Chapter 7

Splitter Theorem and Sequences of Nested Minors

7.1 Overview

Chapters 4, 5, and 6 cover three basic matroid tools: the series-parallel and delta-wye constructions, the path shortening technique, and the separation algorithm. The chapters also include a number of basic matroid results whose proofs rely on these tools. With this foundation, we derive in this chapter and the next one several fundamental results about the decomposition and composition of matroids. Specifically in this chapter, we define matroid splitters, characterize them, and deduce consequences of that characterization.

The concept of splitters and their characterization is due to Seymour. The idea can be summarized as follows. Let $\mathcal{M}$ be a class of binary matroids that is closed under isomorphism and under the taking of minors. Then a 3-connected matroid $N \in \mathcal{M}$ on at least six elements is declared to be a *splitter* of $\mathcal{M}$ if every matroid $M \in \mathcal{M}$ with a proper N minor has a 2-separation. Some researchers define graph or matroid 2-separations to be splits. The term "splitter" is in agreement with that notion.

The concept of splitters may seem rather abstract. But in subsequent chapters, we rely on it a number of times, and without doubt it is one of the central ideas for the decomposition of matroids. We characterize splitters in Section 7.2 in the so-called splitter theorem.

Define two minors of a graph or matroid to be *nested* if one of them is a minor of the other one. In Section 7.3, we derive from the splitter theorem several existence theorems about sequences of nested minors, among them

Tutte's wheel theorem for graphs. A special case of nested minor sequences is used in Section 7.4 to prove Kuratowski's characterization of planar graphs. According to that result, a graph is planar if and only if it does not have $K_{3,3}$ or K_5 minors. In the final section, 7.5, we point out a number of extensions and list references.

The chapter requires familiarity with the material of Chapters 2, 3, 5, and 6.

7.2 Splitter Theorem

Let $\mathcal{M}$ be a class of binary matroids that is closed under isomorphism and under the taking of minors. Recall that a *splitter* of $\mathcal{M}$ is a 3-connected matroid $N \in \mathcal{M}$ such that every matroid $M \in \mathcal{M}$ with a proper N minor is 2-separable. We employ the same terminology for graphs. For example, if $\mathcal{G}$ is a class of graphs that is closed under isomorphism and under the taking of minors, then a *splitter* of $\mathcal{G}$ is a 3-connected graph $G \in \mathcal{G}$ such that every graph of $\mathcal{G}$ with a proper G minor is 2-separable.

In this section, we derive surprisingly simple necessary and sufficient conditions for a given $N \in \mathcal{M}$ to be a splitter. The next theorem stating these conditions is the splitter theorem due to Seymour.

(7.2.1) Theorem (Splitter Theorem). *Let $\mathcal{M}$ be a class of binary matroids that is closed under isomorphism and under the taking of minors. Let N be a 3-connected matroid of $\mathcal{M}$ on at least six elements.*

(a) *If N is not a wheel, then N is splitter of $\mathcal{M}$ if and only if $\mathcal{M}$ does not contain a 3-connected 1-element extension of N.*
(b) *If N is a wheel, then N is a splitter of $\mathcal{M}$ if and only if $\mathcal{M}$ does not contain a 3-connected 1-element extension of N and does not contain the next larger wheel.*

Proof. If N is a splitter of $\mathcal{M}$, then the 3-connected extensions cited in (a) or (b) obviously cannot occur in $\mathcal{M}$. We prove the converse by contradiction. Thus, we suppose that $\mathcal{M}$ does not contain the 3-connected extensions cited in (a) or (b), whichever applies, and that nevertheless N is not a splitter of $\mathcal{M}$. Thus, $\mathcal{M}$ contains a 3-connected matroid M with a proper N minor, and M is not one of the cases excluded under (a) or (b). Since $\mathcal{M}$ is closed under isomorphism, we may assume N itself to be that N minor. To M and N we apply Theorem (6.4.1). According to that theorem, M has a 3-connected minor N' that is a 3-connected 1- or 2-element extension of an N minor. In the 2-element extension case, N' is derived from the N minor by one addition and one expansion. Again, since $\mathcal{M}$ is closed under isomorphism and minor taking, we may take N itself to be that N minor. The 1-element extension case has been ruled out by (a) and (b). Thus, N' is derived from N by one addition and one expansion. Suppose a binary matrix $\overline{B}$ with row index set $\overline{X}$ and column index set $\overline{Y}$ represents N. Then N' can be represented by a binary matrix C that displays $\overline{B}$, and thus N, as follows.

$$\text{(7.2.2)} \qquad C = \begin{array}{c|c|c|} & \overline{Y} & q \\ \hline \overline{X} & \overline{B} & a \\ \hline p & b & \alpha \\ \hline \end{array}$$

Matrix C representing N'

We now show either that N' contains a 3-connected 1-element extension of an N minor, a case ruled out by both (a) and (b), or that N is a wheel and N' is the next larger wheel, a case ruled out by (b). We accomplish this by the following investigation into the structure of C of (7.2.2).

Since N' is 3-connected, the matrix C does not contain zero vectors, unit vectors, or parallel vectors. In particular, the subvectors a and b of C must be nonzero. Furthermore, the submatrices $[\overline{B} \mid a]$ and $[\overline{B}/b]$ of C, which represent 1-element extensions of N, cannot be 3-connected since otherwise we have an eliminated case of (a) and (b). Thus, the subvector a (resp. b) is a unit vector or is parallel to a column (resp. row) of $\overline{B}$.

Because of pivots in $\overline{B}$ and row exchanges in C, we may assume that a is a unit vector with 1 in the topmost position, and that b is parallel to a row of $\overline{B}$. Since C is 3-connected, we necessarily have $\alpha = 1$. We then may partition C as follows.

$$\text{(7.2.3)} \qquad C = \begin{array}{c|c|c|} & \overline{Y} & q \\ \hline & c & 1 \\ \cline{2-3} \overline{X} & 0/1 & 0 \\ \hline p & b & 1 \\ \hline \end{array}$$

Initial partition of C with a = unit vector

In the subsequent processing of C of (7.2.3), we introduce a number of row and column exchanges and pivots that affect the index sets substantially. Since $\mathcal{M}$ is closed under isomorphism, we do not have to keep track of such index changes. So, instead, we just make sure that the matrix obtained from the current C by deletion of the rightmost column and bottom row does represent N up to a relabeling of the elements. We always refer to that matrix as the current $\overline{B}$. With this convention, we can freely introduce new indices or reuse old ones.

Suppose during the subsequent processing of C of (7.2.3), we detect that the current C contains the current $\overline{B}$ plus a nonzero row or column that is not parallel to a row or column of $\overline{B}$ and that is not a unit vector. Then by Lemma (6.2.6), $\overline{B}$ plus that row or column up to indices represents a 3-connected 1-element extension of N, which has been ruled out. Thus, we assume below that this case does not occur. The proof of the theorem is complete once we show N to be a wheel and N' to be the next larger wheel. We are now ready to process C of (7.2.3).

We know that the vector b of C of (7.2.3) is parallel to a row of $\overline{B}$. That vector of $\overline{B}$ cannot be c, for otherwise, C is 2-separable. So assume b is parallel to the second row of $\overline{B}$, say row v. An exchange of rows v and p of C produces

(7.2.4)
$$\begin{array}{c} \\ p \\ \\ v \end{array}\begin{array}{|c|c|} \hline c & 1 \\ \hline b & 1 \\ \hline 0/1 & 0 \\ \hline b & 0 \\ \hline \end{array}$$

Matrix C after exchange of rows p and v

Except for the replacement of the row index p by v, that row exchange does not affect the submatrix $\overline{B}$. The column vector to the right of the current $\overline{B}$ must be parallel to, say, the first column of $\overline{B}$. Exchange the first column and the last column of C. By the 3-connectedness of N', we must have

(7.2.5)
$$\begin{array}{|c|c|c|} \hline 1 & \overline{c} & 1 \\ \hline 1 & \overline{b} & 1 \\ \hline 0 & 0/1 & 0 \\ \hline 0 & \overline{b} & 1 \\ \hline \end{array}$$

Matrix C after exchange of first and last column

Pivot on the circled 1 of (7.2.5). That pivot produces the matrix of (7.2.6) below.

Inductively, assume that the current C is given by (7.2.7) below. Suppose that the subvector b of row x is a unit vector, say with 1 in column $z \in Y_2$, and that column z of the submatrix $\overline{\overline{B}}$ is zero. Then we have $(X_1 \cup Y_1 \cup \{x, y, z\},\ (X_2 \cup Y_2) - \{z\})$ as a 2-separation of C unless $|(X_2 \cup Y_2) - \{z\}| \leq 1$. If $|(X_2 \cup Y_2) - \{z\}| = 1$, then C contains a zero row indexed by X_2, or C has a zero or unit vector column indexed by $Y_2 - \{z\}$. Either case is a contradiction of the 3-connectedness of N'.

$$
\text{(7.2.6)} \qquad
\begin{array}{|c|c|c|}
\hline
1 & \overline{c}\,' & 0 \\
\hline
1 & \overline{b} & 1 \\
\hline
0 & 0/1 & 0 \\
\hline
0 & \overline{b} & 1 \\
\hline
\end{array}
$$

Matrix C after pivot

$$
\text{(7.2.7)} \qquad
\begin{array}{c|cc|c|c|}
 & \multicolumn{2}{c|}{Y_1} & Y_2 & y \\
\hline
 & 1 & 0 & c & 0 \\
\cline{4-5}
X_1 & 1 \ddots & \ddots\ 1 & 0 & 0 \\
\cline{4-5}
 & 0 & 1 & b & 1 \\
\hline
X_2 & \multicolumn{2}{c|}{0} & \overline{\overline{B}} & 0 \\
\hline
x & \multicolumn{2}{c|}{0} & b & 1 \\
\hline
\end{array}
$$

Matrix C for inductive proof

Thus, $|(X_2 \cup Y_2) - \{z\}| = 0$, i.e., $X_2 = \emptyset$ and $Y_2 = \{z\}$, which implies $b = [\,1\,]$. Since the columns z and y of C must be distinct, we also have $c = [\,1\,]$. Then C is

$$
\text{(7.2.8)} \qquad
\begin{array}{c|cccc|c|c|}
 & \multicolumn{4}{c|}{Y_1} & z & y \\
\hline
 & 1 & & & & 1 & \\
 & 1 & 1 & & & & \\
X_1 & & 1 & \ddots & & & 0 \\
 & & & \ddots & 1 & & \\
\cline{7-7}
 & & & & 1 & 1 & 1 \\
\hline
x & \multicolumn{4}{c|}{0} & 1 & 1 \\
\hline
\end{array}
$$

Matrix C displaying wheel case

Evidently, the current $\overline{B}$ is a matrix of type (5.2.9). Accordingly, N is wheel. A pivot in C on the 1 in the bottom right corner confirms that N' is a wheel as well.

Once more, assume that in the matrix C of (7.2.7), the row subvector b is a unit vector with 1 in column z. But this time, suppose that column

z of $\overline{\overline{B}}$ is nonzero. By a pivot in column z of $\overline{\overline{B}}$, we convert the situation to the third possible case, where b is not a unit vector. In that third case, the vector b in row x of C must be parallel to a row of $\overline{\overline{B}}$, say row p. Exchange rows p and x of C. We get

$$
\begin{array}{c|c|c|c|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{y} \\
\cline{2-4}
 & 1 & c & 0 \\
\cline{3-4}
 & 1 \ddots & & \\
 & \ddots\ 1 & 0 & 0 \\
\cline{3-4}
 & 1 & b & 1 \\
\hline
x & & b & 1 \\
\cline{3-4}
 & 0 & 0/1 & 0 \\
\hline
p & 0 & b & 0 \\
\hline
\end{array}
\tag{7.2.9}
$$

Matrix C after exchange of rows p and x

The remaining arguments are analogous to those for (7.2.4)–(7.2.6). They produce an instance of (7.2.7) where $|Y_1|$ has been increased by 1. By induction, the case already discussed must eventually be encountered where N is a wheel and N' is the next larger wheel. $\square$

When specialized to graphs, the splitter Theorem (7.2.1) becomes the following result.

(7.2.10) Corollary. *Let $\mathcal{G}$ be a class of connected graphs that is closed under isomorphism and under the taking of minors. Let H be a 3-connected graph of $\mathcal{G}$ with at least six edges.*

(a) *If H is not a wheel, then H is a splitter of $\mathcal{G}$ if and only if $\mathcal{G}$ does not contain any graph derived from H by one of the following two extension steps:*

  (1) *Connect two nonadjacent nodes by a new edge.*

  (2) *Partition a vertex of degree at least 4 into two vertices, each of degree at least 2, then connect these two vertices by a new edge.*

(b) *If H is a wheel, then H is a splitter of $\mathcal{G}$ if and only if $\mathcal{G}$ does not contain any of the extensions of H described under* (a) *and does not contain the next larger wheel.*

Proof. Let $\mathcal{M}$ be the collection of graphic matroids produced by the graphs of $\mathcal{G}$. Define N to be the graphic matroid of the graph H. Lemma (6.2.7) says that the extensions of H described under (a) are precisely the 3-connected 1-edge extensions of H. Thus, these extensions correspond to the 3-connected 1-element graphic extensions of N. The result then follows from the splitter Theorem (7.2.1). $\square$

Typically, we will specify $\mathcal{M}$ or $\mathcal{G}$ by exclusion of certain minors and of all their isomorphic copies. Clearly, any collection of matroids or graphs so specified is closed under isomorphism and under the taking of minors.

Two graph examples of splitters are given in the next theorem. Recall that W_n is the wheel graph with n spokes.

(7.2.11) Theorem. *W_3 is a splitter of the graphs without W_4 minors, and K_5 is a splitter of the graphs without $K_{3,3}$ minors.*

Proof. There is no 3-connected 1-edge extension of W_3. Thus, by part (b) of Corollary (7.2.10), W_3 is a splitter of the graphs without W_4 minors. For the second part, we note that up to isomorphism there is just one 3-connected 1-edge extension of K_5. To obtain it, one partitions one vertex of K_5 into two vertices of degree 2 and connects the two vertices by a new edge. The resulting graph is readily seen to have a $K_{3,3}$ minor. Thus, by part (a) of Corollary (7.2.10), K_5 is a splitter of the graphs without $K_{3,3}$ minors. □

We will see a number of other splitter examples in subsequent chapters. In the next section, we deduce from the splitter Theorem (7.2.1) certain sequences of nested minors and Tutte's wheel theorem.

7.3 Sequences of Nested Minors and Wheel Theorem

Recall from Section 7.1 that two matroids are *nested* if one of them is a minor of the other one. In this section, we prove the existence of certain sequences of nested minors of binary matroids. As a special case, we establish Tutte's wheel theorem. Main tools are the splitter Theorem (7.2.1) and results proved in Chapter 5 with the path shortening technique.

For a given binary matroid, an arbitrary sequence of nested minors is easy to find. The task becomes difficult and interesting when one imposes conditions on the sequence. We need a few definitions to express such conditions.

Suppose two matroids are nested. Define the *rank gap* between the two matroids to be the absolute difference in rank between them. Analogously, define the *corank gap*. Finally, let the *gap* be the sum of the rank gap and the corank gap. Evidently, the gap is the number of elements that occur only in the larger matroid.

Consider a sequence of nested minors of a given binary matroid. Then the *rank gap* of the sequence is the maximum rank gap among the pairs of successive minors of the sequence. Analogously, define the *corank gap* and the *gap* of the sequence.

We now state the conditions under which we want to find nested minor sequences. Suppose for some $k \geq 2$, we have a k-connected binary matroid M with a k-connected minor N. In the typical situation, we want to find a sequence of nested k-connected minors M_0, M_1, $M_2, \ldots$, $M_t = M$, where M_0 is demanded to be isomorphic to N. Furthermore, the rank gap, or the corank gap, or the gap is to be bounded by some given constant. Other variants are possible. For example, one may require that a given element of M that does not occur in M_0 be present in M_1, that a given element not occurring in M_1 be present in M_2, and so on.

Sequences of the desired kind are readily determined when $k = 2$. The main ingredient for their construction is a recursive application of Lemma (5.2.4). We skip the details of that simple case, and instead turn immediately to the much more complicated case with $k = 3$. Specifically, we prove the existence of three types of sequences for $k = 3$, then point out extensions in Section 7.5.

In each of the cases treated here, we are given a 3-connected binary matroid M with a 3-connected proper minor N on at least six elements. In the first case, we desire a sequence of nested 3-connected minors M_0, $M_1, \ldots$, $M_t = M$, where M_0 is isomorphic to N and where the gap is small. The next theorem shows that the gap can be held to 1 or 2.

(7.3.1) Theorem. *Let M be a 3-connected binary matroid having a 3-connected proper minor N on at least six elements.*

(a) *Assume N is not a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors M_0, $M_1, \ldots$, $M_t = M$, where M_0 is isomorphic to N and where the gap is 1.*
(b) *Assume N is a wheel. Then for some $t \geq 1$, there is sequence of nested 3-connected minors M_0, $M_1, \ldots$, $M_t = M$ with the following features. M_0 is isomorphic to N. For some $0 \leq s \leq t$, the subsequence M_0, $M_1, \ldots$, M_s consists of wheels and has gap 2, and the subsequence M_s, $M_{s+1}, \ldots$, $M_t = M$ has gap 1.*

Proof. We first establish part (a). Thus, we assume that N is not a wheel. Indeed, inductively we assume for some $i \geq 0$, the existence of a sequence of nested 3-connected minors M_0, $M_1, \ldots$, M_i of M, where M_0 is isomorphic to N, where M_i is not a wheel, and where the gap is 1. If $M_i = M$, we are done. So assume that M_i is a proper minor of M.

We rely on the contrapositive statement of part (a) of the splitter Theorem (7.2.1) to find a larger sequence. To this end, we define $\mathcal{M}$ to be the matroid collection containing M, all minors of M, and all matroids isomorphic to these matroids. By this definition, $\mathcal{M}$ is closed under isomorphism and under the taking of minors. Since M_i is a 3-connected proper minor of the 3-connected $M \in \mathcal{M}$, it cannot be a splitter of $\mathcal{M}$. Thus, by part (a) of Theorem (7.2.1), $\mathcal{M}$ contains a matroid M_{i+1} that is a 3-connected 1-element extension of a matroid isomorphic to M_i. Now every 1-element reduction of a wheel with at least six elements is 2-separable. Thus, if M_{i+1} is a wheel, then M_i is 2-separable, a contradiction. We conclude that M_{i+1} is not a wheel.

If necessary, we relabel M_0, $M_1, \ldots, M_i$ so that they constitute a sequence of nested minors of M_{i+1}. These matroids plus M_{i+1} satisfy the induction hypothesis for $i+1$. By induction, the claimed sequence exists for M.

The proof of part (b) is essentially the same, except that we establish M_{i+1} using part (b) of Theorem (7.2.1) when M_i is a wheel. $\square$

We have used the splitter Theorem (7.2.1) for a simple proof of Theorem (7.3.1). Indeed, the two theorems are essentially equivalent, since one may deduce the splitter Theorem (7.2.1) from Theorem (7.3.1) just as easily. We sketch the proof.

Let $\mathcal{M}$ and N be as specified in the splitter Theorem (7.2.1). Suppose N is not a wheel. We must show that N is a splitter of $\mathcal{M}$ if and only if $\mathcal{M}$ does not contain any 3-connected 1-element extension of N. We prove the nontrivial "if" part by contradiction. So let M be a 3-connected matroid of $\mathcal{M}$ with N as proper minor. By Theorem (7.3.1), there is a sequence of nested 3-connected minors M_0, $M_1, \ldots, M_t = M$, where M_0 is isomorphic to N, and where the gap is 1. Since $\mathcal{M}$ is closed under isomorphism, we may assume M to be so chosen that M_0 is equal to N. Then M_1 is a 3-connected 1-element extension of N and $M_1 \in \mathcal{M}$, which contradicts the assumed absence of such extensions. The case where N is a wheel is treated analogously.

A direct translation of Theorem (7.3.1) into graph language results in the following corollary.

(7.3.2) Corollary. *Let G be a 3-connected graph having a 3-connected proper minor H with at least six edges.*

(a) *Assume H is not a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors G_0, $G_1, \ldots, G_t = G$, where G_0 is isomorphic to H, and where each G_{i+1} has exactly one edge beyond those of G_i.*
(b) *Assume H is a wheel. Then for some $t \geq 1$, there is a sequence of nested 3-connected minors G_0, $G_1, \ldots, G_t = G$ with the following features. G_0 is isomorphic to H. For some $0 \leq s \leq t$, the subsequence G_0, $G_1, \ldots, G_s$ consists of wheels, where each G_{i+1} has exactly one additional spoke beyond those of G_i. Furthermore, in the subsequence G_s, $G_{s+1}, \ldots, G_t = G$, each G_{i+1} has exactly one edge beyond those of G_i.*

One may combine Corollary (7.3.2) with Corollary (5.2.15) to obtain Tutte's wheel theorem, which is listed next.

(7.3.3) Theorem (Wheel Theorem). *Let G be a 3-connected graph on at least six edges. If G is not a wheel, then G has some edge z such that at least one of the minors G/z and $G\backslash z$ is 3-connected.*

Proof. Corollary (5.2.15) says that a 3-connected graph with at least six edges, in particular G specified here, has a W_3 minor. Thus, G has a largest wheel minor, say H. Since G is not a wheel, H is a proper minor of G. We apply Corollary (7.3.2) to G and H. Accordingly, G has a sequence of nested 3-connected minors G_0, $G_1, \ldots, G_t = G$, where G_0 is isomorphic to H. Since H is a largest wheel minor of G and since G is not a wheel, the index s of part (b) of Corollary (7.3.2) must be zero, and $t \geq 1$. We also conclude from that part that $G = G_t$ has exactly one edge beyond those of G_{t-1}. Put differently, the 3-connected minor G_{t-1} is for some edge z equal to G/z or $G\backslash z$, which proves the theorem. □

Theorem (7.3.3) can obviously be rewritten so that it becomes a wheel theorem for binary matroids instead of graphs. The proof relies on Theorem (7.3.1) instead of Corollary (7.3.2).

From the sequence of nested 3-connected minors $M_0, M_1, \ldots, M_t = M$ of Theorem (7.3.1), one can derive a number of other interesting sequences. A particular construction utilizes the representation matrices of these minors. We present details following some observations.

By Lemma (3.3.12), a binary matroid M with a given minor $\overline{M}$ has a representation matrix that displays $\overline{M}$. We apply this result inductively to the sequence of nested minors M_0, $M_1, \ldots$, $M_t = M$, and conclude that M has a representation matrix B that simultaneously displays M_0, $M_1, \ldots$, $M_t = M$, say by nested matrices B^0, $B^1, \ldots$, $B^t = B$. Let C^i be the column submatrix of B that has the same column index set as B^i. Let b^i be a row vector of C^i, say with row index x. Assume that b^i is not a row of B^i. Indeed, assume that b^i is nonzero, is not a unit vector, and is not parallel to a row of B^i. Lemma (6.2.6) shows that under these assumptions, B^i plus b^i represent a 3-connected minor $M_i\&x$ of M. There are other minors M_j without x for which $M_j\&x$ is 3-connected. Specifically, let k be the largest index, $i \leq k \leq t$, such that M_j does not contain x. We claim that for each $i < j \leq k$, $M_j\&x$ is 3-connected. The proof consists of the following observation. Let b^j be the row vector of C^j indexed by x. Since $j \leq k$, b^j is not part of B^j. Evidently, B^i is a submatrix of B^j, and b^i is a subvector of b^j. We know that b^i is nonzero, is not a unit vector, and is not parallel to a row of B^i. Then the latter statement must also hold when we use j as superscript instead of i. Accordingly, $M_j\&x$ is 3-connected.

We use these observations as follows. In the sequence M_0, $M_1, \ldots$, $M_t = M$, we redefine each M_j, $i \leq j \leq k$, to be $M_j\&x$. Correspondingly, we redefine the B^j matrices by adjoining the b^j row vectors. The result is a new sequence of nested 3-connected minors. The rank gap of that sequence is larger than that for the original one. But the corank gap is still the same. Indeed, since the gap of the original sequence was 1, the corank gap of the new sequence is 0 or 1.

We repeat the above process, using all possible indices i and x, until no further changes are possible. This occurs when each final B^i cannot be extended to a larger 3-connected matrix within C^i. The final sequence may contain duplicate minors. In that case, we delete just enough minors to eliminate all such duplicates. Then we redefine the indices so that M_0, $M_1, \ldots, M_t = M$ is now the sequence resulting from the above process. Correspondingly, we redefine the indices of the B^i and C^i matrices.

If M_0 and M have same corank, then $t = 0$, and the outcome is $M_0 = M_t = M$, an uninteresting case. If M_0 and M have different corank, we must have $t \geq 1$. Consider the latter case. Evidently, each B^{i+1} is deduced from B^i by first adjoining any number of row vectors b^i to B^i, and then adjoining a column vector a. We claim that each one of these vectors b^i is a unit vector or is parallel to a row of B^i. Indeed, in any other case of a nonzero b^i, the previously described process would have adjoined b^i to B^i and would have redefined M_i prior to termination, a contradiction. In the case of a zero vector b^i, B^{i+1} would have a zero or unit vector, and thus would be 2-separable, again a contradiction.

In terms of the minors of the final sequence, we thus have proved that each M_{i+1} may be derived from M_i by extensions involving any number of series elements, possibly none, followed by a 1-element addition.

The above construction is the main ingredient in the proof of the following variant of Theorem (7.3.1).

(7.3.4) Theorem. *Let M be a 3-connected binary matroid with a 3-connected proper minor N on at least six elements. If M does not contain a 3-connected 1-element expansion (resp. addition) of any N minor, then M has a sequence of nested 3-connected minors $M_0, M_1, \ldots, M_t = M$, where M_0 is an N minor of M, and where each M_{i+1} is obtained from M_i by expansions (resp. additions) involving some series (resp. parallel) elements, possibly none, followed by a 1-element addition (resp. expansion).*

Proof. Clearly, the parenthetic case is dual to the stated one. Thus, we only consider the case where M does not contain any 3-connected 1-element expansion of any N minor. We prove the existence of the claimed sequence of nested minors as follows.

Apply the above construction process to the sequence of nested 3-connected minors $M_0, M_1, \ldots, M_t = M$ of Theorem (7.3.1). The result is a new sequence $M_0, M_1, \ldots, M_t = M$ where each M_{i+1} may be derived from M_i by expansions involving any number of series elements, possibly none, followed by a 1-element addition. Since M does not contain a 3-connected 1-element expansion of any N minor, the construction must have left M_0 unchanged. The sequence so produced is the desired one. $\square$

Theorem (7.3.4) may be specialized to graphs as follows.

(7.3.5) Corollary. *Let G be a 3-connected graph with a 3-connected minor H on at least six edges.*

(a) *Suppose no H minor of G can be extended to another 3-connected minor of G by the following process: Some node of H of degree at least 4 is partitioned into two nodes, each with degree at least 2, and then the two nodes are connected by a new edge. Then G has a sequence of nested 3-connected minors $G_0, G_1, \ldots, G_t = G$ with the following properties. G_0 is an H minor of G. Each G_{i+1} may be obtained from G_i as follows. First, at most two edges of G_i are replaced by paths of length 2. Next, an edge is added so that the new graph has no degree 2 nodes and no parallel edges.*

(b) *Suppose no H minor of G can be extended to another 3-connected minor of G by connecting two nonadjacent nodes by a new edge. Then G has a sequence of nested 3-connected minors $G_0, G_1, \ldots, G_t = G$ with the following properties. G_0 is an H minor of G. Each G_{i+1} may be obtained from G_i as follows. First, some edges, possibly none, incident at some vertex are replaced by two parallel edges each. The new vertex must have degree of at least 4. Next, that vertex is partitioned into two vertices, each with degree at least 2, such that no two edges remain parallel. Finally, the two vertices just created are joined by a new edge.*

Proof. Part (a) is a routine translation of the non-parenthetic part of Theorem (7.3.4) into graph language, in the same sense that Corollary (7.3.2) is the graph version of Theorem (7.3.1). Part (b) is a translation of the parenthetic part of Theorem (7.3.4). □

We conclude this section by proving that the sequences of nested 3-connected minors of the above theorems and corollaries can be readily found. Additional material about nested sequences is contained in Section 7.5.

(7.3.6) Theorem. *There is a polynomial algorithm for the following problem. The input consists of M and N of Theorem (7.3.1) or (7.3.4), or of G and H of Corollary (7.3.2) or (7.3.5). The output is a 2-separation of M or G, or a sequence of nested 3-connected minors with properties as specified in the respective theorem or corollary.*

Proof. The arguments that prove the cited theorems and corollaries can be summarized as follows. Theorem (6.4.1) implies the splitter Theorem (7.2.1), which in turn implies Theorem (7.3.1) and Corollary (7.3.2). The latter results imply Theorem (7.3.4) and Corollary (7.3.5). The polynomial algorithm claimed for the stated problems is essentially an efficient implementation of the constructive proofs of these implications. We first sketch the details for the matroid case with given M and N. With the polynomial algorithm of Theorem (6.4.7), we locate a 2-separation of M or produce the 3-connected 1- or 2-element extensions claimed by Theorem (6.4.1). We use these extensions plus the proof procedure of the splitter Theorem

(7.2.1) to derive the sequence of nested minors of Theorem (7.3.1). From that sequence we construct the sequence of Theorem (7.3.4).

The graph case may be handled by appropriate translation of each step of the above algorithm into graph language. Alternately, one could represent the given G and H by graphic matroids M and N, apply the polynomial algorithm already described, and extract from G the minors $G_0, G_1, \ldots, G_t = G$ corresponding to the sequence $M_0, M_1, \ldots, M_t = M$ found by the algorithm. □

The remainder of this book includes several demonstrations of the power and utility of sequences of nested 3-connected minors. The discussion in the next section may be viewed to be one such demonstration.

7.4 Characterization of Planar Graphs

In this section, we prove Kuratowski's characterization of graph planarity in terms of excluded $K_{3,3}$ and K_5 minors. We state that theorem and describe a proof of beautiful simplicity due to Thomassen.

(7.4.1) Theorem. *A graph is planar if and only if it has no $K_{3,3}$ or K_5 minors.*

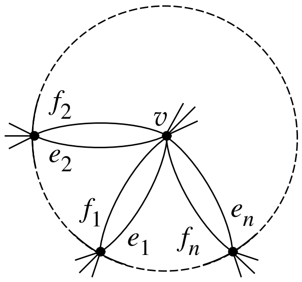
Proof. The "only if" part follows from the fact that planarity is maintained under the taking of minors, and that by Lemma (3.2.48) both $K_{3,3}$ and K_5 are not planar. For a proof of the nontrivial "if" part, let G be a connected nonplanar graph each of whose proper minors is planar. We need to show that G is isomorphic to $K_{3,3}$ or K_5.

We first prove that G cannot be 1- or 2-separable. The case of 1-separability is trivial. Suppose G is 2-separable. Let G be produced by identifying nodes k and l of a graph G_1 with nodes m and n, respectively, of a graph G_2. Now G_1 plus an edge connecting nodes k and l is planar, since that graph, say G_1', is isomorphic to a proper minor of G. The graph G_2' similarly defined from G_2, is also planar. It is easy to combine planar drawings of G_1' and G_2' to one of G.

Thus, G is 3-connected. According to Lemma (5.2.15), G must have a W_3 minor, say H. Since W_3 is also the complete graph K_4, no H minor of G can be extended to another minor of G by addition of an edge that connects two nonadjacent nodes. Under the latter condition, part (b) of Corollary (7.3.5) states that G has a sequence of nested 3-connected minors $G_0, G_1, \ldots, G_t = G$ with the following properties. G_0 is an H minor of G. Each G_{i+1} may be obtained from G_i as follows. First, some edges, possibly none, incident at some vertex are replaced by two parallel edges each. The new vertex must have degree of at least 4. Next, that vertex is partitioned into two vertices, each with degree at least 2, such that no two edges remain parallel. Finally, the two vertices just created are joined by a new edge.

By the minimality of G, the graph G_{t-1} of the sequence is planar, while $G = G_t$ is not. Consider a realization of G_{t-1} in the plane. Note that we do not rely on the fact, not proved here, that the drawing of G_{t-1} is essentially unique. As just described, G may be derived from G_{t-1} by the replacement of some edges incident at a vertex by some parallel edges, followed by a partitioning of that new vertex, etc. Define G'_{t-1} to be the graph on hand when the first step has been carried out — that is, when in G_{t-1} some edges incident at a vertex have been replaced by two parallel edges each. Let v be the vertex of G'_{t-1} to be partitioned, say into vertices v_1 and v_2. By the connectivity conditions, a partial drawing of G'_{t-1} that emphasizes vertex v is either

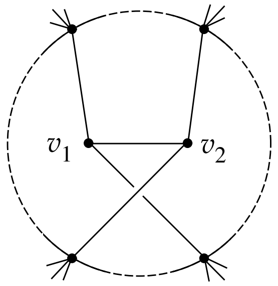
(7.4.2)



Vertex v of G'_{t-1}

or is deduced from that drawing by deletion of any number of the parallel edges $f_1, f_2, \ldots, f_n$. The dashed segments represent internally node-disjoint paths. The vertex v is so partitioned into v_1 and v_2 that the resulting graph G is nonplanar. This implies that we cannot replace v of the assumed drawing for G'_{t-1} by v_1 and v_2, and then join these two nodes by an edge, while retaining a planar drawing. There are two possible causes for the nonplanarity. First, some of the edges $e_1, e_2, \ldots, e_n$ may cross when they are attached to v_1 and v_2. In fact, no matter which particular situation occurs, just four edges can always be selected to produce the graph of (7.4.3) below.

(7.4.3)

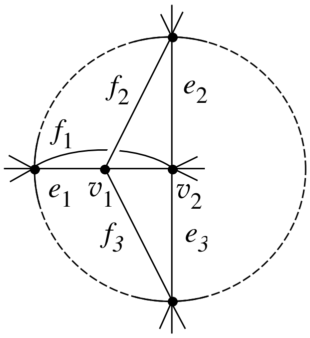


Partition of vertex v producing $K_{3,3}$ minor

The graph contains a subdivision of $K_{3,3}$, so we are done. In the second case, the e_i edges can be attached to v_1 and v_2 so that they do not cross. However, crossing edges are encountered when we attach the f_i edges. We may suppose this to be so even when we relabel edges such that some e_i become f_i, since otherwise we can produce the earlier situation.

This leaves just one case, where $n = 3$, and where for $i = 1, 2, 3$, both e_i and f_i are present. We then have

(7.4.4)



Partition of vertex v producing K_5 minor

which contains a subdivision of K_5. □

Actually, Kuratowski proved for a nonplanar graph G the presence of a subgraph that is a subdivision of $K_{3,3}$ or K_5 and not just a $K_{3,3}$ or K_5 minor. Now a $K_{3,3}$ minor induces a subdivision of $K_{3,3}$. For a proof, carry out the expansion steps that convert the $K_{3,3}$ minor to a subgraph of G. The same argument applies to a K_5 minor, except when an expansion step splits a vertex of degree 4 into two vertices each of which has degree 3 upon insertion of the new edge. The graph so produced has a $K_{3,3}$ minor. Thus, Theorem (7.4.1) is equivalent to Kuratowski's original formulation.

In the final section, we cover extensions and references.

7.5 Extensions and References

The splitter Theorem (7.2.1) and its extension to nonbinary matroids is due to Seymour (1980b); see also Tan (1981) and Truemper (1984). The extension to the nonbinary case requires the introduction of the whirls given by (5.4.1) and (5.4.2). The latter matroids constitute a second special case besides that of wheels. The graph version of Theorem (7.3.1), and thus effectively Corollary (7.2.10), were independently proved in Negami (1982). An early splitter example is implicit in Wagner (1937a). A number of splitters are given in Seymour (1980b), Oxley (1987b), (1987c), (1989a), (1989b), (1990a), and Truemper (1988).

Tutte's wheel Theorem (7.3.3) appeared in Tutte (1966a). The extension of that result to general matroids is also due to Tutte, and is known as the wheels and whirls theorem (Tutte (1966b)). Strengthened versions are in Halin (1969), Oxley (1981b), and Coullard and Oxley (1992).

Theorem (7.3.4) is a binary and slightly weaker version of results in Seymour (1980b) and Truemper (1984). The related graph result appeared for the first time in Barnette and Grünbaum (1969). That graph result has been repeatedly rediscovered.

Kuratowski's original characterization of planar graphs in terms of forbidden subgraphs appeared in Kuratowski (1930). The equivalence of that result to the excluded minor version given by Theorem (7.4.1) is due to Wagner (1937b). The amazingly short proof of Theorem (7.4.1) is from Thomassen (1980). The matroid version of Theorem (7.4.1) is proved in Bixby (1977).

Substantially stronger results about sequences of nested 3-connected minors exist. Bixby and Coullard (1987) contains the most recent and strongest one: Let N be a 3-connected proper minor of a 3-connected matroid M. Then for any element z of M that is not in N, there is a 3-connected minor N' of M that contains z, that has N as a minor, and that has at most four elements beyond those of N. Note that isomorphisms are not involved in this result. When the reference to the element z is dropped, then N' can be guaranteed to have at most three elements beyond those of N instead of four (Truemper (1984)). The latter result can be strengthened when N has no triangles and no triads (= dual triangles). In that situation, N' need to have at most two elements beyond those of N (Bixby and Coullard (1984)). The sequences of nested 3-connected minors that are implied by these theorems may be modified by the construction given in Section 7.3 prior to Theorem (7.3.4). Some examples are worked out in Truemper (1984). For a recursive characterization of 3-connectivity using so-called separating cocircuits, see Bixby and Cunningham (1979).

The original proofs of the results cited in the preceding paragraph are by no means simple. But there is a unified way in which they can be obtained. We sketch the main idea. Recall that the splitter Theorem (7.2.1) is based on the notion of induced 2-separations and on Theorem (6.3.20), which deals with minors called minimal under isomorphism in Section 6.3. Similarly, the theorems cited in the preceding paragraph may be derived using the notion of induced 2-separations plus the following result of Truemper (1986) about minors called minimal in Section 6.3: Suppose that a 2-separation of a matroid N on at least four elements does not induce a 2-separation of a matroid M containing N, and that M is minimal with respect to that condition. Then M has at most five elements beyond those of N.

For $k \geq 4$, characterizations of sequences of nested k-connected minors seem to become very complex. Rajan (1986) and Robertson (1984) contain results about the graph case for various kinds of 4-connectivity.

Work cited in Section 5.4 for minors with specified elements (Bixby (1974), Seymour (1981e), (1985b), (1986a), (1986b), Oxley (1984), (1987a), (1990a), Kahn (1985), Coullard (1986), Coullard and Reid (1988), Oxley and Row (1989), Oxley and Reid (1990), and Reid (1990), (1991a), (1991b), (1993), (1996)) is closely related to the material of this chapter. Several of these results are readily proved with the earlier mentioned generalization of Theorem (7.3.1) due to Bixby and Coullard (1987).

Related to the above results are theorems claiming the following process to be possible. First, a minor N of a matroid M is replaced by an isomorphic copy that still contains a specified subset Z of the elements of N. Second, that isomorphic copy is extended to another minor of M with certain attractive properties. A theorem of this type is given in Tseng and Truemper (1986).