Chapter 6

Separation Algorithm

6.1 Overview

So far, we have described two simple matroid tools: the series-parallel and delta-wye constructions of Chapter 4, and the path shortening technique of Chapter 5. In this chapter, we introduce a third tool called the separation algorithm. Before we summarize that method and some of its uses, let us recall from Section 3.3 some definitions and results concerning matroid separations. Let M be a binary matroid on a set E and with rank function $r(\cdot)$. Furthermore, let B be a binary representation matrix of M with row index set X and column index set Y. Suppose two sets E_1 and E_2 partition E. For $i = 1, 2$, define $X_i = E_i \cap X$ and $Y_i = E_i \cap Y$. Then (E_1, E_2) is a k-separation of M, provided the matrix B when partitioned as

(6.1.1)
$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & B^1 & D^2 \\ \hline X_2 & D^1 & B^2 \\ \hline \end{array}$$

Partitioned version of B

satisfies

(6.1.2)
$$\begin{aligned} &|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq k \\ &\text{GF(2)-rank } D^1 + \text{GF(2)-rank } D^2 \leq k - 1 \end{aligned}$$

The separation is exact if the inequality of (6.1.2) involving the GF(2)-rank of D^1 and D^2 holds with equality. In terms of E_1, E_2, E, and the rank function $r(\cdot)$, the conditions of (6.1.2) are

$$
\begin{aligned}
&|E_1|, |E_2| \geq k \\
&r(E_1) + r(E_2) \leq r(E) + k - 1
\end{aligned}
\tag{6.1.3}
$$

In the case of an exact separation, the inequality involving the rank function holds with equality. Finally, for $k \geq 2$, the matroid M is k-connected if it does not have an l-separation for any $1 \leq l < k$.

We are ready to summarize the material of this chapter. In Section 6.2, we describe and validate the just-mentioned separation algorithm. The scheme solves the following problem. Given is a binary matroid M with a minor N. For some $k \geq 1$, an exact k-separation (F_1, F_2) is known for N. We want to decide whether or not M has a k-separation (E_1, E_2) where for $i = 1, 2$, $E_i \supseteq F_i$. In the affirmative case, we say that (F_1, F_2) *induces* the k-separation (E_1, E_2).

The problem of finding induced separations may seem rather technical. But several important matroid results can be derived from its solution. Two such results are included in Sections 6.3 and 6.4. The first result provides sufficient conditions for the existence of induced separations. In later chapters, we rely upon these conditions to prove the existence of a number of decompositions. The second result builds upon the first one. It concerns the existence of certain extensions of 3-connected minors in 3-connected binary matroids. We use that result in the next chapter to establish the so-called splitter theorem and the existence of some sequences of minors. Finally, in Section 6.5, we sketch extensions of the results to the nonbinary case and provide references.

The chapter relies on the material of Chapters 2, 3, and 5.

6.2 Separation Algorithm

Suppose we are given a binary matroid M on a set E. Let N be a minor of M on a set $F \subseteq E$. Assume that N has, for some $k \geq 1$, an exact k-separation (F_1, F_2). We want to know whether or not M has a k-separation (E_1, E_2) where for $i = 1, 2$, $E_i \supseteq F_i$. If such (E_1, E_2) exists, we declare it to be *induced* by (F_1, F_2). In this section, we describe a simple method called the separation algorithm for deciding the existence of (E_1, E_2).

We begin with an informal discussion that relies on a particular binary representation matrix B^N of N. Let X_2 be a maximal independent subset of N contained in F_2. Then select a subset X_1 from F_1 so that $X_1 \cup X_2$ is a basis of N. For $i = 1, 2$, let $Y_i = F_i - X_i$. The desired representation matrix B^N of N corresponds to the base $X_1 \cup X_2$ of N. By the derivation of that base, B^N is of the form

(6.2.1)
$$B^N = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline X_2 & D & A^2 \\ \hline \end{array}$$

Partitioned version of B^N

for some A^1, A^2, and D. Indeed, the zero submatrix indexed by X_1 and Y_2 is present since X_2 is a maximal independent subset of F_2, and since $Y_2 = F_2 - X_2$. By assumption, (F_1, F_2) is an exact k-separation of N, so we have by (6.1.2)

(6.2.2)
$$\begin{aligned} &|X_1 \cup Y_1|, |X_2 \cup Y_2| \geq k \\ &\text{GF(2)-rank } D = k - 1 \end{aligned}$$

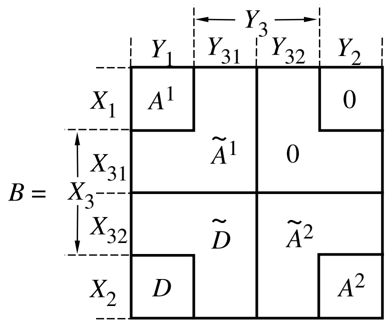
We embed B^N into a representation matrix of M, and thus make the minor N visible, as follows. Since $X_1 \cup X_2$ is independent in N, that set is also independent in M. Thus, we can find a set $X_3 \subseteq E - F$ so that $X_1 \cup X_2 \cup X_3$ is a base of M. Let $Y_3 = (E - F) - X_3$. The representation matrix B of M for this base contains B^N as submatrix. We depict B below. For reasons to become clear shortly, we have placed A^1, A^2, D, and the 0 submatrix of B^N into the corners of B.

(6.2.3)
$$B = \begin{array}{c|c|c|c|} & Y_1 & Y_3 & Y_2 \\ \hline X_1 & A^1 & & 0 \\ \hline X_3 & & 0/1 & \\ \hline X_2 & D & & A^2 \\ \hline \end{array}$$

Matrix B for M displaying partitioned B^N

Recall that we want to find a k-separation (E_1, E_2) of M where for $i = 1, 2$, $E_i \supseteq F_i$, or to prove that no such k-separation exists. In terms of the index sets of (6.2.3), we want to partition the set X_3 into X_{31}, X_{32}, and the set Y_3 into Y_{31}, Y_{32} so that the correspondingly refined matrix B of (6.2.3) is of the form

(6.2.4)

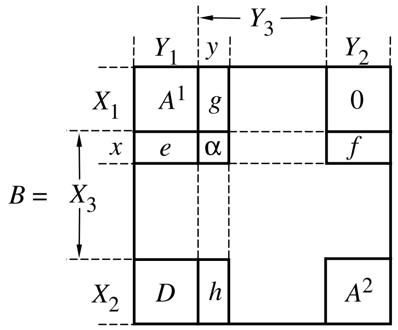


Partition of B induced by that of B^N

with GF(2)-rank $\tilde{D} \leq k-1$, or we are to prove that such partitions of X_3 and Y_3 do not exist. In the affirmative case, D is a submatrix of $\tilde{D}$, and since GF(2)-rank $D = k-1$, we must have GF(2)-rank $\tilde{D} = k-1$. Put differently, if an induced k-separation of M exists at all, then it must be an exact induced k-separation.

We employ a recursive scheme to decide whether or not an induced k-separation exists. As the measure of problem size for the recursion, we use $|X_3 \cup Y_3|$. If $|X_3 \cup Y_3| = 0$, then $M = N$, and for $i = 1, 2$, $E_i = F_i$ gives the desired induced k-separation of M. Suppose $|X_3 \cup Y_3| \geq 0$. Redraw B of (6.2.3) so that an arbitrary row $x \in X_3$ and an arbitrary column $y \in Y_3$ are displayed as follows.

(6.2.5)



Matrix B for M with partitioned B^N,
row $x \in X_3$, and column $y \in Y_3$

The recursive method relies on the analysis of the following three cases of B of (6.2.5). Collectively, these cases cover all situations.

In the first case, we suppose that for some row $x \in X_3$, the subvector e is not spanned by the rows of D. We claim that in any induced k-separation, we must have $x \in X_{31}$. For a proof, take any such separation as depicted by (6.2.4). If $x \in X_{32}$, then the subvector e of row x occurs in $\tilde{D}$. Since e is not spanned by the rows of D, we have GF(2)-rank $\tilde{D} >$ GF(2)-rank D, which contradicts the condition GF(2)-rank $\tilde{D}$ = GF(2)-rank D. Thus, x must be in X_{31} as claimed. We now examine the subvector f of row x. Suppose that subvector is nonzero. Using (6.2.4) once more, we see that in any induced k-separation, the nonzero f forces x to be in X_{32}. But the latter requirement conflicts with the one determined earlier for x. Thus, an induced k-separation cannot exist, and we stop with that conclusion. So we now assume the subvector f of row x to be zero. We know already that x must be in X_{31} in any induced k-separation. Suppose in B of (6.2.5), we adjoin e to A^1 and f to the explicitly shown 0 submatrix, getting a new A^1 and a new 0 submatrix. Correspondingly, we extend N by x to $N\&x$ and redefine N to be the extended matroid. Evidently, $(X_1 \cup \{x\} \cup Y_1, X_2 \cup Y_2)$ is a k-separation of the new N, and that k-separation induces one in M if and only if this is so for the k-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of the original N. Thus, we may replace the original problem by one involving the new N. By our measure of problem size, the new problem is smaller than the original one, and we may apply recursion.

In the second case, we suppose that for some column $y \in Y_3$, the subvector g is nonzero. Arguing analogously to the first case via (6.2.4), we conclude that y must be in Y_{31} in any induced k-separation. Furthermore, suppose that the column subvector h of column y is not spanned by the columns of D. Using (6.2.4) once more, we see that y must also be in Y_{32} in any induced k-separation. Thus, an induced k-separation cannot exist, and we stop with that conclusion. So suppose that h is spanned by the columns of D. Then we adjoin g to A^1, h to D, and correspondingly redefine N to become $N+y$. Then the k-separation $(X_1 \cup Y_1 \cup \{y\}, X_2 \cup Y_2)$ of the new N induces a k-separation of M if and only if this is so for the k-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of the original N. Once more, we may replace the induced k-separation problem involving the original N by one with the new N. The latter problem is smaller, and we may invoke recursion.

For the discussion of the third and final case, we suppose that neither of the above cases applies. Equivalently, for all $x \in X_3$, the subvector e of row x is spanned by the rows of D, and for all $y \in Y_3$, the subvector g of column y is zero. By (6.2.5), $(X_1 \cup Y_1, X_2 \cup X_3 \cup Y_2 \cup Y_3)$ is a k-separation of M induced by the one of N, and we stop with that conclusion.

We call the above recursive method the *separation algorithm*. It clearly has a polynomial implementation. For later reference, we summarize the algorithm below.

Separation Algorithm

1. Suppose B of (6.2.5) has a row $x \in X_3$ with the indicated row subvectors e and f such that GF(2)-rank $[e/D] >$ GF(2)-rank D. Then x must be in X_{31}. Suppose, in addition, that f is nonzero. Then x must also be in X_{32}, i.e., B cannot be partitioned, and we stop with that declaration. On the other hand, suppose $f = 0$. Since x must be in

X_{31}, we adjoin e to A^1, and f to the explicitly shown 0 matrix. Then we start recursively again with the new B and the new B^N.

2. Suppose B of (6.2.5) has a column $y \in Y_3$ with the indicated column subvectors g and h such that g is nonzero. Then y must be in Y_{31}. Suppose, in addition, GF(2)-rank $[D \mid h] >$ GF(2)-rank D. Then y must also be in Y_{32}, i.e., B cannot be partitioned, and we stop with that declaration. On the other hand, suppose GF(2)-rank $[D \mid h] =$ GF(2)-rank D. Since y must be in Y_{31}, we adjoin g to A^1, and h to D. Then we start recursively again with the new B and the new B^N.

3. Finally, suppose that for all rows $x \in X_3$, the row subvector e satisfies GF(2)-rank $[e/D] =$ GF(2)-rank D, and that for all columns $y \in Y_3$, the column subvector g is 0. Then $X_{31} = Y_{31} = \emptyset$, $Y_{32} = Y_3$, $Y_{32} = Y_3$ gives the desired partition of B.

In the next two sections, we put the separation algorithm to good use. Preparatory to that discussion, we establish in the next lemma that certain extensions of binary matroids are 3-connected.

(6.2.6) Lemma. *Let N be a 3-connected binary matroid on at least six elements. Suppose a 1-, 2-, or 3-element binary extension of N, say M, has no loops, coloops, parallel elements, or series elements. Then M is 3-connected.*

Proof. Let C be a binary representation matrix of M that displays a representation matrix, say B, for N. By assumption, B is 3-connected. Suppose C is not connected. A straightforward case analysis proves C to contain a zero vector or unit vector. Thus, M has a loop, coloop, parallel elements, or series elements, a contradiction. Hence, C is connected. If C is not 3-connected, then by Lemma (3.3.20), there is a 2-separation of C with at least five rows/columns on each side. Then, necessarily, the matrix B has a 2-separation with at least two rows/columns on each side, another contradiction. We conclude that C, and hence M, are 3-connected. $\square$

From Lemma (6.2.6), we deduce the following result for 1-edge extensions of 3-connected graphs.

(6.2.7) Lemma. *Let H be a 3-connected graph with at least six edges. Then a connected 1-edge extension of H is 3-connected if and only if it is producible as follows: Either two nonadjacent vertices of H are connected by a new edge, or a vertex of degree at least 4 is partitioned into two vertices, each of degree at least 2, and the two new vertices are connected by a new edge.*

Proof. The described extension steps are precisely the ways in which H can be extended by one edge to a larger connected graph that does not have loops, coloops, parallel edges, or series edges. Thus, the "only if" part is obvious. The "if" part follows from Lemma (6.2.6). □

We are prepared for the next section, where we derive sufficient conditions for the existence of induced separations under various assumptions.

6.3 Sufficient Conditions for Induced Separations

In this section, we employ the separation algorithm to establish sufficient conditions under which induced separations can be guaranteed. Before we begin with the detailed discussion, we describe the general setting in which these conditions will be invoked. To this end, let $\mathcal{M}$ be a class of binary matroids. The class is assumed to be closed under minor-taking and isomorphism.

We select a matroid $N \in \mathcal{M}$, say on set F. Suppose by some method we find, for some $k \geq 2$, an exact k-separation (F_1, F_2) for N. At that point, we would like to claim the following.

(6.3.1) $\left\{\begin{array}{l}\text{Suppose an } M \in \mathcal{M} \text{ has an } N \text{ minor, say } N' \text{. Let } (F_1', F_2') \\ \text{be a } k\text{-separation of } N' \text{ that corresponds to } (F_1, F_2) \text{ under} \\ \text{one of the isomorphisms between } N' \text{ and } N\text{. Then the } k\text{-} \\ \text{separation } (F_1', F_2') \text{ of } N' \text{ induces a } k\text{-separation of } M.\end{array}\right.$

Results of type (6.3.1) are valuable if an $M \in \mathcal{M}$ is known to have an N minor, and if the induced k-separation of M may be employed to effect a useful decomposition of M. In Chapters 10–13, we will see that these two assumptions are satisfied in a number of cases. Thus, nontrivial instances of (6.3.1) are indeed useful.

The technique for proving (6.3.1) is in principle straightforward. With machinery yet to be described, we compute all minimal binary matroids satisfying the assumptions of (6.3.1) but not its conclusion. If no such matroid is in $\mathcal{M}$, then (6.3.1) indeed holds.

Application of the technique entails two difficulties. First, $\mathcal{M}$, N, and (F_1, F_2) must be properly selected. Second, we need structural insight and computational tools to prove that $\mathcal{M}$ has no M for which (6.3.1) fails. In this section, we ignore the first aspect. It will be treated in depth in Chapters 10–13. Instead, we concentrate on the development of the structural insight and of the computational tools.

We break down that development into two phases. In the first one, we accomplish the following task, where N is the previously mentioned matroid with exact k-separation (F_1, F_2).

$$(6.3.2) \quad \begin{cases} \text{Find computationally tractable properties of minimal bi-} \\ \text{nary matroids } M \text{ that have } N \text{ as a minor, but that do} \\ \text{not have a } k\text{-separation induced by the exact } k\text{-separation} \\ (F_1, F_2) \text{ of } N. \end{cases}$$

An M satisfying the conditions of (6.3.2) we simply call *minimal.* In the second phase, we expand the task of (6.3.2) to the task (6.3.3) below. It essentially says that the requirements of (6.3.2) are to hold for some minor isomorphic N, and that M is to be minimal with respect to that condition. The precise statement is as follows.

$$(6.3.3) \quad \begin{cases} \text{Find computationally tractable properties of binary ma-} \\ \text{troids } M \text{ satisfying the following conditions: } M \text{ must have} \\ \text{at least one } N \text{ minor. Some } k\text{-separation of at least one} \\ \text{such minor corresponding to } (F_1, F_2) \text{ of } N \text{ under one of the} \\ \text{isomorphisms must fail to induce a } k\text{-separation of } M. \text{ The} \\ \text{matroid } M \text{ is to be minimal with respect to those condi-} \\ \text{tions.} \end{cases}$$

We say that an M satisfying the conditions of (6.3.3) is *minimal under isomorphism.* Evidently, minimality under isomorphism demands more than the previously defined minimality.

Answers to (6.3.3) give sufficient conditions so that (6.3.1) holds. That is, if none of the binary matroids M with the yet-to-be-determined properties of (6.3.3) is in $\mathcal{M}$, then necessarily (6.3.1) must hold. Thus, answers to (6.3.3) effectively supply sufficient conditions under which (6.3.1) is satisfied.

We begin with the task (6.3.2). We are given a binary matroid N on a set F. In the next lemma, we rely on rank functions instead of representation matrices. Thus, we let $r_N(\cdot)$ be the rank function of N. For some $k \geq 1$, we have an exact k-separation (F_1, F_2) of N. Thus, by (6.1.3),

$$(6.3.4) \qquad \begin{aligned} &|F_1|, |F_2| \geq k \\ &r_N(F_1) + r_N(F_2) = r_N(F) + k - 1 \end{aligned}$$

Let M be any binary matroid on a set E and with rank function $r_M(\cdot)$. Assume that M has N as a minor, and that (F_1, F_2) does not induce a k-separation of M. Thus, the system

$$(6.3.5) \qquad \begin{aligned} &E_1 \supseteq F_1;\ E_2 \supseteq F_2 \\ &r_M(E_1) + r_M(E_2) \leq r_M(E) + k - 1 \end{aligned}$$

has no solution. Assume M to be minimal as defined above. Thus, every proper minor M' of M with N as a minor has a solution for an appropriately adapted (6.3.5). First, we show that there is a unique partition of the set $E - F$ into Z_1 and Z_2 so that $N = M/Z_1 \backslash Z_2$.

(6.3.6) Lemma. *Let M, E, N, F, and (F_1, F_2) be as defined above. Suppose (6.3.5) has no solution and that M is minimal. Then for all $z \in (E - F)$, M/z or $M\backslash z$ does not have N as a minor.*

Proof. Suppose both M/z and $M\backslash z$ have N as a minor. By the minimality of M, they both have induced separations, say (U_1, U_2) for M/z and (W_1, W_2) for $M\backslash z$. Let $r_{M/z}(\cdot)$ and $r_{M\backslash z}(\cdot)$ be the rank functions of M/z and $M\backslash z$. According to (6.3.5),

$$\begin{aligned} &U_1, W_1 \supseteq F_1;\ U_2, W_2 \supseteq F_2 \\ &r_{M/z}(U_1) + r_{M/z}(U_2) \leq r_{M/z}(E - \{z\}) + k - 1 \\ &r_{M\backslash z}(W_1) + r_{M\backslash z}(W_2) \leq r_{M\backslash z}(E - \{z\}) + k - 1 \end{aligned} \tag{6.3.7}$$

By the minimality of M, z cannot be a loop or coloop of M. Thus, the two inequalities of (6.3.7) imply the inequalities

$$\begin{aligned} r_M(U_1 \cup \{z\}) + r_M(U_2 \cup \{z\}) &\leq r_M(E) + k \\ r_M(W_1) \qquad + r_M(W_2) \qquad &\leq r_M(E) + k - 1 \end{aligned} \tag{6.3.8}$$

We add the two inequalities of (6.3.8) and apply submodularity to get

$$\begin{aligned} &r_M(U_1 \cup \{z\} \cup W_1) + r_M((U_1 \cup \{z\}) \cap W_1) \\ &+ r_M(U_2 \cup \{z\} \cup W_2) + r_M((U_2 \cup \{z\}) \cap W_2) \\ &\qquad\qquad \leq 2r_M(E) + 2k - 1 \end{aligned} \tag{6.3.9}$$

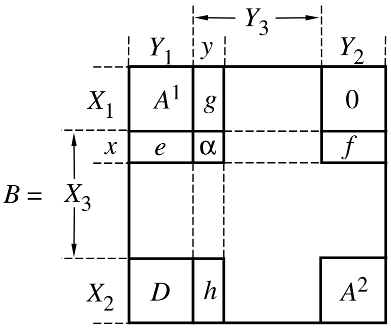
But each one of $(U_1 \cup \{z\} \cup W_1, (U_2 \cup \{z\}) \cap W_2)$ and $(U_2 \cup \{z\} \cup W_2, (U_1 \cup \{z\}) \cap W_1)$ is a pair (E_1, E_2) satisfying $E_1 \supseteq F_1$ and $E_2 \supseteq F_2$. Since (6.3.5) cannot be satisfied, we have

$$\begin{aligned} r_M(U_1 \cup \{z\} \cup W_1) + r_M((U_2 \cup \{z\}) \cap W_2) \geq r_M(E) + k \\ r_M(U_2 \cup \{z\} \cup W_2) + r_M((U_1 \cup \{z\}) \cap W_2) \geq r_M(E) + k \end{aligned} \tag{6.3.10}$$

Summing the latter two inequalities, we obtain a contradiction of (6.3.9). ∎

For further insight into the structure of a minimal M, we employ the separation algorithm of Section 6.2. That is, we have the representation matrix B of (6.2.5) for M, repeated here for ease of reference.

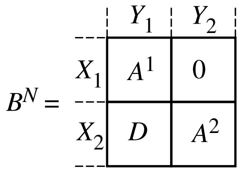
(6.3.11)



Matrix B for M with partitioned B^N,
row $x \in X_3$, and column $y \in Y_3$

The submatrix of B composed of A^1, A^2, D, and the explicitly shown 0 matrix is B^N of (6.2.1), which we also repeat here. The latter matrix represents N.

(6.3.12)



Partitioned version of B^N

We apply the separation algorithm to search for a partition as given by (6.2.4). Since (6.3.5) cannot be satisfied, the algorithm terminates in step 1 or 2 announcing that no partition with the desired properties exist. Below, we list that algorithm again, with references adapted to the just-defined matrices.

Separation Algorithm

1. Suppose B of (6.3.11) has a row $x \in X_3$ with the indicated row subvectors e and f such that GF(2)-rank $[e/D] >$ GF(2)-rank D. Then x must be in X_{31}. Suppose, in addition, that f is nonzero. Then x must also be in X_{32}, i.e., B cannot be partitioned, and we stop with that declaration. On the other hand, suppose $f = 0$. Since x must be in X_{31}, we adjoin e to A^1, and f to the explicitly shown zero matrix. Then we start recursively again with the new B and the new B^N.

2. Suppose B of (6.3.11) has a column $y \in Y_3$ with the indicated column subvectors g and h such that g is nonzero. Then y must be in Y_{31}. Suppose, in addition, GF(2)-rank $[D \mid h] >$ GF(2)-rank D. Then y must also be in Y_{32}, i.e., B cannot be partitioned, and we stop with

that declaration. On the other hand, suppose GF(2)-rank $[D \mid h] =$ GF(2)-rank D. Since y must be in Y_{31}, we adjoin g to A^1, and h to D. Then we start recursively again with the new B and the new B^N.

3. Finally, suppose that for all rows $x \in X_3$, the row subvector e satisfies GF(2)-rank $[e/D] =$ GF(2)-rank D, and that for all columns $y \in Y_3$, the column subvector g is 0. Then $X_{31} = Y_{31} = \emptyset$, $Y_{32} = Y_3$, $Y_{32} = Y_3$ gives the desired partition of B.

By (6.3.11), for any $x \in X_3$ and any $y \in Y_3$, the minors M/x and $M\backslash y$ of M have N as a minor. Thus, by the minimality of M, the separation algorithm does find a partition if we delete any row $x \in X_3$ or any column $y \in Y_3$ from B.

We now prove some results about the structure of a matrix B produced by a minimal M. We start with the special case where B of a minimal M contains just one row or column beyond that of B^N. Consider the case of a single additional row x. In the notation of (6.3.11), $X_3 = \{x\}$ and $Y_3 = \emptyset$. By step 1 of the separation algorithm, the row subvectors e and f of row x satisfy

$$\text{(6.3.13)} \qquad \begin{array}{l} e \text{ is not spanned by the rows of } D\text{, and} \\ f \text{ is nonzero} \end{array}$$

Similarly, we deduce for the case of a single additional column y, i.e., when $X_3 = \emptyset$ and $Y_3 = \{y\}$,

$$\text{(6.3.14)} \qquad \begin{array}{l} g \text{ is nonzero, and} \\ h \text{ is not spanned by the columns of } D \end{array}$$

We now treat the remaining cases, where B has at least two additional rows or columns beyond those of B^N. Thus, $|X_3 \cup Y_3| \geq 2$. We want to show that both X_3 and Y_3 are nonempty, and that there exist $x \in X_3$ and $y \in Y_3$ so that the subvectors e, f of row x, and g, h of column y, as well as the scalar α, obey certain conditions. First we prove the following fact about the subvectors e of the rows $x \in X_3$ and about the subvectors g of the columns $y \in Y_3$.

(6.3.15) Lemma. *Exactly one of the two cases* (i) *and* (ii) *below applies.*

(i) *There is exactly one row $x \in X_3$ such that the subvector e is not spanned by the rows of D. In that row x, the subvector f is zero. Furthermore, for all $y \in Y_3$, the subvector g of column y is zero.*

(ii) *There is exactly one column $y \in Y_3$ such that the subvector g is nonzero. In that column y, the subvector h is spanned by the columns of D. Furthermore, for all $x \in X_3$, the subvector e of row x is spanned by the rows of D.*

Proof. If the condition about f or h does not hold, then we have the smaller case of (6.3.13) or (6.3.14), a contradiction.

To prove the claims about e and g, we apply the separation algorithm to B. Consider each application of steps 1 and 2 except for the last application, when the algorithm stops. In each such application, a row $x \in X_3$ or column $y \in Y_3$ is moved to X_{31} or Y_{31}, and the related row subvector f satisfies $f = 0$, or the column subvector h satisfies GF(2)-rank $[D \mid h] =$ GF(2)-rank D. Exactly one of these conditions is violated in the last iteration. The rows and columns moved to X_{31} and Y_{31}, plus the row or column encountered in the last application, suffice to prove that B has no induced partition. Thus, by the minimality of M, these rows and columns comprise the rows and columns that B has beyond those of B^N. We conclude that in each row $x \in X_3$, the row subvector f is zero except for at most one such vector, and that in each column $y \in Y_3$, the column subvector h is spanned by D except for at most one such vector. Furthermore, if there is a nonzero f, then all vectors h are spanned by D, and if there is an h not spanned by D, then all vectors f are zero. To prove the claims about e and g, we use duality, or equivalently, we apply the above arguments to B^t. Then g plays the role of f above, and e that of h. The conditions just proved for f and h establish the statements about e and g of the lemma. □

We investigate the two cases of Lemma (6.3.15) further. We begin with the situation where a unique row $x \in X_3$ has a subvector e that is not spanned by the rows of D, and where for all $y \in Y_3$, the subvector g of column y is zero. The next lemma tells more about the columns $y \in Y_3$.

(6.3.16) Lemma. *Suppose case* (i) *of Lemma* (6.3.15) *applies. Then there exists a $y \in Y_3$ such that the scalar α of column y is* 1 *and the subvector h is nonzero.*

Proof. As in the proof of Lemma (6.3.15), we apply the separation algorithm to B. By the assumptions, in the first iteration the row x is moved to X_{31}. By Lemma (6.3.15), in the next recursive application of the separation algorithm, step 2 must apply. Thus, a column $y \in Y_3$ is moved to Y_{31}. By Lemma (6.3.15), the subvector g of that column is zero. Then $\alpha = 1$ since otherwise step 2 could not move column y to Y_{31}.

Suppose h is zero. Since g is also zero, we can pivot on $\alpha = 1$ without disturbing the submatrix B^N. This implies that both $M \backslash y$ and M/y have N as a minor, in violation of Lemma (6.3.6). Thus, h must be nonzero. □

Consider now the second case of Lemma (6.3.15). Thus, B has a unique column $y \in Y_3$ with nonzero subvector g, and the subvector h is spanned by the columns of D. Furthermore, for all $x \in X_3$, the subvector

e of row x is spanned by the rows of D. Analogously to Lemma (6.3.16), we have the following result.

(6.3.17) Lemma. *Suppose case* (ii) *of Lemma* (6.3.15) *applies. Then there exists an $x \in X_3$ such that the subvector e is spanned by the rows of D. If the subvector f is zero, then the subvector e is nonzero. Furthermore, the subvector $[e \mid \alpha]$ is not spanned by the rows of $[D \mid h]$.*

Proof. Apply the separation algorithm to B. In the first iteration, the column y is moved to Y_{31}. In the next recursive application of the separation algorithm, step 1 must apply. Thus, a row $x \in X_3$ is moved to X_{31}. The subvector e must be spanned by the rows of D, but the subvector $[e \mid \alpha]$ of row x is not spanned by the rows of $[D \mid h]$.

Suppose f is zero. If e is also zero, then necessarily $\alpha = 1$. A pivot on α does not disturb the submatrix B^N. Thus, both M/x and $M\backslash x$ contain N as a minor, a contradiction of Lemma (6.3.6). Thus, e is nonzero. □

With (6.3.13), (6.3.14), and Lemmas (6.3.16) and (6.3.17), we assemble the following theorem.

(6.3.18) Theorem. *Let M be minimal, and let B be the representation matrix of* (6.3.11) *for M.*

- (a) *Suppose $X_3 = \{x\}$ and $Y_3 = \emptyset$. Then the subvector e of row x is not spanned by the rows of D, and f is nonzero.*
- (b) *Suppose $X_3 = \emptyset$ and $Y_3 = \{y\}$. Then the subvector g of column y is nonzero, and h is not spanned by the columns of D.*
- (c) *Suppose $|X_3 \cup Y_3| \geq 2$. Then either* (c.1) *or* (c.2) *below applies for some $x \in X_3$ and $y \in Y_3$.*
  - (c.1) *The subvector e of row x is not spanned by the rows of D, and f is zero. The subvector g of column y is zero, $\alpha = 1$, and the subvector h is nonzero.*
  - (c.2) *The subvector g of column y is nonzero, and the subvector h is spanned by the columns of D. The subvector e is spanned by the rows of D. If the subvector f is zero, then the subvector e is nonzero. The subvector $[e \mid \alpha]$ is not spanned by the rows of $[D \mid h]$.*

Proof. Statements (6.3.13) and (6.3.14) establish (a) and (b). Lemmas (6.3.15), (6.3.16), and (6.3.17) prove parts (c.1) and (c.2). □

For our purposes, Theorem (6.3.18) suffices as answer for the task (6.3.2). Thus, we turn to the task (6.3.3). That problem demands that we find computationally tractable properties of binary matroids M that are minimal under isomorphism. That is, any such M has a minor isomorphic to N. For at least one such minor the following holds. Some k-separation of that minor corresponds to (F_1, F_2) of N under one of the isomorphisms, and fails to induce a k-separation of M. We want M to be minimal with respect to these conditions.

Let B of (6.3.11) be the representation matrix of an M that is minimal under isomorphism. Since minimality under isomorphism implies the minimality defined for (6.3.2), B observes the conditions of Theorem (6.3.18). Evidently, minimality under isomorphism is a stronger requirement than minimality. Thus, we expect e, f, g, h, and α of Theorem (6.3.18) to obey additional conditions. The next lemma supplies computationally tractable ones. The notation is that of Theorem (6.3.18).

(6.3.19) Lemma.

($\overline{\text{c}}$.1) *If case* (c.1) *of Theorem* (6.3.18) *applies, then the following holds.*

- ($\overline{\text{c}}$.1.1) *The subvector e of row x of B is not parallel to a row of the submatrix A^1.*
- ($\overline{\text{c}}$.1.2) *Suppose column $z \in Y_1$ of A^1 is nonzero. Then the subvector e of row x of B is not a unit vector with* 1 *in column z of B.*

($\overline{\text{c}}$.2) *If case* (c.2) *of Theorem* (6.3.18) *applies, then the following holds.*

- ($\overline{\text{c}}$.2.1) *Suppose $\overline{D}$, the matrix obtained from D by deletion of a column $z \in Y_1$ of D, has the same* GF(2)*-rank as D. Then the subvector $[g/h]$ of column y of B is not parallel to column z of $[A^1/D]$.*
- ($\overline{\text{c}}$.2.2) *Suppose the rows of D do not span a row $z \in X_1$ of A^1. Then $[g/h]$ is not a unit vector with* 1 *in row z.*

Proof. ($\overline{\text{c}}$.1.1): For a proof by contradiction, suppose the subvector e of row x of B is parallel to a row $z \in X_1$ of A^1. In B, we exchange the rows x and z, and appropriately adjust X_1 to $X_1' = (X_1 - \{z\}) \cup \{x\}$ and X_3 to $X_3' = (X_3 - \{x\}) \cup \{z\}$. By (c.1) of Theorem (6.3.18), f is zero. Thus, the submatrix of B indexed by X_1', X_2, Y_1, and Y_2 is B^N except for the change of the index z to x. Let N' be the corresponding minor of M. We know that $N\&x$ does not induce a k-separation of M. Thus, $N'\&z = N\&x$ does not induce one either. The same conclusion applies to N', since row z contains e. Now M is minimal under isomorphism, so M must be minimal with respect to N'. We show that the latter conclusion leads to a contradiction. For the proof, let us examine the effect of the exchange of rows x and z on column $y \in Y_3$. By that exchange, the role of the zero subvector g indexed by X_1 is taken on by a vector g' indexed by X_1'. By (c.1) of Theorem (6.3.18), the entry α in row x is nonzero. Thus, the vector g' is nonzero. Apply Lemma (6.3.15) to N' and the subvectors e and g'. Since e is not spanned by the rows of D and since g' is nonzero, these subvectors violate the conclusions of that lemma, and thus provide the desired contradiction.

($\overline{\text{c}}$.1.2): Suppose that column $z \in Y_1$ of A^1 is nonzero, and that e is a unit vector with 1 in column z of B. Perform a pivot in column $z \in Y_1$ of A^1.

Then the subvector e becomes parallel to a row of A^1, and the above case ($\overline{c}$.1.1) applies.

($\overline{c}$.2.1): Suppose $\overline{D}$, the matrix obtained from D' by deletion of a column $z \in Y_1$ of D, has the same GF(2)-rank as D. Further assume that the subvector $[g/h]$ of column y of B is parallel to column z of $[A^1/D]$. Exchange columns y and z of B. Adjust Y_1 to $Y_1' = (Y_1 - \{z\}) \cup \{y\}$ and Y_3 to $Y_3' = (Y_3 - \{y\}) \cup \{z\}$. The swap of columns effectively replaces N by an isomorphic minor N' and modifies D to D' and e to e'. A simple rank calculation confirms that under the assumption on $\overline{D}$, the rows of D' do not span e'. Arguing analogously to the case ($\overline{c}$.1.1), M is not minimal under isomorphism.

($\overline{c}$.2.2): By a pivot in row z of A^1, this case becomes ($\overline{c}$.2.1), as is readily checked. $\square$

We summarize the preceding conclusions in the next theorem, which finishes the task (6.3.3). The statement of the theorem is rather detailed to simplify its application.

(6.3.20) Theorem. *Let M be minimal under isomorphism. Then one of* (a), (b), *or* (c) *below holds.*

(a) *M is represented by*

(6.3.21)
$$B = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline x & e & f \\ \hline X_2 & D & A^2 \\ \hline \end{array}$$

Matrix B for M minimal
under isomorphism, case (a)

In row x, e is not spanned by the rows of D, and f is nonzero.

(b) *M is represented by*

(6.3.22)
$$B = \begin{array}{c|c|c|c|} & Y_1 & y & Y_2 \\ \hline X_1 & A^1 & g & 0 \\ \hline X_2 & D & h & A^2 \\ \hline \end{array}$$

Matrix B for M minimal
under isomorphism, case (b)

In column y, g is nonzero, and h is not spanned by the columns of D.

(c) *M has a minor $\overline{M}$ with representation matrix*

$$(6.3.23) \qquad \overline{B} = \begin{array}{c|c|c|c|} & Y_1 & y & Y_2 \\ \hline X_1 & A^1 & g & 0 \\ \hline x & e & \alpha & f \\ \hline X_2 & D & h & A^2 \\ \hline \end{array}$$

Matrix $\overline{B}$ for minor $\overline{M}$
of M minimal under isomorphism

Either (c.1) *or* (c.2) *below holds for e, f, g, h, and α.*

(c.1) *e is not spanned by the rows of D; $f = 0$; $g = 0$; $h \neq 0$; $\alpha = 1$; e is not parallel to a row of A^1. If column $z \in Y_1$ of A^1 is nonzero, then e is not a unit vector with* 1 *in column z of $\overline{B}$.*

(c.2) *$g \neq 0$; h is spanned by the columns of D; e is spanned by the rows of D; $f = 0$ implies $e \neq 0$; $[e \mid \alpha]$ is not spanned by the rows of $[D \mid h]$. If $\overline{D}$, the matrix obtained from D by deletion of a column $z \in Y_1$, has the same* GF(2)*-rank as D, then $[g/h]$ is not parallel to column z of $[A^1/D]$. If the rows of D do not span a row $z \in X_1$ of A^1, then $[g/h]$ is not a unit vector with* 1 *in row z.*

Proof. The statements follow directly from Theorem (6.3.18) and Lemma (6.3.19). □

Recall our main goal for this section: We want to determine sufficient conditions for induced separations. In the next corollary, we deduce such conditions from Theorem (6.3.20).

(6.3.24) Corollary. *Let $\mathcal{M}$ be a class of binary matroids that is closed under isomorphism and under the taking of minors. Suppose that N given by B^N of* (6.3.12) *is in $\mathcal{M}$, but that the* 1*- and* 2*-element extensions of N given by* (6.3.21), (6.3.22), (6.3.23), *and by the accompanying conditions are not in $\mathcal{M}$. Assume that a matroid $M \in \mathcal{M}$ has an N minor. Then any k-separation of any such minor that corresponds to $(X_1 \cup Y_1, X_2 \cup Y_2)$ of N under one of the isomorphisms induces a k-separation of M.*

Proof. Take $M \in \mathcal{M}$ satisfying the assumptions. We know $\mathcal{M}$ to be closed under isomorphism. Thus, we may suppose that the N minor of M is N itself. Suppose the k-separation of N does not induce one in M. Then M, or a minor of M containing N, is minimal under isomorphism. By Theorem (6.3.20), M has a minor represented by one of the matrices of (6.3.21), (6.3.22), (6.3.23). Since $\mathcal{M}$ is closed under minor-taking, any such minor of M is in $\mathcal{M}$. But presence of such a minor in $\mathcal{M}$ is ruled out by assumption, a contradiction. □

Sometimes an abbreviated version of Corollary (6.3.24) suffices to produce the desired conclusion of induced separations. The following result is one such version.

(6.3.25) Corollary. *Let $\mathcal{M}$ be a class of binary matroids that is closed under isomorphism and under the taking of minors. Suppose a 3-connected N given by B^N of (6.3.12) is in $\mathcal{M}$. Assume that $N/(X_2 \cup Y_2)$ has no loops and that $N\backslash(X_2 \cup Y_2)$ has no coloops. Furthermore, assume for every 3-connected 1-element extension of N in $\mathcal{M}$, say by an element z, that the pair $(X_1 \cup Y_1, X_2 \cup Y_2 \cup \{z\})$ is a k-separation of that extension. Then for any 3-connected matroid $M \in \mathcal{M}$ with an N minor, the following holds. Any k-separation of any such minor that corresponds to $(X_1 \cup Y_1, X_2 \cup Y_2)$ of N under one of the isomorphisms induces a k-separation of M.*

Proof. Suppose the conclusion is false. By Corollary (6.3.24), $\mathcal{M}$ contains a matroid M represented by one of the matrices (6.3.21), (6.3.22), or (6.3.23). We first dispose of the cases (6.3.21) and (6.3.22). The assumed 3-connectedness of B^N and the conditions of Theorem (6.3.20) on the matrices of (6.3.21) and (6.3.22) imply that these matrices do not contain zero vectors, unit vectors, or parallel vectors. Then by Lemma (6.2.6), these matrices represent 3-connected 1-element extensions of N. By assumption, any 3-connected 1-element extension of N does have an induced k-separation. Hence, the cases (6.3.21) and (6.3.22) cannot occur.

Consider (6.3.23), case (c.1). Delete column y from that matrix. The reduced matrix represents $N\&x$. We claim that $N\&x$ is 3-connected. By (c.1), the subvector e of row x is not spanned by the rows of D. It also is not parallel to a row of A^1. Now $N/(X_2 \cup Y_2)$ has no loop, so A^1 has no zero columns. Then by (c.1), e is not a unit vector. By Lemma (6.2.6), $N\&x$ is 3-connected as claimed. Since e is not spanned by the rows of D, $(X_1 \cup Y_1, X_2 \cup Y_2 \cup \{x\})$ is not a k-separation of $N\&x$. By assumption, $N\&x$ cannot be in $\mathcal{M}$. Yet $N\&x$ is a minor of $M \in \mathcal{M}$, a contradiction.

Consider (6.3.23), case (c.2). We first establish an auxiliary result. Suppose that for some $z \in Y_1$, deletion of column z from D reduces the GF(2)-rank, or that for some $z \in X_1$, the rows of D span row z of A^1. We claim that z is a coloop of $N\backslash(X_2 \cup Y_2)$, contrary to assumption. For a proof, we delete from $[I \mid B^N]$ the columns indexed by $X_2 \cup Y_2$. By a simple rank calculation, every basis of the reduced matrix contains column z. This establishes the claim.

By (c.2) and the auxiliary result, g and h of (6.3.23) satisfy the following conditions: $g \neq 0$, and $[g/h]$ is not parallel to a column of $[A^1/D]$ and is not a unit vector. Then by Lemma (6.2.6) and (6.3.23), $N{+}y$ is 3-connected, and $(X_1 \cup Y_1, X_2 \cup Y_2 \cup \{y\})$ is not a k-separation of $N{+}y$. Thus, $N{+}y$ cannot be in $\mathcal{M}$. Yet $N{+}y$ is a minor of $M \in \mathcal{M}$, a contradiction. □

In Chapter 10, we require the graph version of Corollary (6.3.25) for

$k = 3$. For that situation, we adapt the above matroid language as follows. Suppose we have 3-connected graphs G and H. On hand is a 3-separation (F_1, F_2) for H. Then that 3-separation of H *induces* one for G if the latter graph has a 3-separation (E_1, E_2) where $E_1 \supseteq F_1$ and $E_2 \supseteq F_2$. Here is the special graph version of Corollary (6.3.25) for $k = 3$.

(6.3.26) Corollary. *Let $\mathcal{G}$ be a class of connected graphs that is closed under isomorphism and under the taking of minors. Let a 3-connected graph $H \in \mathcal{G}$ have a 3-separation (F_1, F_2) with $|F_1|, |F_2| \geq 4$. Assume that H/F_2 has no loops and $H\backslash F_2$ has no coloops. Furthermore, assume that for every 3-connected 1-edge extension of H in $\mathcal{G}$, say by an edge z, the pair $(F_1, F_2 \cup \{z\})$ is a 3-separation of that extension. Then for any 3-connected graph $G \in \mathcal{G}$ with an H minor, the following holds. Any 3-separation of any such minor that corresponds to (F_1, F_2) of H under one of the isomorphisms induces a 3-separation of G.*

Proof. By the assumptions and Corollary (6.3.25), (F_1, F_2) is a 3-separation of $M(H)$, and that 3-separation induces (in the matroid sense) a 3-separation (E_1, E_2) in $M(G)$. For $i = 1, 2$, $E_i \supseteq F_i$, and thus $|E_i| \geq |F_i| \geq 4$. By Theorem (3.2.25), part (c), (E_1, E_2) is a 3-separation of G as desired. □

We touch upon the complexity of finding a minor that prevents an induced k-separation. We consider this problem in the following setting. We are given a binary matroid M, a minor N of M, and a k-separation of N. We would like to obtain a k-separation of M induced by that of N. If that is not possible, we want to find a minor M represented up to indices by one of the matrices of (6.3.21)–(6.3.23). The next theorem says that this problem can be solved in polynomial time.

(6.3.27) Theorem. *There is a polynomial algorithm for the following problem. The input consists of a binary matroid M, a minor N of M, and a k-separation of N. The output is to be either a k-separation of M induced by that of N, or a minor of M that is isomorphic to one of the matroids represented by the matrices of* (6.3.21)–(6.3.23).

Proof. If an induced k-separation does exist, then one such k-separation is found by the separation algorithm. Suppose there is no such k-separation of M. Then we use a polynomial implementation of the constructive proofs of Lemmas (6.3.15)–(6.3.17) and (6.3.19) to locate a minor of M represented up to indices by one of the matrices of (6.3.21)–(6.3.23). □

Sometimes a class $\mathcal{M}$ of matroids under investigation is only closed under restricted isomorphism and under special minor-taking. We want sufficient conditions under which the above results for induced decompositions remain valid in the new setting. We state one such instance following a definition.

Let L be a set of elements. Assume that two binary matroids contain the set L. We say that the two matroids are *L-isomorphic* if there is an isomorphism that is an identity on the set L. The conditions on $\mathcal{M}$ are as follows. Each matroid of $\mathcal{M}$ contains L. Furthermore, $\mathcal{M}$ is closed under L-isomorphism and under the taking of minors, provided the minors are connected and contain L. Analogous definitions apply to graphs, or to the term "minimal under L-isomorphism."

As before, let N be a binary matroid with a k-separation (F_1, F_2). We also assume that N is in $\mathcal{M}$, and that $L \subseteq F_2$. The next theorem says that Corollaries (6.3.24) and (6.3.25) remain valid under the additional conditions. Under a suitable change to graph terminology, the same conclusion applies to Corollary (6.3.26). Finally, Theorem (6.3.27) remains valid when L-isomorphisms replace isomorphisms.

(6.3.28) Theorem. *Corollaries* (6.3.24) *and* (6.3.25) *remain valid when $\mathcal{M}$ and N satisfy the following two conditions for some set L contained in the set F_2 of N. First, each matroid of $\mathcal{M}$ contains the set L. Second, $\mathcal{M}$ is closed under L-isomorphism and under the taking of minors, provided the minors are connected and contain L. Corollary* (6.3.26) *remains valid when the above conditions on $\mathcal{M}$ and N are applied to the class $\mathcal{G}$ and to the graph H. Theorem* (6.3.27) *remains valid when L-isomorphisms are claimed instead of isomorphisms.*

Proof. The cited results rely on Theorem (6.3.20), which is nothing but Theorem (6.3.18) plus Lemma (6.3.19). Now Theorem (6.3.18) is a statement about a minimal M, and thus does not involve any isomorphism. But in the proof of Lemma (6.3.19), the matroid N is replaced by an isomorphic matroid N'. However, N' can be derived from N by a relabeling of some elements of $F_1 = X_1 \cup Y_1$. Thus, the elements of F_2 are not affected, and N' is F_2-isomorphic to N. Since $L \subseteq F_2$, N' is also L-isomorphic to N. We apply these observations to rewrite Theorem (6.3.20) so that it becomes a statement about a matroid minimal under L-isomorphism. Indeed, we only need to change the claims about the matrices of (6.3.21), (6.3.22), and (6.3.23) by allowing for a relabeling of indices other than those of L to get the desired theorem. It is now an easy matter to verify that the theorem so derived from Theorem (6.3.20) implies the claimed results for Corollaries (6.3.24), (6.3.25), and (6.3.26), and Theorem (6.3.27). $\square$

An example application of Theorem (6.3.20) is covered in the next section. There we prove the existence of certain extensions of 3-connected binary minors in 3-connected binary matroids. In Chapters 10, 11, and 13, we use Corollaries (6.3.24)–(6.3.26) and Theorem (6.3.28).

6.4 Extensions of 3-Connected Minors

An important matroid problem is as follows. We are given a 3-connected binary matroid M with a 3-connected minor N. The minor has at least six elements. We want to obtain a 3-connected minor N' of M that, for some small $k \geq 1$, is a k-element extension of an N minor of M. In this section, we show that an N' with $k = 1$ or 2 can always be found. Our main tool for establishing this result is Theorem (6.3.20) of the preceding section. In Chapter 7, we refine the conclusion proved here to obtain the so-called splitter theorem.

The precise statement of the above claim about N' and k is as follows.

(6.4.1) Theorem. *Let M be a 3-connected binary matroid with a 3-connected proper minor N. Suppose N has at least six elements. Then M has a 3-connected minor N' that is a 1- or 2-element extension of some N minor of M. In the 2-element case, N' is derived from the N minor by one addition and one expansion.*

Proof. Let z be any element of M that is not in N. Lemma (5.2.4) says that the connected M has a connected minor N' that is a 1-element extension of N by z. Now Theorem (6.4.1) holds for M and N if and only if it holds for M^* and N^*. Hence, by duality, we may assume that the extension is an addition. Let N be represented by a matrix $\overline{B}$. Thus, for some vector a, the minor N' of M is represented by the matrix

$$(6.4.2) \qquad \begin{array}{c|c|c|} & \overline{Y} & z \\ \hline \overline{X} & \overline{B} & a \\ \hline \end{array}$$

Matrix for 1-element extension N' of N

Since N' is connected, the vector a must be nonzero. If a is not a unit vector and is not parallel to a column of $\overline{B}$, then by Lemma (6.2.6), N' is 3-connected and we are done. Otherwise, due to at most one GF(2)-pivot in $\overline{B}$, we may assume a to be a unit vector, say with 1 in row $u \in \overline{Y}$. Let d be the row vector of $\overline{B}$ indexed by u. Partition $\overline{B}$ into d and $\overline{\overline{B}}$, and also partition $\overline{X}$ into $\{u\}$ and $\overline{\overline{X}} = \overline{X} - \{u\}$. We thus can rewrite $[\overline{B} \mid a]$ of (6.4.2) as

$$(6.4.3) \qquad \begin{array}{c|c|c|} & \overline{Y} & z \\ \hline \overline{\overline{X}} & \overline{\overline{B}} & 0 \\ \hline u & d & 1 \\ \hline \end{array}$$

Partitioned version of matrix of (6.4.2) for N'

The partition in (6.4.3) corresponds to the 2-separation $(\overline{\overline{X}} \cup \overline{Y}, \{u, z\})$ of N'. Since M is 3-connected, that 2-separation of N' does not induce one in M. There must be a minor M' of M that proves this fact and that is minimal under isomorphism. The matroid M' has an N' minor. If necessary, we change the element labels of M' so that N' itself is that N' minor.

We apply Theorem (6.3.20). The just-defined M' plays the role of M of the theorem. The submatrices $\overline{\overline{B}}$, d, and $[\,1\,]$ of (6.4.3) correspond to A^1, D, and A^2, respectively, of the theorem. We list enough conditions of parts (a), (b), and (c) of Theorem (6.3.20) to derive the desired conclusion. At the same time, we substitute $\overline{\overline{B}}$, d, and $[\,1\,]$ for A^1, D, and A^2. On the other hand, the indices x and y, the subvectors e, f, g, h, and the scalar α employed below should be interpreted exactly as in Theorem (6.3.20).

(a) M' is represented by

(6.4.4)
$$
\begin{array}{c|c|c|}
 & \overline{Y} & z \\
\hline
\overline{\overline{X}} & \overline{\overline{B}} & 0 \\
\hline
x & e & f \\
\hline
u & d & 1 \\
\hline
\end{array}
$$

Case (a) of Theorem (6.3.20)

In row x, the subvector e is not spanned by d, and $f = 1$.

We evaluate this condition. Evidently, e is nonzero and not parallel to d. Indeed, one easily verifies that the matrix of (6.4.4) has no zero vectors, unit vectors, or parallel vectors. By Lemma (6.2.6), the matroid M' is therefore 3-connected, and hence is a 3-connected 2-element extension of N produced by one addition and one expansion.

(b) M' is represented by

(6.4.5)
$$
\begin{array}{c|c|c|c|}
 & \overline{Y} & y & z \\
\hline
\overline{\overline{X}} & \overline{\overline{B}} & g & 0 \\
\hline
u & d & h & 1 \\
\hline
\end{array}
$$

Case (b) of Theorem (6.3.20)

In column y, the subvector h is not spanned by the columns of d.

This condition is clearly incompatible with the fact that d is a nonzero vector. Thus, this case cannot occur.

(c) M' has a minor $\overline{M}$ with representation matrix

$$\begin{array}{c|c|c|c|}
 & \overline{Y} & y & z \\ \hline
\overline{\overline{X}} & \overline{\overline{B}} & g & 0 \\ \hline
x & e & \alpha & f \\ \hline
u & d & h & 1 \\ \hline
\end{array} \tag{6.4.6}$$

Case (c) of Theorem (6.3.20)

From the conditions (c.1) and (c.2) of Theorem (6.3.20), we extract the following.

(c.1) The vector e is not spanned by d and is not parallel to a row of $\overline{\overline{B}}$. Furthermore, e is not a unit vector with 1 in a column $t \in \overline{Y}$ for which column t of $\overline{\overline{B}}$ is nonzero.

We evaluate these conditions. Since the matrix $\overline{B} = [\overline{\overline{B}}/d]$ for N is 3-connected, each column of $\overline{\overline{B}}$ is nonzero. Thus, the above conditions on e imply that the matrix composed $\overline{\overline{B}}$, d, and e has no zero vectors, unit vectors, or parallel vectors. By Lemma (6.2.6), that matrix represents a 3-connected 1-element extension of N.

(c.2) The vector g is nonzero. If $\overline{d}$, the subvector obtained from d by deletion of an element $t \in \overline{Y}$, has the same GF(2)-rank as d, then $[g/h]$ is not parallel to the column t of $[\overline{\overline{B}}/d]$. If d does not span a row $t \in \overline{\overline{X}}$ of $\overline{\overline{B}}$, then $[g/h]$ is not a unit vector with 1 in row t.

We interpret these conditions. By the 3-connectedness of N, the vector d has at least two 1s and does not span any row of $\overline{\overline{B}}$. Thus, the above conclusions about $[g/h]$ hold for all $t \in \overline{\overline{Y}}$ and all $t \in \overline{\overline{X}}$. Put differently, $[g/h]$ must be nonzero, cannot be a unit vector, and cannot be parallel to a column of $[\overline{\overline{B}}/d]$. By Lemma (6.2.6), the matrix composed of $\overline{\overline{B}}$, d, g, and h represents a 3-connected 1-element extension of N. □

The minor N' of Theorem (6.4.1) may be efficiently found. Indeed, one only needs to implement the preceding constructive proof. The precise complexity claim is as follows.

(6.4.7) Theorem. *There is a polynomial algorithm for the following problem. The input is a connected binary matroid M, and a 3-connected proper minor N of M on at least six elements. The output is either a 2-separation of M, or a minor N' of M that is a 3-connected 1- or 2-element extension of an N minor of M. In the 2-element case, N' is derived from the N minor by one addition and one expansion.*

Proof. We implement the proof of Theorem (6.4.1), using as subroutine the polynomial algorithm of Theorem (6.3.27) to produce either a

2-separation of M or one of the matrices (6.3.21)–(6.3.23). It is easy to see that a polynomial algorithm can thus be assembled for the stated problem. □

In the next section, we discuss extensions of the results of this chapter and include references.

6.5 Extensions and References

The results of Section 6.3 overlap significantly with results of Seymour (1980b) for general matroids, even though the terminology is quite different. The precise relationships are as follows. Lemma (6.3.6) is the binary version of a result taken from Seymour (1980b). That reference proceeds to describe a number of properties of matroids called minimal here. These results are then used to deduce a version of Corollary (6.3.24). In contrast, the approach taken here is based on properties that can be efficiently verified for matroids that are minimal or minimal under isomorphism. Such properties are investigated via abstract matrices for general matroids in Truemper (1986). The sufficiency conditions and testing algorithms so obtained are substantially stronger than those relying on Corollary (6.3.24) or on the even weaker Corollary (6.3.25). The simpler results given here suffice for the proofs in the chapters to come. Truemper (1986) contains additional material about induced decompositions. For example, it is shown that the number of non-isomorphic minimal matroids is finite for a given N, provided all matroids under consideration are representable over a given finite field.

Truemper (1988) treats in detail the case of graphs, which we have skipped here entirely except for the specialized Corollary (6.3.26).

Theorem (6.4.1) may be viewed as a weak version of the splitter theorem of Seymour (1980b). We use Theorem (6.4.1) in the next chapter to prove that result. Upon slight modification, the approach of Section 6.4 yields other important theorems about 3-connected extensions of 3-connected matroids. We sketch the main ideas and provide related references in Section 7.5 of the next chapter.