Chapter 4

Series-Parallel and Delta-Wye Constructions

4.1 Overview

This chapter is the first of three on matroid tools. Here, we construct graphs and binary matroids with elementary procedures. For graphs, the constructions involve addition of a parallel edge, or subdivision of an edge into two series edges, or substitution of a triangle by a 3-star, or substitution of a 3-star by a triangle. We call the first two operations *series-parallel extension steps*, for short SP *extension steps*. Either one of the triangle/3-star substitution steps is a *delta-wye step*, for short ΔY *step*. These operations have a natural translation to operations on binary matroids.

The power of SP extension steps is quite limited. Suppose in the graph case one starts with a cycle with just two edges and applies SP extension steps. Then rather simple graphs are produced. They are usually called *series-parallel graphs*, for short SP *graphs*. In the binary matroid case, let us start with a circuit containing just two parallel elements. Then we produce nothing else but the graphic matroids of the SP graphs. These results and some related material are described in Section 4.2.

The situation changes dramatically when we mix SP extension steps with ΔY steps. In the graph case, suppose we start again with a cycle with two edges. Then we produce all 2-connected planar graphs and more. How much more is a difficult open question. Similarly, suppose that in the binary matroid case we start with a circuit with two edges. Then we produce the graphic matroids of the just-described graphs, as well as nongraphic binary matroids. Here, too, the class of matroids so obtained is not well understood. In Chapter 11, it is proved that every matroid of that class is regular.

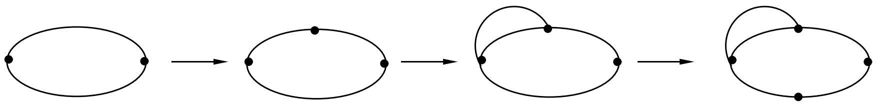
One may, of course, start a sequence of SP extension steps and ΔY steps with any collection of graphs, not just from the cycle with two edges. Little is known about the classes of graphs so obtained. The same goes for binary matroids, with one exception. The class of almost regular matroids, which we define later, is generated when one starts with two binary matroids on seven and eleven elements, respectively.

The cited material on SP extension steps and ΔY steps in graphs and binary matroids is covered in Sections 4.3 and 4.4, respectively. The final Section 4.5 contains applications, extensions of some of the matroid results to general matroids, and relevant references.

The material of this chapter builds upon Chapters 2 and 3.

4.2 Series-Parallel Construction

Start with the cycle with just two edges. In that small graph, replace one edge by two parallel edges or by two series edges. To the resulting graph apply either one of these two operations to get a third graph, and so on. Here is an example with three such extension steps.

(4.2.1) 

Series-parallel extension steps

Each iteration is a *series-parallel extension step*, for short SP *extension step*. The class of graphs producible this way are the *series-parallel graphs*, for short SP *graphs*. The inverse of an SP extension step is an SP *reduction step*.

In this section, we analyze the structure of SP graphs. In particular, we characterize them in terms of forbidden minors. We begin with some elementary lemmas.

(4.2.2) Lemma. *Every SP graph is 2-connected and planar. Any minor of an SP graph is also an SP graph, provided the minor has at least two edges and is 2-connected.*

Proof. The cycle with two edges is 2-connected and planar. An SP extension step in a graph with at least two edges cannot introduce a 1-separation or destroy planarity. By induction, the SP graphs are 2-connected and planar.

For the proof of the second part, paint in a given SP graph the edges of a given 2-connected minor red. Reduce the SP graph by SP reduction steps until the cycle with two edges is obtained. We examine a single reduction step and apply induction. We must consider two cases for that step: deletion of a parallel edge, and contraction of one of two edges with a common degree 2 endpoint. Consider the deletion case. If both edges are red, then the reduction is also an SP reduction in the minor. If exactly one edge is not red, then that edge is deleted. The minor must still be present, since contraction of that edge would turn the red edge into a loop, contrary to the assumption that the minor is 2-connected. If both edges are not red, then the minor is still present after deletion of one of these edges. For if both edges must be contracted to produce the minor, then the second contraction involves a loop, and thus is a deletion. The contraction case is handled analogously. □

Recall that K_4 is the complete graph on four vertices.

(4.2.3) Lemma. *Every 3-connected graph G with at least six edges has a K_4 minor.*

Proof. Take any cycle C of G of minimal length. Since G is 3-connected and has at least six edges, it must have a node that does not lie on C. By Menger's Theorem, there are three internally node-disjoint paths from that additional node to three distinct nodes of C. Suitable deletions and contractions eliminate all other edges and reduce the cycle and three paths to a K_4 minor. □

(4.2.4) Lemma. *K_4 is not an SP graph.*

Proof. K_4 does not have series or parallel edges. □

(4.2.5) Lemma. *No SP graph has a K_4 minor.*

Proof. Presence of a K_4 minor would contradict Lemmas (4.2.2) and (4.2.4). □

We are ready to characterize the SP graphs in terms of excluded minors.

(4.2.6) Theorem. *A 2-connected graph is an SP graph if and only if it has no K_4 minor.*

Proof. The "only if" part is handled by Lemma (4.2.5). We thus prove the converse. Let G be a 2-connected graph without K_4 minors. Simple checking validates the small cases with up to five edges. So assume G has at least six edges. By Lemma (4.2.3), G must be 2-separable. Choose a 2-separation so that for the two corresponding graphs G_1 and G_2, we have G_1 with minimal number of edges.

Suppose G_1 has exactly two edges. These edges must be parallel or incident at a degree 2 node of G. Thus, we can reduce and apply induction. Suppose G_1 has at least three edges. Let k and l be the nodes of G_1 that must be identified with two nodes of G_2 to produce G. Suppose G_1 has an edge z connecting k and l. That edge can be shifted from G_1 to G_2. The corresponding new 2-separation contradicts the minimality assumption on the edge set of G_1. Similarly, the nodes k and l cannot have degree 1 in G_1.

Add an edge e to G_1 connecting nodes k and l. The new graph G_1' is isomorphic to a proper minor of G. By induction, G_1' is an SP graph. By the above discussion, in G_1' the edge e is not parallel to another edge, and it does not have an endpoint of degree 2. Thus, any SP reduction step in G_1' can be carried out in G as well. We perform one such step in G, and invoke induction for the reduced graph. □

One may reformulate the construction of SP graphs as follows. Start with some cycle. Iteratively enlarge the graph on hand as follows. Select a path in the graph where all internal nodes have degree 2. Let k and l be the endpoints of the path. Then add to the graph a path from k to l. Evidently, this construction creates precisely all SP graphs. It also allows a short proof of the following result.

(4.2.7) Lemma. *An SP graph without parallel edges either is a cycle with at least three edges, or has two internally node-disjoint paths with the following properties. Each path has at least two edges. Each intermediate node of the two paths has degree* 2 *in the graph, while the endpoints have degree of at least* 3.

Proof. Consider the alternate construction. The initial cycle must have at least three edges, since otherwise the final graph has parallel edges. When the first path is adjoined to the initial cycle, the lemma is satisfied, or the final graph has parallel edges. The same conclusion holds by induction after each additional path augmentation. □

Lemma (4.2.7) has the following corollary.

(4.2.8) Corollary. *An SP graph with at least four edges and without parallel edges has at least two nonadjacent nodes with degree* 2.

Proof. If the SP graph is a cycle, then the conclusion is immediate. So assume that the SP graph is not a cycle. Then each one of the two paths postulated in Lemma (4.2.7) has at least one intermediate degree 2 node. Thus, the graph has two nonadjacent degree 2 nodes. □

We introduce two interesting subclasses of the class of SP graphs by excluding certain graphs as minors. One of the excluded graphs we already know. It is $K_{2,3}$, the complete bipartite graph with two vertices on one side and three on the other one. The second excluded graph is the *double* *triangle*, obtained from the triangle by replacing each edge by two parallel edges. We denote that graph by C_3^2. We want to characterize first the SP graphs without $K_{2,3}$ minors, and then those without C_3^2 minors. To this end, define a graph to be *outerplanar* if it can be drawn in the plane so that all vertices lie on the infinite face.

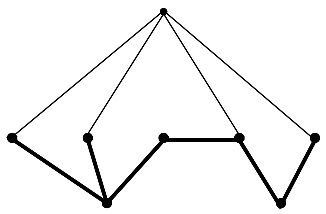
(4.2.9) Theorem. *The following statements are equivalent for a 2-connected graph G with at least two edges.*

(i) *G has no K_4 or $K_{2,3}$ minors.*
(ii) *G is an SP graph without $K_{2,3}$ minors.*
(iii) *G is outerplanar.*

Proof. By Theorem (4.2.6), G is an SP graph if and only if it has no K_4 minors. Thus, (i) $\Longleftrightarrow$ (ii). To show (ii) $\Longrightarrow$ (iii), let G be an SP graph without $K_{2,3}$ minors. Define C to be a cycle of G of maximum length. Suppose G has a node v that does not lie on C. Since G is 2-connected, there exist two paths from node v to distinct nodes i and j on C so that these paths have only the node v in common. If i and j are connected by an edge of C, then C can be extended to a longer cycle using the two paths, a contradiction. If i and j are not joined by an edge of C, then C and the two paths are easily reduced to a $K_{2,3}$ minor of G, another contradiction. Thus, all nodes of G occur on C. For the proof of outerplanarity, we may assume that G has no parallel edges. Draw C in the plane, say using a circle. Then draw the remaining edges, each time placing the edge inside the circle as a straight line segment. If any two such edges cross, then these edges plus C can be reduced to a K_4 minor of G, a contradiction. Thus, no edges cross, and we have produced an outerplanar drawing of G. For (iii) $\Longrightarrow$ (i), we note that K_4 and $K_{2,3}$ are not outerplanar and that outerplanarity is maintained under minor-taking. $\square$

For the second subclass of the class of SP graphs, we define a *suspended tree* to be any graph generated by the following process. Start with a tree. Create an additional vertex called the non-tree vertex. From that vertex, add one arc to each tip node of the tree, plus at most one arc to any other node of the tree. An example is given below. The initial tree is drawn with bold lines.

(4.2.10)



Suspended tree

We permit the degenerate case where the initial tree is just one node.

We have the following theorem, which turns out to be nothing but the dual version of Theorem (4.2.9).

(4.2.11) Theorem. *The following statements are equivalent for a 2-connected graph G with at least two edges.*

(i) *G has no K_4 or C_3^2 minors.*
(ii) *G is an SP graph without C_3^2 minors.*
(iii) *Except for parallel edges, G is 2-isomorphic to a suspended tree.*

Proof. We use graphic matroids and duality to link Theorem (4.2.9) and the one at hand. Specifically, for each of the statements (i)–(iii) of Theorem (4.2.9), we carry out the following process. From that statement, we deduce the matroid version, then dualize that matroid statement, and finally show that the latter statement, when expressed in terms of graphs, yields the statement of Theorem (4.2.11) with the same number. Then one reverses the sequence of arguments, going from each statement of Theorem (4.2.11) to the corresponding one of Theorem (4.2.9). The proof is complete once the following two observations are made. First, the just-described dualization process takes any K_4 (resp. $K_{2,3}$) minor of one graph to a K_4 (resp. C_3^2) minor of the other one. Second, the dualization process takes the property of being an SP graph (resp. of being outerplanar) to the property of being an SP graph (resp. of being 2-isomorphic to a suspended tree up to parallel edges), and vice versa. □

We relate the above material to binary matroids. We begin with the following construction. Start with the 1×1 binary matrix $B = [\,1\,]$. Iteratively enlarge that matrix by adjoining a binary column or row vector parallel to an existing column or row, or by adjoining a column or row unit vector. We call each such iteration a *matroid* SP *extension step*. Define the SP *matroids* to be the binary matroids represented by the matrices that can be so produced. The inverse of a matroid SP extension step is a *matroid* SP *reduction step*. The similarity of terminology with the graph case is no accident, as we shall see next.

(4.2.12) Lemma. *Every SP matroid is the graphic matroid of an SP graph, and vice versa.*

Proof. The key relationships are provided in Section 3.2. The matrix $B = [\,1\,]$ represents the graphic matroid of a cycle with two edges. Suppose after some iterations of the construction process, a matrix B is on hand. Let B represent the graphic matroid of an SP graph G. Then the adjoining of a column z parallel to a given column y of B can be translated to adding an edge z parallel to edge y in G. Similarly, any other extension of B can be translated in G to an addition of parallel edges or to a subdivision of edges into series edges. Thus, every matrix produced by matroid SP extension steps represents the graphic matroid of some SP graph. With similar ease, one proves that the graphic matroid of any SP graph is represented by some B generated by SP extension steps. □

With the aid of Lemma (4.2.12), we translate Theorem (4.2.6) to the following result. Recall that by Corollary (3.2.31), 2-connectedness in a graph G is equivalent to connectedness in the graphic matroid $M(G)$.

(4.2.13) Theorem. *A connected binary matroid is an SP matroid if and only if it has no $M(K_4)$ minor.*

Theorems (4.2.9) and (4.2.11) also have an interesting translation. Since these results are dual to each other, it suffices that we translate Theorem (4.2.11). To simplify matters, we rule out parallel elements. Observe that the matrices

$$(4.2.14) \qquad \boxed{\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}}\ ; \qquad \boxed{\begin{array}{cccc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}}$$

Representation matrices of $M(K_4)$ and $M(C_3^2)$

represent the graphic matroids $M(K_4)$ and $M(C_3^2)$, respectively. In the first case, the corresponding base of $M(K_4)$ is a star of K_4. In the second case, the corresponding base of $M(C_3^2)$ contains any two nonparallel edges of C_3^2.

(4.2.15) Theorem. *The following statements are equivalent for a connected binary matroid M with at least three elements, no two of which are parallel.*

(i) *M has no $M(K_4)$ or $M(C_3^2)$ minors.*
(ii) *M is an SP matroid without $M(C_3^2)$ minors.*
(iii) *No representation matrix of M has any one of the matrices of* (4.2.14) *as submatrix.*
(iv) *M is the graphic matroid of a suspended tree.*
(v) *M is a minor of the matroid M' represented by a binary matrix B' that is the node/edge incidence matrix of a tree.*

Proof. Equivalence of (i), (ii), (iii), and (iv) follows directly from Theorem (4.2.11). To show that (iv) implies (v), let M be the graphic matroid of a suspended tree G. Add edges, if necessary, so that every tree node is connected by exactly one arc with the extra node. The graphic matroid M' of that enlarged graph G' has M as a minor. The edges of G' incident at the extra node form a tree X. Thus, X is a base of M. It is easy to see that the representation matrix B' of M' corresponding to X is nothing but the node/edge incidence matrix of the tree. Hence, (v) holds. The arguments are easily reversed to prove that (v) implies (iv). □

For subsequent reference, we include the following lemma.

(4.2.16) Lemma. *Let M be a connected binary matroid. Then any binary matroid obtained by an SP extension step from M is connected.*

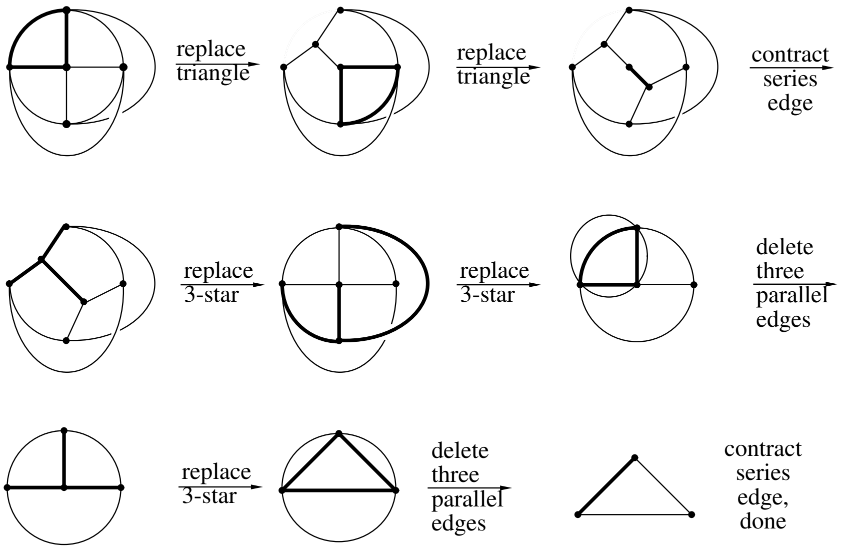
Proof. By Lemma (3.3.19), connectedness of a binary matroid is equivalent to connectedness of any one of its representation matrices. Clearly, any SP extension step maintains connectedness of representation matrices. Thus, the resulting matroid is connected. □

4.3 Delta-Wye Construction for Graphs

The simplicity of SP graphs gives way to far more complicated graphs when we permit two operations in addition to SP extensions. One of them is the replacement of a triangle by a 3-star, and the second one is the inverse of that step. Either operation we call a ΔY *exchange*. Define a sequence of SP extensions and ΔY exchanges to be a ΔY *extension sequence*. The inverse sequence is a ΔY *reduction sequence*. A 2-connected graph is ΔY *reducible* if there is a ΔY reduction sequence that converts the graph to a cycle with just two edges. In this section, we show that ΔY extension sequences applied to such a cycle create all 2-connected planar graphs and more. Any graph so producible is a ΔY *graph*.

As an example for ΔY reduction sequences, we reduce K_5, the complete graph on five vertices, to the cycle with two parallel edges.

(4.3.1)



ΔY reduction sequence for K_5

In each graph of the reduction sequence, the triangle or 3-star involved in a ΔY exchange is indicated by bold lines. Similarly, we emphasize the series or parallel edges of SP reductions. We know from Lemma (3.2.48) that $M(K_5)$ is not cographic. Thus, K_5 is nonplanar, and the example demonstrates that ΔY graphs may be nonplanar.

A ΔY exchange may not preserve 2-connectedness. For example, when a vertex of a triangle in a 2-connected graph has degree 2, then replacement of that triangle by a 3-star produces a 1-separable graph. The next lemma gives the conditions under which 2-connectedness is maintained.

(4.3.2) Lemma. *Let G be a 2-connected graph. Then a triangle to 3-star exchange (resp. 3-star to triangle exchange) in G produces a 2-connected graph G' if and only if the triangle (resp. 3-star) does not contain two edges in series (resp. in parallel).*

Proof. Consider the triangle to 3-star exchange. The "only if" part has been argued above. For proof of the "if" part, suppose a triangle does not contain two series edges. Equivalently, the triangle does not contain a cocycle. Thus, G has a tree that does not include any edges of the triangle. With the aid of this tree, it is easy to see that the graph G' derived from the 2-connected graph G has no 1-separation.

If a 3-star has two edges in parallel, then a 3-star to triangle exchange is not possible. Thus, the "only if" part is trivially satisfied. The "if" part is easily checked, analogously to the case of a triangle to 3-star exchange. □

The asymmetry of arguments in the proof of Lemma (4.3.2) is due to the fact that a triangle is always a cycle of a graph, while a 3-star is not always a cocycle. Note that the conditions of Lemma (4.3.2) are automatically satisfied in ΔY reduction sequences where ΔY exchanges are done only when an SP reduction is not possible.

Our goal is to show that the class of ΔY graphs includes all 2-connected planar graphs. That goal is restated in the next theorem.

(4.3.3) Theorem. *Every 2-connected planar graph is ΔY reducible.*

The proof relies on three lemmas. They show, in fact, that any 2-connected plane graph, i.e., an embedding of a planar graph in the plane, is ΔY reducible under the following restriction. We permit a triangle to 3-star exchange only if the triangle bounds a face.

The first auxiliary lemma is the analogue of Lemma (4.2.2).

(4.3.4) Lemma. *If a 2-connected graph or plane graph G is ΔY reducible, then every 2-connected minor H of G is ΔY reducible as well.*

Proof. We confine ourselves to the plane graph case. By omitting references to the embeddings of G, one obtains the proof for the general situation. We induct on the number of SP reductions and ΔY exchanges that reduce the planar graph G to the cycle with just two edges. We may suppose that H has no series or parallel edges. H is then a minor of G

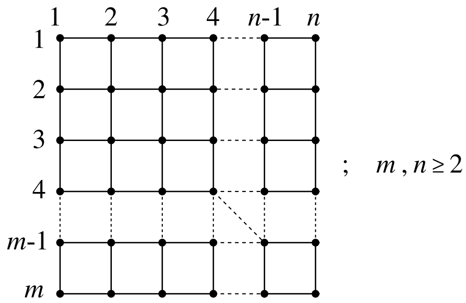
as well as of any graph derived from G by any SP reductions. Thus, by induction, we only need to examine the case where the first step of a given reduction sequence for G is a ΔY exchange.

Consider a triangle to 3-star exchange. By assumption, the triangle, say $\{e, f, g\}$, bounds a face. Let G' be the plane graph resulting from the exchange. Suppose e, f, and g occur in H. Since H is 2-connected, these edges must form a triangle in H that also bounds a face of H. By assumption, H has no series or parallel edges, so we may replace the triangle of H by a 3-star, getting a 2-connected graph H'. Then H' is a 2-connected minor of G', and the conclusion follows by induction. Suppose at least one of the edges of the triangle $\{e, f, g\}$ does not occur in H. It is easily seen that we may delete e, f, or g from G while retaining H as a minor. Regardless of the specific situation, H is isomorphic to a minor of G', and once more induction can be invoked.

The case of a 3-star to triangle exchange is handled analogously. Indeed, in the plane graph case, we may invoke duality. □

The next two lemmas involve *grid graphs*, which are plane graphs of the form

(4.3.5)



Grid graph

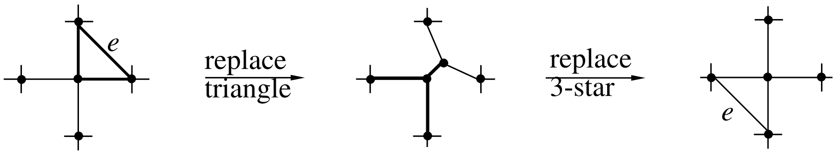
(4.3.6) Lemma. *Every plane graph is a minor of some grid graph.*

Proof. (Sketch) We may assume that the given plane graph is 2-connected, since this can be achieved by the addition of edges. Split each vertex of that plane graph so that a 2-connected plane graph G results where the degree of each vertex is at most 3. By a suitable subdivision of edges, G can be embedded into a grid graph as follows. First embed any one face of G, but not the outer one. Then embed one face at a time so that each one of the successive subgraphs of G so embedded is 2-connected. □

(4.3.7) Lemma. *Every grid graph is ΔY reducible.*

Proof. Two special ΔY reduction subsequences will be used repeatedly. For the first case, suppose we have a grid graph plus one edge e so that e and two edges of a degree 4 vertex of the grid graph form a triangle.

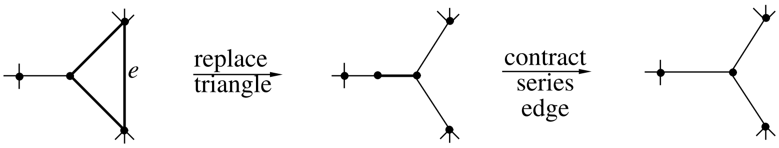
Consider the two ΔY exchanges depicted in (4.3.8) below.

(4.3.8) 

Moving edge e toward lower left-hand corner

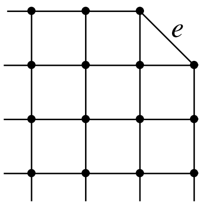
Effectively, the two ΔY exchanges have moved the edge e one step closer toward the lower left-hand corner of the grid graph.

The second situation is even simpler. Suppose an edge e forms a triangle with two edges of a 3-star. Then that edge can be effectively eliminated via one ΔY exchange plus a series reduction as follows.

(4.3.9) 

Removal of edge e

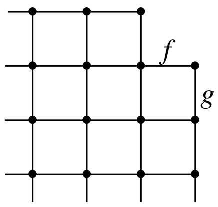
We are ready to describe a ΔY reduction sequence for grid graphs. By (4.3.5), the graph is obviously an SP graph if m or $n = 2$. Hence, suppose $m, n \geq 3$. First we reduce the two pairs of series edges in the upper right corner and lower left corner of the grid graph to one edge each. Thus, the upper right-hand corner has become

(4.3.10) 

Upper right-hand corner after series reduction

By repeated application of (4.3.8), we move the edge e toward the left or bottom boundary of the graph. When that boundary is reached, e can be eliminated either by (4.3.9) or by the fact that it has become parallel to the lower left edge produced in one of the two initial series reductions.

We now have the upper right-hand portion as

(4.3.11) 

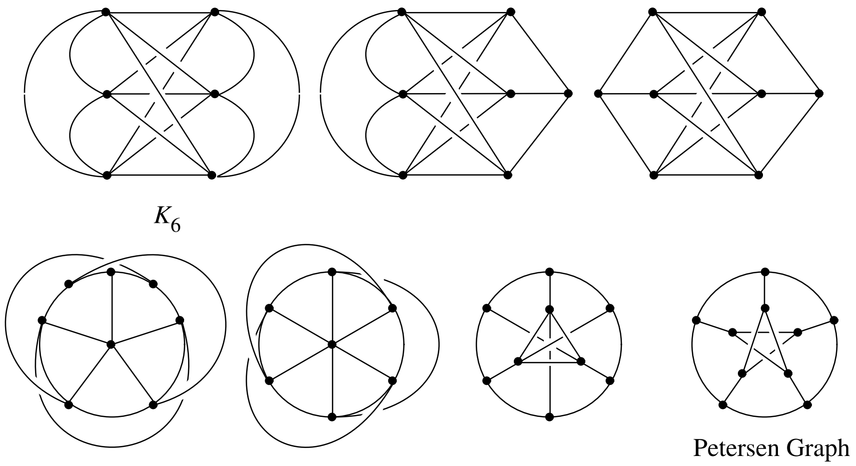
Upper right-hand corner after removal of e

The edge g is in series with f and can be contracted. Then f can be eliminated analogously to the edge e. By repetition of this process and suitable adjustment in the last iteration, we effectively eliminate all nodes of the right-hand boundary of the grid graph. By induction, the lemma follows. □

Proof of Theorem (4.3.3). A given plane graph is by Lemma (4.3.6) a minor of some grid graph. By Lemmas (4.3.4) and (4.3.7), both graphs are ΔY reducible. □

By Lemma (4.3.4), the class of ΔY graphs is closed under the taking of 2-connected minors. Thus, one is tempted to look for a characterization of the ΔY graphs by exclusion of minimal minors that are 2-connected and not ΔY reducible. As a first step toward finding these minors, we introduce the following equivalence relation on the class of 2-connected graphs that are not ΔY reducible and that are minimal with respect to that property. Two such graphs are defined to be related if one can be obtained from the other one by a sequence of ΔY exchanges. The above characterization problem is solved once one finds one member of each equivalence class. It is easy to see that the graph K_6 is one of the desired minimal graphs. By straightforward enumeration, the equivalence class represented by K_6 can be shown to consist of the following graphs.

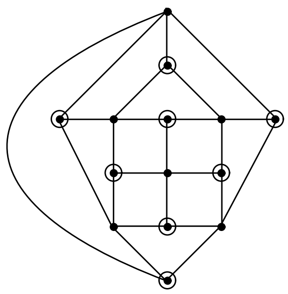
(4.3.12)



Equivalence class of minimal nonreducible graphs represented by K_6

One might conjecture, and would not be the first person to do so, that the graphs of (4.3.12) constitute all minimal nonreducible graphs. The conjecture is appealing but false. A counterexample due to Robertson is the graph G constructed as follows. We start with the planar graph of (4.3.13) below. We add to that graph one vertex v, which is connected to the circled nodes of the planar graph by one edge each. The graph G is evidently 3-connected and has no triangles or 3-stars. Thus, it is not ΔY reducible. We claim that G does not have any one of the graphs of (4.3.12) as a minor. By the construction, G can be reduced to a planar graph by removal of the vertex v. But none of the graphs of (4.3.12) becomes planar when any one its vertices is removed, as is easily verified. Thus, G is a counterexample to the conjecture.

(4.3.13)



Planar graph for counterexample G

In the next section, we adapt the concept of ΔY extension and reduction sequences to binary matroids.

4.4 Delta-Wye Construction for Binary Matroids

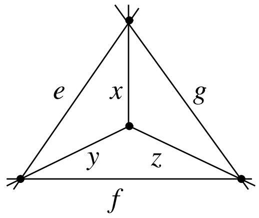
In this section, we translate the definitions and conditions for ΔY graphs given in Section 4.3 to statements about binary matroids. Thus, we obtain ΔY extension steps, ΔY reduction steps, ΔY matroids, etc. Recall from Lemma (4.2.12) that every SP matroid is the graphic matroid of an SP graph, and vice versa. In contrast, the yet-to-be-defined ΔY matroids turn out to be not just the graphic matroids of ΔY graphs.

We start with the definitions. From Section 4.2, we already know how SP extensions are performed in binary matroids. So let us translate the ΔY exchange from graphs to binary matroids. To gain some intuitive insight, we perform a triangle to 3-star step in a 2-connected graph G with edge set E, then translate that step into matroid language for the graphic matroid $M = M(G)$ of G. Let the triangle of G be $C = \{e, f, g\}$. As in Section 4.3, we assume that C does not contain a cocycle.

To carry out the triangle to 3-star exchange in G, we first add the 3-star, say $C^* = \{x, y, z\}$, getting a graph G'. A partial drawing of G' with

C and C^* is shown below.

(4.4.1)



Triangle and 3-star

Then we delete the triangle C, getting the desired graph G''. Since C contains no cocycle, the graph G has a tree X and a cotree $Y = E - X$ so that $C \subseteq Y$. The representation matrix B of M for this X is assumed to be

(4.4.2)

$$B = \begin{array}{c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{\longleftarrow Y \longrightarrow} \\ \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{e} & \multicolumn{1}{c}{f} & \multicolumn{1}{c}{g} \\ \cline{2-5} X & \overline{B} & a & b & c \\ \cline{2-5} \end{array}$$

Matrix B for $M = M(G)$

Clearly, $X \cup \{y\}$ is a tree of the graph G'. We claim that the representation matrix B' for $M' = M(G')$ corresponding to that tree must be

(4.4.3)

$$B' = \begin{array}{c|c|c|c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{4}{c}{\longleftarrow Y \longrightarrow} & \multicolumn{2}{c}{} \\ \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{e} & \multicolumn{1}{c}{f} & \multicolumn{1}{c}{g} & \multicolumn{1}{c}{x} & \multicolumn{1}{c}{z} \\ \cline{2-7} X & \overline{B} & a & b & c & a & b \\ \cline{2-7} y & \multicolumn{4}{c|}{0} & 1 & 1 \\ \cline{2-7} \end{array}$$

Matrix B' for $M' = M(G')$

The proof of this claim is as follows. First, the edges x and z of G' produce the only fundamental circuits with $X \cup \{y\}$ that contain y. This fact justifies the last row of B'. Second, contraction of y in G' makes the edge x (resp. z) parallel to e (resp. f). Correspondingly, upon deletion of row z from B', the columns x and e (resp. z and f) must be parallel. These facts uniquely determine B' as shown in (4.4.3).

Deletion of the columns e, f, and g from B' yields the desired representation matrix B'' for $M'' = M(G'')$, i.e.,

(4.4.4)

$$B'' = \begin{array}{c|c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\widetilde{Y}} & \multicolumn{1}{c}{x} & \multicolumn{1}{c}{z} \\ \cline{2-4} X & \overline{B} & a & b \\ \cline{2-4} y & 0 & 1 & 1 \\ \cline{2-4} \end{array} \quad ; \qquad \widetilde{Y} = Y - \{e, f, g\}$$

Matrix B'' for $M'' = M(G'')$

Instead of $X \cup \{y\}$, we could also have chosen $X \cup \{x\}$ or $X \cup \{z\}$ as tree of G'. Correspondingly, the column vectors a and b explicitly shown in (4.4.4) would have been a and c, or c and b. Each one of the latter matrices may also be obtained by a GF(2)-pivot in row y of B''.

The above translation of a triangle to 3-star exchange in G to a triangle to triad exchange in M may be rephrased as follows. The latter exchange in M is allowed only if the triangle does not contain a cocyle. In the exchange, we first find a representation matrix B where the triangle elements are nonbasic. Next we delete one of the columns corresponding to the triangle. Then we add a row that has 1s in the remaining two columns of the triangle, and 0s elsewhere. Finally, we re-index the two columns formerly indexed by triangle elements. The diagram below summarizes this particular ΔY exchange process. For clarity, the diagram omits indices other than e, f, g and x, y, z.

(4.4.5)
$$\begin{array}{c|c|c|c|} & e & f & g \\ \hline \overline{B} & a & b & c \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{c|c|c|c|} & & x & z \\ \hline & \overline{B} & a & b \\ \hline y & 0 & 1 & 1 \\ \hline \end{array}$$

ΔY exchange, case 1

Note that $a + b + c = 0$ (in GF(2)). There are other ways to display the triangle to triad exchange. Specifically, when we select a B with exactly one element of the triangle, say f, basic, we get a case of the form

(4.4.6)
$$\begin{array}{c|c|c|c|} & & e & g \\ \hline & \overline{B} & b & b \\ \hline f & a & 1 & 0 \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{c|c|c|} & & x \\ \hline & \overline{B} & b \\ \hline z & a & 1 \\ \hline y & a & 0 \\ \hline \end{array}$$

ΔY exchange, case 2

When we select a B with two elements of the triangle basic, say e and f, we get the third possible case

(4.4.7)
$$\begin{array}{c|c|c|} & & g \\ \hline & \overline{B} & 0 \\ \hline e & a & 1 \\ \hline f & b & 1 \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{array}{c|c|} \hline & \overline{B} \\ \hline x & a \\ \hline z & b \\ \hline y & c \\ \hline \end{array}$$

ΔY exchange, case 3

where $a+b+c = 0$ (in GF(2)). Simple checking, analogous to that proving (4.4.5), confirms these claims. We emphasize that in each situation, the symbols $\overline{B}$, a, b, and c refer to different matrices and vectors. However, the relationships between the triangle elements e, f, and g and the triad elements x, y, and z are displayed in agreement with (4.4.1).

The definition of a triangle to triad exchange of a binary but not necessarily graphic matroid M is as follows. We require that the triangle, say $\{e, f, g\}$, does not contain a cocycle. Let B be any representation matrix of M. Then B is one of the left-hand matrices of (4.4.5), (4.4.6), or (4.4.7). The matroid resulting from the exchange is given by the corresponding right-hand matrix. Suppose we select B of (4.4.7) to carry out the triangle to triad exchange, as we always may. Then the cocycle condition on the triangle is equivalent to the requirement that the row vectors a and b of B be nonzero and distinct. Hence, the row vectors a, b, and $c = a + b$ (in GF(2)) in the resulting matrix of (4.4.7) are distinct, and $\{x, y, z\}$ is indeed a triad in the corresponding matroid. The triad trivially does not contain a cycle since the resulting matrix indicates x, y, and z to be part of a basis.

For the definition of a triad to triangle exchange, we invert the above process. We demand that the triad not contain a cycle. The exchange is specified by (4.4.5), (4.4.6), and (4.4.7) when we start with the matrix on the right hand side and derive the one on the left hand side. By taking transposes of the matrices involved in (4.4.5), (4.4.6), and (4.4.7), we see that a triangle to triad exchange in M is precisely a triad to triangle exchange in M^*. Thus, these two operations are dual to each other.

We define ΔY *extension sequence*, ΔY *reduction sequence*, and ΔY *reducibility* for binary matroids by the obvious adaptation of the same terms for graphs. Later, we need the following lemma about ΔY exchanges.

(4.4.8) Lemma. *Let M be a connected binary matroid. Then any ΔY exchange in M produces a connected binary matroid.*

Proof. By duality, we only need to consider the triangle to triad exchange depicted in (4.4.5). By Lemma (3.3.19), M is connected if and only if any representation matrix B of M is connected. The left-hand matrix of (4.4.5) is thus connected. Since the vectors a, b, c of that matrix are nonzero and $a + b + c = 0$ (in GF(2)), the right-hand matrix is easily verified to be connected as well. Thus, the related matroid is connected. □

Suppose we start with the connected, binary, and graphic matroid having just two parallel elements, with representation matrix $B = [\,1\,]$. Let the ΔY *matroids* be the binary matroids that may be produced from that matroid by ΔY extension sequences. Due to Lemmas (4.2.16) and (4.4.8), SP extensions and ΔY exchanges maintain connectedness. Thus, the ΔY matroids are connected. By definition, the smallest ΔY matroid is graphic. Furthermore, all graphic matroids of ΔY graphs are ΔY matroids, as are the duals of these matroids. By (4.3.1), the nonplanar graph K_5 is ΔY reducible, as is, evidently, $K_{3,3}$. Thus, the graphic matroids $M(K_5)$ and

$M(K_{3,3})$ are ΔY-reducible, as well as $M(K_5)^*$ and $M(K_{3,3})^*$. By Lemma (3.2.48), the latter matroids are not graphic. Thus, ΔY matroids need not be graphic. Indeed, there are ΔY matroids that are not graphic and not cographic. For example, the nongraphic and noncographic matroid R_{12} introduced later in Chapter 10 via the representation matrix

(4.4.9)
$$B^{12} = \begin{array}{|cccccc|} \hline 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

Matrix B^{12} for R_{12}

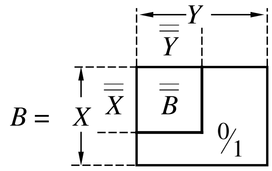
is a ΔY matroid. It would take up too much space to describe the SP reduction and ΔY exchange steps that prove this claim. But with the machinery introduced in Chapter 10, we have deduced $B = [\,1\,]$ from B^{12} of (4.4.9) using eight ΔY exchange steps and ten SP reduction steps.

Analogously to Lemma (4.3.4), we now show that the class of ΔY-matroids is closed under the taking of connected minors. The proof mimics that of Lemma (4.3.4).

(4.4.10) Lemma. *If a connected binary matroid M is ΔY reducible, then every connected minor N of M is ΔY reducible as well.*

Proof. Select a representation matrix B for M that displays the minor N by a submatrix $\overline{\overline{B}}$. Thus,

(4.4.11)



Matrix B for M displaying minor N by $\overline{\overline{B}}$

We induct on the number of SP reductions and ΔY exchanges that reduce M to the matroid with just two parallel elements. We may assume that N has no series or parallel elements. N is then a minor of M as well as of any minor derived from M by SP reductions. Thus, by induction and duality, we only need to examine the case where the first step of a given reduction sequence for M is a triangle to triad exchange. Let the triangle of M be $\{e,f,g\}$.

Three cases are possible, depending on the number of elements of $\{e,f,g\}$ that index rows of B. In the first case, that number is 0. Thus, $e,f,g \in Y$. Suppose at most two of the columns e, f, g of B intersect

$\overline{\overline{B}}$. Then according to (4.4.5), we drop from B one column that does not intersect $\overline{\overline{B}}$, and add a row with two 1s. The matrix $\overline{\overline{B}}$ is still a submatrix of the matrix so derived from B, and we are done by induction. Suppose all three columns e, f, g of B intersect $\overline{\overline{B}}$. Then the change of B given by (4.4.5) may also be viewed as a triangle to triad change involving $\overline{\overline{B}}$, provided we can prove that $\{e, f, g\}$ is a triangle of N that does not contain a cocycle. But the assumption that N is connected and has no series or parallel elements implies these two conditions, and once more we are done by induction. With similar ease one argues the case where one or two elements of $\{e, f, g\}$ index rows of B. □

In Chapter 11, it is shown that the class of ΔY matroids is a subset of the class of regular matroids, which we define in a moment. But otherwise that class is not well understood. Consider a slightly more complicated situation. This time, a class of binary matroids is produced by ΔY extension steps from a given connected binary matroid, or possibly from several such matroids. In the remainder of this section, we cover an interesting instance.

We need a few definitions concerning regular and almost regular matroids. These matroids are treated in depth in Chapters 9–12. We confine ourselves here to a rather terse introduction. A real $\{0, \pm 1\}$ matrix is *totally unimodular* if every square submatrix has real determinant equal to 0 or ± 1. A binary matrix is *regular* if its 1s can be signed to become ± 1s so that the resulting real matrix is totally unimodular. A binary matroid is *regular* if M has a regular representation matrix. It is not so difficult to see, and is also proved in Chapter 9, that for a given binary matroid either none or all representation matrices are regular.

A binary matroid M is *almost regular* if it is not regular, and if for any element z of the matroid, at least one of the minors M/z and $M \backslash z$ is regular. In addition, we demand the existence of a label for each element z so that we can identify at least one of the minors M/z or $M \backslash z$ as regular. The label must be "*con*" or "*del*." If the label for element z is "*con*" (resp. "*del*"), then M/z (resp. $M \backslash z$) must be regular. We still have a rather technical condition that must be satisfied by the labels. They must be so chosen that every circuit (resp. cocircuit) of M has an even number of elements with "*con*" (resp. "*del*") labels. Finally, there must be at least one "*con*" element and at least one "*del*" element.

The reader is likely to be puzzled by the strange parity condition and the existence condition. In Chapter 12, it is shown how they come about, and why one might want to define and investigate almost regular matroids. So for the time being, we suggest that the reader simply accept or at least tolerate these seemingly strange requirements.

One way to construct some almost regular matroids is as follows. We take any square $\{0, \pm 1\}$ real matrix A that is not totally unimodular but all of whose proper submatrices do have that property. Call any such matrix *minimal non-totally unimodular.* There is a rather simple subclass of these matrices where every row and every column has exactly two ± 1s. We assume that A is not of this variety. Let B be the support matrix of A. View B as the binary representation matrix of a binary matroid M. To the elements of M corresponding to the rows (resp. columns) of B, assign "*con*" (resp. "*del*") labels. Then M is an almost regular matroid, a fact proved in Chapter 12. An example is the minimal non-totally unimodular matrix

(4.4.12)
$$A = \begin{array}{|cccc|} \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline \end{array}$$

Minimal non-totally unimodular matrix

The binary support matrix B of A represents an almost regular matroid when labels are assigned as described above. The class of minimal non-totally unimodular matrices is quite rich. Thus, the class of almost regular matroids is interesting as well.

Here are the representation matrices of two almost regular matroids that cannot be produced by the above process. Instead of row and column indices, we record for each row and column the label of the corresponding element of the matroid.

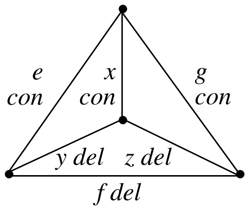
(4.4.13)
$$B^7 = \begin{array}{c|cccc|} & del & con & con & con \\ \hline con & 0 & 1 & 1 & 1 \\ del & 1 & 1 & 0 & 1 \\ del & 1 & 0 & 1 & 1 \\ \hline \end{array} \; ; \qquad B^{11} = \begin{array}{c|cccccc|} & del & con & con & con & con & con \\ \hline con & 0 & 1 & 1 & 1 & 1 & 1 \\ del & 1 & 1 & 0 & 0 & 1 & 1 \\ del & 1 & 1 & 1 & 0 & 0 & 1 \\ del & 1 & 0 & 1 & 1 & 0 & 1 \\ del & 1 & 0 & 0 & 1 & 1 & 1 \\ \hline \end{array}$$

Labeled Matrices B^7 and B^{11}

The matrix B^7 should be familiar. It is, up to column/row permutations and labels, the representation matrix of (3.3.22) of the nonregular Fano matroid. The reader can easily check that the Fano matroid with the given labels does satisfy the above-mentioned parity and existence conditions on labels. The origin of B^{11} is explained in Chapter 12. At any rate, verification of the conditions on labels for B^{11} requires a moderate computational effort.

Due to the labels, we want to restrict SP extension steps and ΔY exchanges a bit. Specifically, we allow a parallel (resp. series) extension of M only if the involved element z of M has a "*con*" (resp. "*del*") label. The new element receives the same label as z. A triangle to triad exchange is permitted only if the triangle, say $\{e, f, g\}$, has exactly two "*con*" elements. The labels of the resulting triad, say $\{x, y, z\}$, can be deduced from the drawing below, which should be interpreted the same way the drawing of (4.4.1) is linked to (4.4.5), (4.4.6), and (4.4.7).

(4.4.14)



Triangle and 3-star with labels

Note that the resulting triad has two "*del*" labels, as demanded by the parity condition. A triad to triangle exchange is just the reverse of the above step. It is permitted only if the triad has exactly two "*del*" labels. A *restricted* ΔY *extension sequence* is a sequence of restricted SP extensions and of restricted ΔY exchanges. It is not difficult to prove that restricted ΔY extension sequences convert almost regular matroids to matroids with the same property. We show this in Chapter 12.

We include a short restricted ΔY extension sequence that starts with B^7 of (4.4.13), and that generates, with appropriate labels, the matrix B deduced earlier from the minimal non-totally unimodular A of (4.4.12).

(4.4.15)

$$
\begin{array}{cc|cccc|}
 & & d & e & f & g \\
 & & del & con & con & con \\
\hline
a & con & 0 & 1 & 1 & 1 \\
b & del & 1 & 1 & 0 & 1 \\
c & del & 1 & 0 & 1 & 1 \\
\hline
\end{array}
\xrightarrow[\text{to } a]{\text{add } h \text{ parallel}}
\begin{array}{cc|ccccc|}
 & & d & e & f & g & h \\
 & & del & con & con & con & con \\
\hline
a & con & 0 & 1 & 1 & 1 & 1 \\
b & del & 1 & 1 & 0 & 1 & 0 \\
c & del & 1 & 0 & 1 & 1 & 0 \\
\hline
\end{array}
\xrightarrow[\{d,g,h\}]{\text{replace triangle}}
$$

$$
\begin{array}{cc|cccc|}
 & & e & f & y & z \\
 & & con & con & del & del \\
\hline
a & con & 1 & 1 & 1 & 1 \\
b & del & 1 & 0 & 1 & 0 \\
c & del & 0 & 1 & 1 & 0 \\
x & con & 0 & 0 & 1 & 1 \\
\hline
\end{array}
\xrightarrow[\text{circled 1s}]{\text{GF(2)-pivot on two}}
\begin{array}{cc|cccc|}
 & & b & c & y & z \\
 & & del & del & del & del \\
\hline
a & con & 1 & 1 & 1 & 1 \\
e & con & 1 & 0 & 1 & 0 \\
f & con & 0 & 1 & 1 & 0 \\
x & con & 0 & 0 & 1 & 1 \\
\hline
\end{array}
$$

Example of restricted ΔY extension sequence

The next theorem states a rather surprising fact about the power of restricted ΔY extension sequences.

(4.4.16) Theorem. *The class of almost regular matroids has a partition into two subclasses. One of the subclasses consists of the almost regular* *matroids producible by ΔY extension sequences from the matroid represented by B^7 of* (4.4.13). *The other subclass is analogously generated by B^{11} of* (4.4.13). *There is a polynomial algorithm that obtains an appropriate ΔY extension sequence for creating any almost regular matroid from the matroid of B^7 or B^{11}, whichever applies.*

Unfortunately, the existing proof of Theorem (4.4.16) is so long that we cannot include it here. In this book, it is one of the few results about binary matroids that we state but do not prove. We outline a proof in Chapter 12, when we restate Theorem (4.4.16) as Theorem (12.4.8). At that time, we use the theorem to establish several matrix constructions.

In the final section, we cover applications, extensions, and references.

4.5 Applications, Extensions, and References

The series-parallel construction is a basic idea of electrical network theory. The characterization of SP graphs in Theorem (4.2.6) in terms of excluded K_4 minors is given in Dirac (1952). Another proof and basic results about SP graphs are provided in Duffin (1965). Theorem (4.2.9) is due to Chartrand and Harary (1967). The dual of that result, Theorem (4.2.11), is proved directly in Truemper and Soun (1979), and Soun and Truemper (1980). Decomposition results for SP graphs and outerplanar graphs are described in Wagner (1987).

One may attempt to generalize SP matroids by dropping the restriction that the matroids be binary. One still starts with the matroid having just two parallel elements. The SP extensions are defined via addition of parallel elements and expansion by series elements. These steps can be nicely displayed by abstract matrices. With that approach, one very easily proves that the supposedly more general procedure generates nothing but the graphic matroids of SP graphs. By Theorems (3.5.2) and (4.2.13), a connected matroid with at least two elements is thus an SP matroid if and only if it has no U_4^2 or $M(K_4)$ minors. Additional material on combinatorial aspects of series-parallel networks may be found in Brylawski (1971).

Akers (1960) and Lehman (1963) contain the following conjecture. Let G be a 2-connected graph that is a plane graph plus one edge called the *return edge*. Then G is conjectured to be ΔY reducible to a cycle with just two edges, one of which must be the return edge. Note that the return edge is not allowed to participate in any ΔY exchange. The conjecture was first proved in Epifanov (1966) by fairly complicated arguments. A simple proof is given in Truemper (1989a), which also contains the proof of Theorem (4.3.3) given here. An interesting but not simple proof of

Theorem (4.3.3) is provided in Grünbaum (1967). That reference relies on Theorem (4.3.3) to derive a very elementary proof of Steinitz's Theorem linking 3-connected graphs and 3-dimensional polytopes. Computational aspects of ΔY graphs are treated in Feo (1985), and Feo and Provan (1988). Gitler (1991) characterizes ΔY reducible planar graphs where k specified nodes, $k \geq 3$, may not be removed. That is, none of the k nodes may be eliminated by Y-to-Δ exchanges or series reductions. The return edge case discussed above is equivalent to the situation with $k = 2$.

The counterexample graph G defined via the planar graph of (4.3.13) is due to Robertson (1988). The class of ΔY graphs may be specialized by demanding that all ΔY exchanges are either Δ-to-Y exchanges or Y-to-Δ exchanges. The ΔY graphs so created, we call Δ-to-Y graphs and Y-to-Δ graphs. The two classes are completely characterized in Politof (1988a), (1988b).

The special structure of SP graphs, ΔY graphs, Δ-to-Y graphs, and Y-to-Δ graphs has been exploited for numerous applications, in particular for combinatorial optimization and reliability problems. Relevant references are Moore and Shannon (1956), Akers (1960), Lehman (1963), Nishizeki and Saito (1975), (1978), Rosenthal and Frisque (1977), Monma and Sidney (1979), Farley (1981), Farley and Proskurowski (1982), Takamizawa, Nishizeki, and Saito (1982), Wald and Colbourn (1983a), (1983b), Agrawal and Satyanarayana (1984), (1985), Satyanarayana and Wood (1985), Arnborg and Proskurowski (1986), Politof and Satyanarayana (1986), (1990), Colbourn (1987), and El-Mallah and Colbourn (1990).

The class of almost regular matroids is defined in Truemper (1992a). Theorem (4.4.16) is taken from that reference.