates of Q (x, y'). Then from the equation of the circle

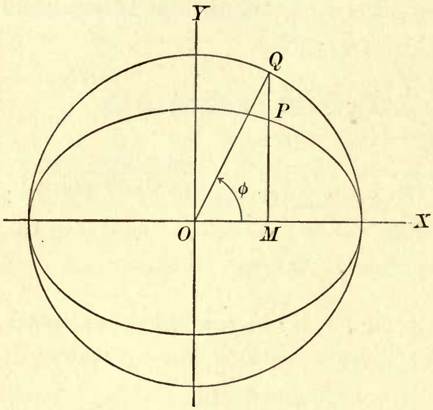

FIG. 171

$$y' = \sqrt{a^2 - x^2},$$

and from the equation of the ellipse

$$y = \frac{b}{a}\sqrt{a^2 - x^2}.$$

Hence $y = \frac{b}{a}y'$.

Draw the line OQ, making the angle $XOQ = \phi$. Then, as in § 165,

$$x = a \cos\phi,$$
$$y' = a \sin\phi.$$

By substituting for y' its value in terms of y, we have

$$x = a\cos\phi, \qquad y = b\sin\phi,$$

the parametric equations of the ellipse.

ϕ is called the *eccentric angle* of a point on the ellipse, and the circle $x^2 + y^2 = a^2$ is called the *auxiliary circle*.

Ex. The parametric equations of an ellipse may be used to find its area. For if A is the area bounded by the ellipse, the axis of y, the axis of x, and any ordinate MP (fig. 171), then (6, § 109)

$$\frac{dA}{dx} = y. \tag{1}$$

But
$$\frac{dA}{dx} = \frac{\frac{dA}{d\phi}}{\frac{dx}{d\phi}} = \frac{\frac{dA}{d\phi}}{-a\sin\phi},$$

and
$$y = b\sin\phi.$$

Therefore (1) is equivalent to

$$\frac{dA}{d\phi} = -ab\sin^2\phi = ab\frac{\cos 2\phi - 1}{2}.$$

Hence
$$A = ab\left(\frac{\sin 2\phi}{4} - \frac{\phi}{2}\right) + c.$$

When $\phi = \frac{\pi}{2}$, $A = 0$; hence $c = \frac{\pi ab}{4}$.

Therefore
$$A = ab\left(\frac{\sin 2\phi}{4} - \frac{\phi}{2} + \frac{\pi}{4}\right).$$

When $\phi = 0$, A is one fourth the area of the ellipse. Therefore the whole area of the ellipse equals πab.

167. The cycloid. If a circle rolls upon a straight line each point of the circumference describes a curve called a cycloid.

Let a circle of radius a roll upon the axis of x and let C (fig. 172) be its center at any time of its motion, N its point of

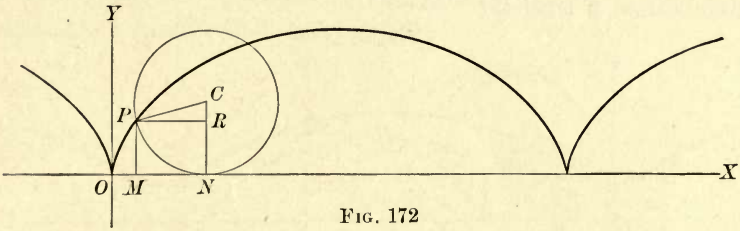

FIG. 172

contact with OX, and P the point on its circumference which describes the cycloid. Take as the origin of coördinates, O, the point found by rolling the circle to the left until P meets OX. Then
$$ON = \text{arc } PN.$$

Draw MP and CN each perpendicular to OX, PR parallel to OX, and connect C and P. Let
$$NCP = \phi.$$

Then
$$\begin{aligned} x &= OM = ON - MN \\ &= \text{arc } NP - PR \\ &= a\phi - a\sin\phi. \\ y &= MP = NC - RC \\ &= a - a\cos\phi. \end{aligned}$$

Hence the parametric representation of the cycloid is
$$x = a(\phi - \sin\phi),$$
$$y = a(1 - \cos\phi).$$

By eliminating ϕ the equation of the cycloid may be written

$$x = a\cos^{-1}\frac{a-y}{a} \pm \sqrt{2ay - y^2},$$

but this is less convenient than the parametric representation.

At each point where the cycloid meets OX a sharp vertex called a *cusp* is formed. The distance between two consecutive cusps is evidently $2\pi a$.

168. The trochoid. When a circle rolls upon a straight line, any point upon a radius, or upon a radius produced, describes a curve called a trochoid.

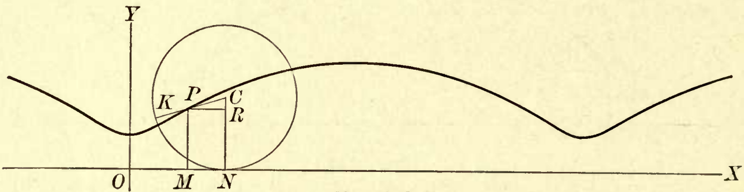

FIG. 173

Let the circle roll upon the axis of x, and let C (figs. 173 and 174) be its center at any time, N its point of contact with the axis of x, $P(x, y)$ the point which describes the trochoid, and

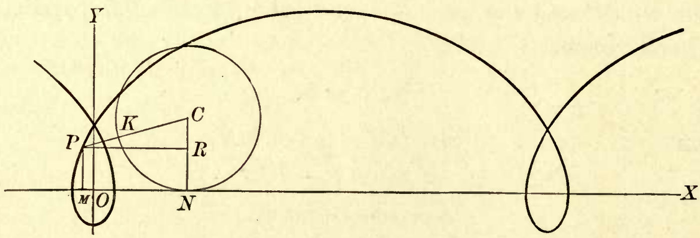

FIG. 174

K the point in which the line CP meets the circle. Take as the origin O the point found by rolling the circle toward the left until K is on the axis of x. Then

$$ON = \text{arc } NK.$$

Draw PM and CN perpendicular to OX, and through P a line parallel to OX, meeting CN or CN produced, in R. Let the radius of the circle be a, CP be h, and NCP be ϕ. Then

$$\begin{aligned} x &= OM = ON - MN \\ &= \text{arc } NK - PR \\ &= a\phi - h\sin\phi. \\ y &= MP = NC - RC \\ &= a - h\cos\phi. \end{aligned}$$

169. The epicycloid. When a circle rolls upon the outside of a fixed circle, each point of the circumference of the rolling circle describes a curve called an epicycloid.

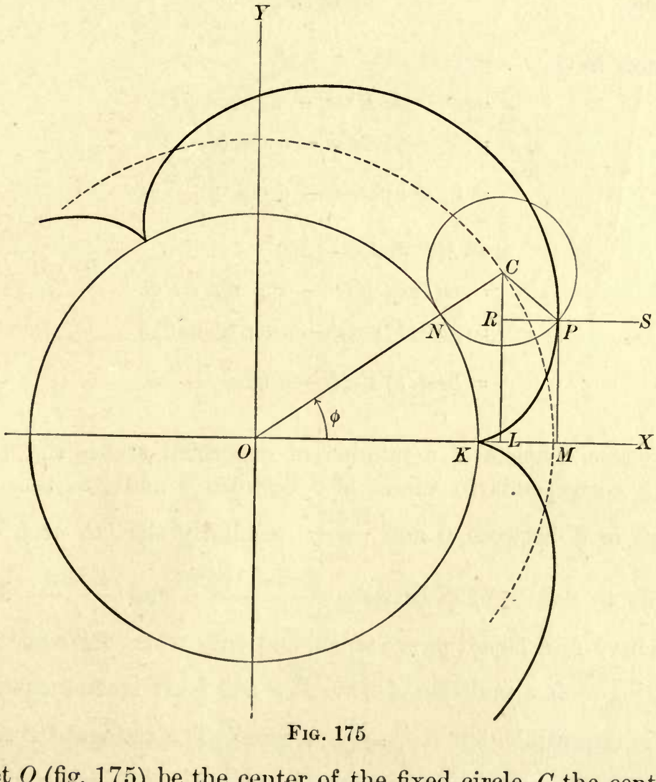

FIG. 175

Let O (fig. 175) be the center of the fixed circle, C the center of the rolling circle, N its point of contact with the fixed circle, and

$P(x, y)$ the point which describes the epicycloid. Determine the point K by rolling the circle C until P meets the circumference of O. Then

$$\text{arc } KN = \text{arc } NP.$$

Take O as the origin of coördinates, and OK as the axis of x. Draw PM and CL perpendicular to OX, PS parallel to OX, meeting CL in R, and connect O and C. Let the radius of the rolling circle be a, that of the fixed circle b, and denote the angle OCP by θ, the angle KOC by ϕ. Then

$$\text{arc } KN = b\phi, \qquad \text{arc } NP = a\theta;$$

whence

$$b\phi = a\theta.$$

We now have

$$\begin{aligned} x &= OM = OL + LM \\ &= OC \cos KOC - CP \cos SPC \\ &= (a+b)\cos\phi - a\cos(\phi+\theta) \\ &= (a+b)\cos\phi - a\cos\frac{a+b}{a}\phi. \end{aligned}$$

$$\begin{aligned} y &= MP = LC - RC \\ &= OC \sin KOC - CP \sin SPC \\ &= (a+b)\sin\phi - a\sin(\phi+\theta) \\ &= (a+b)\sin\phi - a\sin\frac{a+b}{a}\phi. \end{aligned}$$

The curve consists of a number of congruent arches the first of which corresponds to values of θ between 0 and 2π, that is, to values of ϕ between 0 and $\frac{2a\pi}{b}$. Similarly the kth arch corresponds to values of ϕ between $\frac{2(k-1)a\pi}{b}$ and $\frac{2ka\pi}{b}$. Hence the curve is a closed curve when, and only when, for some value of k, $\frac{2ka\pi}{b}$ is a multiple of 2π. If a and b are incommensurable, this is impossible, but if $\frac{a}{b} = \frac{p}{q}$, where $\frac{p}{q}$ is a rational fraction in its lowest terms, the smallest value of $k = q$. The curve then consists of q arches and winds p times around the fixed circle.

170. The hypocycloid. When a circle rolls upon the inside of a fixed circle, each point of the rolling circle describes a curve called the hypocycloid. If the axes and the notation are as in the previous article, the equations of the hypocycloid are

$$x = (b - a)\cos\phi + a\cos\frac{b-a}{a}\phi,$$

$$y = (b - a)\sin\phi - a\sin\frac{b-a}{a}\phi.$$

The proof is left to the student. The curve is shown in fig. 176.

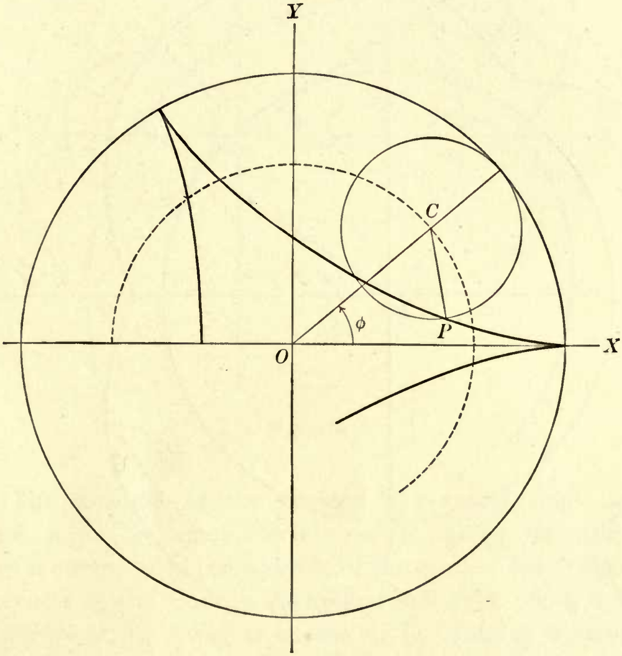

Fig. 176

171. Epitrochoid and hypotrochoid. The epitrochoid and hypotrochoid are generated by the motion of any point on the radius of a circle which rolls upon the outside or the inside of a fixed circle. If h is the distance of the generating point from the center of the moving circle, and the notation is otherwise the same as in the previous articles, the equations of the epitrochoid are

$$x = (a + b) \cos \phi - h \cos \frac{a + b}{a} \phi,$$

$$y = (a + b) \sin \phi - h \sin \frac{a + b}{a} \phi,$$

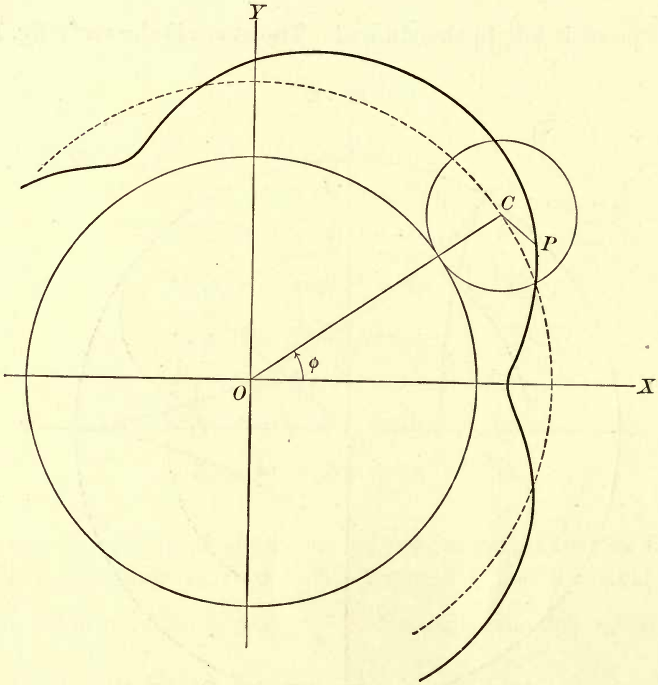

FIG. 177

and of the hypotrochoid are

$$x = (b - a) \cos \phi + h \cos \frac{b - a}{a} \phi,$$

$$y = (b - a) \sin \phi - h \sin \frac{b - a}{a} \phi.$$

The proofs are left to the student. The curves are shown in figs. 177, 178, and 179, 180 respectively.

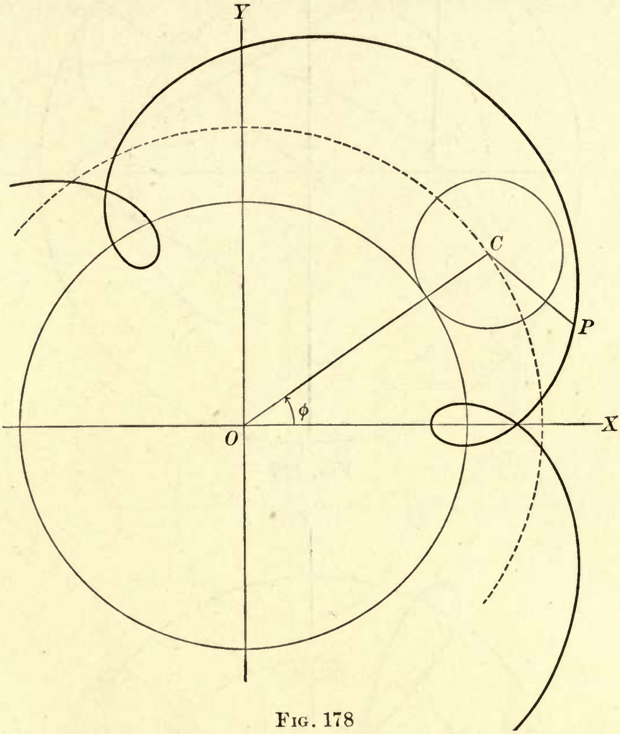

FIG. 178

172. The involute of the circle. If a string, kept taut, is unwound from the circumference of a circle, its extremity describes a curve called the involute of the circle. Let O (fig. 181), be the center of the circle, a its radius, and A the point at which the extremity of the string is on the circle. Take O as the origin of coördinates and OA as the axis of x. Let $P(x, y)$ be a point on the involute, PK the line drawn from P tangent to the circle at K, and ϕ the angle XOK. Then PK represents a portion of the unwinding string, and hence

$$KP = \text{arc}\, AK = a\phi.$$

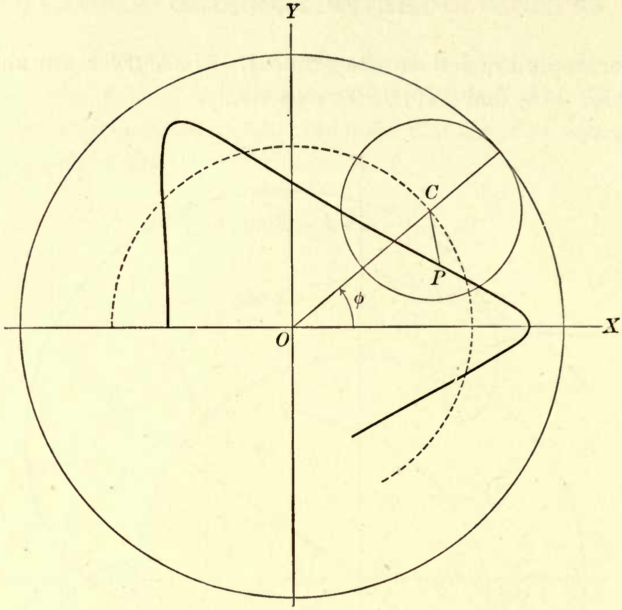

FIG. 179

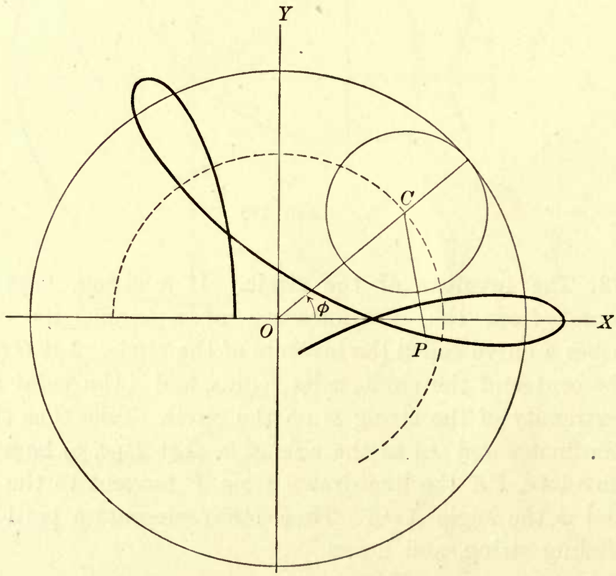

FIG. 180

Now it is clear that for all positions of the point K, OK makes an angle $\phi - 90°$ with OY. Hence the projection of OK on OX is always $OK \cos\phi = a\cos\phi$, and its projection on OY is $OK \cos(\phi - 90°) = a\sin\phi$. Also KP always makes an angle $\phi - 90°$

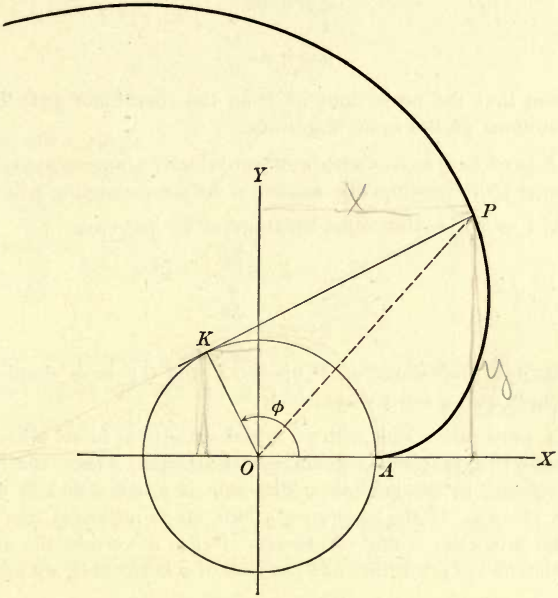

Fig. 181

with OX and $180° - \phi$ with OY. Hence the projection of KP on OX is $KP\cos(\phi - 90°) = a\phi\sin\phi$, and its projection on OY is $KP\cos(180° - \phi) = -a\phi\cos\phi$. The projection of OP on OX is x, and upon OY is y. Hence, by the law of projections, § 15,

$$x = a\cos\phi + a\phi\sin\phi,$$

$$y = a\sin\phi - a\phi\cos\phi.$$

173. Time as the arbitrary parameter. An important use of the parametric representation of curves occurs in mechanics in finding the path of a moving point acted on by known forces. Here the independent parameter is usually the time.

Ex. 1. A particle moves in a circle with uniform velocity, k. Then, if s represents the arc traversed,

$$s = kt \quad \text{and} \quad \phi = \frac{s}{a} = \frac{kt}{a}.$$

Therefore the equations of the circle are (§ 165),

$$x = a \cos \frac{kt}{a},$$

$$y = a \sin \frac{kt}{a}.$$

This shows that the projections of P on the coördinate axes have simple harmonic motions of the same amplitude.

Ex. 2. A particle Q moves with uniform velocity along the auxiliary circle of an ellipse (§ 166); required the motion of its accompanying point, P.

As in Ex. 1, $\phi = \frac{kt}{a}$. Hence the equations of the path are

$$x = a \cos \frac{kt}{a},$$

$$y = b \sin \frac{kt}{a},$$

showing that the projections of P upon OX and OY have simple harmonic motion of amplitudes a and b respectively.

Ex. 3. A projectile is shot with an initial velocity v_0 in an initial direction which makes an angle α with the horizontal direction. Then the initial component of velocity in the horizontal direction is $v_0 \cos \alpha$ and in the vertical direction is $v_0 \sin \alpha$. If the resistance of the air is neglected, the only force acting on the projectile is that of gravity. Hence if we take the origin at the initial position of the projectile, and the axis of x horizontal, we have

$$\frac{d^2x}{dt^2} = 0,$$

$$\frac{d^2y}{dt^2} = -g,$$

which give

$$x = c_1 t + c_2,$$

$$y = -\tfrac{1}{2} gt^2 + c_3 t + c_4.$$

But when $t = 0$, we have

$$x = 0, \quad y = 0, \quad \frac{dx}{dt} = v_0 \cos \alpha, \quad \text{and} \quad \frac{dy}{dt} = v_0 \sin \alpha.$$

Hence the parametric equations of the path of the projectile are

$$x = v_0 t \cos \alpha,$$

$$y = v_0 t \sin \alpha - \tfrac{1}{2} gt^2.$$

Eliminating t from these equations, we have

$$y = x \tan \alpha - \frac{gx^2}{2 v_0^2 \cos^2 \alpha}$$

or

$$2 v_0^2 y \cos^2 \alpha = 2 v_0^2 x \sin \alpha \cos \alpha - gx^2$$

which shows that the curve is a parabola.

174. The derivatives. When a curve is defined by the equations

$$x = f_1(t), \qquad y = f_2(t),$$

we have, by (8), § 96,

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}. \tag{1}$$

Ex. For the cycloid

$$x = a(\phi - \sin\phi),$$
$$y = a(1 - \cos\phi),$$
$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\phi}}{\dfrac{dx}{d\phi}} = \frac{a\sin\phi}{a(1-\cos\phi)} = \cot\frac{\phi}{2}.$$

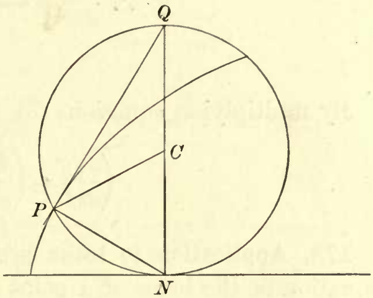

FIG. 182

Now $\frac{dy}{dx}$ is the tangent of the angle made by the tangent with the axis of x. Therefore this angle is $\frac{\pi}{2} - \frac{\phi}{2}$.

From this follows a simple construction of the tangent and normal. For if the line NC (fig. 182) is prolonged until it cuts the circle in Q, and PQ and PN are drawn, the angle $CQP = \frac{\phi}{2}$. Hence PQ makes the angle $\frac{\pi}{2} - \frac{\phi}{2}$ with OX and is therefore the tangent. PN, being perpendicular to PQ, is the normal.

If it is required to find $\frac{d^2y}{dx^2}$, we may proceed as follows:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}. \tag{2}$$

Ex. For the cycloid

$$\frac{dy}{dx} = \cot\frac{\phi}{2},$$
$$\frac{dx}{d\phi} = a(1 - \cos\phi) = 2a\sin^2\frac{\phi}{2}.$$
$$\therefore \frac{d^2y}{dx^2} = \frac{-\dfrac{1}{2}\operatorname{cosec}^2\dfrac{\phi}{2}}{2a\sin^2\dfrac{\phi}{2}} = -\frac{1}{4a\sin^4\dfrac{\phi}{2}}.$$

Formula (1) may be expanded as follows:

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^2}.$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}\frac{dx}{dt} - \frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}. \tag{3}$$

By multiplying equations (3), § 105, by $\left(\frac{ds}{dt}\right)^2$, we have

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2. \tag{4}$$

175. Application to locus problems. In finding the Cartesian equation of the locus of a point which satisfies a given condition, it is often convenient to employ the principles of parametric representation; for by fixing the attention upon a single point of the required locus, it is frequently possible to express its coördinates in terms of a single parameter. The required equation is then found by eliminating the parameter.

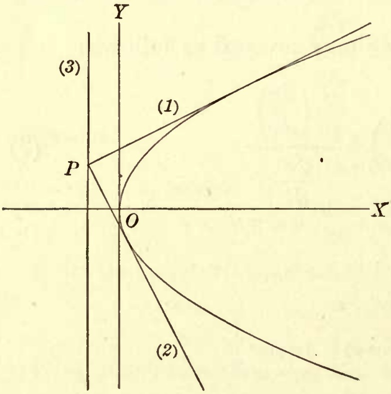

Fig. 183

Ex. 1. *Locus of the point of intersection of perpendicular tangents to a parabola.*

Let the parabola be $y^2 = 4px$ (fig. 183), and let the equation of any tangent to it be written (§ 88)

$$y = mx + \frac{p}{m}. \tag{1}$$

If m is replaced by $-\frac{1}{m}$, we have

$$y = \left(-\frac{1}{m}\right)x + \frac{p}{-\frac{1}{m}},$$

or

$$y = -\frac{x}{m} - pm, \tag{2}$$

as the equation of a tangent perpendicular to (1). Therefore, if $P(x, y)$ is the point of intersection of (1) and (2), P is any point of the locus.

Solving (1) and (2), we find $x = -p$ (3)

and $y = \frac{p}{m}(1 - m^2)$, (4)

which are the parametric representations of the locus, the parameter evidently being m. But for all points of the locus $x = -p$, and (3) is the Cartesian equation of the locus. It is to be noted that in this example the elimination of the parameter is unnecessary, since one of the equations does not contain it.

As (3) is the equation of the directrix, we have the proposition: *Perpendicular tangents to a parabola meet on the directrix.*

Ex. 2. *Locus of the point of intersection of perpendicular tangents to an ellipse.*

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (fig. 184), and let the equation of any tangent to it be written (§ 88)

$$y = mx \pm \sqrt{a^2m^2 + b^2}. \qquad (1)$$

Then the equation of a tangent perpendicular to (1) will be

$$y = -\frac{x}{m} \pm \sqrt{\frac{a^2}{m^2} + b^2}, \qquad (2)$$

and $P(x, y)$, the point of intersection of (1) and (2), will be any point of the locus.

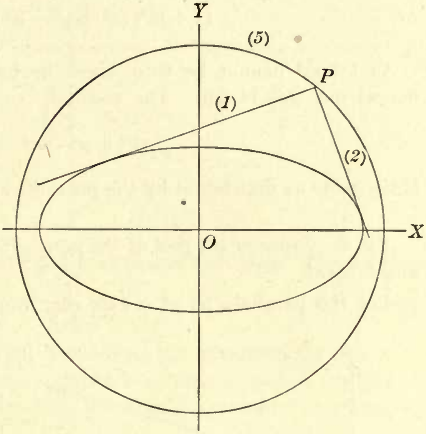

FIG. 184

Solving (1) and (2), we find

$$x = \frac{\pm m\left[\sqrt{\frac{a^2}{m^2} + b^2} - \sqrt{a^2m^2 + b^2}\right]}{m^2 + 1}, \qquad (3)$$

$$y = \frac{\pm\left[m^2\sqrt{\frac{a^2}{m^2} + b^2} + \sqrt{a^2m^2 + b^2}\right]}{m^2 + 1}, \qquad (4)$$

as the parametric representations of the locus in terms of the parameter m.

To eliminate m, we square the respective values of x and y and add, the result being

$$x^2 + y^2 = a^2 + b^2. \qquad (5)$$

The locus is seen to be a circle concentric with the ellipse and having its radius equal to the chord joining the ends of the major and the minor axes of the ellipse.

While (3) and (4) form the *explicit* parametric representation of the locus, x and y being expressed explicitly in terms of the parameter m, (1) and (2) may be regarded as the *implicit* parametric representation of the locus, for x and y, the coördinates of any point of the locus, are expressed implicitly in terms of m.

From this point of view it is evident that we may eliminate m directly from (1) and (2) to find the Cartesian equation of the locus. Accordingly we write (1) and (2) in the forms

$$y - mx = \pm\sqrt{a^2m^2 + b^2},$$
$$my + x = \pm\sqrt{a^2 + b^2m^2},$$

and square and add, the result being

$$(1 + m^2)(x^2 + y^2) = (1 + m^2)(a^2 + b^2),$$

or

$$(1 + m^2)(x^2 + y^2 - a^2 - b^2) = 0.$$

As $1 + m^2$ cannot be zero, since by hypothesis m must be real, we may cancel out this factor. The result,

$$x^2 + y^2 - a^2 - b^2 = 0,$$

is the same as that found by the previous method.

Ex. 3. *Locus of the foot of the perpendicular from the focus of a parabola to any tangent.*

Let the parabola be $y^2 = 4px$ (fig. 185), and let

$$y = mx + \frac{p}{m} \tag{1}$$

be any tangent. Then the perpendicular to the tangent from the focus is

$$y = -\frac{1}{m}(x - p). \tag{2}$$

Their point of intersection, $P(x, y)$, is any point of the locus.

Solving (1) and (2), we find

$$x = 0 \tag{3}$$

and

$$y = \frac{p}{m}. \tag{4}$$

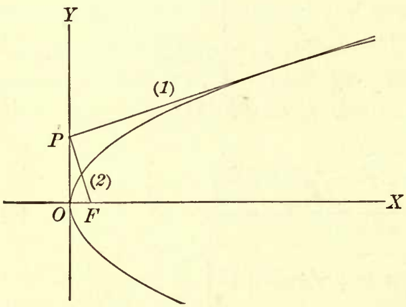

FIG. 185

The locus is therefore $x = 0$, the tangent at the vertex of the parabola.

If we proceed from the implicit parametric representation, we may eliminate the parameter m by substituting in (1) its value found from (2). The result is $x[y^2 + (p - x)^2] = 0$, which breaks up into two equations, i.e. $x = 0$, $y^2 + (x - p)^2 = 0$. As the last equation represents a single point, it is evident by the geometry of the problem that the required equation is $x = 0$, as was found by the other method.

We see then that when we eliminate the parameter from the equations expressing x and y in terms of it, we must examine our result carefully to be sure that no extraneous factor is left in it.

Ex. 4. *Locus of the foot of the perpendicular from the vertex of a parabola to any tangent.*

Let the parabola be $y^2 = 4px$ (fig. 186), and

$$y = mx + \frac{p}{m} \qquad (1)$$

be any tangent. Then the perpendicular to (1) from the vertex is

$$y = -\frac{1}{m}x. \qquad (2)$$

Solving (1) and (2), we find

$$x = \frac{-p}{m^2+1} \qquad (3)$$

$$y = \frac{p}{m(m^2+1)} \qquad (4)$$

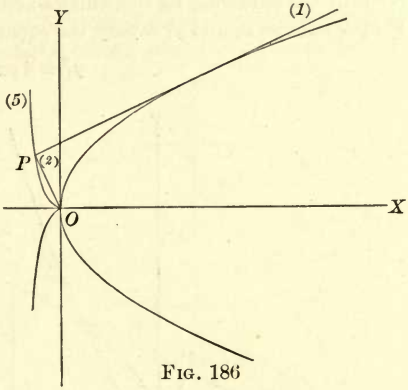

FIG. 186

as the explicit parametric representation of the locus. The Cartesian equation of the locus is most readily found by substituting in (1) the value of m from (2), and reducing. The result is

$$y^2 = -\frac{x^3}{p+x}, \qquad (5)$$

which is the equation of a cissoid (§ 83) situated on the negative axis of x.

The last two loci are special examples of *pedal* curves, i.e. loci of the feet of perpendiculars drawn from any chosen fixed point to tangents to a given curve.

176. In the examples of the last article the parametric representation of the locus was in terms of a single parameter. In the examples of this article the parametric representation, whether implicit or explicit, is in terms of two parameters, which are not independent, however, since they are connected by a single equation. The problem of finding the Cartesian equation of the locus is, then, the elimination of two parameters from three equations.

Ex. 1. *Through the vertex of a parabola a line is drawn perpendicular to any tangent. Required the locus of the intersection of this line and the ordinate through the point of contact of the tangent.*

Let $P_1(x_1, y_1)$ be any point of the parabola $y^2 = 4px$ (fig. 187), P_1T the tangent at P_1, and OT the perpendicular to P_1T from the vertex O. Then the equation of P_1T is

$$y_1y = 2p(x + x_1), \qquad (1)$$

and the equation of OT is

$$y = -\frac{y_1}{2p}x. \qquad (2)$$

The equation of the ordinate M_1P_1 through P_1 is

$$x = x_1. \qquad (3)$$

If $P(x, y)$ is the point of intersection of (2) and (3), P is any point of the locus, and (2) and (3) form the implicit parametric representation of the locus in terms of the parameters x_1 and y_1. Since $P_1(x_1, y_1)$ is by hypothesis any point of the parabola, its coördinates satisfy the equation of the parabola, and the parameters x_1 and y_1 satisfy the equation

$$y_1^2 = 4px_1. \tag{4}$$

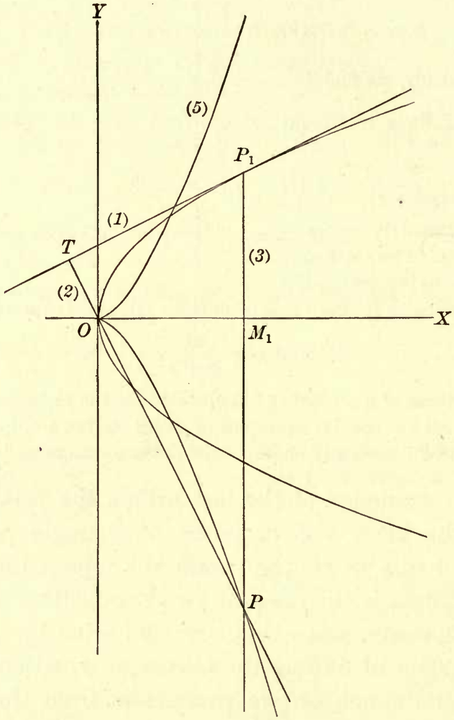

Fig. 187

Solving (2) and (3) for x_1 and y_1 and substituting their values in (4), we thereby eliminate them and have, as the Cartesian equation of the locus,

$$y^2 = \frac{1}{p}x^3. \tag{5}$$

From the form of the equation the locus is seen to be a semicubical parabola.

It may be added that the explicit parametric representation of the locus is readily found to be $x = x_1$ and $y = \frac{-y_1x_1}{2p}$, where $y_1^2 = 4px_1$.

Ex. 2. *Locus of the middle points of chords of an ellipse, drawn through one end of its major axis.*

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (fig. 188), and $P_1(x_1, y_1)$ be any point of the ellipse. Then AP_1 is any chord through A, and $P(x, y)$, its middle point, is any point of the required locus. Since the coördinates of A are $(a, 0)$, by § 18

$$x=\frac{x_1+a}{2} \tag{1}$$

and

$$y=\frac{y_1}{2}. \tag{2}$$

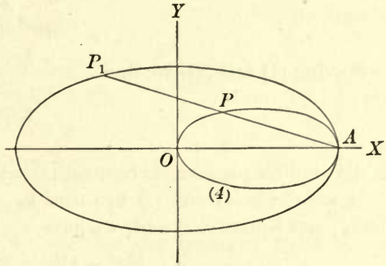

FIG. 188

Then (1) and (2) are the explicit parametric representations of the locus in terms of the parameters x_1 and y_1 which satisfy the equation

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1, \tag{3}$$

since P_1 is any point of the ellipse.

To find the Cartesian equation of the locus, we substitute in (3) the values of x_1 and y_1 from (1) and (2). The result is

$$\frac{\left(x-\frac{a}{2}\right)^2}{\left(\frac{a}{2}\right)^2}+\frac{y^2}{\left(\frac{b}{2}\right)^2}=1. \tag{4}$$

Accordingly the locus is an ellipse with its center at $\left(\frac{a}{2}, 0\right)$ and its semiaxes equal respectively to $\frac{a}{2}$ and $\frac{b}{2}$.

Ex. 3. *Locus of the point of intersection of tangents at the ends of conjugate diameters of an ellipse.*

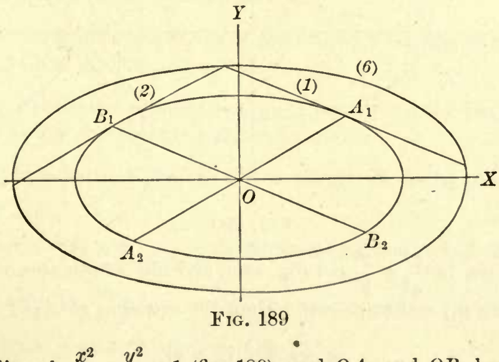

FIG. 189

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (fig. 189), and OA_1 and OB_1 be any two conjugate diameters. If A_1 is (x_1, y_1), B_1 is $\left(-\frac{ay_1}{b}, \frac{bx_1}{a}\right)$ by Ex. 2, § 146.

Then the tangents at A_1 and B_1 will be respectively

$$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1 \tag{1}$$

and

$$\frac{-y_1x}{ab} + \frac{x_1y}{ab} = 1, \tag{2}$$

where

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1. \tag{3}$$

Solving (1) and (2), we find

$$x = \frac{bx_1 - ay_1}{b}, \tag{4}$$

$$y = \frac{bx_1 + ay_1}{a}, \tag{5}$$

as the explicit parametric representations of the locus.

If we write (4) and (5) in forms $bx = bx_1 - ay_1$ and $ay = bx_1 + ay_1$ respectively, and square and add, we have

$$b^2x^2 + a^2y^2 = 2\,(b^2x_1^2 + a^2y_1^2),$$

or

$$b^2x^2 + a^2y^2 = 2\,a^2b^2, \tag{6}$$

by virtue of (3).

As (6) may be written $\dfrac{x^2}{(a\sqrt{2})^2} + \dfrac{y^2}{(b\sqrt{2})^2} = 1$, we see that the required locus is an ellipse, concentric with the given ellipse and with the semiaxes $a\sqrt{2}$ and $b\sqrt{2}$.

Ex. 4. *P_1P_2 is any chord of an ellipse perpendicular to its major axis A_1A_2. Find the locus of point of intersection of A_1P_1 and A_2P_2.*

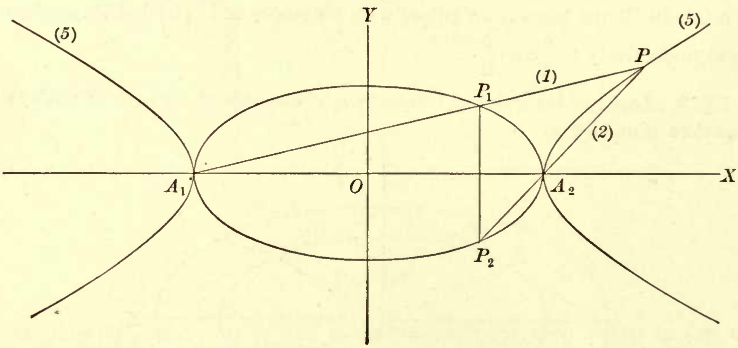

FIG. 190

Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (fig. 190), and the coördinates of P_1 and P_2 be respectively (x_1, y_1) and $(x_1, -y_1)$. Then the equation of A_1P_1 and A_2P_2 are respectively

$$y = \frac{y_1}{x_1 + a}(x + a), \tag{1}$$

$$y = \frac{y_1}{a - x_1}(x - a), \tag{2}$$

which are accordingly the implicit parametric representation of the locus. The parameters x_1 and y_1 satisfy the equation

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1. \tag{3}$$

Taking the product of (1) and (2), we have

$$y^2=\frac{y_1^2}{a^2-x_1^2}(x^2-a^2), \tag{4}$$

which may be written
$$y^2=\frac{b^2}{a^2}(x^2-a^2), \tag{5}$$

by virtue of (3).

As (5) may be written $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we see that the required locus is an hyperbola concentric with the ellipse and having the same semiaxes.

PROBLEMS

1. Show that $x=pt^2$, $y=2pt$ are parametric equations of the parabola.

2. Find the equations of the tangent and the normal to the parabola when the equations of the parabola are as in problem 1.

3. Find the parametric equations of the parabola when the parameter is the slope of a line through the vertex.

4. Find the equations of the tangent and the normal to a parabola when the equations of the curve are as in problem 3.

5. Find the parametric equations of the ellipse when the parameter is the slope of a straight line through the center.

6. Find the parametric equations of the ellipse when the parameter is the slope of a straight line through the left-hand vertex.

7. Find the parametric equations of the cissoid when the parameter is the angle AOP (fig. 91).

8. Show that $x=t$, $y=\frac{a^3}{a^2+t^2}$ are parametric equations of the witch.

9. Show that $x=\frac{2a}{1+t^2}$, $y=\frac{2a}{t(1+t^2)}$ are parametric equations of the cissoid. What is the geometric significance of t?

10. Find the equation of the tangent to the cissoid if the equations of the curve are as in problem 9.

Find the Cartesian equations of each of the following curves:

11. $x = \frac{a}{2} \cdot \frac{t^2 + 2}{t^2 + 1}, \; y = \frac{a}{2} \cdot \frac{t^3 + 2t}{t^2 + 1}.$

12. $x = \frac{3at}{1 + t^3}, \; y = \frac{3at^2}{1 + t^3}.$

13. $x = a + \frac{k^2}{a(1 + t^2)}, \; y = at + \frac{k^2 t}{a(1 + t^2)}.$

14. $x = \frac{1 + 2t}{1 - t^2}, \; y = \frac{1 + t}{1 - t}.$

15. $x = \frac{t + 1}{t - 1}, \; y = \frac{2t}{t^3 - 1}.$

16. $x = \frac{e^t + e^{-t}}{(e^t - 2e^{-t})^2}, \; y = \frac{e^t - e^{-t}}{(e^t - 2e^{-t})^2}.$

17. $x = t^2 + 3t + 2, \; y = t^2 - 1.$

18. $x = \frac{ct}{(a + bt)(1 + t^2)}, \; y = \frac{ct^2}{(a + bt)(1 + t^2)}.$

19. Eliminate t from

$$\frac{x}{a} = \frac{\cos t - \sin t}{e^t}, \qquad \frac{y}{a} = \frac{\cos t + \sin t}{e^t},$$

and prove that the curve represented is a logarithmic spiral (§ 178).

20. Let O be the center of a circle with radius a, A a fixed point, and B a moving point on the circle. If the tangent at B meets the tangent at A in C, and P is the middle point of BC, find the equations of the locus of P in parametric form, using the angle AOB as the arbitrary parameter, OA as the axis of x, and O as the origin. Also find the Cartesian equation of the locus.

21. $OBCD$ is a rectangle with $OB = a$ and $BC = c$. Any line is drawn through C, meeting OB in E, and the triangle EPO is constructed so that the angles CEP and EPO are right angles. Find the parametric equations of the locus of P, using the angle DOP as the parameter, OB as the axis of x, and O as the origin. Find also the Cartesian equation of the locus.

22. Let AB be a given line, O a given point, a units from AB, and k a given constant. Draw any line through O, meeting AB in M, and take P so that $OM \cdot MP = k^2$. Find the parametric equations of the locus of P, using O as the origin, the perpendicular from O to AB as the axis of x, and the angle between OX and OP as the parameter. Also find the Cartesian equation.

23. A and B are two points on the axis of y at a distance $-a$ and $+a$ respectively from the origin. AH is any line through A meeting the axis of x at H. BK is the perpendicular from B on AH, meeting it at K. Through K a line is drawn parallel to the axis of x and through H a line is drawn parallel to the axis of y. These lines meet in P. Find the parametric equations of the locus of P, using the angle BAK as the parameter. Also find the Cartesian equation.

24. Let OA be the diameter of a fixed circle and LK the tangent at A. From O draw any line intersecting the circle at B and LK at C, and let P be the middle point of BC. Find the parametric equations of the locus of P, using the angle AOP as the parameter, OA as the axis of y, and O as the origin. Find also the Cartesian equation.

25. Show that the tangent to the ellipse at any point and the tangent to the auxiliary circle at the corresponding point pass through the same point of the major axis.

26. Prove that the eccentric angles of the ends of a pair of conjugate diameters of an ellipse differ by $\frac{\pi}{2}$.

27. Show that the perpendicular from either focus upon the tangent at any point of the auxiliary circle of an ellipse equals the focal distance of the corresponding point of the ellipse.

28. Q is the point on the auxiliary circle of the ellipse, corresponding to the point P of the ellipse. The straight line through P parallel to OQ meets OX at L and OY at M. Prove $PL = b$, and $PM = a$.

29. Find the equation of the tangent at any point of an ellipse in terms of the eccentric angle at that point.

30. What elevation must be given to a gun to obtain the maximum range on a horizontal line passing through the muzzle of the gun? (In this and the following examples the resistance of the air and the effect of all forces except gravity are neglected.)

31. What elevation must be given to a gun to obtain a maximum range on an oblique line passing through the muzzle of the gun and making an angle β with the horizontal?

32. What elevation must be given to a gun that the projectile should pass through a point in the horizontal line passing through the muzzle and b units from it?

33. A gun stands on a cliff h units above the water. What elevation must be given to the gun that the projectile may strike a point in the water b units from the base of the cliff?

34. Find the parametric equations of the curve described by any point in the connecting rod of a steam engine.

35. If a circle rolls on the inside of a fixed circle of twice its radius, what is the form of the curve generated by a point of the circumference of the rolling circle?

36. Show that the hypocycloid generated when the rolling circle has $\frac{1}{4}$ the radius of the fixed circle has the Cartesian equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = b^{\frac{2}{3}}$.

37. If a wheel rolls with constant angular velocity on a straight line, required the velocity of any point on its circumference; also of any point on one of the spokes.

38. If a wheel rolls with constant angular velocity on the circumference of a fixed wheel, find the velocity of any point on its circumference and on its spoke.

39. Show that the highest point of a wheel rolling with constant velocity on a road moves twice as fast as each of the two points in the rim whose distance from the ground is half the radius of the wheel.

40. If a string is unwound from a circle with constant velocity, find the velocity of the end in the path described.

41. AB and CD are perpendicular diameters of a circle of radius R. AM is a chord of the circle, rotating about A so that the angle BAM varies uniformly. AM is extended to N so that $MN =$ the chord MB. Find the path of N, the velocity of N in its path, and the components of the velocity respectively parallel to AB and CD.

42. O, O', O'' are three points on a straight line and $O''O' = \frac{1}{3}OO'$. LK is drawn through O' perpendicular to OO'', and any point M is taken on LK. From M a straight line is drawn perpendicular to $O''M$, and through O a straight line is drawn parallel to $O''M$. These lines intersect in P. Required the locus of P.

43. O is a fixed point and LK a fixed straight line. Any point M is taken on LK, and the line OM is drawn and prolonged to P so that $OM \cdot OP = k^2$, where k is a constant. Find the locus of P.

44. Show that the locus of points symmetrical to the vertex of a parabola with respect to its tangent lines is a cissoid.

45. Let OA be the diameter of any circle and LK the tangent at A. Through O draw any line intersecting the circle in D and LK in E. Lay off on OE produced the distance $EP = OD$, and find the locus of P.

46. Let a circle with center at O intersect the axis of y at A and the axis of x at C. Take two points G and E on the circle equidistant from A. If the ordinate of G intersects the line CE in P, prove that the locus of P is a cissoid.

47. From a point a units from the axis of x lines are drawn to OX, and from the point where each line meets the axis a line of the same length is drawn at right angles to the first line. Find the equation of the locus of the end of this last line.

48. OA is a diameter of a circle and LK the tangent at A. Through O any line is drawn meeting the circle in B and LK in C. Through B a line is drawn perpendicular to OA and meeting it in M. Finally MB is prolonged to P so that $MP = AC$. Find the locus of P.

49. Find the path described by any point of a tangent line which rolls upon a circle without slipping.

50. CD is perpendicular to OX and distant a units from O. Through A, any point on CD, a straight line OA is drawn, and from A a perpendicular is drawn to OA, intersecting OX at B. From B a straight line is drawn parallel to OY, intersecting OA at P. If m denotes the slope of OA, find the parametric and the Cartesian equations of the locus of P.

51. Prove that the pedal of a parabola with respect to any point is a cubic curve which passes through that point.

52. Prove that the pedal of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to the center is the curve $(x^2+y^2)^2=a^2x^2+b^2y^2$.

53. A line of constant length k moves with its extremities on the two axes of coördinates. Find the locus described by any point of the line.

54. A straight line has its extremities on the coördinate axes and passes through a fixed point. Find the locus of its middle point.

55. If the ordinate NP of an hyperbola be produced to Q, so that $NQ = FP$, find the locus of Q.

56. Find the locus of the points of intersection of normals at corresponding points of the ellipse and the auxiliary circle.

57. P is any point of a parabola, A the vertex, and through A a straight line is drawn perpendicular to the tangent at P. Find the locus of the point of intersection of this line with the diameter through P, and also the locus of the point of intersection of this line with the ordinate through P.

58. Two equal parabolas have their axes parallel and a common tangent at their vertices, and straight lines are drawn parallel to the axes. Show that the locus of the middle points of the parts of the lines intercepted between the curves is an equal parabola.

59. Find the locus of the intersection of the ordinate, produced if necessary, of any point on an ellipse with the perpendicular from the center upon the tangent at that point.

60. Two parabolas have the same axis, and tangents are drawn from points on the first to the second. Prove that the middle points of the chords of contact with the second lie on a parabola.

61. Chords of an ellipse are passed through a fixed point. Find the locus of their middle points.

62. From a point P on an ellipse straight lines are drawn to the vertices A and A', and from A and A' straight lines are drawn perpendicular to AP and $A'P$. Show that the locus of their point of intersection is an ellipse.

63. Show that the locus of the point of intersection of two tangents to a parabola, the ordinates of the points of contact of which are in a constant ratio, is a parabola.

64. If the tangent to the parabola $y^2 = 4px$ meets the axis at T and the tangent at the vertex A at B, and the rectangle $TABQ$ is completed, show that the locus of Q is the parabola $y^2 + px = 0$.

65. Find the locus of the feet of the perpendiculars from the focus to the normals of the parabola $y^2 = 4px$.

66. Show that perpendicular normals to the parabola $y^2 = 4px$ intersect on the curve $y^2 = px - 3p^2$.

67. Find the locus of the intersection of a pair of perpendicular tangents to an hyperbola.

68. Two tangents to an ellipse are so drawn that the product of their slopes is constant. Show that the locus of their point of intersection is an ellipse or an hyperbola according as the product is negative or positive.

69. Prove that the locus of the point of intersection of two tangents to a parabola is a straight line if the product of their slopes is constant.

70. Find the locus of the foot of the perpendicular from either focus of an hyperbola to any tangent.

71. Let AB be the diameter of a circle and O its center. Let NQ be the ordinate of a point Q on the circle and P another point of the circle, so related to Q that OP revolves uniformly from OA through a right angle in the same time that QN travels at a uniform rate from A to O. If OP and QN intersect in R, find the locus of R.

72. Find the equations of the cycloid when the tangent at its highest point is the axis of x, the normal at the vertex is the axis of y, and the angle θ is the angle through which the radius has rotated after passing through the highest point.

73. Prove that the area of an arch of the cycloid above the axis of x is three times the area of the rolling circle.

74. Prove that for a cycloid $\frac{ds}{d\phi} = 2a\sin\frac{\phi}{2}$, and thence find its length from cusp to cusp.

75. Show that for an epicycloid $\frac{ds}{d\phi} = 2(a+b)\sin\frac{b\phi}{2a}$, and thence find its length from cusp to cusp.

CHAPTER XV

POLAR COÖRDINATES

177. Coördinate system. So far we have determined the position of a point in the plane by two distances, x and y. We may, however, use a distance and direction, as follows:

Let O (fig. 191), called the *origin* or *pole*, be a fixed point, and OM, called the *initial line*, be a fixed line. Take P any point in the plane and draw OP. Denote OP by r and the angle MOP by θ. Then r and θ are called the *polar coördinates* of the point $P(r, \theta)$, and when given will completely determine P.

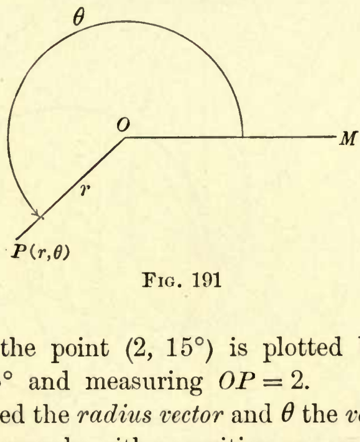

FIG. 191

For example, the point $(2, 15°)$ is plotted by laying off the angle $MOP = 15°$ and measuring $OP = 2$.

OP, or r, is called the *radius vector* and θ the *vectorial angle* of P. These quantities may be either positive or negative. A negative value of θ is laid off in the direction of the motion of the hands of a clock, a positive angle in the opposite direction. After the angle θ has been constructed, positive values of r are measured from O along the terminal line of θ, and negative values of r from O along the backward extension of the terminal line. It follows that the same point may have more than one pair of coördinates.

Thus (2, 195°), (2, − 165°), (− 2, 15°), and (− 2, − 345°) refer to the same point. In practice it is usually convenient to restrict θ to positive values.

Plotting in polar coördinates is facilitated by using paper ruled as in figs. 192 and 193. The angle θ is determined from the numbers at the ends of the straight lines, and the value of r is counted off on the concentric circles, either towards or away from the number which indicates θ, according as r is positive or negative.

When an equation is given in polar coördinates the corresponding curve may be plotted by giving to θ convenient values, computing the corresponding values of r, plotting the resulting points, and drawing a curve through them.

Ex. 1. $r = a \cos \theta$.

a is a constant which may be given any convenient value. We may then find from a table of natural cosines the value of r which corresponds to any value of θ.

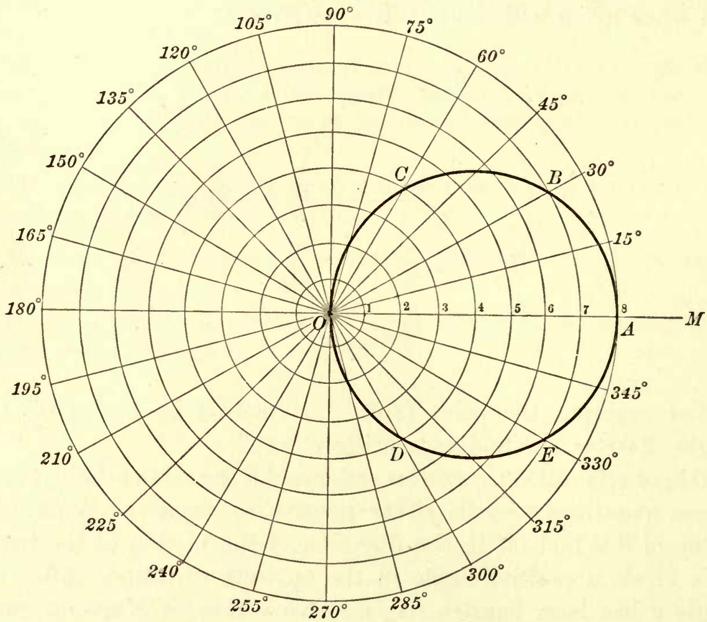

Fig. 192

By plotting the points corresponding to values of θ from 0° to 90° we obtain the arc $ABCO$ (fig. 192). Values of θ from 90° to 180° give the arc $ODEA$. Values of

θ from 180° to 270° give again the arc $ABCO$, and those from 270° to 360° give the arc $ODEA$. Values of θ greater than 360° can clearly give no points not already found. The curve is a circle (§ 184).

Ex. 2. $r = a \sin 3\theta$.

As θ increases from 0° to 30°, r increases from 0 to a; as θ increases from 30° to 60°, r decreases from a to 0; the point $P(r, \theta)$ traces out the loop OAO (fig. 193). As θ increases from 60° to 90°, r is negative and decreases from 0 to $-a$;

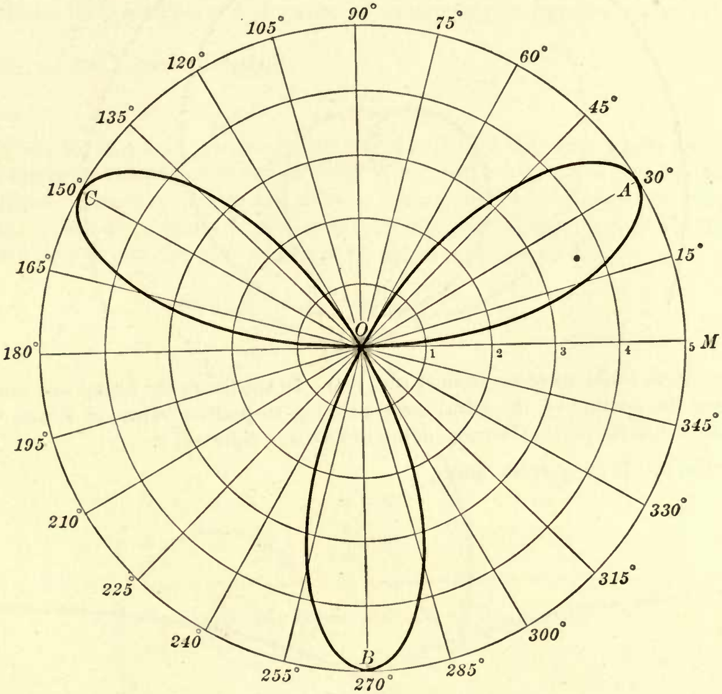

Fig. 193

as θ increases from 90° to 120°, r increases from $-a$ to 0; the point (r, θ) traces out the loop OBO. As θ increases from 120° to 180°, the point (r, θ) traces out the loop OCO. Larger values of θ give points already found, since $\sin 3(180° + \theta) = -\sin 3\theta$. The three loops are congruent because $\sin 3(60° + \theta) = -\sin 3\theta$. This curve is called a rose of three leaves.

178. The spirals. Polar coördinates are particularly well adapted to represent certain curves called spirals, of which the more important follow.

Ex. 1. *The spiral of Archimedes,*

$$r = a\theta.$$

In plotting θ is usually considered in circular measure. When $\theta = 0$, $r = 0$, and as θ increases r increases, so that the curve winds infinitely often around

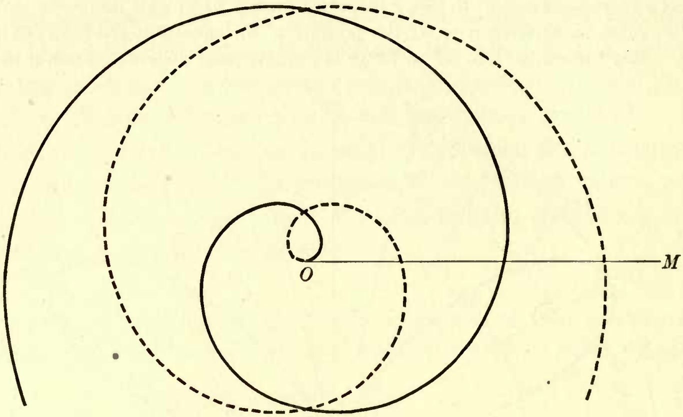

FIG. 194

the origin while receding from it (fig. 194). In the figure the heavy line represents the portion of the spiral corresponding to positive values of θ, and the dotted line the portion corresponding to negative values of θ.

Ex. 2. *The hyperbolic spiral,*

$$r\theta = a,$$

or

$$r = \frac{a}{\theta}.$$

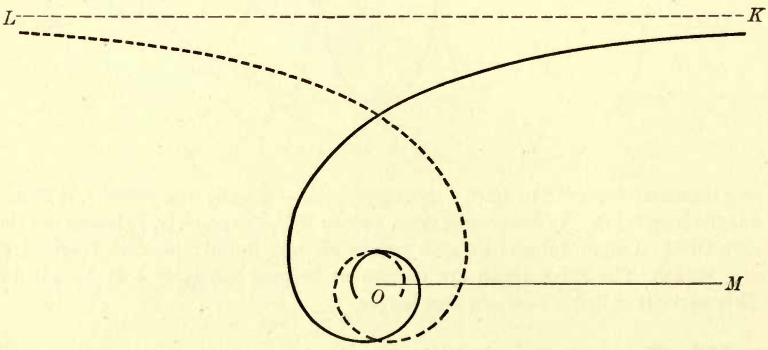

FIG. 195

As θ increases indefinitely r approaches zero. Hence the spiral winds infinitely often around the origin, continually approaching it but never reaching it (fig. 195). As θ approaches zero r increases without limit. If P is a point on the spiral and NP the perpendicular to the initial line,

$$NP = r \sin\theta = a\frac{\sin\theta}{\theta}.$$

Hence as θ approaches zero as a limit, NP approaches a (§ 151). Therefore the curve comes constantly nearer to, but never reaches, the line LK, parallel to OM at a distance a units from it. This line is therefore an asymptote. In the figure the dotted portion of the curve corresponds to negative values of θ.

Ex. 3. *The logarithmic spiral,*

$$r = e^{a\theta}.$$

When $\theta = 0$, $r = 1$. As θ increases r increases, and the curve winds around the origin at increasing distances from it (fig. 196). When θ is negative and increasing numerically without limit, r approaches zero. Hence the curve winds infinitely often around the origin, continually approaching it. The dotted line in the figure corresponds to negative values of θ.

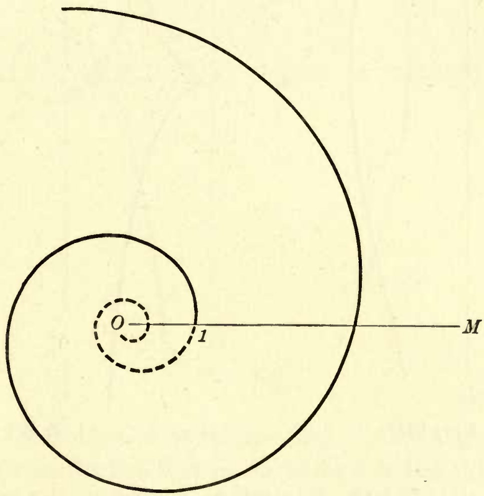

Fig. 196

A property of this spiral is that it cuts the radii vectors at a constant angle. The student may prove this after reading § 187.

We shall now give examples of the derivation of the polar equation of a curve from the definition of the curve.

179. The conchoid. Take a fixed point O (fig. 197) and a fixed straight line BC. Through O draw any line OR intersecting BC in D, and on OR lay off a constant distance DP or DQ, measured from D in either direction. The locus of P and Q is a curve called the *conchoid.*

From the definition the conchoid consists of two parts, one generated by P, the other by Q. We may obtain the whole curve,

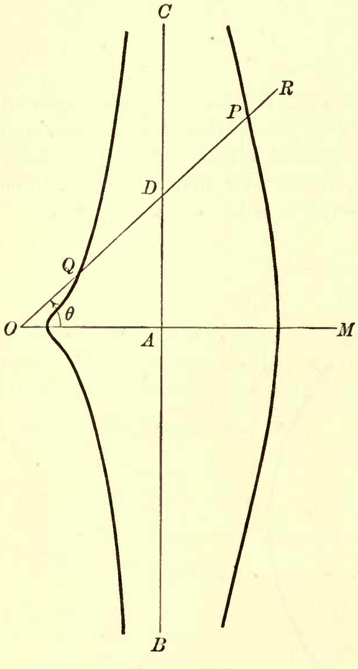

FIG. 197

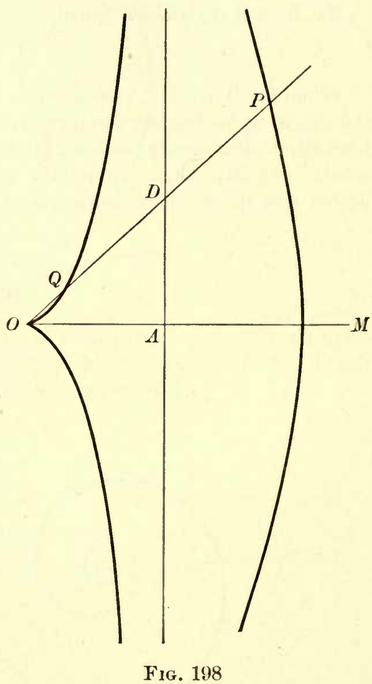

FIG. 198

however, by allowing the line OR to revolve in the positive direction through an angle of $360°$ and always laying off the distance b, measured from D in the direction of the terminal line of the angle AOR. Then if AOR is in the first quadrant, we obtain the upper half of the curve described by P; if AOR is in the second quadrant, we have the lower half of the curve described by Q; if AOR is in the third quadrant, we have the upper half of the curve described by Q; and if AOR is in the fourth quadrant, we have the lower half of the curve described by P.

To find its polar equation, take O as the origin and the line OA perpendicular to BC as the initial line. Let $OA = a$ and the constant distance $DP = b$.

Call the coördinates of P (r, θ), where $\theta = AOR$. When θ is in the first or the fourth quadrant, $r = OD + DP = OD + b$; when θ is in the second or the third quadrant, $r = -OD + DQ = -OD + b$.

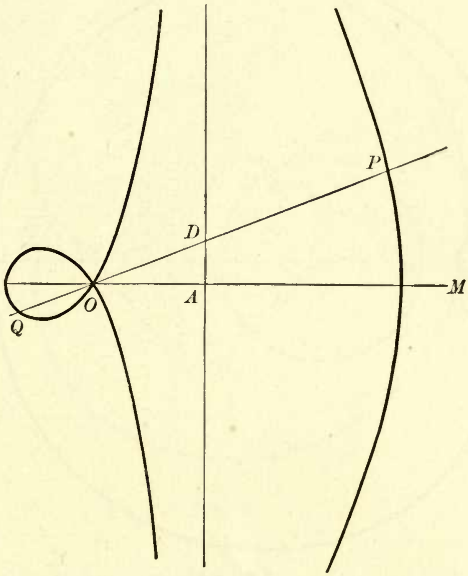

FIG. 199

But $OD = a \sec \theta$ when θ is in the first or the fourth quadrant; and $OD = -a \sec \theta$ when θ is in the second or the third quadrant. Hence for all points on the conchoid

$$r = a \sec \theta + b.$$

The conchoid has three shapes according as $a > b$ (fig. 197), $a = b$ (fig. 198), $a < b$ (fig. 199). If $b = 0$, the conchoid becomes the straight line BC and its equation becomes $r = a \sec \theta$, the equation of the straight line (§ 183).

180. The limaçon. Through any fixed point O (fig. 200) on the circumference of a fixed circle draw any line cutting the circle again at D, and lay off on this line a constant length measured from D in either direction. The locus of the points P and Q thus found is a curve called the *limaçon.*

Take O as the pole, the diameter OA as the initial line of a system of polar coördinates, and call the diameter of the circle a and

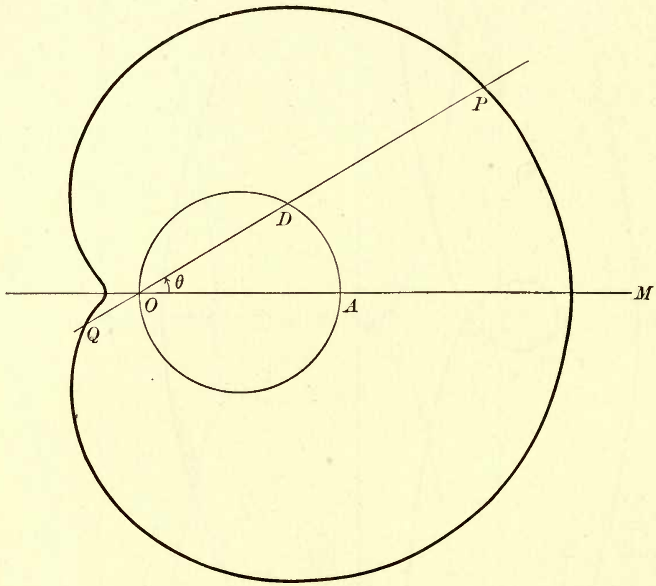

FIG. 200

the constant length b. Then it is clear that the entire locus can be found by causing OD to revolve through an angle of 360° and laying off $DP = b$ always in the direction of the terminal line of AOD.

Let the coördinates of P be (r, θ), where $\theta = AOD$. Then $r = OD + DP$ when θ is in the first or the fourth quadrant, and $r = -OD + DP$ when θ is in the second or the third quadrant. But it appears from the figure that $OD = OA \cos\theta$ when θ is in the first or the fourth quadrant, and $OD = -OA \cos\theta$

when θ is in the second or the third quadrant. Hence for any point on the limaçon

$$r = a \cos \theta + b.$$

In studying the shape of the curve there are three cases to be distinguished.

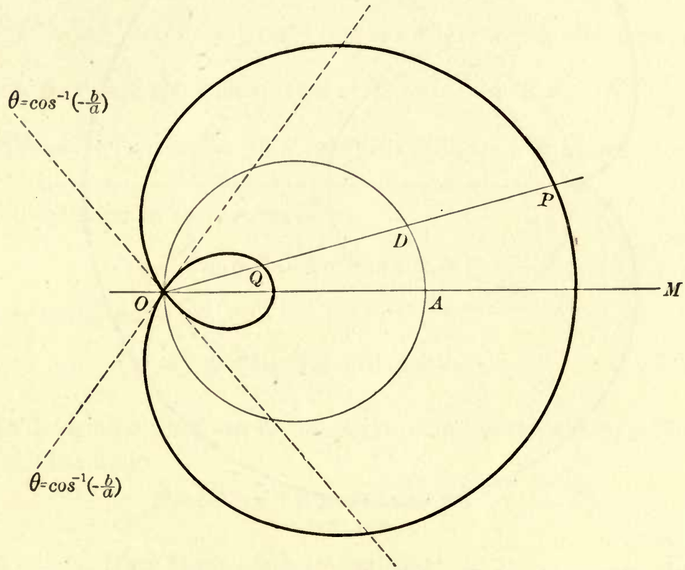

FIG. 201

1. $b > a$. r is always positive and the curve appears as in fig. 200.

2. $b < a$. r is positive when $\cos\theta > -\frac{b}{a}$, negative when $\cos\theta < -\frac{b}{a}$, and zero when $\cos\theta = -\frac{b}{a}$. The curve appears as in fig. 201.

3. $b = a$. The equation now becomes

$$r = a(\cos\theta + 1) = 2a\cos^2\frac{\theta}{2}.$$

r is positive except when $\theta = 180°$, when it is zero. The curve appears as in fig. 202 and is called the *cardioid*.

The cardioid is an epicycloid for which the radii of the fixed and the rolling circles are the same. The proof of this is left to the student.

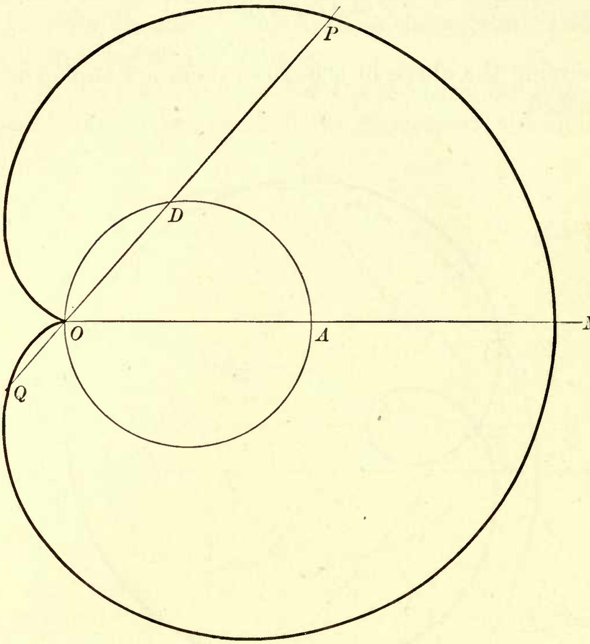

Fig. 202

181. The ovals of Cassini. If a point moves so that the product of its distances from two fixed points is constant, it generates a

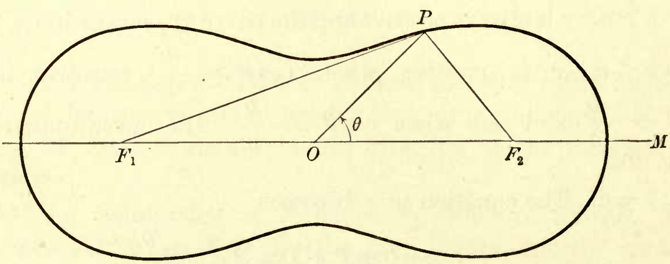

Fig. 203

curve called an *oval of Cassini.* Let F_1 and F_2 (fig. 203) be the two fixed points, called the foci, and b^2 the constant product of the distances of a point of the curve from F_1 and F_2. Take F_1F_2 as the initial line and the point O, halfway between F_1 and F_2, as the pole of a system of polar coördinates, and let P be a point on the curve. Then, by definition,

$$F_1P \cdot F_2P = b^2. \tag{1}$$

By trigonometry,

$$\overline{F_1P}^2 = \overline{OP}^2 + \overline{OF_1}^2 - 2\,OP \cdot OF_1 \cos F_1OP = r^2 + a^2 + 2\,ra\cos\theta,$$

where (r, θ) are the coördinates of P and $2\,a = F_1F_2$. Also

$$\overline{F_2P}^2 = \overline{OP}^2 + \overline{OF_2}^2 - 2\,OP \cdot OF_2 \cos F_2OP = r^2 + a^2 - 2\,ra\cos\theta.$$

Substituting in (1), we have

$$(r^2 + a^2)^2 - 4\,a^2r^2\cos^2\theta = b^4,$$

which is the same as

$$r^4 - 2\,a^2r^2\cos 2\,\theta + a^4 - b^4 = 0. \tag{2}$$

To determine the form of the curve, it is convenient to solve (2) for r^2, obtaining

$$r^2 = a^2\cos 2\,\theta \pm \sqrt{a^4\cos^2 2\,\theta - (a^4 - b^4)}. \tag{3}$$

We have, then, three cases to consider

1. $a^2 < b^2$. The quantity under the radical sign in (3) is positive and greater than $a^4\cos^2 2\,\theta$ for all values of θ. Therefore r^2 in (3) has two real values, one positive and one negative. Consequently r has two, and only two, real values equal in magnitude and opposite in sign. The curve therefore consists of a single oval, symmetric with respect to the origin (fig. 203).

2. $a^2 > b^2$. When $\cos^2 2\,\theta > \dfrac{a^4 - b^4}{a^4}$ the quantity under the radical sign in (3) is positive and less than $a^4\cos^2 2\,\theta$. Hence for these values of θ there are two real positive values of r^2 and therefore four real values of r, two positive and two negative. When $\cos^2 2\,\theta < \dfrac{a^4 - b^4}{a^4}$ the quantity under the radical sign in (3) is

negative, and hence all values of r are imaginary. When $\cos^2 2\theta = \frac{a^4 - b^4}{a^4}$ there are two real values of r, namely $r = \pm\frac{\sqrt{a^4 - b^4}}{a}$. The curve consists of two distinct ovals (fig. 204).

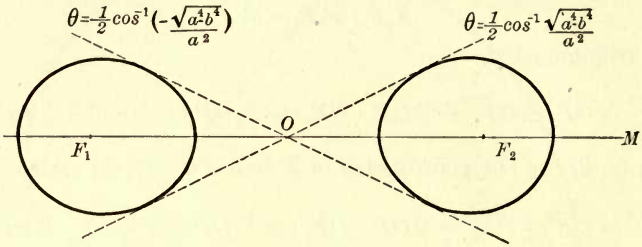

FIG. 204

3. $a^2 = b^2$. Equation (2) then factors into the two equations $r^2 = 0$ and $r^2 - 2a^2 \cos 2\theta = 0$. But $r^2 = 0$ is satisfied only by the origin, which is also a point on the second equation.

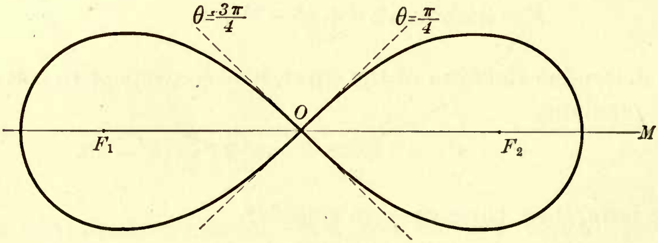

FIG. 205

Hence $$r^2 = 2a^2 \cos 2\theta \tag{4}$$

is the full equation of the locus in this case. From (4) it appears that r has two real values equal in magnitude but opposite in sign when $0 < \theta < \frac{\pi}{4}$, or $\frac{3\pi}{4} < \theta < \frac{5\pi}{4}$, or $\frac{7\pi}{4} < \theta < 2\pi$. Further, $r = 0$ when $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$; and r is imaginary when $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, or $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$. The curve appears as in fig. 205 and is given the special name of the *lemniscate.*

182. Relation between rectangular and polar coördinates. Let the pole O and the initial line OM of a system of polar coördinates be at the same time the origin and the axis of x of a system of rectangular coördinates. Let P (fig. 206) be any point of the plane, (x, y) its rectangular coördinates, and (r, θ) its polar coördinates. Then, by the definition of the trigonometric functions,

$$\cos\theta = \frac{x}{r},$$

$$\sin\theta = \frac{y}{r},$$

whence follows, on the one hand,

$$x = r\cos\theta,\quad y = r\sin\theta, \tag{1}$$

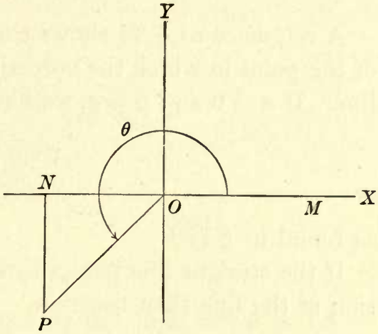

Fig. 206

and, on the other hand,

$$r = \sqrt{x^2+y^2},\qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}},\qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}}. \tag{2}$$

By means of (1) a transformation can be made from rectangular to polar coördinates, and by means of (2) from polar to rectangular coördinates.

Ex. 1. The equation of the cissoid (§ 83) is

$$y^2 = \frac{x^3}{2a - x}.$$

Substituting from (1) and making simple reductions, we have the polar equation

$$r = \frac{2a\sin^2\theta}{\cos\theta}.$$

Ex. 2. The polar equation of the lemniscate is

$$r^2 = 2a^2\cos 2\theta.$$

Placing $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and substituting from (2), we have the rectangular equation

$$(x^2+y^2)^2 = 2a^2(x^2-y^2).$$

183. The straight line. Take the equation of the straight line in the normal form $x \cos \alpha + y \sin \alpha - p = 0$ and substitute the values of x and y from (1), § 182. There results

$$r(\cos \theta \cos \alpha + \sin \theta \sin \alpha) - p = 0;$$

whence

$$r \cos(\theta - \alpha) = p.$$

A reference to § 33 shows that (p, α) are the polar coördinates of the point in which the normal from the origin meets the straight line. If $\alpha = 0$ and $p = a$, we have the special equation

$$r \cos \theta = a,$$

or

$$r = a \sec \theta,$$

as found in § 179.

If the straight line passes through the origin, $p = 0$. The equation of the line then becomes

$$\cos(\theta - \alpha) = 0,$$

or simply

$$\theta = \frac{\pi}{2} + \alpha,$$

which is of the form

$$\theta = c.$$

184. The circle. If (d, e) are the rectangular coördinates of the center of the circle and a its radius, its equation is

$$(x - d)^2 + (y - e)^2 = a^2.$$

If (b, α) are the polar coördinates of the center and (r, θ) those of any point, the pole and the initial line of the polar coördinates being the origin and the axis of x, respectively, of the rectangular system, we have, by (1), § 182,

$$x = r \cos \theta, \qquad y = r \sin \theta,$$
$$d = b \cos \alpha, \qquad e = b \sin \alpha.$$

We obtain, by substitution,

$$r^2 - 2\,rb(\cos \theta \cos \alpha + \sin \theta \sin \alpha) + b^2 - a^2 = 0,$$

or

$$r^2 - 2\,rb \cos(\theta - \alpha) + b^2 - a^2 = 0. \tag{1}$$

This result may also be directly obtained from fig. 207 by noticing that

$$\overline{CP}^2 = \overline{OC}^2 + \overline{OP}^2 - 2\, OP \cdot OC \cos POC.$$

When the origin is at the center of the circle, $b = 0$, and (1) becomes simply

$$r = a. \tag{2}$$

When the origin is on the circle, $b = a$, and (1) becomes

$$r - 2a\cos(\theta - \alpha) = 0,$$

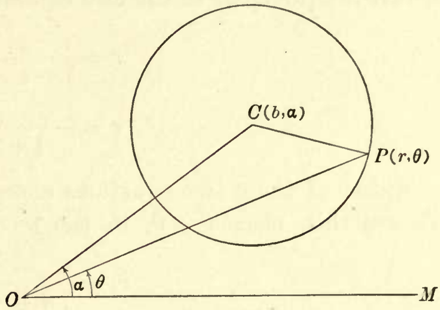

FIG. 207

which may be written

$$r = a_0 \cos\theta + a_1 \sin\theta, \tag{3}$$

where a_0 and a_1 are the intercepts on the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively.

When the origin is on the circle and the initial line is a diameter, (3) becomes

$$r = a_0 \cos\theta. \tag{4}$$

When the origin is on the circle and the initial line is tangent to the circle, (3) becomes

$$r = a_1 \sin\theta. \tag{5}$$

185. The conic, the focus being the pole. From § 81 the equation of a conic, when the axis of x is an axis of the conic and the axis of y is a directrix, is

$$(x - c)^2 + y^2 = e^2x^2.$$

We may transfer to new axes having the origin as the focus and the axis of x as the axis of the conic by placing

$$x = c + x', \qquad y = y',$$

thus obtaining

$$x'^2 + y'^2 = e^2(x' + c)^2.$$

If we now take a system of polar coördinates having the focus as the pole and the axis of the conic as the initial line, we have

$$x' = r\cos\theta, \qquad y' = r\sin\theta.$$

The equation then becomes

$$r^2 = e^2(r\cos\theta + c)^2,$$

which is equivalent to the two equations

$$r = \frac{ce}{1 - e\cos\theta},$$

$$r = -\frac{ce}{1 + e\cos\theta}.$$

Either of these two equations alone will give the entire conic. To see this, place $\theta = \theta_1$ in the second equation, obtaining

$$r_1 = \frac{-ce}{1 + e\cos\theta_1}.$$

Now place $\theta = \pi + \theta_1$ in the first equation, obtaining $r = -r_1$. The points (θ_1, r_1) and $(\pi + \theta_1, -r_1)$ are the same. Hence any point which can be found from the second equation can be found from the first.

Therefore $$r = \frac{ce}{1 - e\cos\theta}$$

is the required polar equation.

186. Examples. We shall now give examples of the use of polar coördinates in solving problems.

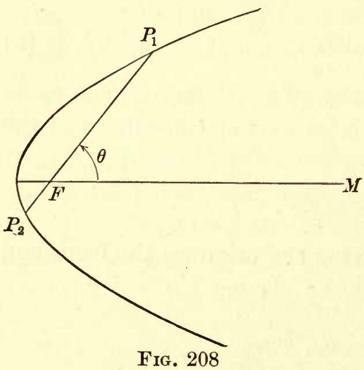

FIG. 208

Ex. 1. Prove that if a secant is drawn through the focus of a conic, the sum of the reciprocals of the segments made by the focus is constant.

Let P_1P_2 (fig. 208) be any secant through the focus F, and let $FP_1 = r_1$ and $FP_2 = r_2$, and the angle $MFP = \theta$. Then the polar coördinates of P_1 are (r, θ) and those of P_2 are $(r, \theta + \pi)$. From the polar equation of the conic we have

$$r_1 = \frac{ce}{1 - e\cos\theta},$$

$$r_2 = \frac{ce}{1 - e\cos(\theta + \pi)} = \frac{ce}{1 + e\cos\theta}.$$

Hence $$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{ce}.$$

Ex. 2. Find the locus of the middle points of a system of chords of a circle all of which pass through a fixed point.

Take any circle with the center C (fig. 209) and let O be any point in the plane. If O is taken for the pole and OC for the initial line of a system of polar coördinates, the equation of the circle is

$$r^2 - 2rb\cos\theta + b^2 - a^2 = 0. \quad (1)$$

Let P_1P_2 be any chord through O and let $OP_1 = r_1$, $OP_2 = r_2$. Then r_1 and r_2 are the two roots of equation (1) which correspond to the same value of θ. Hence

$$r_1 + r_2 = 2b\cos\theta.$$

If Q is the middle point of P_1P_2, and we now place $OQ = r$, we have

$$r = \frac{r_1 + r_2}{2} = b\cos\theta.$$

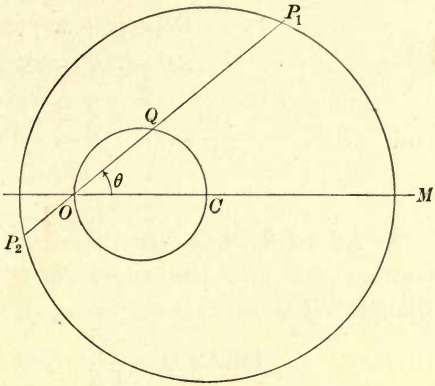

FIG. 209

But this is the polar equation of a circle through the points O and C.

187. Direction of a curve. The direction of a curve expressed in polar coördinates is usually determined by means of the angle between the tangent and the radius vector. Let $P(r, \theta)$ (fig. 210) be any point on the curve, PT the tangent at P, and ψ the angle made by PT and the radius vector OP. Give θ an increment $\Delta\theta = POQ$, expressed in circular measure, thus fixing a second point of the curve $Q(r + \Delta r, \theta + \Delta\theta)$. To determine Δr describe

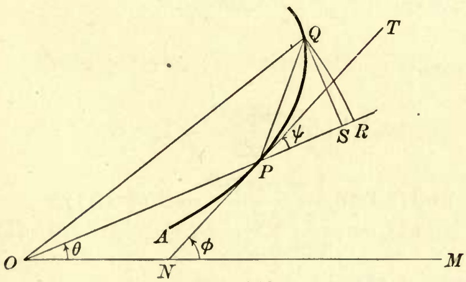

FIG. 210

a circle with center O and radius OQ, intersecting OP produced in R. Then

$$OR = OQ = r + \Delta r,$$

$$PR = \Delta r,$$

and

$$\text{arc } PQ = \Delta s,$$

s being measured from some initial point A.

Draw also the chord PQ and the straight line QS perpendicular to OP and meeting it in S. Then

$$SQ = (r + \Delta r)\sin\Delta\theta,$$
$$OS = (r + \Delta r)\cos\Delta\theta,$$
$$SR = OR - OS$$
$$= (r + \Delta r)(1 - \cos\Delta\theta),$$

and

$$PS = PR - SR$$
$$= \Delta r - (r + \Delta r)(1 - \cos\Delta\theta).$$

As $\Delta\theta$ approaches zero, the chord PQ approaches the limiting position PT and the angle RPQ approaches ψ. But in the triangle SPQ

$$\tan RPQ = \frac{SQ}{PS}$$
$$= \frac{(r + \Delta r)\sin\Delta\theta}{\Delta r - (r + \Delta r)(1 - \cos\Delta\theta)}$$
$$= \frac{(r + \Delta r)\dfrac{\sin\Delta\theta}{\Delta\theta}}{\dfrac{\Delta r}{\Delta\theta} - (r + \Delta r)\dfrac{1 - \cos\Delta\theta}{\Delta\theta}}. \qquad (1)$$

Now as $\Delta\theta$ approaches zero

$$\text{Lim}\,(r + \Delta r) = r, \qquad \text{Lim}\,\frac{\sin\Delta\theta}{\Delta\theta} = 1,$$
$$\text{Lim}\,\frac{\Delta r}{\Delta\theta} = \frac{dr}{d\theta}, \quad \text{and} \quad \text{Lim}\,\frac{1 - \cos\Delta\theta}{\Delta\theta} = 0 \ (\S\ 151).$$

Hence, by taking the limit of (1),

$$\tan\psi = \frac{r}{\dfrac{dr}{d\theta}}. \qquad (2)$$

If it is desired to find the angle $MNP = \phi$, it may be done by the evident relation

$$\phi = \psi + \theta.$$

188. Derivatives with respect to the arc. In the triangle PQS (fig. 210)

$$\sin SPQ = \frac{SQ}{\text{chord}\, PQ}$$

$$= \frac{SQ}{\text{arc}\, PQ} \cdot \frac{\text{arc}\, PQ}{\text{chord}\, PQ}$$

$$= \frac{(r + \Delta r)\sin \Delta\theta}{\Delta s} \cdot \frac{\text{arc}\, PQ}{\text{chord}\, PQ}$$

$$= (r + \Delta r)\frac{\sin \Delta\theta}{\Delta\theta} \cdot \frac{\Delta\theta}{\Delta s} \cdot \frac{\text{arc}\, PQ}{\text{chord}\, PQ}.$$

As $\Delta\theta$ approaches zero, SPQ approaches ψ, $\text{Lim}\,\frac{\sin \Delta\theta}{\Delta\theta} = 1$, and $\text{Lim}\,\frac{\text{arc}\, PQ}{\text{chord}\, PQ} = 1$ (§ 104); hence

$$\sin \psi = r\frac{d\theta}{ds}. \tag{1}$$

By dividing (1) just obtained by (2) of the previous article,

$$\cos \psi = \frac{dr}{ds}. \tag{2}$$

From (1) and (2) we obtain

$$\left(r\frac{d\theta}{ds}\right)^2 + \left(\frac{dr}{ds}\right)^2 = 1. \tag{3}$$

By multiplying (3) by $\left(\frac{ds}{d\theta}\right)^2$ we obtain

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2, \tag{4}$$

and by multiplying (3) by $\left(\frac{ds}{dr}\right)^2$ we obtain

$$\left(\frac{ds}{dr}\right)^2 = \left(r\frac{d\theta}{dr}\right)^2 + 1. \tag{5}$$

189. Area. Let C (fig. 211) be a fixed point and $P\ (r, \theta)$ a variable point on the curve $r = f(\theta)$, and let A denote the area of the figure OCP, bounded by the arc of the curve CP and the radii OC and OP. Then A is a function of θ, since the value of θ fixes the position of the point P. If θ is increased by $\Delta\theta =$ angle POQ, A is increased by $\Delta A =$ area POQ. From O describe arcs of circles PS and QR with radii $OP = r$ and $OQ = r + \Delta r$ respectively. Then in the figure

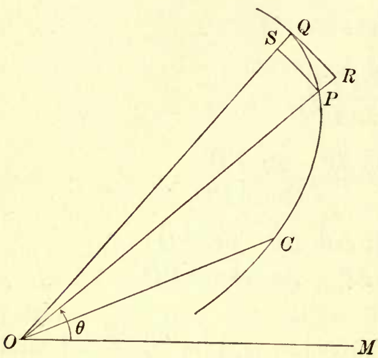

FIG. 211

$$\text{area } POS < \Delta A < \text{area } ROQ.$$

But the area of the sector of the circle POS is

$$\tfrac{1}{2} OP \cdot PS = \tfrac{1}{2} r^2 \Delta\theta,$$

and $$\text{area } ROQ = \tfrac{1}{2} OQ \cdot RQ = \tfrac{1}{2}(r + \Delta r)^2 \Delta\theta.$$

We have then $$\tfrac{1}{2} r^2 \Delta\theta < \Delta A < \tfrac{1}{2}(r + \Delta r)^2 \Delta\theta;$$

whence $$\tfrac{1}{2} r^2 < \frac{\Delta A}{\Delta\theta} < \tfrac{1}{2}(r + \Delta r)^2.$$

Taking now the limit as $\Delta\theta$ approaches zero, we have

$$\frac{dA}{d\theta} = \tfrac{1}{2} r^2.$$

Ex. Find the area of a loop of the lemniscate $r^2 = 2a^2 \cos 2\theta$.

We will take C as the point for which $\theta = 0$, and P as any point for which $0 < \theta < \frac{\pi}{4}$. Then

$$\frac{dA}{d\theta} = a^2 \cos 2\theta;$$

whence $$A = \frac{a^2}{2} \sin 2\theta + c.$$

But when $\theta = 0$, $A = 0$; therefore $c = 0$. Also when $\theta = \frac{\pi}{4}$, $A = \frac{1}{2}$ area of the loop. Hence the area of the loop is $a^2 \sin \frac{\pi}{2} = a^2$.

PROBLEMS

Plot the following curves:

1. $r = a \sin 2\theta$.
2. $r = a \cos 3\theta$.
3. $r = a \tan \theta$.
4. $r = a(1 + \sin\theta)$.
5. $r = a(2 + \sin\theta)$.
6. $r = a(1 + 2\sin\theta)$.
7. $r = a\theta^{-\frac{1}{2}}$.
8. $r = a \sec^2 \frac{\theta}{2}\cdot$
9. $r = \frac{a}{\theta - b}\cdot$
10. $r = a - b\theta$.
11. $r = a \sin \frac{\theta}{2}\cdot$
12. $r = a \cos \frac{\theta}{3}\cdot$
13. $r = a(1 + \cos 2\theta)$.
14. $r = a(1 + 2\cos 2\theta)$.
15. $r = a(1 - \cos 2\theta)$.
16. $r = a(1 + \cos 3\theta)$.
17. $r = a(1 + 2\cos 3\theta)$.
18. $r = 4 + 5\cos 5\theta$.
19. $r = 2 + \sin \frac{3}{2}\theta$.
20. $r = a \tan \frac{\theta}{2}\cdot$
21. $r = a \sin^3 \frac{\theta}{3}\cdot$
22. $r^2 = a^2 \sin\theta$.
23. $r^2 = a^2 \sin 3\theta$.
24. $r \cos\theta = a \cos 2\theta$.
25. $r = \frac{a}{\cos\theta} + \frac{a}{\sin\theta}\cdot$

Find the points of intersection of the following pairs of curves:

26. $r \cos\left(\theta - \frac{\pi}{3}\right) = a,\ r\cos\left(\theta - \frac{\pi}{6}\right) = a$.

27. $r \cos\left(\theta - \frac{\pi}{2}\right) = \frac{3a}{4},\ r = a \sin\theta$.

28. $r^2 = a^2 \sin\theta,\ r^2 = a^2 \sin 3\theta$.

29. $r = a \sin 2\theta,\ r = a(1 - \cos 2\theta)$. [$(r_1, \theta_1)$ and $(-r_1, \theta_1 + \pi)$ are the same points.]

30. O is a fixed point and LK a fixed straight line. If any line through O intersects LK in Q and a point P is taken on this line so that $OP \cdot OQ = k^2$, find the locus of P.

31. A straight line OA of constant length revolves about O. From A a perpendicular is drawn to a fixed straight line through O, intersecting it in B. From B a perpendicular is drawn to OA intersecting it in P. Find the locus of P.

32. MN is a straight line perpendicular to the initial line at a distance a from O. From O a straight line is drawn to any point B of MN. From B a straight line is drawn perpendicular to OB, intersecting the initial line at C. From C a line is drawn perpendicular to BC, intersecting MN at D. Finally, from D a straight line is drawn perpendicular to CD, intersecting OB at P. Find the locus of P.

Transform the following equations to polar coördinates:

33. $y^2 = 4px$.

34. $xy = 7$.

35. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

36. $x^2 + y^2 - 8ax - 8ay = 0$.

37. $x^4 + x^2y^2 - a^2y^2 = 0$.

38. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

39. $x^3 + y^3 - 3axy = 0$.

40. Find the polar equation of the cissoid when the pole is A and the initial line is OA (fig. 91).

41. Find the polar equation of the strophoid (1) when the pole is O and the initial line OA (fig. 92); (2) when the pole is A and the initial line is OA.

42. In the strophoid (fig. 92) show that

$$AP \cdot AP_1 = a^2, \quad \text{and} \quad \frac{1}{AP} + \frac{1}{AP_1} = \frac{2}{AN},$$

where AN is the projection of AO on AD.

Transform the following equations to rectangular coördinates:

43. $r\cos\left(\theta - \frac{\pi}{6}\right) + r\cos\left(\theta + \frac{\pi}{6}\right) = 12$.

44. $r = a\sin\theta$.

45. $r = a(\cos 2\theta + \sin 2\theta)$.

46. $r = a\tan\theta$.

47. $r^2 = a^2\sin\theta$.

48. $r^2 = a^2\sin\frac{\theta}{2}$.

49. Find the Cartesian equation of the rose of four petals $r = a\sin 2\theta$.

50. Find the Cartesian equation of the cardioid $r = a(1 - \cos\theta)$.

51. Find the Cartesian equation of the ovals of Cassini

$$r^4 - 2a^2r^2\cos 2\theta + a^4 - b^4 = 0.$$

52. Find the Cartesian equation of the limaçon $r = a\cos\theta + b$.

53. Find the Cartesian equation of the conchoid $r = a\sec\theta + b$.

54. Find the Cartesian equation of the logarithmic spiral $r = e^{a\theta}$.

55. In a parabola prove that the length of a focal chord which makes an angle of 30° with the axis of the curve is four times the focal chord perpendicular to the axis.

56. A comet is moving in a parabolic orbit around the sun at the focus of the parabola. When the comet is 100,000,000 miles from the sun the radius vector makes an angle of 60° with the axis of the orbit. What is the equation of the comet's orbit? How near does it come to the sun?

57. A comet moving in a parabolic orbit around the sun is observed at two points of its path, its focal distances being 5 and 15 million miles and the angle between them being 90°. What is its distance from the sun when it is nearest it?

58. If a straight line drawn through the focus of an hyperbola, parallel to an asymptote, meets the curve at P, prove that FP is one fourth the chord through the focus perpendicular to the transverse axis.

59. The focal radii of a parabola are extended beyond the curve until their lengths are doubled. Find the equation of the locus of their extremities.

60. If P_1 and P_2 are the points of intersection of a straight line drawn from any point O to a circle, prove that $OP_1 \cdot OP_2$ is constant.

61. If P_1 and P_2 are the points of intersection of a straight line from any point to a fixed circle, and Q is any point on the same straight line such that $OQ = \frac{2\, OP_1 \cdot OP_2}{OP_1 + OP_2}$, find the locus of Q.

62. Secant lines of a circle are drawn from the same point on the circle, and on each secant a point is taken outside the circle at a distance equal to the portion of the secant included in the circle. Find the locus of these points.

63. From a point O a straight line is drawn intersecting a fixed circle at P, and on this line a point Q is taken so that $OP \cdot OQ = k^2$. Find the locus of Q.

64. Find the polar equation of a conic if the pole is a vertex and the initial line an axis.

65. Find the locus of the middle points of the focal chords of a conic.

66. Find the locus of the middle points of the focal radii of a conic.

67. If P_1FP_2 and Q_1FQ_2 are two perpendicular focal chords of a conic, prove that $\frac{1}{P_1F \cdot FP_2} + \frac{1}{Q_1F \cdot FQ_2}$ is constant.

68. Prove that the angle between the normal and the radius vector to any point of the lemniscate is twice the angle made by the radius vector and the initial line.

69. Show that for any curve in polar coördinates the maximum and the minimum values of r occur in general when the radius vector is perpendicular to the tangent.

70. If a straight line drawn through the pole O perpendicular to a radius vector OP meets the tangent in A and the normal in B, show that $OA = \frac{r^2}{\frac{dr}{d\theta}}$ and $OB = \frac{dr}{d\theta}$.
These are called the *polar subtangent* and the *polar subnormal* respectively.

71. If p is the perpendicular distance of a tangent from the pole, prove that $p = \frac{r^2}{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}}$.

72. When a point traverses the curve $r = f(\theta)$ with a uniform angular velocity, find the rate at which r is changing and the rate of the point along the curve.

73. When a point moves along the curve $r = f(\theta)$ at a uniform rate, find the rates at which r and θ are changing.

74. Find the velocity of a point moving in a limaçon when θ changes uniformly.

75. A point moves along the radius vector with a constant velocity a, while the radius vector revolves about O with a constant velocity ω. Find the path of the point.

76. Find the total area bounded by the curve $r^2 = a^2 \sin\theta$.

77. Find the area of a loop of the curve $r^2 = a^2 \sin 3\theta$.

78. Find the area swept over by the radius vector of the spiral of Archimedes as θ changes from 0 to π.

79. Find the area swept over by the radius vector of the logarithmic spiral as θ changes from 0 to π.

80. Find the area swept over by the radius vector of the curve $r = a \sin\frac{\theta}{2}$ as θ changes from 0 to 2π.

81. Find the area swept over by the radius vector of the curve $r = a \tan\theta$ as θ changes from 0 to $\frac{\pi}{4}$.

82. Find the total area of the limaçon.

83. Find the total length of the cardioid.

84. Prove that the length of an arc of the logarithmic spiral is proportional to the difference of the radii vectores drawn to its ends.

85. Show that if the angle between the tangent to a curve and the radius vector to the point of contact is one half the vectorial angle, the curve is a cardioid.

CHAPTER XVI

CURVATURE

190. Definition of curvature. If a point describes a curve the change of direction of its motion may be measured by the change of the angle ϕ (§ 59).

For example, in the curve of fig. 212, if $AP_1 = s$ and $P_1P_2 = \Delta s$, and if ϕ_1 and ϕ_2 are the values of ϕ for the points P_1 and P_2 respectively, then $\phi_2 - \phi_1$ is the total change of direction of the curve between P_1 and P_2. If $\phi_2 - \phi_1 = \Delta\phi$, expressed in circular measure, the ratio $\frac{\Delta\phi}{\Delta s}$ is the average change of direction per linear unit of the arc P_1P_2. Regarding ϕ as a function of s and taking the limit of $\frac{\Delta\phi}{\Delta s}$ as Δs approaches zero as a limit, we have $\frac{d\phi}{ds}$, which is called the *curvature* of the curve at the point P_1. Hence *the curvature of a curve is the rate of change of the direction of the curve with respect to the length of the arc* (§ 109).

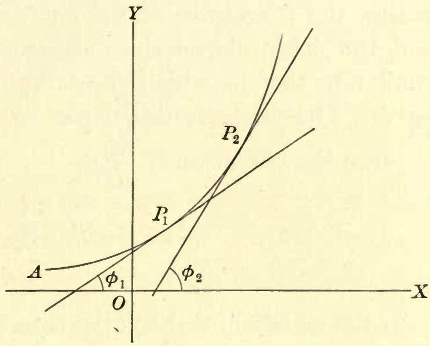

Fig. 212

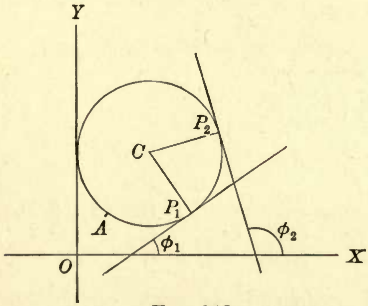

Fig. 213

If $\frac{d\phi}{ds}$ is constant, the curvature is constant or *uniform;* otherwise the curvature is variable. Applying this definition to the circle of fig. 213 of which the center is C and the radius is a, we have $\Delta\phi = P_1CP_2$; and hence $\Delta s = a\Delta\phi$. Therefore $\frac{\Delta\phi}{\Delta s} = \frac{1}{a}\cdot$ Hence $\frac{d\phi}{ds} = \frac{1}{a}$, and *the circle is a curve of constant curvature equal to the reciprocal of its radius.*

191. Radius of curvature. The reciprocal of the curvature is called the *radius of curvature,* and will be denoted by ρ. Through every point of a curve we may pass a circle, with its radius equal to ρ, which shall have the same tangent as the curve at the point, and shall lie on the same side of the tangent. Since the curvature of a circle is uniform and equal to the reciprocal of its radius, the curvatures of the curve and the circle are the same, and the circle shows the curvature of the curve in a manner similar to that in which the tangent shows the direction of the curve. The circle is called the *circle of curvature.*

Since the curvature is $\frac{d\phi}{ds}$,

$$\rho = \frac{1}{\frac{d\phi}{ds}} = \frac{ds}{d\phi}\cdot \qquad \text{(by (6), § 96)}$$

If the equation of the curve is in rectangular coördinates,

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad \text{(by § 105)}$$

and

$$\phi = \tan^{-1}\left(\frac{dy}{dx}\right); \qquad \text{(by § 59)}$$

whence

$$\frac{d\phi}{dx} = \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2}\cdot$$

$$\therefore\ \rho = \frac{ds}{d\phi} = \frac{\frac{ds}{dx}}{\frac{d\phi}{dx}} \qquad \text{(by (8), § 96)}$$

$$= \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\cdot$$

In the above expression for ρ there is an apparent ambiguity of sign, on account of the radical sign. If only the numerical value of ρ is required, a negative sign may be disregarded.

Ex. Find the radius of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Here
$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$
and
$$\frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}.$$
$$\therefore \rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$$

Another formula for ρ, i.e.
$$\rho=\frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}},$$
may be found by defining ϕ as the angle between OY and the tangent, and interchanging x and y in the above derivation.

192. According to the definition, i.e. $\rho=\frac{ds}{d\phi}$, it is evident that ρ is positive when s is measured so that s and ϕ increase at the same time, and is negative when one increases as the other decreases. For convenience we shall assume in the following work that s always increases from left to right * along the curve (figs. 214–217). Then ϕ is always in the first or the fourth quadrant, and hence $\sec\phi$ is always positive.

But $\sec\phi=\sqrt{1+\tan^2\phi}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot$ Therefore in the formula
$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$
the sign of ρ is the same as the sign of $\frac{d^2y}{dx^2}\cdot$ Hence ρ is positive when the curve is concave upward, and negative when the curve is concave downward.

* The results and the proof are the same if s is measured from right to left along the curve; hence the proof is left to the student.

193. Coördinates of center of curvature. The center of the circle described in § 191 is called the *center of curvature* corresponding to the point. Let $C(\alpha, \beta)$ (fig. 214) be the center of curvature corresponding to the point $P(x, y)$ of the curve. Draw CL and PM parallel to OY, and NR through P parallel to OX. Then

$$OL = OM + ML = OM + PN,$$

$$LC = LN + NC = MP + NC.$$

Now $$\angle RPC = \phi + 90°,$$

and $$PC = \rho,$$

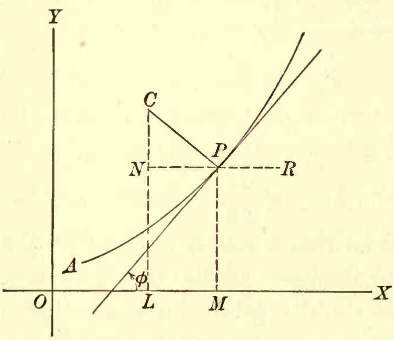

FIG. 214

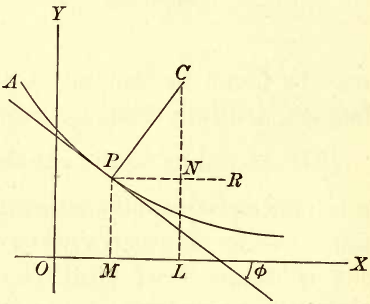

FIG. 215

since $\rho > 0$, the curve being concave upward. Therefore, by the definition of the trigonometric functions,

$$PN = PC \cos RPC = \rho \cos(\phi + 90°) = -\rho \sin\phi,$$

$$NC = PC \sin RPC = \rho \sin(\phi + 90°) = \rho \cos\phi.$$

$$\therefore\ \alpha = x - \rho \sin\phi,$$

$$\beta = y + \rho \cos\phi.$$

There are three other cases represented in figs. 215, 216, 217 respectively. The construction in all these figures is the same as in fig. 214, and the proof from fig. 215 is the same as that just given. The proof from figs. 216 and 217 differs only in that $RPC = \phi - 90°$, and $PC = -\rho$, since $\rho < 0$, the curve being concave downward. Hence the above expressions for α and β are universally true.

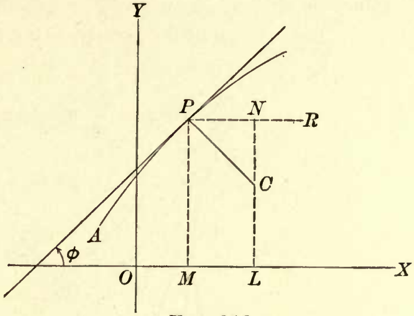

FIG. 216

Since $$\cos\phi = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

and $$\sin\phi = \frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}},$$

the formulas for α and β may be written

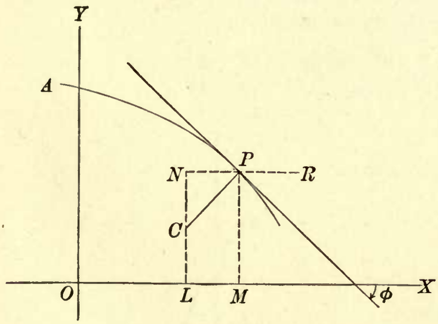

FIG. 217

$$\alpha = x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}},$$

$$\beta = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}.$$

Ex. Find the coördinates of the center of curvature for any point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

In the example of § 191 we found

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$$

Substituting in the above formulas and simplifying, we have

$$\alpha = \left(\frac{a^2 - b^2}{a^4}\right)x^3, \qquad \beta = -\left(\frac{a^2 - b^2}{b^4}\right)y^3.$$

194. Evolute and involute. With the single exception when $\frac{d^2y}{dx^2} = 0$, in which case ρ becomes infinite, there will be a center of curvature corresponding to each point of the curve. The locus of these centers of curvature is a curve called the *evolute* of the given curve, and the given curve is called the *involute*. In fig. 218 (1) is the involute and (2) is the evolute. To find the evolute we find the coördinates of the center of curvature in terms of x and y, and then eliminate x and y from these two equations by the aid of the equation of the curve.

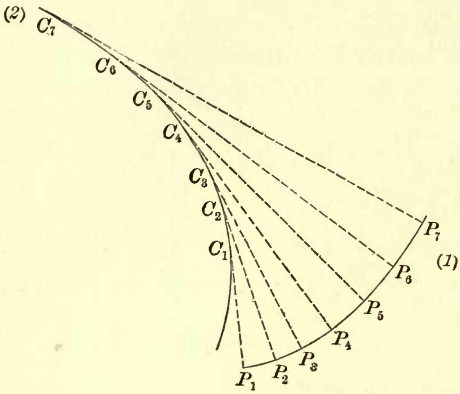

FIG. 218

Ex. To find the evolute of the ellipse, we have then, in the example of the last article, to eliminate x and y from the three equations

$$\alpha = \frac{a^2 - b^2}{a^4} x^3, \qquad \beta = -\frac{a^2 - b^2}{b^4} y^3, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

From the first two equations

$$\frac{x}{a} = \left(\frac{a\alpha}{a^2 - b^2}\right)^{\frac{1}{3}},$$

$$\frac{y}{b} = -\left(\frac{b\beta}{a^2 - b^2}\right)^{\frac{1}{3}}.$$

Substituting these values in the third equation and simplifying, we have

$$(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$$

as the equation of the evolute. The ellipse and its evolute are shown in fig. 219.

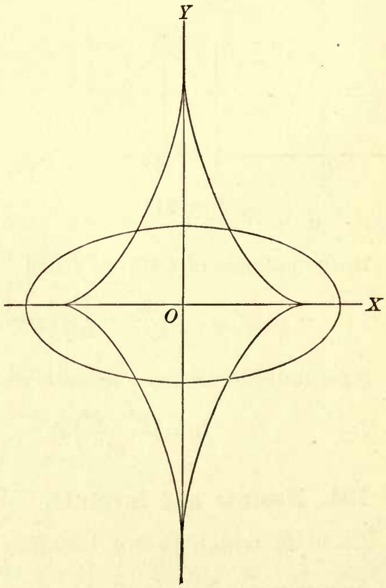

FIG. 219

It may be noted that equations expressing α and β are, in fact, the parametric representation of the evolute, x and y being two independent parameters connected by the equation of the given curve.

195. Properties of evolute and involute. From the equations $\alpha = x - \rho\sin\phi$, $\beta = y + \rho\cos\phi$, we may find the slope of the evolute at any point by assuming α, β, x, y, ρ, and ϕ as functions of s, the length of arc along the involute. Then

$$\begin{aligned}\frac{d\alpha}{ds} &= \frac{dx}{ds} - \rho\cos\phi\frac{d\phi}{ds} - \sin\phi\frac{d\rho}{ds}\\ &= \cos\phi - \rho\cos\phi\left(\frac{1}{\rho}\right) - \sin\phi\frac{d\rho}{ds}\\ &= -\sin\phi\frac{d\rho}{ds}.\end{aligned}$$

$$\begin{aligned}\frac{d\beta}{ds} &= \frac{dy}{ds} - \rho\sin\phi\frac{d\phi}{ds} + \cos\phi\frac{d\rho}{ds}\\ &= \sin\phi - \rho\sin\phi\left(\frac{1}{\rho}\right) + \cos\phi\frac{d\rho}{ds}\\ &= \cos\phi\frac{d\rho}{ds}.\end{aligned}$$

$$\therefore \frac{\dfrac{d\beta}{ds}}{\dfrac{d\alpha}{ds}} = -\operatorname{ctn}\phi; \quad \text{but} \quad \frac{\dfrac{d\beta}{ds}}{\dfrac{d\alpha}{ds}} = \frac{d\beta}{d\alpha},$$

by (8), § 96; and if $\tan\phi'$ is the slope of the evolute at the assumed point, $\dfrac{d\beta}{d\alpha} = \tan\phi'$, and hence $\tan\phi' = -\operatorname{ctn}\phi$. Hence ϕ' and ϕ differ by 90°, and *the tangent to the evolute at any point is perpendicular to the tangent to the involute at the corresponding point* (fig. 218).

If we square and add the above equations, we have

$$\left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2 = \left(\frac{d\rho}{ds}\right)^2.$$

But if we denote the length of arc along the evolute by s', we have $\dfrac{ds'}{d\alpha} = \sqrt{1 + \left(\dfrac{d\beta}{d\alpha}\right)^2}$; and if we regard s', α, β, as expressed in terms of s, the length of arc along the involute, we have

$$\frac{\frac{ds'}{ds}}{\frac{d\alpha}{ds}} = \sqrt{1 + \frac{\left(\frac{d\beta}{ds}\right)^2}{\left(\frac{d\alpha}{ds}\right)^2}};$$

whence
$$\frac{ds'}{ds} = \sqrt{\left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2}.$$

Hence
$$\left(\frac{ds'}{ds}\right)^2 = \left(\frac{d\rho}{ds}\right)^2$$

and
$$\frac{ds'}{ds} = \pm\frac{d\rho}{ds}.$$

$$\therefore\ s' = \pm\rho + c. \qquad \text{(by § 110)}$$

It follows, then, that as the center of curvature moves along the evolute the radius of curvature increases or decreases by exactly the distance traversed by the center (fig. 218).

From these two properties we see that an involute may be described by a pencil attached to the end of a string which is unwound from the evolute, the free portion being kept taut and tangent to the evolute. From any one evolute any number of involutes may be described by changing the length of the string.

196. Radius of curvature in parametric representation. If x and y are expressed in terms of any parameter t, the radius of curvature may be found as follows:

$$\rho = \frac{ds}{d\phi} = \frac{\frac{ds}{dt}}{\frac{d\phi}{dt}}. \qquad \text{(by (8), § 96)}$$

But
$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \qquad \text{(by (4), § 174)}$$

and
$$\phi = \tan^{-1}\frac{dy}{dx} = \tan^{-1}\frac{\frac{dy}{dt}}{\frac{dx}{dt}};$$

whence
$$\frac{d\phi}{dt}=\frac{\left(\frac{dx}{dt}\right)\frac{d^2y}{dt^2}-\left(\frac{dy}{dt}\right)\frac{d^2x}{dt^2}}{\left[1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2\right]\left(\frac{dx}{dt}\right)^2}$$
$$=\frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}.$$

Therefore, by substitution,
$$\rho=\frac{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}.$$

Ex. Find the radius of curvature of the cycloid
$$x=a\phi-a\sin\phi,$$
$$y=a-a\cos\phi.$$

Here the parameter is ϕ.
$$\therefore\ \frac{dx}{d\phi}=a-a\cos\phi,$$
$$\frac{d^2x}{d\phi^2}=a\sin\phi,$$
$$\frac{dy}{d\phi}=a\sin\phi,$$
$$\frac{d^2y}{d\phi^2}=a\cos\phi.$$

Hence, by substitution,
$$\rho=\frac{[a^2(1-\cos\phi)^2+a^2\sin^2\phi]^{\frac{3}{2}}}{a(1-\cos\phi)\cdot a\cos\phi-a\sin\phi\,(a\sin\phi)}$$
$$=-2^{\frac{3}{2}}a(1-\cos\phi)^{\frac{1}{2}}$$
$$=-2^{\frac{3}{2}}a\cdot\left(2\sin^2\frac{\phi}{2}\right)^{\frac{1}{2}}$$
$$=-4a\sin\frac{\phi}{2}.$$

197. Radius of curvature in polar coördinates. The equation of any curve in polar coördinates may always, theoretically, be expressed in the form $r=f(\theta)$. Then, since r may be regarded as a function of θ, the equations $x = r\cos\theta$, $y = r\sin\theta$, are the parametric equations of a curve. From them we may accordingly derive the formula for ρ in polar coördinates by substituting in the formula of § 196 as follows:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta,$$

$$\frac{d^2x}{d\theta^2} = \frac{d^2r}{d\theta^2}\cos\theta - 2\frac{dr}{d\theta}\sin\theta - r\cos\theta,$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta,$$

$$\frac{d^2y}{d\theta^2} = \frac{d^2r}{d\theta^2}\sin\theta + 2\frac{dr}{d\theta}\cos\theta - r\sin\theta.$$

Substituting these values and simplifying, we have, as the required formula,

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

Ex. Find the radius of curvature of the cardioid $r = a(1 - \cos\theta)$.

Here $\qquad \frac{dr}{d\theta} = a\sin\theta$ and $\frac{d^2r}{d\theta^2} = a\cos\theta.$

$$\therefore \rho = \frac{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{\frac{3}{2}}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(1-\cos\theta)a\cos\theta}$$

$$= \frac{[2a^2(1-\cos\theta)]^{\frac{3}{2}}}{a^2(3 - 3\cos\theta)} = \frac{2^{\frac{3}{2}}a}{3}(1-\cos\theta)^{\frac{1}{2}},$$

or $\qquad \rho = \frac{2}{3}(2ar)^{\frac{1}{2}}.$

PROBLEMS

1. Find the radius of curvature of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.

2. Find the radius of curvature of the cissoid $y^2 = \frac{x^3}{2a - x}$.

3. Find the radius of curvature of the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

4. Find the radii of curvature of the curve $a^4y^2 = a^2x^4 - x^6$ at the points (0, 0) and $(a, 0)$.

5. Find the radius of curvature of the curve $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ at the point $(0, b)$.

6. Find the radii of curvature of the curve $y^2 = ax(x - 3a)$ at the points where it crosses the axis of x.

7. Find the radius of curvature of the curve $e^x = \sin y$ at the point (x_1, y_1).

8. Find the slope and the radius of curvature of the curve $y + \log(1 - x^2) = 0$ at the origin of coördinates.

9. Show that the radius of curvature of the curve $r = a(\sin\theta + \cos\theta)$ is constant.

10. Find the radius of curvature of the curve $r = a(2\cos\theta - 1)$.

11. Find the radius of curvature of the curve $r = a\sin^3\frac{\theta}{3}$. Find the greatest and the least values of the radius of curvature.

12. Find the radius of curvature of the lemniscate $r^2 = 2a^2\cos 2\theta$.

13. Given the curve $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$. Find the radius of curvature in terms of t, and show that it will be greatest when $t = \pi$.

14. Find the evolute of the parabola $y^2 = 4px$.

15. Find the radius of curvature of the tractrix

$$y = \frac{a}{2}\log\frac{a + \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}} - \sqrt{a^2 - x^2}.$$

16. Prove that the evolute of the tractrix is the catenary.

17. Prove that the evolute of a cycloid is an equal cycloid.

18. Find the evolute of the four-cusped hypocycloid $x = a\cos^3\phi$, $y = a\sin^3\phi$.

19. Find the evolute of the ellipse from the parametric equations $x = a\cos\phi$, $y = b\sin\phi$.

20. Prove that the center of curvature of any point of the logarithmic spiral is the point of intersection of the normal with the perpendicular to the radius vector.

21. Find the circle of curvature of the curve $y = e^{-x^2}$ when $x = 0$.

22. Show that the catenary $y = \frac{1}{2}(e^x + e^{-x})$ and the parabola $y = 1 + \frac{1}{2}x^2$ have the same tangent and the same circle of curvature at their point of intersection.

23. Find the point of minimum curvature on the curve $y = \log x$.

24. Find the points of greatest and of least curvature of the sine curve $y = \sin x$.

25. Find the points on the ellipse for which the curvature is a maximum or a minimum.

26. Show that the curvature of the parabola $y = ax^2 + bx + c$ is a maximum at the vertex.

27. Find the condition for a maximum or a minimum of the curvature k, where $k = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$.

28. At what points on the curve $y = \log \sin x$ is the radius of curvature unity, and in what direction from the point on the curve is the center of curvature?

29. Show that the product of the radii of curvature of the curve $y = ae^{-\frac{x}{a}}$ at the two points for which $x = \pm a$ is $a^2(e + e^{-1})^3$.

30. If the angle between the radius vector to the point of contact and the straight line drawn from the pole perpendicular to the tangent is either a maximum or a minimum, prove that $\rho = \frac{r^2}{p}$, where p is the length of the perpendicular.

ANSWERS

(The answers to some problems are intentionally omitted.)

CHAPTER I

Page 25

1. 9.

2. $x - y$.

3. $x - x^2$.

4. 4.

5. 17.

6. -18.

7. 1.

8. $2\,abc$.

9. $abc + 2fgh - af^2 - bg^2 - ch^2$.

10. $ab^2 + bc^2 + ca^2 - ac^2 - ba^2 - cb^2$.

11. $7x - 6y - 5$.

12. 3.

13. $2\,a_1a_2c_1c_2 + a_1b_1b_2c_2 + a_2b_1b_2c_1 - a_1^2c_2^2 - a_2^2c_1^2 - a_1b_2^2c_1 - a_2b_1^2c_2$.

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24. $\dfrac{13 \pm \sqrt{61}}{2}$.

25. $0,\ 6 \pm \sqrt{39}$.

26. $8x + y - 13 = 0$.

27. $x^2 + y^2 - x - y = 0$.

28. $x^2 + y^2 - 3x + y - 4 = 0$.

29. $x^2 - (a + b)x + ab - h^2 = 0$.

30. $x^3 - (a + b + c)x^2 + (ab + bc + ca - f^2 - g^2 - h^2)x - abc - 2fgh + af^2 + bg^2 + ch^2 = 0$.

31. $x = 1,\ y = 2$.

32. $x = 2,\ y = -1,\ z = 3$.

33. $x = \frac{2}{3},\ y = -2,\ z = \frac{2}{5}$.

34. $x = -5,\ y = 0,\ z = 4$.

35. $x = -1,\ y = 0,\ z = 0$.

38. $x = \frac{1}{3}(a - 2b + c + d)$, $y = \frac{1}{3}(a + b - 2c + d)$, $z = \frac{1}{3}(a + b + c - 2d)$, $w = \frac{1}{3}(-2a + b + c + d)$.

39. $x_1 : x_2 : x_3 = 3 : -5 : -2$.

40. $x_1 : x_2 : x_3 = 1 : -2 : 3$.

41. $x_3 = 0,\ x_1 : x_2 = -2 : 1$.

42. $x_1 : x_2 : x_3 : x_4 = 4 : -3 : 2 : 5$.

43. $x_1 = 0,\ x_2 = 0,\ x_3 : x_4 = 3 : 2$.

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49. $1,\ \dfrac{-1 \pm \sqrt{-3}}{2}$.

50. $2y^2 + 6y - 3 = 0$.

51. $y^2 + 11y + 12 = 0$.

52. $y - 4 = 0$.

53. $b^2 - (a + c)^2 = 0$.

CHAPTER II

Page 45

1. $5 + 3\sqrt{5}$.

5. $(1\frac{1}{18},\ 3\frac{17}{18})$.

6. $(\frac{17}{18},\ 3\frac{17}{18})$.

7. $(-\frac{1}{3},\ 0)$.

8. $(-2\frac{3}{4},\ -1\frac{3}{8})$, $(-1\frac{3}{8},\ \frac{11}{16})$.

9. $(5,\ 3 \pm \sqrt{7})$.

10. $(-1,\ -3\frac{1}{2})$, $(1,\ -4\frac{1}{2})$.

11. $(-\frac{2}{5},\ 1\frac{2}{5})$.

12. $(-\frac{1}{4},\ -1\frac{1}{4})$.

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14. $(5,\ -1)$.

15. $(-14,\ 17)$.

16. $(-10,\ 31)$.

17. $(2,\ 1)$, $(4,\ -1)$.

18. $\frac{1}{2}\sqrt{89},\ \frac{1}{2}\sqrt{53},\ \sqrt{26}$.

19. $(15,\ -3)$.

CHAPTER III

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21. $\tan^{-1}\frac{7}{4}$. **22.** $\tan^{-1}\frac{4}{7}$. **23.** $\frac{\pi}{4}$. **25.** $5x-4y+40=0$.

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26. $x-y-4=0$.
27. $8x+9y+83=0$.
28. $5x-6y=0$.
29. $7x-2y+8=0$.
30. $3x-y-7=0$.
31. $x+1=0$.
32. $\tan^{-1}\frac{9}{13}$.
33. $2x+8y-17=0$.
34. $9x-6y-2=0$.
35. $x-y+2=0$.
36. $4x-6y+15=0$.
37. $25x+15y-24=0$.
38. $5x-2y-10=0$.
39. $7x+4y+28=0$.
41. $12x-15y-8=0$.
42. $3x-2y-7=0$.
43. $x+4y-4=0$.

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44. $2x+3y+1=0$.
45. $\frac{3}{2}$.
46. $(0, 2), (0, 7), (3, 5)$; $\frac{\pi}{4}, \tan^{-1}\frac{3}{2}, \tan^{-1}5$.
47. $\frac{1}{17}\sqrt{17}$.
48. $\frac{10}{13}\sqrt{13}$.
49. $\frac{b^2-a^2-ab}{\sqrt{b^2+a^2}}$.
50. $\frac{1}{13}\sqrt{13}$.
51. $5\sqrt{2}, \frac{37}{10}\sqrt{2}$.
52. $\frac{2}{5}$.
58. $(3, 0)$.
59. $2x-y-4=0$.
60. $(-2\frac{1}{7}, -1\frac{3}{7})$.
61. $(\pm\frac{21}{13}\sqrt{13}, \pm\frac{14}{13}\sqrt{13})$.

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63. $(1, 1), (-3, 3)$.
64. $5x-y-3=0$; $2\sqrt{26}$.
65. $17x+6y+34=0$, $x+18y+2=0$.
66. $5x+y-12=0$, $x-5y+8=0$.
67. $3x-4y-2=0$, $x-2=0$.

CHAPTER IV

Page 94

7. $a=\frac{9}{28}$, $x=-4\frac{2}{3}$. **8.** $\frac{2a}{c^2}$.

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10. (1) $k=\frac{7}{8}$; (2) $k>\frac{7}{8}$; (3) $k<\frac{7}{8}$.
11. (1) $k=0$ or 4; (2) $k<0$ or $k>4$; (3) $0<k<4$.
12. (1) $k=-3$ or $-\frac{1}{3}$; (2) $-3<k<-\frac{1}{3}$; (3) $k<-3$ or $k>-\frac{1}{3}$.
25. $\frac{3}{2}, \frac{1}{4}(-3\pm 3\sqrt{-3})$.
26. $2, -\frac{1}{2}, -1\pm\sqrt{-3}, \frac{1}{4}(1\pm\sqrt{-3})$.
27. $0, \frac{1}{2}(\pm 1\pm\sqrt{13})$.
28. $0, 0, \pm\sqrt{-3}\pm\sqrt{-6}$.
29. $0, -2a, \frac{a}{13}(-1\pm 5i)$.
30. $\pm(a+1), \pm(a-1)$.
31. $6x^3-13x^2+6x=0$.
32. $x^3-ax^2-(a^2+b)x+a^3-ab=0$.
33. $x^6-4ax^5+(4a^2-b^2-2b)x^4+8abx^3-(8a^2b-2b^3)x^2=0$.
34. $x^2-4x+13=0$.
35. $(2x+2-\sqrt{11})(2x+2+\sqrt{11})$.
36. $(2x+3+\sqrt{-2})(2x+3-\sqrt{-2})$.
37. $(2ax+\frac{1}{2}+\frac{1}{2}\sqrt{-3})(2ax+\frac{1}{2}-\frac{1}{2}\sqrt{-3})$.
38. $(x+a+\sqrt{a})(x+a-\sqrt{a})$.

39. $(ax + b + \sqrt{a + b^2})(ax + b - \sqrt{a + b^2})$.

40. $(ax + b + i\sqrt{b})(ax + b - i\sqrt{b})$.

41. $p^2 - 2q$.

42. $3pq - p^3$.

43. $-\frac{p}{q}$.

44. $\frac{p^2 - 2q}{q^2}$.

45. $\frac{p^2 - 2q}{q}$.

46. $p^2 - 3r$.

47. pr.

48. $\frac{p}{r}$.

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62. $1, \frac{1}{2}(3 \pm \sqrt{29})$.

63. $2, 2, -1$.

64. $2, 2, -\frac{5}{3}$.

65. $-3, \frac{5}{2}, \frac{5}{2}$.

66. $-3, \frac{1}{2}(-1 \pm \sqrt{-3})$.

67. $\frac{1}{2}, \frac{3}{2}, \frac{3}{2}$.

68. $\frac{1}{2}, \frac{7}{2}, -\frac{1}{3}$.

69. $\frac{2}{3}, -2 \pm \sqrt{-2}$.

70. $-\frac{4}{3}, -1 \pm \sqrt{3}$.

71. $-1, -1, \pm\frac{1}{2}$.

72. $3, -2, \frac{3}{2}, -\frac{2}{3}$.

73. $2, \frac{5}{3}, 1 \pm \sqrt{-2}$.

74. $4, -\frac{5}{2}, \frac{1}{2}(3 \pm \sqrt{5})$.

75. $\pm\frac{2}{3}, \frac{1}{4}(3 \pm \sqrt{-7})$.

76. $\frac{1}{2}, -\frac{5}{2}, \frac{1}{2}(1 \pm \sqrt{6})$.

77. $2, 3, -1, -1 \pm \sqrt{-2}$.

78. $-2, \pm\frac{1}{2}, \frac{1}{2}(-1 \pm \sqrt{-3})$.

79. $\frac{1}{3}, \pm\frac{3}{2}, -2 \pm \sqrt{5}$.

80. $1, -2, -\frac{1}{2}, 2 \pm \sqrt{3}$.

87. 1.41.

88. -1.52.

89. 2.05, .59.

90. 1.18, 2.87.

91. .16, 2.93, -2.09.

CHAPTER V

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1. 12.

2. -3.

3. 0.

11. $32x + y + 45 = 0$.

14. $(4\frac{2}{3}, 55\frac{2}{3})$.

15. $\tan^{-1}\frac{6}{17}$.

16. $10\frac{2}{3}$.

17. $(2, 9), (-2, 5)$.

18. $4x + y + 2 = 0$, $108x + 27y + 58 = 0$.

19. $(-1, -6), (\frac{1}{3}, -5\frac{23}{27})$.

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20. Increasing if $x > -2$; decreasing if $x < -2$.

21. Increasing if $x < 0$ or $x > \frac{4}{3}$; decreasing if $0 < x < \frac{4}{3}$.

22. Increasing if $x > -\sqrt[3]{2}$; decreasing if $x < -\sqrt[3]{2}$.

23. Increasing if $x > 1$ or $-1 < x < 0$; decreasing if $0 < x < 1$ or $x < -1$.

24. $(\frac{4}{3}, \frac{5}{3})$.

25. $(0, \frac{1}{4}), (\pm 2, -3\frac{3}{4})$.

26. Maximum value, $\frac{643}{243}$; minimum value, -3.

27. Maximum values, -12, -66; minimum values, -34, -88.

29. $\frac{1}{6}(a + b - \sqrt{a^2 + b^2 - ab})$.

30. Altitude, $\frac{h}{2}$; base, $\frac{b}{2}$.

31. Altitude, $\frac{2}{3}a\sqrt{3}$; radius of base, $\frac{1}{3}a\sqrt{6}$.

32. Altitude is one third the altitude of the cone.

33. $(1\frac{4}{5}, \frac{3}{5})$.

34. The one in which the radius of the circle from which it is cut is one fourth the perimeter of the sector.

35. Altitude is one half a side of the base.

37. (1) Height of the rectangle is equal to the radius of the semicircle.

(2) Semicircle of radius $\frac{a}{\pi}$.

38. $\frac{1}{2}a$.

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39. Length is twice the breadth.
40. $.05\ Cl^4$.
44. Upward if $x > 1$; downward if $x < 1$.
45. Upward if $x > 0$, downward if $x < 0$.
46. $(2, -\frac{1}{2})$.
48. $(0, -8)$.
49. $(1, -27)$.
51. 0.45, 1.80, -1.25.
52. -2.21.
53. 2.09.
54. 1.20, 3.13, -1.33.
55. 1.51, -1.18.
56. 2, 2, -3.
57. 1, 1, $-a \pm \sqrt{a^2 - a}$.
58. $b(b + 4a^3) = 0$.
59. $b^2(27a^4 - b) = 0$.
60. $b^3 - 27a^4 = 0$.
61. See Ex. 23, Chap. I.

CHAPTER VII

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5. $x^2 + y^2 \pm 2ax = 0$.
6. $x^2 + y^2 \pm 2ax \pm 2ay + a^2 = 0$.
7. $x^2 + y^2 + 3x - 2y = 0$.
8. $(-2, 5)$; $\sqrt{65}$.
9. $(-2, 3)$; $2\sqrt{3}$.
10. $(\frac{3}{2}, -1)$; $\frac{1}{6}\sqrt{141}$.
11. $(-\frac{1}{5}, \frac{2}{5})$; 0.
14. $x^2 + y^2 - 3x - 3y = 0$.
15. $x^2 + y^2 - 3x - 4 = 0$.
16. $x^2 + y^2 + 26x + 16y - 32 = 0$.
17. $x^2 + y^2 - 5x + 4y - 46 = 0$.
18. $x^2 + y^2 - 20x - 20y + 100 = 0$, $x^2 + y^2 - 4x - 4y + 4 = 0$.
19. $x^2 + y^2 - 12x - 12y + 36 = 0$, $25x^2 + 25y^2 + 60x - 60y + 36 = 0$.
20. $x^2 + y^2 + 2x + 10y + 1 = 0$, $x^2 + y^2 - 12x - 4y + 15 = 0$.
21. $2x^2 + 2y^2 + 6x + 3y - 10 = 0$.

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22. $x^2 + y^2 + 22x - 34y + 121 = 0$, $x^2 + y^2 - 2x - 10y + 1 = 0$.
23. $x^2 + y^2 - 10x - 28y + 217 = 0$.
24. $x^2 + y^2 + 22x - 44y - 20 = 0$, $x^2 + y^2 + 2x - 4y - 20 = 0$.
25. $4x^2 + 4y^2 \pm 7y - 36 = 0$.
26. $7x^2 + 16y^2 - 112 = 0$.
27. $9x^2 + 5y^2 - 45 = 0$.
28. $5x^2 + 9y^2 - 180 = 0$.
29. $\frac{1}{7}\sqrt{385}$, $\frac{1}{3}\sqrt{165}$.
30. $16x^2 + 25y^2 - 400 = 0$.
31. $5x^2 + 9y^2 - 80 = 0$.
32. $16x^2 + 25y^2 - 400 = 0$.
33. $196x^2 + 132y^2 - 14553 = 0$.
34. 4, 3; $\frac{1}{4}\sqrt{7}$; $(\pm\sqrt{7}, 0)$.
35. $\frac{1}{2}$; $3x^2 + 4y^2 - 3a^2 = 0$.
36. $5x^2 + 9y^2 - 405 = 0$.
37. $x^2 + 4y^2 - a^2 = 0$; $\frac{1}{2}\sqrt{3}$.
38. $\frac{1}{2}\sqrt{2}$.
39. $3x^2 + 5y^2 - 30 = 0$.
40. $\frac{1}{2}\sqrt{2}$, $\frac{1}{3}\sqrt{3}$; $(\pm\frac{1}{6}\sqrt{6}, 0)$; $2x \pm \sqrt{6} = 0$.

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41. $5x^2 - 4y^2 - 20 = 0$.
42. $3y^2 - x^2 - 12 = 0$.
43. $28x^2 - 36y^2 - 175 = 0$.
44. $24x^2 - 25y^2 - 384 = 0$.
46. $3x^2 - y^2 - 3a^2 = 0$.
47. $8x^2 - y^2 - 16 = 0$, $8y^2 - x^2 - 124 = 0$.
48. $x^2 - y^2 - 21 = 0$.
49. $x^2 - 8y^2 + 4 = 0$.
50. $25x^2 - 144y^2 - 3600 = 0$.
51. $25y^2 - 9x^2 - 16 = 0$.
52. $\cos^{-1}\left(\frac{2 - e^2}{e^2}\right)$.
53. $\frac{1}{2}\sqrt{5}$, $\frac{2}{5}\sqrt{5}$.

54. $\frac{1}{5}\sqrt{29}$; $(\pm\sqrt{29}, 0)$; $5x \pm 2y = 0$.

55. $52x^2 - 117y^2 - 576 = 0$.

57. $3x^2 - 4y^2 - 84 = 0$.

58. $\frac{1}{3}\sqrt{13}$; $(\pm\sqrt{13}, 0)$; $13x \pm 9\sqrt{13} = 0$; $2x \pm 3y = 0$.

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61. $p = 1\frac{1}{8}$.

62. $\frac{29}{2}\sqrt{2}$.

63. $38\frac{49}{72}$.

64. $x^2 + y^2 - 5px = 0$.

65. $x^2 + y^2 + 3x - 6y = 0$.

66. $7x - 3y + 2 = 0$.

67. $y \pm 5x = 0$.

68. $x^2 - 8y^2 - 6y + 9 = 0$.

69. $y^2 + 4y - 2x + 11 = 0$.

70. $91x^2 + 84y^2 - 24xy - 364x - 152y + 464 = 0$.

71. $y^2 - 10x + 25 = 0$.

72. $(y - 2)^2 \pm x^3 + x^2 = 0$.

73. $4x + 3y - 31 = 0$, $4x + 3y + 19 = 0$.

74. $5x + y - 5 = 0$, $x - 5y + 7 = 0$.

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80. Circle.

81. Circle.

82. Circle.

83. Concentric circle.

84. Straight line.

85. Straight line.

86. Circle.

87. Two straight lines.

90. Parabola.

91. Parabola.

93. Two parabolas.

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94. Circle.

95. Parabola.

96. Hyperbola.

97. Parabola.

98. Hyperbola.

99. Witch.

101. $8p\sqrt{3}$.

CHAPTER VIII

Page 175

1. $(1, 1)$, $(-2, 3)$.

2. $(0, 1)$.

4. $(0, 0)$, $(-1, -2)$.

5. $(1, 2\frac{1}{2})$.

6. $(1, 3)$, $(\frac{1}{2}, -1)$.

8. $(2 \pm \sqrt{5}, 2 \mp \sqrt{5})$.

9. $\frac{6}{13}\sqrt{13}$.

10. $2x - y + 2 = 0$.

11. -3.

12. $2x + 3y \pm 6\sqrt{2} = 0$.

13. $bx - ay + ab \pm ab\sqrt{2} = 0$.

15. $(0, 0)$, $(\frac{4}{5}, 1\frac{1}{5})$.

16. $(\pm 1\frac{3}{5}, \pm 5)$, $(\pm 1\frac{1}{5}, \pm 3\frac{3}{4})$.

17. $(\frac{3}{2}, 0)$, $(1, -1)$.

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18. $(2, 2)$.

19. $(0, 0)$, $(\pm\sqrt{2}, \pm\frac{1}{2}\sqrt{2})$.

20. $(2, 2)$, $(\frac{4}{5}, -\frac{2}{5})$.

21. $(0, 0)$, $\left(\frac{2am^2}{1 + m^2}, \frac{2am^3}{1 + m^2}\right)$.

22. $(2, 1)$.

23. $(0, 0)$, $(-1, 0)$.

24. $(1, \pm 2\sqrt{3})$, $(6, \pm 6\sqrt{2})$.

25. $\left(\pm\frac{1}{9}\sqrt{48 \mp 3\sqrt{13}}, \frac{-1 \pm \sqrt{13}}{6}\right)$.

26. $(b - a, \pm 2\sqrt{ab})$.

27. $(\pm 2a, a)$.

28. $(0, 0)$, $(-2a + 2a\sqrt{3}, \pm 2a\sqrt{2\sqrt{3} - 2})$.

29. $(0, 0)$, $(\frac{4}{3}a, \pm\frac{4}{3}a\sqrt{2})$.

30. $(2a, a)$.

31. $(\pm 2a, a)$.

32. $(\pm 2a, a)$.

33. $5x + 4y + 6 = 0$.

34. $2x + 5y - 13 = 0$.

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35. $3x + y - 1 = 0$.

36. $5x^2 + 5y^2 + 28x + 42y = 0$.

38. $3x^2 + 3y^2 + 13x + 13y - 4 = 0$.

39. $x^2 + 8y^2 - 9 = 0$.

CHAPTER IX

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1. $9x^2 + 14x + 5.$
2. $20(x+1)(3x^2 + 6x + 2).$
3. $\frac{2a}{(x+a)^2}.$
4. $-\frac{6x^2}{(x^3-1)^2}.$
5. $\frac{14(1-x)}{(3x^2 - 6x + 1)^2}.$
6. $-\frac{1}{(x+1)^2}.$
7. $\frac{1}{3}\left(\frac{4}{x^{\frac{1}{3}}} + \frac{3}{x^{\frac{2}{3}}} + \frac{2}{x^{\frac{4}{3}}} - \frac{10}{x^{\frac{5}{3}}}\right).$
8. $2(4x-3) + \frac{1}{x^3}(6-7x).$
9. $\frac{x+1}{2x\sqrt{x}}.$
10. $\frac{1}{3}\left(\frac{2}{\sqrt[3]{x}} - \frac{1}{\sqrt[3]{x^2}} - \frac{1}{x\sqrt[3]{x}} + \frac{2}{x\sqrt[3]{x^2}}\right).$
11. $2(3x^2 - 5x + 6)(6x - 5).$
12. $6x(x^2+1)^2.$
13. $\frac{8x+5}{2\sqrt{4x^2 + 5x - 6}}.$
14. $\frac{2x+1}{3\sqrt[3]{(x^2+x-1)^2}}.$
15. $-\frac{2x}{(x^2+1)^2}.$
16. $-\frac{5(3x^2+2x)}{(x^3+x^2+1)^2}.$
17. $-\frac{5x}{\sqrt[4]{(x^2+1)^5}}.$
18. $5(2x-1)(2x-3)(x+1)^2.$
19. $(3x-5)(12x^2 - 55x + 31).$
20. $\frac{2x^2+x+1}{\sqrt{x^2+1}}.$
21. $\frac{3x^4 - 10x^3 + 6x^2 + x - 2}{(x^2-4x+3)^{\frac{1}{2}}(x^3+1)^{\frac{1}{3}}}.$
22. $\frac{1}{2}\left(\frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{x-1}}\right).$
23. $1 + \frac{x}{\sqrt{x^2+1}}.$
24. $2x\left(\frac{1}{\sqrt[3]{(3x^2+1)^2}} - \frac{1}{\sqrt[3]{(3x^2+1)^4}}\right).$
25. $\frac{1}{(x+1)\sqrt{x^2-1}}.$
26. $-\frac{x+1}{(x-1)^2\sqrt{x^2+1}}.$
27. $\frac{x-x^2}{(x^2+1)^{\frac{1}{2}}(x^3+1)^{\frac{4}{3}}}.$
28. $\frac{2x^2+1}{\sqrt{1+x^2}} - 2x.$
29. $\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}.$
30. $\frac{x - \sqrt{a^2+x^2}}{a^2\sqrt{a^2+x^2}}.$
31. $\frac{4x(2y^2-x^2)}{y(3y-8x^2)}.$
32. $\frac{5x^4 - 3x^2}{5y^4 - 1}.$
33. $\frac{3[x^2y^4 + (x-y)^2]}{3(x-y)^2 - 4x^3y^3}.$
34. $\frac{3x^2 - 5x^4}{4y^3 - 1}.$
35. $-\frac{x + \sqrt{x^2-y^2}}{y}.$
36. $\frac{y}{x - y(x-y)^2}.$
37. $-\frac{5x}{2y};\ -\frac{25}{2y^3}.$
38. $-\frac{x^6}{y^6};\ -\frac{6a^7x^5}{y^{13}}.$
39. $-\frac{b^2x}{a^2y};\ -\frac{b^4}{a^2y^3}.$
40. $\frac{a^2}{3y^2 - a^2};\ \frac{6a^4y}{(a^2-3y^2)^3}.$
41. $\frac{3x^2}{3y^2+1};\ \frac{6x(1-3y^2)}{(1+3y^2)^3}.$
42. $\frac{2(3x^2-2y)}{3y^2+4x};\ \frac{128xy}{(3y^2+4x)^3}.$

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43. tan., $3x + y + 5 = 0$;
nor., $x - 3y + 5 = 0$:
tan., $3x - y + 9 = 0$;
nor., $x + 3y - 7 = 0$.
44. $x - 6y + 17 = 0$, $6x + y - 9 = 0$.
45. $x + 2y - 2 = 0$.
46. $4x - 3y - 1 = 0$.
47. $x + 2y - 1 = 0$.
48. $(3y_1^2 - x_1)y - y_1x - x_1y_1 - 3a = 0$.
49. $x - 3y_1^2y + 2x_1 - 3 = 0$.
50. $2y_1y - 3x_1^2x + x_1^3 = 0$.
51. $(2 - x_1^3)x + x_1^3y - 3x_1 = 0$.
52. $x_1^{-\frac{1}{2}}x + y_1^{-\frac{1}{2}}y = a^{\frac{1}{2}}$.
53. $x_1^{-\frac{1}{3}}x + y_1^{-\frac{1}{3}}y = a^{\frac{2}{3}}$.
54. $5y + \sqrt{10}\,x - 5\sqrt{5} = 0$,
$10y - 5\sqrt{10}\,x + 4\sqrt{5} = 0$.
55. tan., $4x - y - 6 = 0$;
nor., $x + 4y - 10 = 0$:
tan., $4x - y + 6 = 0$;
nor., $x + 4y + 10 = 0$.
56. $\dfrac{y}{\dfrac{dy}{dx}}$, $y\dfrac{dy}{dx}$.
57. $\dfrac{y}{\dfrac{dy}{dx}}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$, $x\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$.
61. $(-0.57, 2.08)$.

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62. $(\pm \frac{1}{2}a\sqrt{2}, \pm \frac{1}{2}b\sqrt{2})$.
63. $\left(\pm \dfrac{a^2}{\sqrt{a^2 + b^2}}, \pm \dfrac{b^2}{\sqrt{a^2 + b^2}}\right)$.
73. $\sqrt{p(p + x_1)}$.
76. $a^2y_1x - b^2x_1y = 0$; $\dfrac{a^2b^2}{\sqrt{b^4x_1^2 + a^4y_1^2}}$.
77. $\dfrac{\pi}{4}$, $\tan^{-1}\frac{1}{3}$.

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78. 0, $\tan^{-1}\frac{9}{2}$.
79. $\dfrac{\pi}{2}$.
80. $\tan^{-1}\sqrt{3}$.
81. $\tan^{-1}\frac{5}{3}\sqrt{3}$.
85. $\dfrac{\pi}{2}$, $\tan^{-1}\frac{3}{4}$.
86. $\tan^{-1}3$.
88. $\dfrac{\pi}{2}$, $\tan^{-1}\sqrt{2}$.
89. $\tan^{-1}\frac{9}{13}$.
90. $\tan^{-1}\frac{3}{4}$.
91. 0, $\tan^{-1}3\sqrt{3}$.
92. 0, $\dfrac{\pi}{2}$, $\tan^{-1}\frac{1}{2}\sqrt[3]{2}$.
93. The length is twice the breadth.
94. He walks 2.86 mi.

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95. 8 rd., 12 rd.
96. Cross section is a square.
97. Of equal length.
98. 4 mi. from nearest point on bank to A.
99. $FD = (\sqrt{2} - 1)AB$.
103. Area of ellipse is $\dfrac{\pi}{2}$ area of rectangle.
104. Central angle of sector is $\frac{2}{3}\pi\sqrt{6}$.
105. Breadth a, depth $a\sqrt{3}$.
106. Breadth, $\frac{2}{3}a\sqrt{3}$; depth, $\frac{2}{3}a\sqrt{6}$.

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107. Velocity in still water $\frac{3}{2}a$ mi. per hour.
108. Radius of base equals altitude.
109. Altitude is $\frac{4}{3}$ radius of sphere.
110. $a - \dfrac{bm}{\sqrt{n^2 - m^2}}$ mi. on land, $\dfrac{bn}{\sqrt{n^2 - m^2}}$ mi. in water.

111. Altitude is $\frac{1}{2}\sqrt{2}$ radius of semicircle.

112. Altitude is $\frac{2}{3}$ distance between vertex of parabola and bounding straight line.

113. (b, a).

114. $(\pm \frac{1}{3}\sqrt{3}, \frac{3}{4})$.

115. maximum value when $x = \frac{1}{5}$, minimum value when $x = 1$; points of inflection when $x = -1$ or $\frac{1 \pm \sqrt{6}}{5}$.

116. minimum ordinate, $x = \frac{a}{\sqrt{3}}$; maximum ordinate, $x = -\frac{a}{\sqrt{3}}$; points of inflection, $(0, 0)$, $(\pm a, 0)$.

117. $(\pm \frac{2}{3} a\sqrt{3}, \frac{3}{2}a)$.

119. $\left(\pm \frac{a}{6}\sqrt{27 - 3\sqrt{33}}, \pm \frac{a}{12}\sqrt{5\sqrt{33} - 21}\right)$.

120. $\left(\pm \frac{a}{2}\sqrt{2}, \pm \frac{b}{4}\sqrt{2}\right)$.

121. $(1, 3)$, $(5, -5)$.

122. Stationary when $t = 0$, 4, or 8; maximum velocity when $t = 1.69$; moving backward when $4 < t < 8$.

123. 20, $10\sqrt{5}$; (100, 20).

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124. When $x = \frac{1}{2}$; parallel.

125. Velocity of top : velocity of bottom = distance of bottom from wall : distance of top from ground.

126. 54π ft. per hour.

127. 389 ft. per minute.

128. 41.9 ft. per second.

129. 15 ft. per second.

130. 0.2 in. per second.

131. 17.9 mi. per hour.

132. $3y = x^3 + 9x - 19$.

133. $3x - 2xy - 2 = 0$.

134. $3y = 2x^{\frac{3}{2}} + 11$.

136. $\frac{1}{y} - 1 = k(1 - x)$.

137. 90,000, $677\frac{1}{12}$.

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138. 108. **140.** $\frac{hk}{n+1}$. **141.** $\frac{1}{6}a^2$. **142.** $85\frac{1}{3}$. **143.** $\frac{16}{3}a^2$. **144.** $10\frac{2}{3}$. **145.** $8\frac{8}{15}$.

CHAPTER X.

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1. $(-1, 5)$, $(-7, 7)$, $(2, -5)$.

2. $x^2 + 4y^2 - 4 = 0$.

3. $y^3 - 15y^2 + 3x^2 + 75y - 6x - 106 = 0$.

4. $b^2x^2 + a^2y^2 - 2a^2by = 0$.

5. $b^2x^2 + a^2y^2 - 2ab^2x = 0$.

6. $b^2x^2 - a^2y^2 - 2ab^2x = 0$.

7. $y^2 = -\frac{x(a + x)^2}{2a + x}$.

8. $y^2 = \frac{(x - a)^2(2a - x)}{x}$.

9. $y = -\frac{2ax^2}{x^2 + 4a^2}$.

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10. $y = \frac{4a^3 - ax^2}{x^2 + 4a^2}$.

11. $y^2 = -\frac{(2a + x)^3}{x}$

12. $y^2 = \frac{(a + x)^3}{a - x}$.

13. $y^2 = 4px + 4p^2$.

14. $y^2 = 4px - 4p^2$.

15. $y^2 = 8x$.

16. $2x^2 + 3y^2 - 6 = 0$.

17. $ab - c$.

18. $\dfrac{c^2}{4a} + \dfrac{d^2}{4b} - e$.

20. $x^2 + 9y^2 + 6x - 36y + 36 = 0$.

21. $196x^2 + 900y^2 + 784x + 5400y + 8875 = 0$.

22. $x^2 - 4y^2 - 2x - 16y - 19 = 0$.

23. $2x^2 - 6y^2 - 8x + 36y - 47 = 0$.

24. $y^2 + 4y - 8x + 28 = 0$.

25. $x^2 - 8x + 16y - 64 = 0$.

26. $3x^2 + 4y^2 - 12x - 24y - 27 = 0$.

27. $8x^2 + 9y^2 - 16x - 64 = 0$.

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28. $5x^2 - 4y^2 + 10x - 16y - 31 = 0$.

29. $y^2 + 4y - x = 0$.

30. $\frac{1}{3}\sqrt{5}$; $(-1, 2)$; $(2, 2)$, $(-4, 2)$; $(-1 \pm \sqrt{5}, 2)$; $5x + 5 \pm 9\sqrt{5} = 0$.

31. $\frac{1}{5}\sqrt{10}$; $(-3, 2)$; $(-3 \pm \sqrt{5}, 2)$; $(-3 \pm \sqrt{2}, 2)$; $2x + 6 \pm 5\sqrt{2} = 0$.

32. $\frac{1}{2}\sqrt{13}$; $(3, -4)$; $(5, -4)$, $(1, -4)$; $(3 \pm \sqrt{13}, -4)$; $13x - 39 \pm 4\sqrt{13} = 0$; $3x - 2y - 17 = 0$, $3x + 2y - 1 = 0$.

33. $\frac{1}{2}\sqrt{10}$; $(-1, 2)$; $(-1 \pm \sqrt{2}, 2)$; $(-1 \pm \sqrt{5}, 2)$; $5x + 5 \pm 2\sqrt{5} = 0$; $\sqrt{3}(x + 1) \pm \sqrt{2}(y - 2) = 0$.

34. $(-\frac{1}{3}, \frac{1}{4})$; $(-\frac{1}{3}, -\frac{3}{8})$; $3x + 1 = 0$; $8y - 7 = 0$.

35. $(-2, -3)$; $(-\frac{3}{4}, -3)$; $y + 3 = 0$; $4x + 13 = 0$.

36. $(\frac{1}{2}\sqrt{3}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2}\sqrt{3})$, $\left(\dfrac{1 + \sqrt{3}}{2}, \dfrac{1 - \sqrt{3}}{2}\right)$.

37. $x^2 - 4y^2 - 4 = 0$.

38. $y^2 = \dfrac{x^2(3a\sqrt{2} - 2x)}{3a\sqrt{2} + 6x}$.

40. $x^2 + 14y^2 - 14 = 0$.

42. $xy = -18$, or $xy = 18$.

43. $17x^2 + 7y^2 - 2 = 0$, or $7x^2 + 17y^2 - 2 = 0$.

44. $x^2 \pm y = 0$, or $y^2 \pm x = 0$.

45. $2x^2 - y^2 - 1 = 0$.

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46. $5x^2 + 8y^2 = 40$.

47. $4xy = 7$.

48. $\left(\dfrac{x + y}{x - y}\right)^2 = \dfrac{a\sqrt{2} - x + y}{a\sqrt{2} + x - y}$.

49. $5x^2 - 6y^2 = 30$.

CHAPTER XI

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1. Hyperbola; center, $(-7, 2)$; slopes of axes, 2 and $-\frac{1}{2}$.
2. Parabola; slope of axis, $\frac{1}{3}$; vertex, $(\frac{22}{35}, -4\frac{16}{35})$.
3. No curve.
4. Hyperbola; center, $(-1, 0)$; slopes of axes, 1 and -1.
5. Hyperbola; center, $(2, -\frac{3}{2})$; slopes of axes, 1 and -1.
6. The line $x - y + 1 = 0$ taken twice.
7. Ellipse; center, $(-1, 2)$; slopes of axes, 1 and -1.
8. A pair of straight lines intersecting at $(3, -2)$, and having the slopes $-1 \pm \frac{1}{2}\sqrt{6}$.
9. A pair of straight lines intersecting at $(3, -2)$, and having the slopes $\frac{2}{3}$ and $-\frac{3}{2}$.
10. Parabola; slope of axis, 1; vertex, $(-\frac{47}{32}, -\frac{19}{32})$.

11. The parallel straight lines $x + 3y - 5 = 0$, $x + 3y - 1 = 0$.
12. Ellipse; center, $(2, -1)$; slopes of axes, $\frac{4}{3}$ and $-\frac{3}{4}$.
13. Point, $(0, 2)$.

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20. $\tan^{-1}\dfrac{2\sqrt{H^2 - AB}}{A + B}$.
23. $2x^2 + 3xy + y^2 + 12x - 13y - 50 = 0$.
24. $xy - 2y^2 - 2x + 4y = 0$.
25. $x^2 - xy + y^2 - a^2 = 0$.
26. $6x^2 + 5xy + y^2 - 29x - 13y + 30 = 0$.
27. $9x^2 - 12xy + 4y^2 - 117x + 78y + 380 = 0$,
or $49x^2 - 56xy + 16y^2 - 621x + 354y + 1964 = 0$.
28. $\sqrt{1 - \tan^2\frac{\beta}{2}}$ if $\tan\frac{\beta}{2} < 1$, $\dfrac{\sqrt{\tan^2\frac{\beta}{2} - 1}}{\tan\frac{\beta}{2}}$ if $\tan\frac{\beta}{2} > 1$ (β the angle between the lines).

CHAPTER XII

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1. $5x - y + 3 = 0$; $(0, 3)$, $(-1, -2)$.
2. $x + y - 3 = 0$; $(0, 3)$, $(1, 2)$.
3. $x - y + 1 = 0$; $(-\frac{1}{4}, \frac{3}{4})$.
4. $y - 2x + 5 = 0$; $(1, -3)$, $(2, -1)$.
5. $(0, 3)$.
6. $(\frac{4}{9}, -\frac{1}{3})$.
7. $(2, 3)$.
8. $(1, -2)$.

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9. $3x - 2y = 0$, $x - y + 1 = 0$.
10. $x = 0$, $x - y + 1 = 0$.
11. $3x + y - 1 = 0$.
12. $2x - y = 0$, $x + 2y - 10 = 0$.
13. $x + 2y - 2 = 0$, $x - 3y - 2 = 0$.
14. $x - 3y - 2 = 0$, $2x - y + 1 = 0$.
21. $\dfrac{b^2}{a^2}$.
24. At infinity.
25. $bex - ay = 0$, $bx + aey = 0$.

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30. $-e$.
31. $a^2 = 2b^2$.

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47. $\left(ae, \pm\dfrac{b^2}{a}\right)$; $\tan^{-1}(\pm e)$.

CHAPTER XIII

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21. $a\cos 2ax$.
22. $a[\sec^2(ax + b)\operatorname{ctn}(ax + c) - \tan(ax + b)\csc^2(ax + c)]$.
23. $-8\csc^2 4x$.
24. $\dfrac{2(2\operatorname{ctn} 2x - 1)}{\csc 2x}$.
25. $\dfrac{3\sec 3x(\tan 3x - 1)}{(\tan 3x + 1)^2}$.
26. $\sec^2 x$.
27. $mn\sec^m nx\csc^n mx(\tan nx - \operatorname{ctn} mx)$.
28. $2\sec^2 2x(2\tan 2x + 1)$.
29. $-2\csc 2x(2\csc^2 4x + \operatorname{ctn} 4x\operatorname{ctn} 2x)$.
30. $\cos(x\cos x)(\cos x - x\sin x)$.

31. $5 \sin^2 x \cos^3 x$.

32. $8 \sec^5 x - 3 \sec x$.

33. $\frac{1}{\sqrt{2}} \cos x$.

34. $\frac{x}{\sqrt{1-x^2}} \sin \sqrt{1-x^2}$.

35. $\sec x \tan x$.

36. $\frac{1}{1+x^2}$.

37. $\frac{1}{\sqrt{1-x^2}}$.

38. $-\frac{1}{(1+x)\sqrt{x}}$.

39. $\frac{1}{\sqrt{a^2-x^2}}$.

40. $-\frac{1}{a+x}\sqrt{\frac{a}{x}}$.

41. $\frac{1}{(x+1)\sqrt{x}}$.

42. $-\frac{2}{(x^2+2x)\sqrt{x^2+2x-1}}$.

43. 0.

44. $\frac{1}{a+b\cos x}$.

45. $\frac{2}{\sqrt{1-x^2}}$.

46. $\frac{1}{(2x-1)\sqrt{x^2-x}}$.

47. $\frac{1}{2x^2+2x+1}$.

48. $-\frac{4x}{x^4+1}$.

49. $2(x+1)e^{x^2+2x}$.

50. $\frac{2a^2x}{x^4-a^4}$.

51. $e^{\sqrt{1-x^2}} a^{\frac{x}{\sqrt{1-x^2}}}\left(\frac{\log a}{(1-x^2)^{\frac{3}{2}}} - \frac{x}{\sqrt{1-x^2}}\right)$.

52. $\frac{1}{\sqrt{x^2+x}}$.

53. $e^x \sqrt[x]{a}\left(1 - \frac{\log a}{x^2}\right)$.

54. $a^{\tan x} \log a \cdot \sec^2 x$.

55. $2x(1 + 2\log x)$.

56. $\log a \cdot \sec^2 x(\sec^2 x + 2\tan^2 x)\, a^{\tan x \sec^2 x}$.

57. $2\sec 2x(\sec 2x + \log a \cdot \tan^2 2x)\, a^{\sec^2 x}$.

58. $[2(a+x)\sin mx + m\cos mx]\, e^{(a+x)^2}$.

59. -2.

60. $\left(1 + \frac{1}{2}\sqrt{\frac{a}{x}}\right) e^{\tan^{-1}\sqrt{\frac{x}{a}}}$.

61. $\frac{2}{e^{2x}+e^{-2x}}$.

62. $\frac{8}{3+5\sin 2x}$.

63. $\frac{2}{e^x+e^{-x}}$.

64. $e^{x\cos a}\cos(a + x\sin a)$.

65. $\frac{4x}{\sin(2x^2+2a^2)}$.

66. $\cos^{-1} x$.

67. $\sec ax$.

68. $(a^2+1)\, e^{a \sin^{-1} x}$.

69. $\operatorname{ctn}^{-1} x$.

70. $\frac{\cos^{-1} x}{\sqrt{(1-x^2)^3}}$.

71. $e^{ax} \sin mx$.

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72. $2x \operatorname{ctn}^{-1}\frac{a}{x}$.

73. $\frac{\sin^{-1} x}{x^2}$.

74. $\frac{1}{\sqrt{x^2-a^2}} - \frac{1}{\sqrt{a^2-x^2}}$.

75. $\frac{1}{x}\sqrt{\frac{x+a}{x-a}}$.

76. $\frac{(x-1)\log(x-1)}{(x^2-2x)^{\frac{3}{2}}}$.

77. $\frac{1}{a+b\cos x}$.

78. $4\csc(4x+2)[1 - \operatorname{ctn}(4x+2)]$.

79. $-\frac{x}{\sqrt{2ax-x^2}}$.

80. $2\sqrt{a^2+x^2}$.

81. $\dfrac{\sec^{-1} 2\sqrt{x}}{\sqrt{x(4x-1)^3}}$.

82. $\sqrt{2ax - x^2}$.

83. $\dfrac{2 - \sqrt{x^2-1}}{2x(x^2-1)}$.

84. $-\dfrac{1}{(e^x+2)\sqrt{e^{2x}+2e^x-1}} - \dfrac{(e^{2x}+e^x)\log(e^x+2)}{\sqrt{(e^{2x}+2e^x-1)^3}}$.

85. $x \tan^{-1}\sqrt{1-x^2}$.

86. $\dfrac{\sqrt{1+x^2}}{2+x^2}$.

87. $\dfrac{y}{2\sqrt{x}}(\sec^2\sqrt{x}\log\sin\sqrt{x}+1)$.

88. $\dfrac{y}{x^2}(x \operatorname{ctn} x - \log\sin x)$.

89. $yx^x\left[\dfrac{1}{x} + \log x + (\log x)^2\right]$.

90. $ye^x\left(\dfrac{1}{x} + \log x\right)$.

91. $yx^x(1+\log x)$.

92. $y\left[\dfrac{\tan^{-1}(a+x)}{a+x} + \dfrac{\log(a+x)}{1+(a+x)^2}\right]$.

93. $\dfrac{y\left(\tan xy - \dfrac{1}{x}\right)}{\log x - x\tan xy}$.

94. $\dfrac{2x + \dfrac{y}{1+x^2}}{2y - \tan^{-1}x}$.

95. $\dfrac{y\cos x + \sin(x-y)}{\sin(x-y) - \sin x}$.

96. $\dfrac{my}{(ny+1)x}$.

97. $\dfrac{e^y\sin x + e^x\sin y}{e^y\cos x - e^x\cos y}$.

98. $\dfrac{y(1-\cos x) - \cos y}{\sin x - x(1+\sin y)}$.

99. $\dfrac{x\sin y - y}{x\log x - x^2\cos y}$.

100. $\dfrac{y(1-x^2-y^2)}{x(1+x^2+y^2)}$.

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101. $\dfrac{x+y}{x-y}$, $\dfrac{2(x^2+y^2)}{(x-y)^3}$, $\dfrac{4(x+2y)(x^2+y^2)}{(x-y)^5}$.

102. $-e^{x-y}$, $-e^{x-y}(1+e^{x-y})$, $-e^{x-y}(1+e^{x-y})(1+2e^{x-y})$.

103. $\dfrac{1-y^2}{x^2-1}$, $\dfrac{2(x+y)(y^2-1)}{(x^2-1)^2}$, $\dfrac{6(x+y)^2(1-y^2)}{(x^2-1)^3}$.

104. $\dfrac{x+y-1}{x+y+1}$, $\dfrac{4(x+y)}{(x+y+1)^3}$, $\dfrac{8(x+y)(1-2x-2y)}{(x+y+1)^5}$.

105. $\dfrac{y-x}{x(1-\log x)}$, $\dfrac{y(1+\log x)-2x}{[x(1-\log x)]^2}$, $\dfrac{2y[1+\log x+(\log x)^2]-3x(1+\log x)}{[x(1-\log x)]^3}$.

106. $x = \cos^{-1}\dfrac{-1\pm\sqrt{33}}{8}$.

107. 1 or 2.

108. $\tan^{-1}2\sqrt{2}$.

109. $\tan^{-1}\left(2\tan\dfrac{a}{2}\sec\dfrac{a}{2}\right)$.

110. $\tan^{-1}\frac{1}{3}$, $\tan^{-1}3$.

112. $x = k\pi$, $x = 2k\pi \pm \dfrac{\pi}{3}$.

113. $(ae^{\frac{1}{2}}, \frac{3}{2}ae^{-\frac{3}{2}})$.

114. $\left(\pm\dfrac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$.

116. Maxima when $x = (2k+1)\dfrac{\pi}{2}$, minima when $x = k\pi$; points of inflection when $x = (2k+1)\dfrac{\pi}{4}$.

117. Maxima when $x = (k+\frac{7}{4})\pi$, minima when $x = (k+\frac{3}{4})\pi$; points of inflection when $x = k\pi$.

118. Maximum when $x = n$, minimum when $x = 0$ and n is an even integer; points of inflection when $x = n \pm \sqrt{n}$.

119. Circle.

120. $2\sqrt{(s-3)(5-s)},\ 4(4-s)$.

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122. a.

123. $\dfrac{ab}{2\sqrt{b^2 - a^2t^2}}$.

126. $-b\sin\theta - \dfrac{b^2 \sin\theta\cos\theta}{\sqrt{a^2 - b^2\sin^2\theta}}$ times velocity of AB, where $\theta = CAB$.

127. $\dfrac{500}{\sqrt{10{,}000 - x^2}}$, where x is the distance from the center.

128. $\dfrac{500 \sin\alpha}{\sqrt{10{,}000 - x^2\sin^2\alpha}}$, where x is the distance from the center.

129. 15 sq. ft.; 9.03 sq. ft. per second.

130. $\dfrac{\pi}{2} - \dfrac{\alpha}{2}$ each.

131. 23.7.

132. At an angle $\tan^{-1}k$ with the ground.

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133. 8 in.

134. $1 : \sqrt{2}$.

135. $5\sqrt{5}$ ft.

136. 2.

137. $\sqrt{2} - 1$.

138. 2.

139. $e - 1$.

140. $\dfrac{a^2}{e}(e^2 - 1)$.

141. $\log 2$.

142. $x = 2a$.

143. $\dfrac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) + c$.

144. $y - 3 = k \log \dfrac{x}{2}$.

145. $\log \dfrac{y}{b} = k(x - a)$.

146. $s = ce^{kt}$.

147. $k\pi + (-1)^k .6661$.

148. $2k\pi \pm .567,\ 2k\pi \pm 2.206$.

149. $k\pi$.

150. $k\pi,\ (2k+1)\dfrac{\pi}{4},\ 2k\pi \pm \dfrac{\pi}{3}$.

151. $k\pi \pm \dfrac{\pi}{4},\ k\pi \pm \dfrac{\pi}{3}$.

152. 4.4934.

153. 4.275.

154. 0.199.

155. -0.7035.

156. 1.857, 4.54.

CHAPTER XIV

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2. tan., $x - ty + pt^2 = 0$; nor., $tx + y - 2pt - pt^3 = 0$.

3. $x = \dfrac{4p}{t^2},\ y = \dfrac{4p}{t}$.

4. tan., $t^2x - 2ty + 4p = 0$; nor., $2t^2x + t^3y - 4pt^2 - 8p = 0$.

5. $x = \pm \dfrac{ab}{\sqrt{b^2 + a^2t^2}},\ y = \pm \dfrac{abt}{\sqrt{b^2 + a^2t^2}}$.

6. $x = \dfrac{a(b^2 - m^2a^2)}{b^2 + m^2a^2},\ y = \dfrac{2ab^2m}{b^2 + m^2a^2}$.

7. $x = 2a\sin^2\phi,\ y = \dfrac{2a\sin^3\phi}{\cos\phi}$.

10. $(1 + 3t^2)x - 2t^3y - 2a = 0$.

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11. $y^2 = \dfrac{2x^2(a-x)}{2x-a}$.

12. $x^3 + y^3 - 3axy = 0$.

13. $y^2(ax - a^2) = x^2(a^2 + k^2 - ax)$.

14. $3y^2 - 4xy + 2y - 1 = 0$.

15. $y = \dfrac{(x+1)(x-1)^2}{3x^2+1}$.

16. $(3y - x)^2 = 2\sqrt{x^2 - y^2}$.

17. $9y = (x-y)^2 - 6(x-y)$.

18. $(x^2 + y^2)(ax + by) = cxy$.

19. $\sqrt{x^2+y^2} = a\sqrt{2}\, e^{\frac{\pi}{4} - \tan^{-1}\frac{y}{x}}$.

20. $x = a\cos^2\dfrac{\theta}{2}$, $y = \dfrac{a}{2}\left(\sin\theta + \tan\dfrac{\theta}{2}\right)$; $y = \dfrac{2x+a}{2}\sqrt{\dfrac{a-x}{x}}$.

21. $x = (a - c\tan\theta)\sin^2\theta$, $y = (a\cos\theta - c\sin\theta)\sin\theta$; $y(x^2+y^2) = x(ay - cx)$.

22. $x = \dfrac{1}{a}(a^2 + k^2\cos^2\theta)$, $y = \dfrac{1}{a}(a^2\tan\theta + k^2\sin\theta\cos\theta)$;
$a(x-a)(x^2+y^2) = k^2x^2$.

23. $x = a\tan\theta$, $y = a\cos 2\theta$; $y = \dfrac{a(a^2 - x^2)}{a^2 + x^2}$.

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24. $x = a\sin\theta(\cos\theta + \sec\theta)$, $y = a\cos\theta(\cos\theta + \sec\theta)$;
$y^2(x^2+y^2)^2 = a^2(x^2 + 2y^2)^2$.

29. $bx\cos\phi + ay\sin\phi - ab = 0$.

30. $\dfrac{\pi}{4}$.

31. $\dfrac{\pi}{4} + \dfrac{\beta}{2}$.

32. $\dfrac{1}{2}\sin^{-1}\dfrac{gb}{v_0^2}$.

33. $\tan^{-1}\dfrac{v_0^2 \pm \sqrt{v_0^4 + 2hgv_0^2 - g^2b^2}}{gb}$.

34. $x = a\cos\theta - l\sqrt{b^2 - a^2\sin^2\theta}$, $y = (1-l)\,a\sin\theta$, where the center of the driving wheel is the origin, a the length of the radius of the driving wheel, b the length of the connecting rod, and lb the distance from the wheel to the point.

35. Straight line.

37. $2a\omega\sin\dfrac{\phi}{2}$; $\omega\sqrt{a^2 - 2ah\cos\phi + h^2}$, where ω is the constant angular velocity.

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38. $2\dfrac{a}{b}(a+b)\,\omega\sin\dfrac{\theta}{2}$, $\dfrac{a+b}{b}\omega\sqrt{a^2 - 2ah\cos\theta + h^2}$.

40. $a\theta\omega$, where a is the radius of the circle, $a\theta$ the distance through which the point of the string in contact with the wheel has moved along the rim of the wheel, and ω the constant angle of velocity.

41. $x = a(\cos 2\theta + \sin 2\theta)$, $y = a(1 + \sin 2\theta - \cos 2\theta)$;
$2\sqrt{2}\,a\omega$, $2a(\cos 2\theta - \sin 2\theta)\,\omega$, $2a(\sin 2\theta + \cos 2\theta)\,\omega$.

42. $y^2 = \dfrac{x^2(3a+x)}{a-x}$.

43. Circle.

45. $y^2(x^2+y^2)^2 = 4a^2(x^2+2y^2)^2$.

47. $y = x - a$.

48. The witch.

49. $x = a(\cos\theta + \theta\sin\theta)$,
$y = a(\sin\theta - \theta\cos\theta)$.

50. $x = a(1+m^2)$, $y = ma(1+m^2)$;
$ay^2 = x^2(x-a)$.

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53. Ellipse.
54. Hyperbola.
55. Straight line.
56. Concentric circle.
57. $x + 2p = 0, \; py^2 = x^3$.
59. Concentric ellipse.
61. Ellipse.

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65. Parabola.
67. Concentric circle.
70. Concentric circle.
71. $y = x \operatorname{ctn} \dfrac{\pi x}{2a}$.
72. $x = a(\phi + \sin\phi)$, $y = a(-1 + \cos\phi)$.
74. $8a$.
75. $\dfrac{8a(a+b)}{b}$.

CHAPTER XV

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26. $\left(1.0353\,a, \dfrac{\pi}{4}\right)$.
27. $\left(\dfrac{a}{2}\sqrt{3}, \dfrac{\pi}{3}\right), \; \left(\dfrac{a}{2}\sqrt{3}, \dfrac{2\pi}{3}\right)$.
28. $(0,0), \left(\pm\dfrac{a}{\sqrt[4]{2}}, \dfrac{\pi}{4}\right), \left(\pm\dfrac{a}{\sqrt[4]{2}}, \dfrac{3\pi}{4}\right)$.
29. $(0, 0), \; \left(a, \pm\dfrac{\pi}{4}\right)$.
30. Circle.
31. $r = a\cos^2\theta$.
32. $r = \dfrac{a\cos 2\theta}{\cos^3\theta}$.

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40. $r^3\sin^2\theta\cos\theta + (2a + r\cos\theta)^3 = 0$.
41. $r\cos\theta = a\cos 2\theta, \; (r^2 + a^2)\cos\theta + 2ar = 0$.
49. $(x^2 + y^2)^3 - 4a^2x^2y^2 = 0$.
50. $(x^2 + y^2 + ax)^2 - a^2(x^2 + y^2) = 0$.
51. $(x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 - b^4 = 0$.
52. $(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2)$.
53. $(x - a)^2(x^2 + y^2) = b^2x^2$.
54. $\log(x^2 + y^2) = 2a\tan^{-1}\dfrac{y}{x}$.
56. 25,000,000.
57. 1,200,000, or 4,800,000.

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59. $r = \dfrac{2c}{1 - \cos\theta}$.
61. Straight line.
62. Circle.
63. Circle.
64. $r = \dfrac{2b^2\cos\theta}{a(1 - e^2\cos^2\theta)}$.
65. $r = \dfrac{ce^2\cos\theta}{1 - e^2\cos^2\theta}$.
66. $2r = \dfrac{ce}{1 - e\cos\theta}$.
72. $\omega f'(\theta), \; \omega\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}$.
73. $\dfrac{\omega f'(\theta)}{\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}}, \; \dfrac{\omega}{\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}}$.

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74. $\omega\sqrt{a^2 + b^2 + 2\,ab\cos\theta}$.

75. Spiral of Archimedes.

76. $2\,a^2$.

77. $\frac{1}{3}a^2$.

78. $\frac{1}{6}\pi^3a^2$.

79. $\frac{1}{4\,a}(e^{2\pi a} - 1)$.

80. $\frac{1}{2}\pi a^2$.

81. $\frac{1}{8}a^2(4 - \pi)$.

82. $\frac{1}{2}\pi(a^2 + 2\,b^2)$.

83. $8\,a$.

CHAPTER XVI

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1. $\frac{y^2}{a}$.

2. $\frac{ax^{\frac{1}{2}}(8\,a - 3\,x)^{\frac{3}{2}}}{3(2\,a - x)^2}$.

3. $3(axy)^{\frac{1}{3}}$.

4. $\frac{1}{2}a,\ a$.

5. $\frac{a^2}{3\,b}$.

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6. $\frac{3}{2}a^2,\ \frac{3}{2}a^2$.

7. e^{-x_1}.

8. $0,\ \frac{1}{2}$.

10. $\frac{a(5 - 4\cos\theta)^{\frac{3}{2}}}{9 - 6\cos\theta}$.

11. $\frac{3}{4}a\sin^2\frac{\theta}{3}$; greatest value, $\frac{3}{4}a$; least value, 0.

12. $\frac{2\,a^2}{3\,r}$.

13. $\frac{8}{3}\sin\frac{t}{2}$.

14. $y^2 = \frac{4}{27\,p}(x - 2\,p)^3$.

15. $\frac{a\sqrt{a^2 - x^2}}{x}$

18. $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2\,a^{\frac{2}{3}}$

21. $x^2 + y^2 - y = 0$.

23. Minimum curvature when $x = \frac{1}{\sqrt{2}}$.

24. Maximum curvature when $x = (2\,k + 1)\frac{\pi}{2}$; minimum curvature when $x = k\pi$.

25. Maximum curvature at ends of major axis; minimum curvature at ends of minor axis.

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27. $\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0$.

28. $(2\,k + 1)\frac{\pi}{2}$.

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