THE UNIVERSITY OF AKRON
The Department of Mathematical Sciences

Article: Continuous Functions

Directory

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Continuous Functions

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1. Introduction

Prerequisite: Limits.

In the article Limits, we had some discussion of the notion of continuous function. In that article, we introduced the definition and classified certain collections as having the property of being continuous. In this article we expand and extend the ideas introduced in Limits.

Continuity is a property of a function. In analysis, of which *Calculus* is a part, certain facts (called *theorems*) are true when the functions being dealt with are continuous, while other facts are not true (or may not be true) when dealing with noncontinuous functions; therefore, it is incumbent on us to be able to understand continuity and to be able to identify functions that are continuous — in this way, we can utilize the facts from analysis *with confidence and authority*.

2. What is a Continuous Function?

Let's begin with some definitions — the love of a mathematician's life.

Definition 2.1. Let f be a function and $a \in \mathrm{Dom}(f)$. We say that f is *continuous at* $x = a$ provided

$$\lim_{x \to a} f(x) = f(a), \tag{1}$$

or, more precisely, for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$x \in \mathrm{Dom}(f) \text{ and } |x - a| < \delta \text{ implies } |f(x) - f(a)| < \epsilon. \tag{2}$$

Corollary Definition 2.2. Let f be a function defined on a set A. We say that f is a *continuous on* A if f is continuous *at each point* $a \in A$. If $A = \mathbb{R}$, sometimes we say that f is continuous *everywhere*.

Definition Notes: Here are some thoughts on the definition, given in the form of bulleted paragraphs.

■ The phrase "f is continuous at $x = a$" is designated to those functions for which the evaluation of the limit is done simply by evaluating the function, f at the limiting point, a. This is the content of (1). *Not every limit can be evalulated this way.* In a sense, the continuous functions easiest kind of function to deal with (yet very important). Of course, we must *first prove* a given function is continuous before evaluating limits so easily.

■ There is a difference between *continuous at* $x = a$, and just *continuous*: the former is a *local property*, and the latter is a *global property*. A given function can be continuous *at* one point but not *at* another. For example, the function

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 17 & \text{if } x = 0 \end{cases}$$

Now this is a rather artificial example but it serves the point. Look at the limit of f at $x = 2$ and $x = 0$:

$$\lim_{x \to 2} f(x) = 4 = f(2)$$

but,

$$\lim_{x \to 0} f(x) = 0 \neq f(0)$$

This function is *continuous at* $x = 2$, but *not continuous*, or (*discontinuous*) *at* $x = 0$. Thus, the property of being continuous is relative to the particular point in its domain (hence is a *local property*). On the other hand, if a function is continuous at each point in its domain, we refer to that function as being *continuous* – at every point understood.

Theorem 2.3. (Continuity of Linear Functons) *Let $f(x) = mx + b$ be a linear function, and $a \in \mathbb{R}$. Then f is continuous at $x = a$. Thus, f is a continuous function.*

Proof. Let $\epsilon > 0$, choose $\delta = \dfrac{\epsilon}{1 + |m|}$. Note that the chosen δ is positive (which is a requirement of the definition). Now, suppose

$$|x - a| < \delta,$$

then,

$$\begin{aligned}
|f(x) - f(a)| &= |(mx + b) - (ma + b)| \\
&= |mx - ma| = |m(x - a)| \\
&= |m||x - a| \\
&\leq |m|\frac{\epsilon}{1 + |m|} \\
&= \frac{|m|}{1 + |m|}\epsilon \\
&< \epsilon. \qquad (3)
\end{aligned}$$

Note that in line (3), we used the fact that

$$\frac{|m|}{1+|m|} < 1.$$

We have shown that, for any $\epsilon > 0$, there is a $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. This is precisely what is needed to prove that f is continuous at $x = a$. $\square$

Note: Compare this choice of δ with the choice of δ for the same proof in the article on Limits. Why did I take $\delta = \epsilon/(1 + |m|)$?

Definition 2.1 is a low-level tool for showing that a given function is continuous. Working with a function at this definitional level can be quite difficult and challenging. In the next section we develop some high-level tools for arguing that a function is continuous; however, as with a higher level computer language, certain things can be done more easily at the high level, but other things cannot be done at all — as a result, you must maintain our low level hooks. You must always keep in back of your mind the definitions — it is the definitions that ultimately give meaning to any concept.

⇓ For Those Who Want to Know More. This rest of this section is devoted to some esoteric details. In the discussion below, I assume that your have already For Those Who Want to Know More.

In Definition 2.1, the two conditions (1) and (2) are not, in general, equivalent.

Condition (2) is, in fact, the standard definition of continuity at $x = a$. Now, if a is an accumulation point, then (1) is equivalent to (2).

EXERCISE 2.1. Define a function f by

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 5 & x = 2 \end{cases}$$

Utilizing condition (2) as the definition of continuity, show that f is continuous at $x = 2$.

If a point $a \in \mathrm{Dom}(f)$ is *not* an accumulation point of the domain, then by definition (condition (2)) f is continuous at $x = a$. If $a \in \mathrm{Dom}(f)$ *is* an accumulation point of the domain, then continuity is a nontrivial concept. The law for continuity is now governed by the equation

$$\lim_{x \to a} f(x) = f(a).$$

In a standard course, continuity of a function is only discussed for points in the domain of a function that are accumulation points of the domain. Usually, the function in question is defined for an interval of numbers. In this case, all points of the interval are accumulation points of the interval.

Summary: The standard definition of continuity is given by condition (2). When $a \in \mathrm{Dom}(f)$ is an accumulaton point, then continuity is equivalent to (1):

$$\lim_{x \to a} f(x) = f(a).$$

When $a \in \mathrm{Dom}(f)$ and is not an accumulation point of $\mathrm{Dom}(f)$, then by definition (i.e. by (2)) and as EXERCISE 2.1 illustrates, f is continuous at $x = a$.

3. The Algebra of Continuous Functions

We have already seen the *Algebra of Functions*, and the *Algebra of Limits*. We now present the same type theorem for continuous functions. This theorem would represent a "high-level tool" for classifying functions as continuous.

Theorem 3.1. (Local Version) *Let f and g be functions and let a and c be number. Suppose f and g are continuous at $x = a$. Then*

(1) *the function $f + g$ is continuous at $x = a$;*
(2) *the function cf is continuous at $x = a$;*
(3) *the function fg is continuous at $x = a$;*
(4) *the function $\dfrac{f}{g}$ is continuous at $x = a$, provided, $g(a) \neq 0$.*

Proof.

Corollary 3.2. (Global Version) *Let f and g be functions that are continuous on a common domain A, and let c be a constant. Then each of the functions are continuous on the domain A: $f+g$, cf, and fg. In the case of the quotient function, f/g is continuous on the domain $B = \{ x \in A \mid g(x) \neq 0 \}$.*

Proof.

One can think of Theorem 3.1 as a building block result. There are many building block results in mathematics; you have seen several of them already, even though I didn't tell you. Theorem 3.1 enables us to take functions that are known to be continuous, and use them to *build* more complex continuous functions.

What functions do we *know* are continuous? Answer: Linear Functions are continuous everywhere. This was very easy to prove. (See the proof of Theorem 2.3.) A linear function is a function of the form:

$f(x) = mx + b$. Take the simplest cases are $m = 1$ and $b = 0$, and $m = 0$ and $b = 1$. Thus we know that

$$f(x) = x \quad \text{and} \qquad g(x) = 1 \tag{1}$$

is continuous on their domains: $\mathrm{Dom}(f) = \mathrm{Dom}(g) = \mathbb{R}$.

A polynomial is constructed by taking the little functions in (1) and multiplying them together (with themselves and with each other), multiplying them by constants, and adding them together. For example, polynomial $p(x) = 3x^4 - 7x^3 + 9x - 10$ is nothing more than

$$p(x) = 3f(x)^4 - 7f(x)^3 + 9f(x) - 10g(x)$$

or, without the arugments

$$p = 3f^4 - 7f^3 + 9f - 10g.$$

You can see that p is the sum and product of the functions f and g. Since f and g are continuous on $\mathbb{R}$, and in light of Corollary 3.2, we conclude that p is continuous on $\mathbb{R}$ as well.

Do you see the building block of effect. The blocks are the functions $f(x) = x$ and $g(x) = 1$, and Corollary 3.2 supplies the "mortar" to stack the blocks. Thus, we have argued, informally, ...

Continuity of Polynomials
Let p be a polynomial function, then p is continuous on $\mathbb{R}$.

Now we have new building blocks! Polynomials. Let $N(x)$ and $D(x)$ be two polynomials, and define

$$r(x) = \frac{N(x)}{D(x)}.$$

You will recall that r is a rational function. The domain of r is given by

$$\operatorname{Dom}(r) = \{\, x \in \mathbb{R} \mid D(x) \neq 0 \,\}.$$

Now applying Corollary 3.2 again, we conclude that r is continuous on its domain $\operatorname{Dom}(r)$. Let's highlight this observation.

Continuity of Rational Functions
Let r be a rational function, then r is continuous on $\mathrm{Dom}(r)$.

4. Continuity of the Trigonometric Functions

In this section we develop the continuity properties of the trigonometric functions. In the article on *Limits*, we discussed some basic limit problems of the trigonometric functions. In particular, recall, we have

$$\lim_{x \to 0} \sin(x) = 0 \qquad \lim_{x \to 0} \cos(x) = 1. \tag{1}$$

The pair of equations (1) implies the *sine* and *cosine* functions are everywhere continuous!

Theorem 4.1. *The functions* $\sin(x)$ *and* $\cos(x)$ *are continuous on* $\mathbb{R}$*.*

Proof. We want to prove, for any $a \in \mathbb{R}$, that

$$\lim_{x \to a} \sin(x) = \sin(a) \qquad \lim_{x \to a} \cos(x) = \cos(a).$$

I will only prove the first equation.

Recall the *additive formula* for the sine function:

$$\sin(x + h) = \sin(x)\cos(h) + \cos(x)\sin(h).$$

This identity and (1) is all we need — oh yes, there is one more profound fact we need: $x = a + (x - a)$.

$$
\begin{aligned}
\lim_{x \to a} \sin(x) &= \lim_{x \to a} \sin(a + (x - a)) \\
&= \lim_{x \to a} (\sin(a) \cos(x - a) + \cos(a) \sin(x - a)) \\
&= \sin(a) \lim_{x \to a} \cos(x - a) + \cos(a) \lim_{x \to a} \sin(x - a) \\
&= \sin(a)(1) + \cos(a)(0) \qquad \text{by (1)} \\
&= \sin(a)
\end{aligned}
$$

We have shown,

$$\lim_{x \to a} \sin(x) = \sin(a).$$

This is what we wanted to prove.

EXERCISE 4.1. Prove, for any $a \in \mathbb{R}$, $\lim_{x \to a} \cos(x) = \cos(a)$.

Now, having accomplished the establishment of the continuity of the $\sin(x)$ and $\cos(x)$, we now turn to the other four trig functions. But these are trivial because of the *Algebra of Continuous Functions*. Recall,

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \sec(x) = \frac{1}{\cos(x)}$$
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \qquad \csc(x) = \frac{1}{\sin(x)}$$

The $\tan(x)$ function, for example, is the ratio of two continuous functions, $\sin(x)$ and $\cos(x)$. Therefore, $\tan(x)$ is continuous at all points where the denominator is different from zero. Thus, $\tan(x)$ is continuous at all $x \neq \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \pm 7\pi/2, \ldots$.

EXERCISE 4.2. For each of the trig functions $\tan(x)$, $\sec(x)$, $\cot(x)$, and $\csc(x)$, determine all x for which these functions are *discontinuous*,

5. Continuity of Composite Functions

When we discussed *The Limit of Composite Functions*, we introduced the definitive theorem on how the limit process interacts with composition. In this section, that theorem is illustrated within the context of continuous functions.

Basically, the content of the The Composite Limit Theorem is that *continuous functions preserve limits*. Let's restate this theorem using slightly different terminology.

Theorem 5.1. (The Composite Limit Theorem) *Let f and g be functions that are compatible for composition, let $a \in \mathbb{R}$. Suppose,*

(1) $\lim_{x \to a} g(x)$ *exists, let* $b = \lim_{x \to a} g(x)$*;*
(2) f *is continuous at* $b \in \operatorname{Dom}(f)$*.*

Then

$$\lim_{x \to a} f(g(x)), \ \textit{exists}$$

and,

$$\lim_{x \to a} f(g(x)) = f(b),$$

or,

$$\boxed{\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)).} \tag{1}$$

Proof.

Corollary 5.2. (Composition of Continuous Functions) *Let f and g be functions that are compatible for composition, let $a \in \mathbb{R}$. Suppose, g is continuous at $x = a$, and f is continuous at $g(a)$. Then $f \circ g$ is continuous at $x = a$; that is,*

$$\lim_{x \to a} f(g(x)) = f(g(a))$$

Proof.

6. Discontinuities

Let f be a function and $a \in \mathrm{Dom}(f)$. We say that f is discontinuous at $x = a$ if f is *not* continuous at $x = a$. Functions can be discontinuous in a number of different ways. In this section is briefly survey these different ways and introduce some terminology.

6.1. Types of Discontinuities

There are four types of discontinuities discussed in this section: removable, jump, infinite, and oscillatory. Each of these is addressed in turn.

Removable Discontinuities. We say that the function f has a *removable discontinuity* at $x = a$ provided,

$$\lim_{x \to a} f(x) \qquad \text{exists} \tag{1}$$

but

$$\lim_{x \to a} f(x) \neq f(a).$$

Thus, the limit must exist but not be equal to what it's supposed to be equal if it were continuous.

As the name suggests, these kind of discontinuities can be removed. See the section Remove them Discontinuities

A useful criteria for testing for a removable discontinuity is to bring in one-sided limits: f has a removable discontinuity at $x = a$ provided,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a). \tag{2}$$

That is, the left-hand and the right-hand limits exist and are equal to each other (hence, the two-sided limit exists), but does not equal to $f(a)$ — what it's supposed to equal if it is going to be continuous.

In the article on *Limits*, we say many examples of functions having removable discontinuities.

Example 6.1. Define the function f by

$$f(x) = \begin{cases} \dfrac{x^3 + 2x^2 - x - 2}{(x-1)(x+2)} & x \neq -2, 1 \\ 0 & \text{otherwise} \end{cases}$$

Argue that f has a removable discontinuity at $x = 1$.

Exercise 6.1. For the function f in Example 6.1, show that f has a removable discontinuity at $x = -2$.

Exercise 6.2. Define the function f by

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x \neq 2 \\ 0 & x = 2 \end{cases}$$

Argue that f has a removable discontinuity at $x = 2$.

EXAMPLE 6.2. Define the function

$$f(x) = \begin{cases} x+1 & x < 1 \\ 6 & x = 1 \\ x^2+1 & x > 1 \end{cases}$$

Argue that f has a removable discontinuity at $x = 1$.

Jump Discontinuities. We say that the function f has a *jump discontinuity* at $x = a$ provided,

$$\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x). \tag{3}$$

That is, the left-hand and the right-hand limits exist but are *not* equal to each other (hence, the two-sided limit exists), does not equal exist. The amount of the jump is

$$\mathrm{jmp} = |\lim_{x \to a^+} f(x) - \lim_{x \to a^-} f(x)|.$$

Examples of functions having jump discontinuities are easy to find. Here is a few.

EXAMPLE 6.3. **Jump Discontinuity** Show that the function

$$f(x) = \begin{cases} x+1 & x \leq 3 \\ x^3 & x < 3 \end{cases}$$

has a jump discontinuity at $x = 3$, and calculate the amount of the jump at that point.

EXAMPLE 6.4. **Greatest Integer Function.** Define the greatest integer function by

$$\lfloor x \rfloor = \begin{array}{l} \text{the greatest integer less than} \\ \text{or equal to } x. \end{array}$$

Show that $\lfloor x \rfloor$ has a jump discontinuity at each integer value of x.

Infinite Discontinuities. Let f be a function and $a \in \mathbb{R}$ such that f has a vertical asymptote at $x = a$; i.e. either

$$\lim_{x \to a^-} f(x) = \pm\infty$$

or,

$$\lim_{x \to a^+} f(x) = \pm\infty$$

The point $a \in \mathbb{R}$ may or may not belong to $\mathrm{Dom}(f)$.

We have seen many examples of infinite discontinuities.

Example 6.5. Consider the following two functions.

$$f(x) = \frac{1}{x} \quad x \neq 0, \quad \text{and} \quad g(x) = \begin{cases} \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Argue that each has an infinite discontinuity at $x = 0$.

Oscillatory Discontinuity. This type of discontinuity is not easy, at this level of play, to give a rigorous definition. Perhaps, definition by example.

Example 6.6. **Oscillatory Discontinuity.** The function

$$f(x) = \sin(\frac{1}{x}), \qquad x \neq 0.$$

has an oscillatory discontinuity at $x = 0$.

6.2. Remove them Discontinuities!

Functions with removable discontinuities can be *redefined* in such a way that the redefined function is now continuous.

The procedure is obvious.

EXAMPLE 6.7. Consider the function:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & x \neq 2 \\ 0 & x = 2 \end{cases}$$

It was seen that in EXERCISE 6.2 that f has a removable discontinuity at $x = 2$. REMOVE IT!

EXAMPLE 6.8. Consider the function

$$f(x) = \begin{cases} \dfrac{\sin(2x)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The function f has a removable discontinuity at $x = 0$. Remove it.

7. One-Sided Continuity

I will not devote very many electronic pages to this topic. It has been discussed rather extensively already. See the sections: *Left-Hand Limits*, *Right-Hand Limits*, and *Two-Sided and One-Sided Limits Related*.

When trying to argue that a function of a more pathological type is or is not continuous, it is sometimes useful to consider one-sided limits. The pivotal result from the article *Two-Sided and One-Sided Limits Related* is reproduced below — modified to reflect the topic of continuity.

Theorem 7.1. *Let f be a function and $a \in \mathrm{Dom}(f)$. Then, f is continuous at $x = a$ if and only if*

$$\lim_{x \to a^-} f(x) = f(a) \tag{1}$$

and

$$\lim_{x \to a^+} f(x) = f(a) \tag{2}$$

EXAMPLE 7.1. Define a function f as follows:

$$f(x) = \begin{cases} x^3 & x \leq 2 \\ 3x^2 - 2x & x > 2 \end{cases}$$

Problem: Prove that f is continuous at $x = 2$.

Definition 7.2. Let f be a function and $a \in \mathrm{Dom}(f)$. We say the f is *left-continuous at* $x = a$, provided

$$\lim_{x \to a^-} f(x) = f(a). \tag{3}$$

We say that f is *right-continuous at* $x = a$, provided

$$\lim_{x \to a^+} f(x) = f(a). \tag{4}$$

Theorem 7.1 can be recast utilizing the terminology just introduced:

Let f be a function and $a \in \mathrm{Dom}(f)$. Then, f is continuous at $x = a$ if and only if f is left-continuous and right-continuous at $x = a$

8. The Intermediate Value Theorem

After you have graphed several continuous functions, *and* graphed several noncontinuous functions, you might eventually allow your mind to dwell upon the differences in the appearance of the two types of functions. You might think: "Continuous functions have non-broken (or connected) graphs, discontinuous functions have graphs that are broken (or disconnected) in certain locations."

One can use this interpretation of a continuous function to reason intuitively in the physical realm. Suppose at a certain time, designated as time $t = 0$, a *bird* is seen sitting on a light pole, known to be 10 feet high. Later, 30 seconds later, the same bird is observed sitting atop an electrical pole, known to be 18 feet high. Apparently the bird flew from the light pole to the electrical pole during the time interval $[\,0, 30\,]$, where time t is measured is seconds.

At any time $t \in [\,0, 30\,]$, the bird has some height, y, off the ground. This establishes a functional relationship between t and y. Let

$$y = f(t), \qquad 0 \le t \le 30,$$

symbolically represent y, the height of bird from ground at time t, as a function of t.

Now, think about the graph of the function $y = f(t)$. What does it look like? The bird can fly from the light pole to the electric pole in any of infinitely many ways; we don't, therefore, know the exact graph.

What very general observations can we make about the flight of the bird? Or, what general observation can we make about the graph of f?

We have two points on the graph on the graph of f

$$L(\,0, 10\,) \qquad E(\,30, 18\,), \tag{1}$$

but what else do we know or willing to assume?

Can we assume that the graph of $y = f(t)$ is a continuous function? No matter what flight path the bird filed with the flight controller, its graph must surely be unbroken, connected. Because the bird exists in the real world, we would discount as impossible for the bird to suddenly and instantaneously, at some time t_0, change its altitude.

EXERCISE 8.1. Express mathematically the concept of sudden and instantaneous change in altitude at time $t = t_0$.

We feel, therefore, safe in assuming that the function $y = f(t)$ is a continuous function.

What else can we say about the function $y = f(t)$. At time $t = 0$, the bird has altitude $y = 10$ feet, and at time $t = 30$, the bird has altitude $y = 18$. When the bird leaves the light pole, it either flies horizontally, flies upward, or flies downward. Whatever it does, by the time 30 seconds are up, it must have an altitude of 18 feet. What else can be said about the flight of the bird?

Because of the infinitely many options available to the bird, there is not a whole lot more we can say except for the following obvious statement.

During the time interval $[\,0, 30\,]$, *the bird must surely attain all heights between* 10 *feet and* 18 *feet.*

That is, the bird must go from a height of 10 feet to an height of 18 feet (off the ground), in the process, the bird must surely attain all altitudes in between, for otherwise, the graph of f would be broken.

This rather long-winded discussion, ending in a statement of the obvious, was a way of introducing the *Intermediate Value Theorem*. This theorem is a formal statement that all continuous function have the property of the bird: they must take on all values between two altitudes on their graph.

Theorem 8.1. (The Intermediate Value Theorem) *Let f be a function defined on a closed interval $[\,a, b\,]$. Suppose M is a number strictly between $f(a)$ and $f(b)$. Then there is a number $c \in (\,a, b\,)$, such that $f(c) = M$.*

Proof.

Exercise 8.2. Let $A(a, b)$ and $B(c, d)$ be two points in the first quadrant such that A and B are *different* distances from the origin, $O(0, 0)$. Assume $c < a$. Form the triangle $\triangle OAB$. (Draw a diagram.) Let α be the angle formed by the positive x-axis and the line segment OA and let β be the angle formed by the positive x-axis and the line segment OB. The condition $c < a$ implies $\alpha < \beta$. (Verify?) Let θ be a symbol representing an angle such that $\alpha \leq \theta \leq \beta$. For any θ, draw a line from the origin making an angle of θ with the positive x-axis; the point at which this line intersects the line segment AB will be referred to as $P(\theta)$ and the distance $P(\theta)$ is away from the origin will be designated by $r(\theta)$. **The Problem.** Argue, using the Intermediate Value Theorem, that there is an angle θ_0, $\alpha < \theta_0 < \beta$, such that the area of the triangle $\triangle OAB$ is equal to the area of the sector of circle whose central angle is $\beta - \alpha$ and radius is $r(\theta_0)$. (Did you ever hear the phrase, "A picture is worth a thousand words?")

8.1. Root Hunting

One of the important applications to Theorem 8.1 is to *root hunting*! Let $f(x)$ be a given function. A *root* (or, a *zero*) of f is any number $a \in \mathrm{Dom}(f)$ such that $f(a) = 0$; or, in other words, a root of f is any solution to the equation $f(x) = 0$. Graphically, roots of f are the x-intercepts: The points at which the graph of the function crosses the x-axis.

In mathematics, it is not unusual to want to solve the equation:

$$f(x) = 0, \tag{2}$$

that is, we want to find the roots or zeros of the function f. It would be a waste of time to try to solve (2) if there are no solutions; therefore, the first step towards solving an equation is to determine whether there are any solutions at all! The **Intermediate Value Theorem** can be very useful in this regard.

The following example illustrates the basic reasoning pattern.

EXAMPLE 8.1. Consider the equation,

$$3x^3 - 4x^2 + 8x - 1 = 0 \tag{3}$$

Argue that there is at least one solution to this equation.

How to Argue Existence of Roots:
To argue that the equation $f(x) = 0$ has solutions, where f is a continuous function, you need to find two numbers a, $b \in \text{Dom}(f)$ such that $f(a)f(b) < 0$. (Note: the latter condition is a fancy way of saying that $f(a)$ and $f(b)$ have opposite signs — one positive and one negative, get it?)

The last point can be rephrased slightly, to argue that f has a root, we need to find in interval within the domain of f such that the value of f at the two endpoints of this interval are opposite (one positive and one negative).

EXERCISE 8.3. (Graphing Calculator Exercise) In EXAMPLE 8.1, we argued that there is a solution to the equation

$$3x^3 - 4x^2 + 8x - 1 = 0$$

between 0 and 1. Using your graphing calculator, approximate this root accurate to six decimal places.

EXERCISE 8.4. The equation $x^3 - 3x^2 + 1 = 0$ has three roots. Use the **Intermediate Value Theorem** to argue that this equation has *three* roots. Do this by finding three distinct intervals such that the endpoints of each interval have different signs.

9. Presentation of the Theory

In this section I have accumulated all theorems whose proofs had hypertext links. If you wish a truly deeper understanding of the concepts of this article, Click Here for a stand alone presentation of theory.

Solutions to Exercises

2.1. Let $\epsilon > 0$, choose $\delta = 1/2$. Now suppose $x \in \mathrm{Dom}(f)$, and $|x - 2| < \delta = 1/2$. We want to argue that $|f(x) - f(2)| < \epsilon$. Indeed, if $x \in \mathrm{Dom}(f)$, and $|x - 2| < 1/2$, then it *must* be true that $x = 2$; thus $|f(x) - f(2)| = |f(2) - f(2)| = 0 < \epsilon$. Thus, f is continuous at $x = a$.

Think about the sudden deduction of the previous paragraph. Why does $x \in \mathrm{Dom}(f)$ and $|x - 2| < 1/2$ imply $x = 2$?

In this example, $x = 2$ is not an accumulation point of $\mathrm{Dom}(f)$. The condition (2) does not require $0 < |x - a| < \delta$, but only requires $|x - a| < \delta$. Because we were not required to satisfy $x \neq a$ that enabled us to prove that this function was continuous at $x = 2$.

Exercise 2.1. ■

4.1. Mimicry is the highest form of flattery. Study my proof of Theorem 4.1 and use the identity

$$\cos(x + h) = \cos(x)\cos(h) - \sin(x)\sin(h).$$

Good luck. Aloha. Exercise 4.1. ■

4.2. I'll start you off, $\tan(x)$ is continuous at all points except,

$$x \neq \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \pm 7\pi/2, \dots,$$

or, in a more tersely,

$$x \neq (2n-1)\frac{\pi}{2} \qquad n \in \mathbb{Z}.$$

Construct a nice table showing all the trig functions and listing their continuity properties. I'll check on your work later. Exercise 4.2. ■

6.1. Factor the numerator, for $x \neq -2,\ 1$,

$$\begin{aligned} \frac{x^3 + 2x^2 - x - 2}{(x-1)(x+2)} &= \frac{(x-1)(x+1)(x+2)}{(x-1)(x+2)} \\ &= x + 1 \end{aligned}$$

Thus,

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} (x+1) = -1 \neq 0 = f(-2),$$

by (1), f has a removable discontinuity at $x = -2$.

Exercise 6.1. ■

6.2. The function f has the same definition on either side of $x = 2$, so there is no need to use one-sided limits.

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2} (x+2) \\ &= 4\end{aligned}$$

Thus we have shown that

$$\lim_{x\to 2} f(x) = 4 \neq 0 = f(2)$$

Exercise 6.2. ■

8.1. This sudden and instantaneous change in altitude at time $t = t_0$ can be expressed mathematically as either

$$f(t_0) \neq \lim_{t \to t_0^-} f(t) \tag{A-1}$$

or

$$f(t_0) \neq \lim_{t \to t_0^+} f(t). \tag{A-2}$$

That is, the bird's altitude at time t_0 is different form the altitude determined by its past history (A-1), or, the bird's altitude is different from the altitude determined by its future flight (A-2).

Exercise 8.1. ■

8.2. The statement of the problem is longer than its solution. The function $r(\theta)$ is a continuous function of θ.

Let $\alpha \leq \theta \leq \beta$. Sweep a circular arc of radius $r(\theta)$ across the triangle $\triangle OAB$. The area of that triangle is

$$A(\theta) = \frac{1}{2}(\theta - \alpha)(r(\theta))^2.$$

(Why?)

The function $A(\theta)$, $\alpha \leq \theta \leq \beta$, is a continuous function of θ. (Why?)

I am assuming A and B are different distances from the origin; for definiteness, assume A is closer to the origin than B. This assumption this implies

$$A(\alpha) \leq \mathcal{A} \leq A(\beta)$$

where $\mathcal{A}$ is the area of the triangle $\triangle OAB$. (Why?)

I conclude, from the INTERMEDIATE VALUE THEOREM that there is a θ_0, $\alpha < \theta_0 < \beta$ such that

$$A(\theta_0) = \mathcal{A}.$$

(Why?)

Questions. Can you justify each step of this solution. Can you answer the "Why's?"

Exercise 8.2. ■

8.3. I've approximated the root on my desk supported computer, and the approximation is

$$\boxed{x_{\text{root}} = 0.1329574493}$$

and furthermore, this is the only root!

Exercise 8.3. ■

8.4. Just go root hunting. Start calculating values of f until sign changes are observed. This exercise is easy if you have a graphing calculator — simply graph the function and you can see where the roots are.

One choice of calculations yields

$$f(-2) = -19 \quad f(0) = 1 \quad f(2) = -3 \quad f(4) = 17,$$

from this, we can conclude that there is a root inside the interval $(-2, 0)$, another inside the interval $(0, 2)$, and a third root within the interval $(2, 4)$¿

Exercise 8.4. ■

Solutions to Examples

6.1. Factor the numerator, for $x \neq -1, 1$,

$$\begin{aligned}\frac{x^3 + 2x^2 - x - 2}{(x-1)(x+2)} &= \frac{(x-1)(x+1)(x+2)}{(x-1)(x+2)} \\ &= x + 1\end{aligned}$$

Thus,

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} (x+1) = 2 \neq 0 = f(1),$$

by (1), f has a removable discontinuity at $x = 1$.

Example 6.1. ■

6.2. First note that $f(1) = 2$. Now, calculate, in the usual way, the one-sided limits.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 1) = 2)$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 1) = 2$$

We have shown that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 2 \neq 6 = f(1).$$

This means, by (2), that f has a removable discontinuity at $x = 1$.

Example 6.2. ■

6.3. Calculate the one-sided limits.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x+1) = 3$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x^3 = 27$$

The left-hand limit does not equal the right-hand limit, therefore from (3), this function has a jump discontinuity of $x = 3$. The amount of the jump is

$$jmp = |\lim_{x\to 3^+} f(x) - \lim_{x\to 3^-} f(x)| = |27 - 3| = 24.$$

That's quite a jump!

This function is otherwise, for $x \neq 3$, continuous.

Example 6.3. ■

6.4. Let $n \in \mathbb{N}$ ($n = 1, 2, 3, 4, \dots$). Then $\lfloor n \rfloor = n$. Now if $n-1 < x < n$, then $\lfloor x \rfloor = n - 1$, since $n-1$ would be the largest integer less than or equal to x. This implies, at least in my mind, that

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1. \tag{S-1}$$

Now look at the $\lfloor x \rfloor$ for x a little larger than n. Let $n < x < n+1$, then $\lfloor x \rfloor = n$, since n is the greatest integer less then or equal to x. This implies,

$$\lim_{x \to n^+} \lfloor x \rfloor = n. \tag{S-2}$$

We have shown that the left-hand limit, (S-1), is not equal to the right-hand limit, (S-2). Thus, the function $\lfloor x \rfloor$ has a jump discontinuity at $x = n$. The amount of the jump is 1.

We made the argument for n a positive integer. We leave the other case of $n \le 0$ to you.

Notes: Otherwise, i.e. for $n \notin \mathbb{Z}$, the function $\lfloor x \rfloor$ is continuous. ($\mathbb{Z} =$ the set of all integers.)

Example 6.4. ■

6.5. The major point I wanted to make in offering up the two functions

$$f(x) = \frac{1}{x} \quad x \neq 0, \quad \text{and} \quad g(x) = \begin{cases} \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

was it doesn't really matter whether the function is defined at a particular point or not. What determines the infinite discontinuity is how the function behaves as you approach the point of interest.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{1}{x} = +\infty$$

This is enough to argue that f and g both have an infinite discontinuity at $x = 0$.

For completeness, we include,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Example 6.5. ■

6.6. An examination of the graph of the function is descriptive of the phrase "oscillatory discontinuity." As the graph approach the y-axis, the function oscillates more and more frequently. Example 6.6. ■

6.7. We say in Exercise 6.2 that

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x+2) = 4.$$

The old function is

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 0 & x = 2 \end{cases}$$

Define a new function g as

$$g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$$

Notice we have changed the value at $x = 2$ from 0 to 4. The only difference between f and g is their values at $x = 2$.

Now show that g is continuous at $x = 2$. Indeed,

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} (x+2) = 4 = g(2).$$

Example Notes: With this redefinition, the function g can be written more concisely as $g(x) = x + 2$. Thus the function $g(x) = x + 2$ is the continuous redefinition of the function f. In this case, we have the analytical explanation of cancelling a common factor between numerator and denominator in an algebraic expression. But the technique of removing the discontinuity extends beyond merely explaining the cancellation of common factors.

Example 6.7. ■

6.8. In an example in the section in *Limits* giving a General Discussion of Limits, we looked at the limit

$$\lim_{x \to 0} \frac{\sin(2x)}{x}$$

numerically and showed

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = 2.$$

Now if this result is also true analytically (which it is), we can say that f has a removable discontinuity at $x = 0$. Define a new function g by

$$g(x) = \begin{cases} \dfrac{\sin(2x)}{x} & x \neq 0 \\ 2 & x = 0 \end{cases}$$

Now, we have

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{\sin(2x)}{x} = 2 = g(0).$$

This means that g is continuous at $x = 0$.

Example Notes: Now the removal of the discontinuity was a nontrivial event in this example. Unlike EXAMPLE 6.7, there was no common factors to divide out.

Example 6.8. ■

7.1. This is a job for Theorem 7.1. First note that $f(2) = 8$. We reason as follows.

Calculate the left-hand limit.

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} x^3 \\ &= 2^3 = 8 \\ &= f(2) \end{aligned} \tag{S-3}$$

Calculate the right-hand limit.

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} 3x^2 - 2x \\ &= 3(2^2) - 2(2) = 8 \\ &= f(2) \end{aligned} \tag{S-4}$$

The left-hand limit equals the right-hand limit and they, in turn, equal the value $f(2)$; therefore, by Theorem 7.1, f is continuous at $x = 2$.

Example 7.1. ■

8.1. The left-hand side of (3) can be thought of as the defining formula of a function. Define, therefore,

$$f(x) = 3x^3 - 4x^2 + 8x - 1.$$

The equation (3) is now of the form: $f(x) = 0$. Noting that f is a continuous function, we see that solutions (3), if any exist, represent roots of a *continuous function*.

By examining the values of this function, we come across the following two calculations:

$$f(0) = -1 \text{ and } f(1) = 6.$$

That is, f is negative at $x = 0$ and f is positive at $x = 1$. By the **Intermediate Value Theorem**, Theorem 8.1 the function f must attain all altitudes in between -1 and 6. Here's the critical observation: The number 0 is between -1 and 6! Somewhere between 0 and 1, f *must* take on a value of 0.

Here's the details of the assertions of the previous paragraph. Think of $a = 0$ and $b = 1$, and $M = 0$ in Theorem 8.1. Observe that

$$-1 = f(0) < M < f(1) = 6.$$

Now, by the **Intermediate Value Theorem**, Theorem 8.1, there is some number $c \in (\,0, 1\,)$ such that $f(c) = M = 0$.

This means that there is a root, c, of f somewhere between 0 and 1; that is, there is at least one solution to the equation (3) and furthermore, this root lives between 0 and 1. This is all the **Intermediate Value Theorem** tells us. It tells us that there is a root, but does not compute it for us. The theorem only says "there exists a number $c \in (\,a, b\,)$ . . . ," no more.

Example Notes: Now that we know that there is a root between 0 and 1, we can gather our resources and set off to find it. (Left to the reader, good luck!)

Example 8.1. ■