Chapter 5

Path Shortening Technique

5.1 Overview

In this chapter, we introduce a matroid tool called the path shortening technique. It is an adaptation of an elementary graph method to matroids. In Section 5.2, we first motivate that technique, then use it to prove several results concerning the existence of certain separations and of certain minors in binary matroids. In subsequent chapters, we rely a number of times on these results.

In Section 5.3, we employ the technique to devise a polynomial algorithm that solves the following problem. Given are two matroids M_1 and M_2 on a common set E. One must find a maximum cardinality set $Z \subseteq E$ that is independent in both M_1 and M_2. This problem is called the cardinality intersection problem, for short, intersection problem. Correspondingly, the algorithm is called the intersection algorithm. That scheme also solves the following problem, which is called the partitioning problem. As before, two matroids M_1 and M_2 on a common set E are given, say with rank functions $r_1(\cdot)$ and $r_2(\cdot)$. One must partition E into two sets, say E_1 and E_2, such that $r_1(E_1) + r_2(E_2)$ is minimized. The intersection algorithm also provides a constructive proof of a max-min theorem that links the intersection problem with the partitioning problem. The results of Section 5.3 and many more on the intersection and partitioning of matroids are almost entirely due to Edmonds. They constitute some of the most profound results of matroid theory.

The results of Section 5.3 are related to the remaining chapters as follows. There we frequently must find certain separations of matroids.

The intersection algorithm may be employed to do this. But one may also locate the desired separations with the separation algorithm of the next chapter. Thus, in principle, we do not require the intersection algorithm for the remainder of the book. As a consequence, the material of Section 5.3 may be just scanned or even skipped on a first reading without loss of continuity.

The final section, 5.4, contains extensions and references. The chapter utilizes the material of Chapters 2 and 3.

5.2 Shortening of Paths

Consider the following simple graph problem. Given is a 2-connected graph G with at least two edges, among them edges e and f. We are asked to prove that G has a 2-connected minor consisting of just e and f. The solution is straightforward. As stated in Section 2.2, any two edges of a 2-connected graph lie on some cycle. In particular, the edges e and f lie on some cycle C of G. Evidently, C consists of e, f, and two node-disjoint paths, say P_1 and P_2. We obtain the desired minor by deleting all edges not in C and contracting all edges of P_1 and P_2. One could call the second step a shortening of the paths P_1 and P_2.

The path shortening operation has numerous uses in graph theory involving far more complicated situations than the trivial example treated above. The method also has an interesting translation to matroid theory, where we will refer to it as the *path shortening technique*. In this section, we describe that technique while proving the matroid analogue of the just-used fact that in a 2-connected graph any two edges lie on some cycle. Subsequently, we rely on the path shortening technique to establish other matroid results that will be repeatedly invoked in later chapters. We begin with the matroid version of the cited graph result.

(5.2.1) Lemma. *A binary matroid M is connected if and only if for every two elements x and z of M, there is a circuit containing both x and z.*

Proof. Let B be a binary representation matrix of M. First we prove the "if" part by contradiction. If M is not connected, then by Lemma (3.3.19), B is not connected. Evidently, M then has elements x and z that cannot both be in any circuit.

We turn to the "only if" part. Thus, we assume M, and hence B, to be connected. If the rank of M is 0, then due to the connectedness assumption, M has at most one element, and the desired conclusion holds vacuously. So assume that the rank of M is at least 1. Correspondingly, B has at least one row. Let x and z be two elements of M. We must show that some circuit of M contains both x and z.

Since B is connected, by at most one GF(2)-pivot we can assure that x indexes a row of B. Suppose that z indexes a column of B. If the element B_{xz} of B is a 1, then x is in the fundamental circuit of M given by the column z of B, and we are done. If $B_{xz} = 0$, then any GF(2)-pivot in column z produces the case where z also indexes a row. Thus, from now on we may assume that both x and z index rows of B.

At this point, we switch from the connected matrix B to the connected bipartite graph BG(B). The analysis of BG(B) consists of the following elementary step. We examine a shortest path from row node x to row node z. Suppose that the row nodes of the path are x, r_1, $r_2, \ldots,$ r_m, z, for some $m \geq 1$, and that the column nodes are y, s_1, $s_2, \ldots,$ s_m. The nodes are encountered in the given order as one moves along the path from x to z. We claim that B can be partitioned as

(5.2.2)
$$B = \begin{array}{c|cccc|c}
 & y & s_1 & \cdots & s_m & \\
\hline
x & 1 & & & & \\
r_1 & 1 & 1 & & & \\
\vdots & & 1 & \ddots & & \\
r_m & & & 1 & 1 & \\
z & & & & 1 & \\
\hline
 & & & & 0/1 &
\end{array}$$

Shortest path from x to z displayed by B

In the submatrix $\overline{B}$ of B indexed by x, r_1, $r_2, \ldots,$ r_m, z and y, s_1, $s_2, \ldots,$ s_m, the 1s, circled or not, correspond to the edges of the path. Recall a convention of Section 2.3 about the display of matrices: If a submatrix or region of a matrix contains explicitly shown 1s while leaving the remaining entries unspecified, then the latter entries are to be taken as 0s. Accordingly, the display of the submatrix $\overline{B}$ implies that $\overline{B}$ does not contain any 1s beyond those corresponding to the edges of the path. We must prove, of course, that this is the case. So suppose there are additional 1s in $\overline{B}$. Then one readily confirms that the path has a chord, and thus is not shortest, a contradiction.

By GF(2)-pivots on the circled 1s of B of (5.2.2), we derive from B the following matrix B'.

(5.2.3)
$$B' = \begin{array}{c|cccc|c}
 & x & r_1 & \cdots & r_m & \\
\hline
y & 1 & & & & \\
s_1 & & 1 & & & \\
\vdots & & & \ddots & & \\
s_m & \text{all} & & & 1 & \\
z & \text{1s} & & & 1 & \\
\hline
 & & & & 0/1 &
\end{array}$$

Matrix B' obtained by path shortening pivots

The matrix B' proves that z is contained in the fundamental circuit that x forms with the base producing B', and we have proved x and z to be in some circuit of M as desired. □

Note that in BG(B), the path connecting x and z has at least two edges. In contrast, the graph BG(B') has an edge connecting x and z. Thus, the GF(2)-pivots transforming B to B' have reduced the path of BG(B) to a shorter path in BG(B'). For this reason, we call the above method the *path shortening technique*. Evidently, the proof procedure of Lemma (5.2.1) can be implemented in a polynomial algorithm that determines a circuit containing two given elements in any connected binary matroid.

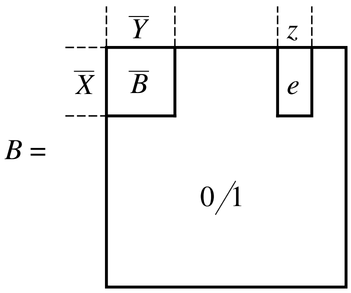
We now prove additional matroid results using the path shortening technique. They concern the existence of certain separations or of certain minors. In each case, the proof procedure has an obvious translation to a polynomial algorithm that very effectively locates one of the claimed separations or minors.

The first case concerns the existence of connected 1-element extensions of a given minor in a binary matroid.

(5.2.4) Lemma. *Let N be a connected minor of a connected binary matroid M. Define z to be an element of M not present in N. Then M has a connected minor N' that is a 1-element extension of N by z.*

Proof. By Lemma (3.3.12), M has a representation matrix B that displays N via a submatrix $\overline{B}$. Let the rows of $\overline{B}$ be indexed by $\overline{X}$ and the columns by $\overline{Y}$. Since M and N are connected matroids, both B and $\overline{B}$ are connected matrices. Suppose the element z indexes a column in B. We thus have B given by (5.2.5) below. If the subvector e of column z of B is nonzero, then the connected submatrix $[\overline{B} \mid e]$ represents the desired connected minor N' with z.

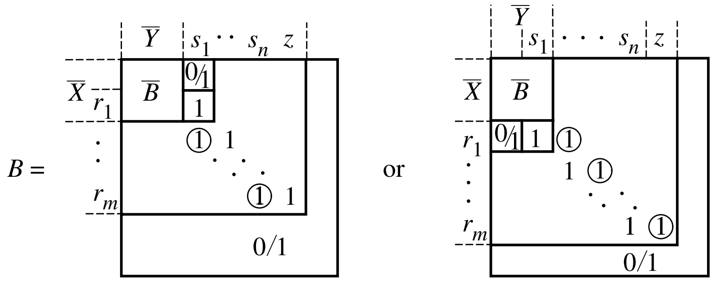
(5.2.5)



Matrix B displaying minor N by $\overline{B}$

So suppose $e = 0$. We know that the bipartite graph BG(B) is connected. Thus, that graph contains a path from $\overline{X} \cup \overline{Y}$ to z. Take a shortest path, say with row nodes $r_1, r_2, \ldots, r_m$ and column nodes $s_1, s_2, \ldots, s_n, z$, for some $m, n \geq 1$. The increasing indices indicate the order in which the nodes are encountered as one moves along the path from $\overline{X} \cup \overline{Y}$ to z. The endpoint of the path in $\overline{X} \cup \overline{Y}$ is r_1 or s_1, whichever applies. Below, we display B. The explicitly shown 1s correspond to the edges of the path. Two cases are possible, depending on whether r_1 or s_1 is the endpoint of the path in $\overline{X} \cup \overline{Y}$.

(5.2.6)



Case 1 of B: r_1 = endpoint of path

Case 2 of B: s_1 = endpoint of path

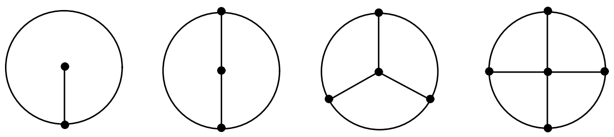
In each matrix of (5.2.6), the unspecified entries, which by our convention are 0s, are justified as follows. Suppose such an entry is instead a 1, for example in a row $x \in \overline{X}$ and column s_j, $j \geq 2$. Then the path does not have minimal length since there is a path starting at node x that is shorter. All other unspecified entries are argued similarly.

Assume case 1 of (5.2.6) applies. Then pivots on the circled 1s of B produce the already-discussed instance with $e \neq 0$. For case 2 of (5.2.6), pivots on the circled 1s produce a B' with a submatrix $[\overline{B}/d]$, where the vector d is nonzero and indexed by z. That submatrix represents the desired N'. The case where z indexes a row of B is handled by duality. □

So far, we have explained each step of the path shortening technique in detail. In the proofs to follow, we skip such details since the arguments are identical or at least very similar to those above.

The next result concerns the presence of minors that are isomorphic to the graphic matroids of wheels. Recall from Section 2.2 that for $n \geq 1$, the wheel W_n is constructed from a cycle with n nodes as follows. Add one extra node and link it with one edge each to the nodes of the cycle. Small wheels are as follows.

(5.2.7)



W_1 W_2 W_3 W_4

Small wheels

The spokes of each wheel W_n form a tree. With that tree as base of $M(W_n)$, we obtain from (5.2.7) the following representation matrices for $M(W_1), \ldots, M(W_4)$.

(5.2.8)

$$\underset{M(W_1)}{\boxed{0}} \qquad \underset{M(W_2)}{\boxed{\begin{matrix}1 & 1\\ 1 & 1\end{matrix}}} \qquad \underset{M(W_3)}{\boxed{\begin{matrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{matrix}}} \qquad \underset{M(W_4)}{\boxed{\begin{matrix}1 & 0 & 0 & 1\\ 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\end{matrix}}}$$

Representation matrices

Indeed, for $n \geq 3$, $M(W_n)$ is represented by the $n \times n$ matrix

(5.2.9)

$$\boxed{\begin{matrix}1 & & & 1\\ 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \;\; 1\end{matrix}}$$

Representation matrix for $M(W_n)$, $n \geq 3$

Clearly for $n \geq 3$, W_n is 3-connected. Thus, by Theorem (3.2.36), W_n is the unique graph producing $M(W_n)$.

The graphic matroids of wheels are cited very often in the remainder of this book. We adopt an abbreviated terminology and simply call them *wheels*. The double meaning of "wheel" should not cause a problem. The context invariably makes it clear whether we mean a matroid or a graph. At any rate, we use *wheel graph* and *wheel matroid* if there is even a slight chance of confusion.

A basic result about wheels is as follows.

(5.2.10) Lemma. *Let M be a binary matroid with a binary representation matrix B. Suppose the graph* BG(B) *contains at least one cycle. Then M has an $M(W_2)$ minor.*

Proof. Since the graph BG(B) has at least one cycle, it has a cycle C without chords. Now BG(B) is bipartite, so C has at least four edges. The submatrix $\overline{B}$ of B corresponding to C evidently is up to indices either the 2×2 matrix for $M(W_2)$ of (5.2.8), or for some $n \geq 3$, the $n \times n$ matrix of (5.2.9) for $M(W_n)$. Path shortening pivots on 1s of $\overline{B}$ convert the latter case to the former one. □

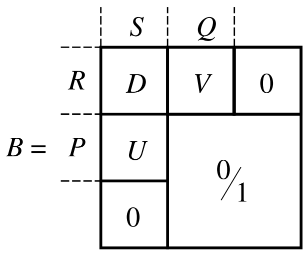
Lemma (5.2.10) and another application of the path shortening technique lead to a proof of the following result.

(5.2.11) Lemma. *Let M be a connected binary matroid with at least four elements. Then M has a 2-separation or an $M(W_3)$ minor.*

Proof. Let B be a binary representation matrix of M. Since M is connected, the bipartite graph $\mathrm{BG}(B)$ is connected. Suppose $\mathrm{BG}(B)$ does not contain a cycle. Thus, that graph is a tree. Any tip node of the tree corresponds to a row or column unit vector in B. Thus, M has parallel or series elements. Since M has at least four elements, M has a 2-separation.

We turn to the remaining case where $\mathrm{BG}(B)$ has a cycle. By Lemma (5.2.10), M has an $M(W_2)$ minor. We may assume that B displays that minor via the 2×2 matrix of (5.2.8), with four 1s. Enlarge that 2×2 matrix to a maximal submatrix of B containing only 1s. Let D be that submatrix, say with rows indexed by a set R and columns indexed by a set S. We use these sets to partition B as shown in (5.2.12) below. In B, each row of the submatrix U and each column of the submatrix V is assumed to be nonzero. By the maximality of D, at least one 0 must be contained in each row of U and in each column of V.

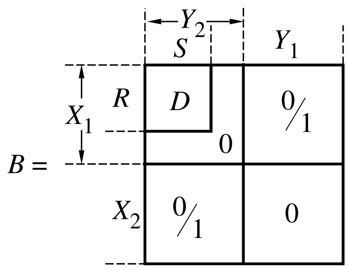
(5.2.12)



Partitioned matrix B

Let F be the graph obtained from $\mathrm{BG}(B)$ by deletion of the edges corresponding to the 1s of the submatrix D. Suppose no path of F connects a node of R with one of S. Let X_1 (resp. Y_1) be the row (resp. column) nodes of F that are connected by some path with some node of R. Since $X_1 \supseteq R$, we have $|X_1 \cup Y_1| \geq 2$. Define $X_2 = X - X_1$ and $Y_2 = Y - Y_1$. Since $Y_2 \supseteq S$, we have $|X_2 \cup Y_2| \geq 2$. By derivation of F from $\mathrm{BG}(B)$, any arc of $\mathrm{BG}(B)$ connecting a node of $X_1 \cup Y_1$ with one of $X_2 \cup Y_2$ must correspond to a 1 of D. But $X_1 \supseteq R$ and $Y_2 \supseteq S$, so the partitioning of B according to X_1, X_2, Y_1, and Y_2 must result in

(5.2.13)



Matrix B with 2-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$

Since GF(2)-rank $D = 1$, $(X_1 \cup Y_1, X_2 \cup Y_2)$ is a 2-separation of B, and we are done. So suppose a path in F does connect a node of R with one of S. Choose a path of minimal length. Evidently, the length must be odd and at least 3.

Assume the length to be 3. Then the center edge of the path corresponds in B of (5.2.12) to a 1 in some row $p \in P$ and some column $q \in Q$. Now row p of U contains a 0 and a 1, say in columns $s_1 \in S$ and $s_2 \in S$, respectively. Similarly, column q of V has a 0 and a 1, say in rows $r_1 \in R$ and $r_2 \in R$, respectively. Then in B of (5.2.12), the submatrix indexed by r_1, r_2, p and s_1, s_2, q is

(5.2.14)
$$\begin{array}{c|ccc|} & s_1 & s_2 & q \\ \cline{2-4} r_1 & 1 & 1 & 0 \\ r_2 & 1 & 1 & 1 \\ p & 0 & 1 & 1 \\ \cline{2-4} \end{array}$$

Submatrix of B representing $M(W_3)$ minor

A GF(2)-pivot on the 1 in row p and column q produces up to indices the matrix for $M(W_3)$ of (5.2.8). Thus, M has an $M(W_3)$ minor.

Finally, suppose the shortest path in F from R to S has length greater than 3. With the path shortening technique, we reduce that path to one of length 3. The related GF(2)-pivots in B of (5.2.12) do not affect the submatrices D, U, or V. Thus, the pivots produce the earlier situation with length 3. □

Lemma (5.2.11) supports the following conclusion about 3-connected binary matroids.

(5.2.15) Corollary. *Every 3-connected binary matroid M with at least six elements has an $M(W_3)$ minor.*

Proof. By Lemma (5.2.11), M has a 2-separation or an $M(W_3)$ minor. Since M is 3-connected, the former case is not possible. □

At times, one would like to claim that a given matroid M has a certain minor containing a specified element. A simple case is given in the following result.

(5.2.16) Lemma. *Let M be a connected binary matroid with an $M(W_3)$ minor. Then for every element z of M, there is an $M(W_3)$ minor of M that contains z.*

Proof. Let N be the assumed $M(W_3)$ minor of M, and z be any element of M. If z is an element of N, we are done. Otherwise, by Lemma (5.2.4), M has a connected minor of the form $N{+}z$ or $N\&z$. Consider the first case. We may assume $N{+}z$ to be represented by the matrix of (5.2.8) for $M(W_3)$, plus one nonzero column indexed by z. A straightforward case analysis proves that for each possible column z, the matroid $N{+}z$ has an $M(W_3)$ minor with z. Thus, M has an $M(W_3)$ minor containing z as well. The situation for $N\&z$ is handled analogously, or by duality. □

In the next section, we investigate the intersection and partitioning problems. The subsequent chapters do not rely on the material of the section, so it may be skipped without loss of continuity.

5.3 Intersection and Partitioning of Matroids

Let M_1 and M_2 be two arbitrary matroids defined on a common set E and with rank functions $r_1(\cdot)$ and $r_2(\cdot)$, respectively. The *cardinality intersection problem* demands that we find a maximum cardinality set $Z \subseteq E$ that is independent in M_1 and M_2. The *partitioning problem* requires that we locate a partition of E into E_1 and E_2 such that $r_1(E_1) + r_2(E_2)$ is minimized. In this section, we present an algorithm that simultaneously solves both problems. We call it the *intersection algorithm*. The method consists of repeated applications of the path shortening technique, though carried out in a rather unusual fashion. The algorithm also provides a constructive proof of a max-min theorem that links the two problems in an unexpected way.

We have elected to describe the intersection algorithm for general matroids instead of just binary ones, since in many, if not most, applications, at least one of the two matroids M_1, M_2 is nonbinary. Correspondingly, the algorithm makes use of abstract matrices. Thus, the reader should be familiar with the material on abstract matrices in Section 3.4, in particular with Lemma (3.4.11). That result says that any triangular submatrix of an abstract matrix is nonsingular if and only if the submatrix has only 1s on its diagonal.

We begin with the intersection problem. As stated above, we have two matroids M_1 and M_2 on a common set E. The algorithm must find a maximum cardinality set Z that is independent in both M_1 and M_2. The scheme begins with any set Z that is independent in both matroids. For example, $Z = \emptyset$ will do. The method iteratively replaces the given set Z by a larger one that is also independent in both matroids, until a set of maximum cardinality is found. It suffices that we describe one iteration. It consists of three steps, which we summarize next.

In step 1, we deduce two matrices from certain abstract matrices for M_1 and M_2. We combine the two matrices to a new matrix C that has a strange form, but that actually is a handy encoding of the initial abstract matrices of M_1 and M_2.

In step 2, we derive a graph from C and search in that graph for a certain path. Suppose a path of the desired kind can be located. We interpret that path in terms of the abstract matrices for M_1 and M_2 and deduce a set Z' that is larger than Z and independent in M_1 and M_2. With the set Z' in hand, we terminate the iteration. If a path of the desired kind cannot be found, we go to step 3.

In step 3, we conclude that the absence of paths of the desired kind implies a certain partition of the matrix C. We interpret that partition in terms of the abstract matrices for M_1 and M_2 and conclude that Z is optimal. Thus, we stop. The proof of optimality for Z also shows that the partition of C implies a partition of the set E into two sets, say E_1 and E_2, that solve the partitioning problem. In addition, the proof establishes the previously mentioned max-min theorem that connects the intersection problem with the partitioning problem.

We begin the detailed description. In step 1, we first find for $i = 1, 2$, a base X_i of M_i that contains the set Z. This is possible since Z is independent in M_1 and M_2. Let B^i be the abstract matrix of M_i corresponding to the base X_i. Thus, B^i has row index set X_i and column index set $E - X_i$. We adjoin an identity to B^i, getting $[I \mid B^i]$. In agreement with the indexing rules introduced in Section 2.3, we index the columns of the submatrix I of $[I \mid B^i]$ by X_i. Next we permute the columns of $[I \mid B^1]$ and $[I \mid B^2]$ such that the columns of the two matrices in same position have the same column index. Furthermore, the columns indexed by Z are to become the leftmost columns. Finally, we add zero rows if necessary, so that both matrices have the same number of rows. For $i = 1, 2$, let A^i be the matrix so obtained from $[I \mid B^i]$, and define $Y = E - Z$. Then the matrix A^i is of the form

(5.3.1)
$$A^i = \begin{array}{c|c|c|} & Z & Y \\ \hline Z & \begin{matrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix} & \\ \cline{1-2} & 0 & 0/1 \\ \hline \end{array} \quad ; i = 1, 2$$

Matrix A^i obtained from $[I \mid B^i]$

The rows of A^i without index either are rows of $[I \mid B^i]$ indexed by $X_i - Z$, or are added zero rows.

Before going on, we would like to establish a simple lemma about the matrix A^i of (5.3.1). The result will allow easy verification that certain subsets of M_i are independent.

(5.3.2) Lemma. *For some $k \geq 1$, let $\overline{Z}$ be a subset of E with k elements. If the column submatrix of A^i indexed by $\overline{Z}$ contains a $k \times k$ triangular* *submatrix that has only 1s on the diagonal, then $\overline{Z}$ is independent in M_i.* □

Proof. Except possibly for column permutations and additional zero rows, A^i is the matrix $[I \mid B^i]$. Thus, the postulated $k \times k$ triangular submatrix of A^i has block triangular form, where one of the blocks is an identity submatrix of I, and where the other block is a triangular submatrix of B^i with only 1s on the diagonal. By Lemma (3.4.11), the set $\overline{Z}$ must be a subset of a base of M_i, and thus is independent in M_i. □

We continue with step 1. In the matrix A^1 (resp. A^2), we replace each 1 by α (resp. β), getting a matrix $\tilde{A}^1$ (resp. $\tilde{A}^2$). We compute a matrix $C = \tilde{A}^1 + \tilde{A}^2$ by adding entries termwise according to the following rule: $0+0=0$, $\alpha+0=\alpha$, $0+\beta=\beta$, and $\alpha+\beta=\gamma$. By (5.3.1), the matrix C is of the form

$$
C = \begin{array}{c|c|c|}
 & Z & Y \\ \hline
Z & \begin{matrix}\gamma & & \\ & \gamma & \\ & & \ddots \\ & & & \gamma\end{matrix} & \\ \cline{1-2}
P & 0 & 0/\alpha/\beta/\gamma \\ \hline
\end{array}
\tag{5.3.3}
$$

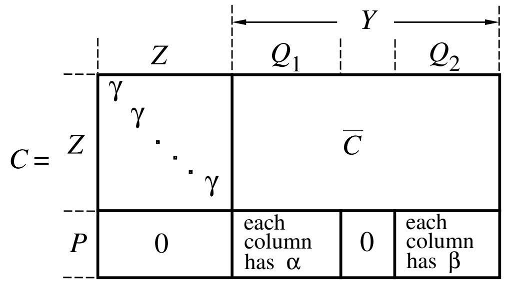
Matrix $C = \tilde{A}^1 + \tilde{A}^2$

Note the row index subset P in (5.3.3). The rows of P arise from the rows of A^1 and A^2 shown in (5.3.1) without index. We consider P to be a new index set that is disjoint from the index sets of A^1 and A^2. This concludes step 1.

In step 2, we first examine C of (5.3.3) for a trivial way of augmenting the set Z to a larger set Z' that is independent in M_1 and M_2. Specifically, assume that C contains a column z that in rows indexed by P has both an α and a β, or a γ. Then in the matrices A^1 and A^2, the column submatrices $\overline{A}^1$ and $\overline{A}^2$ indexed by $Z' = Z \cup \{z\}$ have triangular submatrices that via Lemma (5.3.2) prove Z' to be independent in M_1 and M_2. Thus, we can stop the iteration. So from now on, we assume that C has no such column z. Thus, each column of C contains in the rows indexed by P either just 0s and αs, or just 0s and βs, or just 0s. Let Q_1 (resp. Q_2) index the columns of the first (resp. second) kind. Using these two index sets, we partition C of (5.3.3) further as shown in (5.3.4) below. From the submatrix $\overline{C}$ defined by the row index set Z and the column index set Y of C of (5.3.4), we construct the following directed bipartite graph G. We start with the undirected bipartite graph $\mathrm{BG}(\overline{C})$. Let (i, j) be an edge of $\mathrm{BG}(\overline{C})$ where i is a row node of Z and j is a column node of Y. If the entry of $\overline{C}$ producing that edge is an α (resp. β), then we direct that edge from i to j (resp. j

to i). If that edge corresponds to a γ, then we replace it by two directed edges of opposite direction.

(5.3.4)

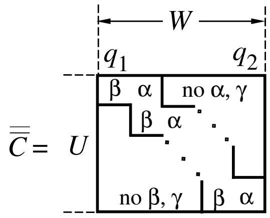


Matrix C partitioned by Q_1 and Q_2

Using any convenient shortest route algorithm, we locate a shortest path from Q_1 to Q_2, or determine that no such path exists. A case of the latter variety is on hand, for example, if Q_1 or Q_2 is empty.

If a shortest path does not exist, we go to step 3, to be covered shortly. Otherwise, let such a path connect a node $q_1 \in Q_1$ with a node $q_2 \in Q_2$. Let $\overline{\overline{C}}$ be the submatrix of $\overline{C}$ defined by the nodes of that path. We claim that $\overline{\overline{C}}$ either is the matrix

(5.3.5)



Matrix $\overline{\overline{C}}$ defined by the nodes of the path

or is obtained from the matrix of (5.3.5) by replacing any number of the explicitly shown αs and βs by γs. In $\overline{\overline{C}}$ of (5.3.5), the statements "no α, γ" and "no β, γ" are valid since any violating entry would permit a shorter path from Q_1 to Q_2, a contradiction. For the same reason, we have for $i = 1, 2$, $Q_i \cap W = \{q_i\}$. We use the index sets U and W of $\overline{\overline{C}}$ to define $Z' = (Z - U) \cup W$. By (5.3.5), $|W| = |U| + 1$, so $|Z'| = |Z| + 1$. We claim that Z' is independent in M_1 and M_2. For a proof, we examine the column submatrix C' of C indexed by Z'. By (5.3.4) and (5.3.5), the corresponding column submatrices of A^1 and A^2 contain triangular matrices that confirm the claim. Thus, we have completed the iteration.

As an aside, consider one pivot in B^1 and B^2, each time on a particular 1 in column q_1. Specifically, in B^1 the 1 corresponds to an α entry of C in column q_1 and in a row of P, and in B^2 to the β entry of C explicitly shown in column q_1 and in the first row, say x, of $\overline{\overline{C}}$. Correspondingly, we drop the element x from Z and add the element q_1, getting, say, a set $\tilde{Z}$. The sets Z and $\tilde{Z}$ have the same cardinality, but if we repeat the above process for $\tilde{Z}$ instead of Z, we discover a shorter path. By suitable repetition of the above procedure, the path becomes ever shorter until we finally just add an element to obtain the set Z'.

Finally, we discuss step 3. We enter that step when a path from Q_1 to Q_2 does not exist. Let $Z_2 \subseteq Z$ and $Y_2 \subseteq Y$ be the nodes reachable from the nodes of Q_1, and define $Z_1 = Z - Z_2$ and $Y_1 = Y - Y_2$. The sets Z_1, Z_2, Y_1, Y_2 induce the following partition in C of (5.3.3).

(5.3.6)
$$C = \begin{array}{cc|c|c|cc|cc|}
 & & \multicolumn{2}{c}{\longleftarrow Z \longrightarrow} & \multicolumn{4}{c}{\longleftarrow Y \longrightarrow} \\
 & & Z_2 & Z_1 & \multicolumn{2}{c}{Y_2} & \multicolumn{2}{c}{Y_1} \\
\hline
Z & Z_2 & \gamma \ddots \gamma & 0 & \multicolumn{2}{c|}{0/\alpha/\beta/\gamma} & \multicolumn{2}{c|}{\text{no } \alpha, \gamma} \\
\hline
 & Z_1 & 0 & \gamma \ddots \gamma & \multicolumn{2}{c|}{\text{no } \beta, \gamma} & \multicolumn{2}{c|}{0/\alpha/\beta/\gamma} \\
\hline
 & P & 0 & 0 & \text{each column has } \alpha & 0 & 0 & \text{each column has } \beta \\
\hline
\end{array}$$

Matrix C when a path does not exist

In particular, the statements "no α, γ" and "no β, γ" in the submatrices indexed by Z_2, Y_1 and Z_1, Y_2, respectively, are correct since otherwise at least one additional node could be reached from Q_1. For $i = 1, 2$, define $E_i = Z_i \cup Y_i$. By definition, E_1 and E_2 partition E. From C of (5.3.6), it is obvious that E_i as subset of the matroid M_i has rank equal to $|Z_i|$. Furthermore, since $Z = Z_1 \cup Z_2$, we have

(5.3.7)
$$|Z| = r_1(E_1) + r_2(E_2)$$

Now for any set Z independent in M_1 and M_2, and for any partition of E into any sets E_1 and E_2, we must have for $i = 1, 2$, $|Z \cap E_i| = r_i(Z \cap E_i) \leq r_i(E_i)$. Adding over $i = 1, 2$, we obtain

(5.3.8)
$$|Z| = |Z \cap E_1| + |Z \cap E_2| \leq r_1(E_1) + r_2(E_2)$$

By (5.3.7) and (5.3.8), the set Z on hand in step 3 solves the intersection problem, and the sets E_1 and E_2 found at that time solve the partition problem. Thus, the algorithm has solved both problems simultaneously. Clearly, the algorithm is polynomial — indeed, very efficient.

The preceding arguments also prove the following theorem due to Edmonds.

(5.3.9) Theorem (Matroid Intersection Theorem). *Let M_1 and M_2 be two matroids on a set E and with rank functions $r_1(\cdot)$ and $r_2(\cdot)$. Then*

(5.3.10)
$$\max |Z| = \min\{r_1(E_1) + r_2(E_2)\}$$

where the maximization is over all sets Z that are independent in both M_1 and M_2, and where the minimization is over all partitions of E into sets E_1 and E_2.

We describe two representative example applications. The first example involves an undirected bipartite graph G, say with edge set E connecting the nodes of a set V_1 with those of a set V_2. Define a *matching* to be a subset Z of E such that every node of G has at most one edge of Z incident. For $i = 1, 2$, let M_i be the matroid on E where a subset Z is independent if the nodes of V_i have at most one edge of Z incident. It is easily checked that M_i is the disjoint union of $|V_i|$ uniform matroids of rank 1, and thus is a matroid. Clearly, the matchings of G are precisely the subsets of E that are independent in both M_1 and M_2. Define a *node cover* to be a node subset such that every edge has at least one endpoint in that subset. We now reformulate the matroid intersection Theorem (5.3.9) to a basic result of graph theory due to König.

(5.3.11) Theorem. *The cardinality of a maximum matching of a bipartite graph is equal to the cardinality of a minimum node cover.*

We leave it to the reader to prove that Theorem (5.3.11) follows from Theorem (5.3.9).

The second application concerns separations in matroids. Suppose we want to know whether a given matroid M on a set E has a k-separation with at least $k + l$ elements on each side, for some nonnegative integers k and l. How can we find such a separation or prove that none exists? Suppose we can efficiently solve the following, more restricted, problem. We are given two disjoint subsets F_1 and F_2 of E, each of cardinality $k + l$. We must decide whether M has a k-separation (E_1, E_2) such that $E_1 \supseteq F_1$ and $E_2 \supseteq F_2$. If we can solve the restricted problem, then we can solve the original one by enumerating all possible choices of the sets F_1 and F_2. The overall algorithm is polynomial if the possible values of $k + l$ can be uniformly bounded by some constant.

We analyze the restricted problem. Define $r(\cdot)$ to be the rank function of M. We need to find a pair (E_1, E_2) such that

$$\begin{aligned} &E_1 \supseteq F_1;\ E_2 \supseteq F_2 \\ &r(E_1) + r(E_2) \leq r(E) + k - 1 \end{aligned} \tag{5.3.12}$$

or prove that such a pair does not exist. An answer to the following problem,

$$\min_{\substack{(E_1,E_2) \\ E_1 \supseteq F_1, E_2 \supseteq F_2}} \{r(E_1) + r(E_2)\} \tag{5.3.13}$$

obviously suffices. We want to eliminate the conditions $E_1 \supseteq F_1$ and $E_2 \supseteq F_2$ from (5.3.13) by some matroid construction. Let $M_1 = M/F_1\backslash F_2$ and

$M_2 = M/F_2\backslash F_1$, with respective rank functions $r_1(\cdot)$ and $r_2(\cdot)$. Define $\overline{E} = E - (F_1 \cup F_2)$, and for $i = 1,2$, $\overline{E}_i = E_i \cap \overline{E}$. Then for $i = 1,2$, $r(E_i)$ is nothing but $r(F_i) + r_i(\overline{E}_i)$. Thus, $r(E_1) + r(E_2) = r(F_1) + r_1(\overline{E}_1) + r(F_2) + r_2(\overline{E}_2) = r_1(\overline{E}_1) + r_2(\overline{E}_2) +$ a constant. Thus, equivalent to (5.3.13) is the problem

$$\min_{(\overline{E}_1,\overline{E}_2)} \{r_1(\overline{E}_1) + r_2(\overline{E}_2)\} \tag{5.3.14}$$

which can be solved by the intersection algorithm. For uniformly bounded $k + l$, the above scheme is polynomial. But from a practical standpoint, even small bounds on $k + l$, such as 4 or 5, are likely to make the scheme practically unusable. So one might want to explore other avenues to find k-separations. In the next chapter, we learn about a method for a particular case that turns out to be very important for several settings. Specifically, we will have $k = 3$ and $l \leq 3$.

In the next section, we point out additional material about intersection and partitioning problems, and list references.

5.4 Extensions and References

The path shortening technique is introduced in Truemper (1984). As we have seen in Section 5.3, the technique fully applies to abstract representation matrices of general matroids. Thus, virtually all results of Section 5.2 can be translated to almost identical ones for general matroids. One additional class of nonbinary matroids must be introduced, though. It is the class of *whirls* $\mathcal{W}_n$. We define these matroids next. A *whirl* of rank $n \geq 1$ is derived from the wheel matroid with same rank by declaring the circuit containing the elements of the rim edges to be independent. Small whirls are represented over GF(3) by the matrices of (5.4.1) below. In general, for $n \geq 3$, $\mathcal{W}_n$ is represented over GF(3) by the matrix of (5.4.2) below, where $\alpha \in \{+1,-1\}$ is so chosen that the GF(3)-determinant of the matrix is -1. Equivalently, the real sum of the entries must be $2 (\text{mod}\, 4)$. Note that $\mathcal{W}_2$ is also U_4^2, the uniform matroid of rank 2 on four elements.

(5.4.1)

$$\underset{\mathcal{W}_1}{\boxed{1}} \qquad \underset{\mathcal{W}_2}{\boxed{\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}}} \qquad \underset{\mathcal{W}_3}{\boxed{\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}}} \qquad \underset{\mathcal{W}_4}{\boxed{\begin{array}{cccc} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}}}$$

Representation matrices over GF(3) for whirls $\mathcal{W}_1$–$\mathcal{W}_4$

(5.4.2)

$$\begin{array}{|ccccc|}\hline 1 & & & & \alpha \\ 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \\ \hline\end{array}$$

Representation matrix over GF(3) for whirl $\mathcal{W}_n$, $n \geq 3$

Lemmas (5.2.1) and (5.2.4) are valid for general matroids. Lemmas (5.2.10) and (5.2.11) and Corollary (5.2.15) are readily extended to the general case by allowing occurrence of the whirl $\mathcal{W}_2$ as possible minor besides the given wheel matroids. All these results, including the extensions to nonbinary matroids, are implicit in Whitney (1935) and Tutte (1958), (1965), (1971).

We have no reference for Lemma (5.2.16), but that result is well known. The nonbinary version of that lemma is proved in Bixby (1974). It involves U_4^2 instead of $M(W_3)$, and claims the existence of a U_4^2 minor with z, as follows.

(5.4.3) Lemma. *Let M be a connected matroid with a U_4^2 minor. Then for every element z of M, there is a U_4^2 minor of M that contains z.*

The proof is virtually identical to that of Lemma (5.2.16) except that abstract matrices are used here. The result motivated a long series of papers concerning the presence of specified elements in given minors (Seymour (1981e), (1985b), (1986a), (1986b), Oxley (1984), (1987a), (1990a), Kahn (1985), Coullard (1986), Coullard and Reid (1988), Oxley and Row (1989), Oxley and Reid (1990), and Reid (1990), (1991a), (1991b), (1993), (1996)).

The matroid intersection and partitioning results of Section 5.3 are just a small sampling of a wealth of material. The roots of these problems can be traced back to several matching results of which Theorem (5.3.11), due to König (1936), is an example. Lovász and Plummer (1986) give a very complete account of these developments. Other early results related to matroid intersection and partitioning are the solution of the problem of partitioning a graph into forests in Nash-Williams (1961), (1964) and Tutte (1961), and the solution of the so-called optimum branching problem, first in Chu and Liu (1965), and later in Edmonds (1967a), Bock (1971), and Karp (1971).

Edmonds (1965a), (1970), (1979) introduced and solved the matroid intersection and partitioning problems, as well as generalizations to matroids with weighted elements and to so-called polymatroids. He proved that these problems can be converted to structurally simple linear programs since certain polytopes have only integer vertices. In the terminology of linear programming, the intersection problem is then the linear programming dual of the partitioning problem. This work and Edmonds's profound results for matching problems (for a complete coverage, see Lovász and Plummer (1986)) establish Edmonds as the founder of polyhedral combinatorics. For applications, generalizations, and other algorithms, see Lehman (1964), Edmonds (1965b), (1967b), Edmonds and Fulkerson (1965), Lawler (1975), (1976), Frank (1981), Hassin (1982), Lawler and Martel (1982a), (1982b), Grötschel, Lovász, and Schrijver (1988), and Fujishige (1989), (1991). Schrijver (1984) houses many of the problems treated in those references under one roof.

As far as we know, the graph formulation of the intersection algorithm described in Section 5.3 is due to Krogdahl (see Lawler (1976)), except for our use of abstract matrices to simplify the arguments. In Cunningham (1986) a considerably faster version of the algorithm is given. The improvement rests on Menger's Theorem and the observation that the path used to derive the set Z' from the given set Z should be chordless, but need not be shortest. Incidentally, Menger's Theorem may also be used to determine, for the given set Z, a set $R \subseteq (E-Z)$ of minimum cardinality so that Z becomes an independent set of maximum cardinality in the minors $M_1 \backslash R$ and $M_2 \backslash R$ of M_1 and M_2. The graph approach may also be used to improve the already very appealing algorithm of Frank (1981) for the intersection problem with weighted elements.

The second application cited at the end of Section 5.3, which involves certain k-separations of a matroid, is due to Cunningham (1973), and Cunningham and Edmonds (1978).

Finally, we should mention that the intersection case involving at least three matroids, is in general $\mathcal{NP}$-hard since it includes the $\mathcal{NP}$-complete Hamiltonian cycle problem (see Garey and Johnson (1979)).