Chapter 8

Matroid Sums

8.1 Overview

In this chapter, we describe ways of decomposing or composing binary matroids, using a class of constructs called k-sums, where k ranges over the positive integers. Thus, there are 1-sums, 2-sums, 3-sums, etc. In subsequent chapters, we use k-sums frequently, in particular to analyze or construct certain graphs or matroids, for example the graphs without K_5 minors, the regular matroids, and the max-flow min-cut matroids. The setting in which k-sums are then invoked is as follows. One wants to understand or construct a given class of graphs or matroids. Already available is some insight into several proper subclasses. One conjectures that each graph or matroid not in any one of those subclasses can be recursively constructed by composition steps where the elementary building blocks are taken from the subclasses. It turns out that the k-sums defined in this chapter are well suited for such a composition process, as well as for the inverse decomposition process.

Generally, the structural complexity of k-sums grows as k increases. Thus, 1-sums represent the simplest, indeed trivial, case of decomposition or composition. We cover that case in Section 8.2. In the same section, we also discuss the more interesting but still elementary case of 2-sums.

For 3-sums, or generally for k-sums with $k \geq 3$, the simplicity of 1- and 2-sums gives way to a setting of rich structure that permits many interesting conclusions. In Section 8.3, we explore these k-sums, especially 3-sums. In Section 8.4, we acquire an efficient method for finding 1-, 2-, and 3-sums. Two alternatives of 3-sums, called Δ-sum and Y-sum, are covered in Section 8.5.

In the final section, 8.6, we summarize applications of k-sums, list extensions of the k-sum concept to general matroids via abstract matrices, mention other ways to decompose or compose matroids, and provide references.

We close this section with a review of the exact k-separations defined in Section 3.3. They turn out to be the key ingredient for k-sums. Suppose a binary matroid M on a set E has rank function $r(\cdot)$. Then a pair (E_1, E_2) partitioning E is an exact k-separation if $|E_1|, |E_2| \geq k$ and $r(E_1)+r(E_2) = r(E) + k - 1$. Given such a separation, let X_2 be a maximal independent subset of E_2, then enlarge X_2 by a subset X_1 of E_1 to a base of M. Define for $i = 1, 2$, $Y_i = E_i - X_i$. The representation matrix B of M corresponding to the base $X_1 \cup X_2$ must be of the form

(8.1.1)
$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline X_2 & D & A^2 \\ \hline \end{array}$$

Matrix B with exact k-separation

where $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq k$ and GF(2)-rank $D = k - 1$. Below, we repeatedly utilize B of (8.1.1) when we work with exact k-separations.

The chapter requires knowledge of Chapters 2, 3, 5, 6, and 7. We also use the process of ΔY exchanges of Chapter 4.

8.2 1- and 2-Sums

In this section, we learn how to deduce and manipulate 1- and 2-sum decompositions and compositions. We start with the 1-sum case. Let M be a binary matroid on a set E and with a representation matrix B. Lemma (3.3.19) says that M has a 1-separation if and only if B is not connected. Assume the latter case. Then B can clearly be partitioned as shown in (8.2.1) below, with $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 1$. The latter condition is equivalent to demanding that the submatrices A^1 and A^2 of B are nonempty, i.e., they are not 0×0 matrices. We declare that the binary matroids represented by A^1 and A^2, say M_1 and M_2, are the two *components* of a *1-sum decomposition* of M. The decomposition is reversed in the obvious way, giving a *1-sum composition* of M_1 and M_2 to M. We mean either process when we say that M is a 1-*sum* of M_1 and M_2, denoted by $M = M_1 \oplus_1 M_2$.

(8.2.1)

$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline X_2 & 0 & A^2 \\ \hline \end{array}$$

Matrix B of 1-separation

Under the assumption of graphicness, the 1-sum has a straightforward graph interpretation given by the next lemma. We omit the elementary proof via Theorem (3.2.25), part (a).

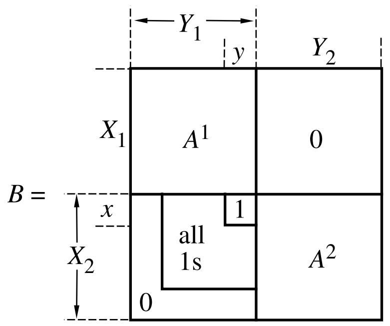
(8.2.2) Lemma. *Let M be a binary matroid. Assume M to be a 1-sum of two matroids M_1 and M_2.*

(a) *If M is graphic, then there exist graphs G, G_1, and G_2 for M, M_1, and M_2, respectively, such that identification of a node of G_1 with one of G_2 creates G.*
(b) *If M_1 and M_2 are graphic (resp. planar), then M is graphic (resp. planar).*

Analogously to the matroid case, we call the graph G of Lemma (8.2.2) a 1-*sum* of G_1 and G_2, and denote this by $G = G_1 \oplus_1 G_2$.

We move to the more interesting case of 2-sums. We assume that the given binary matroid M is connected and has a 2-separation (E_1, E_2). Since M is connected, the 2-separation must be exact. Thus, in any matrix B of (8.1.1) corresponding to that exact 2-separation, the submatrix D must have GF(2)-rank 1. Hence, B is the following matrix, where for the moment the indices $x \in X_2$ and $y \in Y_1$ are to be ignored.

(8.2.3)

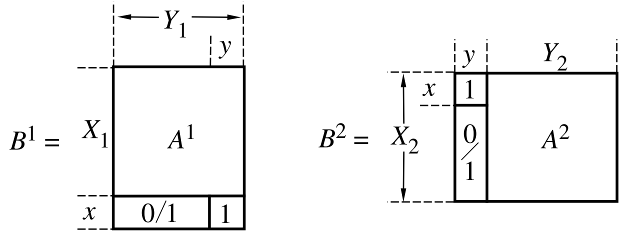


Matrix B of exact 2-separation

We refer to the submatrix of B of (8.2.3) indexed by X_2 and Y_1 as D, in agreement with (8.1.1). We want to extract from B two submatrices B^1 and B^2 that contain A^1 and A^2, respectively, and that also contain enough information to reconstruct B. Evidently, the latter requirement is equivalent to the condition that we must be able to compute D from B^1 and B^2. Now D has GF(2)-rank 1. Thus, knowledge of one nonzero row of D and of one nonzero column of D suffices to compute D.

With this insight, we choose B^1 to be A^1 plus one nonzero row of D, say row x, and B^2 to be A^2 plus one nonzero column of D, say column y. The two indices x and y are shown in (8.2.3). Below, we display B^1 and B^2 so selected.

(8.2.4)



Matrices B^1 and B^2 of 2-sum

We reconstruct D, and thus implicitly B, from B^1 and B^2 by computing

$$D = (\text{column } y \text{ of } B^2) \cdot (\text{row } x \text{ of } B^1) \tag{8.2.5}$$

Let M_1 and M_2 be the minors of M represented by B^1 and B^2. We call these minors the *components* of a 2-*sum decomposition* of M. The reverse process, which corresponds to a reconstruction of B from B^1 and B^2, is a 2-*sum composition* of M_1 and M_2 to M. Both cases are handled by saying that M is a 2-*sum* of M_1 and M_2, denoted by $M = M_1 \oplus_2 M_2$. For future reference, we record in the next lemma that 2-separations of connected binary matroids produce 2-sums with connected components.

(8.2.6) Lemma. *Any 2-separation of a connected binary matroid M produces a 2-sum with connected components M_1 and M_2. Conversely, any 2-sum of two connected binary matroids M_1 and M_2 is a connected binary matroid M.*

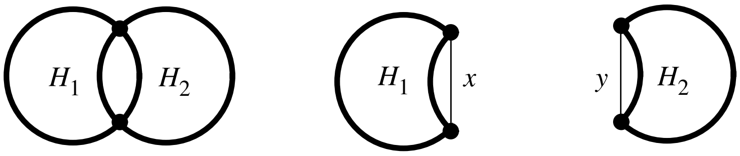
Proof. The above definitions establish the lemma except for the connectedness claims. It is easily verified that connectedness of B of (8.2.3) implies connectedness of B^1 and B^2 of (8.2.4), and vice versa. By Lemma (3.3.19), connectedness of representation matrices is equivalent to connectedness of the corresponding matroids. Thus, connectedness of B, B^1, and B^2 is equivalent to connectedness of M, M_1, and M_2, respectively. $\square$

The 2-sum decomposition or composition has the following appealing graph interpretation.

(8.2.7) Lemma. *Let M be a connected binary matroid that is a 2-sum of M_1 and M_2, as given via B, B^1, and B^2 of (8.2.3) and (8.2.4).*

(a) *If M is graphic, then there exist 2-connected graphs G, G_1, and G_2 for M, M_1, and M_2, respectively, with the following feature. The graph G is produced when one identifies the edge x of G_1 with the edge y of G_2, and when subsequently the edge so created is deleted. The drawing below depicts the general case.*

(8.2.8)



Graph G Graph G_1 Graph G_2

(b) *If M_1 and M_2 are graphic (resp. planar), then M is graphic (resp. planar).*

Proof. (a) Given B of (8.2.3) for M, let $E_1 = X_1 \cup Y_1$ and $E_2 = X_2 \cup Y_2$. We know that the pair (E_1, E_2) is a 2-separation of M. Select any graph G for M. Since M is connected, G is 2-connected. Let H_1 and H_2 be the subgraphs of G with edge sets E_1 and E_2, respectively. By Theorem (3.2.25) part (b), the 2-separation (E_1, E_2) of M implies that the connected components of H_1 and H_2 are connected in cycle fashion. By the switching operation of Section 3.2, we may rearrange G so that both H_1 and H_2 are connected. We may assume that G already is that graph. Thus, G is as given by (8.2.8), with connected H_1 and H_2. By Lemma (8.2.6), the matrices B^1 and B^2 of (8.2.4) are connected. By (8.2.3) and the connectedness of B^1, contraction in G of the edges of $X_2 - \{x\}$ and deletion of the edges of Y_2 must produce the graph G_1 of (8.2.8). Similarly, contraction of the edges of X_1 and deletion of the edges of $Y_1 - \{y\}$ must produce the graph G_2 of (8.2.8). Thus, G_1 and G_2 are graphs for the graphic matroids given by B^1 and B^2 of (8.2.4). The two graphs are 2-connected since the corresponding matrices are connected.
(b) Let G_1 and G_2 be 2-connected graphs for M_1 and M_2. For some H_1 and H_2, the drawings of (8.2.8) correctly depict G_1 and G_2. Identify the edge x of G_1 with the edge y of G_2, then delete the so-created edge. There are two ways to accomplish the identification, but either way is acceptable and produces a graph G as depicted by (8.2.8). Elementary checking of fundamental cycles of G versus fundamental circuits of M confirms that G represents the matroid M defined by B of (8.2.3). Thus, M is graphic. If G_1 and G_2 are plane graphs, we can carry out the edge identification so that G becomes a plane graph. Thus, M is planar if M_1 and M_2 are planar. $\square$

We turn to the more challenging case of general k-sums, with $k \geq 3$.

8.3 General k-Sums

We are given a 3-connected binary matroid M on a set E. For some $k \geq 3$, we also know an exact k-separation (E_1, E_2). We want to decompose M in some useful way. We investigate this problem using the matrix B of (8.1.1). Slightly enlarged, we repeat that matrix below.

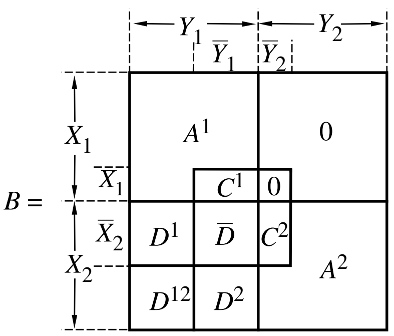
$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline X_2 & D & A^2 \\ \hline \end{array} \qquad (8.3.1)$$

Matrix B with exact k-separation

Recall that the submatrix D of B has GF(2)-rank $k-1$. We want to decompose M into two matroids M_1 and M_2 that correspond to two submatrices B^1 and B^2 of B. As in the 2-sum case, we postulate that B^1 and B^2 include A^1 and A^2, respectively. Furthermore, B^1 and B^2 must permit a reconstruction of B. The latter requirement can be satisfied by including in B^1 (resp. B^2) a row (resp. column) submatrix of D with the same rank as D, i.e., with GF(2)-rank $k-1$. Indeed, the submatrix D of B can be computed from these row and column submatrices. We provide the relevant formulas in a moment. Last but not least, we want B^1 and B^2 to be proper submatrices of B.

There are numerous ways to satisfy these requirements. In the most general case, both B^1 and B^2 intersect all four submatrices A^1, A^2, D, and 0 of B, and thus induce the following rather complicated-looking partition of B. We explain the partition momentarily.

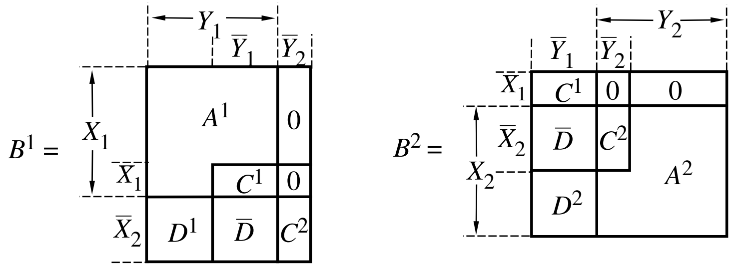
(8.3.2)



Partition of B displaying k-sum

In the notation of (8.3.2), the submatrix B^1 of B, which is not explicitly indicated, is indexed by $X_1 \cup \overline{X}_2$ and $Y_1 \cup \overline{Y}_2$. Furthermore, the submatrix B^2 is indexed by $\overline{X}_1 \cup X_2$ and $\overline{Y}_1 \cup Y_2$. Hence, B^1 contains A^1, intersects A^2 in C^2, and intersects D in $[D^1 \mid \overline{D}]$. The submatrix B^2 contains A^2, intersects A^1 in C^1, and intersects D in $[\overline{D}/D^2]$. We assume that C^1 (resp. C^2) is a proper submatrix of A^1 (resp. A^2). This implies that both B^1 and B^2 are proper submatrices of B. Observe that $\overline{D}$ is the submatrix of D contained in both B^1 and B^2. We assume that both submatrices $[D^1 \mid \overline{D}]$ and $[\overline{D}/D^2]$ of D have GF(2)-rank equal to $k-1$. By Lemma (2.3.14), this implies that $\overline{D}$ has GF(2)-rank $k-1$. We display the matrices B^1 and B^2 below.

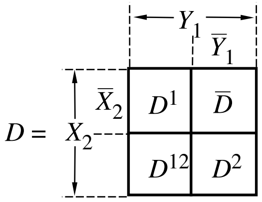
(8.3.3)



Matrices B^1 and B^2 of k-sum

The decomposition of B into B^1 and B^2 corresponds to a decomposition of M into two matroids, say M_1 and M_2, that are represented by B^1 and B^2. We call that decomposition of M a *k-sum decomposition* and declare M_1 and M_2 to be the *components* of the k-sum. The decomposition process is readily reversed. All submatrices of B except for the submatrix D^{12}, which is indexed by $X_2 - \overline{X}_2$ and $Y_1 - \overline{Y}_1$, are present in B^1 and B^2, and thus are already known. For the computation of D^{12} from B^1 and B^2, we first depict D and its submatrices.

(8.3.4)



Partitioned version of D

Since GF(2)-rank $D =$ GF(2)-rank $\overline{D}$, there is a matrix F so that $[D^{12} \mid D^2] = F \cdot [D^1 \mid \overline{D}]$. Thus,

$$D^2 = F \cdot \overline{D} \tag{8.3.5}$$

and

$$D^{12} = F \cdot D^1 \tag{8.3.6}$$

We first solve (8.3.5) for F. The solution may not be unique, but any solution is acceptable. Then we use F in (8.3.6) to obtain D^{12}.

Suppose $\overline{D}$ is square and nonsingular. Then from (8.3.5), we deduce $F = D^2 \cdot (\overline{D})^{-1}$. With that solution, we compute D^{12} according to (8.3.6) as

$$D^{12} = D^2 \cdot (\overline{D})^{-1} \cdot D^1. \tag{8.3.7}$$

The reconstruction of B from B^1 and B^2 corresponds to a *k-sum composition* of M_1 and M_2 to M. Both the k-sum decomposition of M and the k-sum composition of M_1 and M_2 to M, we call a *k-sum* and denote it by $M = M_1 \oplus_k M_2$.

It is a simple exercise to prove that the 1- and 2-sums of Section 8.2 are special instances of the above situation. In the 1-sum case, D is a zero matrix. Thus, we may select C^1, C^2, D^1, D^2, and $\overline{D}$ to be trivial or empty, whichever applies. Then by (8.3.3), we have $B^1 = A^1$ and $B^2 = A^2$, which is the 1-sum of Section 8.2. In the 2-sum situation, $\overline{D}$ is the 1×1 matrix $[\,1\,]$, C^1 and C^2 are trivial matrices, $[D^1 \mid \overline{D}]$ is row x of D, and $[\overline{D}/D^2]$ is column y of D. The formula (8.2.5) for D is nothing but (8.3.7) with the just-defined D^1, D^2, and $\overline{D}$.

The reader is well justified to wonder why we consider such complex k-sums. We argue in favor of our approach as follows. Suppose we intend to investigate a certain matroid property that is inherited under minor-taking, say graphicness. To be even more specific, let us assume that we want to obtain the minor-minimal matroids that are not graphic. Suppose the above general conditions for k-sums admit a particular k-sum case where graphicness of the components implies graphicness of the k-sum. Then no minor-minimal nongraphic matroid can be such a k-sum. For if such a k-sum M exists, then its components, smaller as they are, are graphic. But by the just assumed feature of the k-sum, M is graphic as well, a contradiction. Thus, we can restrict our search for the minimal nongraphic matroids to instances that are not k-sums, a possibly very attractive insight. A second situation is as follows. We want to find a construction for matroids having a certain property. Indeed, we are willing to allow composition steps to be part of the construction. In that case, we need composition rules that preserve the property of interest.

The k-sum compositions described here, complex as they may be, do preserve a number of interesting matroid properties. Thus, for investigations into these properties, the k-sums are very useful. The evidence supporting this claim will be presented in subsequent chapters.

So far we have expressed the k-sum decomposition or composition in terms of representation matrices. It is interesting to consider k-sums in terms of matroid minors as well. By the very derivation of B^1 and B^2, the components M_1 and M_2 of the k-sum are minors of M. It is easy to see from (8.3.3) that B^1 and B^2 share precisely the submatrix $\overline{B}$ given by (8.3.8) below. Define $\overline{M}$ to be the matroid represented by $\overline{B}$. Clearly, $\overline{M}$ is a minor of M, but more importantly, of M_1 and M_2 as well. Indeed, one may view the composition of M_1 and M_2 to be an identification of the minor $\overline{M}$ of M_1 with the minor $\overline{M}$ of M_2. One might also say that $\overline{M}$ as minor of M forms the connection between the components M_1 and M_2 in M. In agreement with the latter terminology, we call $\overline{M}$ the *connecting matroid* or *connecting minor* of the k-sum. By (8.3.8) and the fact that D and $\overline{D}$ have the same rank, the pair $(\overline{X}_1 \cup \overline{Y}_1,\ \overline{X}_2 \cup \overline{Y}_2)$ is an exact k-separation of $\overline{M}$, provided that $|\overline{X}_1 \cup \overline{Y}_1|, |\overline{X}_2 \cup \overline{Y}_2| \geq k$. The latter condition is satisfied, for example, if both C^1 and C^2 are nonempty and nontrivial.

(8.3.8)

$$\overline{B} = \begin{array}{c|c|c|} & \overline{Y}_1 & \overline{Y}_2 \\ \hline \overline{X}_1 & C^1 & 0 \\ \hline \overline{X}_2 & \overline{D} & C^2 \\ \hline \end{array}$$

Submatrix $\overline{B}$ representing the connecting minor $\overline{M}$ of the k-sum

Recall that we want important matroid properties to be preserved under k-sum composition. Evidently, $\overline{M}$ is the minor of M that decides whether or not we achieve that goal. Thus, the selection of $\overline{M}$ requires great care when one desires k-sums suitable for the investigation of matroid properties. Note that a particular choice of $\overline{M}$ imposes constraints on the exact k-separations that are to be converted to a k-sum. For example, any k-separation (E_1, E_2) capable of producing a k-sum with a given $\overline{M}$ minor must satisfy for $i = 1, 2$, $|E_i| \geq |\overline{X}_i \cup \overline{Y}_i| + 1$, since otherwise the submatrices C^1 and C^2 of B are not proper submatrices of A^1 and A^2. The selection of $\overline{M}$ becomes even more complex when computational aspects are considered. For example, one might demand that the k-sums defined by $\overline{M}$ can be located in polynomial time.

For this book, 3-sums are of considerable importance. A good choice for the matrix $\overline{B}$ representing $\overline{M}$ turns out to be

(8.3.9)

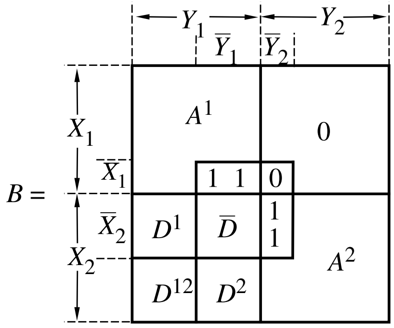
$$\overline{B} = \begin{array}{|c|c|} \hline C^1 & 0 \\ \hline \overline{D} & C^2 \\ \hline \end{array} = \begin{array}{|cc|c|} \hline 1 & 1 & 0 \\ \hline \multicolumn{2}{|c|}{\multirow{2}{*}{$\overline{D}$}} & 1 \\ & & 1 \\ \hline \end{array}$$

Matrix $\overline{B}$ of 3-sum

where $\overline{D}$ is any 2×2 GF(2)-nonsingular matrix. If $\overline{D}$ is the 2×2 identity matrix, then by (5.2.8), $\overline{B}$ represents up to indices $M(W_3)$, which is the graphic matroid of the wheel with three spokes. If $\overline{D}$ contains exactly three

1s, the only other choice, then by one GF(2)-pivot, say in $C^1 = [1\ 1]$, we obtain the former case. Thus, in all instances, $\overline{M}$ is an $M(W_3)$ minor of M. With $\overline{B}$ of (8.3.9), the matrix B of (8.3.2) becomes the following matrix.

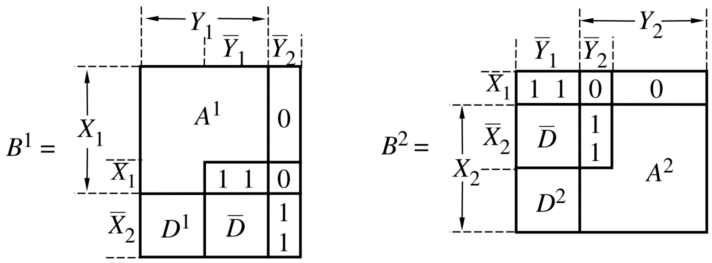
(8.3.10)



Matrix B of 3-sum

The representation matrices B^1 and B^2 of the components M_1 and M_2 are

(8.3.11)



Matrices B^1 and B^2 of 3-sum

When does an exact 3-separation (E_1, E_2) of a binary matroid M lead to a 3-sum with this $\overline{M}$? Appealing sufficient conditions turn out to be 3-connectedness of M and the restriction that $|E_1|, |E_2| \geq 4$. We state this result in the next lemma.

(8.3.12) Lemma. *Let M be a 3-connected binary matroid on a set E. Then any 3-separation (E_1, E_2) of M with $|E_1|, |E_2| \geq 4$ produces a 3-sum, and vice versa.*

Proof. Let (E_1, E_2) be a 3-separation of M with $|E_1|, E_2| \geq 4$. Since M is 3-connected, the 3-separation must be exact. Hence, by (8.3.1), M has a binary representation matrix B given by (8.3.13) below, with GF(2)-rank $D = 2$. In particular, any column submatrix of D containing four or more columns must contain a zero column or two parallel columns. We claim that the row index subset X_1 of B of (8.3.13) is nonempty. Assume otherwise. By $|X_1 \cup Y_1| \geq 4$, we have $|Y_1| \geq 4$. But then D has a zero column or two parallel columns, and M is not 3-connected, a contradiction. Thus, $X_1 \neq \emptyset$, and trivially $Y_1 \neq \emptyset$.

(8.3.13)

$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline X_2 & D & A^2 \\ \hline \end{array}$$

Matrix B with 3-separation

Let A^{11} be a connected block of A^1. There must be at least one such block, since otherwise M has coloops. To exhibit A^{11}, we partition B of (8.3.13) further as follows.

(8.3.14)

$$B = \begin{array}{cc|c|c|c|} & & \multicolumn{2}{c|}{Y_1} & \\ & & & Y_{11} & Y_2 \\ \hline X_1 & & 0/1 & 0 & 0 \\ \cline{2-4} & X_{11} & 0 & A^{11} & \\ \hline X_2 & & 0/1 & D^{11} & A^2 \\ \hline \end{array}$$

Matrix B of (8.3.13) with partitioned submatrix A^1

Note the column submatrix D^{11} of D corresponding to A^{11}. We claim that GF(2)-rank $D^{11} = 2$. If this is not the case, then the pair $(X_{11} \cup Y_{11}, E - (X_{11} \cup Y_{11}))$ constitutes a 1- or 2-separation of M, a contradiction.

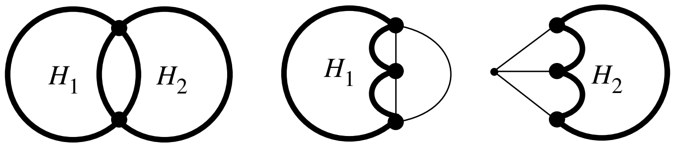
Let us examine the submatrix $[A^{11}/D^{11}]$ of B of (8.3.14) more closely. Consider the paths in the bipartite graph BG(A^{11}) between all pairs of nodes y and z in Y_{11} for which the columns y and z of D^{11} are GF(2)-independent. Since A^{11} is connected and GF(2)-rank $D^{11} = 2$, there is at least one path. In a shortest path, all intermediate nodes in Y_{11} correspond to zero columns of D^{11}. Suppose a shortest path has at least four arcs. With the path shortening technique of Chapter 5, we reduce that path by pivots in A^{11} to one with exactly two arcs. Thus, we may assume that a shortest path of length 2 exists, say connecting nodes y and z in BG(A^{11}). Put differently, we may assume that A^{11} contains a row with two 1s in columns indexed by $y, z \in Y_{11}$ such that the columns of D^{11} indexed by y and z are GF(2)-independent. The path shortening pivots, if any, can also be carried out in B of (8.3.14) without affecting the entries of D and A^2. Thus, we may assume A^1 to contain a row with two 1s in columns y and z such that the columns y and z of D are GF(2)-independent.

By duality, we may suppose that A^2 contains a column having two 1s in rows u and v such that the rows u and v of D are GF(2)-independent. By Lemma (2.3.14), GF(2)-rank $D = 2$ implies that the rows u and v of D and the columns y and z of D must intersect in a 2×2 GF(2)-nonsingular submatrix $\overline{D}$ of D. When we partition B of (8.3.13) to exhibit the two 1s of A^1, the two 1s of A^2, and $\overline{D}$, we get an instance of (8.3.10), with the connecting minor given by $\overline{B}$ of (8.3.9). Thus, M is a 3-sum.

The converse is obvious, since any 3-sum given by (8.3.10) produces the 3-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ with $|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq 4$. $\square$

Let us translate the 3-sum operation of binary matroids to graphs. For simplicity, we confine ourselves to the case where a given graph G with edge set E is 3-connected. In agreement with Lemma (8.3.12), we assume that G has a 3-separation (E_1, E_2) where $|E_1|, |E_2| \geq 4$. By that lemma, the graphic matroid $M = M(G)$ has a 3-sum decomposition induced by the 3-separation (E_1, E_2). The previously discussed results for the connecting minor $\overline{M}$ of that 3-sum can be restated for the case at hand as follows: The graph G has a minor $\overline{G}$ on edge set $\overline{E}$ and with a 3-separation $(\overline{E}_1, \overline{E}_2)$ such that $\overline{E}_1 \subseteq E_1$ and $\overline{E}_2 \subseteq E_2$. Up to indices, $\overline{G}$ is the wheel W_3.

The deletions and contractions reducing G to $\overline{G}$ evidently involve the edges of $E_1 - \overline{E}_1$ and $E_2 - \overline{E}_2$. Suppose in G we carry out these deletions and contractions just for the edges of $E_2 - \overline{E}_2$. Declare G_1 to be the resulting graph. The process producing G_1 corresponds to the reduction of the 3-sum M to the component M_1. Thus, $M_1 = M(G_1)$. By analogous reductions, this time confined to the edges of $E_1 - \overline{E}_1$, we obtain a graph G_2 for which $M_2 = M(G_2)$. Examine the representation matrices B^1 and B^2 of (8.3.11) for M_1 and M_2. Clearly, the set $\overline{X}_2 \cup \overline{Y}_2$ is a triangle in M_1, and the set $\overline{X}_1 \cup \overline{Y}_1$ is a triad of M_2. Correspondingly, $\overline{E}_2$ must be a triangle in G_1, and $\overline{E}_1$ must be a cocircuit of G_2 of size 3. Indeed, by the structure of G and G_2, that cocircuit of G_2 must be a 3-star. The drawing below summarizes these conclusions for G, G_1, and G_2.

(8.3.15) 

Graph G Graph G_1 Graph G_2

So far we have described the 3-sum decomposition of G into component graphs G_1 and G_2. The composition is even easier. We identify the three nodes of G_1 explicitly shown in (8.3.15) with the three nodes of G_2 so that the triangle of G_1 and the 3-star of G_2 form a copy of $\overline{G}$. The resulting graph is G plus that copy of $\overline{G}$. We delete the edges of that copy of $\overline{G}$ and have the desired G.

We use the term *3-sum* to describe the decomposition of G into G_1 and G_2, as well as the composition of the latter graphs to G. In agreement with the matroid case, we denote this by $G = G_1 \oplus_3 G_2$ and call $\overline{G}$ the *connecting graph* or *connecting minor* of the 3-sum of G_1 and G_2.

We saw earlier that certain separations can be converted to 1-, 2-, or 3-sums. In the next section, we discuss how one may locate such separations and thus 1-, 2-, and 3-sums.

8.4 Finding 1-, 2-, and 3-Sums

Suppose that for a given binary matroid M, we want to either find a 1-, 2-, or 3-sum decomposition, or conclude that there is no such decomposition. In this section, we solve this seemingly difficult problem. Main tools are the separation algorithm of Section 6.2, some results about sequences of nested minors of Section 7.3, and the lemmas of Section 8.3 that assure us of a 1-, 2-, or 3-sum decomposition when certain separations are present.

We start with the simplest case, where the given binary matroid M is not connected. By Lemma (3.3.19), every representation matrix of M is disconnected. Thus, we easily detect this situation using an arbitrarily selected representation matrix B of M. From that B, we may deduce a 1-sum decomposition of M as described in Section 8.2. Thus, from now on, we may assume M to be connected. We also suppose that M has at least four elements.

To find a 2- or 3-sum decomposition, if it exists, we first rely on Lemma (5.2.11). That result says that under the above assumptions, M has a 2-separation or an $M(W_3)$ minor. Implicit in the proof of the lemma is a polynomial algorithm that decides which case applies. Suppose a 2-separation is determined. Then Lemma (8.2.6) tells us that this 2-separation can be easily converted to a 2-sum decomposition of M. So we now assume that we detect an $M(W_3)$ minor, say N.

To M and N we apply Theorem (7.3.6), which in turn cites Theorem (7.3.1). Accordingly, there is a polynomial algorithm that finds a 2-separation of M, or that establishes a sequence of nested 3-connected minors $M_0, M_1, \ldots, M_t = M$ with the following features. M_0 is isomorphic to N. For some s, $1 \le s \le t$, the subsequence $M_0, M_1, \ldots, M_s$ consists of wheel matroids and has gap 2, and the subsequence $M_s, M_{s+1}, \ldots, M_t$ has gap 1. We know already how to handle the 2-separation case. Thus, we may assume that the prescribed sequence of nested 3-connected minors is found. The description of that sequence can be simplified by saying that the sequence starts with an $M(W_3)$ minor and has gap 1 or 2. We use that sequence to find a 3-sum decomposition for M or to prove that there is no such decomposition. The details are as follows.

First, we observe that by Lemma (8.3.12), M has a 3-sum decomposition if and only if it has a 3-separation (E_1, E_2) with $|E_1|, |E_2| \geq 4$. Indeed, the decomposition can be efficiently determined from such a 3-separation using the algorithm implicit in the proof of that lemma. Thus, our task is finding such a 3-separation or proving that none exists.

From Section 5.3, we know that the intersection algorithm can solve the latter problem in polynomial time. However, the order of the algorithm is so large that the scheme is practically unusable unless the matroid M is quite small. In addition, it presently is not known how one might reduce the order to an acceptable level. For this reason, we outline below another algorithm that by suitable refinements and appropriate implementation does become very efficient. The method is based on the following observations, where for the moment we assume that M does have a 3-separation (E_1, E_2) with the desired property. For $j = 0, 1, \ldots, t$, let E^j be the groundset of M_j. For each j, and for $i = 1, 2$, define $E_i^j = E^j \cap E_i$. Clearly, each pair (E_1^j, E_2^j) satisfying $|E_1^j|, |E_2^j| \geq 4$ is a 3-separation of M_j. By its very derivation, any such 3-separation of M_j induces a 3-separation of M with at least four elements on each side, for example (E_1, E_2).

For the moment, let j be the smallest index so that $|E_1^j|, |E_2^j| \geq 4$. There is such an index since $(E_1^t, E_2^t) = (E_1, E_2)$ satisfies $|E_1^t|, |E_2^t| \geq 4$. Note that M_j has necessarily at least eight elements. Since M_0 has only six elements, we conclude that $j \geq 1$. Because of the minimality of j, we must have $|E_1^{j-1}|$ or $|E_2^{j-1}| \leq 3$. Since the gap of the sequence is at most 2, we deduce from the latter fact that $|E_1^j|$ or $|E_2^j| \leq 5$. There are only polynomially many pairs (E_1^j, E_2^j) for M_j where $|E_1^j|, |E_2^j| \geq 4$ and $|E_1^j|$ or $|E_2^j| \leq 5$. Indeed, there are only polynomially many pairs satisfying that condition when we allow j to range over $1, 2, \ldots, t$.

We use these observations as follows. Note that we still assume M to have a 3-separation (E_1, E_2) where $|E_1|, |E_2| \geq 4$. To find such a 3-separation, we first locate for $j = 1, 2, \ldots, t$ all pairs (E_1^j, E_2^j) satisfying $|E_1^j|, |E_2^j| \geq 4$ and $|E_1^j|$ or $|E_2^j| \leq 5$. For each pair, we test whether it is a 3-separation of M_j. We discard the cases where the answer is "no." For each remaining pair, say (E_1^j, E_2^j) of M_j, we check with the separation algorithm of Section 6.2 whether it induces a 3-separation of M. For at least one such pair, say (E_1^j, E_2^j) of M_j, we must obtain an affirmative answer. The 3-separation of M so found, say (E_1', E_2'), satisfies for $i = 1, 2$, $E_i' \supseteq E_i^j$ and thus $|E_i'| \geq |E_i^j| \geq 4$. Hence, (E_1', E_2') is the sought-after 3-separation of M.

So far, we have assumed that M has a 3-separation (E_1, E_2) where $|E_1|, |E_2| \geq 4$. If that is not the case, we can still carry out the above polynomial process, without success of course. But the lack of success proves that M has no 3-separation (E_1, E_2) with $|E_1|, |E_2| \geq 4$.

The next theorem summarizes the above discussion.

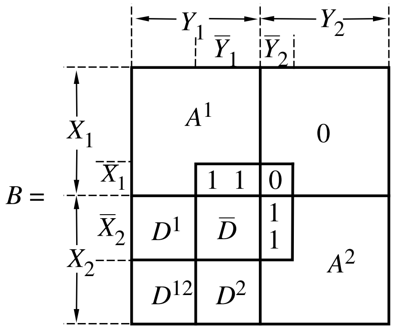
(8.4.1) Theorem. *There is a polynomial algorithm that accepts any binary matroid M as input, and that outputs the conclusion that M is not connected, or that M is connected but not 3-connected, or that M is 3-connected. In the first case, the algorithm also supplies a 1-sum decomposition, in the second case, a 2-sum decomposition, and in the third case, a 3-sum decomposition if it exists.*

In the next section, we meet close relatives of 3-sums, called Δ-sums and Y-sums.

8.5 Delta-Sum and Wye-Sum

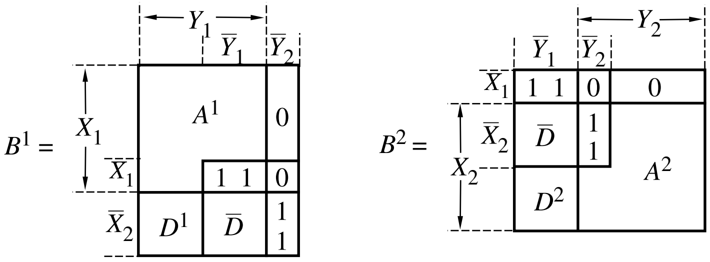
There are two variations of the 3-sum. We call them delta-sum and wye-sum, for short Δ-sum and Y-sum. The relationships among 3-, Δ-, and Y-sums are very elementary. That does not imply that they may be employed interchangeably. Indeed, for some applications one definitely prefers one type over another. Both Δ-sum and Y-sum are derived from the 3-sum, so for convenient reference we repeat to related matrices of (8.3.10) and (8.3.11) below.

(8.5.1)



Matrix B of 3-sum

(8.5.2)

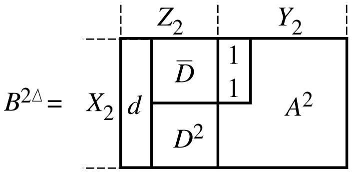


Matrices B^1 and B^2 of 3-sum

We proceed as follows. We first define Δ- and Y-sums. Then we show that the components of these sums are isomorphic to some minors of the sum, provided the sum is 3-connected. Finally, we indicate why one would want to use one type of these sums over another in certain applications.

We derive the Δ-sum from the 3-sum as follows. Let M be a 3-connected binary matroid with a 3-sum decomposition into M_1 and M_2, with representation matrices B, B^1, and B^2 given by (8.5.1) and (8.5.2). Suppose in M_2 we perform a ΔY exchange that replaces the triad $\overline{X}_1 \cup \overline{Y}_1$ by a triangle Z_2 as specified in (4.4.5). The new matroid, say $M_{2\Delta}$, is represented by

(8.5.3)



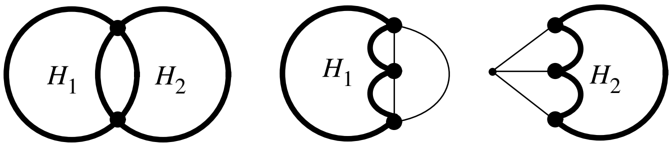
Matrix $B^{2\Delta}$ for $M_{2\Delta}$

where the three columns of $B^{2\Delta}$ indexed by Z_2 sum to 0 (in GF(2)).

We say that M has been decomposed in a *Δ-sum decomposition* into *components* M_1 and $M_{2\Delta}$. The reverse process we call a *Δ-sum composition* of M_1 and $M_{2\Delta}$ to M. We refer to both cases as a *Δ-sum* and denote it by $M = M_1 \oplus_\Delta M_{2\Delta}$. The relevant triangle in M_1 or $M_{2\Delta}$, i.e., $\overline{X}_2 \cup \overline{Y}_2$ or Z_2, is the *connecting triangle* of M_1 or $M_{2\Delta}$. It is not difficult to check, say using the circuits of M, that the $\oplus_\Delta$ operator is commutative. Thus, $M_1 \oplus_\Delta M_{2\Delta} = M_{2\Delta} \oplus_\Delta M_1$. We omit the details of the proof, since we will make no use of this result.

The graph interpretation of the Δ-sum is as follows. Recall that by (8.3.15), the 3-sum decomposition of a graph G into component graphs G_1 and G_2 can be depicted as

(8.5.4)

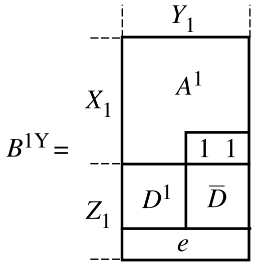


Graph G Graph G_1 Graph G_2

The ΔY exchange transforming M_2 to $M_{2\Delta}$ becomes a replacement of the explicitly shown 3-star of G_2 by a triangle. Let $G_{2\Delta}$ be the resulting graph. We may view the composition of G_1 and $G_{2\Delta}$ to G to be an identification of the triangle of G_1 with that of G_2, followed by a deletion of the so-created triangle from the resulting graph. Borrowing from the terminology of the matroid case, we call G a *Δ-sum* of G_1 and $G_{2\Delta}$, and denote this by $G = G_1 \oplus_\Delta G_{2\Delta}$.

The Y-sum of binary matroids is defined in an analogous fashion. As for the Δ-sum, we start with a 3-sum $M = M_1 \oplus_3 M_2$. But now we convert the triangle $\overline{X}_2 \cup \overline{Y}_2$ of M_1 by a ΔY exchange to a triad Z_1, getting a matroid $M_{1\mathrm{Y}}$. By B^1 of (8.5.2) and the ΔY exchange of (4.4.7), the following matrix $B^{1\mathrm{Y}}$ represents $M_{1\mathrm{Y}}$.

(8.5.5)



Matrix $B^{1\mathrm{Y}}$ for $M_{1\mathrm{Y}}$

Here, the three rows of $B^{1\mathrm{Y}}$ indexed by Z_1 sum to 0 (in GF(2)). We call M a Y-*sum* with *components* $M_{1\mathrm{Y}}$ and M_2, and write $M = M_{1\mathrm{Y}} \oplus_{\mathrm{Y}} M_2$. The relevant triad in $M_{1\mathrm{Y}}$ or M_2 is the *connecting triad* of $M_{1\mathrm{Y}}$ or M_2. The Y-sum operator can be verified to be commutative, as one would expect.

In the graph case of a Y-sum, we replace the explicitly shown triangle of G_1 of (8.5.4) by a 3-star, getting a graph $G_{1\mathrm{Y}}$. G is obtained from $G_{1\mathrm{Y}}$ and G_2 as follows. Let i, j, k (resp. p, q, r) be the three nodes of attachment of the 3-star in $G_{1\mathrm{Y}}$ (resp. G_2). Assume that i (resp. j, k) corresponds to p (resp. q, r). Now connect the node i (resp. j, k) of $G_{1\mathrm{Y}}$ minus its 3-star with the node p (resp. q, r) of G_2 minus its 3-star, say using an edge e (resp. f, g). The edges e, f, and g form a cutset in the new graph, and contraction of that cutset produces the graph G. We call G a Y-*sum* of $G_{1\mathrm{Y}}$ and G_2, and denote this by $G = G_{1\mathrm{Y}} \oplus_{\mathrm{Y}} G_2$.

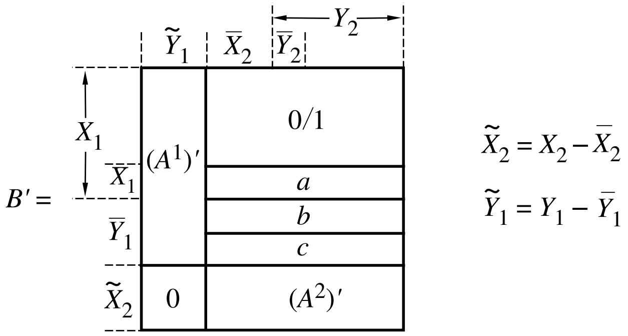
The above Y-sum operation may seem cumbersome. In fact, the reader most likely has an alternate Y-sum composition in mind where the 3-star of $G_{1\mathrm{Y}}$ is identified with that of G_2, and where the edges of the resulting 3-star are deleted. Both Y-sum processes produce the same outcome, but they do differ in the way in which the composition is carried out. We chose the above, more complicated description because then the Δ-sum and Y-sum composition steps are dual operations. The reader may want to try to dualize the alternate Y-sum process to a new Δ-sum process for graphs. This should turn out to be a difficult task. Indeed, one can show that such a dual composition process cannot exist for graphs. At any rate, we know that if M is the Δ-sum $M_1 \oplus_\Delta M_{2\Delta}$, then M^* is the Y-sum $M_1^* \oplus_{\mathrm{Y}} M_{2\Delta}^*$.

The matrices B^1 and B^2 of (8.5.2) are submatrices of B of (8.5.1). Thus, by definition, M_1 and M_2 are minors of M. One cannot expect $M_{1\mathrm{Y}}$ or $M_{2\Delta}$ also to be minors of M due to the assigned index sets Z_1 and Z_2. However, one would hope that $M_{1\mathrm{Y}}$ and $M_{2\Delta}$ are isomorphic to minors of M. The next result supports this notion.

(8.5.6) Lemma. *Let M be the 3-connected matroid represented by the* *binary matrix B of (8.5.1). Then M has minors isomorphic to $M_{1\mathrm{Y}}$ and $M_{2\Delta}$.*

Proof. By duality, it suffices that we prove the claim for $M_{2\Delta}$. For this, we may assume that the nonsingular submatrix $\overline{D}$ of B of (8.5.1) is a 2×2 identity matrix, since this can always be achieved by at most one GF(2)-pivot in B on one of the 1s indexed by $\overline{X}_1$ and $\overline{Y}_1$. Given $\overline{D}$ as a 2×2 identity matrix, we first perform in B two GF(2)-pivots on the 1s of $\overline{D}$. These pivots convert B to the matrix B' below. We explain the structure of B' momentarily.

(8.5.7)

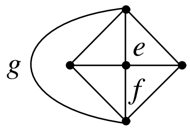


Matrix B' derived from B of (8.5.1) by two pivots

Note that the pivots in B are made within the submatrix B^2. Thus, the submatrix of B' indexed by the rows of $\overline{X}_1 \cup \overline{Y}_1 \cup \tilde{X}_2$ and the columns of $\overline{X}_2 \cup Y_2$ represents M_2. Since $(X_1 \cup Y_1, X_2 \cup Y_2)$ is a 3-separation of M, the submatrix D' of B' indexed by $X_1 \cup \overline{Y}_1$ and $\overline{X}_2 \cup Y_2$ has GF(2)-rank 2. Indeed, since $\overline{X}_1 \cup \overline{Y}_1$ is a triad of M_2, the row subvectors a, b, c are nonzero and add (in GF(2)) to the zero vector. By Lemma (8.3.12), pivots in $(A^1)'$ of B' are possible so that the new $(A^1)'$ of the new B' has two 1s in some column w and in rows u and v for which the rows u and v of D' are GF(2)-independent. The submatrix of the new B' indexed by $\tilde{X}_2 \cup \{u, v\}$ and $\overline{X}_2 \cup Y_2 \cup \{w\}$ then is isomorphic to $M_{2\Delta}$ by the rule (4.4.7) for ΔY exchanges. ☐

The simple relationships among 3-sum, Δ-sum, and Y-sum may deceive one into thinking that each such sum can be substituted for another one in all applications. A first warning of the incorrectness of this conclusion comes from applications involving planar graphs. That is, a Y-sum of two planar graphs must be planar, while a Δ-sum of two planar graphs need not be planar. Planarity of the Y-sum is most easily seen when one embeds each component on a sphere and combines the two drawings to one drawing on another sphere. The Δ-sum need not be planar. For example, the nonplanar graph $K_{4,3}$ can be created from two copies of the planar graph given below.

(8.5.8)



Planar component of nonplanar Δ-sum

The triangle $\{e, f, g\}$ of one copy is identified with that of the other copy. Subsequently, the triangle created by the identification process is deleted, and $K_{4,3}$ results. Other significant applications where caution is in order involve the max-flow min-cut matroids and the convex hull of the disjoint unions of circuits of a binary matroid (the so-called cycle polytope). On the other hand, when one examines matroid regularity, one may freely switch among the three types of sums without penalty. Validity of these claims is proved in Chapters 11 and 13.

In the final section, we link the material of this chapter to related results and provide references.

8.6 Extensions and References

Basic aspects of the composition or decomposition of matroids have been explored in a number of references, e.g. in Edmonds and Fulkerson (1965), Edmonds (1965a), (1979), Nash-Williams (1961), (1964), (1966), Bixby (1972), (1975), Brylawski (1972), (1975), Cunningham (1973), (1979), (1982a), (1982b), Cunningham and Edmonds (1978), (1980), Iri (1979), Nakamura and Iri (1979), Seymour (1980b), Tomizawa and Fujishige (1982), Fujishige (1983), (1985), and Conforti and Laurent (1988), (1989). Decomposing highly connected matroids and composing them again has not been treated extensively. Related to the approach taken here are the decomposition and composition of the graphs without K_5 minors in Wagner (1937a), the modular constructions of Brylawski (1975), and the decomposition and composition of the regular matroids in Seymour (1980b). The concept of the connecting minor is taken from Truemper (1985a), where it is developed for general matroids using abstract matrices, instead of just for binary matroids as done here. The material of Sections 8.2 and 8.3 is also derived from general matroid results of that reference. The methods of Section 8.4 for finding 1-, 2-, and 3-sums are from Truemper (1985a), (1990).

Locating k-sums for general $k \geq 4$ is much more difficult since one must identify a k-separation and an *a priori* specified connecting minor. We do not know how this can be efficiently accomplished for binary matroids, let alone for general matroids. Truemper (1985a) treats a particular 4-sum case where the connecting minors have eight elements.

Most of the material of Section 8.5 is based on Grötschel and Truemper (1989b). That reference examines the Δ-sum and Y-sum in much more detail. The Δ-sum of Section 8.5 is popular in graph theory (e.g., see Wagner (1937a)) and is used in Seymour (1980b) to effect the decomposition of the regular matroids. Lemma (8.5.6) is proved by a quite different method in Seymour (1980b).

We have omitted entirely some interesting decomposition results of Truemper (1985a),(1985b). These references contain a number of basic results about k-sums and three decomposition classification theorems plus applications. The theorems cover general matroids, binary matroids, and graphs, respectively. Each of them says that any given matroid is decomposable, or a number of elements can be removed in any order without loss of 3-connectivity, or the matroid belongs to a small class of matroids with few elements. The proofs are quite long and are the main reason that we have omitted these results. Significant applications of the theorems are rather short proofs of the profound excluded minor theorems for planar graphs, graphic matroids, and regular matroids due to Kuratowski (1930) and Tutte (1958), (1965). The latter theorems are proved in this book using different machinery in Chapters 7, 9, and 10.